# A Poor Man's Guide to Lie Theory

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The goal of this guide is to give a feel about how Lie groups, Lie algebras, and their representations interact as quickly as possible. It is not meant to be very precise.

## I. LIE GROUPS AND LIE ALGEBRAS

The relation between Lie groups and Lie algebras is encapsulated in this statement: *the categories of connected, simply-connected Lie groups and finite-dimensional Lie algebras are the same.* (5,4,2) It cannot be the category of all Lie groups because Lie algebras cannot know the global topology of Lie groups. So we often study connected, simply-connected covering groups instead of the groups that are of interest to us.

1. A **Lie group** $G$ is a group that is also a smooth manifold. Roughly speaking, a **manifold** $M$ is a space that is locally Euclidean i.e. the chart $\varphi$ mapping an open set $U$ to a subset of $\mathbb{R}^n$ gives a point $p \in M$ coordinates in $\mathbb{R}^n$. The **dimension** of a Lie group is $n$, the dimension of the manifold.

2. A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $k$, equipped with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the **Lie bracket**, which is skew-symmetric $[x, y] = -[y, x]$ (or equivalently, $[x, x] = 0$) and satisfies the Jacobi identity

   $$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

   To understand the Jacobi identity, let us look at the **adjoint map** that takes an element $x \in \mathfrak{g}$ and turns it into an endomorphism (homomorphism to $\mathfrak{g}$ itself) of $\mathfrak{g}$.

   $$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathrm{End}\,\mathfrak{g} \\ \mathrm{ad}x\,(y) &= [x, y]\,. \end{aligned}$$

   One can check that it really is a homomorphism:

   $$\mathrm{ad}\,([x, y]) = [\mathrm{ad}x, \mathrm{ad}y]\,.$$

   So it is a representation of the Lie algebra called an **adjoint representation** of $\mathfrak{g}$. It will play an important role in the structure theory of semisimple Lie algebras.

   Fixing the element $x$, the Jacobi idensity precisely means that $\mathrm{ad}x$ is a **derivation** (obeying Leibniz rule):

   $$\mathrm{ad}x\,([y, z]) = [\mathrm{ad}x\,(y)\,, z] + [y, \mathrm{ad}x\,(z)]\,.$$

   This ties in with another definition of a Lie algebra of a Lie group as the tangent space at the identity $T_eG = \left\{X | e^{tX} \in G, t \in \mathbb{R}\right\}$ of $G$, which is isomorphic to the set of partial differential operators at the identity. Thus, the **dimension** of the Lie group and that of its Lie algebra are the same.

   Every Lie group has a Lie algebra. The best converse of this statement is **Lie's third theorem**: any abstractly defined Lie algebra has a corresponding connected, simply-connected Lie group.

3. The standard mathematical definition of the Lie algebra of a Lie group $G$ is the set of left-invariant vector fields on $G$. The set of all vector fields can be given the structure of a Lie algebra by the Lie bracket, but only left-invariant vector fields are determined by their values at the identity, hence the definition in 2. We prove this statement here.

   To prepare ourselves, think of a group element $g \in G$ as an automorphism of the algebra $C^\infty(M)$ of smooth functions on a manifold $M$ that translates functions:

   $$(gf)\,(p) = f\left(g^{-1}p\right).$$

   This induces an action on vector fields. Define the **Adjoint map** (with a capital "A") as ($X$ is a vector field)

   $$\mathrm{Ad}_g X = gXg^{-1}.$$

The value of $\mathrm{Ad}_g X$ at point $q = gp$ can be computed at point $p$.

$$\left(\mathrm{Ad}_g X\right)_q = dg_p\left(X_p\right).$$

Let us unpack this little formula. $g : M \to M$ is an automorphism of $M$. $dg_p : T_p M \to T_{gp} M = T_q M$ is the differential of the map $g$ at point $p$. So both the right hand side $dg_p\left(X_p\right)$ and the left hand side are in the tangent space at $q$.

