Two-level Systems

Examples

Left- and right-circular polarization

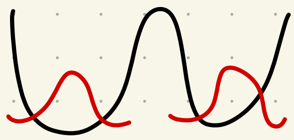
① Polarization of light $|L\rangle, |R\rangle$

② Spin-1/2 $|\uparrow\rangle, |\downarrow\rangle$

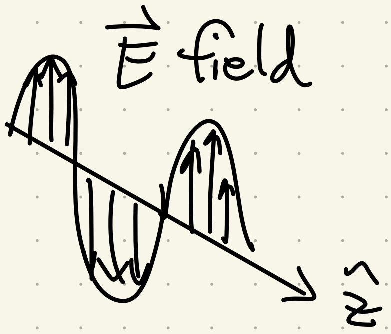
Ground Excited

③ Atomic energy levels $|g\rangle, |e\rangle$

④ Double-well potential



Isomorphic classical system: Polarization of light



$$\vec{E}_x = E_x \cos(kz - \omega t - \phi_x)\hat{e}_x$$

$$= \mathrm{Re}\left(E_x e^{i(kz-\omega t - \phi_y)}\hat{e}_x\right)$$

Same for $\vec{E}_y$

The total $\vec{E}$ field is the vector sum:

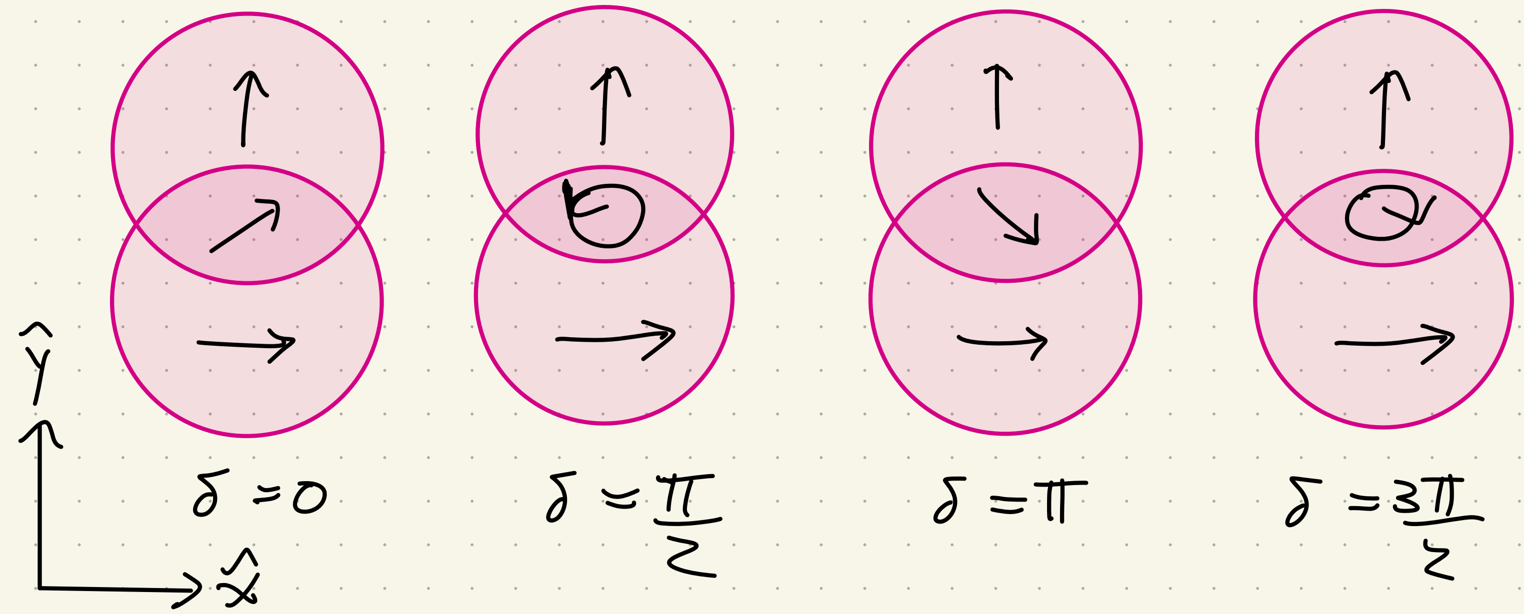
$$\vec{E} = \mathrm{Re}\left[e^{i(kz-\omega t)}\left(E_x e^{-i\phi_x}\hat{e}_x + E_y e^{-i\phi_y}\hat{e}_y\right)\right]$$

