Angular Momentum

First non-trivial example of classification of states according to their symmetries, in this case in particular according to how the quantum states transform under rotations in 3D.

$$\vec{L} = \vec{r} \times \vec{p}$$

Orbital angular momentum operator

$$\left.\begin{aligned}\hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x\end{aligned}\right\} \quad \hat{L}_j = \epsilon_{jkl}\,\hat{r}_k\,\hat{p}_j$$

No ordering problem

$$[\hat{r}_j, \hat{p}_k] = i\hbar\,\delta_{jk}$$

Vector operator $\vec{L} = \hat{L}_x\hat{e}_x + \hat{L}_y\hat{e}_y + L_z\hat{e}_z$

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$$

$$= \hat{y}[\hat{p}_z, \hat{z}]\hat{p}_x + \hat{x}[\hat{z}, \hat{p}_z]\hat{p}_y$$

$$= i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) = i\hbar\,\hat{L}_z$$

$\therefore\ [\hat{L}_j, \hat{L}_k] = i\hbar\, \epsilon_{jk\ell} L_\ell$ same as Pauli matrices

($\{\hat{L}_j\}_j$ & $\{\hat{\sigma}_j\}_j$ are two reps of the same Lie algebra)

9 equations (3 independent)

To derive all of them in one go, we index notation

$$\vec{A} \times \vec{B} = \hat{e}_j\, \epsilon_{jk\ell} A_k B_\ell = \det\begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$$

$$\delta_{jk}\delta_{k\ell} = \delta_{j\ell}$$

$$\delta_{jk}\delta_{jk} = \delta_{jj} = \mathrm{Tr}\, \mathbb{1}_{d\times d} = d$$

(Dimension of the space)

$$\epsilon_{jk\ell}\epsilon_{jmn} = \delta_{km}\delta_{\ell n} - \delta_{kn}\delta_{\ell m}$$

$$\epsilon_{jk\ell}\epsilon_{jkm} = \delta_{kk}\delta_{\ell m} - \delta_{km}\delta_{\ell m}$$

$$= d\,\delta_{\ell m} - \delta_{\ell m} = (d-1)\,\delta_{\ell m}$$

$$\hat{L}_j = \epsilon_{jkl}\hat{r}_k\hat{p}_l$$

$$[\hat{L}_j, \hat{L}_k] = \epsilon_{jlm}\epsilon_{knq}\underbrace{[\hat{r}_l\hat{p}_m, \hat{r}_n\hat{p}_q]}$$

$$\hat{r}_l[\hat{p}_m, \hat{r}_n]\hat{p}_q + \hat{r}_n[\hat{r}_l, \hat{p}_q]\hat{p}_m$$

$$= -i\hbar\,\delta_{mn}\hat{r}_l\hat{p}_q + i\hbar\,\delta_{lq}\hat{r}_n\hat{p}_m$$

$$= i\hbar\left(-\epsilon_{jlm}\epsilon_{kmq}\hat{r}_l\hat{p}_q + \epsilon_{jlm}\epsilon_{knl}\hat{r}_n\hat{p}_m\right)$$

m l l n m n

$$= i\hbar\left(-\epsilon_{jml}\epsilon_{kln}\hat{r}_m\hat{p}_n + \epsilon_{jlm}\epsilon_{knl}\hat{r}_n\hat{p}_m\right)$$

Swapping introduces a minus sign twice

$$= i\hbar\,\underbrace{\epsilon_{jlm}\epsilon_{knl}}_{\delta_{mk}\delta_{jn} - \delta_{mn}\delta_{jk}}(\hat{r}_n\hat{p}_m - \hat{r}_m\hat{p}_n)$$

$$= i\hbar\left(\hat{r}_j\hat{p}_k - \cancel{\hat{r}_m\hat{p}_m} - \hat{r}_k\hat{p}_j + \cancel{\hat{r}_m\hat{r}_m}\right)$$

