M564 Non-Relativistic Quantum Mechanics Ninnat Dangniam

Homework Assignment 1

DUE: Monday 19 Sep 2022 (Tentative) 50+15 points

Potentially useful identities:

Simplified BCH identity: $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}$ if both $\hat{A}$ and $\hat{B}$ commute with $[\hat{A},\hat{B}]$

Braiding identity: $e^{\hat{B}}Ae^{-\hat{B}} = \hat{A} + [\hat{B},\hat{A}] + \frac{1}{2!}[\hat{B},[\hat{B},\hat{A}]] + \frac{1}{3!}[\hat{B},[\hat{B},[\hat{B},\hat{A}]]] + \cdots$

$$\cosh r = \sum_{\text{even}\, k\geq 0} \frac{r^k}{k!} = \frac{e^r + e^{-r}}{2}, \qquad \sinh r = \sum_{\text{odd}\, k\geq 1} \frac{r^k}{k!} = \frac{e^r - e^{-r}}{2}$$

1. Anisotropic oscillator (10 points).

Consider a three-dimensional harmonic potential,

$$V(x,y,z) = \frac{m\omega^2}{2}\left[\left(1+\frac{2\lambda}{3}\right)(x^2+y^2) + \left(1-\frac{4\lambda}{3}\right)z^2\right], \tag{1}$$

where $0 \leq \lambda \leq 3/4$.

(a) What are the eigenstates of the Hamiltonian and the corresponding energy eigenvalues?

(b) Compute and discuss the energy, the parity, and the degree of degeneracy of the ground state and the first two excited states as λ varies.

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2. More properties of coherent states (10 points).

The coherent state associated to a complex number α is given by

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|n\rangle = D(\alpha)|0\rangle, \tag{2}$$

where $|n\rangle$ is the nth energy eigenstate of the harmonic oscillator, and $D(a) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the displacement operator.

(a) Compute and discuss the overlap squared $|\langle\alpha|\beta\rangle|^2$ between two coherent states $|\alpha\rangle$ and $|\beta\rangle$. Can a pair of coherent states be orthogonal?

(b) As a preparation for **(c)**, compute the integral

$$\int_0^\infty dr\, e^{-r^2} r^{2n+1}. \tag{3}$$

You may use the Gaussian integral $\int_0^\infty e^{-\alpha r^2} = \sqrt{\pi/\alpha}$. **Hint:** Consider differentiating $\int_{-\infty}^\infty e^{-\alpha r^2}$ with respect to α.

(c) Show that the coherent-state projectors, properly normalized, form a resolution of the identity:

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{1}, \tag{4}$$

where $\int d^2\alpha$ is an integral over the entire complex plane.

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3. Squeezed states (15 points).

We have seen from the lectures that, for a state $|\psi\rangle$ to be a minimum-uncertainty state, $|\psi\rangle$ must satisfy the equation

$$\left(\widehat{\Delta X} + \lambda\widehat{\Delta P}\right)|\psi\rangle = 0, \tag{5}$$

for a purely imaginary λ, where

$$\hat{X} := \sqrt{\frac{m\omega}{\hbar}}\hat{x}, \qquad\qquad \hat{P} := \frac{\hat{p}}{\sqrt{\hbar m\omega}}, \tag{6}$$

are the dimensionless position and momentum operators respectively. In the lectures, we have noted that $|\lambda| = \Delta X/\Delta P$ signifies the trade-off in the variances of X and P. By choosing $\lambda = i$, (5) becomes precisely the condition that $|\psi\rangle$ is an eigenstate of the annihilation operator a. This exercise is about what happens if we choose other values of λ.

For simplicity, let us omit the zero-point energy and write

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}, \qquad\qquad \hat{U}(t,0) = \exp\left(-\frac{i\hat{H}t}{\hbar}\right) = \exp(-i\omega\hat{a}^\dagger\hat{a}t). \tag{7}$$

Consider λ to be a decaying exponential function.

$$(\hat{a}\cosh r + \hat{a}^\dagger\sinh r)|\psi_r\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(e^r\hat{x} + ie^{-r}\frac{\hat{p}}{m\omega}\right)|\psi_r\rangle = 0, \tag{8}$$

where r is a real number called the *squeeze parameter*.

(a) Evaluate the expectation values $\langle\hat{x}\rangle, \langle\hat{p}\rangle, \langle\hat{x}^2\rangle, \langle\hat{p}^2\rangle$, and $\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle$ with respect to the state $|\psi_r\rangle$.

