Two-level Systems

Examples

Left- and right-circular polarization

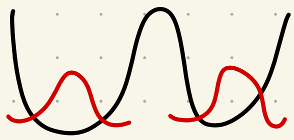
① Polarization of light $|L\rangle, |R\rangle$

② Spin-1/2 $|\uparrow\rangle, |\downarrow\rangle$

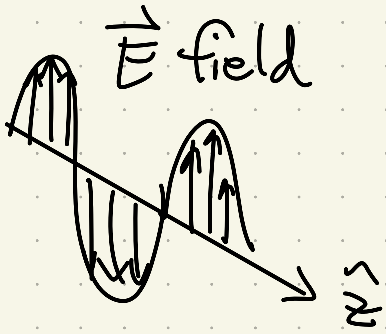
Ground Excited

③ Atomic energy levels $|g\rangle, |e\rangle$

④ Double-well potential



Isomorphic classical system: Polarization of light



$$\vec{E}_x = E_x \cos(kz - \omega t - \phi_x)\hat{e}_x$$

$$= \mathrm{Re}\left(E_x e^{i(kz-\omega t-\phi_y)}\hat{e}_x\right)$$

Same for $\vec{E}_y$

The total $\vec{E}$ field is the vector sum:

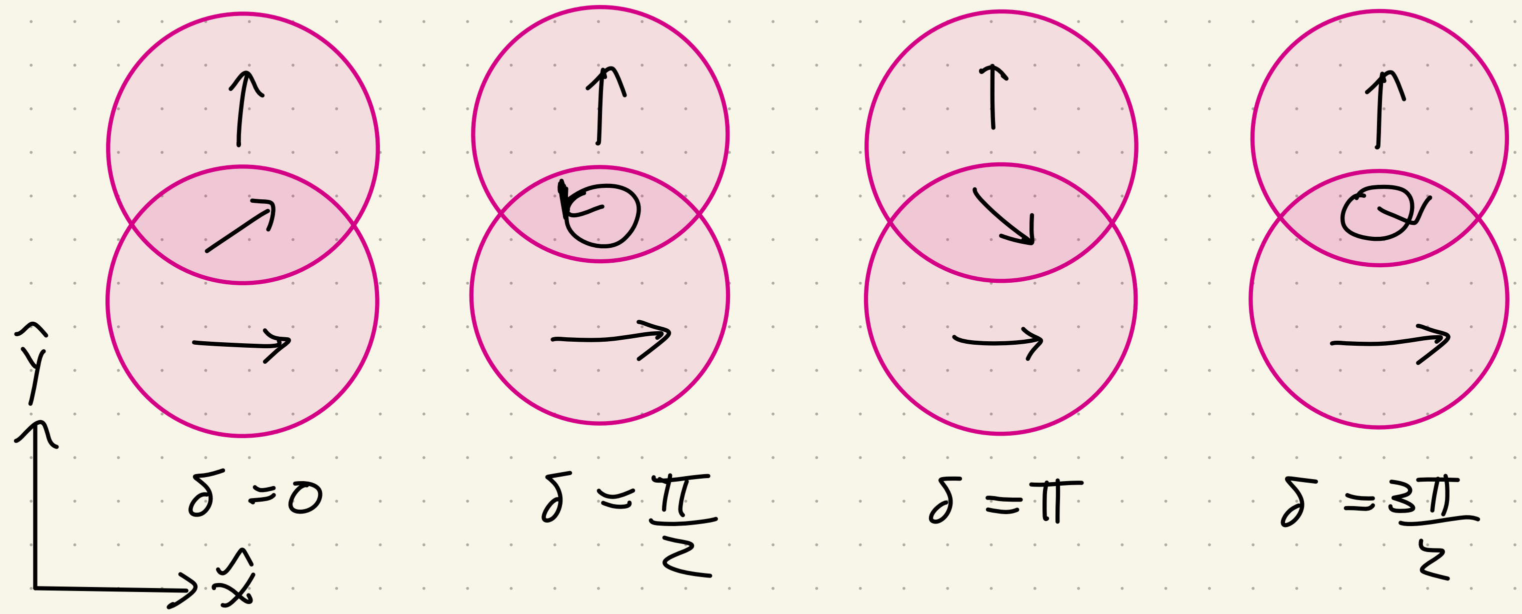
$$\vec{E} = \mathrm{Re}\left[e^{i(kz-\omega t)}\left(E_x e^{-i\phi_x}\hat{e}_x + E_y e^{-i\phi_y}\hat{e}_y\right)\right]$$

