M525/P622 Quantum Information	Ninnat Dangniam
Homework Assignment 6	
50 points	DUE: 26 Sep 2024

1. Statistical distance on the probability simplex (10 points).

Consider a random variable with D possible values, and let p_j denote the probability for value j. The space of probabilities, defined by

$$1 = \sum_{j=1}^{D} p_j, \qquad\qquad p_j \geq 0,\ j = 1, \ldots, D,$$

is a $(D-1)$-dimensional regular polyhedron in D-dimensional Euclidean space.

(a) Consider a fiducial probability distribution p_j and a nearby distribution $q_j = p_j + \delta p_j$. By expanding the relative entropy $H(\mathbf{q}||\mathbf{p})$ about the fiducial distribution, show that the first non-vanishing term in the relative entropy is

$$H(\mathbf{q}||\mathbf{p}) = \frac{2}{\ln 2} \sum_j \frac{(\delta p_j)^2}{4p_j}.$$

The sum on the right-hand side defines a Riemannian metric on the probability simplex, which is called the *Bhattacharya-Wootters statistical distance* or the *Fisher information*.

(b) By introducing new coordinates, $r_j = \sqrt{p_j}$, show that according to the statistical distance, the probability simplex is a part of a unit sphere.

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2. Minimum error probability for mixed states (10 points).

Consider two mixed states, ρ_1 and ρ_2, occurring with probabilities q_1 and q_2. Let $\{E_1, E_2\}$ be a two-outcome POVM such that E_j corresponds to the decision that the state was ρ_j.

(a) Show that the error probability is

$$P_e = q_1 - \mathrm{Tr}\left[E_1(q_1\rho_1 - q_2\rho_2)\right].$$

(b) Show that

$$\max_{0 \leq E \leq 1} \mathrm{Tr}\left[E_1(q_1\rho_1 - q_2\rho_2)\right] = \frac{1}{2}\mathrm{Tr}\left(|q_1\rho_1 - q_2\rho_2|\right) + \frac{1}{2}(q_1 - q_2).$$

(c) Use the results of parts **(a)** and **(b)** to show that the minimum error probability is

$$(P_e)_{\mathrm{min}} = \frac{1}{2} - \frac{1}{2}\mathrm{Tr}\left(|q_1\rho_1 - q_2\rho_2|\right),$$

and find a POVM that gives the minimum error probability.

(d) Determine the minimum error probability when the two states are pure.

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3. Unambiguous state discrimination with different prior probabilities (20 points).

When a quantum state is in one of the two states, $|\psi_1\rangle$ or $|\psi_2\rangle$, but you don't know which one, the task of unambiguous state discrimination asks for a three-outcome POVM, two of whose outcomes conclusively identify one state or the other, but whose third outcome is inconclusive. If the states have equal probabilities of occurrence, the minimum probability for the "no decision" outcome is

$$\left(P_{\text{nd}}\right)_{\text{min}} = |\langle\psi_1|\psi_2\rangle|,$$

When the two probabilities, now denoted by q_1 and q_2, are unequal, find the minimum inconclusive probability $\left(P_{\text{nd}}\right)_{\text{min}}$. Without loss of generality, we may assume that $q_1 \geq q_2$.

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4. Approximate cloning by the isotropic POVM (10 points).

The isotropic POVM for a D-dimensional Hilbert space[1] has a POVM element $dE_\varphi = \alpha\,|\varphi\rangle\langle\varphi|\; d\Gamma_\varphi$ for every ray $|\varphi\rangle$ in the projective Hilbert space, where $d\Gamma_\varphi$ is the unitarily invariant measure, the *Haar measure*, on the projective Hilbert space, and α is a constant. The POVM satisfies the completeness relation

$$\mathbb{1} = \int dE_\varphi = \alpha \int |\varphi\rangle\langle\varphi|\, d\Gamma_\varphi.$$

The integration measure on the projective Hibert space, conveniently derived in Appendix A below, is

$$d\Gamma_\varphi = \sin^{2D-3}\theta\; \cos\theta\, d\theta\, d\mathcal{S}_{2D-3},$$

where $d\mathcal{S}_{2D-3}$ is the standard measure on the unit sphere $\mathcal{S}_{2D-3}$ in $2D-2$ real dimensions (the number of real parameters required to identify a pure state). This form of the measure is useful for doing integrals over functions of the Hilbert-space angle $|\langle\varphi|\psi\rangle| = \cos\theta$.

