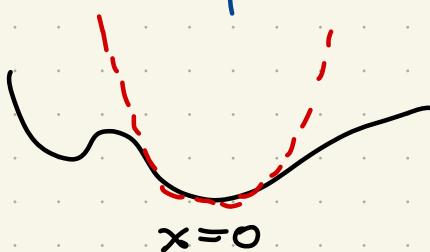


The harmonic oscillator

Importance of simple harmonic oscillator (SHO) in physics:

- Approximation of a bounded potential around an equilibrium point



Set zero of energy

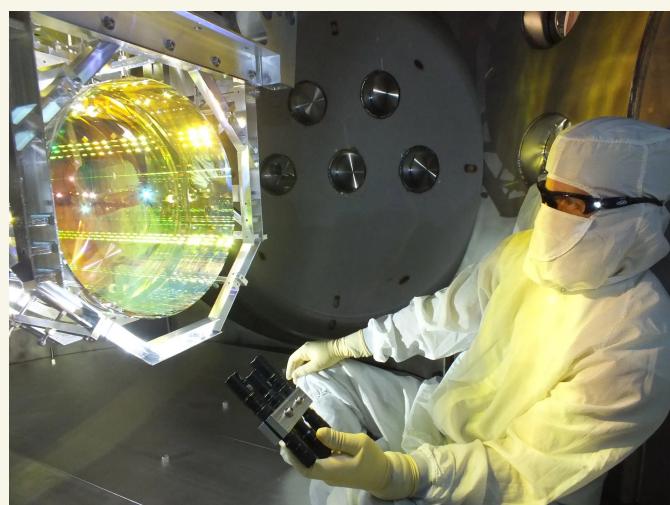
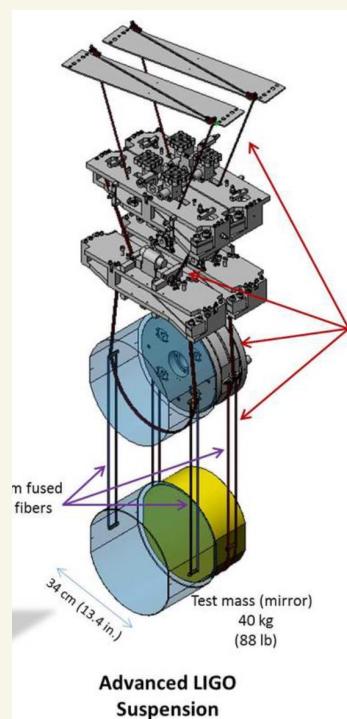
$$V(x) = V(0) + x \frac{dV}{dx} \Big|_{x=0} + \frac{x^2}{2!} \frac{d^2V}{dx^2} \Big|_{x=0} + O(x^3)$$

$$F = -kx = -\frac{dV}{dx} \Rightarrow V(x) = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2$$

$$\text{Angular frequency } \omega = \sqrt{\frac{k}{m}}$$

- Mechanical vibration: phonons

The center-of-mass mode of a 10 kg mirror in LIGO is treated as a harmonic oscillator



(Effective temp.)
 $\gg \hbar k$

Whittle et al.,
Science (2021)

- Field vibration: photons and other elementary particles

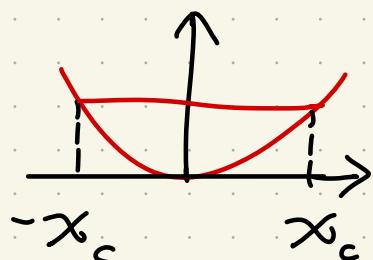
$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

\hbar has the unit of action = [energy][time] ← Conjugate pairs

$$\begin{aligned} S &= \int dt L \\ &= [\text{length}] [\text{momentum}] \\ &= \underline{[\text{length}]^2 [m\omega S]} \\ &\quad / [\text{time}] \end{aligned}$$

Characteristic length

$$x_c = \sqrt{\frac{\hbar}{m\omega}}$$



Classical meaning:

If the particle has energy $E = \frac{\hbar\omega}{2}$
(Ground energy of SHO)

$$x_c = \sqrt{\frac{2E}{m\omega^2}} \Leftrightarrow E = \frac{1}{2} m\omega^2 x_c^2 = \text{P.E.}$$

Classical turning point
(Max. displacement)