Normalized $E_0 = \sqrt{E_x^2 + E_y^2}$, $\tilde{E}_x = \dfrac{E_x}{E_0}$, $\tilde{E}_y = \dfrac{E_y}{E_0}$

$$\vec{E} = \mathrm{Re}\left[E_0 e^{i(kz-\omega t-\phi_x)}\left(\tilde{E}_x\hat{e}_x + \tilde{E}_y e^{i(\phi_x-\phi_y)}\hat{e}_y\right)\right]$$

Examples

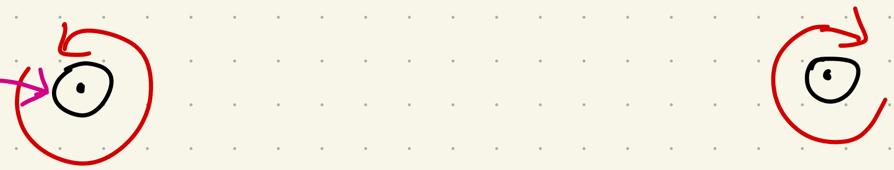
Superposition ($\delta = \phi_x - \phi_y$)



Right-circular polarization

Left-circular polarization

$\hat{z}$ pointing out of the screen



Suppose that $E_x = E_y$, $\phi_x = 0$

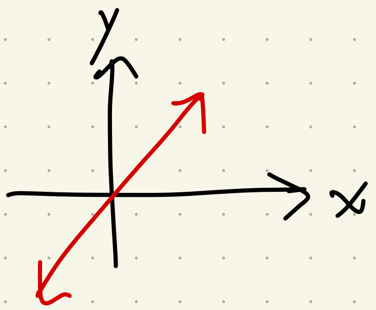
At $z = 0$, $\vec{E}(0,t) = \begin{pmatrix} Re(e^{-i\omega t}) \\ Re(e^{-i\omega t} e^{i\delta}) \end{pmatrix}$

- What states of polarization are represented by real linear combinations (L.C.) of $\hat{e}_x$ and $\hat{e}_y$?
- What about complex L.C.?

Real L.C. $\Rightarrow$ Linear polarization

Example $E_x = E_y$

$$\delta = 0 \Rightarrow \vec{E}(0,t) = \begin{pmatrix} Re\,(e^{-i\omega t}) \\ Re\,(e^{-i\omega t}) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ \cos(\omega t) \end{pmatrix}$$



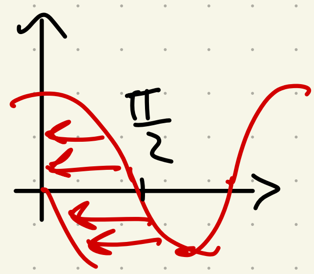
Diagonal polarization

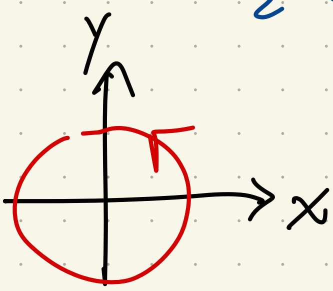
Complex L.C. $\Rightarrow$ Elliptical polarization

Example $E_x = E_y$

$$\delta = \frac{\pi}{2} \Rightarrow \vec{E}(0,t) = \begin{pmatrix} Re\,(e^{-i\omega t}) \\ Re\,(i e^{-i\omega t}) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega t) \\ \cos(\omega t + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{pmatrix}$$



