Tensor product

Suppose that we have two quantum systems described by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.
What is the Hilbert space of the joint system?

ONBs $\{|e_j\rangle\}_j$ for $\mathcal{H}_A$ dim $= k$, $\{|f_k\rangle\}_k$ for $\mathcal{H}_B$ dim $= m$

Cartesian product $\{|e_j\rangle\}_j \times \{|f_k\rangle\}_k$ is a set of size km

of pairs $(e_j, f_k) \Rightarrow \begin{cases} \text{Particle A is in state } |e_j\rangle \\ \text{—"— B —"— } |f_k\rangle \end{cases}$

Not an inner product.
I ran out of notations.

But in quantum theory, we are allowed to take a superposition of states.

$$\sum_{jk} c_{jk}(e_j, f_k), \quad \forall c_{jk} \in \mathbb{C}$$

The vector space that contains these superpositions is the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ and we write $\sum_{jk} c_{jk} |e_j\rangle_A \otimes |f_k\rangle_B$ or $\sum_{jk} c_{jk} |e_j f_k\rangle$

Example

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\otimes\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \begin{array}{c} \begin{pmatrix}\alpha\gamma\\ \alpha\delta\\ \beta\gamma\\ \beta\delta\end{pmatrix} \end{array} \begin{array}{l} 00\\ 01\\ 10\\ 11\end{array}$$

However, tensor product is not unique to quantum theory.

$$\vec{p} = \begin{pmatrix}p\\ 1-p\end{pmatrix},\ \vec{q} = \begin{pmatrix}q\\ 1-q\end{pmatrix}$$

Joint probability

$$\begin{pmatrix}pq\\ p(1-q)\\ (1-p)q\\ (1-p)(1-q)\end{pmatrix}$$

(But here we restrict ourselves to positive coeffients (convex combination) only)

Product state $|v\rangle\otimes|w\rangle$, $\quad |v\rangle\in V,\ |w\rangle\in W$

Scalar multiplication

$$\lambda(|v\rangle\otimes|w\rangle) = (\lambda|v\rangle)\otimes|w\rangle = |v\rangle\otimes(\lambda|w\rangle)$$

Vector addition

$$(|v_1\rangle+|v_2\rangle)\otimes|w\rangle = |v_1\rangle\otimes|w\rangle + |v_2\rangle\otimes|w\rangle$$

$$|v\rangle\otimes(|w_1\rangle+|w_2\rangle) = |v\rangle\otimes|w_1\rangle + |v\rangle\otimes|w_2\rangle$$

Unlike direct sum

$$\begin{array}{ccc} V & \oplus & W\\ \uparrow & & \uparrow\\ |v\rangle & + & |w\rangle \end{array}$$

$$\lambda(|v\rangle+|w\rangle) \neq \lambda|v\rangle+|w\rangle$$
$$\neq |v\rangle+\lambda|w\rangle$$

Linear combinations of product vectors with no common factor are genuinely new objects in $V\otimes W$. These are correlated distribution in the classical theory and entangled states in quantum theory

(Another characterization of the tensor product is an "optimal" space in which any bilinear map $V \times W \to Z$ is represented as a linear map $V \otimes W \to Z$.)

Some arbitrary vector space

Bilinear maps

Inner product

$$\big(\langle v_1 | \otimes \langle w_1 | \big) \big(|v_2\rangle \otimes |w_2\rangle \big) = \langle v_1 | v_2 \rangle \langle v_2 | w_2 \rangle \in \mathbb{C}$$

then extend to whole $V \otimes W$ by linearity

Linear operators

$$A_V \otimes B_W |vw\rangle = \big(A_V |v\rangle \big) \otimes \big(B_W |w\rangle \big)$$

Composition of linear operators

$$(A_V \otimes B_W)(C_V \otimes D_W) = AC_V \otimes BD_W$$

Special case: local operations

$$A_V \otimes \mathbb{1}_W |vw\rangle = \big(A_V |v\rangle \big) \otimes |w\rangle$$

Local operations on separated subsystems commute

$$[A_V \otimes \mathbb{1}_W, \mathbb{1}_V \otimes B_W]$$

$$= A_V \otimes B_W - A_V \otimes B_W = 0$$

The matrix form of a tensor product of linear operators is given by the Kronecker product

$$\begin{matrix} A_{00}B_{00} & A_{00}B_{01} \\ A_{00}B_{10} & A_{00}B_{11} \end{matrix} = A_{00}B$$

$$A \otimes B = \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{pmatrix} A_{j_0}B_{l_0} & A_{j_0}B_{l_1} & A_{j_1}B_{l_0} & A_{j_1}B_{l_1} \\ & & & \\ & & & \\ & & & \end{pmatrix} = \begin{pmatrix} A_{00}B & A_{01}B \\ A_{10}B & A_{11}B \end{pmatrix}$$

Linear functionals on the space of operators

A is $k \times k$. B is $m \times m$.

$$\operatorname{tr}(A \otimes B) = (\operatorname{tr} A)(\operatorname{tr} B)$$

ไขว้

$$\det(A \otimes B) = (\det A)^m (\det B)^k$$

Unitary transformations of the form $U_A \otimes U_B$ can't generate entanglement.

But watch out!

① Dynamics

Entangled operation in general

Hamiltonian $H_A \otimes H_B \Rightarrow U = e^{-iH_A \otimes H_B t/\hbar} = \sum_{k=0}^{\infty} \left(-\frac{t}{\hbar}\right)^k H_A^k \otimes H_B^k$

Example

$$e^{-i\theta X \otimes Y}|00\rangle = (\cos\theta\, \mathbb{1}\otimes\mathbb{1} - i\sin\theta\, X\otimes Y)|00\rangle$$

Kronecker

$$\begin{pmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{pmatrix}$$

$$= \cos\theta|00\rangle + \sin\theta|11\rangle$$

$\theta = \frac{\pi}{4} \Rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ Entangled state! (But can show $e^{A\otimes\mathbb{1}} = e^{A}\otimes\mathbb{1}$)

② Measurement Suppose we make a $Z\otimes Z$ measurement on the state $|++\rangle$ and get outcome "1"

$$|++\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \xrightarrow{\text{Collapse}} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$ Entangled state!

Entanglement vs classical correlation

It may come as a surprise that both classical probability theory and quantum theory feature tensor product as a method to combine independent systems (random variables in classical theory). So what is the difference?

Classical correlation $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} p \\ 1-p \end{pmatrix} \otimes \begin{pmatrix} q \\ 1-q \end{pmatrix}$ for any $\vec{p}$ and $\vec{q}$.