$$= i\hbar\,\epsilon_{jkl}\hat{L}_l$$

$\hat{L}_j \leftarrow$ Angular momentum operator in 3D

$\hat{J}_j \leftarrow$ Abstract angular momentum operator

$$[\hat{J}_j, \hat{J}_k] = i\epsilon_{jkl}\hat{J}_l$$

$$\vec{J}^2 := \hat{J}_x^2 + J_y^2 + J_z^2$$

$$
\begin{aligned}
[\vec{J}^2, \hat{J}_j] &= [\hat{J}_k\hat{J}_k, \hat{J}_j] \\
&= \hat{J}_k[\hat{J}_k, \hat{J}_j] + [\hat{J}_k, \hat{J}_j]\hat{J}_k \\
&= i\hbar \underbrace{\epsilon_{kjl}}_{\text{Antisymmetric}} \underbrace{(\hat{J}_k\hat{J}_l + \hat{J}_l\hat{J}_k)}_{\text{Symmetric}} = 0
\end{aligned}
$$

under interchange of j and k

$\vec{J}^2$ and $\hat{J}_z$ form an CSCO (Complete set of commuting observables)

We can also choose $\{\vec{J}^2, \hat{J}_x\}$ or $\{\vec{J}^2, \hat{J}_y\}$ to be a CSCO, but once we include one of the $\hat{J}$'s, we cannot include other $\hat{J}$'s since they don't commute.

$$\vec{J}^2 |\beta, m\rangle = \beta\hbar^2 |\beta, m\rangle$$
$$\vec{J}_z |\beta, m\rangle = m\hbar |\beta, m\rangle$$

I follow Ballentine Section 7.2

Fact 1 $\boxed{\beta \geq m^2}$

- $\underbrace{\langle \beta, m | \vec{J}^2 | \beta, m\rangle}_{\beta\hbar^2} = \langle \beta, m | \hat{J}_x^2 | \beta, m\rangle + \langle \beta, m | J_y^2 | \beta, m\rangle + \underbrace{\langle \beta, m | \hat{J}_z^2 | \beta, m\rangle}_{m\hbar^2}$ $\square$

Define $\hat{J}_\pm := \hat{J}_x \pm i\hat{J}_y$

$$[\hat{J}_z, \hat{J}_\pm] = \pm\hbar \hat{J}_\pm$$
$$[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$$
$$\hat{J}_- \hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_x, \hat{J}_y]$$
$$= \vec{J}^2 - \hat{J}_z^2 - \hbar J_z$$

$\leftarrow$ Simple action on $|\beta, m\rangle$

$$\hat{J}_z(\hat{J}_+|\beta,m\rangle) = ([\hat{J}_z,\hat{J}_+] + \hat{J}_+\hat{J}_z)|\beta,m\rangle$$

$$= \hbar(m+1)|\beta,m\rangle$$

$\Longrightarrow$ Either $\hat{J}_+|\beta,m\rangle$ is an eigenvector with eigenvalue $(m+1)\hbar$ or $\hat{J}_+|\beta,m\rangle = 0$

Suppose that we call the value of m for which the equation $\hat{J}_+|\beta,m\rangle = 0$ is true $m=j$

$$\hat{J}_+|\beta,j\rangle = 0$$

$$0 = \hat{J}_-\hat{J}_+|\beta,j\rangle = (\vec{J}^2 - \vec{J}_z^2 - \hbar\vec{J}_z)|\beta,j\rangle$$

$$= \hbar^2[\beta - j(j+1)]|\beta,j\rangle$$

$$\Longrightarrow \boxed{\beta = j(j+1)}$$

Since β is positive (Proof: $\langle\psi|\vec{J}^2|\psi\rangle$ is a sum of positive numbers $\langle\psi|\hat{J}_j^2|\psi\rangle = (\langle\psi|\hat{J}_j)(\hat{J}_j|\psi\rangle) \geq 0$) for a given β, we can solve for a unique j. Thus, we can label the eigenstates by j instead of β.

$$|j,m\rangle$$

Similarly, $\hat{J}_z(\hat{J}_-|j,m\rangle) = \hbar(m-1)(\hat{J}_-|j,m\rangle)$

$\implies$ Either $\hat{J}_-|\beta,m\rangle$ is an eigenvector with eigenvalue $(m-1)\hbar$ or $\hat{J}_-|\beta,m\rangle = 0$

