Group

A set G with a associative binary operation $g \cdot h = gh$ ("multiplication") is said to be a group if

$g, h \in G$

⓪ The multiplication is close: $gh \in G \ \forall g, h \in G$

① There exists an identity element e s.t.

$$eg = ge = g, \quad \forall g \in G$$

② For all $g \in G$, there exists $g^{-1} \in G$ s.t.

$$gg^{-1} = g^{-1}g = e$$

Examples

- Finite groups: permutation, dihedral
- Infinite groups: rotation, unitary

Representation

A (unitary) representation of G on a vector space V is a map $\varphi : G \to U(V)$ s.t.

Group of unitary matrices on V

$$\boxed{\varphi(g)\varphi(h) = \varphi(gh)}$$

$$\to \varphi(e) = \mathbb{1}$$

$$\to \varphi(g^{-1}) = [\varphi(g)]^{-1} = [\varphi(g)]^{\dagger}$$

Group action on functions

If G is able to act on a set X, G also acts on functions over X as follow

$$g f(x) = f(g^{-1}x)$$

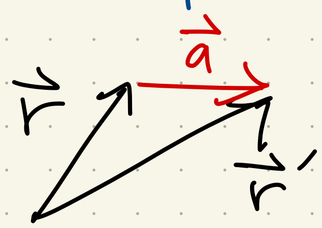
Proof that this definition preserves group composition

$$(gh) f(x) = g(h f(x)) = g f(h^{-1}x)$$

Define $\tilde{f}(x) = f(h^{-1}x)$

$$g\tilde{f}(x) = \tilde{f}(g^{-1}x) = f(h^{-1}g^{-1}x)$$
$$= f[(gh)^{-1}x]$$

Examples Translating wave functions $X = \mathbb{R}^3$



$$|\psi\rangle \mapsto |\psi'\rangle = \hat{T}_{\vec{a}}|\psi\rangle$$

$$\psi(\vec{r}-\vec{a}) = \langle\vec{r}|\hat{T}_{\vec{a}}|\psi\rangle$$

Rep of translation group

$$= \langle\hat{T}^{\dagger}_{\vec{a}}\vec{r}|\psi\rangle$$

$\hat{T}_{\vec{a}}\hat{T}_{\vec{b}} = \hat{T}_{\vec{a}+\vec{b}}$

$$= \langle\vec{r}-\vec{a}|\psi\rangle$$

$\hat{T}^{-1}_{\vec{a}} = \hat{T}^{\dagger}_{\vec{a}} = \hat{T}_{-\vec{a}}$

$$\therefore \hat{T}_{\vec{a}}|\vec{r}\rangle = |\vec{r}+\vec{a}\rangle$$

(Active vs Passive transformations)

$$\psi(\vec{r}-\vec{a}) = \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!}\left(-a\vec{\nabla}\right)^n}_{e^{-a\vec{\nabla}}} \langle \vec{r}|\psi\rangle$$

$$\boxed{\vec{p} = \frac{\hbar}{i}\vec{\nabla}}$$

$$= \langle \vec{r}| e^{-i\hat{p}\cdot\vec{a}/\hbar} |\psi\rangle$$

$$e^A e^B = e^{A+B+[A,B]/2+\left([A,[A,B]]-[B,[A,B]]\right)/12+\cdots}$$

Group multiplication only depends on commutators of generators $\Rightarrow$ Lie algebra. A vector space V equipped with a (non-associative) binary operation $[A,B]$ s.t.

$$[A,B] = -[B,A] \qquad \text{bilinear too}$$

$$[A,[B,C]]+[C,[A,B]]+[B,[C,A]]=0$$

(Jacobi identity)

$$T^\dagger_{\vec{a}}\,\hat{x}\,T_{\vec{a}} = \left(\mathbb{1}+\frac{ia\hat{p}}{\hbar}\right)\hat{x}\left(\mathbb{1}-\frac{ia\hat{p}}{\hbar}\right)+\mathcal{O}(a^2)$$

$$= \hat{x}+\frac{ia}{\hbar}\left(\hat{p}\hat{x}-\hat{x}\hat{p}\right)+\mathcal{O}(a^2)$$

$$= \hat{x}-\frac{ia}{\hbar}[\hat{x},\hat{p}]+\mathcal{O}(a^2)$$

But we know this is $\hat{x}+a \Rightarrow [\hat{x},\hat{p}] = -\frac{\hbar}{i} = i\hbar$

What's important is to notice which properties are rep-dependent and which are rep-independent.

