Addition of Angular Momenta

Motivating example: Spin-orbit coupling (relativistic effect)

$$H_{SO} = \xi(r)\, \hat{\vec{L}} \cdot \hat{\vec{S}}$$

Neither $\hat{L}_j$ nor $\hat{S}_j = \frac{\hbar}{2} \hat{\sigma}_j$ commute with H_{SO}, but their sum $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ does.

$$[\hat{L}_j, \hat{H}_{SO}] = \xi(r)\,[\hat{L}_j, \hat{L}_k \hat{S}_k]$$

$$= \xi(r)\,[\hat{L}_j, \hat{L}_k]\, \hat{S}_k$$

$$= \xi(r)\, i\hbar\, \epsilon_{jkl}\, \hat{L}_l \hat{S}_k$$

$$[\hat{S}_j, \hat{H}_{SO}] = \xi(r)\,[\hat{S}_j, \hat{L}_k \hat{S}_k]$$

$$= \xi(r)\, \hat{L}_k\, [\hat{S}_j, \hat{S}_k]$$

$$= \xi(r)\, i\hbar\, \epsilon_{jkl}\, \hat{L}_k \hat{S}_l$$

$$\therefore\ [\hat{L}_j + \hat{S}_j, \hat{H}_{SO}] = 0$$

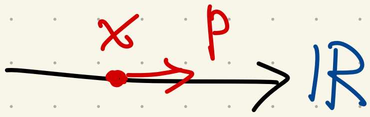
Thus, the $\hat{J}$'s are constants of motion; instead of constructing a basis from eigenvectors of $\hat{L}^2$, $\hat{L}_z$ and $\hat{S}^2$, $\hat{S}_z$, $\hat{H}$ will have a much simpler block-diagonal structure in a basis constructed from eigenvectors of $\hat{\vec{J}}^2, \hat{J}_z$. The general problem of constructing eigenbasis of $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$ from eigenbasis of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ is called the addition of angular momenta.

But before we talk about we need to talk about the idea of tensor product.

Tensor Product

One degree of freedom

A particle moving in 1D



The Hilbert space of wave function is $L^2(\mathbb{R})$

ONB $\{|u_n\rangle\}_{n=1}^{\infty}$

Completeness

$$\sum_{n=1}^{\infty} |u_n\rangle\langle u_n| = 1$$

$\Rightarrow$ Basis of orthonormal functions

$$\langle x|x'\rangle = \sum_{n=1}^{\infty} \langle x|u_n\rangle\langle u_n|x'\rangle = \sum_{n=1}^{\infty} u_n(x)\, u_n(x')^*$$

$$\langle x|x'\rangle = \delta(x-x')$$

$$\boxed{[\hat{x}, \hat{p}] = i\hbar}$$

Multiple degrees of freedom (say 3)

While the coordinates are the Cartesian product

$\mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ ← Elements are tuples (x,y,z)

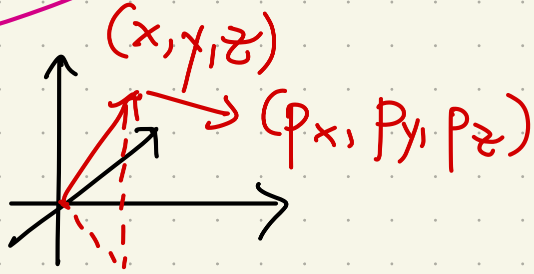
The Hilbert space of wave functions are the tensor product

Allow superposition

$$L^2(\mathbb{R}^3) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$$

ONBs

$\{|u_n\rangle\}_n, \; \{|v_m\rangle\}_m, \; \{|w_k\rangle\}_k$



imply that the following forms an ONB for the tensor product space

$$|u_n\rangle \otimes |v_m\rangle \otimes |w_k\rangle =: |u_n v_m w_k\rangle$$

⇒ Basis of orthonormal functions

$$u_n(x)\, v_m(y)\, w_k(z)$$

Separable functions

$$\boxed{\begin{aligned} [\hat{x}_j, \hat{x}_k] &= 0 \\ [\hat{p}_j, \hat{p}_k] &= 0 \\ [\hat{x}_j, \hat{p}_k] &= i\hbar\,\delta_{jk} \end{aligned}}$$

Uncoupled Hamiltonians

Example $\hat{H} = \hbar\omega(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y + \hat{a}_z^\dagger \hat{a}_z + \frac{3}{2})$
have as its eigenstates separable states

$$|n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle =: |n_x n_y n_z\rangle$$

