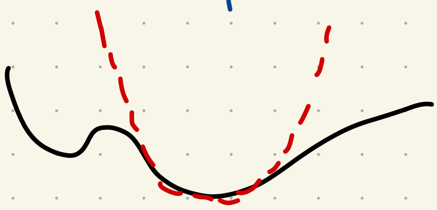
The harmonic oscillator

Importance of simple harmonic oscillator (SHO) in physics:

- Approximation of a bounded potential around an equilibrium point



$x=0$

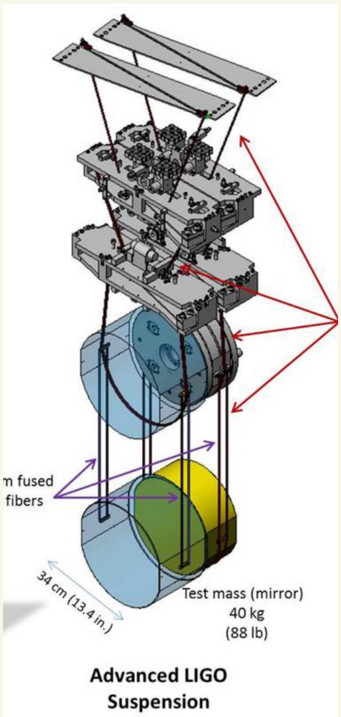
Set zero of energy   0 at minima

$$V(x) = \cancel{V(0)} + x \cancel{\frac{dV}{dx}}\Big|_{x=0} + \boxed{\frac{x^2}{2!}\frac{d^2V}{dx^2}\Big|_{x=0}} + O(x^3)$$

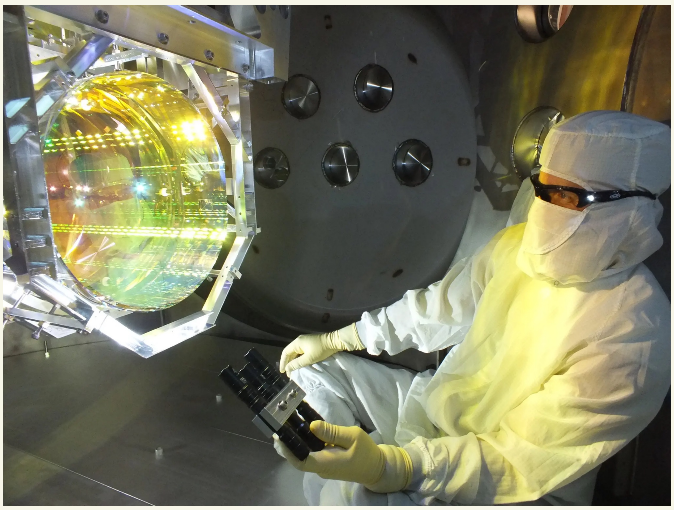
$$F = -kx = -\frac{dV}{dx} \Rightarrow V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

Angular frequency $\omega = \sqrt{\frac{k}{m}}$

- Mechanical vibration: phonons



The center-of-mass mode of a 10 kg mirror in LIGO is treated as a harmonic oscillator



(Effective temp. $\gg$ nK)

Whittle et al., Science (2021)

- Field vibration: photons and other elementary particles

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$\hbar$ has the unit of action $= [\text{energy}][\text{time}]$ ← Conjugate pairs

$S = \int dt\, L$

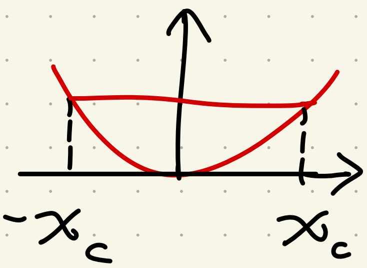
$$= [\text{length}][\text{momentum}]$$

$$= \frac{[\text{length}]^2[\text{mass}]}{[\text{time}]}$$

Characteristic length

$$x_c = \sqrt{\frac{\hbar}{m\omega}}$$

Classical meaning:
If the particle has energy $E = \frac{\hbar\omega}{2}$
(Ground state energy of SHO)



Classical turning point
(Max. displacement)

$$x_c = \sqrt{\frac{2E}{m\omega^2}} \Leftrightarrow E = \frac{1}{2} m\omega^2 x_c$$