Now we can understand what left-invariant vector fields mean. Take $M = G$ the Lie group itself. Let $L_g$ be the left action on the group itself: $L_g(h) = gh$. A vector field is **left-invariant** if $L_g X = X L_g$ i.e. $\mathrm{Ad}_g X = X$. As promised, the left-invariant vector field at any point $g \in G$ can be seen to be determined by its value at the origin $X_e$:

$$X_g = \left(\mathrm{Ad}_g X\right)_g = dg_e\left(X_e\right)$$

4. The Adjoint map $\mathrm{Ad}_g : G \to \mathrm{End}\mathfrak{g}$ (3) is a homomorphism

$$\mathrm{Ad}_g \mathrm{Ad}_h = \mathrm{Ad}_{gh}.$$

So it is a representation of the group called the **Adjoint representation** of $G$. Its differential is the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\mathfrak{g}$ from 2. They are related by

$$\mathrm{Ad}_{e^x} = e^{\mathrm{ad}x}.$$

More concretely, suppose that $g = e^{tx} \in G$ and $y \in \mathfrak{g}$.

$$\left.\frac{d}{dt}\mathrm{Ad}_g(y)\right|_{t=0} = \left.\frac{d}{dt}\left(e^{tx} y e^{-tx}\right)\right|_{t=0} = [x, y] = \mathrm{ad}x(y)$$

This is a special case of the relation between any Lie group and Lie algebra homomorphism

$$\Pi\left(e^x\right) = e^{d\Pi(x)},$$

which is the manifestation of **Lie's second theorem**: $\mathrm{Hom}\left(G, G'\right) = \mathrm{Hom}\left(\mathfrak{g}, \mathfrak{g}'\right)$ if $G$ is simply connected. The corresponding statement for representations $\Pi$ is that $\Pi \to d\Pi$ gives the equivalence of the categories of representations of $G$ and representations of $\mathfrak{g}$. Moreover, the vector spaces of intertwining operators (morphisms of representations) are isomorphic: $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathfrak{g}}(V, W)$.

5. A **Lie subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A Lie subalgebra of $\mathfrak{g}$ corresponds to a subgroup of $G$ which is also an immersed submanifold, called a **Lie subgroup** of $G$. A map $F : M \to N$ is a **smooth immersion** if its differential $dF$ is injective at every point: $\mathrm{rank}F = \dim M$. (To remember the names, an **i**mmersion is **i**njective, whereas a **s**ubmersion is **s**urjective.) An **immersed submanifold** $S$ of $M$ is a topological manifold (not necessary having the topology of $M$) together with the inclusion map $S \to M$ which is a smooth immersion. **Lie's first theorem** identifies every Lie subalgebra with a connected Lie subgroup and vice versa.

6. Just be aware that there is no standard definition of a Lie subgroup. Another notion of a submanifold is that of an **embedded submanifold**, which can be obtained for instance by setting some coordinates to zero. The key difference is that embedding has to be a homeomorphism preserving the global topology. Two classic examples of immersions that do not preserve the global topology are the "figure-eight" map and the irrational winding of a torus. [3] The former maps an open interval in $\mathbb{R}$ to a closed set in $\mathbb{R}^2$. The latter takes a line with an irrational slope in $\mathbb{R}^2$ and map it to a torus. The line will densely wind around the entire torus. We will call a subgroup which is an embedded submanifold an **embedded Lie subgroup**. Fulton and Harris [4], for example, give the opposite definitions; their Lie subgroups are our embedded Lie subgroup, whereas our Lie subgroups are their immerse Lie subgroups.

   By the **closed subgroup theorem**, a subgroup is an embedded Lie subgroup if and only if it is a closed set. Given a closed subgroup $H \subset G$, this makes $G/H$ a **homogeneous space**, a smooth manifold with a transitive group action.

## II. COMPLEX SEMISIMPLE LIE ALGEBRAS AND THEIR REPRESENTATIONS

**From now on, a Lie algebra is over the complex field $k = \mathbb{C}$ unless stated otherwise**. A Lie algebra is semisimple if and only if it has a root decomposition. (7) The tool to study semisimple Lie algebras and their representations is a generalization of the theory of angular momentum in quantum mechanics (11) familiar to every physicist.

