M526/P623 Quantum Computation *Lecturer: Ninnat Dangniam*

Homework Assignment 6

DUE: 7 Mar 2025 50 points

1. Commutator weight (10 points)

Consider an n-qubit systems. In this problem, we denote by $\sigma_i^x, \sigma_i^y, \sigma_i^z$ the Pauli operators acting on the ith qubit. For an index $\alpha \in \{x, y, z\}$, we write σ_i^α to indicate a Pauli operator on qubit i. The set $\{\mathbb{1}, \sigma^x, \sigma^y, \sigma^z\}$ of single-qubit operators is closed, up to a constant prefactor, under multiplication, taking the commutator, and taking the anticommutator. (A pair of Pauli operators either commute or anticommute.) The weight $\mathrm{wt}(A)$ of an operator A is the number of qubits on which the operator acts nontrivially. For instance, a product of Pauli operators on k distinct qubits (omitting the tensor product symbols and the identity operators acting $n-k$ other qubits)

$$O_k = \sigma_{i_1}^{\alpha_{i_1}} \sigma_{i_2}^{\alpha_{i_2}} \cdots \sigma_{i_k}^{\alpha_{j_k}}$$

has weight k. We typically call an operator of this form a *k-point (spin) operator*. Find the weight of the commutator of an arbitrary k-point spin operator, O_k, and an arbitrary l-point spin operators, O_l. That is, find $\mathrm{wt}([O_k, O_l])$. You may assume $k \le l$ without loss of generality.

(**Hint:** Doing this problem is mostly a matter of having an efficient notation. Denote the set of ordered indices $I \equiv \{i_1, \ldots, i_k\}$ and $J \equiv \{j_1, \ldots, j_l\}$, both subsets of $[n] \equiv \{1, 2, \ldots, n\}$. Partition the set I into the piece that does not overlap with J, denoted by $I \cap \overline{J}$, and the piece $I \cap J$ that overlaps with J. The answer will depend on the size of these sets and certain subsets of them.)

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2. Bosonic codes (10 points).

A bosonic mode is an infinite-dimensional Hilbert space with a standard basis labeled by non-negative integers, i.e., $|0\rangle, |1\rangle, |2\rangle, \ldots$, representing the eigenstates of a harmonic oscillator. For instance, for light, $|n\rangle$ is a state with $|n\rangle$ photons in this mode. (Here the mode is specified by a particular wave number, polarization, and possibly spatial profile). For bosonic modes, there is a natural generalization of the amplitude damping channel to

$$\rho \mapsto \sum_k A_k \rho A_k^\dagger,$$

where

$$A_k = \sum_{j \ge k} \sqrt{\binom{j}{k} \gamma^k (1-\gamma)^{j-k}}\, |j-k\rangle\langle j|,$$

representing loss of k photons from a mode with the loss rate γ. In particular,

$$A_0 = \sum_j (1-\gamma)^{j/2} |j\rangle\langle j|,$$

$$A_1 = \sum_{j \ge 1} \sqrt{j\gamma(1-\gamma)^{j-1}}\, |j-1\rangle\langle j|.$$

Note that, as with amplitude damping, A_0 is not proportional to the identity as more highly excited states are more likely to emit photons.

For this problem, we will look at codes encoding a single qubit in m bosonic modes to correct for loss of a single photon from one mode. Let $B_0 = A_0^{\otimes m}$ be the no-loss operator and $B_i = A_0^{\otimes(i-1)} \otimes A_1 \otimes A_0^{\otimes(m-i)}$ be the operator which has loss of one photon from the ith mode and no loss from the other modes. The set of errors that we are trying to correct is thus $\mathcal{E} = \{B_0, B_1, \ldots, B_m\}$.

(a) Consider the following encoding

$$\left|\overline{0}\right\rangle = \frac{|4,0\rangle + |0,4\rangle}{\sqrt{2}}, \qquad\qquad \left|\overline{1}\right\rangle = |2,2\rangle\,.$$

Show that this is a QECC correcting the error set $\mathcal{E}$ for two modes.

(b) Consider the following encoding

$$\left|\overline{0}\right\rangle = \frac{|3,0,0\rangle + |0,3,0\rangle + |0,0,3\rangle}{\sqrt{3}}, \qquad\qquad \left|\overline{1}\right\rangle = |1,1,1\rangle\,.$$

Show that this is a QECC correcting the error set $\mathcal{E}$ for three modes.

(c) A multimode basis state $|n_1, n_2, \ldots, n_m\rangle$ has a well defined total photon number $\sum_i n_i$. A superposition of such states would still have a well defined total photon number if every term in the superposition has the same total photon number. For instances, the codewords for the code in part **(a)** have total photon number 4 and the code in part **(b)** has total photon number 3. Show that there is no bosonic code for any number of modes that has total photon number 1.

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3. Spot the stabilizers (10 points).

For each of the following sets of Paulis, determine if they generate a valid stabilizer group. If so, give their parameters $[[n, k, d]]$.

(a)
$$\begin{matrix} X & X & Z & Y & \mathbb{1} \\ Z & Y & \mathbb{1} & \mathbb{1} & X \\ X & \mathbb{1} & X & Z & Z \end{matrix}$$

(b)
$$\begin{matrix} X & X & X & X & X & X \\ Y & Y & Y & Y & Y & Y \\ Z & Z & Z & Z & Z & Z \end{matrix}$$

(c) In the binary symplectic form (X | Z):

$$\left(\begin{array}{cccccc|cccccc} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

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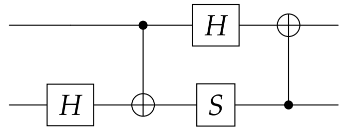
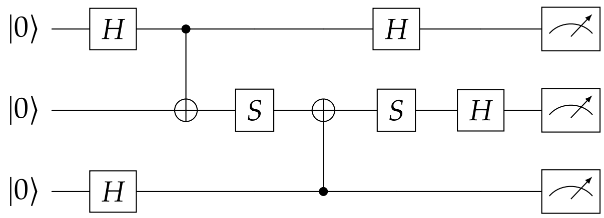
4. Simulation of Clifford circuits (10 points).

(a) Compute the overall action of the circuit below on Paulis, and use the results to write down the 4×4 unitary representation of the circuit.



(b) Use Clifford simulation techniques to compute the full probability distribution of the 8 possible classical outputs of the following circuit after measuring all qubits in the computational basis.



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5. Teleporting a non-Clifford gate (10 points).

The controlled-S operation

$$\Lambda(S) \longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

lies outside the Clifford group and belongs to the third level of the Clifford hierarchy. (The kth level of the Clifford hierarchy is defined as

$$\mathcal{C}^{(k)} = \{U \in \mathrm{U}(2^n) | UPU^\dagger \in \mathcal{C}^{(k-1)}\},$$

where $\mathrm{U}(2^n)$ is the unitary group on n qubits, P is a Pauli operator, and the Pauli group is defined to be $\mathcal{C}^{(1)}$. Thus, $\mathcal{C}^{(k)}$ generalizes the Clifford group, which is $\mathcal{C}^{(2)}$.)

(a) Find $\Lambda(S) P \Lambda(S)^\dagger$ for $P = X \otimes \mathbb{1}$, $\mathbb{1} \otimes X$, $Z \otimes \mathbb{1}$, and $\mathbb{1} \otimes Z$.

(b) Suppose that we want to perform $\Lambda(S)$ on an arbitrary two-qubit state $|\psi\rangle$. Show that this can be done with two ancilla qubits and a (four-qubit) circuit consisting of *Clifford* gates, Pauli measurements, and classical feed-forward of measurement results.