Eigenstates of Hamiltonians that couple motion in x, y, and z direction would not give uncoupled stationary states (Although separable functions $u_n(x) v_m(y) w_k(z)$ still form a perfectly fine basis; it's just that the Hamiltonian would not be diagonal in this basis)

Formal definition (finite dim for simplicity)

Suppose that $\{|u_j\rangle\}_{j=1}^{d_A}$ is an ONB for $\mathcal{H}_A$

and $\{|v_k\rangle\}_{k=1}^{d_B}$ is an ONB for $\mathcal{H}_B$

then the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ is the linear span of formal products of the form

$$|u_j\rangle \otimes |v_k\rangle, \quad \begin{array}{l} j=1,2,\dots,d_A \\ k=1,2,\dots,d_B \end{array}$$

Rules for manipulation

1 Scalar multiplication

$$\lambda(|u\rangle \otimes |v\rangle) = (\lambda|u\rangle)\otimes|v\rangle = |u\rangle \otimes (\lambda|v\rangle)$$

2 Addition of products with a common factor

$$|u\rangle\otimes|v\rangle + |u'\rangle\otimes|v\rangle = (|u\rangle + |u'\rangle)\otimes|v\rangle$$

3 Otherwise, vectors of the form

$$|u\rangle\otimes|v\rangle + |u'\rangle\otimes|v'\rangle \quad \text{(Entangled states)}$$

are genuinely new objects in $\mathcal{H}_A\otimes\mathcal{H}_B$ that are not products of elements from $\mathcal{H}_A$ and $\mathcal{H}_B$

Tensor product of linear operators

$$(A \otimes B)|u\rangle \otimes |v\rangle = (A|u\rangle) \otimes (B|v\rangle)$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

Tensor product is used to combine independent degrees of freedom.

State view

$$Pr(x_1, x_2) = |\psi(x_1, x_2)|^2 = |\psi_1(x_1)|^2 |\psi_2(x_2)|^2$$

$$= Pr(x_1)\, Pr(x_2)$$

(Uncorrelated random variables)

Operator view

Operators that act on different Hilbert spaces always commute! "Local operators"

$$[A \otimes \mathbb{1}, \mathbb{1} \otimes B] = (A \otimes \mathbb{1})(\mathbb{1} \otimes B) - (\mathbb{1} \otimes B)(A \otimes \mathbb{1})$$

$$= A \otimes B - A \otimes B = 0$$

Tensor product of two (possibly non-square) matrices are represented in matrix form by their Kronecker product

$$\underbrace{A}_{\substack{m\times n \\ \text{matrix}}} \otimes B = \begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{pmatrix}$$

Examples

Outer product of $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|+\rangle\langle -| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$ (Standard matrix multiplication)

$$\|$$

$$|+\rangle \otimes \langle -| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix} 1 & 0 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\|$$

$$\langle -| \otimes |+\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} & 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\vec{J}} = \hat{\vec{J}}_1 \otimes \hat{\mathbb{1}}_2 + \hat{\mathbb{1}}_1 \otimes \hat{\vec{J}}_2$$

(Acts on $\mathcal{H}_1$) (Acts on $\mathcal{H}_2$)

Recall the reason we're considering the sum $\hat{\vec{J}}$ instead of the individual $\hat{\vec{J}}_1, \hat{\vec{J}}_2$

$$[\hat{\vec{J}}_1 + \hat{\vec{J}}_2, \hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2] = 0 \text{ while } \begin{cases} [\hat{\vec{J}}_1, \hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2] \neq 0 \\ [\hat{\vec{J}}_2, \hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2] \neq 0 \end{cases}$$

What forms an CSCO in $\mathcal{H}_1 \otimes \mathcal{H}_2$?

$$[J_j, J_k] = [J_j^{(1)} + J_j^{(2)}, J_k^{(1)} + J_k^{(2)}]$$

$$= [J_j^{(1)}, J_k^{(2)}] + [J_j^{(2)}, J_k^{(2)}]$$

$$= i\hbar \epsilon_{jkl} (J_l^{(1)} + J_l^{(2)})$$

$$= \boxed{i\hbar \epsilon_{jkl} J_l}$$

So $\hat{\vec{J}}$ is still an angular momentum operator.