Why?

$$
\begin{aligned}
& pq \\
0 = \ & p(1-q) \Rightarrow p=0 \text{ or } q=1 \\
0 = \ & (1-p)q \Rightarrow p=1 \text{ or } q=0 \\
& (1-p)(1-q)
\end{aligned}
$$

Bell states

$$|\Phi_\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\Psi_\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

Denote $|\Phi_+\rangle$ by $|\Omega\rangle$. All other Bell states can be obtained by acting only on one half of $|\Omega\rangle$ by Pauli operators.

$$\mathbb{1} \otimes \mathbb{1} |\Omega\rangle = |\Omega\rangle := |\Omega_{00}\rangle$$

$$\mathbb{1} \otimes Z |\Omega\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = |\Omega_{01}\rangle$$

$$\mathbb{1} \otimes X |\Omega\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = |\Omega_{10}\rangle$$

$$\mathbb{1} \otimes XZ |\Omega\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = |\Omega_{11}\rangle$$

$|\Omega_{ab}\rangle$

Parity bit $\rightarrow a$, Phase bit $\rightarrow b$

The Bell states can be uniquely specified by the eigenvalues under the parity measurement $Z \otimes Z$ and the phase — " — $X \otimes X$.

$$Z \otimes Z |x_1 x_2\rangle = (-1)^{x_1 + x_2} |x_1 x_2\rangle, \quad X \otimes X |x_1 x_2\rangle = |\bar{x}_1 \bar{x}_2\rangle$$

$$\Rightarrow \left. \begin{aligned} Z \otimes Z |\Phi_\pm\rangle &= |\Phi_\pm\rangle \\ Z \otimes Z |\Psi_\pm\rangle &= -|\Psi_\pm\rangle \end{aligned} \right\} Z \otimes Z |\Omega_{ab}\rangle = (-1)^a |\Omega_{ab}\rangle$$

$$\left.\begin{array}{l} X\otimes X\,|\Phi_\pm\rangle = |\Phi_\pm\rangle \\ X\otimes X\,|\Psi_\pm\rangle = -|\Psi_\pm\rangle \end{array}\right\} X\otimes X\,|\Omega_{ab}\rangle = (-1)^b\,|\Omega_{ab}\rangle$$

$$Y\otimes Y = -(Z\otimes Z)(X\otimes X) \text{ so } Y\otimes Y\,|\Omega_{ab}\rangle = \boxed{-(-1)^{a+b}\,|\Omega_{ab}\rangle}$$

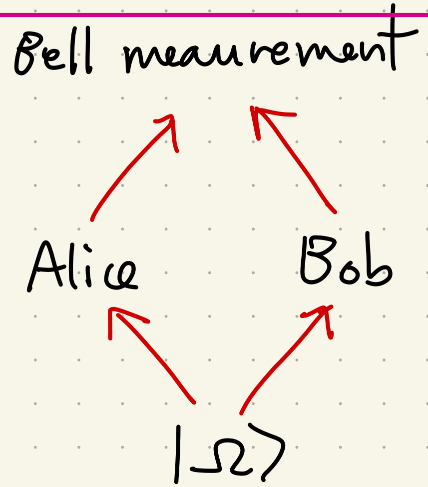
We will pay special attention to the singlet state $|\Psi_-\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

You will prove in the tutorial that it has a very special property that

$|\Psi_-\rangle$ can be written as $\boxed{\dfrac{|\hat{n}\rangle|-\hat{n}\rangle - |-\hat{n}\rangle|\hat{n}\rangle}{\sqrt{2}}}$ i.e. the spins are anti-correlated in any direction.

In a photonic system, for example, spontaneous parametric down-conversion (SPDC) can be used to produce a pair of photons with opposite polarization.

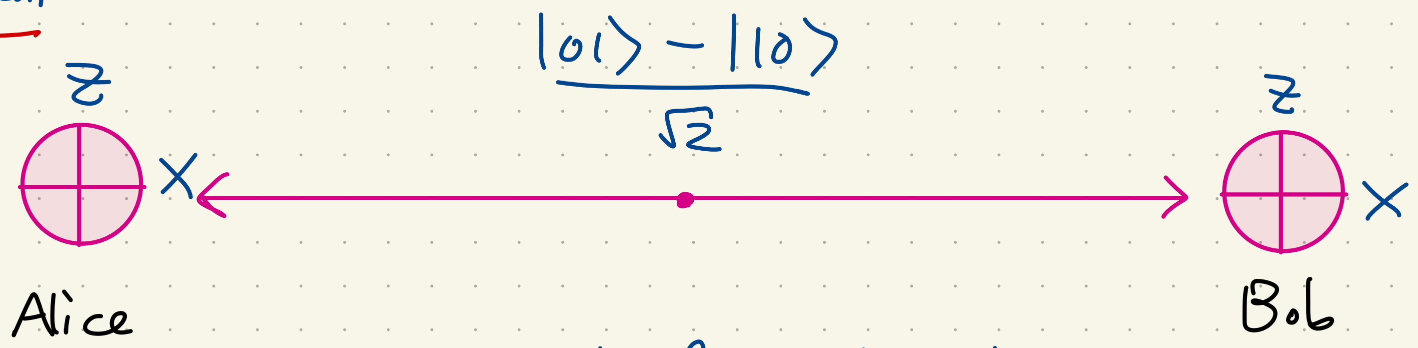
Dense coding



Encode 2 classical bits in the choice of local operations

$$xy \rightarrow \begin{cases} \mathbb{1} \Rightarrow |\Omega_{00}\rangle \\ Z \Rightarrow |\Omega_{01}\rangle \\ X \Rightarrow |\Omega_{10}\rangle \\ Y \Rightarrow |\Omega_{11}\rangle \end{cases}$$

EPR argument



A philosophically troubling aspect of quantum theory is that one cannot in general think of an act of measuring as revealing a pre-existing value of the system's property. (The reason is "contextuality" which we won't talked about in these lectures.) But when the system is in an eigenstate of an observable A with eigenvalue λ, subsequent measurements of A do not alter the state, hence there is a tendency to think of the value λ as pre-existed. Einstein, Padolsky, and Rosen (EPR) devised a clever argument using quantum correlation and the principle of relativity to argue that values of incompatible (non-commuting) observables can simultaneously pre-exist.

$|\psi_-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = |\Omega_{11}\rangle$ is a simultaneous eigenstate of ZZ and XX.

(Verify that they commute) with eigenvalues -1 for both observables.

