M525/P622 Quantum Information	Ninnat Dangniam
Homework Assignment 5b	
30 points	DUE: 17 Sep 2024

3. Graphical representations of quantum operations. (10 points).

Consider a linear superoperator

$$\begin{aligned}\mathcal{A} : L(\mathcal{H}_Q) &\to L(\mathcal{H}_Q),\\ \rho &\mapsto \mathcal{A}(\rho)\end{aligned}$$

for a quantum system Q. The space of linear operators $L(\mathcal{H}_Q)$ inherits the Hilbert-space structure from $\mathcal{H}$ though the Hilbert-Schmidt inner product $\mathrm{Tr}\big(A^\dagger B\big)$. In particular, the vectorization,

$$A \mapsto |A) \equiv A_Q \otimes \mathbb{1}_R \left|\overline{\Omega}\right\rangle,$$

where $\left|\overline{\Omega}\right\rangle = \sum_j |j\rangle_Q |j\rangle_R$ is the unnormalized, phase-less entangled state across the original Hilbert space $\mathcal{H}_Q$ and the isomorphic, reference Hilbert space $\mathcal{H}_R$, turns the Hilbert-Schmidt inner product to the inner product on $\mathcal{H}_Q \otimes \mathcal{H}_R$: $(A|B) = \mathrm{Tr}\big(A^\dagger B\big)$. The advantage of this faux bra-ket notation is that we can now contemplate another way that a linear superoperator can act besides the *ordinary* action of a superoperator

$$\mathcal{A} = \sum_j X_j \odot Y_j^\dagger,$$

where an input operator is dropped into the middle $\odot$. Now writing

$$\mathcal{A} = \sum_j |X_j)(Y_j|$$

suggests the *left-right* action

$$(A|\mathcal{A} = \sum_j \mathrm{Tr}\Big(A^\dagger X_j\Big) Y_j^\dagger, \qquad\qquad \mathcal{A}|B) = \sum_j \mathrm{Tr}\Big(Y_j^\dagger B\Big) X_j.$$

Notice that when vectorized properly,

$$\sum_j |X_j)(Y_j| = \sum_j (X_j \otimes \mathbb{1}) \sum_k |kk\rangle \sum_l \langle ll|\,(Y_j^\dagger \otimes \mathbb{1}) = \mathcal{A} \otimes \mathcal{I}(\left|\overline{\Omega}\middle\rangle\middle\langle\overline{\Omega}\right|)$$

is nothing but the **Choi matrix** of $\mathcal{A}$, so perhaps we should give the form $\sum_j |X_j)(Y_j|$ another name, say, $\breve{\mathcal{A}}$.[1]

(a) The inverse of the vectorization map is often expressed in terms of matrix elements:

$$A_{jk} = {}_Q\langle j|_R\langle k|\,|A).$$

[1]There are already standard notations for a Choi state and a Choi matrix, which are $|\Phi_{\mathcal{A}}\rangle$ and $\Lambda_{\mathcal{A}}$ respectively. Here I just want to minimize the uses of subscripts and not having to write Φ and Λ everywhere.

We can define "matrix elements" of a superoperator $\mathcal{A}$ as

$$\mathcal{A}_{jk;lm} = \langle l| \, \mathcal{A}(|j\rangle\langle k|) \, |m\rangle \, .$$

Find an expression for $\mathcal{A}_{jk;lm}$ in terms of $\breve{A}$.

(b) The difference between the ordinary action and the left-right action is that while the former only contracts indices in the reference system R, the left-right action always contract indices across the two Hilbert spaces. The superoperator that acts as the identity is also different for each action. For the ordinary action, the identity superoperator is

$$\mathcal{I} \equiv \mathbb{1} \odot \mathbb{1} = \sum_{jk} |j\rangle\langle j| \odot |k\rangle\langle k| \, .$$

For the left-right action, the identity superoperator is

$$\mathbf{I} \equiv \sum_{\alpha} |\tau_\alpha)(\tau_\alpha| = \sum_{jk} |j\rangle\langle k| \otimes |k\rangle\langle j| \, ,$$

where the $\tau_\alpha \equiv \tau_{jk} \equiv |j\rangle\langle k|$'s form an orthonormal basis for $L(\mathcal{H})$. Show graphically (i.e. using tensor diagrams) that $\mathbf{I}$ is indeed the identity superoperator with respect to the left-right action.