Normalized $E_0 = \sqrt{E_x^2 + E_y^2}$, $\tilde{E}_x = \dfrac{E_x}{E_0}$, $\tilde{E}_y = \dfrac{E_y}{E_0}$

$$\vec{E} = \mathrm{Re}\left[E_0 e^{i(kz-\omega t - \phi_x)}\left(\tilde{E}_x\hat{e}_x + \tilde{E}_y e^{i(\phi_x - \phi_y)}\hat{e}_y\right)\right]$$

Examples

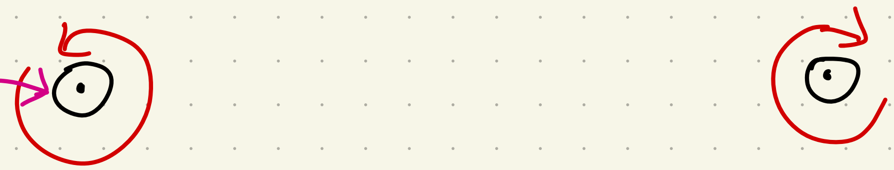
Superposition ($\delta = \phi_x - \phi_y$)



Right-circular polarization

Left-circular polarization

$\hat{z}$ pointing out of the screen



Suppose that $E_x = E_y$, $\phi_x = 0$

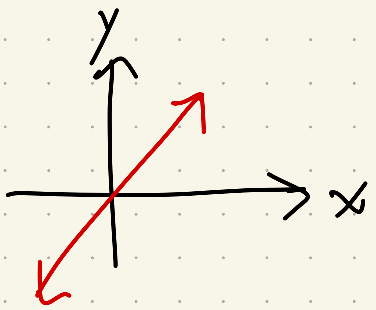
At $z = 0$, $\vec{E}(0,t) = \begin{pmatrix} Re(e^{-i\omega t}) \\ Re(e^{-i\omega t} e^{i\delta}) \end{pmatrix}$

- What states of polarization are represented by real linear combinations (L.C.) of $\hat{e}_x$ and $\hat{e}_y$?
- What about complex L.C.?

Real L.C. $\Rightarrow$ Linear polarization

Example $E_x = E_y$

$$\delta = 0 \Rightarrow \vec{E}(0,t) = \begin{pmatrix} Re\,(e^{-i\omega t}) \\ Re\,(e^{-i\omega t}) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ \cos(\omega t) \end{pmatrix}$$



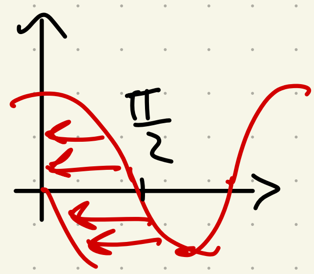
Diagonal polarization

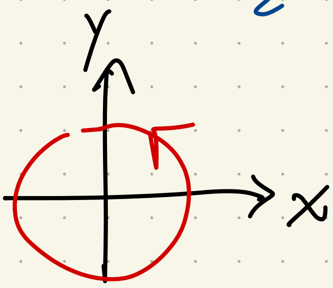
Complex L.C. $\Rightarrow$ Elliptical polarization

Example $E_x = E_y$

$$\delta = \frac{\pi}{2} \Rightarrow \vec{E}(0,t) = \begin{pmatrix} Re\,(e^{-i\omega t}) \\ Re\,(i e^{-i\omega t}) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega t) \\ \cos(\omega t + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{pmatrix}$$