Now, while Alice can't make a Z meas. and an X meas. at the same time on her particle, she can nevertheless choose to measure one. If Alice were to measure Z_A and find the value $z_A = \pm 1$, then the spin of Bob's particle would have the opposite value $z_B = z_A$ to satisfy $z_A z_B = -1$. Since the anti-correlation holds also for the spin X component, the same conclusion follows if Alice were to measure X_A. But, EPR argued, the act of measurement by Alice over here can't effect Bob's particle over there.

Therefore, the fact that Alice would be able to infer z_B or x_B without disturbing Bob's particle means that those values already existed before the measurement.

We can summarize the argument as follows.

$$\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right) + \left(\text{Locality}\right) = \left(\begin{array}{c}\text{Pre-existing values} \\ \text{for X and Z measurements} \\ \text{"Hidden variable"}\end{array}\right)$$

So in this scenario, the quantum correlation revealed by the measurements is no different from a classical correlation; only that quantum theory forbids a simultaneous measurement of incompatible observers.

Comment

The original EPR article uses $\psi_{AB} = \delta(x_1 - x_2 - L)\,\delta(p_1 + p_2)$

$$\left(\begin{array}{c}\text{Center of mass} \\ \text{coordinate}\end{array}\right) \quad x_1 - x_2 \qquad \left(\begin{array}{c}\text{Total} \\ \text{momentum}\end{array}\right) \quad p_1 + p_2$$

$$[x_1 - x_2, p_1 + p_2] = [x_1, p_1] + [x_1, p_2] - [x_2, p_1] - [x_2, p_2] = i\hbar - i\hbar = 0.$$

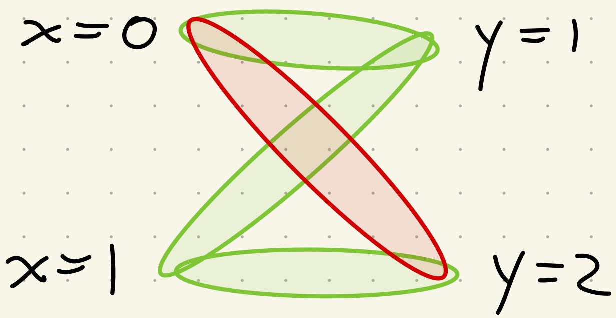
Bell-CHSH inequality

Non-local game

	Alice	Bob
Choice of measurement	$x=0,1$	$y=0,1$
Measurement outcome	$a=0,1$	$b=0,1$

Winning if $\boxed{a \oplus b = x \wedge y}$

x	y	$x \wedge y$	a,b?
0	0	0	same
0	1	0	same
1	0	0	same
1	1	1	opp



Frustrated graph

We can enumerate all classical deterministic strategies; there are

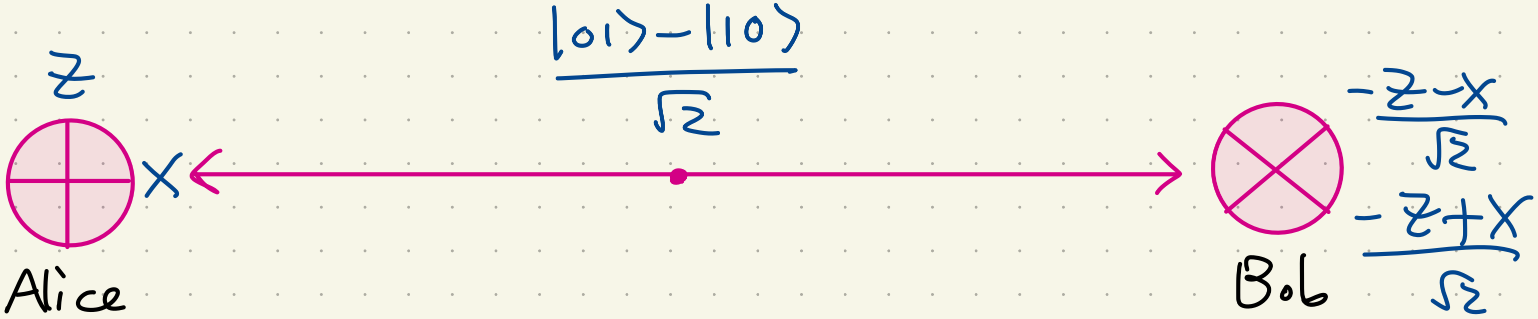
$$\underset{\text{Alice}}{2^2} \times \underset{\text{Bob}}{2^2} = 16 \text{ of them.}$$

\# Input values

\# Output values

$f(x) = a$

It turns out that ones of the best classical strategies are to always answer $a=b=0$ or $a=b=1 \Longrightarrow Pr_{win} = 3/4$. This only fails when $x \wedge y = 1$.



Let's play the game using the singlet state.

Shorthand $(\hat{n}_x \cdot \vec{\sigma}_A)(\hat{n}_y \cdot \vec{\sigma}_B) = \sigma_{xy}$ and $P^{(\pm)}_{xy}$ are the projectors onto the ± 1 eigenspaces of σ_{xy}

Bits that indicate choice of measurement. Not Cartesian axes.

$$Pr_{win} = \left(p^{(+)}_{00} + p^{(+)}_{01} + p^{(+)}_{10} + p^{(-)}_{11}\right)/4$$

$$= \left\langle P^{(+)}_{00} + P^{(+)}_{01} + P^{(+)}_{10} + P^{(-)}_{11} \right\rangle / 4$$

$$\left.\begin{aligned} P^{(+)} + P^{(-)} &= \mathbb{1} \\ P^{(+)} - P^{(-)} &= \sigma_{xy} \end{aligned}\right\} \Longrightarrow P^{(\pm)} = \frac{\mathbb{1} \pm \sigma_{xy}}{2}$$

$$\Rightarrow \Pr_{win} = \langle \mathbb{1} + \sigma_{00} + \mathbb{1} + \sigma_{01} + \mathbb{1} + \sigma_{10} + \mathbb{1} - \sigma_{11} \rangle / 8$$

Expectation value w.r.t. some state

$$= \frac{1}{2} + \langle \sigma_{00} + \sigma_{01} + \sigma_{10} - \sigma_{11} \rangle / 8$$

Specialize to the angles specified in the previous page.