Characteristic momentum

$$p_c = \sqrt{\hbar m\omega} \leftarrow p_c = \sqrt{2mE} \Leftrightarrow E = \frac{p_c^2}{2m} = \text{K.E.}$$

(Max. momentum)

Dimensionless

$$X = \frac{x}{x_c} = \sqrt{\frac{m\omega}{\hbar}} x,$$

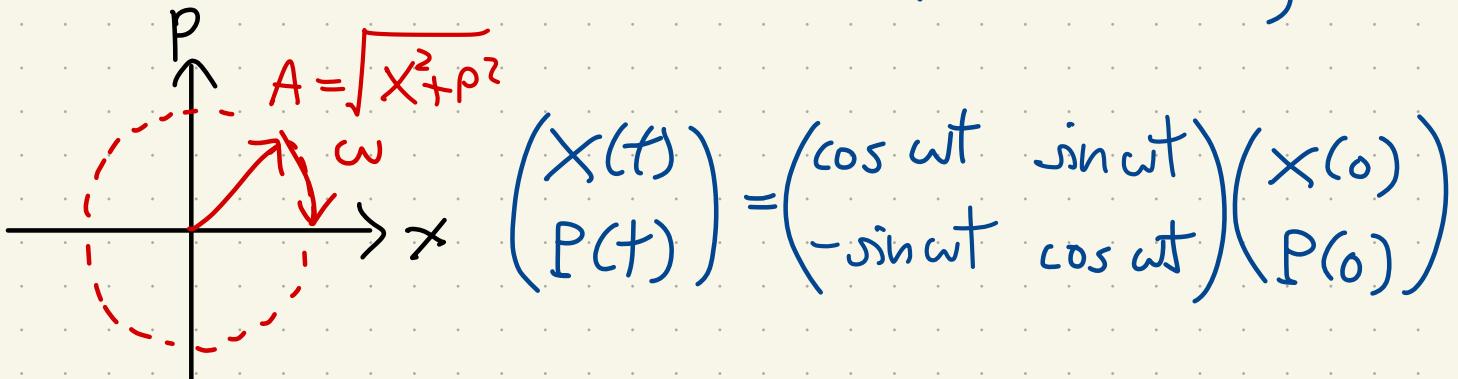
$$P = \frac{p}{p_c} = \frac{p}{\sqrt{\hbar m\omega}}$$

$$H = \frac{\hbar\omega}{2}(X^2 + P^2)$$

Recall classical eq. of motion

$$\begin{aligned} \dot{X} &= \omega P \\ \dot{P} &= -\omega X \end{aligned} \quad \Rightarrow \quad \ddot{X} = -\omega^2 X$$

$$\Rightarrow \begin{cases} X(t) = X(0) \cos(\omega t) + P(0) \sin(\omega t) \\ P(t) = P(0) \cos(\omega t) - X(0) \sin(\omega t) \end{cases}$$



[[Note on how to obtain the classical eq. of motion]]

Classically there is no \hbar , so our change of variables shouldn't involve \hbar .

$$X = \sqrt{m\omega} x, \quad P = \frac{P}{\sqrt{m\omega}} \quad (\text{Not dimensionless})$$

$$\Rightarrow H = \frac{\omega}{2}(X^2 + P^2) \Rightarrow \begin{cases} \dot{X} = \frac{\partial H}{\partial P} = \omega P \\ \dot{P} = -\frac{\partial H}{\partial X} = -\omega X \end{cases}$$

(SHO is a linear system)
in the sense that the time evolution transforms x and p linearly

Define new coördinates (Still classical! ∇)

$$\alpha = \frac{X + iP}{\sqrt{2}}, \quad \alpha^* = \frac{X - iP}{\sqrt{2}}$$

$$\begin{aligned} \alpha(t) &= \frac{1}{\sqrt{2}} [X(0) \cos(\omega t) + P(0) \sin(\omega t) \\ &\quad + iP(0) \cos(\omega t) - iX(0) \sin(\omega t)] \\ &= \frac{X(0) + iP(0)}{\sqrt{2}} [\cos(\omega t) - i \sin(\omega t)] \\ &= \alpha(0) e^{-i\omega t} \quad \Leftrightarrow \dot{\alpha} = -i\omega \alpha \end{aligned}$$

This can be done completely analogously at the quantum level.