(c) It is evident that $\breve{A}|\rho) \neq \mathcal{A}(\rho)$. We can define another representation, the *Liouville representation* $\widehat{\mathcal{A}}$, in which $\mathcal{A}$ act on a vectorized density operator $|\rho)$ by a standard matrix multiplication; that is, $\widehat{\mathcal{A}}|\rho) = \mathcal{A}(\rho)$. If $\mathcal{A} = \sum_j X_j \odot Y_j^\dagger$, what is $\widehat{\mathcal{A}}$?

The Liouville representation is useful, for example, if we want to find operators that are fixed points of a quantum operation; they would simply be eigenvectors of $\widehat{\mathcal{A}}$.

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4. Lindblad master equation. (10 points).

A density operator of a quantum system Q in contact with an environment E changes in time according to

$$\rho(t + dt) = \rho(t) + dt\mathcal{L}(\rho) = (\mathcal{I} + dt\mathcal{L})(\rho), \tag{1}$$

where $\mathcal{I} = \mathbb{1} \odot \mathbb{1}$ is the identity superoperator and $\mathcal{L}$ is assumed to be a time-independent superoperator. The assumption required to write down Eq.(1) is a strong one, namely that the time evolution is *Markovian* or *memoryless*; that is, the time evolution in each infinitesimal time step from t to $t + dt$ depends only on the density operator of Q at time t. Importantly, this means that $\mathcal{I} + dt\mathcal{L}$ at each time step is a trace-preserving quantum operation.

Eq.(1) can be written as a differential equation called the *master equation*,

$$\frac{d\rho}{dt} = \mathcal{L}(\rho), \tag{2}$$

which has the solution

$$\rho(t) = e^{\mathcal{L}t}(\rho) = \mathcal{C}_t(\rho),$$

where $\mathcal{C}_t$ is a trace-preserving, time-dependent quantum operation with initial condition $\mathcal{C}_{t=0} = \mathcal{I}$. The goal of this problem is to find a general form of the differential equation for ρ given the complete-positivity and trace-preservation constraints.

$\mathcal{C}_t$ has a Kraus decomposition

$$\mathcal{C}_t = \sum_{\alpha=0}^{N-1} B_\alpha(t) \odot B_\alpha^\dagger(t),$$

with

$$\sum_{\alpha=0}^{N-1} B_\alpha^\dagger(t) B_\alpha(t) = \mathbb{1}.$$

At $t = 0$,

$$\mathcal{C}_0 = \sum_{\alpha=0}^{N-1} B_\alpha(0) \odot B_\alpha^\dagger(0) = \mathcal{I}.$$

Since this gives two different Kraus decompositions of the identity superoperator, the decomposition theorem for completely positive maps (which follows from an application of the HJW theorem to the Choi matrix $\Lambda_{\mathcal{I}}$) tells us that

$$B_\alpha(0) = V_{\alpha 0}\mathbb{1},$$

where the $V_{\alpha 0}$'s are the zeroth column of a unitary matrix.

Now consider an infinitesimal time interval dt, for which we have

$$\mathcal{C}_{dt} = +\mathcal{L}dt = \sum_{\alpha=0}^{N-1} B_\alpha(dt) \odot B_\alpha^\dagger(dt). \tag{3}$$

Decomposition of the operators $B_\alpha(dt)$ separates into two classes.

1. The first class consists of those Kraus operators that go to a (nonzero) multiple of $\mathcal{I}$ as $dt \to 0$, i.e., those for which $V_{\alpha 0} \neq 0$. There must be at least one such operator to produce the identity contribution to $\mathcal{C}_{dt}$, but there can be more than one. Suppose there are m of these operators; assign them the indices $\alpha = 0, \ldots, m-1$. To produce terms linear in dt in Eq.(3), the decomposition operators in the first classmust have the form

   $$B_\alpha(dt) = V_{\alpha 0}\mathbb{1} + b_\alpha dt,$$

   for all $\alpha = 0, \ldots, m-1$.