$\hat{\vec{J}}^2$ still commute with $\hat{J}_z$, $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, but not $\hat{J}_{1z}$, $\hat{J}_{2z}$.

$$\hat{\vec{J}}^2 = \left(\hat{\vec{J}}_1 + \hat{\vec{J}}_2\right)^2 = \hat{\vec{J}}_1^2 + \hat{\vec{J}}_2^2 + 2\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2$$

[Transform into the form in which we know the action on $|j_1, j_2, m_1, m_2\rangle$

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$$\Rightarrow \hat{\vec{J}}^2 = \hat{\vec{J}}_1^2 + \hat{\vec{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$$]

$$[J^2, J_j] = [J_1^2 + J_2^2 + 2J_{1k}J_{2k}, J_{1j} + J_{2j}]$$

$$= 2\left([J_{1k}J_{2k}, J_{1j}] + [J_{1k}J_{2k}, J_{2j}]\right)$$

$$= 2i\hbar\, \epsilon_{kjl}\left(J_{1l}J_{2k} + J_{1k}J_{2l}\right)$$

$$= \boxed{0}$$

swap and get a minus sign

$$[J^2, J_1^2] = [J_1^2 + J_2^2 + 2J_{1j}J_{2j}, J_{1k}J_{1k}]$$

$$= 2[J_{1j}, J_{1k}J_{1k}]J_{2j}$$

$$= 2\left(J_{1k}[J_{1j}, J_{1k}] + [J_{1j}, J_{1k}]J_{1k}\right)J_{2j}$$

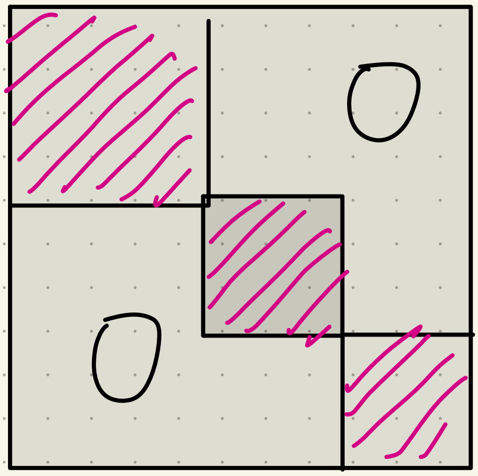
$$= 2\left(J_{1k}[J_{1j}, J_{1k}] + [J_{1j}, J_{1k}]J_{1k}\right) J_{2j}$$

$$= 2i\hbar\, \epsilon_{jkl}\left(J_{1k}J_{1l} + J_{1l}J_{1k}\right) J_{2j} = \boxed{0}$$

$\therefore \{\hat{\vec{J}}^2, \hat{\vec{J}}_1^2, \hat{\vec{J}}_2^2, \vec{J}_z\}$ is a commuting set.

We shall find out later that it is sufficient to pick $\{\hat{\vec{J}}^2, \hat{J}_z\}$ as an CSCO if $\{\hat{\vec{J}}_1^2, \hat{J}_{1z}, \hat{\vec{J}}_2^2, J_{2z}\}$ is an CSCO.

* We know that $\mathcal{H}_1 \otimes \mathcal{H}_2$ breaks up into subspaces $\mathcal{H}_{J,\alpha}$ of $2J+1$ dimension (α labels the degeneracy of J^2 eigenvalues)



* $\hat{\vec{J}}^2, \hat{J}_z, \hat{J}_\pm$ and functions of them ($e^{i\theta J_z}$) only have nonzero matrix elements inside each of the subspaces $\mathcal{H}_{J,\alpha}$
* Matrix elements of these functions are independent of α

We want to go from the basis $\{|\underset{\hat{\vec{J}}_1^2}{j_1}, \underset{\hat{J}_{1z}}{m_1}, \underset{\hat{\vec{J}}_2^2}{j_2}, \underset{\hat{J}_{2z}}{m_2}\rangle\}$

to $\{|\underset{\hat{\vec{J}}_1^2}{j_1}, \underset{\hat{\vec{J}}_2^2}{j_2}, \underset{\hat{\vec{J}}}{J}, \underset{\hat{J}_z}{m}\rangle\}$

Example $\frac{1}{2} \otimes \frac{1}{2}$

Recall
$$\hat{J}^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle$$
$$\hat{J}_z |j,m\rangle = m\hbar |j,m\rangle$$
$$\hat{J}_\pm |j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle$$

