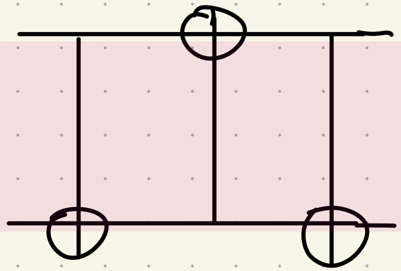
① Circuit identities

(1.1) What does this sequence of controlled-NOT gates do?



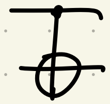
The definition of CNOT is

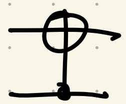
Control qubit　　Target qubit　　　Flip the value of the bit x

$$|0\rangle|x\rangle \mapsto |0\rangle|x\rangle$$

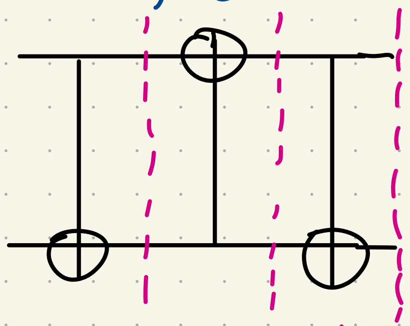
$$|1\rangle|x\rangle \mapsto |1\rangle|x\oplus 1\rangle = |1\rangle X|x\rangle$$

Matrix form

 $= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$

 $= \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix}$

The action of any gate (linear op.) is determined by its action on a basis set.



$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |11\rangle \rightarrow |10\rangle$

$|10\rangle \rightarrow |11\rangle \rightarrow |10\rangle \rightarrow |10\rangle$

$|11\rangle \rightarrow |10\rangle \rightarrow |01\rangle$

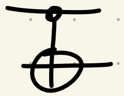
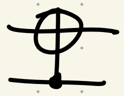
The circuit acts as a SWAP gate

$$\begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

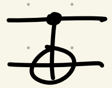
$$|\psi\rangle|\varphi\rangle \mapsto |\varphi\rangle|\psi\rangle$$

The action of the circuit on all 4 computational basis states can also be computed in one fell swoop using the XOR ($\oplus$) definition of CNOT (Credit to น้องไอซ์)

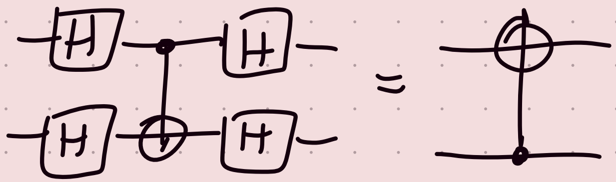
Because $x \oplus x = 0 \mod 2$

$$|x\rangle|y\rangle \longmapsto |x\rangle|y \oplus x\rangle \longmapsto |x \oplus y \oplus x\rangle|y \oplus x\rangle = |y\rangle|y \oplus x\rangle$$



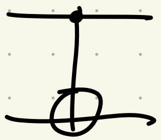
$$\longmapsto |y\rangle|y \oplus x \oplus y\rangle = |y\rangle|x\rangle \quad \text{SWAP gate}$$

(1.2) Show 

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow H^{\otimes 2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

There are several ways to do this without explicitly multiplying the matrix representation of the gates. I will show 2 ways.

Way 1 (Algebraic)

-  $= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$

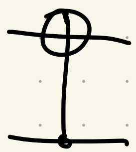
$$(H^{\otimes 2})\, \mathrm{CNOT}\, (H^{\otimes 2}) = |+\rangle\langle +| \otimes \mathbb{1} + |-\rangle\langle -| \otimes Z$$

$$= \frac{1}{2}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big) \otimes \mathbb{1}$$

$$+ \frac{1}{2}\big(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\big) \otimes Z$$

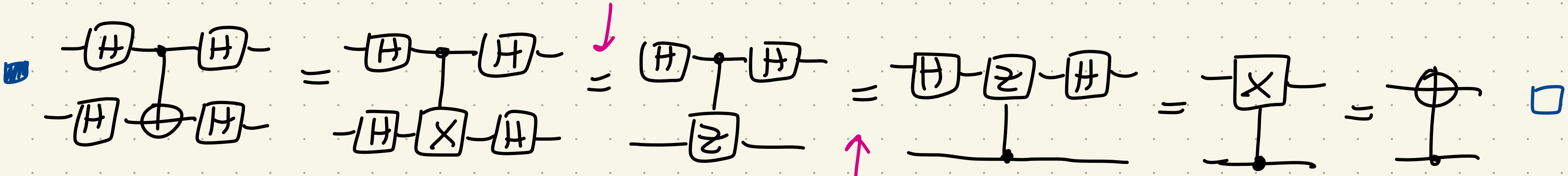
$$= \mathbb{1} \otimes \left(\frac{\mathbb{1}+Z}{2}\right) + X \otimes \left(\frac{\mathbb{1}-Z}{2}\right) = \mathbb{1} \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|$$