Throughout this problem, you can leave the area S_{2D-3} of the unit sphere $\mathcal{S}_{2D-3}$ as a normalization constant without having to calculate its explicit value. If you do **(c)** correctly, you should find that S no longer appears in the final answer.

(a) Using the completeness relation, determine the value of the constant α.

(b) Compute the integral $\int |\langle\varphi|\psi\rangle|^2\, d\Gamma_\varphi$

(c) One strategy for approximate cloning is to perform the isotropic POVM measurement on $|\psi\rangle$ and then make a copy of the result $|\varphi\rangle$. Find the average fidelity between $|\psi\rangle$ and the copy $|\varphi\rangle$.

[1] I use an uppercase D here to distinguish the dimension from differentials.

A Metric on the projective Hilbert space

Consider the infinitesimal distance ds between a normalized vector $|\psi\rangle$ and a nearby vector $|\psi'\rangle = |\psi\rangle + |d\psi\rangle$, also normalized. The Hilbert-space angle between the two is given by

$$\cos ds = \left|\langle \psi | \psi' \rangle\right| = |1 + \langle \psi | d\psi \rangle|.$$

The *Fubini-Study line element* is given by

$$\begin{aligned} ds^2 &= \sin^2 ds = 1 - \cos^2 ds = 1 - |1 + \langle \psi | d\psi \rangle|^2 \\ &= -2\mathrm{Re}\, \langle \psi | d\psi \rangle - |\langle \psi | d\psi \rangle|^2 \end{aligned}$$

The normalization of $|\psi'\rangle$ implies that the terms in red vanishes:

$$\langle \psi' | \psi' \rangle = 1 + 2\,\mathrm{Re}\, \langle \psi | d\psi \rangle + \langle d\psi | d\psi \rangle,$$

which gives

$$\begin{aligned} ds^2 &= \langle d\psi | d\psi \rangle - |\langle \psi | d\psi \rangle|^2 \\ &= \langle d\psi | \left(\mathbb{1} - |\psi\rangle\langle\psi| \right) |d\psi\rangle \equiv \left\langle d\psi^\perp \middle| d\psi^\perp \right\rangle, \end{aligned} \tag{1}$$

where $|d\psi^\perp\rangle$ is the projection of the small displacement $|d\psi\rangle$ orthogonal to $|\psi\rangle$. $\langle d\psi | d\psi \rangle$ is the standard metric on the unit sphere.$\langle \psi | d\psi \rangle$ is the component of the small displacement along $|\psi\rangle$; the real part describes changes in normalization and the imaginary parts describes changes in phase.

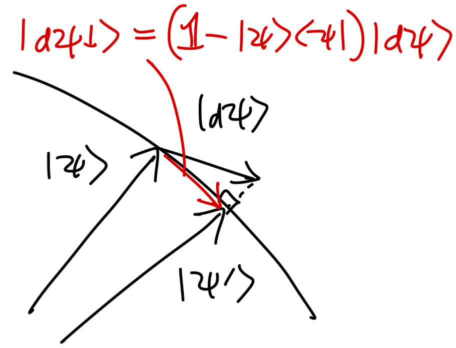

Figure 1: An attempt to geometrically depict the situation. It doesn't look quite right because both $|\psi\rangle$ and $|\psi'\rangle = |\psi\rangle + |d\psi\rangle$ are normalized vectors, hence $|d\psi\rangle$ should already be orthogonal to $|d\psi\rangle$ (in addition to $\langle d\psi | d\psi \rangle$ being zero to first order in small displacements). But we need to remember that these are complex vectors. In particular, the component of $|d\psi\rangle$ along $|\psi\rangle$ is purely imaginary to first order in small displacements.

