M525/P622 Quantum Information Ninnat Dangniam

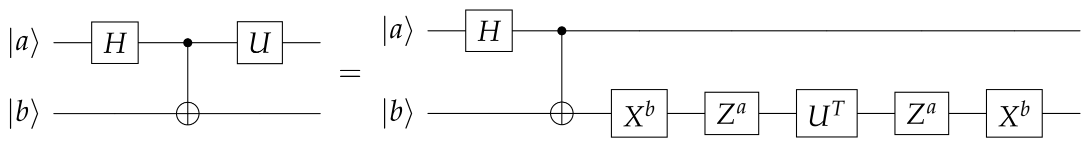
Homework Assignment 4

40 points DUE: 15 August (Thursday)

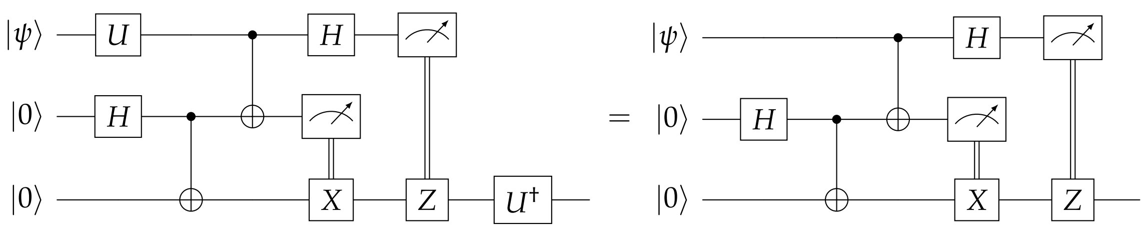
There are countless approaches to showing that the teleportation protocol works as advertised. In the first two problems, you will work through some of these approaches based on quantum circuit diagrams.

1. Teleportation circuit I (10 points).

(a) Let $|a\rangle$ denotes a qubit state in the computation basis, i.e. $a = 0$ or 1. Show the following circuit identity:



(b) Using the result of **(a)**, show that the circuit on the left hand side is equivalent to the teleportation circuit for an arbitrary single-qubit unitary U.



(c) Use the result of **(b)** to show that the teleportation circuit does indeed transfer the input state of the top qubit to the output state of the bottom qubit.

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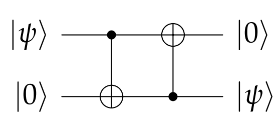
2. Teleportation circuit II (10 points).

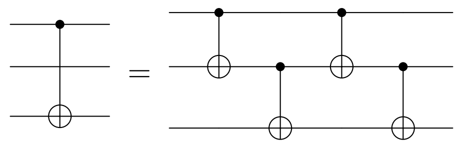
One thing that a teleportation circuit does is transferring the input state of the top qubit to the output state of the bottom qubit. The SWAP gate, which is equivalent to the following circuit when the bottom qubit is initialized in the state $|0\rangle$, does the same thing.



(1)

But this is not teleportation because the two qubits have to interact coherently. What we will do is introducing a “middleman" qubit.

(a) Show the following circuit identity:



(b) Start from (1), add a middle qubit, and use the result of **(a)** to transform the circuit into the teleportation circuit.

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3. Neumark extension of rank-one POVMs (10 points).

Consider a POVM for a D-dimensional quantum system consisting of $N \geq D$ rank-one POVM elements. This POVM can be thought of as a measurement of one-dimensional orthogonal projectors (an "ODOP") on an N-dimensional, extended Hilbert space. The aim of this exercise is to develop such a viewpoint, called the *Neumark extension*.[1]

Let the (rank-one) POVM elements be denoted by $E_\alpha = |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha|$, $\alpha = 1, ..., N$, where the vectors $|\overline{\psi}_\alpha\rangle = \sqrt{\mu_\alpha}\,|\psi_\alpha\rangle$ could be subnormalized. That is, the norm squared $\mu_\alpha = \langle\overline{\psi}_\alpha|\overline{\psi}_\alpha\rangle$ satisfies $0 < \mu_\alpha \leq 1$.

The POVM obeys the completeness relation

$$P = \sum_{\alpha=1}^{N} E_\alpha = \sum_{\alpha=1}^{N} |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha| = \sum_{\alpha=1}^{N} \mu_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|,$$

where P denotes the identity operator on the D-dimensional Hilbert space.