$$[X, P] = \frac{1}{\hbar c p_c} [x, p] = \frac{1}{\hbar} i\hbar = i$$

$a = \frac{X + iP}{\sqrt{2}}$	$a^+ = \frac{X - iP}{\sqrt{2}}$
$X = \frac{a + a^+}{\sqrt{2}}$	$P = \frac{a - a^+}{\sqrt{2}i}$

$$\begin{aligned} [a, a^+] &= \frac{1}{\sqrt{2}} [X + iP, X - iP] \\ &= -\frac{i}{\sqrt{2}} [X, P] + \frac{i}{\sqrt{2}} [P, X] = -i[X, P] = 1 \end{aligned}$$

The commutation relation can be used to switch the order of operators $a^+a = 1 + a a^+$

$$H = \frac{\hbar\omega}{2}(a^\dagger a + a a^\dagger) = \boxed{\hbar\omega(a^\dagger a + \frac{1}{2})}$$

Define the number operator $N = a^\dagger a$

$$[a, N] = [a, a^\dagger a] = [a, a^\dagger]a = a$$

$$[a^\dagger, N] = [a^\dagger, a^\dagger a] = a^\dagger[a^\dagger, a] = -a^\dagger$$

Heisenberg eq. of motion

$$\dot{A} = \frac{1}{i\hbar}[A, H] + \frac{\partial A}{\partial t}$$

$$\dot{X} = \frac{\omega}{2i}[X, X^2 + P^2] = \frac{\omega}{2i}[X, P^2]$$

$$= \frac{\omega}{2i}([X, P]X + X[X, P]) = \omega X$$

$$\dot{P} = \frac{\omega}{2i}[P, X^2 + P^2] = \frac{\omega}{2i}[P, X^2]$$

$$= \frac{\omega}{2i}([P, X]P + P[P, X]) = -\omega P$$

$$\Rightarrow \ddot{X} = -\omega X \Rightarrow \begin{cases} X(t) = X(0) \cos(\omega t) + P(0) \sin(\omega t) \\ P(t) = P(0) \cos(\omega t) - X(0) \sin(\omega t) \end{cases}$$

$$\Rightarrow a(t) = e^{-i\omega t} a(0)$$

Or solve directly

$$\dot{a} = \frac{1}{i\hbar}[a, H] = \frac{\omega}{i}[a, N + \frac{1}{2}] = \frac{\omega}{i}[a, N] = -i\omega a$$

The spectrum

There're mainly two ways to solve for the spectrum

- ① 2nd order DE in the position representation
- ② Algebraic using ladder operators

We will do the latter. (But we'll still have to solve a **first order** DE to remove the degeneracy for the ground state.)

Let $|v\rangle$ be a normalized eigenstate of N with eigenvalue v

Lemma I (Lemma I, C-T p. 491) $v > 0$

$$\blacksquare v = \langle v | a^\dagger a | v \rangle = \|a|v\rangle\|^2 \geq 0 \quad \square$$

Lemma II (Lemma III, C-T p. 492)

① $a|v\rangle$ is an eigenstate of H with eigenvalue $v-1$

② $a^\dagger|v\rangle \xrightarrow{\text{?}} v+1$

$$\begin{aligned} \blacksquare N(a|v\rangle) &= (aa^\dagger - 1)a|v\rangle \\ &= a(a^\dagger a)|v\rangle - a|v\rangle = (v-1)(a|v\rangle) \end{aligned}$$

$$\begin{aligned} N(a^\dagger|v\rangle) &= a^\dagger a a^\dagger|v\rangle = a^\dagger(a^\dagger a + 1)|v\rangle \\ &= (v+1)(a^\dagger|v\rangle) \quad \square \end{aligned}$$

The operator a removes one unit of energy, so it's called the annihilation operator (and a^\dagger is the creation operator).

Normalization

$$\|a|\nu\rangle\|^2 = \langle\nu|a^\dagger a|\nu\rangle = \nu \Rightarrow a|\nu\rangle = \sqrt{\nu}|\nu-1\rangle$$

$$\|a^\dagger|\nu\rangle\|^2 = \langle\nu|a^\dagger a^\dagger|\nu\rangle = \langle\nu|(a^\dagger a + 1)|\nu\rangle = \nu + 1$$

$$\Rightarrow a^\dagger|\nu\rangle = \sqrt{\nu+1}|\nu+1\rangle$$

Since the lowering process can't continue indefinitely, there need to be an eigenstate $|\nu\rangle$ such that $a|\nu\rangle = 0$.

