M525/P622 Quantum Information	Ninnat Dangniam

Homework Assignment 3

65 points — DUE: 6 August (Tuesday)

1. Pure-state density operator (10 points).

For both **(a)** and **(b)**, assume that $\rho^\dagger = \rho$.

(a) Show that the condition $\mathrm{Tr}(\rho) = \mathrm{Tr}(\rho^2) = 1$ is **not** sufficient to make ρ a density operator.

(b) Show that the condition $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho^3) = 1$ suffices to make ρ a pure-state density operator.

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2. Qubit ensemble decomposition (15 points).

Consider the qubit state

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\,. \tag{1}$$

(a) Consider the set of all states with $\langle Z\rangle = 1/2$. Give ensemble decompositions for ρ comprises of **(i)** two states from the set; **(ii)** three states from the set; **(ii)** all states in the set.

(b) Give an ensemble decomposition for ρ that includes all pure states, i.e., is an integral over the surface of the Bloch sphere

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3. Schmidt decomposition for three qubits? (10 points).

Consider three qubits, A, B, and C.

(a) Show that an arbitrary pure state $|\psi\rangle$ of the three qubits can be transformed to the following Schmidt-like form using local unitary operators on A, B, and C:

$$\cos\theta|0\rangle\otimes\underbrace{\big(\cos\chi|0\rangle\otimes|0\rangle+\sin\chi|1\rangle\otimes|1\rangle\big)}_{|\phi_0\rangle}+$$

$$\sin\theta|1\rangle\otimes\underbrace{\Big[\cos\xi\,(\sin\chi|0\rangle\otimes|0\rangle-\cos\chi|1\rangle\otimes|1\rangle)+e^{i\delta}\sin\xi\,(\cos\eta|0\rangle\otimes|1\rangle+\sin\eta|1\rangle\otimes|0\rangle)\Big]}_{|\phi_1\rangle}\,.$$

The states $|\phi_0\rangle$ and $|\phi_1\rangle$ are orthonormal states of BC. Five parameters, θ,χ,ξ,η, and δ are necessary to specify an arbitrary three-qubit pure state; determine the range of these five parameters. [**Hint**: Schmidt decompose $|\psi\rangle$ with respect to the division A vs. BC. Then Schmidt decompose one of the resulting Schmidt states of BC with respect to the division B vs. C, writing the other BC Schmidt state in the resulting Schmidt bases of B and C.]

The presence of four terms in $|\phi_1\rangle$, instead of just the first two terms or the last two terms, prevents this from being a genuine three-qubit Schmidt decomposition. This illustrates why there is generally no three-particle Schmidt decomposition.

(b) Find all the marginal density operators of the three qubits, that is, $\rho_{AB}, \rho_{BC}, \rho_{AC}, \rho_A, \rho_B$, and ρ_C.

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4. Quantum nonlocality without probabilities (10 points).

Consider the following state of three qubits:

$$|\psi\rangle = \frac{|+++\rangle - |---\rangle}{\sqrt{2}}.$$

This is, up to a sign flip, the *Greenberger-Horne-Zeilinger* (GHZ) state.

(a) Show that $|\psi\rangle$ is a $+1$ eigenstate of $X \otimes Y \otimes Y$, $Y \otimes X \otimes Y$, and $Y \otimes Y \otimes X$.

(b) Use the results of part **(a)** to argue that each qubit has well-defined values of X and Y. For qubit j, denote these values by x_j and y_j. We say that these values are *elements of reality*. What does local realism, i.e., the assumption of realistic values that are undisturbed by measurements on other spins, predict for the product of the outcomes of measurements of X on each qubit?

(c) What does quantum mechanics predict for the product of the outcomes of X measurements on each qubit?

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5. Maximal Bell-CHSH violation. (20 points).

Consider two qubits, X and Y. Let $A = \boldsymbol{\sigma}_X \cdot \mathbf{a}$, $B = \boldsymbol{\sigma}_Y \cdot \mathbf{b}$, $C = \boldsymbol{\sigma}_X \cdot \mathbf{c}$, and $D = \boldsymbol{\sigma}_Y \cdot \mathbf{d}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are unit vectors in three dimensions. We omit the subscripts X and Y on the Pauli operators in the following because ordering in tensor products indicates which system the Pauli operators apply to. Now let

$$\begin{aligned}
\mathcal{B} &= A \otimes B + C \otimes B + C \otimes D - A \otimes D \\
&= \boldsymbol{\sigma} \cdot \mathbf{a} \otimes \boldsymbol{\sigma} \cdot (\mathbf{b} - \mathbf{d}) + \boldsymbol{\sigma} \cdot \mathbf{c} \otimes \boldsymbol{\sigma} \cdot (\mathbf{b} + \mathbf{d}) \\
&= |\mathbf{b} - \mathbf{d}| \boldsymbol{\sigma} \cdot \mathbf{a} \otimes \boldsymbol{\sigma} \cdot \mathbf{f} + |\mathbf{b} + \mathbf{d}| \boldsymbol{\sigma} \cdot \mathbf{c} \otimes \boldsymbol{\sigma} \cdot \mathbf{g}
\end{aligned}$$

be the *Bell operator*, where $\mathbf{f}$ and $\mathbf{g}$ are unit vectors which lie along the directions of $\mathbf{b} - \mathbf{d}$ and $\mathbf{b} + \mathbf{d}$. The quantity we called S in our discussion of the CHSH inequality is the expectation value of the Bell operator, i.e., $S = \langle \mathcal{B} \rangle$.

(a) Show the *Tsirelson's bound*: $|S| = |\langle \mathcal{B} \rangle| \leq 2\sqrt{2}$, which gives the maximal violation of the Bell-CHSH inequality.

(b) Find the conditions for equality in Tsirelson's bound.