1. Glossary of notions borrowed from group and ring theories:

   An **ideal** $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace such that $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. An ideal is automatically a Lie subalgebra. An ideal of $\mathfrak{g}$ corresponds to a **normal subgroup** $H$ of $G$: $gH = Hg$ for all $g \in G$.

   The **idealizer** of a set $S \in \mathfrak{g}$ consists of all elements of $\mathfrak{g}$ that preserve $S$ through the commutator: $I(\mathfrak{h}) = \{x | \mathrm{ad}x(S) \subset S\}$. Sometimes the word "normalizer" is used instead.

   The **centralizer** of a set $S \in \mathfrak{g}$ consists of all elements of $\mathfrak{g}$ that commute with all elements of $S$: $C(S) = \{x | \mathrm{ad}x(S) = 0\}$. The **center** is the centralizer of the whole Lie algebra. A commutative Lie algebra is called **abelian**.

   Both idealizers and centralizers are subalgebras.

   A Lie algebra $\mathfrak{g}$ is **solvable** if its **derived series** of ideals $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i]$ terminates. ($\mathfrak{g}^1 = \mathfrak{g}$.) In an algebraically closed field, we can always choose a basis such that ad$x$ matrix is upper-triangular for all $x \in \mathfrak{g}$ by the **Lie theorem**. Every abelian Lie algebra is solvable.

   A Lie algebra $\mathfrak{g}$ is **nilpotent** if its **lower central series** of ideals $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]$ terminates. ($\mathfrak{g}_1 = \mathfrak{g}$.) A nilpotent Lie algebra is automatically solvable. We can always choose a basis such that ad$x$ matrix is strictly upper-triangular (i.e. nilpotent) for all $x \in \mathfrak{g}$ by **Engel theorem**.

2. The standard equivalent definition of a **semisimple Lie algebra** is a Lie algebra that has no nonzero solvable ideal. This implies that semisimple Lie algebras do not have a center because 0 is contained in every Lie subalgebra. Thus, being semisimple is as far from being abelian as possible. A Lie algebra is **simple** if it is not abelian and has no nontrivial ideal. We will see below that every semisimple Lie algebra is made up of simple Lie algebras. (If abelian Lie algebras are not excluded, simple Lie algebras may not be semisimple.)

   With five exceptions, every simple complex Lie algebra is isomorphic to one of the followings: $\mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C})$ or $\mathfrak{sp}(2n, \mathbb{C})$.

   A connection to semisimple (diagonalizable) matrices is that a semisimple Lie algebra always contain at least one nonzero **semisimple element**, $x$ whose ad$x$ matrix is semisimple. This follows from a generalization of the Jordan decomposition in linear algebra: every element in a semisimple Lie algebra can be written as a sum of commuting semisimple and nilpotent element:

   $$x = x_s + x_n,$$
   $$[x_s, x_n] = 0.$$

   If $x_s = 0$ for any $x \in \mathfrak{g}$, then by Engel theorem, $\mathfrak{g}$ is nilpotent hence solvable, contradicting the semisimplicity of $\mathfrak{g}$.

3. **Levi theorem** states that every Lie algebra can be written as the sum of its unique largest solvable ideal, the **radical** rad $(\mathfrak{g})$, and a semisimple Lie subalgebra $\mathfrak{s}$:

   $$\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{s}.$$

   Recall that solvability is a generalization of abelianity, so every Lie algebra is a sum of one part that is close to being abelian and the other part that is far from being abelian. If the first part is actually abelian, we call the Lie algebra **reductive**. Any ideal $I$ in a reductive Lie algebra has a complementary ideal $I^\perp$ such that $\mathfrak{g} = I \oplus I^\perp$.

4. A **toral subalgebra** is an abelian subalgebra that consists of semisimple elements. A **Cartan subalgebra** is a maximal toral subalgebra which corresponds to a **maximal torus** in $G$, hence the name "toral".