Characteristic momentum

$$p_c = \sqrt{\hbar m\omega} \longleftarrow p_c = \sqrt{2mE} \Leftrightarrow E = \frac{p_c^2}{2m}$$

(Max. momentum)

Dimensionless $\boxed{X = \frac{x}{x_c} = \sqrt{\frac{m\omega}{\hbar}}\, x}$, $\boxed{P = \frac{p}{p_c} = \frac{p}{\sqrt{\hbar m\omega}}}$

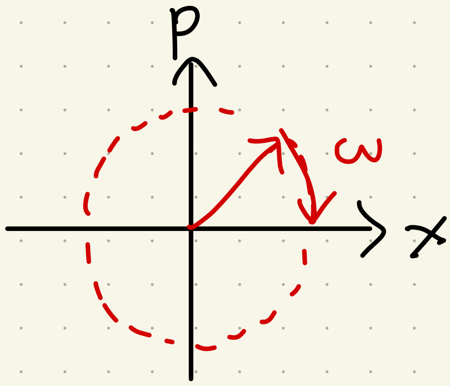
$$H = \frac{\hbar\omega}{2}(X^2 + P^2)$$

(SHO is a <u>linear</u> system in the sense that the time evolution transforms x and p linearly)

Recall classical eq. of motion

$$\left.\begin{aligned} \dot{X} &= \omega P \\ \dot{P} &= -\omega X \end{aligned}\right\} \Rightarrow \ddot{X} = -\omega^2 X$$

$$\Rightarrow \begin{cases} X(t) = X(0)\cos(\omega t) + P(0)\sin(\omega t) \\ P(t) = P(0)\cos(\omega t) - X(0)\sin(\omega t) \end{cases}$$



$$\begin{pmatrix} X(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} X(0) \\ P(0) \end{pmatrix}$$

[[Note on how to obtain the classical eq. of motion]]

Classically there is no $\hbar$, so our change of variables shouldn't involve $\hbar$.

$$X = \sqrt{m\omega}\, x, \qquad P = \frac{p}{\sqrt{m\omega}} \quad \text{(Not dimensionless)}$$

$$\Rightarrow H = \frac{\omega}{2}(X^2 + P^2) \Rightarrow \begin{cases} \dot{X} = \dfrac{\partial H}{\partial P} = \omega P \\[2ex] \dot{P} = -\dfrac{\partial H}{\partial X} = -\omega X \end{cases}$$

Define new coördinates (still classical!)

$$\alpha = \frac{X + iP}{\sqrt{2}}, \qquad \alpha^* = \frac{X - iP}{\sqrt{2}}$$

$$\alpha(t) = \frac{1}{\sqrt{2}}\Big[X(0)\cos(\omega t) + P(0)\sin(\omega t)$$

$$+ iP(0)\cos(\omega t) - iX(0)\sin(\omega t)\Big]$$

$$= \frac{X(0) + iP(0)}{\sqrt{2}}\left[\cos(\omega t) - i\sin(\omega t)\right]$$

$$= \alpha(0)\, e^{-i\omega t} \iff \dot{\alpha} = -i\omega\alpha$$

This can be done completely analogously at the quantum level.

$$[X, P] = \frac{1}{x_c p_c}[x, p] = \frac{1}{\hbar} i\hbar = i$$

$$\boxed{\begin{aligned} a &= \frac{X + iP}{\sqrt{2}}, & a^\dagger &= \frac{X - iP}{\sqrt{2}} \\ X &= \frac{a + a^\dagger}{\sqrt{2}}, & P &= \frac{a - a^\dagger}{\sqrt{2}\, i} \end{aligned}}$$

$$[a, a^\dagger] = \frac{1}{2}[X + iP, X - iP]$$

$$= -\frac{i}{2}[X, P] + \frac{i}{2}[P, X] = -i[X, P] = 1$$

The commutation relation can be used to switch the order of operators $a^\dagger a = 1 + a a^\dagger$

$$H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a) = \boxed{\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)}$$

Define the number operator $N = a^\dagger a$

$$[a, N] = [a, a^\dagger a] = [a, a^\dagger]a = a$$

$$[a^\dagger, N] = [a^\dagger, a^\dagger a] = a^\dagger[a^\dagger, a] = -a^\dagger$$