Hence by lemma 2, ν needs to be a non-negative integer $\nu = n = 0, 1, 2, \dots$

Summary The number states, eigenstates $|n\rangle$ of $N=a^\dagger a$ corresponding to eigenvalues $n=0, 1, 2, \dots$, are stationary states of the SHO with eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

a priori there could be many $n=0$ states. We'll establish that there is no such degeneracy by solving the DE for the unique ground state.

Not the zero vector. 0 is just the n label.

$$\begin{aligned}
 0 &= \langle x | a | 0 \rangle = \langle x | \left(\frac{x + i p}{\sqrt{2}} \right) | 0 \rangle \\
 &= \langle x | \left(\sqrt{\frac{m\omega}{2\hbar}} x + \frac{i}{\sqrt{2\hbar m\omega}} p \right) | 0 \rangle \\
 &= \sqrt{\frac{m\omega}{2\hbar}} x \langle x | 0 \rangle + \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx} \langle x | 0 \rangle
 \end{aligned}$$

$$\Leftrightarrow \frac{d}{dx} \langle x | 0 \rangle = -\frac{m\omega}{\hbar} x \langle x | 0 \rangle$$

$$\Rightarrow \langle x | 0 \rangle = C \exp\left(-\frac{m\omega}{2\hbar} x^2\right)$$

Normalization

$$\begin{aligned}
 1 &= |\langle x | 0 \rangle|^2 = C^2 \int_{-\infty}^{\infty} dx \exp\left(-\frac{m\omega}{\hbar} x^2\right) \\
 &= C^2 \sqrt{\frac{\pi \hbar}{m\omega}} \Leftrightarrow C = \left(\frac{m\omega}{\pi \hbar}\right)^{1/4} \quad \begin{array}{l} \text{(Choose } C \\ \text{to be real)} \end{array}
 \end{aligned}$$

Gaussian integral $\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

Lemma III The 1D SHO spectrum has no degeneracy.

- Let's write the eigenstates as $|n, k\rangle$ where k labels orthogonal degenerate states ($\langle n, j | n, k \rangle = \delta_{jk}$)

Then

$$a^n |n, k\rangle = \sqrt{n!} |0, k\rangle$$

$$\Rightarrow \langle n, j | (a^\dagger)^n a^n |n, k\rangle = (n!) \delta_{jk}$$

But there is only one ground state given by the Gaussian wave function that we derived, therefore there can't exist orthogonal degenerate states for any n . \square (2D and 3D SHO can easily have degeneracies)

$$\begin{aligned} a^k |n\rangle &= \sqrt{n(n-1) \dots (n-k+1)} |n-k\rangle \\ &= \sqrt{\frac{n!}{(n-k)!}} |n-k\rangle = n^{\underline{k}} |n-k\rangle \end{aligned}$$

Falling factorial

$$\begin{aligned} (a^\dagger)^k |n\rangle &= \sqrt{(n+1)(n+2) \dots (n+k)} |n+k\rangle \\ &= \sqrt{\frac{(n+k)!}{n!}} |n+k\rangle = (n+1)^{\overline{k}} |n+k\rangle \end{aligned}$$

Rising factorial

In particular, $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$.

The number state wave functions

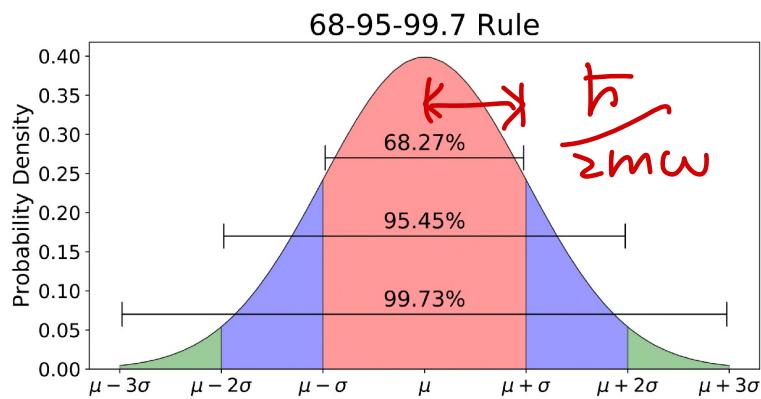
$$\langle x|_0 \rangle = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left(-\frac{m\omega}{2\hbar} x^2 \right)$$

