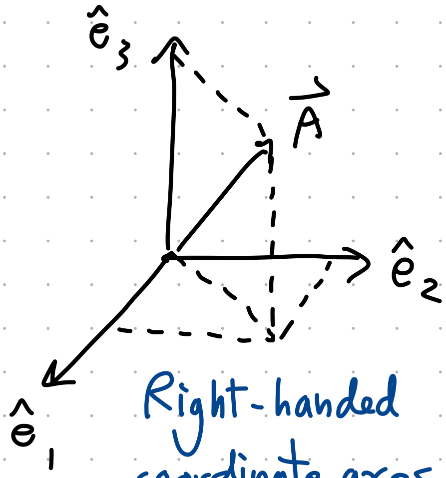


Right-handed coordinate axes

$$\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = \sum_{i=1}^{3} A_i \hat{e}_i$$

Implicitly using an orthonormal basis

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

Dot product and cross product (See also Tutorial 24 Nov)

You can define "cross product" in arbitrary dim (exterior product), but it will be a vector only in 3 dim

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3, \quad \vec{A}\times\vec{B} = \begin{pmatrix} A_2B_3 - B_2A_3 \\ A_3B_1 - B_3A_1 \\ A_1B_2 - B_1A_2 \end{pmatrix}$$

Einstein summation convention: we sum over a repeated index

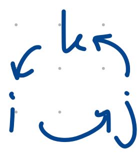
$$\vec{A} = A_i\hat{e}_i \qquad \vec{A}\cdot\vec{B} = (A_i\hat{e}_i)\cdot(B_j\hat{e}_j) = A_iB_j\,\hat{e}_i\cdot\hat{e}_j = A_iB_j\delta_{ij}$$
$$= A_iB_i$$

dummy indices (หุ่นจำลอง)

> Don't do this!
>
> $\vec{x}\cdot\vec{y} = (x_i\hat{e}_i)\cdot(y_i\hat{e}_i)$ ✗
>
> Don't know what are included in which summation

Length $\sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_iA_i} = \sqrt{A_1^2 + A_2^2 + A_3^2}$

$\hat{e}_i \times \hat{e}_j = \hat{e}_k$ where i,j,k are cyclic



$(i,j,k) = (1,2,3)$ or $(3,1,2)$ or $(2,3,1)$

Antisymmetry

$\hat{e}_i \times \hat{e}_j = -\hat{e}_j \times \hat{e}_i$ (In particular $\hat{e}_i \times \hat{e}_i = 0$)

More generally, we can compactly define the cross product as

$$\vec{A}\times\vec{B} = \epsilon_{ijk}A_iB_j\hat{e}_k$$

Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) \text{ is cyclic} \\ -1 & (i,j,k) \text{ is anticyclic} \\ 0 & \text{otherwise} \end{cases}$$

The kth component

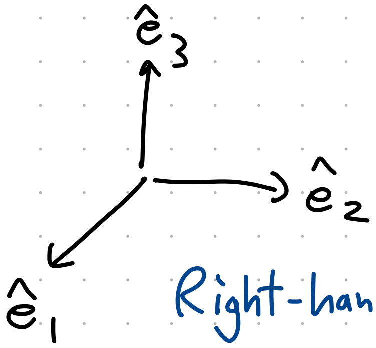
$$(\vec{A}\times\vec{B})_k = \epsilon_{ijk}A_iB_j$$

Fixed index Dummy

Examples $\epsilon_{312} = 1$, $\epsilon_{321} = -1$

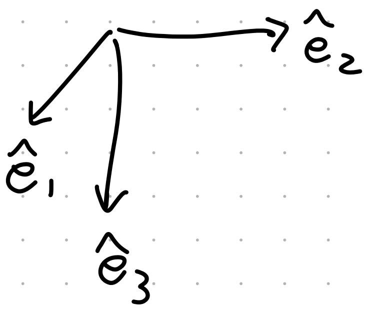
$\epsilon_{112} = 0$

Handedness

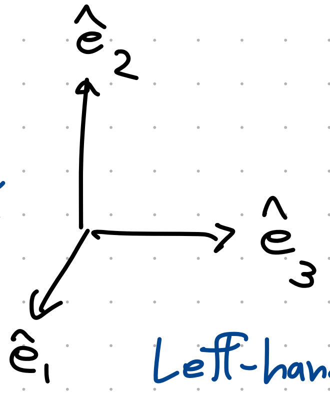


Right-handed

vs



$\cong$



Left-handed

The difference is encapsulated in the triple product $\hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k)$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \epsilon_{ijk} A_i B_j C_k = \det \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