Examples

Rep-independent

$$[\hat{J}_j, \hat{J}_k] = i\hbar \epsilon_{jkl} \hat{J}_l$$

$$[\hat{\sigma}_j, \hat{\sigma}_k] = 2i\epsilon_{jkl}\hat{\sigma}_l$$

$$\Rightarrow \left[\frac{\hbar}{2}\hat{\sigma}_j, \frac{\hbar}{2}\hat{\sigma}_k\right] = \frac{\hbar^2}{4} 2i\epsilon_{jkl}\hat{\sigma}_l = i\hbar\epsilon_{jkl}\frac{\hbar}{2}\hat{\sigma}_l$$

Rep-dependent

$$\left(\frac{\hbar}{2}\hat{\sigma}_j\right)^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\hat{\mathbb{1}} \text{ for any } j$$

$$\hat{L}_x^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \hat{L}_y^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\hat{L}_z^2 = \hbar^2\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 1 \end{pmatrix}, \qquad \hat{\vec{L}}^2 = 1(1+1)\hbar^2\mathbb{1}$$

The usual matrix multiplication is not defined in the Lie algebra. Thus, multiplication of matrix rep of Lie algebra elements needs not agree in different representations.

Theorem The group $SO(3)$ is a representation of $SU(2)$.

- Let φ denote the defining rep of $SU(2)$ (as a det 1, 2×2 unitary matrices). The action of $\varphi(g)$ on a two-dim Hilbert space induces an action on the space of traceless Hermitian matrices by conjugation.

Indeed the fact that this is also a representation of $SU(2)$ follows from the representation property of φ:

$$\varphi(g)\varphi(h)\, H\, [\varphi(h)]^{-1}[\varphi(g)]^{-1}$$
$$\parallel$$
$$\varphi(gh)\, H\, [\varphi(gh)]^{-1}$$

But we have learned that any 2×2 traceless Hermitian matrix is in one-to-one correspondence with a 3D unit vector

$$H = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \longleftrightarrow \hat{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

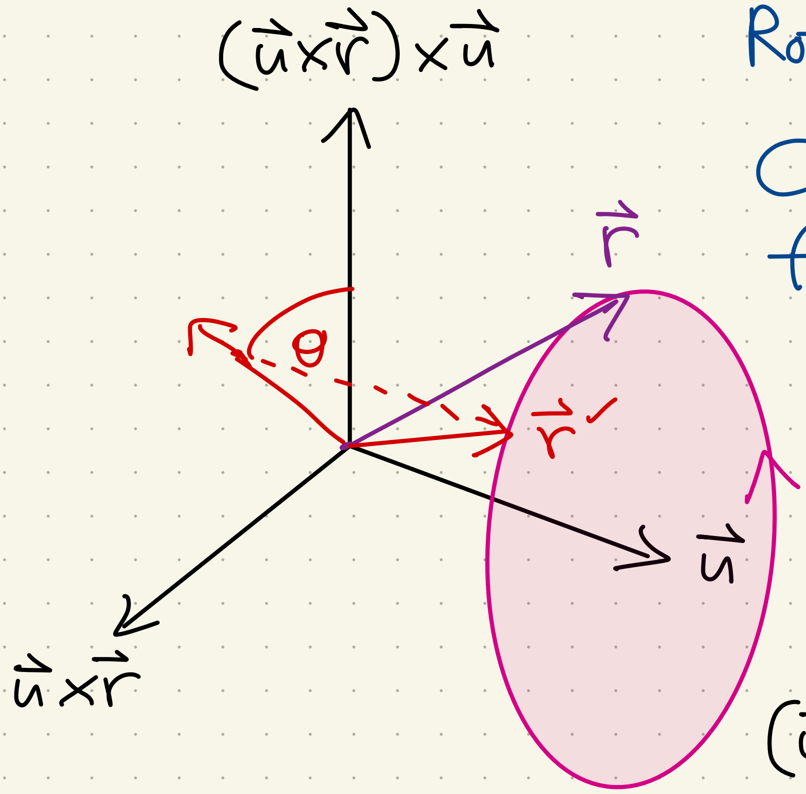
$$\varphi(g) \longleftrightarrow R(g)$$

Thus, the conjugation action of $SU(2)$ can represent all 3D rotations R $\square$