Prob. $|\langle x|_0 \rangle|^2 = \sqrt{\frac{m\omega}{\pi\hbar}} \exp \left(-\frac{m\omega}{\hbar} x^2 \right)$

Gaussian

$$\frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(x-\mu)^2}{2\sigma^2} \right]$$

$$\Delta x^2 = \sigma^2 = \frac{\hbar}{2m\omega}$$



To find the wave function of other number states

① Use $a^\dagger |n-1\rangle = \sqrt{n} |n\rangle \Rightarrow$ Recursion relation

② Use $|n\rangle = (a^\dagger)^n |0\rangle / \sqrt{n!}$

① $\langle x|a^\dagger|n-1\rangle = \sqrt{n} \langle x|n\rangle$

$$\sqrt{\frac{m\omega}{2\hbar}} \langle x| \left(x - \frac{i\hbar}{m\omega} \right) |n-1\rangle = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{d}{dx} \right) \langle x|n-1\rangle$$

$$\Rightarrow \langle x|n\rangle = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{d}{dx} \right) \langle x|n-1\rangle$$

$$② \langle x|n\rangle = \frac{1}{\sqrt{n!}} \left(\frac{m\omega}{2\hbar}\right)^{n/2} \left(x - \frac{\hbar}{m\omega} \frac{d}{dx}\right)^n \langle x|0\rangle$$

$$= \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\underbrace{\sqrt{\frac{m\omega}{\hbar}} x - \sqrt{\frac{\hbar}{m\omega}} \frac{d}{dx}}_{} \right)^n \exp\left(-\frac{m\omega}{2\hbar} x^2\right)$$

$$\text{Let } z = \sqrt{\frac{m\omega}{\hbar}} x$$

$$\left(z - \frac{d}{dz}\right)^n e^{-\frac{1}{2}z^2}$$

$$\text{Hermite polynomials } H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}$$

C-T Complement B_V

$$\left(z - \frac{d}{dz}\right) e^{\frac{1}{2}z^2} f(z) = \cancel{ze^{\frac{1}{2}z^2} f(z)} - \cancel{ze^{\frac{1}{2}z^2} f(z)} - e^{\frac{1}{2}z^2} \frac{d}{dz} f(z)$$

$$\Rightarrow \left(z - \frac{d}{dz}\right)^n e^{\frac{1}{2}z^2} f(z) = (-1)^n e^{\frac{1}{2}z^2} \frac{d^n}{dz^n} f(z)$$

$$\text{Therefore, setting } f(z) = e^{-z^2}, \quad (-1)^n e^{-z^2} H_n(z)$$

$$\langle x|n\rangle = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} (-1)^n e^{\frac{1}{2}z^2} \underbrace{\frac{d^n}{dz^n} e^{-z^2}}_{}$$

$$= \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}z^2} H_n(z)$$

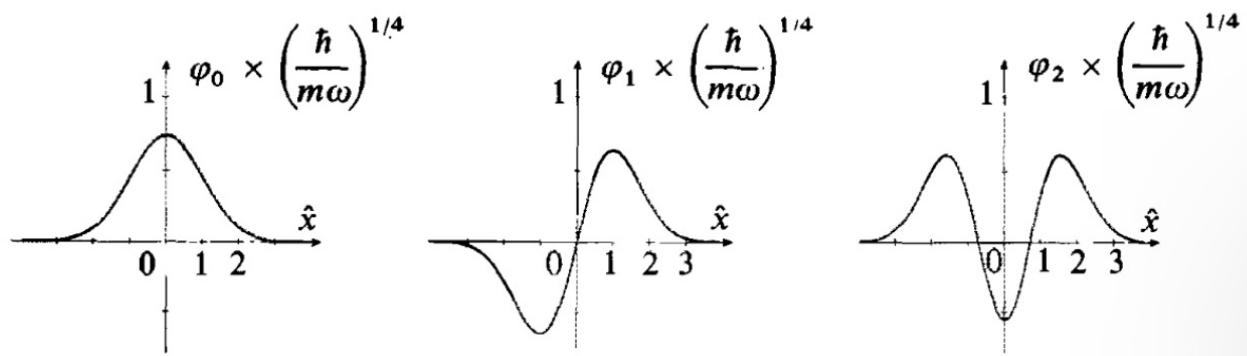


FIGURE 4

Wave functions associated with the first three levels of a harmonic oscillator.