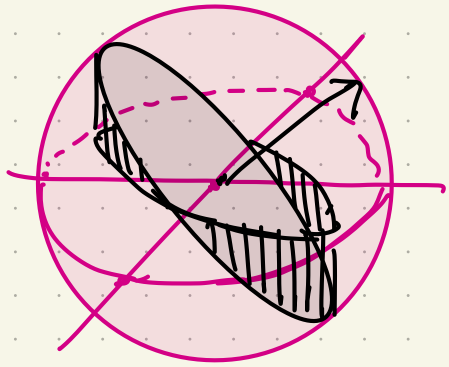


Right-circular polarization

If $E_x \neq E_y \Rightarrow$ Elliptical polarization

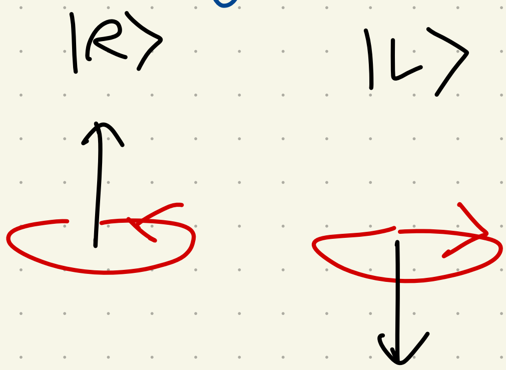
Poincaré sphere

Visual representation of all possible polarization states (all possible ellipses) as 3D vectors



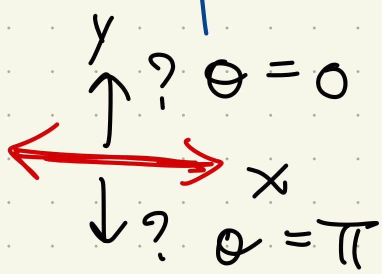
- Draw an ellipse as the projection of a great circle onto the horizontal plane, and then associate to the ellipse the vectorial area of the great circle.

Left- and right-circular polarizations are the south and north poles respectively.



- There remains the problem with all other states which is that antipodal points represent the same state

Example: horizontal polarization



$\Longrightarrow$ Solution: double the polar angle

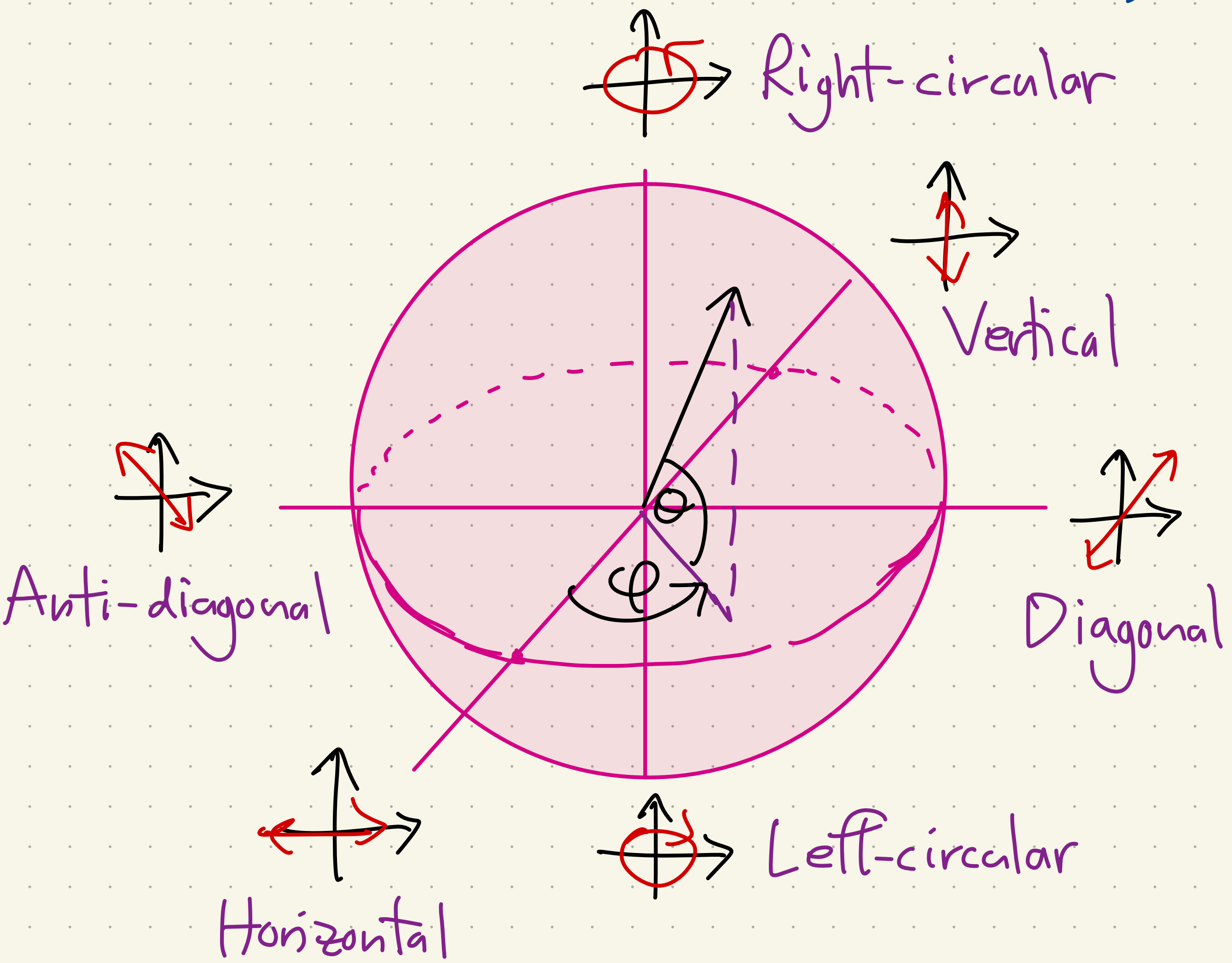
$$\vec{E} = \mathrm{Re}\Big(E_0\, e^{i(kz-\omega t)} \underbrace{\frac{\hat{e}_x + i\hat{e}_y}{\sqrt{2}}}_{\hat{f}_R}\Big)$$