2. The second class consists of those Kraus operators that go to zero as $dt \to 0$, i.e., those for which $V_{\alpha 0} = 0$. These operators contribute only to the $\mathcal{L}dt$ part of $\mathcal{C}_{dt}$. Suppose there are n of these operators, with the indices $\alpha = m, \ldots, m+n-1 = N-1$. To produce terms linear in dt in Eq.(3), the decomposition operators in this second class must have the form

   $$B_\alpha(dt) = b_\alpha \sqrt{dt},$$

   for all $\alpha = m, \ldots, N-1$. (Don't be too startled by the $\sqrt{dt}$; it can be made rigorous through the notion of a *Wiener process*, which makes stochastic calculus possible. You only need to know that $\sqrt{dt}\sqrt{dt} = dt$ and $\sqrt{dt}\,dt = 0$.)

(a) Show that the master equation (2) can be brought to the form

$$\frac{d\rho}{dt} = -i[h,\rho] + \frac{1}{2}\sum_{\alpha=1}^{n}(2a_\alpha\rho a_\alpha^\dagger - a_\alpha^\dagger a_\alpha\rho - \rho a_\alpha^\dagger a_\alpha), \tag{4}$$

where h is a Hermitian operator which can be thought of as the system Hamiltonian, and a_α describes the effect of the environment.

(b) Show that a more general *Lindblad form* of the master equation

$$\frac{d\rho}{dt} = -i[h,\rho] + \frac{1}{2}\sum_{\alpha,\beta} A_{\alpha\beta}(2c_\alpha\rho c_\beta^\dagger - c_\beta^\dagger c_\alpha\rho - \rho c_\beta^\dagger c_\alpha),$$

where $A_{\alpha\beta}$ is a positive matrix and the sums can run to arbitrary large integers, can be converted to the form in Eq.(4).

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5. Unital qubit quantum operations (10 points).

An arbitrary trace-preserving qubit quantum operation can be written as

$$\mathcal{A}(\rho) = V\mathcal{B}\left(U\rho U^\dagger\right)V^\dagger,$$

where

$$\mathcal{B}(\rho) = \frac{1}{2}\left(1 + \boldsymbol{\sigma}\cdot(T\mathbf{S} + \mathbf{d})\right),$$

and where

$$T = \begin{pmatrix} t_x & 0 & 0 \\ 0 & t_y & 0 \\ 0 & 0 & t_z \end{pmatrix}$$

is a diagonal matrix that contracts (and perhaps inverts or reflects, depending on the number of negative diagonal entries) the Bloch sphere and $\mathbf{d}$ is a displacement of the contracted sphere. The initial and final unitaries, U and V, represent initial and final rotations of the Bloch sphere. Since these unitary transformations are completely positive, the complete positivity of $\mathcal{A}$ is determined by the complete positivity of $\mathcal{B}$.

In this problem we deal with the special case where there is no displacement of the Bloch sphere, i.e., $\mathbf{d} = 0$. In this case the operation takes the maximally mixed state to itself, i.e., $\mathcal{B}(1) = 1$. Such a quantum operation is called *unital*. Positivity of $\mathcal{B}$ requires the output states to be in the Bloch sphere, which means that the diagonal entries of T must satisfy $|t_j| \leqslant 1$. Thus positivity alone allows the diagonal entries to lie anywhere within a cube of side length 2 centered at the origin. The goal of this problem is to explore the further restrictions imposed by complete positivity.

(a) Find the restrictions imposed by complete positivity, and describe or draw the region of allowed values of t_x, t_y, and t_z.

(b) What kinds of maps of the Bloch sphere are allowed by complete positivity when one axis is left unchanged, say, $t_z = 1$?

(c) What are the extreme points of the convex set of unital qubit quantum operations?.
