M525/P622 Quantum Information	Ninnat Dangniam

Homework Assignment 2

60 points	DUE: 23 July (Tuesday)

1. Hilbert-Schmidt inner product (10 points).

The set of all linear operators on a Hilbert space $\mathcal{H}$, denoted by $L(\mathcal{H})$,[1] is obviously a vector space with respect to addition of operators and multiplication by complex scalars. What's more, it can be given a natural inner product structure, turning it into a Hilbert space.

(a) Show that the function $(\cdot, \cdot)$ on $L(\mathcal{H}) \times L(\mathcal{H})$ defined by

$$(A, B) \equiv \mathrm{tr}\Big(A^\dagger B\Big)$$

is an inner product function.

(b) If $\mathcal{H}$ has d dimensions, show that $L(\mathcal{H})$ has dimensions d^2.

(c) Find an orthonormal basis of Hermitian matrices for $L(\mathcal{H})$.

(d) Find an operator basis of d^2 pure states, i.e., one-dimensional projectors. Is it possible for such a basis to be orthonormal?

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2. Normal operators for qubits (10 points).

Consider an arbitrary qubit operator

$$A = A_\alpha \sigma_\alpha = A_0 \mathbb{1} + \mathbf{A} \cdot \boldsymbol{\sigma},$$

where the coefficients A_α are arbitrary complex numbers. We can write $\mathbf{A} = \mathbf{B} + i\mathbf{C}$.

(a) Show that the condition for A to be a normal operator is that $0 = \mathbf{A} \times \mathbf{A}^* = -2i\mathbf{B} \times \mathbf{C}$, i.e., that $\mathbf{C} = \mu \mathbf{B}$ for some (real) constant μ.

(b) Find the conditions on A_0 and $\mathbf{A}$ for A to be **(i)** a Hermitian operator and **(ii)** a unitary operator

(c) Show that A has a spectral decomposition if and only if A is normal.

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3. Non-normal operators for qubits (10 points).

A good example of a non-normal operator is the qubit raising operator

$$\sigma_+ = \frac{\sigma_x + i\sigma_y}{2} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(a) Find the eigenvectors and eigenvalues of σ_+.

[1] In the quantum information literature, this space is sometimes also denoted by $\mathrm{Lin}(\mathcal{H})$, or $\mathfrak{B}(V)$, where $\mathfrak{B}$ stands for *bounded* linear operators.

(b) Find the left and right polar decomposition of σ_+ (PU and VQ respectively, where P and Q are positive operators, and U and V are unitary operators). What is the relationship between the eigenvalues and eigenvectors of σ_+ and the polar decomposition?

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4. Qubit rotations (15 points).

An arbitrary unitary operator in a two-dimensional vector space can be written in the form

$$U = e^{i\delta} e^{-i\mathbf{n}\cdot\boldsymbol{\sigma}\theta/2}.$$

The phase $e^{i\delta}$ produces a global phase change, so we can discard it and write the general unitary operator as

$$U_R = e^{-i\mathbf{n}\cdot\boldsymbol{\sigma}\theta/2}.$$

(a) Give an eigendecomposition of U_R. Then from the eigendecomposition, show that

$$U_R = \cos(\theta/2) - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\theta/2)$$

without explicitly writing the Taylor series for the exponential function.

(b) Show the operator analogue of the *Rodrigues' rotation formula*:

$$\begin{aligned} U_R^\dagger \boldsymbol{\sigma} U_R &= \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\sigma}) - \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\sigma})\cos\theta + \mathbf{n}\times\boldsymbol{\sigma}\sin\theta \\ &= \boldsymbol{\sigma}\cos\theta + \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\sigma})(1-\cos\theta) + \mathbf{n}\times\boldsymbol{\sigma}\sin\theta \equiv \mathbf{R_n}(\theta)\boldsymbol{\sigma}. \end{aligned}$$

Here $\mathbf{R_n}(\theta)$ is the 3-dimensional orthogonal matrix that describes a rotation by angle θ about axis $\mathbf{n}$.

