M564 Non-Relativistic Quantum Mechanics	Ninnat Dangniam
Homework Assignment 3	
DUE: Wednesday 19? Oct 2022	65 points

1. (10 points). Cohen-Tannoudji (C-T) $\mathrm{F_{VI}}$ 3.

These calculations show, in particular, that the three components of the position operator $\hat{\mathbf{r}}$ and those of the momentum operator $\hat{\mathbf{p}}$ are *vector operators*, whereas $\hat{\mathbf{r}}^2$, $\hat{\mathbf{p}}^2$, and $\hat{\mathbf{r}} \cdot \hat{\mathbf{p}}$ are *scalar operators* with respect to rotations.

2. (10 points). C-T $\mathrm{F_{VI}}$ 5.

3. (10 points). C-T $\mathrm{F_{VI}}$ 8.

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4. Angular Momentum of an Isotropic Oscillator (10 points).

The Hamiltonian of an isotropic, three-dimensional harmonic oscillator is given by

$$\widehat{H}(t) = \hbar\omega\left(\hat{a}_x^\dagger a_x + \hat{a}_y^\dagger a_y + \hat{a}_z^\dagger a_z + \frac{3}{2}\right)' = \hbar\omega\left(\hat{a}_R^\dagger a_R + \hat{a}_L^\dagger a_L + \hat{a}_z^\dagger a_z + \frac{3}{2}\right), \tag{1}$$

where $\hat{a}_x$, $\hat{a}_y$, and $\hat{a}_z$ are annihilation operators for linear oscillators along the three Cartesian axes and

$$\hat{a}_R = \frac{\hat{a}_x - i\hat{a}_y}{\sqrt{2}}, \qquad \hat{a}_L = \frac{\hat{a}_x + i\hat{a}_y}{\sqrt{2}}, \tag{2}$$

are annihilation operators for right- and left-circular oscillators. The energy eigenstates are

$$|\chi_{n_R,n_L,n_z}\rangle = \frac{(\hat{a}_R^\dagger)^{n_R}(\hat{a}_L^\dagger)^{n_L}(\hat{a}_z^\dagger)^{n_z}}{\sqrt{n_R!n_L!n_z!}}\,|\chi_{0,0,0}\rangle\,. \tag{3}$$

One can show that the three angular momentum operators can be written in terms of the creation and annihilation operators as follow.

$$\hat{L}_x = i\hbar(\hat{a}_z^\dagger\hat{a}_y - \hat{a}_y^\dagger\hat{a}_z), \tag{4}$$
$$\hat{L}_y = i\hbar(\hat{a}_x^\dagger\hat{a}_z - \hat{a}_z^\dagger\hat{a}_x), \tag{5}$$
$$\hat{L}_z = i\hbar(\hat{a}_y^\dagger\hat{a}_x - \hat{a}_x^\dagger\hat{a}_y), \tag{6}$$

or, written compactly,

$$\hat{L}_j = \epsilon_{jkl}\hat{x}_k\hat{p}_l = i\hbar\epsilon_{jkl}\hat{a}_l^\dagger\hat{a}_k. \tag{7}$$

The raising and lowering operators can also be written in the form

$$\hat{L}_+ = \sqrt{2}\hbar(\hat{a}_z^\dagger\hat{a}_L - \hat{a}_R^\dagger\hat{a}_z), \tag{8}$$
$$\hat{L}_- = \sqrt{2}\hbar(\hat{a}_L^\dagger\hat{a}_z - \hat{a}_z^\dagger\hat{a}_R). \tag{9}$$

Consider, in this problem, putting two quanta of energy into the oscillators. That is, we are considering the six-dimensional subspace $\mathcal{H}_2$ spanned by the energy eigenstates with energy $7\hbar\omega/2$.

(a) Show that the energy eigenstates $|\chi_{n_R,n_L,n_z}\rangle$ in $\mathcal{H}_2$ are eigenstates of $\hat{L}_z$ and find their eigenvalues.

(b) In $\mathcal{H}_2$, find eigenstates and the corresponding eigenvalues of $\hat{\vec{L}}^2$ and $\hat{L}_z$ in terms of the states $|\chi_{n_R,n_L,n_z}\rangle$. Explain the physical meaning of the eigenvalues you assign to $\hat{\vec{L}}^2$.

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5. (10 points). C-T G_X 5.

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6. Applications of the Wigner-Eckart theorem (15 points).