(b) Squeezed states can be obtained by applying the unitary *squeeze operator*

$$\hat{S}(r,\theta) := \exp\left[\frac{r}{2}\left(\hat{a}^2 e^{-2i\theta} - (\hat{a}^\dagger)^2 e^{2i\theta}\right)\right] \tag{9}$$

to the oscillator's ground states $|0\rangle$:

$$|\varphi_{r,\theta}\rangle := \hat{S}(r,\theta)|0\rangle. \tag{10}$$

Show that

$$\hat{S}(r,\theta)\,\hat{a}\,\hat{S}^\dagger(r,\theta) = \hat{a}\cosh r + \hat{a}^\dagger e^{2i\theta}\sinh r. \tag{11}$$

(c) Show that the state $|\psi_r\rangle$ can be taken to be a squeezed state $|\varphi_{r,0}\rangle$.

(d) Suppose that the initial state of the oscillator is $|\psi(0)\rangle = |\varphi_{r,0}\rangle$. Show that the state

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle, \tag{12}$$

at an arbitrary time t remains a squeezed state $|\varphi_{r(t),\theta(t)}\rangle$ and determine the degree of squeezing $r(t)$ and the squeezing angle $\theta(t)$ as functions of t. Show that at time $t = \pi/(2\omega)$, the sign of the squeeze parameter reverses: $r \mapsto -r$.

(e) Find the wave function $\psi_r(x) = \langle x|\psi_r\rangle$ up to an irrelevant global phase.

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4. Oscillator driven by a time-dependent force. (15+15 points)

Consider a harmonic oscillator acted on by a generalized force $f(t)$, leading to the Hamiltonian with an explicit time dependence

$$\hat{H}(t) = \hat{H}_0 + i\hbar\left[\hat{a}^\dagger f(t) - \hat{a} f^*(t)\right]. \tag{13}$$

Solving the operator Schrödinger equation

$$i\hbar\frac{d\hat{U}(t,0)}{dt} = \hat{H}(t)\hat{U}(t,0), \tag{14}$$

yields the solution

$$\hat{U}(t,0) = e^{i\delta(t)}\hat{U}_0(t,0)\hat{D}(\alpha(t)), \tag{15}$$

where $\alpha(t)$ and $\delta(t)$ are functions to be determined, $\hat{D}(\alpha(t)) = \exp(\alpha(t)\hat{a}^\dagger - \alpha^*(t)\hat{a})$ is the displacement operator, and $\hat{U}_0(t,0) = \exp(-i\omega\hat{a}^\dagger\hat{a}t)$ is the free time-evolution operator.

(a) Show that $\hat{D}(\alpha(t))$ satisfies the following differential equation:

$$\frac{d\hat{D}(\alpha(t))}{dt} = \left[-\frac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + (\dot{\alpha}\hat{a}^\dagger - \dot{\alpha}^*\hat{a})\right]\hat{D}(\alpha(t)). \tag{16}$$

Hint: Use the Baker-Campbell-Hausdorff identity to write the displacement operator in the normal order first.

(b) Use the result of **(a)** to show that the Schrödinger equation implies two ODEs for $\alpha(t)$ and $\delta(t)$, the solutions of which are

$$\alpha(t) = \int_0^t dt' f(t')e^{i\omega t'}, \qquad \delta(t) = \frac{i}{2}\int_0^t dt'(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*). \tag{17}$$

(c) Suppose that the oscillator is initially in the ground state $|\psi(0)\rangle = |0\rangle$. Find the expectation values $\langle\hat{a}\rangle, \langle\hat{a}^\dagger\rangle, \langle\hat{a}^\dagger\hat{a}\rangle$, and $\langle\hat{a}^2\rangle$ in terms of $\alpha(t)$ and $\delta(t)$. Use these expectation values to calculate the expectation values and variances of $\hat{x}$ and $\hat{p}$.

(d) Derive the Heisenberg equations of motion for $\hat{a}, \hat{a}^\dagger, \hat{x}$ and $\hat{p}$.

4(e) and 4(f) are optional bonus problems.

(e) Solve the resulting Heisenberg equations of motion for $\hat{a}$ and $\hat{a}^\dagger$ with appropriate initial conditions at $t = 0$. You can employ any technique at your disposal (for example, Green's functions).

(f) Repeat **(c)** using the form of the Heiserberg operators obtained in **(e)**. That is, show that the expectation values calculated in the Schrödinger picture and the Heisenberg picture are the same.