Call this lowest value of $m = k$

$$0 = \hat{J}_+\hat{J}_-|j,k\rangle = \hbar^2[j(j+1) - k(k-1)]|j,k\rangle$$

$$\implies j(j+1) = k(k-1)$$

$$= (-k)(-k+1)$$

$$\implies \quad k = -j$$

Thus, we have that $\boxed{-j \leq m \leq j}$

Now we don't know yet the range of values m or j can take, but we do know that an application of $\hat{J}_+$ (resp. $\hat{J}_-$) increases (resp. decreases) the value of m by 1. Therefore,

\# of steps to hit the highest rung from m

① $m + p = j$

② $m - q = -j$

\# of steps to hit the lowest rung from m

① − ② ;

$$\frac{p+q}{2} = j$$

$\therefore$ $\boxed{j \text{ is either an integer or half-integer}}$

	$j=0$	$j=\frac{1}{2}$	$j=1$	$j=\frac{3}{2}$	...
m	0	1/2 −1/2	1 0 −1	3/2 1/2 −1/2 −3/2	

- j labels an $(2j+1)$-dimensional subspace $\mathcal{H}_j$
- m labels a specific element of the ONB $|j,m\rangle$ for $\mathcal{H}_j$

Normalization of $\hat{J}_\pm |j,m\rangle$

$$\| \hat{J}_+ |j,m\rangle \|^2 = \langle j,m| \hat{J}_- \hat{J}_+ |j,m\rangle$$

$$= \hbar^2 [j(j+1) - m(m-1)] |j,m\rangle$$

$$\Rightarrow \quad \hat{J}_+ |j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\ |j,m\rangle$$
$$= \hbar\sqrt{(j+m+1)(j-m)}\ |j,m\rangle$$

Similarly

$$\hat{J}_- |j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\ |j,m\rangle$$
$$= \hbar\sqrt{(j-m+1)(j+m)}\ |j,m\rangle$$

Example

① $j = \frac{1}{2}$ $\quad \hat{J}_+ |\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar\sqrt{\frac{3}{4} - \frac{3}{4}}\, |\tfrac{1}{2},\tfrac{1}{2}\rangle = 0$

$$\hat{J}_+ |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{\frac{3}{4}+\frac{1}{4}}\, |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\, |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

Ordered basis $\{ |\tfrac{1}{2},\tfrac{1}{2}\rangle, |\tfrac{1}{2},-\tfrac{1}{2}\rangle \}$

$$\hat{J}_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \hat{J}_- = \hat{J}_+^\dagger = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_x$$

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_y$$

$\hat{J}_z$ just has $m\hbar$ in the diagonal: $\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_z$

② $j = 1$

$$\hat{J}_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{J}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{array}{ll|c|cc|ccc|cccc}
 & j = & 0; & \tfrac{1}{2} & ; & & 1 & ; & & \tfrac{3}{2} & & \\
 & m = & 0; & \tfrac{1}{2}, & -\tfrac{1}{2}; & 1, & 0, & -1; & \tfrac{3}{2}, & \tfrac{1}{2}, & -\tfrac{1}{2}, & -\tfrac{3}{2} \\
\hline
j' = 0,\ m' = & 0 & 0 & & & & & & & & & \\
\hline
j' = \tfrac{1}{2},\ m' = & \tfrac{1}{2} & & 0 & 1 & & & & & & & \\
 & -\tfrac{1}{2} & & 0 & 0 & & & & & & & \\
\hline
j' = 1,\ m' = & 1 & & & & 0 & \sqrt{2} & 0 & & & & \\
 & 0 & & & & 0 & 0 & \sqrt{2} & & & & \\
 & -1 & & & & 0 & 0 & 0 & & & & \\
\hline
j' = \tfrac{3}{2},\ m' = & \tfrac{3}{2} & & & & & & & 0 & \sqrt{3} & 0 & 0 \\
 & \tfrac{1}{2} & & & & & & & 0 & 0 & \sqrt{4} & 0 \\
 & -\tfrac{1}{2} & & & & & & & 0 & 0 & 0 & \sqrt{3} \\
 & -\tfrac{3}{2} & & & & & & & 0 & 0 & 0 & 0
\end{array} \qquad (7.16)$$

Matrix representation of $\hat{J}_+$ in $\mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{\frac{3}{2}} \oplus \cdots$ where the subscript indicates the j eigenvalue.

(Ballentine p.164)