General

Spin-$\frac{1}{2}$ $\Rightarrow$ $\hat{S}_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\hat{S}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$\hat{\vec{S}}^2 = \hat{\vec{S}}_1^2 + \hat{\vec{S}}_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

$$\hat{\vec{S}}^2 |+,+\rangle = \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + \cancel{2}\cdot\frac{\hbar}{\cancel{2}}\frac{\hbar}{2}\right)|+,+\rangle$$
$$= 2\hbar^2 |+,+\rangle$$

$$\hat{\vec{S}}^2 |+,-\rangle = \left[\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + \cancel{2}\cdot\frac{\hbar}{\cancel{2}}\left(-\frac{\hbar}{2}\right)\right]|+,-\rangle$$
$$+ S_{1-}S_{2+}|+-\rangle \leftarrow \hbar^2|-,+\rangle$$
$$= \hbar^2\left(|+,-\rangle + |-,+\rangle\right)$$

$$\hat{\vec{S}}^2 |-,+\rangle = \hbar^2\left(|-,+\rangle + |+,-\rangle\right)$$
$$\hat{\vec{S}}^2 |-,-\rangle = 2\hbar^2 |-,-\rangle$$

$$\therefore \hat{\vec{S}}^2 = \hbar^2 \begin{pmatrix} 2 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 2 \end{pmatrix} \Big\} \text{ spanned by } |+,-\rangle, |-,+\rangle$$

In this subspace $\hat{\vec{S}}^2 = \hbar^2(\hat{\mathbb{1}} + \hat{\sigma}_x)$

Eigenvectors	Eigenvalues
$\frac{\|+,-\rangle + \|-,+\rangle}{\sqrt{2}}$	$2\hbar^2$
$\frac{\|+,-\rangle - \|-,+\rangle}{\sqrt{2}}$	0

	S	m_s
Triplet states	$\|+,+\rangle$	1
	$\frac{\|+,-\rangle + \|-,+\rangle}{\sqrt{2}}$	0
	$\|-,-\rangle$	-1
Singlet state	$\frac{\|+,-\rangle - \|-,+\rangle}{\sqrt{2}}$	0

Spin-1 normalization constant

$$\frac{\hat{J}_-}{\sqrt{2}\hbar}|++\rangle = \frac{(\hat{J}_{1-} + \hat{J}_{2-})}{\sqrt{2}\hbar}|++\rangle$$

General theory — $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$

Eigenvalues of $\hat{\vec{J}}$: $J(J+1)\hbar^2 \rightarrow$ Subspace $\mathcal{H}_{J,\alpha}$

—— " —— $\hat{J}_z$: $M\hbar$

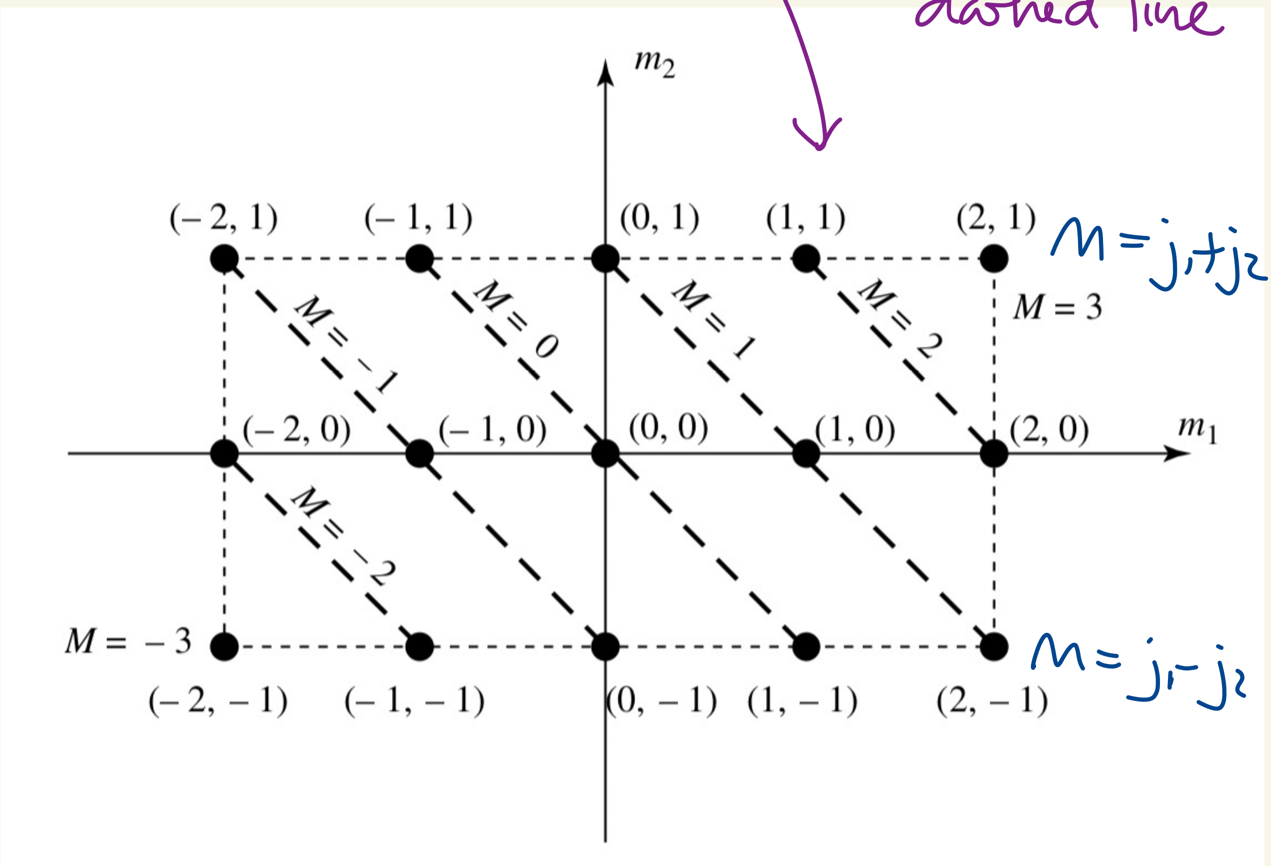
In each subspace $\mathcal{H}_{J,\alpha}$, each value of M, $|M| \leq J$, appears only once.