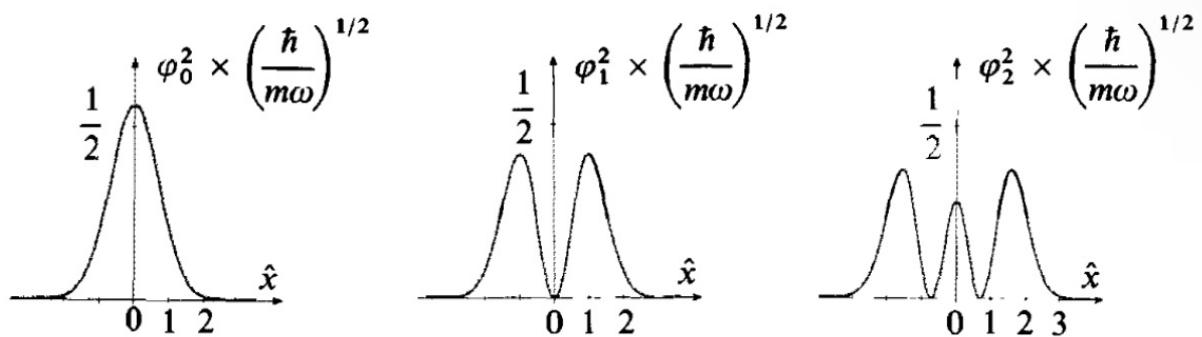


FIGURE 5

Probability densities associated with the first three levels of a harmonic oscillator.

$\hbar \uparrow$ nodes \uparrow

Ground state
↓

$$a = \begin{pmatrix} 0 & 1 & 0 & \sqrt{2} & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & -1 & 0 & -\sqrt{2} & 0 & \dots \\ -1 & 0 & -\sqrt{2} & 0 & -\sqrt{3} & \dots \end{pmatrix}$$

Statistical moments in stationary states $|n\rangle$

I'll write expectation values as $\langle \psi | A | \psi \rangle = \langle A \rangle_\psi$ with a subscript ψ to indicate the state. So for the stationary states, I'll use both $|n\rangle$ and the more dense notation $|a_n\rangle$ that the book uses interchangeably.

$$\langle a \rangle_n = \sqrt{n} \langle n | n-1 \rangle = 0$$

$$\langle a^\dagger \rangle_n = \sqrt{n+1} \langle n | n+1 \rangle = 0$$

$$\langle X \rangle_n = \left\langle \frac{a + a^\dagger}{\sqrt{2}} \right\rangle_n = 0$$

$$\langle P \rangle_n = \left\langle \frac{a - a^\dagger}{\sqrt{2}i} \right\rangle_n = 0$$

Not what we expect classically from an SHO.

$$\begin{aligned} \langle X^2 \rangle_n &= \frac{1}{2} \langle (a^\dagger)^2 + aa^\dagger + a^\dagger a + a^2 \rangle_n \\ &= \frac{1}{2} \langle aa^\dagger + a^\dagger a \rangle_n = n + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \langle P^2 \rangle_n &= -\frac{1}{2} \langle (a^\dagger)^2 - aa^\dagger - a^\dagger a + a^2 \rangle_n \\ &= +\frac{1}{2} \langle aa^\dagger + a^\dagger a \rangle_n = n + \frac{1}{2} \end{aligned}$$

$$\langle (\Delta X)^2 \rangle_n = \langle X^2 \rangle_n - \cancel{\langle X \rangle_n^2} = n + \frac{1}{2}$$

$$\langle (\Delta P)^2 \rangle_n = n + \frac{1}{2}$$

$$\Rightarrow \Delta X \Delta P = n + \frac{1}{2}$$

Minimum for the ground state where it saturates the HUP

$$\Delta X \Delta P = \frac{1}{2}.$$

Cohesive states

canonical

There are a few characterizations of coherent states.

- ① Minimum-uncertainty states (Equal uncertainties in the two quadratures)
- ② Eigenstates of the annihilation operation a .
- ③ The states obtained by applying "displacement operators" to $|0\rangle$.