$$x \begin{cases} 0 \Rightarrow Z \\ 1 \Rightarrow X \end{cases} \qquad y \begin{cases} 0 \Rightarrow \frac{-Z+X}{\sqrt{2}} \\ 1 \Rightarrow \frac{-Z-X}{\sqrt{2}} \end{cases}$$

$$\Pr_{win} = \frac{1}{2} + \frac{1}{8} \left\langle Z \otimes \frac{-Z+X}{\sqrt{2}} + Z \otimes \frac{-Z-X}{\sqrt{2}} + X \otimes \frac{-Z+X}{\sqrt{2}} - X \otimes \frac{-Z-X}{\sqrt{2}} \right\rangle$$

$$= \frac{1}{2} + \frac{1}{8\sqrt{2}} \langle -ZZ + \cancel{ZX} - ZZ - \cancel{ZX} - \cancel{XZ} + XX + \cancel{XZ} + XX \rangle$$

$$= \frac{1}{2} - \frac{\sqrt{2}}{8} \langle XX + ZZ \rangle_{\psi^-}$$

Take the expectation value w.r.t. the singlet state

$$= \frac{1}{2} + \frac{\sqrt{2}}{4} \approx 85\% > \text{classical } \Pr_{win} = 3/4$$

Comments

① $S = \langle \sigma_{00} \rangle + \langle \sigma_{01} \rangle + \langle \sigma_{10} \rangle - \langle \sigma_{11} \rangle$ is the main quantity in the original CHSH inequality $|S|_{\text{classical}} \leq 2$.

The maximal violation in quantum theory is given by Tsirelson's bound: $|S|_{\text{quantum}} \leq 2\sqrt{2}$ the value we got in the previous example. In fact, we don't need the singlet state. For instance, since $\langle XX + ZZ \rangle_{\Phi^+} = 2$, if we flip the sign of the observables on system B (equivalently, re-labeling the outcomes), then the state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ provides maximal violation of the inequality.

$Pr_{\text{win}} = 1$ corresponds to $|S| = 4$. This is achievable in a super-quantum, but still non-signaling theory. Such a theory makes communication complexity trivial. (Any communication task can be solved by sending a single bit.)

(2) The first inequality derived by Bell has only three measurement angles, not four, and makes use of the fact that Alice and Bob would find their results to be perfectly anti-correlated when they make the same spin measurement, a condition that is unrealistic for an experimental test of the inequality

(3) "Loophole free" Bell test has been performed by Hanson's group at Delft (Netherlands) in 2015 using entanglement between NV-center qubits mediated by photons via entanglement swapping (which we will see is just a variant of quantum teleportation).

Density operators / Density matrices (DMs)

Problems with our current formulation of quantum theory.

① No quantum state of a subsystem of an entangled system.

Suppose that we want to measure just parts of the whole system, we need to be able to say what happens to the unmeasured parts.

Alice measures in an ONB $\{|e_j\rangle\}_{j=1,\dots,d=\dim\mathcal{H}_A}$

$$|\psi\rangle \underset{\text{to write}}{\overset{\text{Find a way}}{=\!=}} \sum_{j=1}^{d} c_j\, |e_j\rangle_A \otimes |f_k\rangle_B$$ where $\{|f_k\rangle\}_k$ may not be orthogonal or even linearly independent. ($\dim\mathcal{H}_B$ may be less than d.)

The only thing Bob can say is that system B is in one of the states $|f_k\rangle$ with probability $|c_k|^2$ but we don't know which one.

Only when Alice communicates the result to Bob, Bob can update his state to a particular $|f_k\rangle$. (Conditional state)

(2) Since the number of states in the ensemble $\{|\phi_k\rangle\}_k$ is not limited by the dimension of the Hilbert space $\mathcal{H}_B$, there are infinitely many ways to form an ensemble. Do we need an infinite number of parameters to describe every possible ensemble?

Suppose that we have an ensemble $\{|\psi_j\rangle, p_j\}_{j=1,\dots,m}$ i.e. the system is in state $|\psi_j\rangle$ with probability p_j. Note that the $|\psi_j\rangle$'s don't have to be linearly independent and m can be as large as we'd like.

$$Pr(j) = \sum_{k=1}^{m} p_k |\langle e_j|\psi_k\rangle|^2 = \langle e_j| \underbrace{\left(\sum_k p_k |\psi_k\rangle\langle\psi_k|\right)}_{\rho} |e_j\rangle$$

Density matrix (DM)

Properties of DMs

① $\mathrm{tr}\rho = 1$

② $\rho \geq 0 \iff \langle\psi|\rho|\psi\rangle \geq 0, \ \forall \psi \in \mathcal{H}$

We say that ρ is positive-semidefinite or that ρ is a positive operator.

A positive $\iff A = B^\dagger B$ for some matrix B

(Cholesky decomposition)

The $\Leftarrow$ is easy.

▪ $\langle\psi|A|\psi\rangle = (\langle\psi|B^\dagger)(B|\psi\rangle) = \|(B|\psi\rangle\|^2 \geq 0.$ □

Lemma $\rho \geq 0$ implies $\rho^\dagger = \rho$. This is not true in real vector space!

▪ $\langle\psi|\rho|\psi\rangle \geq 0$ means that $\langle\psi|\rho|\psi\rangle$ is real. So $\langle\psi|\rho|\psi\rangle = \langle\psi|\rho|\psi\rangle^*$

$= \langle\psi|\rho^\dagger|\psi\rangle \ \forall\psi \Rightarrow \rho^\dagger = \rho.$ □

Counterexample: Any anti-symmetric matrix

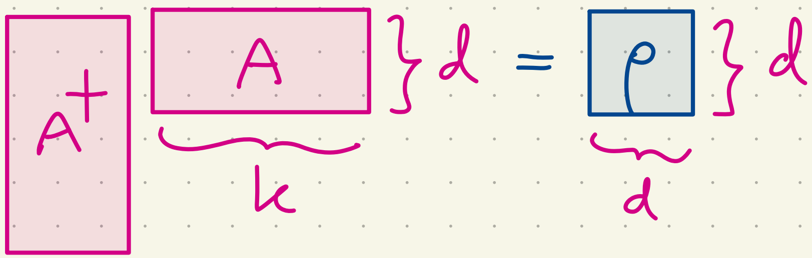
$$\begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \ \forall a,b \in \mathbb{R}$$

Therefore, if ρ is positive and trace-one, it can be put into the spectral form with positive diagonal elements that sum to 1 i.e. ρ can be written as an ensemble of quantum states.

Proofs that any mixture $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ have properties ① & ②.