But note the double-valuedness $\left.\begin{matrix} R(\mathbb{1}_{2\times 2}) \\ R(-\mathbb{1}_{2\times 2}) \end{matrix}\right\} \to \mathbb{1}_{3\times 3}$

Rotation of $\vec{r}$ about $\vec{u}$

Construct orthogonal axes from $\vec{u}$, $\vec{u}\times\vec{r}$, $(\vec{u}\times\vec{r})\times\vec{u}$



Can check that

$$(\vec{u}\times\vec{r})\times\vec{u} = \vec{r} - (\vec{u}\cdot\vec{r})\,\vec{u}$$

$$\downarrow$$

$$R_{\vec{u}}(\theta)\,\vec{r} = (\vec{u}\cdot\vec{r})\,\vec{u} + \cos\theta\,(\vec{u}\times\vec{r})\times\vec{u} + \sin\theta\,\vec{u}\times\vec{r}$$

$$\|$$

$$R_{\vec{u}}(\theta)\,x_j\hat{e}_j = x_j\left[R_{\vec{u}}(\theta)\,\hat{e}_j\right]$$

$$= x_j\,\hat{e}_k\underbrace{\left[\hat{e}_k R_{\vec{u}}(\theta)\,\hat{e}_j\right]}_{\text{Matrix element}}$$

Insert the identity

Compare

Infinitesimal rotation $d\theta$

$$R_{\vec{u}}(d\theta)\,\vec{r} = \vec{r} + d\theta\,\vec{u}\times\vec{r}$$

$$\boxed{\left[R_{\vec{u}}(d\theta)\right]_{jk} = \delta_{jk} + d\theta\,\epsilon_{jkl}\,u_l}$$

What are generators of rotations?

$$R_{\vec{u}}(d\theta)\psi(\vec{r}) = \langle R_{\vec{n}}(-d\theta)\vec{r}|\psi\rangle$$
$$= \langle \vec{r} - d\theta\,\vec{u}\times\vec{r}|\psi\rangle$$

Special case $\vec{u} = \hat{e}_z$

$\Rightarrow$ $\hat{e}_z \times \vec{r} = \epsilon_{3jk}\, x_j\, \hat{e}_k = x\hat{e}_y - y\hat{e}_x$

$$\therefore \langle \vec{r}|R_{\hat{e}_z}(d\theta)|\psi\rangle = \langle \vec{r} - d\theta(x\hat{e}_y - y\hat{e}_x)|\psi\rangle$$
$$= \psi(x + y\,d\theta, y - x\,d\theta, z)$$
$$= \psi(x,y,z) + \left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right)d\theta$$
$$= \langle \vec{r}|\left[\hat{\mathbb{1}} - \frac{i}{\hbar}\hat{L}_z(d\theta)\right]|\psi\rangle$$

To obtain the commutation relation, proceed as in the case of translations. Generally, suppose that we have vector operators $\vec{V}$

$$\hat{R}^\dagger(d\theta)\,\hat{\vec{V}}\,\hat{R}(d\theta) = \hat{\vec{V}} + d\theta\,\vec{u}\times\hat{\vec{V}}$$

$$= \left(\mathbb{1} + \frac{i}{\hbar}d\theta\,\hat{J}_j\right)\hat{\vec{V}}\left(\mathbb{1} - \frac{i}{\hbar}d\theta\,\hat{J}_j\right)$$

$$= \vec{V} + \frac{i}{\hbar}d\theta\left(\hat{J}_j\hat{\vec{V}} - \hat{\vec{V}}\hat{J}_j\right)$$

compare

$$\therefore \quad \boxed{[\hat{J}_j, \hat{V}_k] = i\hbar\,\epsilon_{jkl}\hat{V}_l}$$

Thus, the angular momentum commutation relation can be understood as a consequence of the $\hat{J}_j$'s being vector operators.