Heisenberg eq. of motion

$$\dot{A} = \frac{1}{i\hbar}[A, H] + \frac{\partial A}{\partial t}$$

$$\dot{X} = \frac{\omega}{2i}[X, X^2 + P^2] = \frac{\omega}{2i}[X, P^2]$$

$$= \frac{\omega}{2i}\big([X, P]X + X[X, P]\big) = \omega X$$

$$\dot{P} = \frac{\omega}{2i}[P, X^2 + P^2] = \frac{\omega}{2i}[P, X^2]$$

$$= \frac{\omega}{2i}\big([P, X]P + P[P, X]\big) = -\omega P$$

$$\Rightarrow \ddot{X} = -\omega X \Rightarrow \begin{cases} X(t) = X(0)\cos(\omega t) + P(0)\sin(\omega t) \\ P(t) = P(0)\cos(\omega t) - X(0)\sin(\omega t) \end{cases}$$

$$\Rightarrow \boxed{a(t) = e^{-i\omega t}a(0)}$$

Or solve directly

$$\dot{a} = \frac{1}{i\hbar}[a, H] = \frac{\omega}{i}\left[a, N + \frac{1}{2}\right] = \frac{\omega}{i}[a, N] = -i\omega a$$

The spectrum

There're mainly two ways to solve for the spectrum

1. 2nd order DE in the position representation
2. Algebraic using ladder operators

We will do the latter. (But we'll still have to solve a first order DE to remove the degeneracy for the ground state.)

Let $|\nu\rangle$ be a normalized eigenstate of N with eigenvalue ν

Lemma I (Lemma I, C-T p. 491) $\nu \geq 0$

■ $\nu = \langle\nu|a^\dagger a|\nu\rangle = \| a|\nu\rangle \|^2 \geq 0$ □

Lemma II (Lemma III, C-T p. 492)

1. $a|\nu\rangle$ is an eigenstate of H with eigenvalue $\nu - 1$
2. $a^\dagger|\nu\rangle$ ——— " ——— $\nu + 1$

■ $N(a|\nu\rangle) = (aa^\dagger - 1)a|\nu\rangle$

$$= a(a^\dagger a)|\nu\rangle - a|\nu\rangle = (\nu-1)(a|\nu\rangle)$$

$N(a^\dagger|\nu\rangle) = a^\dagger a a^\dagger |\nu\rangle = a^\dagger(a^\dagger a + 1)|\nu\rangle$

$$= (\nu+1)(a^\dagger|\nu\rangle)$$ □

The operator a removes one unit of energy, so it's called the annihilation operator (and $a^\dagger$ is the creation operator).

Normalization

$$\| a|\nu\rangle \|^2 = \langle \nu | a^\dagger a | \nu \rangle = \nu \Rightarrow a|\nu\rangle = \sqrt{\nu}\, |\nu - 1\rangle$$

$$\| a^\dagger |\nu\rangle \|^2 = \langle \nu | a a^\dagger | \nu \rangle = \langle \nu | (a^\dagger a + 1) | \nu \rangle = \nu + 1$$

$$\Rightarrow a^\dagger |\nu\rangle = \sqrt{\nu+1}\, |\nu+1\rangle$$

Since the lowering process can't continue indefinitely, there need to be an eigenstate $|\nu\rangle$ such that $a|\nu\rangle = 0$. Hence by lemma 2, ν needs to be a non-negative integer $\nu = n = 0, 1, 2, \ldots$

Summary The number states, eigenstates $|n\rangle$ of $N = a^\dagger a$ corresponding to eigenvalues $n = 0, 1, 2, \ldots$, are stationary states of the SHO with eigenvalues $\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}$

a priori there could be many $n=0$ states. We'll establish that there is no such degeneracy by solving the DE for the unique ground state.