- ① $\mathrm{tr}\,\rho = \sum_k p_k \,\mathrm{tr}(|\psi_k\rangle\langle\psi_k|) = \sum_k p_k = 1.$

② Define $A = \sum_{k=1}^{m} p_k |\psi_k\rangle\langle e_k|$, $\{|e_k\rangle\}_k$ an ONB.
Note that A doesn't have to be square. Then $\rho = A^\dagger A$ showing $\rho \geq 0$. □



$$\boxed{\text{Ensemble} \Longleftrightarrow \text{Density operator}}$$

Caution: Positive operators do not form a vector space.
($\rho - \sigma$ is not positive in general.)

Measurements and dynamics

We have already seen this, but let's summarize the meas. rule here again.

$$\Pr(j) = \langle e_j|\rho|e_j\rangle = \mathrm{tr}(|e_j\rangle\langle e_j|\rho)$$

In general, for a possibly degenerate outcome associated to a projector P_j

$$\boxed{\Pr(j) = \mathrm{tr}(P_j\rho)}$$

Post-measurement state

$$\boxed{\frac{P_j\rho P_j}{\mathrm{tr}(P_j\rho)}}$$

$$\rho \mapsto \rho(t) = U(t)\rho U^\dagger(t)$$

H and U commute

$$\frac{d\rho}{dt} = \frac{dU(t)}{dt}(\rho U^\dagger) + (U\rho)\frac{dU^\dagger}{dt} = \frac{1}{i\hbar}\left(HU\rho U^\dagger - U\rho \overbrace{U^\dagger H}\right)$$

$$\boxed{i\hbar\frac{d\rho}{dt} = [H,\rho]}$$

Opposite sign to Heisenberg picture evolution $i\hbar\dot{A} = [A,H]$

Qubit DMs

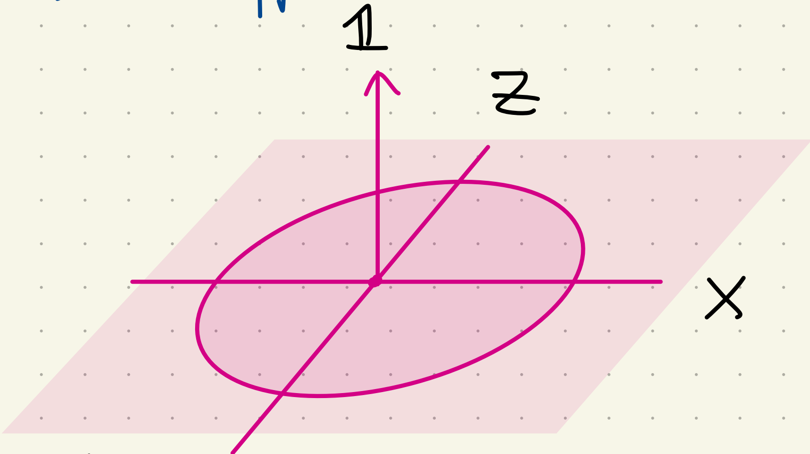
Bloch vector $\vec{r}$

$$\sum_k p_k |\hat{n}_k\rangle\langle\hat{n}_k| = \frac{1}{2}\Big(\mathbb{1} + \underbrace{\sum_k p_k \hat{n}_k}_{\vec{r}} \cdot \vec{\sigma}\Big)$$

only real coefficients

$$|\hat{n}\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$

Bloch ball lives in the operator space (without trace-one constraint) which is 4D, so suppress one dimension to visualize (say Y so that we have a "rebit".)

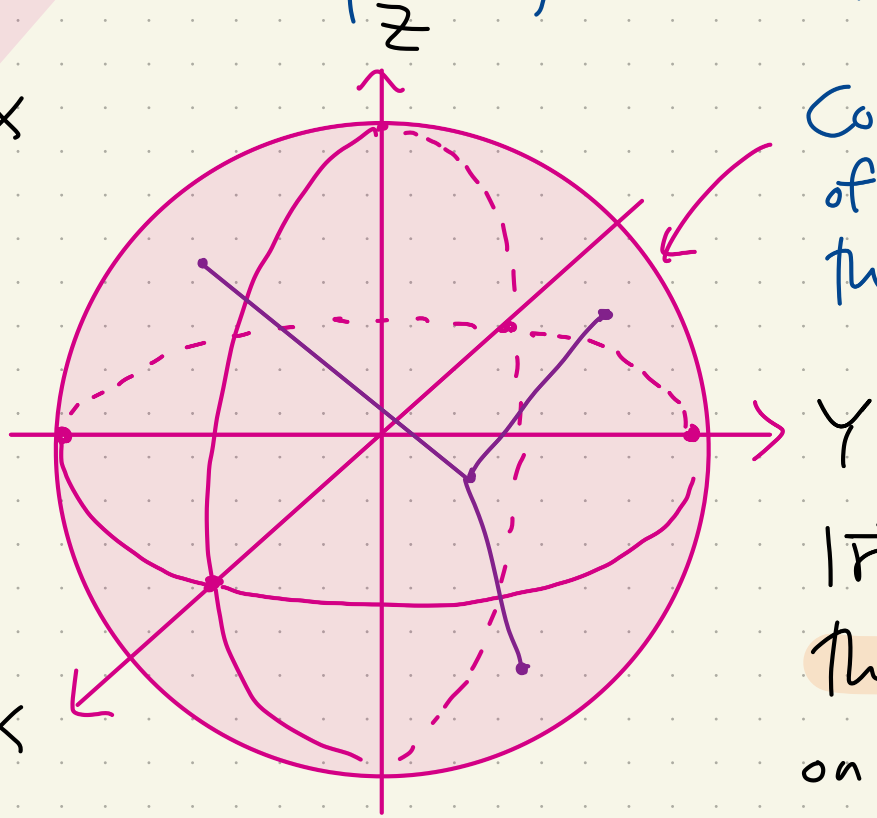


$\mathrm{tr}\,\rho = 1$ hyperplane

$(A_{00} = 1/2)$

Eigenvalues of $\vec{r}\cdot\vec{\sigma}$ is $\pm|\vec{r}|$

so positivity constraint $\sqrt{r_x^2 + r_y^2 + r_z^2} \leq 1$



Convex combinations of $|\hat{n}\rangle\langle\hat{n}|$ live inside the Bloch sphere

$|\vec{r}|^2$ is nothing but the purity we introduced on probability simplices.

Comments

① This simple Bloch-vector picture doesn't exist in higher dimension (qudit for $d>2$). Pure states still live on the surface of a hypersphere but not all points on the hypersphere are pure states by dimension counting.

Ambient operator space in which the hypersphere is embedded is d^2-1 dim. The manifold of pure states is $2d-2$ dim. (We are talking about real manifolds here. So 1 complex dim = 2 real dim.)

