Tutorial problems & solutions

① Show that the state $|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is entangled.

- What "entangled" means is that the state can't be written in the product form $|\psi\rangle|\varphi\rangle$. So suppose for the sake of the proof that it has a product form.

$$(\alpha|0\rangle_A + \beta|1\rangle_A) \otimes (\gamma|0\rangle_B + \delta|1\rangle_B) = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

We wish to show that this is inconsistent with the state being $|\Phi^+\rangle$.

$$\left.\begin{aligned} \alpha\gamma = \frac{1}{\sqrt{2}} &\Rightarrow \gamma = \frac{1}{\sqrt{2}\alpha} \\ \beta\delta = \frac{1}{\sqrt{2}} &\Rightarrow \delta = \frac{1}{\sqrt{2}\beta} \end{aligned}\right\} |\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + \frac{\alpha}{\beta}|01\rangle + \frac{\beta}{\alpha}|10\rangle + |11\rangle\right)$$

We see that there is no way to make both $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ zero at the same time. □

(We can also start from $\alpha\delta = \beta\gamma = 0$ and examine case-by-case

$$\begin{cases} \alpha = \beta = 0 \Rightarrow 0 \\ \alpha = \gamma = 0 \Rightarrow |11\rangle \\ \delta = \beta = 0 \Rightarrow |00\rangle \\ \delta = \gamma = 0 \Rightarrow 0 \end{cases} \quad |\psi\rangle_{AB}$$

② Show that the singlet state $\dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$ can be written as

$$\frac{|\hat{n}\rangle|-\hat{n}\rangle - |-\hat{n}\rangle|\hat{n}\rangle}{\sqrt{2}}$$ for any spin direction $|\hat{n}\rangle$

- We won't need the explicit form of $|\hat{n}\rangle$ (in terms of the Bloch sphere angles).

Just need $|\hat{n}\rangle = \alpha|0\rangle + \beta|1\rangle$, $|-\hat{n}\rangle = -\beta^*|0\rangle + \alpha^*|1\rangle$ $\Rightarrow \langle\hat{n}|-\hat{n}\rangle = (-\beta\ \alpha)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

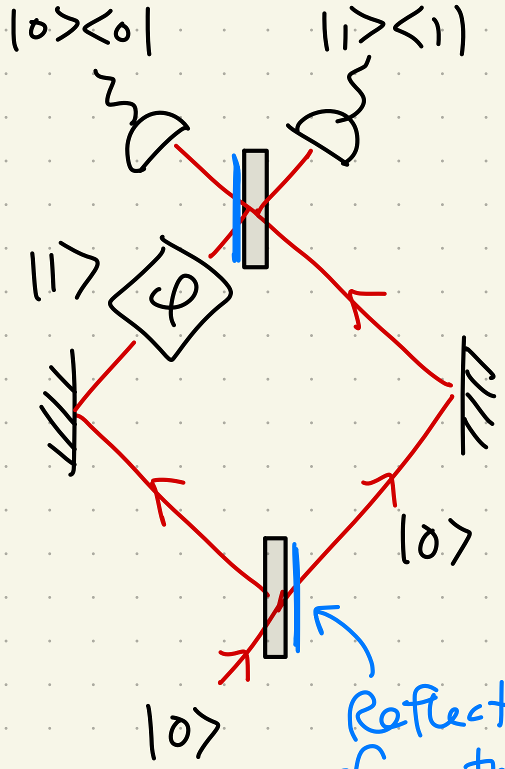
$= -\alpha\beta + \alpha\beta = 0$

$$|\hat{n}\rangle|-\hat{n}\rangle - |-\hat{n}\rangle|\hat{n}\rangle = (\alpha|0\rangle + \beta|1\rangle)(-\beta^*|0\rangle + \alpha^*|1\rangle) - (-\beta^*|0\rangle + \alpha^*|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$$

$$= \cancel{-\alpha\beta^*|00\rangle} + |\alpha|^2|01\rangle - |\beta|^2|10\rangle + \cancel{\alpha^*\beta|11\rangle}$$
$$\cancel{+\alpha\beta^*|00\rangle} + |\beta|^2|01\rangle - |\alpha|^2|10\rangle \cancel{-\alpha^*\beta|11\rangle}$$

$$= (|\alpha|^2 + |\beta|^2)|01\rangle - (|\alpha|^2 + |\beta|^2)|10\rangle = |01\rangle - |10\rangle \quad \square$$