Not the zero vector. 0 is just the n label.

$$0 = \langle x | a | 0 \rangle = \langle x | \left(\frac{X + iP}{\sqrt{2}} \right) | 0 \rangle$$

$$= \langle x | \left(\sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{i}{\sqrt{2\hbar m\omega}}\, p \right) | 0 \rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\, x \langle x | 0 \rangle + \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx} \langle x | 0 \rangle$$

$$\Longleftrightarrow \quad \frac{d}{dx} \langle x | 0 \rangle = -\frac{m\omega}{\hbar} x \langle x | 0 \rangle$$

$$\Longrightarrow \quad \langle x | 0 \rangle = C \exp\left(-\frac{m\omega}{2\hbar} x^2 \right)$$

Normalization

$$1 = |\langle x | 0 \rangle|^2 = C^2 \int_{-\infty}^{\infty} dx \, \exp\left(-\frac{m\omega}{\hbar} x^2 \right)$$

$$= C^2 \sqrt{\frac{\pi\hbar}{m\omega}} \quad \Longleftrightarrow \quad C = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4}$$

(Choose C to be real)

Gaussian integral $\displaystyle \int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

Lemma III The 1D SHO spectrum has no degeneracy.

- Let's write the eigenstates as $|n,k\rangle$ where k labels orthogonal degenerate states ($\langle n,j|n,k\rangle = \delta_{jk}$)

Then
$$a^n|n,k\rangle = \sqrt{n!}\,|0,k\rangle$$

$$\Rightarrow \langle n,j|(a^\dagger)^n a^n|n,k\rangle = (n!)\,\delta_{jk}$$

But there is only one ground state given by the Gaussian wave function that we derived, therefore there can't exist orthogonal degenerate states for any n. $\square$ (2D and 3D SHO can easily have degeneracies)

$$a^k|n\rangle = \sqrt{n(n-1)\cdots(n-k+1)}\,|n-k\rangle$$
$$= \sqrt{\frac{n!}{(n-k)!}}\,|n-k\rangle = n^{\underline{k}}\,|n-k\rangle$$ ← Falling factorial

$$(a^\dagger)^k|n\rangle = \sqrt{(n+1)(n+2)\cdots(n+k)}\,|n+k\rangle$$
$$= \sqrt{\frac{(n+k)!}{n!}}\,|n+k\rangle = (n+1)^{\overline{k}}\,|n+k\rangle$$ ← Rising factorial

In particular, $$|n\rangle = \frac{(a^\dagger)^n|0\rangle}{\sqrt{n!}}.$$

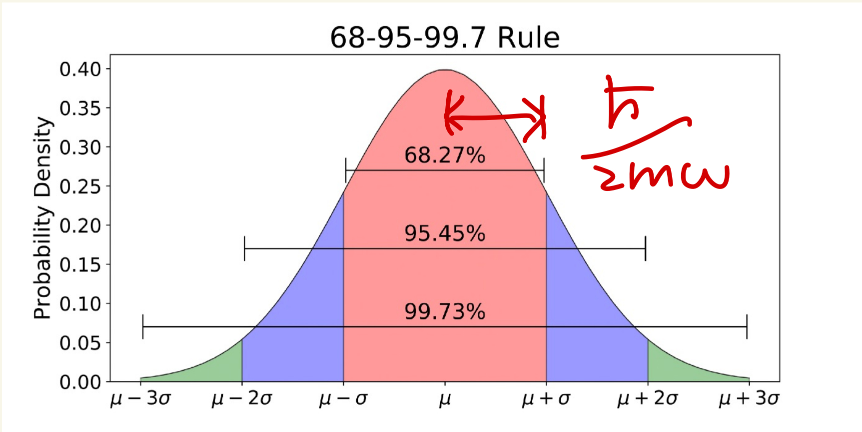
The number state wave functions

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

Prob. $|\langle x|0\rangle|^2 = \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{\hbar}x^2\right)$

Gaussian $\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

$$\Delta x^2 = \sigma^2 = \frac{\hbar}{2m\omega}$$



To find the wave function of other number states

① Use $a^\dagger|n-1\rangle = \sqrt{n}\,|n\rangle \Rightarrow$ Recursion relation

② Use $|n\rangle = (a^\dagger)^n|0\rangle/\sqrt{n!}$

①
$$\langle x|a^\dagger|n-1\rangle = \sqrt{n}\,\langle x|n\rangle$$
$$\|$$
$$\sqrt{\frac{m\omega}{2\hbar}}\langle x|\left(x-\frac{ip}{m\omega}\right)|n-1\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|n-1\rangle$$

$$\Rightarrow \quad \langle x|n\rangle = \sqrt{\frac{m\omega}{2n\hbar}}\left(x-\frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|n-1\rangle$$