② The physical difference between superpositions of pure states and mixtures is that the former can interfere whereas the latter cannot. → Interferometer example

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{01}^* & \rho_{11} \end{pmatrix}$$

Coherence

Population

superposition

The notion of "coherence" is basis-dependent. The off-diagonal elements can be made to disappear by going into the eigenbasis of ρ.

So ρ_{01} displays coherence in the specific basis in which ρ is written.

③ Proper vs Improper mixtures

Every mixture has infinitely many ensemble decompositions (EDs).
Example: maximally mixed qubit state $\frac{\mathbb{1}}{2} = \frac{|\hat{n}\rangle\langle\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}|}{2}$
for any $\hat{n}$.

Can we say that one ED is more "real" than the others? For instance, if I "know" that there is a machine produces $|0\rangle$ or $|1\rangle$ with 50/50 probability, can I say that ρ is really $\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$ and not, say $\frac{|+\rangle\langle +| + |-\rangle\langle -|}{2}$ even though I can't distinguish them by any measurement performed locally on the qubit?

"Improper mixture"

It turns out that you cannot interpret a quantum mixed state as reflecting classical ignorance and nothing else, and this is why the measurement "Proper mixture" problem is not solved.
(Although it is debatable whether it is a scientific or a philosophical problem.)

To understand improper mixtures, we finally come to the topic of reduced states / reduced density matrices (RDMs) and purification.

$$\Pr(j) = \mathrm{tr}\left(|e_j\rangle_A\langle e_j| \otimes \mathbb{1}_B \, \rho_{AB}\right) \qquad \rho_{AB} \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$$

$$= \mathrm{tr}\left(|e_j\rangle_A\langle e_j| \otimes \sum_k |f_k\rangle\langle f_k| \, \rho_{AB}\right)$$

$$= \langle e_j| \underbrace{\left(\sum_k \langle f_k| \rho_{AB} |f_k\rangle\right)}_{} |e_j\rangle = \mathrm{Tr}(|e_j\rangle\langle e_j| \, \rho_A)$$

(Partial trace) $\boxed{\mathrm{tr}_B(\rho_{AB}) = \rho_A}$ RDM of system A

What we have shown is that any measurement statistics on system A can be calculated from the RDM which is an operator on $\mathcal{H}_A$ alone.

Comment look vs not look at meas. on other system

Easy to verify that an RDM is actually a DM.

① $\operatorname{tr} \rho_A = \operatorname{tr}(\operatorname{tr}_B \rho_{AB}) = \operatorname{tr} \rho_{AB} = 1$

② $\langle\psi|\rho_A|\psi\rangle = \langle\psi|\sum_k \left(\langle f_k|\rho_{AB}|f_k\rangle\right)|\psi\rangle = \sum_k \underbrace{\langle\psi|\langle f_k|}\rho_{AB}\underbrace{|f_k\rangle|\psi\rangle} \geq 0$ vector in $\mathcal{H}_A \otimes \mathcal{H}_B$

$|\psi\rangle \in \mathcal{H}_A$

Schmidt decomposition

We will be able to clearly see the connection between the rank ("mixedness") of an RDM to the amount of entanglement of the joint state.

General state on $\mathcal{H}_A \otimes \mathcal{H}_B$ $\quad |\psi\rangle = \sum_{jk} c_{jk} |e_j\rangle|f_k\rangle$

$\{|e_j\rangle\}_{j=1,\dots,m}$ ONB for $\mathcal{H}_A$

$\{|f_k\rangle\}_{j=1,\dots,\ell}$ —"— $\mathcal{H}_B$

c_{jk} is a matrix (possibly non-square)

We want to "diagonalize" it.

Importantly, $\mathcal{H}_A$ and $\mathcal{H}_B$ may have different dimensions!

Singular value decomposition (SVD)

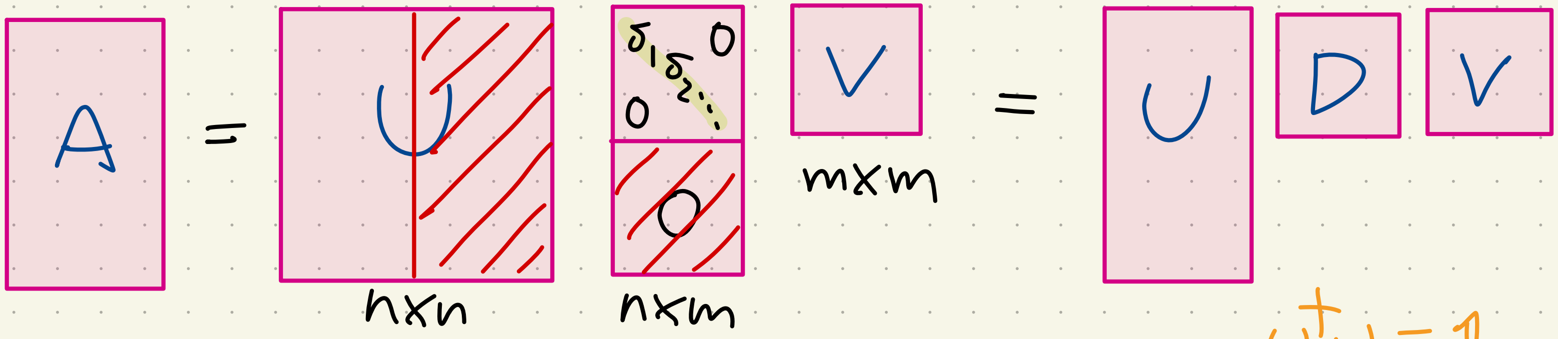
Unitary

Arbitrary $n\times m$ matrix $A = UDV^\dagger$, $A_{jk} = \sum_r U_{jr} D_{rr} V^\dagger_{rk}$, where

Diagonal

$r \le \operatorname{rank} A$ and $D_{jj} = \delta_j$, $\delta_1 \ge \delta_2 \ge \dots \ge \delta_r \ge 0$.