$\hat{f}_R$ Right circular

$$\frac{\hat{e}_x - i e_y}{\sqrt{2}} = \hat{f}_L$$

Every polarization state can be written as

$$\cos\left(\frac{\theta}{2}\right)\hat{f}_R + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\hat{f}_L \quad \begin{cases} \theta \in [0, \pi] \\ \varphi \in [0, 2\pi) \end{cases}$$



Use left- and right-circular polarization states as basis vectors

Same algebra $\hat{f}_R, \hat{f}_L \longleftrightarrow |L\rangle, |R\rangle$

Horizontal

$$|H\rangle = |R\rangle + |L\rangle \leftrightarrow \hat{e}_x + i\hat{e}_y + \hat{e}_x - i\hat{e}_y \propto \hat{e}_x$$

Vertical

$$|V\rangle = |R\rangle + e^{i\pi}|L\rangle$$

$$= |R\rangle - |L\rangle \leftrightarrow \hat{e}_x + i\hat{e}_y - \hat{e}_x + i\hat{e}_y \propto \hat{e}_y$$

Diagonal

$$|D\rangle = |R\rangle + e^{i\pi/2}|L\rangle$$

$$= |R\rangle + i|L\rangle \leftrightarrow \hat{e}_x + i\hat{e}_y + i\hat{e}_x + \hat{e}_y$$

$$= (1+i)(\hat{e}_x + \hat{e}_y) \propto \hat{e}_x + \hat{e}_y$$

Anti-diagonal

$$|A\rangle = |R\rangle + e^{i3\pi/2}|L\rangle$$

$$= |R\rangle - i|L\rangle \leftrightarrow \hat{e}_x + i\hat{e}_y - i\hat{e}_x - \hat{e}_y$$

$$= (1-i)\hat{e}_x - (1-i)\hat{e}_y$$

$$\propto \hat{e}_x - \hat{e}_y$$

Transformations: Birefringent (Light with different polarizations travel with different speed through the material) → Quarter/Half waveplate

- Rotations in 3D are specified by the axis and the angle of rotation (not true in higher dimension)

  How many axes of rotation do we need?

  → 2 (Euler angles)

(Selective) measurements: Polarizer

Spin-1/2

Gyromagnetic ratio

$$\hat{H} = \vec{\mu}\cdot\vec{B} = -\gamma\hbar\,\frac{\vec{\sigma}\cdot\vec{B}}{2} \Rightarrow \hat{U}(t,0) = e^{-i\hat{H}t/\hbar} = e^{i\gamma\vec{B}\cdot\vec{\sigma}/2}$$

Magnetic (dipole) moment

Pauli matrices

Remarks

① States differ only by a global phase ($|\psi\rangle$ vs $e^{i\delta}|\psi\rangle$) are represented by the same point on the Poincaré/Bloch sphere. Thus this representation of states is more accurate than the representation by normalized vectors.