② $\langle x|n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{n/2}\left(x-\frac{\hbar}{m\omega}\frac{d}{dx}\right)^n\langle x|0\rangle$

$$= \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\underbrace{\left(\sqrt{\frac{m\omega}{\hbar}}x-\sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)^n\exp\left(\frac{-m\omega}{2\hbar}x^2\right)}_{\left(z-\frac{d}{dz}\right)^n e^{-\frac{1}{2}z^2}}$$

Let $z=\sqrt{\frac{m\omega}{\hbar}}x$

Hermite polynomials $H_n(z)=(-1)^n e^{z^2}\frac{d^n}{dz^n}e^{-z^2}$

C-T Complement B_V

$$\left(z-\frac{d}{dz}\right)e^{\frac{1}{2}z^2}f(z) = \cancel{ze^{\frac{1}{2}z^2}f(z)}-\cancel{ze^{\frac{1}{2}z^2}f(z)} - e^{\frac{1}{2}z^2}\frac{d}{dz}f(z)$$

$$\Rightarrow \left(z-\frac{d}{dz}\right)^n e^{\frac{1}{2}z^2}f(z) = (-1)^n e^{\frac{1}{2}z^2}\frac{d^n}{dz^n}f(z)$$

Therefore, setting $f(z)=e^{-z^2}$,

$$\langle x|n\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}(-1)^n e^{\frac{1}{2}z^2}\overbrace{\frac{d^n}{dz^n}e^{-z^2}}^{(-1)^n e^{-z^2}H_n(z)}$$

$$= \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{1}{2}z^2}H_n(z)$$

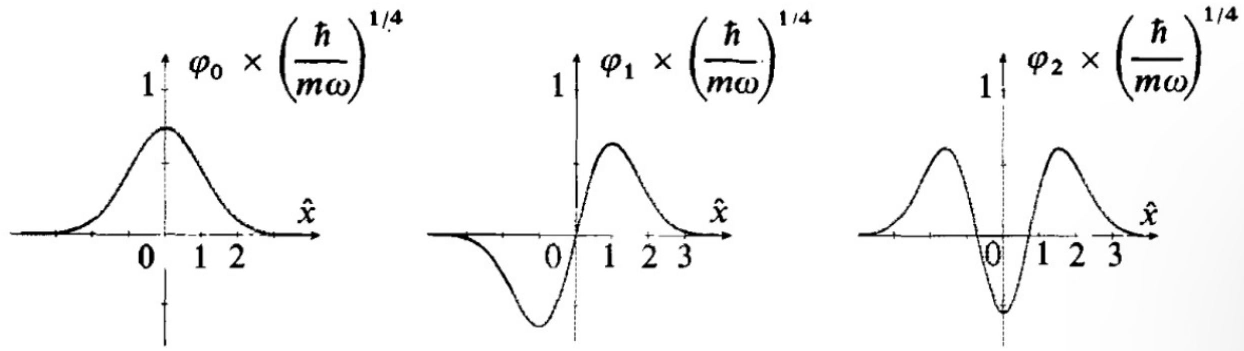


FIGURE 4

Wave functions associated with the first three levels of a harmonic oscillator.

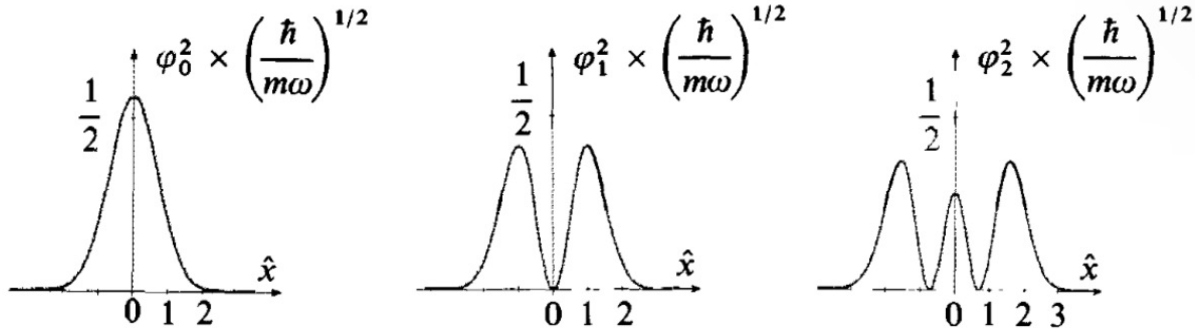


FIGURE 5

Probability densities associated with the first three levels of a harmonic oscillator.