Note that all three equivalent notions of coherent states are independent of the Hamiltonian. But the time evolution under the harmonic potential preserves these properties.

Recall Variance $(\Delta A)^2 = (A - \langle A \rangle)^2$

$$\langle (\Delta A)^2 \rangle = \langle A^2 - 2\langle A \rangle A + \langle A \rangle^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = (\langle \psi | \Delta x | \psi \rangle) (\langle \psi | \Delta p | \psi \rangle)$$

$$\geq |\underbrace{\langle \psi | \Delta x \Delta p | \psi \rangle}|^2 \quad (\text{Cauchy-Schwarz})$$

$$\frac{1}{2} \left(\{ \Delta x, \Delta p \} + [\Delta x, \Delta p] \right) \\ = [x, p] = i$$

$$= \frac{1}{4} \left| \langle \psi | \{ \Delta x, \Delta p \} | \psi \rangle + i \right|^2 = \frac{c^2 + 1}{4} \geq \frac{1}{4}$$

(Standard uncertainty relation (Schrödinger-Robertson)
(ignores the anticommutator term $\{ \Delta x, \Delta p \}$).

To saturate the inequality, we have to optimize two parts.

① C-S $\Rightarrow \Delta x | \psi \rangle$ and $\Delta p | \psi \rangle$ are collinear

$$(\Delta x + \lambda \Delta p) | \psi \rangle = 0, \quad \lambda \in \mathbb{C}$$

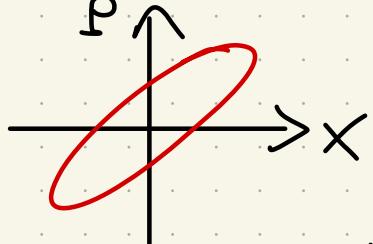
$$\left(\langle (\Delta x)^2 \rangle = |\lambda|^2 \langle (\Delta p)^2 \rangle \right)$$

$|\lambda|$ is the ratio $\frac{\Delta x}{\Delta p}$

$$\textcircled{2} \quad \langle \{\Delta X, \Delta P\} \rangle = 0$$

Plugging \textcircled{1} $\Delta X | \psi \rangle = -i \Delta P | \psi \rangle$ in \textcircled{2}

$$0 = (\langle \psi | \Delta X) \Delta P | \psi \rangle + \langle \psi | \Delta P (\Delta X | \psi \rangle) \\ = -2 \operatorname{Re} i \langle \psi | (\Delta P)^2 | \psi \rangle$$



$\Rightarrow \lambda$ is purely imaginary other λ gives squeezed states

Let's choose $\lambda = i$. (equal uncertainty $\Delta X = \Delta P = \frac{1}{\sqrt{2}}$)

$$0 = (\Delta X + i \Delta P) | \psi \rangle$$

$$= \sqrt{2} a | \psi \rangle - (\langle X \rangle + i \langle P \rangle) | \psi \rangle$$

$$\Leftrightarrow a | \psi \rangle = \langle a \rangle | \psi \rangle$$

$| \psi \rangle$ is an eigenvector of a .

Does $|\alpha\rangle$ exist such that $a |\alpha\rangle = \alpha |\alpha\rangle$?

$$a |\alpha\rangle = a \left(\sum_{n=0}^{\infty} c_n | n \rangle \right) = \sum_{n=1}^{\infty} c_n \sqrt{n} | n-1 \rangle \\ = \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} | n \rangle \\ \Rightarrow \alpha c_{n-1} = c_n \sqrt{n}$$

$$c_n = \frac{\alpha}{\sqrt{n}} \frac{\alpha}{\sqrt{n-1}} \dots \alpha c_0 = \frac{\alpha^n}{\sqrt{n!}} c_0$$

c_0 is determined from normalization.

$$1 = |c_0|^2 \sum_{n=0}^{\infty} \frac{(\alpha)^n}{n!} = |c_0|^2 e^{-|\alpha|^2} \Rightarrow c_0 = e^{-|\alpha|^2/2}$$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Poisson distribution
in the number basis