Proof (optional)

Determinant in N dimension

$$\det M = \sum_{\sigma \in S_N} \left(\operatorname{sgn} \sigma \prod_{i=1}^{N} M_{i,\sigma(i)} \right)$$

S_N: Permutation group of N objects

$|S_N| = N!$

$$3 \text{ dim} \Rightarrow \operatorname{sgn} \sigma = \begin{cases} 1 & \text{cyclic} \\ -1 & \text{anticyclic} \end{cases}$$

$|S_3| = 3! = 6$

and specialize the notation to $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$ (I can write this because $\det M^T = \det M$)

$\sigma = (i, j, k)$ means $\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ i & j & k \end{matrix}$

σ	$\prod M_{i,\sigma(i)}$	$\Rightarrow$ New notation	$\operatorname{sgn} \sigma$
$(1,2,3)$	$M_{11} M_{22} M_{33}$	$A_1 B_2 C_3$	$+1$
$(1,3,2)$	$M_{11} M_{23} M_{32}$	$A_1 B_3 C_2$	-1
	and so on		
	$\vdots$		

$$\epsilon_{ijk} A_i B_j C_k = \vec{A} \cdot (\vec{B} \times \vec{C})$$

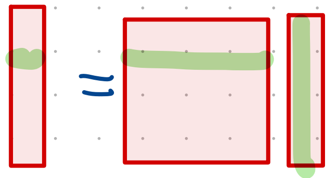
Linear transformations

Matrix $M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$

$\left(\begin{array}{l} \text{Matrix multiplication} \\ (MN)_{ij} = M_{ik} N_{kj} \end{array} \right)$

Multiplication with a vector

ith entry →



Let $\vec{B} = M\vec{A}$, $B_i = M_{ij} A_j$

If $\vec{C} = M\vec{e}_i$

$C_j = M_{ji} \Leftrightarrow \vec{C} = \vec{M}_i$, the ith column of M.

So if M maps the set of coordinate axes to a new one, $\det M$ determines whether M changes the handedness of the axes.

$\hat{e}_1' = M\hat{e}_1$

$\hat{e}_2' = M\hat{e}_2$

$\hat{e}_3' = M\hat{e}_3$

$$\hat{e}_1' \cdot (\hat{e}_2' \times \hat{e}_3') = \underbrace{\vec{M}_1 \cdot (\vec{m}_2 \times \vec{m}_3)}_{\text{(columns)}} = \det M$$

Transformations that preserve the dot product (lengths, angles)

$$\vec{A} \cdot \vec{B} \overset{?}{=} (O\vec{A}) \cdot (O\vec{B}) = (O\vec{A})_i (O\vec{B})_i = O_{ij} A_j O_{ik} B_k$$

$$= A_j B_k O_{ij} O_{ik} = A_j B_k \underbrace{O^T_{ji} O_{ik}}_{\text{Want } \delta_{jk}} \Leftrightarrow \boxed{O^T O = \mathbb{1}}$$

The definition of an orthogonal transformation

$\left(\begin{array}{l} \text{Equivalently} \\ O O^T = \mathbb{1} \end{array} \right)$