$n\uparrow$ nodes $\uparrow$

Ground state ↓

$$a = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

Statistical moments in stationary states $|n\rangle$

I'll write expectation values as $\langle\psi|A|\psi\rangle = \langle A\rangle_\psi$ with a subscript ψ to indicate the state. So for the stationary states, I'll use both $|n\rangle$ and the more dense notation $|\varphi_n\rangle$ that the book uses interchangeably.

$$\langle a\rangle_n = \sqrt{n}\,\langle n|n-1\rangle = 0$$

$$\langle a^\dagger\rangle_n = \sqrt{n+1}\,\langle n|n+1\rangle = 0$$

$$\langle X\rangle_n = \left\langle \frac{a+a^\dagger}{\sqrt{2}}\right\rangle_n = 0$$

$$\langle P\rangle_n = \left\langle \frac{a-a^\dagger}{\sqrt{2}i}\right\rangle_n = 0$$

Not what we expect classically from an SHO.

$$\langle X^2\rangle_n = \frac{1}{2}\langle (a^\dagger)^2 + aa^\dagger + a^\dagger a + a^2\rangle_n$$

$$= \frac{1}{2}\langle aa^\dagger + a^\dagger a\rangle_n = n + \frac{1}{2}$$

$$\langle P^2\rangle_n = -\frac{1}{2}\langle (a^\dagger)^2 - aa^\dagger - a^\dagger a + a^2\rangle_n$$

$$= +\frac{1}{2}\langle aa^\dagger + a^\dagger a\rangle_n = n + \frac{1}{2}$$

$$\langle(\Delta X)^2\rangle_n = \langle X^2\rangle_n - \cancel{\langle X\rangle_n^2}^{\,0} = n + \frac{1}{2}$$

$$\langle(\Delta P)^2\rangle_n = n + \frac{1}{2}$$

$$\Rightarrow \Delta X \Delta P = n + \frac{1}{2}$$

Minimum for the ground state where it saturates the HUP $\Delta X \Delta P = \frac{1}{2}$.

<u>Coherent states</u>

There are a few characterizations of canonical coherent states.

① Minimum-uncertainty states (Equal uncertainties in the two quadratures)

② Eigenstates of the annihilation operation a.

③ The state obtained by applying "displacement operators" to $|0\rangle$.

(Note that all three equivalent notions of coherent states are independent of the Hamiltonian. But the time evolution under the harmonic potential preserves these properties.)

Recall Variance $(\Delta A)^2 = (A - \langle A \rangle)^2$

$$\langle (\Delta A)^2 \rangle = \langle A^2 - 2\langle A \rangle A + \langle A \rangle^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

$$\langle (\Delta X)^2 \rangle \langle (\Delta P)^2 \rangle = (\langle \psi | \Delta X)(\Delta X | \psi \rangle)\,(\langle \psi | \Delta P)(\Delta P | \psi \rangle)$$

$$\geq |\langle \psi | \underbrace{\Delta X \Delta P}_{\frac{1}{2}\left(\{\Delta X, \Delta P\} + \underbrace{[\Delta X, \Delta P]}_{=[X,P]=i}\right)} | \psi \rangle|^2$$ (Cauchy-Schwarz)

$$= \frac{1}{4} \left| \langle \psi | \{\Delta X, \Delta P\} | \psi \rangle + i \right|^2 = \frac{c^2+1}{4} \geq \frac{1}{4}$$

(Standard uncertainty relation (Schrödinger-Robertson) ignores the anticommutator term $\{\Delta X, \Delta P\}$.)

To saturate the inequality, we have to optimize two parts.