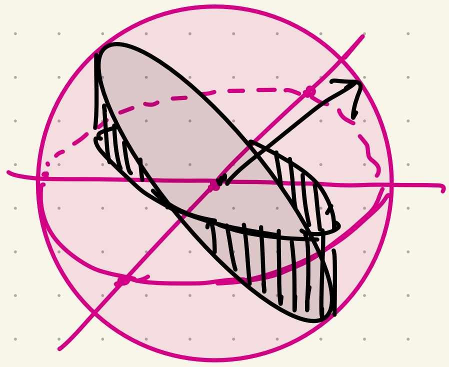




Right-circular polarization

If $E_x \neq E_y \Rightarrow$ Elliptical polarization

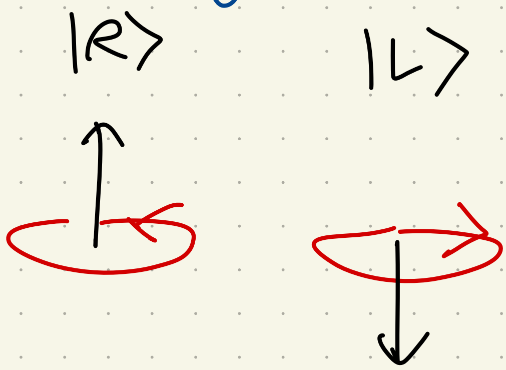
Poincaré sphere



Visual representation of all possible polarization states (all possible ellipses) as 3D vectors

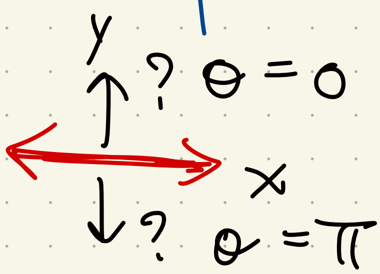
- Draw an ellipse as the projection of a great circle onto the horizontal plane, and then associate to the ellipse the vectorial area of the great circle.

Left- and right-circular polarizations are the south and north poles respectively.



- There remains the problem with all other states which is that antipodal points represent the same state

Example: horizontal polarization



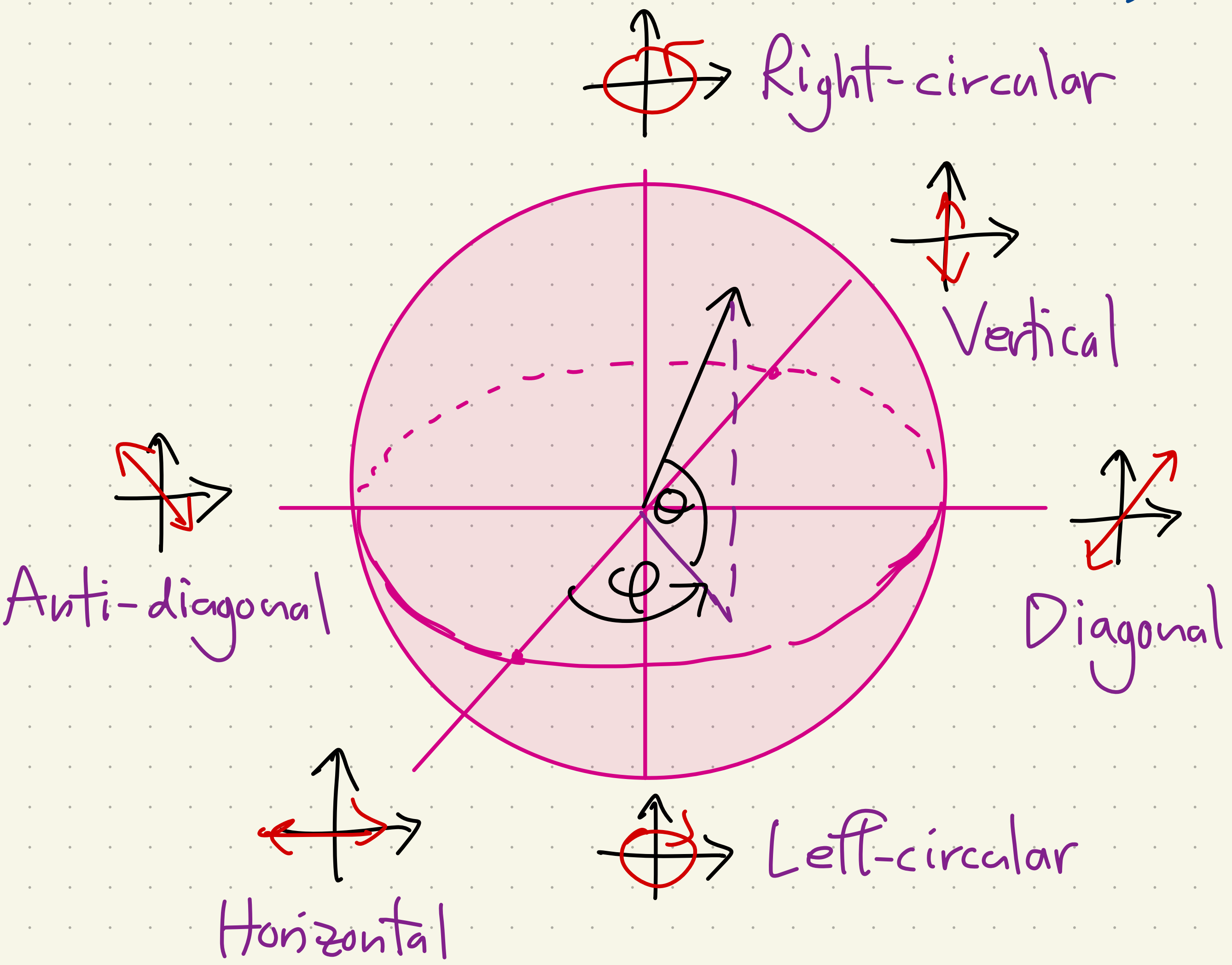
$\Longrightarrow$ Solution: double the polar angle