Compare

$=$  $\square$

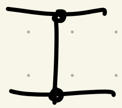
Way 2 (Graphical)

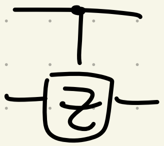
H switches btw X and Z bases



Controlled-Z (CZ; also called CPHASE) is symmetric btw the control and target qubits

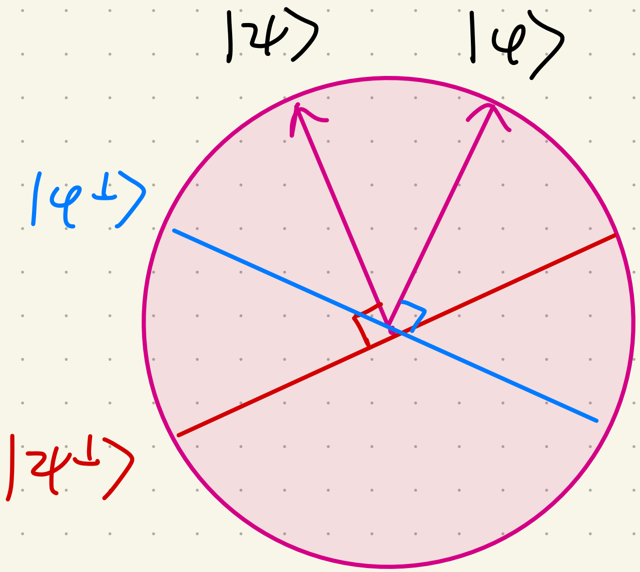
สัญลักษณ์ เผ CPHASE →





$$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} =$$

② Unambiguous state discrimination



$$E_1 = a|\psi^\perp\rangle\langle\psi^\perp|, \quad E_2 = a|\varphi^\perp\rangle\langle\varphi^\perp|,$$

$$E_3 = \mathbb{1} - E_1 - E_2$$

2.1 Find values of a that make E_3 a positive operator. Also find the value of a that makes E_3 positive and rank-one.

- We can choose to work in an ONB formed by $|\psi\rangle$ and $|\psi^\perp\rangle$. From the assumption given in the problem statement (see pdf), I can write

$$|\varphi\rangle = \langle\psi|\varphi\rangle|\psi\rangle + \langle\psi^\perp|\varphi\rangle|\psi^\perp\rangle = \Delta|\psi\rangle + \sqrt{1-\Delta^2}\,|\psi^\perp\rangle.$$

Can choose positive square root without loss of generality. Δ can take negative value, so we can still have a relative phase.

Now $E_3 = \mathbb{1} - a\left(|\psi^\perp\rangle\langle\psi^\perp| + |\varphi^\perp\rangle\langle\varphi^\perp|\right)$

$$= (1-2a)\mathbb{1} + a\left(|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|\right)$$

$$|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi| = (1+\Delta^2)|\psi\rangle\langle\psi| + (1-\Delta^2)|\psi^\perp\rangle\langle\psi^\perp| + \Delta\sqrt{1-\Delta^2}\left(|\psi\rangle\langle\psi| + |\psi^\perp\rangle\langle\psi^\perp|\right)$$

$$= \mathbb{1} + \Delta^2 Z_\psi + \Delta\sqrt{1-\Delta^2}\, X_\psi$$

where Z_ψ and X_ψ have the matrix form $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ respectively in the ONB $\{|\psi\rangle, |\psi^\perp\rangle\}$.

Unit vector $\hat{n} = \begin{pmatrix} \sqrt{1-\Delta^2} \\ 0 \\ \Delta \end{pmatrix}$

$$= \mathbb{1} + \Delta\, \hat{n}\cdot\vec{\sigma}$$

$\vec{\sigma} = \begin{pmatrix} X_\psi \\ Y_\psi \\ Z_\psi \end{pmatrix}$ Vector of Pauli observables

The key point is that we know that $\hat{n}\cdot\vec{\sigma}$ has eigenvalues ± 1.
(It is a spin observable in direction $\hat{n}$.)