Let us work with a concrete parametrization. Given a particular normalized vector $\{|e_1\rangle\}$, which can be completed to an ONB $\{|e_j\rangle\}, j = 1 \ldots, D$, any other normalized vector can be written as

$$|\psi\rangle = e^{i\delta}(\cos\theta\, |e_1\rangle + \sin\theta\, |\eta\rangle),$$

where $0 \leq \theta \leq \pi/2$, and $|\eta\rangle = \sum_{j=2}^{D} c_j \left|e_j\right\rangle$ is a normalized vector in the subspace orthogonal to $|e_1\rangle$. In this parametrization, a small displacement of $|\psi\rangle$ takes the form

$$|d\psi\rangle = i\, d\delta\, |\psi\rangle + e^{i\delta}[(-\sin\theta\, |e_1\rangle + \cos\theta\, |\eta\rangle)d\theta + \sin\theta\, |d\eta\rangle],$$

which gives

$$\begin{aligned}\langle\psi|d\psi\rangle &= i\, d\delta - \cancel{\sin\theta\cos\theta\, d\theta} + \cancel{\sin\theta\cos\theta\, d\theta} + \sin^2\theta\, \langle\eta|d\eta\rangle \\ &= i\, d\delta + \sin^2\theta\, \langle\eta|d\eta\rangle\end{aligned}$$

We move to work in the projective Hilbert space by making a particular phase choice $\delta = 0$, making

$$\langle\psi|d\psi\rangle = \sin^2\theta\, \langle\eta|d\eta\rangle\,,$$

$$\langle d\psi|d\psi\rangle = d\theta^2 + \sin^2 \langle d\eta|d\eta\rangle - \cancelto{0}{\sin\theta\cos\theta\, d\theta\, \langle d\eta|d\eta\rangle},$$

where the rightmost term vanishes by virtue of being third order in small displacements. Thus, the line element (1) becomes

$$\begin{aligned}ds^2 &= d\theta^2 + \sin^2\theta(\langle d\eta|d\eta\rangle - \sin^2\theta|\langle\eta|d\eta\rangle|^2) \\ &= d\theta^2 + \sin^2\theta(\langle d\eta|d\eta\rangle - |\langle\eta|d\eta\rangle|^2 + \cos^2\theta|\langle\eta|d\eta\rangle|^2) \\ &= d\theta^2 + \sin^2\theta\left(\left\langle d\eta^\perp \middle| d\eta^\perp\right\rangle + \cos^2\theta|\langle\eta|d\eta\rangle|^2\right).\end{aligned} \tag{2}$$

The first term of the line element

$$\left\langle d\eta^\perp \middle| d\eta^\perp\right\rangle + \cos^2\theta|\langle\eta|d\eta\rangle|^2$$

is that of a $(D-1)$-dimensional Hilbert space (the subspace orthogonal to $|\eta\rangle$), while the second term reflects a phase change scaled by a factor of $\cos\theta$. Recall that the line element on the unit $(N-1)$-sphere is

$$ds^2 = d\theta^2 + \sin^2\theta d\Omega^2_{N-2}, \tag{3}$$

with the corresponding volume element

$$d\mathcal{S}_{N-1} = \sin^{N-2}\theta\, d\theta\, d\mathcal{S}_{N-2}.$$

By comparing (2) and (3), we see that the overall volume element is that of a $(2D-2)$-sphere whose lengths in one dimension is scaled by the factor $\cos\theta$. That is,

$$d\Gamma_\varphi = \sin^{2D-3}\theta\cos\theta\, d\theta\, d\mathcal{S}_{2D-3}$$

is the unitarily invariant measure as given in Problem 4.