NAS×IF Summer School 2022	**Quantum Information**

Problem set I

June, 2 2022 — Ninnat Dangniam

1. Mach-Zehnder interferometer. Consider the setup of a single-photon Mach-Zehnder interferometer in the lectures. Each beamsplitter (gray bar) puts an incoming photon in an equal su-

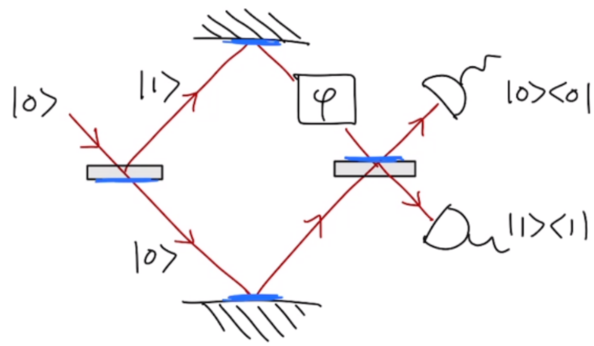

Figure 1: A Mach-Zehnder interferometer with a phase-shifter

perposition of being transmitted and being reflected. In addition, any time a photon is reflected from the blue side of a mirror, it gains an extra π phase-shift. Therefore, the unitary matrices that correspond to a beamsplitter and the mirror in the lower arm ("$|0\rangle$") and the upper arm ("$|1\rangle$") are

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad M_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

respectively. Finally, the phase-shifter (the box with symbol φ) is simply a Z rotation

$$R_Z(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}, \tag{2}$$

in the Bloch sphere picture. Show that the probability of detecting a photon in arm ("$|0\rangle$") and ("$|1\rangle$") of the interferometer are $\cos^2(\varphi/2)$ and $\sin^2(\varphi/2)$ respectively.

2. Bell states

(a) Convince yourself that the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is entangled.

(b) Show that the state $(|01\rangle - |10\rangle)/\sqrt{2}$ can be written as $(|\hat{\mathbf{n}}\rangle\,|-\hat{\mathbf{n}}\rangle - |-\hat{\mathbf{n}}\rangle\,|\hat{\mathbf{n}}\rangle)/\sqrt{2}$ for any spin direction $\hat{\mathbf{n}}$. That is, the singlet state is *rotationally-invariant* under any unitaries of the form $U \otimes U$.

3. Entangling unitaries.

(a) Show that if H is an operator that squares to the identity: $H^2 = \mathbb{1}$, then

$$e^{iH\theta} = \cos(\theta)\mathbb{1} + i\sin(\theta)H. \tag{3}$$

In particular, H does not have to be a single-qubit operator. Any Pauli string on the Hilbert space of n qubits squares to the identity, for example.

(b) Unitary operators of the form $U \otimes V$ are local. But when it comes to observables and Hamiltonians, the local ones are of the form $A \otimes \mathbb{1} + \mathbb{1} \otimes B$. Show that $\exp(A \otimes \mathbb{1}) = \exp(A) \otimes \mathbb{1}$. Therefore, the exponential of local observables factorizes into local unitaries.

(c) Show that the time evolution $\exp(-i\theta X \otimes Y)$ is able to produce an entangled state when acting on a product state $|00\rangle$. Can you get the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$? What about $(|01\rangle + |10\rangle)/\sqrt{2}$

4. The partial trace. Consider the following three-qubit states

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}}, \qquad \frac{|100\rangle + |010\rangle + |001\rangle}{\sqrt{3}}. \tag{4}$$

Are they entangled states? Try to see what happens if you trace out one of the qubits. What if you trace out two of the qubits? Is there a local unitary that transforms these two states into each other? (Think about what properties are invariant under local unitaries.)

5. Teleportation. Verify that the teleportation process in the lectures actually work; that is, show that Charlie's final state is the original state $|\psi\rangle$ that Bob wanted to send.