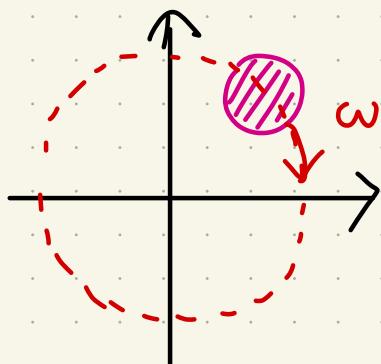
Time evolution

Heisenberg picture

$$a(t) = e^{-i\omega t} a(0)$$

Schrödinger picture

$$U(t, 0) |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega(n+\frac{1}{2})t} |n\rangle$$



$$\begin{aligned} &= e^{-i\omega t/2} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle \\ &= e^{-i\omega t/2} |\alpha e^{-i\omega t}\rangle \end{aligned}$$

Classical-like
motion

Statistical moments in coherent states $|\alpha\rangle$

The expectation value of monomials of a, a^\dagger in normal order can be computed easily.

$$\langle (a^\dagger)^n a^m \rangle_\alpha = (\alpha^*)^n \alpha^m$$

Normal order = a^\dagger to the left, a to the right

$$:a a^\dagger: = a a^\dagger$$

$$:a a^\dagger: = a a^\dagger - ,$$

$$\text{Antinormal order} \quad .. a a .. = a a^\dagger$$

$$.. a a .. = a a^\dagger + ,$$

Inner product

$$\langle \alpha | \beta \rangle = \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right) \sum_{n,m} \underbrace{\frac{(\alpha^*)^n \beta^m}{\sqrt{n! m!}}}_{\langle n|m \rangle}$$

$$\sum_n \underbrace{\frac{(\alpha^* \beta)^n}{n!}}_{\langle n|m \rangle} = e^{\alpha^* \beta}$$

$$|\langle \alpha | \beta \rangle|^2 = e^{-|\alpha - \beta|^2} \quad (\text{Not orthogonal?})$$

Displacement operator

$$D(\alpha) := \exp(\alpha a^\dagger - \alpha^* a)$$

Unitary $D^{-1}(\alpha) = D^\dagger(\alpha) = D(-\alpha)$

BCH formula

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2!} [B, [B, A]] + \dots$$

Here $[\alpha a^\dagger - \alpha^* a, a] = \alpha [a^\dagger, a] = -\alpha$

So $D^{-1}(\alpha) a D(\alpha) = a + \alpha$

$$\begin{aligned}\therefore a(D(\alpha)|0\rangle) &= D(\alpha) D^{-1}(\alpha) a D(\alpha)|0\rangle \\ &= D(\alpha)(a + \alpha)|0\rangle \\ &= \underbrace{\alpha(D(\alpha)|0\rangle)}_{|\alpha\rangle}\end{aligned}$$

\Rightarrow Another definition of a coherent state $|\alpha\rangle$ is the vacuum $|0\rangle$ displaced by α on the phase space.

Braided identity (general)

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2!} [B, [B, A]] + \frac{1}{3!} [B, [B, [B, A]]] + \dots$$

More compactly if I define $\text{Com}_B(A) = [B, A]$,

I can write $e^B A e^{-B} = \sum_{k=0}^{\infty} \frac{(\text{Com}_B)^k}{k!}(A)$.

Proof① (Taylor expansion)

Consider expanding $f(\lambda) = e^{\lambda B} A e^{-\lambda B}$ around $\lambda=0$.

$$\frac{d f(\lambda)}{d \lambda} = B f(\lambda) - f(\lambda) B = \text{Com}_B f(\lambda)$$

$$\frac{d^k f(\lambda)}{d \lambda^k} = (\text{Com}_B)^k f(\lambda)$$

$$\therefore f(\lambda) = A + \lambda \underset{\substack{\parallel \\ f(0)}}{\text{Com}_B} f(\lambda) + \frac{\lambda^2}{2!} (\text{Com}_B)^2 f(\lambda) + \dots$$

Then set $\lambda=1 \quad \square$

■ Proof ② (Algebraic) (Michael Müger's note)

Define left- and right-multiplication operators

$$L_B(A) = BA$$

L_A and R_B commute

$$R_B(A) = AB$$

$$(ABC = L_A R_C(B) = R_C L_A(B))$$

$$\Rightarrow \text{Com}_B(A) = L_B(A) - R_B(A)$$

$$\text{Now } e^B A e^{-B} = L_{e^B} R_{e^{-B}}(A)$$

$$= e^{L_B} e^{R_{-B}}(A) = e^{L_B - R_B}(A) = e^{\text{Com}_B(A)} \square$$