② The density matrix formalism also eliminates the global phase:

$$|\psi\rangle \mapsto |\psi\rangle\langle\psi|$$
$$\parallel$$
$$e^{i\delta}|\psi\rangle \mapsto e^{i\delta}|\psi\rangle\langle\psi|e^{-i\delta}$$

In fact, it is more natural to think of points on the Bloch sphere as projection operators/density matrices $|\hat{n}\rangle\langle\hat{n}|$ rather than state vectors $|\hat{n}\rangle$, as we shall see now.

Why the one half in $\theta/2$? Look at the operator space.

$$|\hat{n}\rangle = \cos\left(\tfrac{\theta}{2}\right)|R\rangle + \sin\left(\tfrac{\theta}{2}\right)e^{i\varphi}|L\rangle$$

$$|-\hat{n}\rangle = -\sin\left(\tfrac{\theta}{2}\right)e^{-i\varphi}|R\rangle + \cos\left(\tfrac{\theta}{2}\right)|L\rangle$$

$$|\hat{n}\rangle\langle\hat{n}| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & 1-\cos\theta \end{pmatrix}$$

(Angle doubling)

$$= \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix},$$

$$\boxed{\begin{aligned} x &= \sin\theta\cos\varphi \\ y &= \sin\theta\sin\varphi \\ z &= \cos\theta \end{aligned}}$$

Constrained to radius 1

$$=: \boxed{\frac{\mathbb{1} + \hat{n}\cdot\vec{\sigma}}{2}} \quad \text{where } \vec{\sigma} = \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix}$$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|-\hat{n}\rangle\langle-\hat{n}| = \frac{1}{2}\begin{pmatrix} 1-\cos\theta & -\sin\theta\, e^{-i\varphi} \\ -\sin\theta\, e^{i\varphi} & 1+\cos\theta \end{pmatrix} = \frac{\mathbb{1} - \hat{n}\cdot\vec{\sigma}}{2}$$

$\theta \mapsto \pi - \theta$

$\varphi \mapsto \pi + \varphi$

Spin observable in any direction $\hat{n}$

$$\hat{\sigma}_{\hat{n}} = |\hat{n}\rangle\langle\hat{n}| - |-\hat{n}\rangle\langle-\hat{n}|$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & \cos\theta \end{pmatrix} = \boxed{\hat{n}\cdot\vec{\sigma}}$$

How are the geometry of the Hilbert space and that of the real space related (metric $\Rightarrow$ geometry)

Pauli-matrix algebra

Properties

$$\left.\begin{array}{ll} ① & \hat{\sigma}_j^\dagger = \hat{\sigma}_j \quad \text{(Hermiticity)} \\ ② & \hat{\sigma}_j^2 = \hat{\mathbb{1}} \end{array}\right\} \text{True for } \hat{\sigma}_{\hat{n}} \text{ for arbitrary direction } \hat{n}$$

③ $\hat{\sigma}_1\hat{\sigma}_2 = i\hat{\sigma}_3, \quad \hat{\sigma}_2\hat{\sigma}_3 = i\hat{\sigma}_1, \quad \hat{\sigma}_3\hat{\sigma}_1 = i\hat{\sigma}_2$

$$\boxed{\sigma_j\sigma_k = \delta_{jk}\hat{\mathbb{1}} + i\epsilon_{jkl}\sigma_l}$$

(Einstein summation convention)

$$[\hat{\sigma}_j, \hat{\sigma}_k] = 2i\epsilon_{jkl}\hat{\sigma}_l$$

$$\{\hat{\sigma}_j, \hat{\sigma}_k\} := \hat{\sigma}_j\hat{\sigma}_k + \hat{\sigma}_k\hat{\sigma}_j = 2\delta_{jk}\hat{\mathbb{1}}$$

$$\mathrm{Tr}\, A := \sum_j A_{jj}$$

$$\mathrm{Tr}(|\psi\rangle\langle\varphi|) := \langle\varphi|\psi\rangle$$

(Representation-independent definition)