(c) Use the result of part **(c)** to show that U_R rotates any state $|\mathbf{m}(\theta,\varphi)\rangle \equiv |\mathbf{m}\rangle$, i.e., that

$$U_R|m\rangle = e^{i\phi(\mathbf{R},\mathbf{m})}\,|\mathbf{R}\,\mathbf{m}\rangle\,,$$

where $\phi(\mathbf{R},\mathbf{m})$ is a phase.

(d) Show that the unitary operator $\mathbf{n}\cdot\boldsymbol{\sigma}$ produces a 180° rotation about $\mathbf{n}$.

(e) The Hadamard transform

$$H \equiv i e^{-i\mathbf{n}\cdot\boldsymbol{\sigma}\pi/2},$$

where $\mathbf{n} = \frac{\mathbf{e}_x+\mathbf{e}_z}{\sqrt{2}}$, rotates by 180° about the axis midway between the x and z axes. Show that the matrix representation of H in the computational basis is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

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5. Two-level atom interacting with a classical field mode (15 points).

Consider a two-level atom with energy-level spacing $\hbar\Omega$ interacting with a single classical field mode with frequency $\omega = \Omega + \Delta$, where Δ is the detuning. The Hamiltonian for the two-level atom is

$$H = \frac{1}{2}\hbar\Omega\sigma_3 + \hbar g_0 \left(\sigma_- e^{i\omega t} + \sigma_+ e^{-i\omega t}\right),$$

where $g_0 > 0$ is a coupling constant. The second part of the Hamiltonian describes transitions from the upper atomic level to the lower (σ_-) and from the lower level to the upper (σ_+). The explicit time dependences $e^{\pm i\omega t}$ come from the harmonic time dependence of the classical field.

A general way of solving the Schrödinger equation is to find the unitary operator $U(t)$, which satisfies an operator Schrödinger equation

$$i\hbar \frac{dU(t)}{dt} = HU(t).$$

(a) Show that the evolution operator for this Hamiltonian can be written as

$$U(t) = U_R U_{R'},$$

where $\mathbf{R} = \mathbf{R}_{\mathbf{e}_3}(\omega t)$ is a rotation by angle ωt about $\mathbf{e}_3$, and $\mathbf{R}' = \mathbf{R}_{\mathbf{n}}(rt)$ is a rotation by angle rt about the unit vector

$$\mathbf{n} = \frac{2g_0}{r}\mathbf{e}_1 - \frac{\Delta}{r}\mathbf{e}_3,$$

with $r = \sqrt{4g_0^2 + \Delta^2}$.

(b) Use the result of part **(a)** to write the evolution of the Bloch vector $\mathbf{S}(t) = \langle \boldsymbol{\sigma}(t) \rangle$ in terms of $\mathbf{R}$ and $\mathbf{R}'$.

(c) Suppose now that the atom is initially in the upper state $|0\rangle$, i.e., $\mathbf{S}(0) = \mathbf{e}_3$. Use the result of part **(b)** to derive an explicit expression for $\mathbf{S}(t)$; describe the motion of $\mathbf{S}(t)$ on the Bloch sphere.

(d) Continuing to assume that the atom is initially in the upper state, find the amplitudes $c_0(t) = \langle 0|U(t)|0\rangle$ and $c_1(t) = \langle 1|U(t)|0\rangle$ to be in the upper and lower states at time t. Derive directly from these amplitudes the components of the Bloch vector $\mathbf{S}(t)$, and show that they agree with the results of part **(c)**.

(e) Specialize your expression for $\mathbf{S}(t)$ to the case of no detuning ($\Delta = 0$), and for this case describe or draw the motion of $\mathbf{S}(t)$ on the Bloch sphere. (The frequency $2g_0$ is called the *Rabi frequency*.)