Can be proven using only $\hat{e}_i$

$$\delta_{ij} = \hat{e}_i \cdot \hat{e}_j \overset{?}{=} (O\hat{e}_i) \cdot (O\hat{e}_j)$$

$$= (O\hat{e}_i)_k (O\hat{e}_j)_k$$

$$= O_{ki} O_{kj} = O^T_{ik} O_{kj}$$

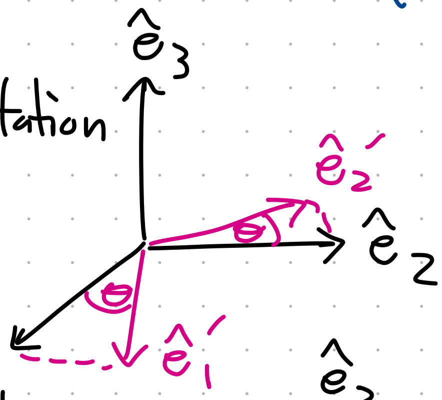
$$\Leftrightarrow O^T O = \mathbb{1}$$

What is the det of an orthogonal transform. ?

$$1 = \det \mathbb{1} = \det(O^T O) = \det(O^T)\det(O)$$
$$= \det(O)\det(O) = (\det O)^2$$

$$\Rightarrow \det O = \begin{cases} +1 & \text{Preserves the handedness} \\ -1 & \text{Switches ——— " ———} \end{cases}$$

Rotation

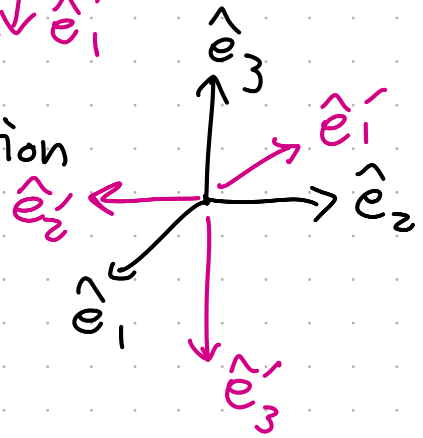


$$\hat{e}_1' = \cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2$$
$$\hat{e}_2' = -\sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_2$$

$$O = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det O = 1$$

Reflection



$$O = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \det O = -1$$

Let us agree that $\det(\vec{A}\,\vec{B}\,\vec{C})$ means the det of a 3×3 matrix with columns $\vec{A}$, $\vec{B}$, and $\vec{C}$.

Under the reflection

$$\left.\begin{aligned} \vec{A} &\mapsto \vec{A}' = -\vec{A} \\ \vec{B} &\mapsto \vec{B}' = -\vec{B} \\ \vec{C} &\mapsto \vec{C}' = -\vec{C} \end{aligned}\right\}$$

① $\det(\vec{A}'\vec{B}'\vec{C}') = \det[(-\vec{A})(-\vec{B})(-\vec{C})] = -\det(\vec{A}\vec{B}\vec{C})$

$\Rightarrow$ Pseudoscalars change signs under reflections

② Let $\vec{A}\times\vec{B} = \vec{D}$, then $\vec{A}'\times\vec{B}' = (-\vec{A})\times(-\vec{B}) = \vec{D}$

Pseudovectors remain unchanged under reflections

(A product of two vectors act like a scalar in the same way the dot product is a scalar)

Examples

Scalars	$\vec{A}\cdot\vec{B}$
Pseudoscalars	$\vec{A}\cdot(\vec{B}\times\vec{C})$
Vectors	$\vec{A}$, $\vec{A}\times(\vec{B}\times\vec{C})$
Pseudovectors	$\vec{A}\times\vec{B}$

$$\vec{F}_{\text{magnetic}} = q\underbrace{\vec{v}}_{\text{Vector}}\times \overset{\text{Pseudo vector}}{\vec{B}}$$

because $\vec{B}$ is a cross product

$\vec{B} = \vec{\nabla}\times\vec{A}$ (Curl of a vector potential)

or $d\vec{B}(\vec{r}) = \frac{I}{c}d\vec{\ell}\times\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$

from Biot-Savart

It is worth emphasizing that these notions are only defined with respect to a given class of transformations

(Here, they are orthogonal transformations in 3 dimensions)

As an example, unitary transformations are privilaged in QM

$U^\dagger U = UU^\dagger = \mathbb{1}$ where $U^\dagger = (U^T)^*$ ← complex conjugation

Tensors

Now we will look at matrices (and more generally multi-index objects) from the point of view of their transformation properties.