③ Mach-Zehnder interferometer



Reflection from this side gives a π phase shift

3.1 Show that the probability of detecting a photon is $\cos^2(\varphi/2)$ for the left ("0") detector and $\sin^2(\varphi/2)$ for the right ("1") detector.

$$M_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \hat{\varphi} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

Beamsplitter $B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- Final state $= B\hat{\varphi}M_1M_0B|0\rangle = B\hat{\varphi}M_1M_0\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$

$$= -B\hat{\varphi}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = -B\left(\frac{|0\rangle+e^{i\varphi}|1\rangle}{\sqrt{2}}\right)$$

$$= -\frac{|0\rangle+|1\rangle+e^{i\varphi}|0\rangle-e^{i\varphi}|1\rangle}{2} = -\frac{(1+e^{i\varphi})|0\rangle+(1-e^{i\varphi})|1\rangle}{2}$$

$$= -e^{i\varphi/2}\left[\left(\frac{e^{-i\varphi/2}+e^{i\varphi/2}}{2}\right)|0\rangle + i\left(\frac{e^{-i\varphi/2}-e^{i\varphi/2}}{2i}\right)|1\rangle\right] = e^{i\varphi/2}\left(\cos\left(\frac{\varphi}{2}\right)|0\rangle + i\sin\left(\frac{\varphi}{2}\right)|1\rangle\right)$$

take modulus square ให้ได้ความน่าจะเป็น □

There is a somewhat quicker way if you know that the beam splitter matrix is just the Hadamard gate which switches between the X and Z bases:

$$BZB = B$$

$$B\hat{\varphi}M_0M_1B = -B\hat{\varphi}B = -e^{i\varphi/2}B\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}B$$

$$= -e^{i\varphi/2}Be^{-i\varphi Z/2}B = -e^{i\varphi/2}e^{-i\varphi BZB/2}$$

$$= -e^{i\varphi/2}e^{-i\varphi X/2} = -e^{i\varphi/2}\left(\cos\left(\frac{\varphi}{2}\right)\mathbb{1} - i\sin\left(\frac{\varphi}{2}\right)X\right)$$

So $B\hat{\varphi}M_0M_1B|0\rangle = -e^{i\varphi/2}\left[\cos(\varphi/2)|0\rangle - i\sin\left(\frac{\varphi}{2}\right)|1\rangle\right]$

$$= e^{i\varphi/2}\left[\cos(\varphi/2)|0\rangle + i\sin\left(\frac{\varphi}{2}\right)|1\rangle\right]$$

4 Interaction Hamiltonian

4.1 Show that if H is an operator that squares to the identity $H^2 = \mathbb{1}$, then

$$e^{iH\theta} = \cos\theta\,\mathbb{1} + i\sin\theta\, H$$

- By definition

$$e^{iH\theta} = \sum_{k=0}^{\infty} \frac{(i\theta H)^k}{k!} = \mathbb{1} + i\theta H - \frac{(\theta H)^2}{2} - i\frac{(\theta H)^3}{3!} + \frac{(\theta H)^4}{4!} + \dots$$

$$= \overbrace{\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right)}^{\text{Even terms}} \mathbb{1} + i\overbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}^{\text{Odd terms}} H$$

$$= \cos\theta\,\mathbb{1} + i\sin\theta\, H \qquad \square$$

4.2 Show that $e^{iA\otimes\mathbb{1}} = e^{iA}\otimes\mathbb{1}$ (Perhaps too boring and can be skipped)

$$e^{iA\otimes\mathbb{1}} = \sum_{k=0}^{\infty}\frac{(iA\otimes\mathbb{1})^k}{k!} = \mathbb{1} + iA\otimes\mathbb{1} - \frac{A^2\otimes\mathbb{1}}{2!} - i\frac{A^3\otimes\mathbb{1}}{3!} + \dots = e^{iA}\otimes\mathbb{1}$$

(4.2) Show that $e^{-i\theta X_A Y_B}$ produces an entangled state when acting on the product state $|00\rangle$. Can you get the Bell state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$?

$$e^{-i\theta X_A Y_B}|00\rangle = \left(\cos\theta\, \mathbb{1}_A \mathbb{1}_B - i\sin\theta\, X_A Y_B\right)|00\rangle$$

Kronecker product

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Y = \begin{pmatrix} 0Y & 1Y \\ 1Y & 0Y \end{pmatrix} = \begin{pmatrix} 0 & & & -i \\ & & i & \\ & -i & & \\ i & & & 0 \end{pmatrix}$$

Only need to know the first column

$$= \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix}|00\rangle$$

$$= \cos\theta|00\rangle + \sin\theta|11\rangle$$

Yes, you can get to $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ by setting $\theta = \frac{\pi}{4}$. □

⑤ Teleportation

Confirm that the teleportation process in the lectures actually work. That is, Charlie's final state is the state that Bob wants to send.