Example

$$[\hat{L}_j, \hat{x}_k] = [\epsilon_{jlm}\hat{x}_l\hat{p}_m, \hat{x}_k]$$

$$= \epsilon_{jlm}\hat{x}_l[\hat{p}_m, \hat{x}_k]$$

$$= -i\hbar\,\epsilon_{jlk}\hat{x}_l$$

$$= i\hbar\,\epsilon_{jkl}\hat{x}_l$$

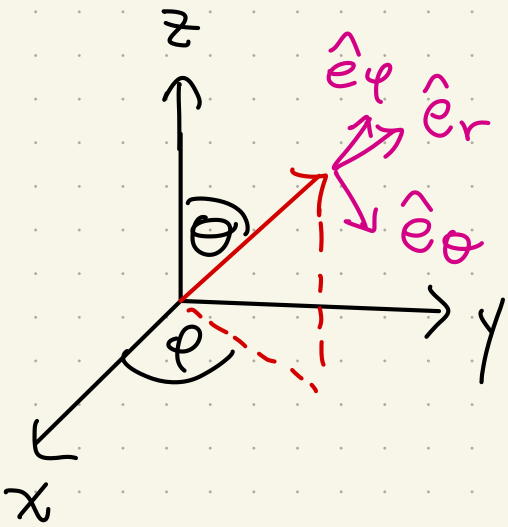
Orbital angular momentum

We will now show that the Hilbert space of wave functions $\psi(\vec{r})$ over $\mathbb{R}^3$ decomposes into a direct sum of all irreps of $SO(3)$ with integral values of j.

(C-T VI.D pp. 663–664)

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

$$\hat{L}_\pm = \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$



ℓ and m must be integral

$$\frac{\hbar}{i}\frac{\partial}{\partial\varphi}Y_\ell^m(\theta,\varphi) = m\hbar Y_\ell^m(\theta,\varphi)$$

Assuming separable solutions

$$Y_\ell^m(\theta,\varphi) = F_\ell^m(\theta)e^{im\varphi}$$

Then the continuity at $\varphi = 0$ and $\varphi = 2\pi$ implies that m is an integer $\Rightarrow$ ℓ is also an integer

All integral values of l appear

$$0 = \hat{L}_+ Y_l^l(\theta,\varphi) = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi}\right) Y_l^l(\theta,\varphi)$$

$$= \hbar e^{i\varphi}\left\{\left[\frac{dF_l^l(\theta)}{d\theta}\right] e^{il\varphi} + i\cot\theta\, F_l^l(\theta)\,\frac{d e^{il\varphi}}{d\varphi}\right\}$$

$$\Longleftrightarrow \left(\frac{d}{d\theta} - l\cot\theta\right) F_l^l(\theta) = 0$$

$$dF = l\cot\theta\, d\theta\, F$$

$$\frac{dF}{F} = l\,\frac{d(\sin\theta)}{\sin\theta}$$

$$d(\ln F) = l\, d[\ln(\sin\theta)]$$

$$\ln F = l\ln(\sin\theta) + C$$

$$F \propto e^{l\ln\sin\theta} = (\sin\theta)^l$$

Clearly, for each l, there exists a unique

$$Y_l^l(\theta,\varphi) \propto (\sin\theta)^l e^{il\varphi}$$

The rest of Y_l^m can be obtained from applying $\hat{L}_-$ to Y_l^l. (C-T Complement A_{VI})

Orthonormality

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, Y_{\ell'}^{m'*}(\theta,\varphi) Y_\ell^m(\theta,\varphi) = \delta_{\ell'\ell}\delta_{m'm}$$

Completeness (Closure relation)

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\theta',\varphi') Y_\ell^m(\theta,\varphi) = \frac{\delta(\theta-\theta')}{\sin\theta}\delta(\varphi-\varphi')$$

3D delta function

$$\delta(x-x')\delta(y-y')\delta(z-z') = \frac{\delta(r-r')}{r}\frac{\delta(\theta-\theta')}{r\sin\theta}\delta(\varphi-\varphi')$$

Parity

Under reflection $\begin{cases} \theta \mapsto \pi-\theta \\ \varphi \mapsto \pi+\varphi \end{cases}$

$$Y_\ell^m(\pi-\theta,\pi+\varphi) = (-1)^\ell Y_\ell^m(\theta,\varphi)$$

$$Y_\ell^{m*}(\theta,\varphi) = (-1)^m Y_\ell^{-m}(\theta,\varphi)$$