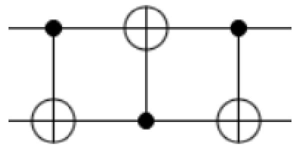
NAS×IF Summer School 2022	**Quantum Information**

Problem Set II

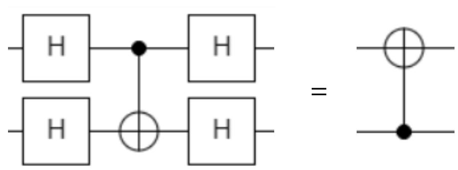
June, 9 2022 — Ninnat Dangniam

1. Circuit identities.

(a) What does this sequence of CNOT gates do?



(b) Show the following circuit identity.



That is, switching between the Z and the X bases also switches the role of the control and the target qubits.

2. Unambiguous state discrimination. Suppose that we are given either a qubit state $|\psi\rangle$ or $|\varphi\rangle$ with equal probability. It is possible to construct a three-outcome POVM,

$$E_1 = a \left|\psi^\perp\right\rangle\!\left\langle\psi^\perp\right|, \qquad E_2 = a \left|\varphi^\perp\right\rangle\!\left\langle\varphi^\perp\right|, \qquad E_3 = \mathbb{1} - E_1 - E_2. \tag{1}$$

that lets us conclude with certainty that we *do not* have one of the states (when we obtain either E_1 or E_2), but with some probability of obtaining an uninformative outcome (E_3).

For simplicity, you may assume that $\Delta := \langle\psi|\varphi\rangle$ and $\langle\psi^\perp|\varphi\rangle$ are real, since we can always rotate the two states that we want to distinguish to be in the X-Z plane of the Bloch sphere.

(a) What a has to be to ensure that E_3 is a positive operator? What a has to be for E_3 to be a rank-one positive operator?

(b) From now on, assume that E_3 is a rank-one positive operator and specialize to the case where

$$|\psi\rangle = |1\rangle\,, \qquad |\varphi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\,. \tag{2}$$

Being rank-one, E_3 can be written as an outer product $E_3 = |E_3\rangle\langle E_3|$ of a *subnormalized* vector $|E_3\rangle$ (i.e. $\langle E_3|E_3\rangle < 1$). Your goal is to find $|E_3\rangle$. This measurement is unitarily equivalent to what is called a *Mercedes-Benz* POVM for obvious reason.

(c) The POVM in **(b)** can be realized by measuring in an ONB of a three-dimensional Hilbert space (spanned by $\{|0\rangle, |1\rangle, |2\rangle\}$) and projecting down to the two-dimensional subspace spanned by $\{|0\rangle, |1\rangle\}$. Find this higher-dimensional measurement. That is, extend

$$|E_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle, \tag{3}$$
$$|E_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle, \tag{4}$$
$$|E_3\rangle = \alpha_3 |0\rangle + \beta_3 |1\rangle, \tag{5}$$

to an ONB

$$\left|\widetilde{E}_1\right\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle + \gamma_1 |2\rangle, \tag{6}$$
$$\left|\widetilde{E}_2\right\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle + \gamma_2 |2\rangle, \tag{7}$$
$$\left|\widetilde{E}_3\right\rangle = \alpha_3 |0\rangle + \beta_3 |1\rangle + \gamma_3 |2\rangle. \tag{8}$$
$$\left\langle \widetilde{E}_j \middle| \widetilde{E}_k \right\rangle = \delta_{jk} \tag{9}$$

This process is called *Naimark/Neumark extension*.[1]

3. Kraus operators. The action of a dephasing channel is defined in the lectures via the action

$$|0\rangle_A \mapsto \sqrt{1-p}\, |0\rangle_A |0\rangle_B + \sqrt{p}\, |0\rangle_A |1\rangle_B, \tag{10}$$
$$|1\rangle_A \mapsto \sqrt{1-p}\, |1\rangle_A |0\rangle_B + \sqrt{p}\, |1\rangle_A |2\rangle_B, \tag{11}$$

of a unitary U acting on a joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$ are orthonormal vectors in $\mathcal{H}_B$. Find the three Kraus operators

$$K_j = {}_B\langle j|U|0\rangle_B, \qquad j = 0, 1, 2, \tag{12}$$

for this quantum channel.

[1] Naimark and Neumark are two common transliterations of the same Russian name.