From this point onward, assume $j_1 \geq j_2$

Eigenvalues of $\hat{J}_z$ are all possible sums of j_1 and j_2.

Degeneracy $g_{j_1,j_2}(M)$ of M is the number of distinct m_1, m_2 such that $M = m_1 + m_2$.

Length of each dashed line



Example when $j_1 = 2$, $j_2 = 1$

* If $|M| > j_1+j_2 \Rightarrow$ $\boxed{g(M)=0}$

* If $|j_1-j_2| \leq |M| \leq j_1+j_2 \Rightarrow$ The degeneracy of M increases by one for every step down from $M=j_1+j_2$

$$\therefore \boxed{g(M) = j_1+j_2-|M|+1}$$

Symmetry
$g(-M)=g(M)$

Note that this is correct for negative M values as well.

* If $0 \leq |M| \leq |j_1-j_2| \Rightarrow$ Constant $g(M)$ equal to $g(|j_1-j_2|)$ in the previous case

$$\begin{aligned} &\parallel \\ &j_1+j_2-|j_1-j_2|+1 \\ &\parallel \\ &\cancel{j_1}+j_2-\cancel{j_1}+j_2+1 = 2j_2+1 \end{aligned}$$

Positive since we assume $j_1 \geq j_2$

Generally, $\boxed{g(M) = 2\min(j_1,j_2)+1}$

But what we want are the multiplicities $p(J)$ of the irreps $\Rightarrow$ Obtained from inverting the relation

$$g(M) = \sum_{J\geq|M|} p(J) = p(J=|M|)+p(J=|M|+1)$$
$$+\,p(J=|M|+2)+\ldots$$

("In how many irreps this value of M can appear")

$$\Rightarrow p(J) = g(M=J) - g(M=J+1) \quad \left(\begin{matrix}\text{J can't}\\ \text{be negative}\end{matrix}\right)$$

$$= g(M=-J) - g(M=-J+1)$$

$$\therefore \ p(J) = \begin{cases} 1 & \text{if } |j_1 - j_2| \leq J \leq j_1 + j_2 \\ 0 & \text{otherwise} \end{cases}$$

In other words,

$$\boxed{\mathcal{H}_1 \otimes \mathcal{H}_2 \cong \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_J} \quad \left(\begin{matrix}\text{No}\\ \text{multiplicity}\\ \alpha\end{matrix}\right)$$

Clebsh-Gordan series for SU(2)

For fixed j_1 and j_2,

$$|J,M\rangle = \sum_{m_1,m_2} \underbrace{\langle j_1, j_2, m_1, m_2 | J, M\rangle}_{\text{C-G coefficients}} |j_1 j_2 m_1 m_2\rangle$$

(Conventionally chosen to be real) → C-T Complement B_X

The C-G coefficient vanishes unless

1. $m_1 + m_2 = M$
2. $|j_1 - j_2| \leq J \leq j_1 + j_2$
3. $j_1 + j_2 + J = \text{integer}$

$$(-1)^{2J} \{\ \} = (-1)^{2(j_1+j_2)} \{\ \}$$

$$\Downarrow$$

$2(J + j_1 + j_2)$ is even