- There are various ways one can go about the calculation (e.g. Wikipedia, textbooks etc.) I'll just copy my way from the lecture notes here:

$$ {}_{AB}\langle\Omega_{ab}|\left(|\psi\rangle_A|\Omega_{00}\rangle_{BC}\right) = \frac{1}{2}\left[\sum_j \langle j_A j_B|(Z^b X^a)^\dagger_A \otimes \mathbb{1}_B\right]|\psi\rangle \sum_k |k_B k_C\rangle $$

You need this easy fact in the proof

$$ = \frac{1}{2}\sum_{jk} \langle j|\underbrace{X^a Z^b|\psi\rangle}_{|\psi'\rangle} \overbrace{\langle j_B|k_B\rangle}^{\delta_{jk}}|k_C\rangle $$

$$ = \frac{1}{2}\sum_j \langle j|\psi'\rangle|j\rangle_C = \frac{1}{2}|\psi'\rangle_C $$

This is the state Charlie has to correct (by an instruction from Bob's message) to recover the original state $|\psi\rangle$.

$\frac{1}{2}$ comes from the probability $\frac{1}{4}$ for each outcome of the Bell measurement

$|\Omega_{00}\rangle$

$|\Omega_{01}\rangle = \mathbb{1}\otimes Z|\Omega\rangle = Z\otimes\mathbb{1}|\Omega\rangle$

$|\Omega_{10}\rangle = \mathbb{1}\otimes X|\Omega\rangle = X\otimes\mathbb{1}|\Omega\rangle$

$|\Omega_{11}\rangle = \mathbb{1}\otimes XZ|\Omega\rangle = ZX\otimes\mathbb{1}|\Omega\rangle$

⑥ Partial trace

Consider these two three-qubit states

a) $\dfrac{|000\rangle + |111\rangle}{\sqrt{2}}$ b) $\dfrac{|100\rangle + |010\rangle + |001\rangle}{\sqrt{3}}$

Are they entangled? Try to see what happens if you trace out one of the qubits. What if you trace out two of the qubits? Can they transform into each other under local unitaries?

- A multipartite pure state is entangled if its reduced state is mixed. (More accurately, the state is "genuinely entangled" = entangled across all qubits iff the reduced state across any bipartition is mixed.)

a) $\mathrm{tr}_C(|\psi\rangle\langle\psi|) = \frac{1}{2}\mathrm{tr}_C\big(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\big)$

$= \dfrac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$ Mixture of product states ("classically correlated state")

Cross terms in the third qubit are killed off by the partial trace

$$\frac{1}{2}\mathrm{tr}_B\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$$ Maximally mixed state

Pink = terms that survive

b) $$\mathrm{tr}_C(|\psi\rangle\langle\psi|) = \frac{1}{3}\mathrm{tr}\Big[|100\rangle\langle 100| + |100\rangle\langle 010| + |100\rangle\langle 001|$$
$$+ |010\rangle\langle 100| + |010\rangle\langle 010| + |010\rangle\langle 001|$$
$$+ |001\rangle\langle 100| + |001\rangle\langle 010| + |001\rangle\langle 001|\Big]$$

$$= \frac{(|10\rangle + |01\rangle)(\langle 10| + \langle 01|) + |00\rangle\langle 00|}{3}$$

$$= \frac{2|\psi_+\rangle\langle\psi_+| + |00\rangle\langle 00|}{3}$$

Mixture of an entangled state and a product state

$\Rightarrow$ States a) and b) are not interconvertible because, for example, their two-qubit RDMs ρ_{AB} have different spectra.

$$\frac{1}{3}\mathrm{tr}_B\left(2|\psi_+\rangle\langle\psi_+| + |00\rangle\langle 00|\right)$$

$$= \frac{1}{3}\left(2\,\mathrm{tr}_B|\psi_+\rangle\langle\psi_+| + \mathrm{tr}_B|00\rangle\langle 00|\right)$$

$$= \frac{\mathbb{1} + |0\rangle\langle 0|}{3} = \frac{2|0\rangle\langle 0| + |1\rangle\langle 1|}{3}$$

Note that in these two examples, it doesn't matter at any step which qubit we trace out, because of permutation symmetry