Suppose that we shine a monochromatic light described by the following vector and scalar potentials

$$\mathbf{A}(bfr,t) = A_0 \mathrm{Re}[e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\hat{\boldsymbol{\epsilon}}], \tag{10}$$
$$\phi(bfr,t) = 0, \tag{11}$$

where the unit vector $\hat{\boldsymbol{\epsilon}}$ indicates the polarization state of the light. Let $q = -e = -1.6202\cdots \times 10^{-19}$ be the electron's charge and m_e be the mass of an electron. The Hamiltonian of the atom is

$$\widehat{H} = \frac{1}{2m_e}\left[\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}(\mathbf{x},t)\right]^2 + \frac{e}{m_e c}\mathbf{B}(\mathbf{x},t)\cdot\hat{\mathbf{S}} + V(r) + \text{Relativistic corrections} \tag{12}$$
$$= \hat{H}_0 - \frac{q}{m_e c}\left[\mathbf{A}(\mathbf{x},t)\cdot\hat{\mathbf{p}} + \mathbf{B}(\mathbf{x},t)\cdot\hat{\mathbf{S}}\right] + O(A_0^2), \tag{13}$$

where H_0 is the free Hamiltonian of the atom (including relativistic corrections).

A rough estimate of the order of magnitude of the frequency of light that could excite the atom from one state to another can be given using dimensional analysis. The estimate is ($k = \omega c$)

$$ka_0 \sim \alpha = \frac{e^2}{\hbar c} = \frac{1}{137}. \tag{14}$$

In this problem, we use the fact that ka_0 is small to consider only the first two terms in the Taylor expansion,

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 1 + i\mathbf{k}\cdot\mathbf{r} + \cdots. \tag{15}$$

(a) Find terms in the Hamiltonian that are independent of $\mathbf{k}$ and call them V_{E1}. For the reason that will be clear later on, V_{E1} induces *electric dipole (E1) transitions*. Determine the E1 selection rules. What states are allowed if the initial state is $|n=3, j=\frac{1}{2}, m=\frac{1}{2}, l=0\rangle$?

(b) Show that if we ignore the relativistic corrections,

$$[\hat{\mathbf{r}}, H_0] = \frac{i\hbar\hat{\mathbf{p}}}{m_e}. \tag{16}$$

Then use this identity to show that

$$\langle E_f|V_{\mathrm{E1}}|E_i\rangle \propto \langle E_f|\mathbf{E}\cdot q\hat{\mathbf{r}}|E_i\rangle, \tag{17}$$

where $|E_i\rangle$ and $|E_f\rangle$ are some eigenstates of $\widehat{H}_0$, and $q\hat{\mathbf{r}}$ is the electric dipole operator. Suppose that the light propagates in the z direction. Find the polarization vector $\hat{\boldsymbol{\epsilon}}$ of the light which would add an $m = \pm 1$ unit of angular momentum to the atom.

(c) Now we move on to terms that are linear in $\mathbf{k}$ in the Hamiltonian. In particular, separate out the terms of the form

$$V_{\mathrm{M1}} \propto \mathbf{B} \cdot (\hat{\mathbf{L}} + 2\hat{\mathbf{S}}). \tag{18}$$

V_{M1} induces *magnetic dipole (M1) transitions*. Determine the selection rules for such a transition.

(d) Show that the matrix elements of the rest of the linear terms that we ignored in **(c)** (that is, terms that are not proportional to $\mathbf{B} \cdot (\hat{\mathbf{L}} + 2\hat{\mathbf{S}})$) can be written as

$$\langle E_f | V_{\mathrm{E2}} | E_i \rangle \propto \sum_{q=0,\pm1,\pm2} c_q \langle E_f | r^2 Y_2^q(\theta, \varphi) | E_i \rangle \tag{19}$$

For simplicity, assume that $\mathbf{k}$ is in the x direction: $\mathbf{k} = k\hat{\mathbf{e}}_x$, and that $\hat{\epsilon} = \hat{\mathbf{e}}_y$. V_{E2} induces *electric quadrupole (E2) transitions*. Determine the selection rules for such a transition. You may need the definitions of the following spherical harmonics.

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2\theta, \quad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \sin\theta\cos\theta, \quad Y_2^0 = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right). \tag{20}$$

(e) The atomic transition probabilities between energy eigenstates $|i\rangle$, $|f\rangle$ of the free Hamiltonian can be computed using the so-called *Fermi's golden rule* (which is actually due to Dirac):

$$\Pr(i \leftrightarrow f) \propto \left| \langle E_f | V | E_i \rangle \right|^2. \tag{21}$$

Give an order-of-magnitude estimate of how rare the M1 and E2 transitions are compared to E1 transitions.