M564 Non-Relativistic Quantum Mechanics	Ninnat Dangniam
Homework Assignment 2	
DUE: Tuesday 4 Oct 2022	40 points

1. (10 points). Cohen-Tannoudji J_{IV} 3.

2. (10 points). Cohen-Tannoudji J_{IV} 5.

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3. Nuclear Magnetic Resonance (10 points).

Consider a spin-1/2 particle in a constant external field $\mathbf{B} = B_0\hat{\mathbf{e}}_\mathbf{z}$ under the influence of some time-dependent Hamiltonian $\widehat{W}(t)$, leading to the total Hamiltonian

$$\widehat{H}(t) = \widehat{H}_0 + \widehat{W}(t); \tag{1}$$

$\widehat{H}_0 = -\boldsymbol{\mu}\cdot\mathbf{B} = -\Omega\,\hat{\sigma}_z/2$, where $\Omega = \gamma B_0$ is the Larmor frequency. In order to study the dynamics governed by the time-dependent part, it is useful to move into the rotating frame of reference in which the spin no longer precesses about the z axis. Let $\widehat{U}_0 = \exp(i\Omega\hat{\sigma}_z t/2)$ be the free time-evolution operator, and define

$$|\psi_I(t)\rangle := \widehat{U}_0^\dagger(t,0)\,|\psi(t)\rangle\,, \qquad \widehat{H}_I(t) := \widehat{U}_0^\dagger(t,0)\widehat{H}\,\widehat{U}_0(t,0). \tag{2}$$

The subscript I indicates that we are in the *interaction picture* (also known as the *Dirac picture*) in which both states and operators evolve in time.

(a) Show that the Schrödinger equation in the interaction picture takes on the familiar form

$$i\frac{d}{dt}\,|\psi_I(t)\rangle = \widehat{H}_I(t)\,|\psi_I(t)\rangle\,. \tag{3}$$

(b) Show that

$$\begin{aligned}
\widehat{U}_0^\dagger\hat{\sigma}_z\widehat{U}_0 &= \hat{\sigma}_z,\\
\widehat{U}_0^\dagger\hat{\sigma}_x\widehat{U}_0 &= \cos(\Omega t)\hat{\sigma}_x + \sin(\Omega t)\hat{\sigma}_y,\\
\widehat{U}_0^\dagger\hat{\sigma}_y\widehat{U}_0 &= \cos(\Omega t)\hat{\sigma}_y - \sin(\Omega t)\hat{\sigma}_x.
\end{aligned} \tag{4}$$

Since any arbitrary $\widehat{W}(t)$ can be decomposed in the Pauli basis

$$\widehat{W}(t) = A_0\hat{\mathbb{1}} + A_x\hat{\sigma}_x + A_y\hat{\sigma}_y + A_z\hat{\sigma}_z, \tag{5}$$

we have that

$$\widehat{H}_I(t) = A_0\hat{\mathbb{1}} + \big[A_x\cos(\Omega t) - A_y\sin(\Omega t)\big]\,\hat{\sigma}_x + \big[A_x\sin(\Omega t) + A_y\cos(\Omega t)\big]\,\hat{\sigma}_y + A_z\hat{\sigma}_z. \tag{6}$$

If the Schrödinger-picture Hamiltonian is constant or varies slowly with time, the $\hat{\sigma}_x$ and $\hat{\sigma}_y$ terms in (6) will oscillate rapidly around zero, and thus the effect of these terms will be canceled out on average, leaving the net effect in the rotating frame being just an extra precession about the z axis.

(c) Now consider a specific oscillatory Hamiltonian

$$\widehat{W}(t) = \cos(\Omega' t - \phi)\hat{\sigma}_x - \sin(\Omega' t - \phi)\hat{\sigma}_y, \tag{7}$$

which could have been turned on by shining a circularly-polarized light onto the spin-1/2 particle. Use (6) to write down the interaction-picture Hamiltonian $\widehat{H}_I(t)$. You can ignore the identity component $A_0\hat{\mathbb{1}}$, since it only contributes an overall global phase. Discuss what happens in the rotating frame if $\Omega - \Omega'$ is close to zero.

(d) We can use this phenomenon of *nuclear magnetic resonance* to generate an arbitrary rotation of the quantum state of the spin-1/2 particle. Suppose that Ω' exactly matches the Larmor frequency: $\Omega - \Omega' = 0$. Describe how one would generate an arbitrary Z rotation in the rotating frame using a time-sequence of Hamiltonians of the form (7) with varying values of ϕ.

Magnetic Resonance Imaging (MRI) is a technique based on NMR that allows us to distinguish various kinds of tissues in the human body. In a typical MRI machine, the magnetic field's magnitude B_0 is a few Tesla, and radio-frequency pulses (on the order of one to several hundred megahertz) are used to match the Larmor frequency.

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4. Unpolarized Light (10 points).

The goal of this exercise is to develop a basic appreciation of the notion of mixed quantum states.

When people talk about "unpolarized" light, they mean that the polarization state can point in any direction and all directions are equally probable.

(a) One consequence of this is that the expectation of all polarization observables must vanish: $\langle\hat{\sigma}_x\rangle = \langle\hat{\sigma}_y\rangle = \langle\hat{\sigma}_z\rangle = 0$. If we assume that the quantum state is of the form

$$|\psi\rangle = \alpha\,|R\rangle + \beta\,|L\rangle\,, \tag{8}$$

where $|R\rangle$ and $|L\rangle$ are the orthonormal right- and left-circularly polarization state as in the lectures. Calculate $\langle\hat{\sigma}_x\rangle$, $\langle\hat{\sigma}_y\rangle$, $\langle\hat{\sigma}_z\rangle$ in terms of α and β. Is it possible for all of them to vanish simultaneously?

(b) As a first attempt to describe a statistical mixture of polarization states with equal weights, consider averaging the state

$$|\psi(\varphi)\rangle = \frac{|R\rangle + e^{i\varphi}\,|L\rangle}{\sqrt{2}}, \tag{9}$$

over all possible relative phases φ:

$$\int_0^{2\pi} \frac{d\varphi}{2\pi}\,|\psi(\varphi)\rangle\,, \tag{10}$$

over all possible relative phases φ. Does such an averaging process produce a valid state vector? Does the value of $\langle\hat{\sigma}_z\rangle$ in the "state" (10) make sense?

(c) The correct averaging process relies on the density matrix formalism. Compute the average density matrix

$$\hat{\rho} = \int_0^{2\pi} \frac{d\varphi}{2\pi}\,|\psi(\varphi)\rangle\,\langle\psi(\varphi)|\,, \tag{11}$$

and show that the expectation values $\langle \hat{\sigma}_j \rangle = \mathrm{Tr}(\hat{\rho}\, \hat{\sigma}_j)$ vanish for all directions x,y, and z. (The same density matrix can be obtained by averaging the density matrix associated to the state

$$|\hat{\mathbf{n}}\rangle = |\hat{\mathbf{n}}(\theta, \varphi)\rangle = \cos(\theta/2)\, |R\rangle + e^{i\varphi} \sin(\theta/2)\, |L\rangle \tag{12}$$

over all values of θ and φ, that is, over the entire Poincaré sphere and not just along the equator.)