Therefore, $E_3 = (1-2a)\mathbb{1} + a\,(\underbrace{|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|}_{\mathbb{1}+\Delta\hat{n}\cdot\vec{\sigma}})$

$$= (1-a)\mathbb{1} + a\Delta\,\hat{n}\cdot\vec{\sigma}$$

The eigenvalues of E_3 are $1-a \pm a\Delta$. Thus, to make E_3 a positive operator,

$$1 - a(1\pm\Delta) \geq 0$$

$$a \leq \frac{1}{1\pm\Delta}$$

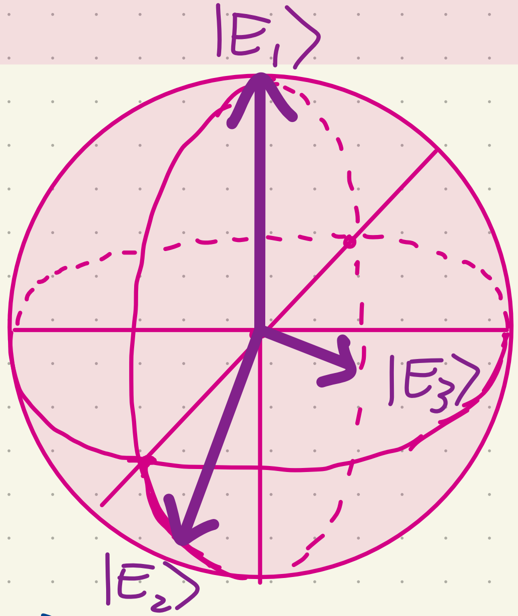
Remember that Δ is real but can be negative.

$$\boxed{a \leq \frac{1}{1+|\Delta|}}$$

In addition, to make E_3 rank-one (has only one non-zero eigenvalue), we simply set

$$\boxed{a = \frac{1}{1+|\Delta|}} \quad \square$$

(2.2) Specialize to $|\psi\rangle = |1\rangle$ and $|\varphi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$, and let the value of the constant a be such that E_3 is a rank-one, positive operator. In particular, we can write $E_3 = |E_3\rangle\langle E_3|$. Find the subnormalized vector $|E_3\rangle$.



- Since we now know $|\psi\rangle$ and $|\varphi\rangle$, we can figure out $\Delta = \langle\psi|\varphi\rangle$ and $a = \dfrac{1}{1+|\Delta|}$

$$\Delta = -\frac{1}{2} \implies a = \frac{2}{3}$$

$$E_3 = (1-a)\mathbb{1} + a\Delta\left(\Delta Z + \sqrt{1-\Delta^2}\,X\right)$$

$$= \frac{1}{3}\mathbb{1} - \frac{1}{3}\left(-\frac{Z}{2} + \frac{\sqrt{3}}{2}X\right) = \frac{2}{3}\frac{\mathbb{1}}{2} + \frac{2}{3}\left(\frac{Z}{2} - \frac{\sqrt{3}}{2}X\right)/2$$

$$= \frac{2}{3}\left(\frac{\mathbb{1} + \hat{n}\cdot\vec{\sigma}}{2}\right) \text{ where } \hat{n} = \begin{pmatrix} +\sqrt{3}/2 \\ 0 \\ -1/2 \end{pmatrix} \implies |E_3\rangle = \sqrt{\frac{2}{3}}\,|\hat{n}\rangle$$

$\theta = +60^\circ$ $-+2\pi/3$

$$= \boxed{\sqrt{\frac{2}{3}}\left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)} \quad \square$$

(2.3) Can you implement this measurement as a measurement of an observable in a three-dimensional Hilbert space? That is, can you extend

$$\begin{aligned}|E_1\rangle &= \alpha_1|0\rangle + \beta_1|1\rangle\\ |E_2\rangle &= \alpha_2|0\rangle + \beta_2|1\rangle\\ |E_3\rangle &= \alpha_3|0\rangle + \beta_3|1\rangle\end{aligned} \quad \text{to an } \underline{\text{ONB}} \quad \begin{aligned}|\tilde{E}_1\rangle &= \alpha_1|0\rangle + \beta_1|1\rangle + \gamma_1|2\rangle\\ |\tilde{E}_2\rangle &= \alpha_2|0\rangle + \beta_2|1\rangle + \gamma_2|2\rangle\\ |\tilde{E}_3\rangle &= \alpha_3|0\rangle + \beta_3|1\rangle + \gamma_3|2\rangle\end{aligned} \; ?$$

- Enlarge the qubit Hilbert space by extending the ONB to $\{|0\rangle, |1\rangle, |2\rangle\}$.

$$\begin{array}{cc} & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \\ |2\rangle \end{array} & \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ ? & ? & ? \end{pmatrix} \end{array}$$

$$|\tilde{E}_1\rangle = \sqrt{\frac{2}{3}}|0\rangle + \alpha_1|2\rangle$$

$$|\tilde{E}_2\rangle = \frac{1}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle + \alpha_2|2\rangle$$

$$|\tilde{E}_3\rangle = \frac{1}{\sqrt{6}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle + \alpha_3|2\rangle$$