   For semisimple Lie algebras, the standard definition of a Cartan subalgebra is a toral subalgebra and self-centralizing: $C(\mathfrak{h}) = \{x | \mathrm{ad}x(\mathfrak{h}) = 0\} = \mathfrak{h}$. For the curious reader, the general definition that works for any Lie algebra is that a subalgebra is Cartan if it is nilpotent and self-idealizing: $I(\mathfrak{h}) = \{x | \mathrm{ad}x(\mathfrak{h}) \subset \mathfrak{h}\} = \mathfrak{h}$. (The reduction to centralizing in the semisimple case is, of course, due to the toricity.) But since we are not going to pursue the general definition here, we take the semisimple definition to be *the* definition of a Cartan subalgebra, which can be shown to be equivalent to the definition as a maximal toral subalgebra in the semisimple case.

The **rank** of a Lie algebra is the dimension of its Cartan subalgebra. Even though a Cartan algebra is not unique, this notion makes sense because all Cartan subalgebras are conjugated, hence have the same dimension.

5. A bilinear form (or sesquilinear form for a complex vector space) $\langle , \rangle$ is **invariant** under the action of $G$ if
$$\left\langle \mathrm{Ad}_g y, \mathrm{Ad}_g z \right\rangle = \langle y, z \rangle$$
or equivalently, by setting $g = e^{tx}$ and differentiate w.r.t. $t$ at the origin,
$$\langle \mathrm{ad}x\,(y)\,, z \rangle + \langle y, \mathrm{ad}x\,(z) \rangle = 0$$
i.e. $\mathrm{ad}x$ is antisymmetric w.r.t. the form. Any representation gives an invariant bilinear form $\langle x, y \rangle = \mathrm{Tr}\,(\rho\,(x)\,, \rho\,(y))$. It turns out to be very useful to take $\rho$ to be the adjoint representation itself. Then we have the **Killing form**
$$\langle x, y \rangle = \mathrm{Tr}\,(\mathrm{ad}x, \mathrm{ad}y)\,.$$
A bilinear form can be degenerate so it may not be an inner product, but the Killing form on $\mathfrak{g}$ is nondegenerate if and only if $\mathfrak{g}$ is semisimple (**Cartan's criteria for semisimplicity**). So it identifies a Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$.

6. A Lie algebra of a compact real Lie group is reductive and has a negative semidefinite Killing form. Conversely, a semisimple real Lie algebra with a negative definite Killing form is a Lie algebra of a compact real Lie group. (There is no nontrivial real Lie algebra with a positive definite Killing form.) A Lie algebra of a compact Lie group is called a **compact Lie algebra**.

   The matrices of the adjoint representation can be made antisymmetric i.e. the structure constant can be made totally antisymmetric, if the Lie algebra is a direct sum of simple compact Lie algebras. Example: $\mathfrak{su}(2)$. Counterexample: the adjoint representation of the Heisenberg Lie algebra $\mathfrak{h}$ is nilpotent and thus not antisymmetric. This is consistent with the fact that $\mathfrak{h}$ is not simple because it has a one-dimensional ideal: $[\mathbb{1}, x] = 0$ for all $x$ in $\mathfrak{h}$.

7. Let $\mathfrak{h}$ be a Cartan subalgebra. It is toral, so $\mathrm{ad}h$ for all $h \in \mathfrak{h}$ are simultaneously diagonalizable. Nonzero eigenvalues of arbitary linear combinations in $\mathfrak{h}$ define a set (not necessarily a vector space) of **roots** $\alpha \in R \subset \mathfrak{h}^* - \{0\}$. Each root labels a **root space** $\mathfrak{h}_\alpha = \{x \in \mathfrak{g} | \mathrm{ad}h\,(x) = \langle \alpha, h \rangle\, x\}$. By an abuse of notation, a root can also refer to the number $\alpha\,(h) := \langle \alpha, h \rangle$.

   (a) **Root decomposition**: the Lie algebra splits into eigenspaces of $\mathrm{ad}h$:
   $$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{h}_\alpha$$
   (b) $\left[\mathfrak{h}_\alpha, \mathfrak{h}_\beta\right] \subset \mathfrak{h}_{\alpha+\beta}$
   (c) If $\alpha + \beta \neq 0$, then $\mathfrak{h}_\alpha$ and $\mathfrak{h}_\beta$ are orthogonal w.r.t. the Killing form. This is because
   $$\langle h_\gamma |\, \mathrm{ad}h_\alpha \mathrm{ad}h_\beta \,| h_\gamma \rangle = \langle h_\gamma | h_{(\alpha+\beta)+\gamma} \rangle = 0$$
   for all $\gamma$. That is, $\mathrm{ad}h_\alpha \mathrm{ad}h_\beta$ has no diagonal element. Therefore, the trace is zero.