④ $\mathrm{Tr}\,\sigma_j = 0 \longleftarrow$ True for $\hat{\sigma}_{\hat{n}}$

Inner product using Pauli-matrix algebra

$$|\langle \hat{n}|\hat{m}\rangle|^2 = \mathrm{Tr}(\hat{\rho}_{\hat{n}}\hat{\rho}_{\hat{m}}) = \frac{1}{4}\mathrm{Tr}\left[(\hat{\mathbb{1}}+\hat{n}\cdot\vec{\sigma})(\hat{\mathbb{1}}+\hat{m}\cdot\vec{\sigma})\right]$$

$$= \frac{1}{4}\mathrm{Tr}\left[\hat{\mathbb{1}} + \underbrace{\hat{n}\cdot\vec{\sigma} + \hat{m}\cdot\vec{\sigma}}_{\text{Traceless}} + (\hat{n}\cdot\vec{\sigma})(\hat{m}\cdot\vec{\sigma})\right]$$

$$= \frac{1}{2} + \frac{1}{4}\mathrm{Tr}\left[(\hat{n}\cdot\vec{\sigma})(\hat{m}\cdot\vec{\sigma})\right]$$

What is the trace of $(\hat{n}\cdot\vec{\sigma})(\hat{m}\cdot\vec{\sigma})$?

$$(\hat{n}\cdot\vec{\sigma})(\hat{m}\cdot\vec{\sigma}) = n_j m_k \hat{\sigma}_j \hat{\sigma}_k$$

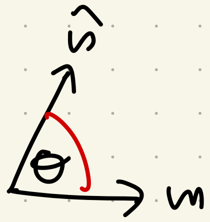
$$= n_j m_k \left(\delta_{jk}\hat{\mathbb{1}} + i\epsilon_{jkl}\hat{\sigma}_l\right)$$

$$= n_j m_j \hat{\mathbb{1}} + i\,\underbrace{\epsilon_{jkl} n_j m_k}_{(\hat{n}\times\hat{m})_l}\hat{\sigma}_l$$

$$= \hat{n}\cdot\hat{m}\,\hat{\mathbb{1}} + \underbrace{(\hat{n}\times\hat{m})\cdot\vec{\sigma}}_{\text{Traceless}}$$

$$\therefore \boxed{|\langle \hat{n}|\hat{m}\rangle|^2 = \frac{1+\hat{n}\cdot\hat{m}}{2}}$$

Relate the geometry of the Hilbert & real space



$$|\langle \hat{n}|\hat{m}\rangle|^2 = \frac{1+\cos\theta}{2} = \cos^2\left(\frac{\theta}{2}\right)$$

Real

Hilbert

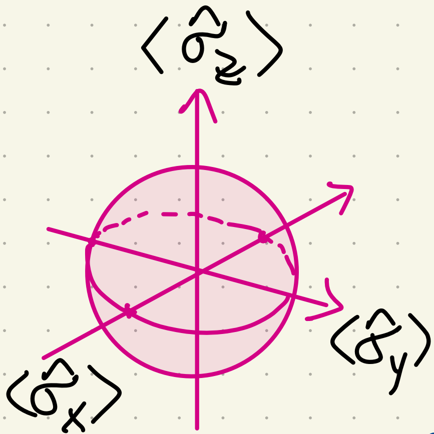
(Angle doubling again)

Expectation values

$$\langle \hat{n} | \hat{\sigma}_j | \hat{n} \rangle = \mathrm{Tr}(|\hat{n}\rangle\langle\hat{n}|\,\hat{\sigma}_j)$$
$$= \frac{1}{2}\mathrm{Tr}\left[(\mathbb{1} + \hat{n}\cdot\vec{\sigma})\hat{\sigma}_j\right]$$
$$= \frac{1}{2}\mathrm{Tr}(n_k \hat{\sigma}_k \hat{\sigma}_j) = n_j \qquad \begin{pmatrix}\text{Nonzero trace} \\ \text{only if } j = k\end{pmatrix}$$