$$\vec{E} = \mathrm{Re}\, E_0 e^{i(kz-\omega t)} \underbrace{\frac{\hat{e}_x + i\hat{e}_y}{\sqrt{2}}}_{\hat{f}_R}$$ Right circular

$$\frac{\hat{e}_x - ie_y}{\sqrt{2}} = \hat{f}_L$$

Every polarization state can be written as

$$\cos\left(\frac{\theta}{2}\right)\hat{f}_R + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\hat{f}_L \quad \begin{cases} \theta \in [0,\pi] \\ \varphi \in [0, 2\pi) \end{cases}$$



Use left- and right-circular polarization states as basis vectors

Same algebra $\hat{f}_R, \hat{f}_L \longleftrightarrow |L\rangle, |R\rangle$

Horizontal

$$|H\rangle = |R\rangle + |L\rangle \leftrightarrow \hat{e}_x + i\hat{e}_y + \hat{e}_x - i\hat{e}_y \propto \hat{e}_x$$

Vertical

$$|V\rangle = |R\rangle + e^{i\pi}|L\rangle$$
$$= |R\rangle - |L\rangle \leftrightarrow \hat{e}_x + i\hat{e}_y - \hat{e}_x + i\hat{e}_y \propto \hat{e}_y$$

Diagonal

$$|D\rangle = |R\rangle + e^{i\pi/2}|L\rangle$$
$$= |R\rangle + i|L\rangle \leftrightarrow \hat{e}_x + i\hat{e}_y + i\hat{e}_x + \hat{e}_y$$
$$= (1+i)(\hat{e}_x + \hat{e}_y) \propto \hat{e}_x + \hat{e}_y$$

Anti-diagonal

$$|A\rangle = |R\rangle + e^{i3\pi/2}|L\rangle$$
$$= |R\rangle - i|L\rangle \leftrightarrow \hat{e}_x + i\hat{e}_y - i\hat{e}_x - \hat{e}_y$$
$$= (1-i)\hat{e}_x - (1-i)\hat{e}_y$$
$$\propto \hat{e}_x - \hat{e}_y$$