   Roots are special cases of weights. (10)

8. A **root system** is a finite set of vectors (roots) of a Euclidean space $E$ with rigid geometrical relations:

   (a) For any two roots $\alpha$ and $\beta$, the projection of $\beta$ onto $\alpha$ is a half-integer multiple of $\alpha$.
   (b) The reflection of $\beta$ around the hyperplane $p \cdot \alpha = 0$ gives another root. The group generated by these reflections is called the **Weyl group**.

   Associated with each root $\alpha$ is a **coroot** $\alpha^\vee \in E$ defined by
   $$\left\langle \alpha^\vee, \beta \right\rangle = \frac{2\alpha \cdot \beta}{\alpha \cdot \alpha}.$$
   There is a set of **simple roots** $\alpha_j$ such that every root $\alpha$ can be uniquely written as an integral linear combination of simple roots
   $$\alpha = \sum n_j \alpha_j,$$
   where $n_j \in \mathbb{Z}$. Because of the Weyl reflection, we may consider only the set $R_+$ of **positive roots** ($\forall n_j > 0$).

9. Root systems are in one-one correspondence to semisimple Lie algebras. The point of introducing them is that they can be classified completely according to their **Dynkin diagrams**.

10. We enlarge the root system $R$ to a **root lattice** $Q$, an abelian group generated by $R$ (by vector addition in $E$). Similarly, the **coroot lattice** $Q^\vee$ is generated by $\alpha^\vee$. The **weight lattice** $P$ is the dual lattice of $Q^\vee$:

$$P = \left\{ \lambda \in E | \left\langle \lambda, \alpha^\vee \right\rangle \in \mathbb{Z}, \forall \alpha^\vee \in Q^\vee \right\}.$$

(This is simply the definition of a dual lattice.) Let $\pi$ be a representation of $\mathfrak{g}$ on $V$. A vector $v \in V$ is called a vector of **weight** $\lambda \in \mathfrak{h}^*$ if $hv = \langle \lambda, h \rangle v$ for all $h \in \mathfrak{h}$. Then $V$ decomposes into weight spaces:

$$V = \bigoplus_\lambda V_\lambda := \bigoplus_\lambda \{ v \in V | hv = \langle \lambda, h \rangle v, \forall h \in \mathfrak{h} \}.$$

The root lattice is contained in the weight lattice $Q \subset P$ because $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$. (They may coincide.) A weight $\lambda$ is **dominant integral** if

$$\left\langle \lambda, \alpha^\vee \right\rangle \in \mathbb{Z}_+, \forall \alpha \in R_+.$$

The **highest weight theorem** then guarantees that each and every dominant integral weight $\lambda$ gives distinct finite dimensional irreps of $\mathfrak{g}$ with the highest weight $\lambda$ and every finite dimensional irrep of $\mathfrak{g}$ arises in this way. A weight is a rank$\mathfrak{g}$-tuple, so one needs a partial ordering of weights when rank$\mathfrak{g} > 1$ to identify the highest weight.

Weights are also related to the logarithm of the characters of a representation of $G$ restricted to a maximal torus.

11. Let us look at a rank-1 example (rank$\mathfrak{su}(n) = n-1$): $\mathfrak{su}(2)$. For concreteness, also let us work in a specific representation.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is a *real* vector space whose basis can be chosen to be is $\{iX, iY, iZ\}$,

$$[iX, iY] = -2iZ, \qquad [iY, iZ] = -2iX, \qquad [iZ, iX] = -2iY. \tag{1}$$

The Lie algebra of $\mathfrak{so}(3, \mathbb{R})$ is

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[J_x, J_y] = J_z, \qquad [J_y, J_z] = J_x, \qquad [J_z, J_x] = J_y. \tag{2}$$

