M525/P622 Quantum Information	Ninnat Dangniam

Homework Assignment 5a

20 points — DUE: TBA

1. Noisy measurement (10 points).

(a) Suppose you make a von Neumann measurement of the spin component of a spin-1/2 particle along one of the three Cartesian axes, tossing a fair three-sided die to determine which of the three measurements to make. Determine the Kraus operators, quantum operations, and POVM elements for this six-outcome measurement.

(b) Suppose now that all you know after the measurement of part **(a)** is whether the the result is $+1$ or -1 , but not which axis was used. Determine the Kraus operators, quantum operations, and POVM elements for this two-outcome measurement. This is a model of a very noisy measurement, the noise coming from not knowing how the measuring apparatus is oriented.

(c) Find at least one other pair of quantum operations that corresponds to the POVM of part **(b)**.

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2. From Neumark extension to a measurement model (10 points).

The goal of this problem is to convert the Neumark extension of a rank-one POVM in **P3** of **HW4** to a measurement model. Recall that in the original setting, we considered an N-element, rank-one POVM for a qubit system, and denoted the POVM elements by $E_\alpha = |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha|$, $\alpha = 1, ..., N$, where the vectors $|\overline{\psi}_\alpha\rangle = \sqrt{\mu_\alpha}\,|\psi_\alpha\rangle$ are subnormalized, with the norm squared $\mu_\alpha = \langle\overline{\psi}_\alpha|\overline{\psi}_\alpha\rangle$. The Neumark-extended vectors, $|\widehat{\psi}_\alpha\rangle$, live in a Hilbert space of dimension N, which is the *direct sum* of the qubit Hilbert space and a Hibert space of $N-2$ additional dimensions. The Neumark-extended vectors satisfy $P|\widehat{\psi}_\alpha\rangle = |\overline{\psi}_\alpha\rangle$, where $P = \sum_{\alpha=1}^{N} E_\alpha$ is the projector onto the qubit Hilbert space.

The immediate issue we run into when we want to convert the Neumark extension to a measurement model is that, in the latter, the total Hilbert space is the *tensor product* of the qubit Hilbert space with the ancillary Hilbert space. So now it does not make sense to tak about a projector P onto the qubit Hilbert space anymore.

We begin by choosing the ancillae to consist of $n-1$ additional qubits, where n is the smallest integer such that $2^n \geq N$. We then add $2^n - N$ zero POVM elements (that have zero probability of occurring) so that we formally have a 2^n-outcome, rank-one POVM. The index α now takes on 2^n values, so we replace it by a bitstring $\mathbf{a} = a_1 \ldots a_n$. The idea is to choose a two-dimensional subspace embedded in the n-qubit tensor product space to be our system Hilbert space. Consider the Hilbert space spanned by $|0\rangle \otimes |0\rangle^{\otimes(n-1)}$ and $|1\rangle \otimes |0\rangle^{\otimes(n-1)}$. The POVM elements take the form $E_{\mathbf{a}} = |\overline{\psi}_{\mathbf{a}}\rangle\langle\overline{\psi}_{\mathbf{a}}| \otimes (|0\rangle\langle 0|)^{\otimes(n-1)}$. The projector onto the system Hilbert space is

$$P = \sum_{\mathbf{a}} E_{\mathbf{a}} = \sum_{\mathbf{a}} |\overline{\psi}_{\mathbf{a}}\rangle\langle\overline{\psi}_{\mathbf{a}}| \otimes (|0\rangle\langle 0|)^{\otimes(n-1)} = \mathbb{1} \otimes (|0\rangle\langle 0|)^{\otimes(n-1)}$$

The Neumark-extended vector $|\widehat{\psi}_{\mathbf{a}}\rangle$ live in the n-qubit tensor product space and satisfy $P\,|\widehat{\psi}_{\mathbf{a}}\rangle = |\overline{\psi}_{\mathbf{a}}\rangle \otimes |0\rangle^{\otimes(n-1)}$.

(a) By considering the way the Neumark extension is constructed from the computational basis $|\mathbf{x}\rangle = |x_1 \dots x_n\rangle$, draw an n-qubit quantum circuit that corresponds to a measurement of the POVM.

(b) The circuit of part **(a)** does not fit our definition of a measurement model because it involves a direct measurement on the original qubit, instead of just measurements on the ancilla qubits. Construct a proper measurement model by adding one more ancilla qubit to your circuit. Draw the resulting $(n+1)$-qubit circuit, and determine the Kraus operators for the measurement model.

(c) By controlling on the outputs of the n measurements in part **(b)**, modify the circuit so that the Kraus operators are $A_{\mathbf{a}} = \sqrt{E_{\mathbf{a}}} = \sqrt{\mu_{\mathbf{a}}}\,|\psi_{\mathbf{a}}\rangle\langle\psi_{\mathbf{a}}| = |\psi_{\mathbf{a}}\rangle\langle\overline{\psi}_{\mathbf{a}}|$.
