**M564 Non-Relativistic Quantum Mechanics** — Ninnat Dangniam

# Homework Assignment 1

DUE: 15 Sep 2022 (Tentative) — 50+15 points

**1. Anisotropic oscillator (10 points).**

Consider the three-dimensional potential,

$$V(x,y,z) = \frac{m\omega^2}{2}\left[\left(1+\frac{2\lambda}{3}\right)(x^2+y^2) + \left(1-\frac{4\lambda}{3}\right)z^2\right], \tag{1}$$

where $0 \leq \lambda \leq 3/4$.

**(a)** What are the eigenstates of the Hamiltonian and the corresponding energy eigenvalues?

**(b)** Compute and discuss the energy, the parity, and the degree of degeneracy of the ground state and the first two excited states as $\lambda$ varies.

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**2. More properties of coherent states (10 points).**

**(a)** Show that any two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are never orthogonal by computing $|\langle\alpha|\beta\rangle|^2$.

**(b)** As a preparation for **(c)**, compute the integral

$$\int_0^\infty dr\, e^{-r^2} r^{2n+1}. \tag{2}$$

You may use the Gaussian integral $\int_0^\infty e^{-\alpha r^2} = \sqrt{\pi/\alpha}$. **Hint:** Consider differentiating $\int_{-\infty}^\infty e^{-\alpha r^2}$ with respect to the parameter $\alpha$.

**(c)** Show that

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{1}, \tag{3}$$

where $\int d^2\alpha$ is an integral over the entire complex plane.

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**3. Squeezed states (15 points).**

We have seen from the lectures that, for a state $|\psi\rangle$ to be a minimum-uncertainty state, $|\psi\rangle$ must satisfy the equation

$$(\Delta X + \lambda \Delta P)|\psi\rangle = 0, \tag{4}$$

for a purely imaginary $\lambda$, where

$$X := \sqrt{\frac{m\omega}{\hbar}}x, \qquad P := \frac{p}{\sqrt{\hbar m\omega}}, \tag{5}$$

are the dimensionless position and momentum operators respectively. By choosing $\lambda = i$, (4) becomes precisely the condition that $|\psi\rangle$ is an eigenstate of the annihilation operator $a$. This exercise is about what happens if we choose other values of $\lambda$.

In the lectures, we have noted that

$$|\lambda| = \frac{\Delta X}{\Delta P} \tag{6}$$

signifies the trade-off in the variances of $X$ and $P$. For simplicity, let us omit the zero-point energy and write

$$H = \hbar\omega a^\dagger a, \qquad U(t,0) = \exp\left(-\frac{iHt}{\hbar}\right) = \exp(-i\omega a^\dagger a t). \tag{7}$$

Consider $\lambda$ to be a decaying exponential function.

$$(a\cosh r + a^\dagger \sinh r)\,|\psi_r\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(e^r x + ie^{-r}\frac{p}{m\omega}\right)|\psi_r\rangle = 0, \tag{8}$$

where $r$ is a real number called the *squeeze parameter*.

**(a)** Evaluate the expectation values $\langle x\rangle, \langle p\rangle, \langle x^2\rangle, \langle p^2\rangle$, and $\langle xp + px\rangle$ with respect to the state $|\psi_r\rangle$.

**(b)** Squeezed states can be obtained by applying the unitary *squeeze operator*

$$S(r,\theta) := \exp\left[\frac{r}{2}\left(a^2 e^{-2i\theta} - (a^\dagger)^2 e^{2i\theta}\right)\right] \tag{9}$$

to the oscillator's ground states $|0\rangle$:

$$|\varphi_{r,\theta}\rangle := S(r,\theta)\,|0\rangle\,. \tag{10}$$

Show that

$$S(r,\theta)\, a\, S^\dagger(r,\theta) = a\cosh r + a^\dagger e^{2i\theta}\sinh r. \tag{11}$$

**(c)** Show that the state $|\psi_r\rangle$ can be taken to be a squeezed state $|\varphi_{r,0}\rangle$.

**(d)** Suppose that the initial state of the oscillator is $|\psi(0)\rangle = |\varphi_{r,0}\rangle$. Show that the state

$$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle\,, \tag{12}$$

at an arbitrary time $t$ remains a squeezed state $|\varphi_{r(t),\theta(t)}\rangle$ and determine the degree of squeezing $r(t)$ and the squeezing angle $\theta(t)$ as functions of $t$. Show that at time $t = \pi/(2\omega)$, the sign of the squeeze parameter reverses: $r \mapsto -r$.

**(e)** Find the wave function $\psi_r(x) = \langle x|\psi_r\rangle$ up to an irrelevant global phase.

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**4. Oscillator driven by a time-dependent force. (15+15 points)**

Consider a harmonic oscillator acted on by a generalized force $f(t)$, leading to the Hamiltonian with an explicit time dependence

$$H(t) = H_0 + i\hbar\left[a^\dagger f(t) - a f^*(t)\right]. \tag{13}$$

Solving the operator Schrödinger equation

$$i\hbar\frac{dU(t,0)}{dt} = H(t)U(t,0), \tag{14}$$

yields the solution

$$U(t,0) = e^{i\delta(t)}U_0(t,0)D(a,\alpha(t)), \tag{15}$$

where $\alpha(t)$ and $\delta(t)$ are functions to be determined, $D(a,\alpha(t)) = \exp(\alpha(t)a^\dagger - \alpha^*(t)a)$ is the displacement operator, and $U_0(t,0) = \exp(-i\omega a^\dagger a t)$ is the free time-evolution operator.

**(a)** Show that $D(a,\alpha(t))$ satisfies the following differential equation:

$$\frac{dD(a,\alpha(t))}{dt} = \left[-\frac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + (\dot{\alpha}a^\dagger - \dot{\alpha}^* a)\right] D(a,\alpha(t)). \tag{16}$$

**Hint:** Use the Baker-Campbell-Hausdorff identity to write the displacement operator in the normal order first.

**(b)** Use the result of **(a)** to show that the Schrödinger equation implies two ODEs for $\alpha(t)$ and $\delta(t)$, the solutions of which are

$$\alpha(t) = \int_0^t dt' f(t')e^{i\omega'}, \qquad \delta(t) = \frac{i}{2}\int_0^t dt'(\alpha^*\dot{\alpha} - \alpha\dot{\alpha^*}). \tag{17}$$

**(c)** Suppose that the oscillator is initially in the ground state $|\psi(0)\rangle = |0\rangle$. Find the expectation values $\langle a\rangle, \langle a^\dagger\rangle, \langle a^\dagger a\rangle$, and $\langle a^2\rangle$ in terms of $\alpha(t)$ and $\delta(t)$. Use these expectation values to calculate the expectation values and variances of $x$ and $p$.

**(d)** Derive the Heisenberg equations of motion for $a, a^\dagger, x$ and $p$.

**4(e) and 4(f) are optional bonus problems.**

**(e)** Solve the resulting Heisenberg equations of motion for $a$ and $a^\dagger$ with appropriate initial conditions at $t = 0$.

**(f)** Repeat **(c)** using the form of the Heiserberg operators obtained in **(e)**. That is, show that the expectation values calculated in the Schrödinger picture and the Heisenberg picture are the same.