(a) Show that by adding $N - D$ dimensions to the Hilbert space, you can find an orthonormal set of vectors $|\widehat{\psi}_\alpha\rangle$, $\alpha = 1, ..., N$, that project to the POVM elements in the original D dimensions, i.e., $P|\widehat{\psi}_\alpha\rangle = |\overline{\psi}_\alpha\rangle$. (**Hint**: Expand the POVM elements in an orthonormal basis, and show that the expansion coefficients form part of a unitary matrix.)

(b) Consider the three Bloch vectors,

$$\begin{aligned}
\hat{\mathbf{n}}_1 &= \hat{\mathbf{e}}_x, \\
\hat{\mathbf{n}}_2 &= -\frac{1}{2}\hat{\mathbf{e}}_x + \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_y, \\
\hat{\mathbf{n}}_3 &= -\frac{1}{2}\hat{\mathbf{e}}_x - \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_y,
\end{aligned}$$

pointing to the vertices of an equilateral triangle in the equatorial planes, and satisfying $\hat{\mathbf{n}}_\alpha \cdot \hat{\mathbf{n}}_\beta = -1/2$ for any pair $\alpha \neq \beta$. The corresponding state vectors, $|\hat{\mathbf{n}}_1\rangle$, $|\hat{\mathbf{n}}_2\rangle$, and $|\hat{\mathbf{n}}_3\rangle$, can be used to form a three-outcome, rank-one POVM called the *trine*. Write out the trine POVM elements, and construct a Neumark extension for the POVM.

[1]There is an alternative spelling, "Naimark", based on a romanization "Naĭmark" of the Russian name.

(c) The four Bloch vectors,

$$\begin{aligned}
\hat{\mathbf{n}}_1 &= \hat{\mathbf{e}}_3, \\
\hat{\mathbf{n}}_2 &= \sqrt{\frac{8}{9}}\hat{\mathbf{e}}_1 - \frac{1}{3}\hat{\mathbf{e}}_3, \\
\hat{\mathbf{n}}_3 &= -\sqrt{\frac{2}{9}}\hat{\mathbf{e}}_1 + \sqrt{\frac{2}{3}}\hat{\mathbf{e}}_2 - \frac{1}{3}\hat{\mathbf{e}}_3, \\
\hat{\mathbf{n}}_4 &= -\sqrt{\frac{2}{9}}\hat{\mathbf{e}}_1 - \sqrt{\frac{2}{3}}\hat{\mathbf{e}}_2 - \frac{1}{3}\hat{\mathbf{e}}_3,
\end{aligned}$$

pointing to the vertices of a tetrahedron, and satisfying $\hat{\mathbf{n}}_\alpha \cdot \hat{\mathbf{n}}_\beta = -1/3$ for any pair $\alpha \neq \beta$. The corresponding state vectors, $|\hat{\mathbf{n}}_1\rangle$, $|\hat{\mathbf{n}}_2\rangle$, $|\hat{\mathbf{n}}_3\rangle$, and $|\hat{\mathbf{n}}_4\rangle$ can be used to form a four-outcome POVM called the *tetrahedron*. Write out the tetrahedron POVM elements, and construct a Neumark extension for the POVM.

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4. Space of POVM elements (10 points).

POVM elements for a D-dimensional quantum system are positive operators E that satisfy $0 \leq E \leq \mathbb{1}$. Equivalently, POVM elements are Hermitian operators whose eigenvalues lie between 0 and 1, inclusive. Therefore, POVM elements form a convex set; if E and F are POVM elements, then for $0 \leq \lambda \leq 1$, so is $\lambda E + (1-\lambda)F$.

(a) The POVM elements for a qubit can be written in the Pauli representation as

$$E = a\mathbb{1} + b\mathbf{n} \cdot \boldsymbol{\sigma},$$

where $a, b \in \mathbb{R}$. Find the allowed values of a and b, and describe the convex set of qubit POVM elements. By suppressing one of the “spatial" dimensions in the Pauli representation, draw the convex set of qubit POVM elements. Identify the extreme points of the convex set from the picture.

(b) For a D-dimensional system, show that the extreme points of the convex set of POVM elements are the projection operators of all ranks, including the zero operator and the identity operator. (**Hint**: To show that any POVM element E can be written as a convex combination of projectors, start with the eigendecomposition of E, and rewrite it as a convex combination of projectors by including the zero operator in the convex combination.)
