M526/P623 Quantum Computation Ninnat Dangniam

Homework Assignment 1

30 points DUE: 9 Dec 2024

1. Busy Beaver (5 points).

Let $\mathrm{BB}(n)$ be the nth *Busy Beaver number*, the maximum number of steps that an n-state Turing machine can make on an initially blank tape before halting. Note that the maximum here is taken over all n-state Turing machines that eventually halt. Prove that $\mathrm{BB}(n)$ grows faster than any computable function. (**Hint:** You may use the undecidability of the halting problem.)

---

2. Two-bit reversible gates (10 points).

(a) Show that the most general two-bit classical gate can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \oplus M \begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} p \\ q \end{pmatrix} xy,$$

where a, b, p, q and elements of the 2×2 matrix M can be either 0 or 1.

(b) Show that the most general **reversible** two-bit classical gate can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \oplus M \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

where a, b can be either 0 or 1. Additionally, M is restricted to be one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2}$$

From your answer, argue that no reversible two-bit classical gate can be universal on its own.

(c) Asuming that $a = b = 0$, draw the (quantum) circuit for the six reversible M of part **(b)**.

---

3. Three-bit reversible gates (15 points).

The most general three-bit classical gate can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \oplus M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus N \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} \oplus \begin{pmatrix} p \\ q \\ r \end{pmatrix} xyz, \tag{3}$$

where a, b, c, p, q, r and elements of the 3×3 matrices M and N can be either 0 or 1. Introduce a vector notation,

$$\mathbf{M}_j = \begin{pmatrix} M_{1j} \\ M_{2j} \\ M_{3j} \end{pmatrix}, \qquad \mathbf{N}_j = \begin{pmatrix} N_{1j} \\ N_{2j} \\ N_{3j} \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

(a) Show that to be reversible, the three columns of M must be linearly independent or be the three nonzero vectors in a two-dimensional subspace.

Throughout the rest of this problem, we will concentrate on the first scenario i.e. when the columns of M are linearly independent.

(b) Show that $\mathbf{p} = 0$. (**Hint:** Think in terms of the vectors $\mathbf{V}_1 = \mathbf{M}_2 \oplus \mathbf{M}_3$, $\mathbf{V}_1 = \mathbf{M}_1 \oplus \mathbf{M}_3$, $\mathbf{V}_3 = \mathbf{M}_1 \oplus \mathbf{M}_2$, and $\mathbf{V}_4 = \mathbf{M}_1 \oplus \mathbf{M}_2 \oplus \mathbf{M}_3$.)

(c) Characterize all transformations that can be achieved by the columns of N. (**Hint:** Think of permutations of the vectors introduced in part **(b)**. There are seven classes of transformations up to permutations of $\mathbf{V}_1, \mathbf{V}_2$, and $\mathbf{V}_3$.)

(d) Show how FREDKIN and TOFFOLI fit into your characterization from part **(c)**.