Since each tilde vector must be normalized, we know immediately that

$|\alpha_1| = |\alpha_2| = |\alpha_3| = \frac{1}{\sqrt{3}}$. Then we can start

from $\langle 0|0\rangle$ $\langle 1|1\rangle$ $\langle 2|2\rangle$

$$\langle \tilde{E}_2 | \tilde{E}_3 \rangle = \frac{1}{6} - \frac{1}{2} + \alpha_2^* \alpha_3 \Rightarrow \alpha_2^* \alpha_3 - \frac{1}{3} = 0 \Rightarrow \text{Can choose } \boxed{\alpha_2 = \alpha_3 = \frac{1}{\sqrt{3}}}$$

$\langle 0|0\rangle$ $\langle 2|2\rangle$

$$\langle \tilde{E}_1 | \tilde{E}_2 \rangle = \langle \tilde{E}_1 | \tilde{E}_3 \rangle = \frac{1}{3} + \frac{\alpha_1}{\sqrt{3}} \Rightarrow \boxed{\alpha_1 = -\frac{1}{\sqrt{3}}}$$

$\square$

③ Kraus operators

From the lectures, the action of a dephasing channel which couples a qubit to a meter Hilbert space $\mathcal{H}_M$ with orthonormal states $|0\rangle, |1\rangle, |2\rangle$ is defined as follows.

$$|0\rangle_A |0\rangle_M \mapsto \sqrt{1-p}\, |0\rangle_A |0\rangle_M + \sqrt{p}\, |0\rangle_A |1\rangle_M$$

$$|1\rangle_A |0\rangle_M \mapsto \sqrt{1-p}\, |1\rangle_A |0\rangle_M + \sqrt{p}\, |1\rangle_A |2\rangle_M$$

The action is realized by a unitary U in the joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_M$. Find the three Kraus operators defined as $K_0 = {}_M\langle 0|U|0\rangle_M$, $K_1 = {}_M\langle 1|U|0\rangle_M$, $K_2 = {}_M\langle 2|U|0\rangle_M$

- The given action partially determines the 6×6 unitary U as follows.

Output of $|00\rangle$ $= \sqrt{1-p}\,|00\rangle + \sqrt{p}\,|01\rangle$

Output of $|10\rangle$ $= \sqrt{1-p}\,|10\rangle + \sqrt{p}\,|12\rangle$

$$U = \begin{array}{c} \begin{array}{cccccc} 00 & 01 & 02 & 10 & 11 & 12 \end{array} \\ \begin{bmatrix} \sqrt{1-p} & & & 0 & & \\ \sqrt{p} & & & 0 & & \\ 0 & & & 0 & & \\ 0 & & & \sqrt{1-p} & & \\ 0 & & & 0 & & \\ 0 & & & \sqrt{p} & & \end{bmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 02 \\ 10 \\ 11 \\ 12 \end{array}$$

Now to read out the matrix elements of ${}_m\langle \alpha | U | 0\rangle_m$, $\alpha = 0,1,2$, we have to think about what it means to sandwich an operator in $\mathcal{H}_A \otimes \mathcal{H}_m$ by vectors in $\mathcal{H}_m$:

${}_m\langle 0 | U | 0\rangle_m$ is an operator on $\mathcal{H}_A$ that, when sandwiched between states $|x\rangle, |y\rangle \in \mathcal{H}_A$, $x,y = 0,1$

$${}_A\langle y| \otimes {}_m\langle 0 | U | 0\rangle_m \otimes |x\rangle_A,$$

Now each row/column is labelled by a pair of numbers, not just one. So a matrix element is labelled by 4 numbers.

returns the matrix element $(y0, x0)$. So ${}_m\langle 0 | U | 0\rangle_m$ should be the 2×2 matrix

$$\begin{pmatrix} U_{00,00} & U_{00,10} \\ U_{10,00} & U_{10,10} \end{pmatrix} = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} = \sqrt{1-p}\,\mathbb{1} = K_0$$

The remaining cases work the same:

$$K_1 := {}_m\langle 1|U|0\rangle_m = \begin{pmatrix} U_{01,00} & U_{01,10} \\ U_{11,00} & U_{11,10} \end{pmatrix} = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}$$

$$K_2 := {}_m\langle 2|U|0\rangle_m = \begin{pmatrix} U_{02,00} & U_{02,10} \\ U_{12,00} & U_{12,10} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \quad \square$$