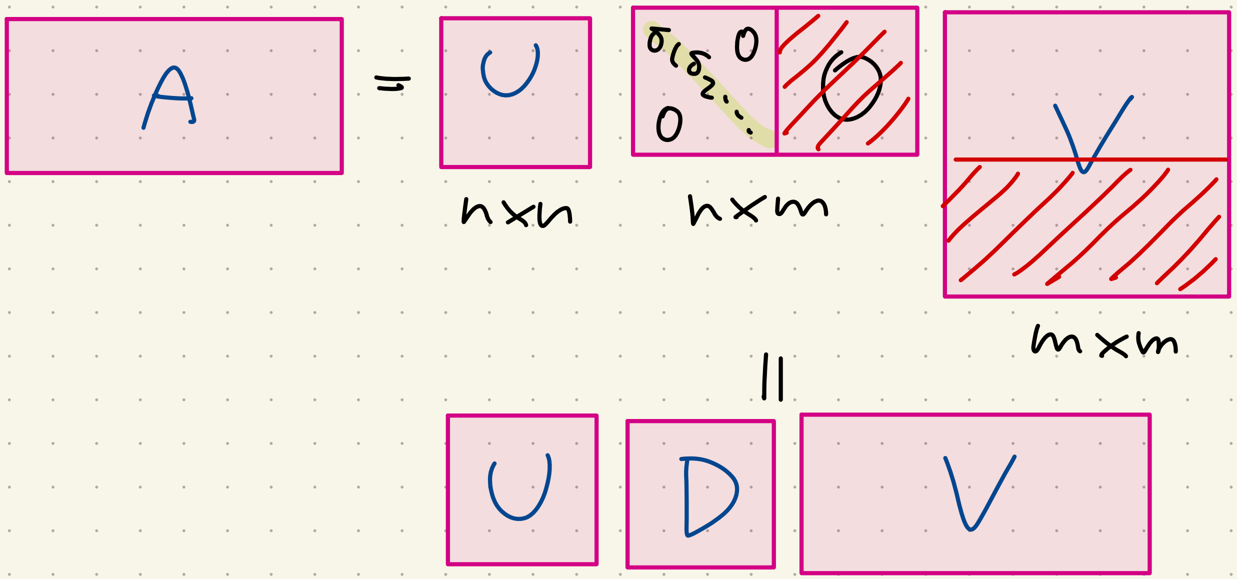
$n>m$



"Skinny SVD"

$U^\dagger U = 1_{m\times m}$, $\cancel{UU^\dagger = 1_{n\times n}}$

$n<m$



$\cancel{V^\dagger V = 1_{m\times m}}$, $VV^\dagger = 1_{n\times n}$

To obtain the SVD, start from the polar decomposition.

$$A = \underset{\text{unitary}}{W}\sqrt{A^\dagger A}, \qquad \text{Correctness: } A^\dagger A = \left(\sqrt{A^\dagger A}\, W^\dagger\right)\left(W\sqrt{A^\dagger A}\right)$$

$$= \sqrt{A^\dagger A}\,(\underbrace{W^\dagger W}_{\mathbb{1}})\sqrt{A^\dagger A} = A^\dagger A$$

$A^\dagger A$ and $AA^\dagger$ are positive operators (that's why the positive square root can be defined), so they have the spectral form

$$A^\dagger A = \sum_{j=1}^{m} p_j |e_j\rangle\langle e_j|, \qquad AA^\dagger = \sum_{j=1}^{n} q_j |f_j\rangle\langle f_j|$$

The sets of nonzero $\{p_j\}_j$ and $\{q_j\}_j$ coincide. Why? We can construct eigenvectors of $AA^\dagger$ from those of $A^\dagger A$:

$$AA^\dagger(A|e_j\rangle) = A(A^\dagger A|e_j\rangle) = p_j(A|e_j\rangle) \Rightarrow A|e_j\rangle \text{ is an eigenvector of } AA^\dagger \text{ with eigenvalue } p_j$$

$$\| A|e_j\rangle \| = \langle e_j|A^\dagger A|e_j\rangle = p_j \Rightarrow \underbrace{\frac{A}{\sqrt{p_j}}|e_j\rangle =: |f_j\rangle}_{\text{ONB}}$$

$$\left.\begin{matrix}|e_1\rangle\\|e_2\rangle\\\vdots\\|e_m\rangle\end{matrix}\right\} \underset{A^\dagger}{\overset{A}{\rightleftarrows}} \left\{\begin{matrix}|f_1\rangle\\|f_2\rangle\\\vdots\\|f_n\rangle\end{matrix}\right.$$

Thus, the only consistent choice of W must be $W = \sum_j |f_j\rangle\langle e_j|$

- $A|e_j\rangle = W\sqrt{A^\dagger A}\,|e_j\rangle = \sqrt{p_j}\,W|e_j\rangle = \sqrt{p_j}\,|f_j\rangle$ which we know is $A|e_j\rangle$. $\square$

Therefore, $\boxed{A = \sum_{j=1}^{r} \sqrt{p_j}\,|f_j\rangle\langle e_j|}$. That is, A can be "diagonalized" by mapping two bases to the computational basis $\{|j\rangle\}_j$ as follows.

Define a unitary $V|j\rangle = |e_j\rangle$ and $U = WV \Rightarrow U|j\rangle = |f_j\rangle$. Then

$$U^\dagger A V = \sum_{j=1}^{r} \sqrt{p_j}\,|j\rangle\langle j| = D \iff \boxed{A = UDV^\dagger} \quad \text{SVD}$$

Back to entangled states. (Here $\{|e_j\rangle\}_j$ and $\{|f_k\rangle\}_k$ are arbitrary ONBs for $\mathcal{H}_A$ and $\mathcal{H}_B$.)

$$|\psi\rangle = \sum_{jk} c_{jk} |e_j\rangle |f_k\rangle = \sum_{jkr} \big(\underbrace{U_{jr}\sqrt{p_r}\, V^{\dagger}_{rk}}^{\text{SVD}} \big) |e_j\rangle |f_k\rangle$$

$$= \sum_r \sqrt{p_r} \Big(\sum_j U_{jr} |e_j\rangle \Big) \Big(\sum_k V^*_{kr} |f_k\rangle \Big)$$

$$= \sum_j \sqrt{p_j}\, |\mu_j\rangle |\mu_j'\rangle \quad \text{(Schmidt decomposition)}$$

ONBs from unitarity of U and V

$$\sum_{jk} \langle e_k | U^*_{mk} U_{jl} | e_j \rangle = \sum_j U^*_{mj} U_{jl} = \mathbb{1}_{ml} = \delta_{ml}$$

Comments

1. The number of nonzero terms is called the Schmidt rank, which measures the amount of entanglement between the two systems, which cannot be changed by any local unitaries

2. Note that the Schmidt rank can only be as large as the dimension of the smaller subsystem.

③

$$\rho_{AB} = |\psi\rangle\langle\psi| = \sum_{jk} \sqrt{p_j p_k}\, |\mu_j \mu_j'\rangle\langle \mu_k \mu_k'|$$

RDM

$$\rho_A = \mathrm{tr}_B\, \rho_{AB} = \sum_j p_j |\mu_j\rangle\langle\mu_j|$$

(Already in the spectral form with the singular value squared being the ensemble probabilities)

Symmetry

$$\rho_B = \mathrm{tr}_A\, \rho_{AB} = \sum_j p_j |\mu_j'\rangle\langle\mu_j'|$$

Therefore, the rank of the RDM of either subsystem quantifies entanglement; it exactly equals the Schmidt rank.