These two can be identified via the isomorphism $iX \to -2J_x$, etc. that lifts to a homomorphism between the groups. Physicists' commutation relations of $\mathfrak{su}(2)$ are in fact those of $\mathfrak{su}(2)_\mathbb{C}$, the complexified $\mathfrak{su}(2)$.

$$[X, Y] = 2iZ, \qquad [Y, Z] = 2iX, \qquad [Z, X] = 2iY, \tag{3}$$

No two of them can be simultaneously diagonalized, so we pick one of them, say $Z$, to be a basis of the one-dimensional Cartan subalgebra. The complexification allows us to use the **Chevalley** or **Cartan basis** which incorporates the raising and lowering operators:

$$H = Z, \qquad E = \frac{X + iY}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \frac{X - iY}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

isomorphic to $\mathfrak{sl}(2, k)$. When $k = \mathbb{C}$, it is the lowest-dimensional complex semisimple Lie algebra. (In dimensions one and two, we get either an abelian Lie algebra or $gl(2, \mathbb{C})$ which is reductive (3) but not semisimple.) For finite dimensional representations, representations of $\mathfrak{sl}(2, \mathbb{C})$ corresponds to unitary representations of

$su(2)_{\mathbb{C}}$ and hence its real form $\mathfrak{su}(2) = \mathfrak{so}(3, \mathbb{R})$. $SL(2, \mathbb{C})$ is not compact, so it does not have any finite-dimensional unitary irrep and the correspondence is more complicated, but we will not need it in any way.

We know from quantum mechanics that these commutation relations completely determine all $n+1$-dimensional irreps $V_n$ of $SU(2)$, each with the highest weight $n$. $V_n$ splits into the direct sum of all one-dimensional weight spaces

$$
\begin{aligned}
V_n &= \bigoplus_k V_k, \\
EV_k &\subset V_{k+2}, \\
FV_k &\subset V_{k-2},
\end{aligned}
$$

and $V_n$ can be constructed by applying the lowering operator $F$ to the highest weight vector $|n, k\rangle \subset V_k$:

$$E\,|n, k\rangle = 0.$$

(A potential point of confusion: some math books define $|n, k\rangle$ to be $k$ steps *below* the highest weight vector i.e. $F^k\,|j, j\rangle$.) Irreps can be projected out of a representation as eigenspaces of the **Casimir operator**. Given an orthonormal operator basis $\{|E_\alpha)\}$ of $\mathfrak{g}$, the (quadratic) Casimir operator is the identity operator

$$\sum_\alpha |E_\alpha)(E_\alpha|,$$

where

$$(E_\alpha| = |E_\alpha)^\dagger$$

For $\mathfrak{sl}(2, \mathbb{C})$, it is

$$X^2 + Y^2 + Z^2 = H^2 + EF + FE.$$

Note that the Casimir operator does not lie in the Lie algebra since only the Lie bracket is defined there. (It lies in the **universal enveloping algebra**.)

The two- and three-dimensional representations above are $V_1$ and $V_2$ with weights -1,1 and -2,0,2 respectively. $V_1$ is a natural representation of the Lie algebra of $SU(2)$. $V_2$ is the adjoint representation and the natural representation of the Lie algebra of $SO(3, \mathbb{R})$. The simple root 2 generates the root lattice isomorphic to even integers $2\mathbb{Z}$ strictly contained in the weight lattice $\mathbb{Z}$. Representations with the highest weights outside the root lattice are (projective) **spin** or **spinor representations** of $SO(3, \mathbb{R})$.

## III. COMPACT AND SEMISIMPLE LIE GROUPS

Fundamental groups, homotopy groups...

Non-connected, non-simply connected semisimple Lie groups, maximal compact subgroups, QR decomposition, Gram-Schmidt process as deformation retraction to a maximal compact subgroup, Iwasawa decomposition as a generalized QR decomposition,...

Real forms, compact real form of classical Lie groups, the Lorentz group SO(1,3), conformal groups for $p+q>2$...

Measure and Riemannian metric on compact groups, symmetric spaces...

Hilgert and Neeb could be a really good reference for this section.

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[2] Brian C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, Springer, 2003.
[3] John M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Springer, 2012.
[4] William Fulton and Joe Harris, *Representation Theory: A First Course*, Springer, 1991.