① C-S $\Rightarrow$ $\Delta X|\psi\rangle$ and $\Delta P|\psi\rangle$ are collinear

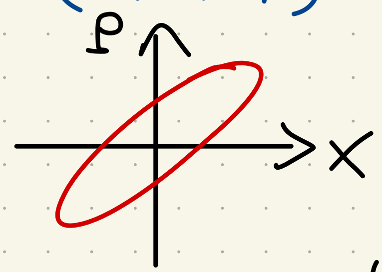
$$(\Delta X + \lambda \Delta P)|\psi\rangle = 0, \quad \lambda \in \mathbb{C}$$

$$\left(\begin{array}{c} \langle (\Delta X)^2 \rangle = |\lambda|^2 \langle (\Delta P)^2 \rangle \\ |\lambda| \text{ is the ratio } \dfrac{\Delta X}{\Delta P} \end{array} \right)$$

② $\langle \{\Delta X, \Delta P\} \rangle = 0$

Plugging ① $\Delta X |\psi\rangle = -\lambda \Delta P |\psi\rangle$ in ②

$$0 = (\langle\psi|\Delta X)\Delta P|\psi\rangle + \langle\psi|\Delta P(\Delta X|\psi\rangle)$$
$$= -2\,\mathrm{Re}\,\lambda \langle\psi|(\Delta P)^2|\psi\rangle$$



$\Longrightarrow$ λ is purely imaginary

Other λ gives squeezed states

Let's choose $\lambda = i$. (equal uncertainty $\Delta X = \Delta P = \frac{1}{\sqrt{2}}$)

$$0 = (\Delta X + i\Delta P)|\psi\rangle$$
$$= \sqrt{2}\, a|\psi\rangle - (\langle X\rangle + i\langle P\rangle)|\psi\rangle$$
$$\Longleftrightarrow \quad a|\psi\rangle = \langle a\rangle |\psi\rangle$$

$\Longleftrightarrow$ $|\psi\rangle$ is an eigenvector of a.

Does $|\alpha\rangle$ exist such that $a|\alpha\rangle = \alpha|\alpha\rangle$?

$$a|\alpha\rangle = a\left(\sum_{n=0}^{\infty} c_n |n\rangle\right) = \sum_{n=1}^{\infty} c_n\sqrt{n}\,|n-1\rangle$$
$$= \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle$$
$$\Longrightarrow \quad \alpha c_{n-1} = c_n\sqrt{n}$$

$$c_n = \frac{\alpha}{\sqrt{n}}\frac{\alpha}{\sqrt{n-1}}\cdots\alpha c_0 = \frac{\alpha^n}{\sqrt{n!}}c_0$$

c_0 is determined from normalization.

$$1 = |c_0|^2 \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!} = |c_0|^2 e^{-|\alpha|^2} \Rightarrow c_0 = e^{-|\alpha|^2/2}$$

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Poisson distribution in the number basis

Time evolution

Heisenberg picture

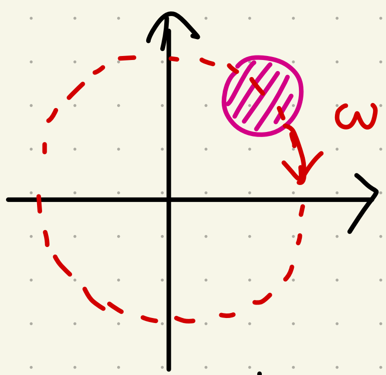
$$a(t) = e^{-i\omega t} a(0)$$

Schrödinger picture

$$U(t,0)|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega(n+\frac{1}{2})t} |n\rangle$$

$$= e^{-i\omega t/2} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-i\omega t/2} |\alpha e^{-i\omega t}\rangle$$



Classical-like motion

Statiscal moments in coherent states $|\alpha\rangle$

The expectation value of monomials of $a, a^\dagger$ in normal order can be computed easily.

$$\langle (a^\dagger)^n a^m \rangle_\alpha = (\alpha^*)^n \alpha^m$$

Normal order = $a^\dagger$ to the left, a to the right

$$:a a^\dagger: = a^\dagger a$$
$$:a a^\dagger: = a a^\dagger - 1$$

Antinormal order $\quad \vdots a^\dagger a \vdots = a a^\dagger$

$$\vdots a^\dagger a \vdots = a^\dagger a + 1$$

Inner product

$$\langle \alpha | \beta \rangle = \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right) \underbrace{\sum_{n,m} \frac{(\alpha^*)^n \beta^m}{\sqrt{n!m!}} \langle n | m \rangle}$$

$$\sum_n \frac{(\alpha^* \beta)^n}{n!} = e^{\alpha^* \beta}$$

$$|\langle \alpha | \beta \rangle|^2 = e^{-|\alpha - \beta|^2}$$ (Not orthogonal !)