④ Purification is sort of the converse process.

$$\rho = \sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j| \qquad \{|\psi_j\rangle\}_j \text{ not necessarily an ONB}$$

Define $\rho_{AB} = \sum_j \sqrt{p_j}\, |\psi_j\rangle |j\rangle$ $\{|j\rangle\}_j$ an ONB for $\mathcal{H}_B$, $\dim \mathcal{H}_B = m$

Then $\mathrm{tr}_B\, \rho_{AB} = \rho$. This tells us that any mixed state can be thought of as a reduced state of a part of an entangled system.

Discussion: quantum eraser

Tracing out

$|\psi\rangle^{(1)}_{AB}$

$|\psi\rangle^{(2)}_{AB}$

$\vdots$

same ρ_A

Purification

$|\psi\rangle^{(1)}$

$|\psi\rangle^{(2)}_{AB}$

$\vdots$

same ρ_A

Related by (possibly non-square) unitary matrices on B

Schrödinger proved in 1936 !

(Schrödinger–HJW theorem)

This man was ahead of his time.

Singlet $\longleftrightarrow$ Maximally mixed

$$|\hat{n}\rangle|-\hat{n}\rangle + |-\hat{n}\rangle|\hat{n}\rangle/\sqrt{2} \longleftrightarrow |\hat{n}\rangle\langle\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}| \,/2$$

After Bob measure his spin and found $|\hat{n}\rangle$, Bob knows that Alice's spin must be in the state $|-\hat{n}\rangle$. FTL? No, Bob has to tells Alice his meas. direction and result. Otherwise, Alice only sees $\mathbb{1}/2$.

(Contrapositively, if we can distinguish diffent ensembles that give rise to the same density operates, we can violate relativity.)

Coherent $\quad$ Subsystem incoherent

$$|0\rangle+|1\rangle \longmapsto |00\rangle+|11\rangle$$

Bob measures X: $|0\rangle\dfrac{(|+\rangle+|1\rangle)}{\sqrt{2}} + |1\rangle\dfrac{(|+\rangle-|1\rangle)}{\sqrt{2}} \longrightarrow \begin{cases} + & \dfrac{|0\rangle+|1\rangle}{\sqrt{2}} \\ - & \dfrac{|0\rangle-|1\rangle}{\sqrt{2}} \end{cases}$

The leaked info is "erased"

Decoherence

Teleportation

Shared entanglement can be consumed to "move" the state of one particle to the other.

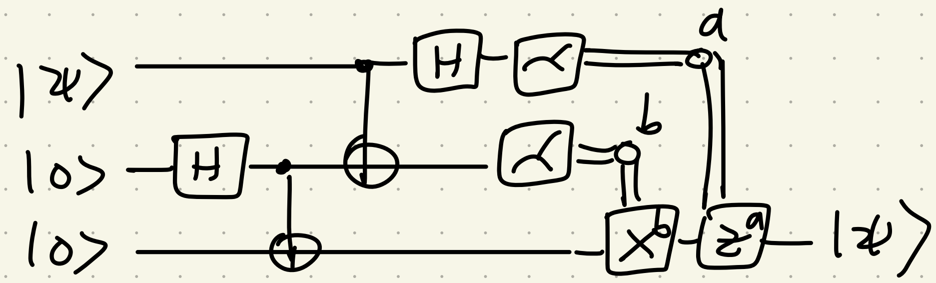
① Suppose that Bob want to teleport the state $|\psi\rangle_A$ to Charlie (C) who share the Bell state $|\Omega\rangle_{BC} = \frac{|0_B 0_C\rangle + |1_B 1_C\rangle}{\sqrt{2}}$.

② Make a joint measurement on AB in the Bell basis $\{|\Omega_{ab}\rangle\}_{\substack{a=0,1\\ b=0,1}}$

I claim that now C collapse to the state $\sigma_{ab}|\psi\rangle$ depending on the measurement outcomes a, b

$$\begin{cases} \sigma_{00} = \mathbb{1} \\ \sigma_{01} = Z \\ \sigma_{10} = X \\ \sigma_{11} = XZ \end{cases}$$

③ Bob communicates the bits a, b which tells Charlie which Pauli he needs to apply to recover the state $|\psi\rangle$ on his end.



$$
{}_{AB}\langle \Omega_{ab}| \left(|\psi\rangle_A |\Omega_{00}\rangle_{BC} \right) = \frac{1}{2}\left[\sum_j \langle j_A j_B | (Z^a X^b)^\dagger_A \otimes \mathbb{1}_B \right] |\psi\rangle \sum_k |k_B k_C\rangle
$$

$$
= \frac{1}{2} \sum_{jk} \langle j| X^b Z^a |\psi\rangle \overbrace{\langle j_B|k_B\rangle}^{\delta_{jk}} |k_C\rangle
$$

$$
= \frac{1}{2}\sum_j \underbrace{\langle j|X^b Z^a|\psi\rangle}_{\text{Amplitudes of the state } X^b Z^a|\psi\rangle} |j\rangle_C
$$

$\frac{1}{2}$ comes from the probability $\frac{1}{4}$ for each outcome of the Bell measurement

$$
= \frac{1}{2}|\psi\rangle_C
$$

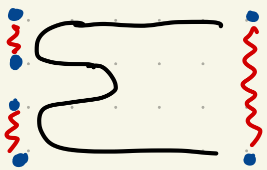
$|\Omega_{00}\rangle$

$|\Omega_{01}\rangle = \mathbb{1}\otimes Z|\Omega\rangle = Z\otimes\mathbb{1}|\Omega\rangle$

$|\Omega_{10}\rangle = \mathbb{1}\otimes X|\Omega\rangle = X\otimes\mathbb{1}|\Omega\rangle$

$|\Omega_{11}\rangle = \mathbb{1}\otimes XZ|\Omega\rangle = ZX\otimes\mathbb{1}|\Omega\rangle$

Entanglement swapping



Comments

1. Doesn't violate no-cloning
2. The protocol reveals no information about the state $|\psi\rangle$ (otherwise we'll destroy the state).

Bell meas. outcome is completely random.

3. No FTL. Need to send classical bits to recover $|\psi\rangle$.