Outer product $\vec{A}\vec{B}^T = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}(B_1 B_2 B_3) = \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_3B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{pmatrix}$

(Also called dyadics and can be written as $\vec{A}\otimes\vec{B}^T$ or $|A\rangle\langle B|$ for QM aficionado) or $T_{ij} = A_iB_j$ Example of a rank-2 tensor

Recall $A'_i = O_{ij} A_j$

$$T'_{ij} = O_{ik} O_{jl} A_k B_l = O_{ik} O_{jl} T_{kl} \swarrow$$ want to have j as the last index

$$= O_{ik} T_{kl} (O^T)_{lj} = (O T O^T)_{ij}$$

$$\Leftrightarrow T' = O T O^T = O T O^{-1}$$

So it transforms under O likes an ordinary matrix.
But the transformation is reducible.

vector $\mapsto$ vector' mixes all component = irreducible
scalar $\mapsto$ scalar' —— " —— (trivially)

You will show in the homework a special case of the fact that any rank-2 tensor can be decompose into 3 irreducible parts:

The scalar part: $T^{(0)}_{ij} = \frac{\mathrm{Tr}\, T}{3} \delta_{ij}$

The vector part: $T^{(1)}_{ij} = \frac{T_{ij} - T_{ji}}{2}$

The rest: $T^{(2)}_{ij} = \frac{T_{ij} + T_{ji}}{2} - \frac{\mathrm{Tr}\, T}{3} \delta_{ij}$

where the trace $\mathrm{Tr}\, T = T_{ii} = T_{11} + T_{22} + T_{33}$

More generally, the components of a rank-n tensor transform as

$$T'_{i_1, i_2, \ldots, i_n} = O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_n j_n} T_{j_1, j_2, \ldots, j_n}$$

A contraction by setting a pair of indices to be the same turns a rank n tensor to a rank $(n-2)$ tensor

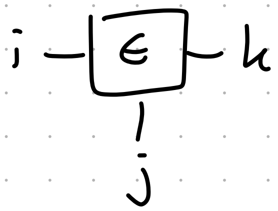
Example: Rank-2 $\quad T_{ij} \mapsto T_{ii} = \mathrm{Tr}\, T \quad$ scalar (rank-0)

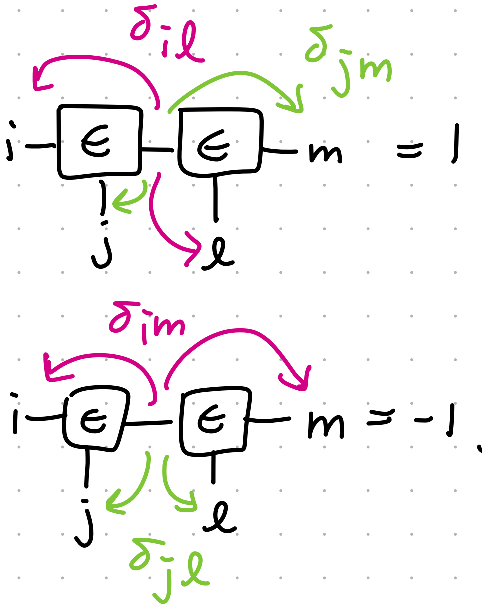
Rank-3 $\quad T_{ijk} = A_i B_j C_k \mapsto T_{iik} = A_i B_i C_k$

$$T \mapsto (\vec{A} \cdot \vec{B}) \vec{C} \quad \text{vector}$$

How to remember the product-of-Levi-Civita identity (Personal note)

Levi-Civita





$$\Rightarrow \epsilon_{ijk} \epsilon_{k\ell m} = \delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}$$

Can we this graphical notation to contract any two indices

Ultimately, it's a contraction of two special 3-tensors. This kind of diagrammatical notations is used frequently in many-body quantum physics (and to some extent in QI/HEP)