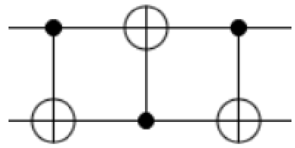
NAS×IF Summer School 2022 **Quantum Information**

Problem Set II

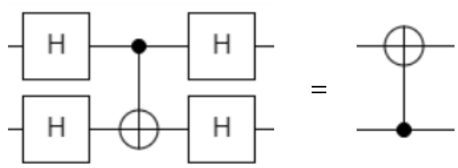
June, 9 2022 Ninnat Dangniam

1. Circuit identities.

(a) What does this series of CNOTs do?



(b) Show the following circuit identity.



That is, switching between the Z and the X bases also switches the role of the control and the target qubits.

2. Unambiguous state discrimination. Suppose that we are given either a qubit state $|\psi\rangle$ or $|\varphi\rangle$ with equal probability. It is possible to construct a three-outcome POVM,

$$E_1 = a\left|\psi^{\perp}\right\rangle\!\left\langle\psi^{\perp}\right|, \qquad E_2 = a\left|\varphi^{\perp}\right\rangle\!\left\langle\varphi^{\perp}\right|, \qquad E_3 = \mathbb{1} - E_1 - E_2. \tag{1}$$

that lets us conclude with certainty that we *do not* have one of the states (when we obtain either E_1 or E_2), but with some probability of obtaining an uninformative outcome (E_3).

For simplicity, you may assume that $\Delta := \langle\psi|\varphi\rangle$ and $\left\langle\psi^{\perp}|\varphi\right\rangle$ are real, since we can always rotate the two states that we want to distinguish to be in the X-Z plane of the Bloch sphere.

(a) What a has to be to ensure that E_3 is a positive operator? What a has to be for E_3 to be a rank-one positive operator?

(b) From now on, assume that E_3 is a rank-one positive operator and specialize to the case where

$$|\psi\rangle = |1\rangle\,, \qquad |\varphi\rangle = \frac{\sqrt{3}}{2}\,|0\rangle - \frac{1}{2}\,|0\rangle\,. \tag{2}$$

Being rank-one, E_3 can be written as an outer product $E_3 = |E_3\rangle\langle E_3|$ of a *subnormalized* vector $|E_3\rangle$ (i.e. $\langle E_3|E_3\rangle < 1$). Your goal is to find $|E_3\rangle$. This measurement is sometimes called a *Mercedes-Benz POVM* for obvious reason.

(c) The POVM in **(b)** can be realized by measuring in an ONB of a three-dimensional Hilbert space (spanned by $\{|0\rangle, |1\rangle, |2\rangle\}$) and projecting down to the two-dimensional subspace spanned by $\{|0\rangle, |1\rangle\}$. Find this higher-dimensional measurement. That is, extend

$$|E_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle, \tag{3}$$
$$|E_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle, \tag{4}$$
$$|E_3\rangle = \alpha_3 |0\rangle + \beta_3 |1\rangle, \tag{5}$$

to an ONB

$$\left|\widetilde{E}_1\right\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle + \gamma_1 |2\rangle, \tag{6}$$
$$\left|\widetilde{E}_2\right\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle + \gamma_2 |2\rangle, \tag{7}$$
$$\left|\widetilde{E}_3\right\rangle = \alpha_3 |0\rangle + \beta_3 |1\rangle + \gamma_3 |2\rangle. \tag{8}$$

3. Kraus operators. From the lectures, the action of a dephasing channel is defined by a unitary U that acts on the joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ in the following way.

$$|0\rangle_A \mapsto \sqrt{1-p}\, |0\rangle_A |0\rangle_B + \sqrt{p}\, |0\rangle_A |1\rangle_B, \tag{9}$$
$$|1\rangle_A \mapsto \sqrt{1-p}\, |1\rangle_A |0\rangle_B + \sqrt{p}\, |1\rangle_A |2\rangle_B, \tag{10}$$

where $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$ are orthonormal vectors in $\mathcal{H}_B$. Find the three Kraus operators

$$K_j = {}_B\langle j|U|0\rangle_B, \qquad j = 0, 1, 2, \tag{11}$$

for this quantum channel.