For any spin observable in the direction $\hat{m}$,
(Also an alternative derivation of the above result)

$$\mathrm{Tr}(|\hat{n}\rangle\langle\hat{n}|\hat{\sigma}_{\hat{m}}) = \mathrm{Tr}\left[|\hat{n}\rangle\langle\hat{n}|\left(|\hat{m}\rangle\langle\hat{m}| - |-\hat{m}\rangle\langle-\hat{m}|\right)\right]$$
$$= |\langle\hat{n}|\hat{m}\rangle|^2 - |\langle\hat{n}|-\hat{m}\rangle|^2$$
$$= \frac{\hat{n}\cdot\hat{m} - \hat{n}\cdot(-\hat{m})}{2} = \hat{n}\cdot\hat{m}$$



$$\therefore \begin{pmatrix} \langle\hat{\sigma}_x\rangle \\ \langle\hat{\sigma}_y\rangle \\ \langle\hat{\sigma}_z\rangle \end{pmatrix} = \begin{pmatrix} n_x = \sin\theta\cos\varphi \\ n_y = \sin\theta\sin\varphi \\ n_z = \cos\theta \end{pmatrix} = \hat{n}$$

completely specifies any pure state.

Convenient basis for operators (C-T B_{2V})

$\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ forms a basis for the 2x2 matrix space.

Orthogonal w.r.t. the Hilbert-Schmidt inner product

$$\hat{\sigma}_0 := \hat{\mathbb{1}}, \quad \mathrm{Tr}(\hat{\sigma}_\alpha^\dagger \hat{\sigma}_\beta) = \mathrm{Tr}(\hat{\sigma}_\alpha \hat{\sigma}_\beta) = 2\delta_{\alpha\beta}$$

$\Longrightarrow \{\sigma_\alpha/\sqrt{2}\}_{\alpha=0,1,2,3}$ forms an ONB

Fixes $\mathrm{Tr}(A)$

$$A = A_0 \mathbb{1} + A_1 \hat{\sigma}_1 + A_2 \hat{\sigma}_2 + A_3 \hat{\sigma}_3$$

Complex $A_\alpha \Rightarrow$ Most general 2x2 matrix

Real $A_\alpha \Rightarrow$ —— " —— 2x2 Hermitian matrix

$\Downarrow$

Hamiltonians/Observables

Density matrices have trace 1 and non-negative eigenvalues.

$$\hat{\rho} = \frac{\hat{\mathbb{1}}}{2} + |A| \underbrace{(\hat{n} \cdot \hat{\sigma})}_{\text{Eigenvalues } \pm 1}$$

Positivity of the eigenvalue constraints $|A| \leq \frac{1}{2}$

Can't easily tell unitarity $\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \hat{\mathbb{1}}$ from the coefficients A_0, A_1, A_2, A_3 but we can write down the general form (careful!) of a 2x2 unitary matrix given an arbitrary Hermitian generator:

$$|A| := \sqrt{A_1^2 + A_2^2 + A_3^2}$$

$$\hat{U}(t,0) = e^{-i\hat{A}t/\hbar} = e^{-iA_0 t/\hbar}\, e^{-i\hat{n}\cdot\vec{\sigma}\,|A|t/\hbar}$$

For θ real,
$$e^{-i\theta\hat{n}\cdot\vec{\sigma}} = e^{-i\theta}|\hat{n}\rangle\langle\hat{n}| + e^{+i\theta}|-\hat{n}\rangle\langle-\hat{n}|$$
$$= \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)\hat{\mathbb{1}} + \left(\frac{e^{-i\theta} - e^{+i\theta}}{2}\right)\hat{n}\cdot\vec{\sigma}$$
$$= \cos\theta\,\hat{\mathbb{1}} - i\sin\theta\,\hat{n}\cdot\vec{\sigma}$$

(Alternatively, and for general Hilbert-space dimension, the "Euler form" $e^{-i\theta\hat{A}} = \cos\theta\,\hat{\mathbb{1}} - i\sin\theta\,\hat{A}$ can be obtained whenever $\hat{A}^2 = \hat{\mathbb{1}}$ by explicitly writing out the Taylor series for the matrix expansion

$$e^{\hat{A}} := \sum_{k=0}^{\infty} \frac{\hat{A}^k}{k!}$$
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