CHAPTER 1

Whence the Quantum?

SECTION 1.1

What happens when nobody is looking

Imagine throwing a ball through a barrier with two openings. In classical mechanics, if the ball passes through one of the openings, it will land behind the opening in a predictable manner; it may not land perfectly behind the opening because the ball may bounces off a side of the opening, but statistically speaking, directly behind the opening is the place that the ball most likely will land.

Suppose now that, instead of throwing a ball, we shine a beam of monochromatic light on a barrier with two narrow openings, or "slits", and a screen is set up some distance away from the barrier to collect the light. Huygen's principle tells us that the two slits act as sources of secondary, partial waves, ϕ_1 and ϕ_2, which then interfere with one another, creating a pattern of bright and dark bands on the screen known as *interference fringes* (Fig. 1).

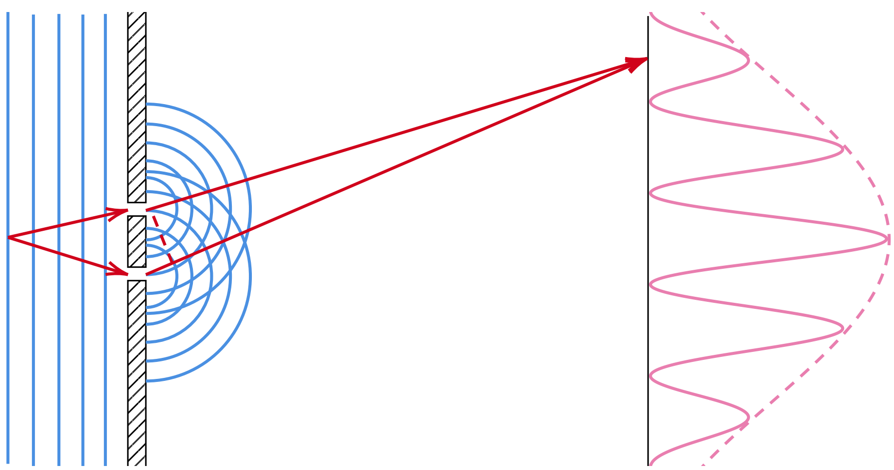

Figure 1. Thomas Young's double-slit experiment with light. The pink solid curve represents the double-slit interference fringes, while the pink dashed curve represents the single-slit envelope.

This is not what one would expect if light were a stream of little ball-like particles. If the intensity of the light when either slit is opened has some value I, then the bright fringe at the center has an intensity of $4I$ instead of $2I$. What is even more bizarre from the particle point-of-view is that combining light from two sources can also *decrease* the brightness. Spots that are bright when the barrier only has one slit (the envelope in Fig. 1), can become dark when there are multiple slits.

Indeed, this behavior is explained by the fact that the intensity is proportional to the modulus square of the wave amplitude $I \propto |\phi|^2 = |\phi_1 + \phi_2|^2$, where each partial wave acquires a complex phase as it travels. So not only that the two partial waves can reinforce each other when they are "in phase"—when a crest of one partial wave coincides with a crest of the other—in which case we say that the waves *constructively interfere*, they can also cancel each other out when they are "out of phase", creating dark fringes by *destructive interference*.

Then quantum mechanics enters the story. It was discovered that the behavior of a closed system[1] such as a neutron in a vacuum chamber, can be predicted by postulating the existence of a complex *probability amplitude* ψ associated to each *event* or *process*, the modulus square of which expresses the probability of that event or process happening $p = |\psi|^2$.

[1] *The meaning of "closed" will be clarified at the end of this section.*

Let us imagine performing the double-slit experiment using neutrons. Consider the probability p that the particle will be found at some point $\mathbf{r}$ on the screen. Suppose that ψ_1 (resp. ψ_2) is the probability amplitude in the case that only slit #1 (resp. #2) is opened. In ordinary probability theory, the probability that at least one of multiple mutually exclusive events happens is the sum of the probability of said events. Thus, if both slits are opened, we would predict that the probability density of a neutron that has passed through either one of the slits is

$$p = p_1 + p_2 \equiv |\psi_1|^2 + |\psi_2|^2. \tag{1.1}$$

Experimentally, however, we find that

$$\begin{aligned} p &= |\psi_1 + \psi_2|^2 && (1.2)\\ &= (\psi_1^* + \psi_2^*)(\psi_1 + \psi_2) && (1.3)\\ &= |\psi_1|^2 + |\psi_2|^2 + \underbrace{2\mathrm{Re}(\psi_1^*\psi_2)}_{\text{Quantum interference}} \neq |\psi_1|^2 + |\psi_2|^2 && (1.4) \end{aligned}$$

In other words, the probability amplitudes *interfere* to give the final probability for the particle to be detected at a particular location on the screen. Note that this holds true even if the particles are fired one at a time at the barrier at a time. The interference is not between the probability amplitudes of different particles. Colloquially, we could say that the single-particle probability amplitude *interferes with itself*.

There is another twist to the story. If we now try to find out which slit the particle went through, for example by shining light and placing a little camera behind the slits, we would indeed find that the particle passed though slit #1 (resp. #2) with probability $|\psi_1|^2$ (resp. $|\psi_2|^2$), but *no interference fringe would be developed on the screen*. The rule to manipulate probabilities returns to the ordinary one:

$$p = |\psi_1|^2 + |\psi_2|^2. \tag{1.5}$$

The act of observing which slit the particle passes through destroys the interference pattern (Fig. 2)[2].

[2] *It is important to note that any process that can distinguish the two alternatives counts as an "observation". The process can even be automated and computerized; no conscious observer needs to be present in a quantum measurement.*

The Mach-Zehnder interferometer provides another model that demonstrates the interference phenomenon, while being easier to analyzed than the double-slit experiment. Consider a *beamsplitter*, a device that takes a beam of light and divides it into two beams of lower intensity. Suppose that we have a

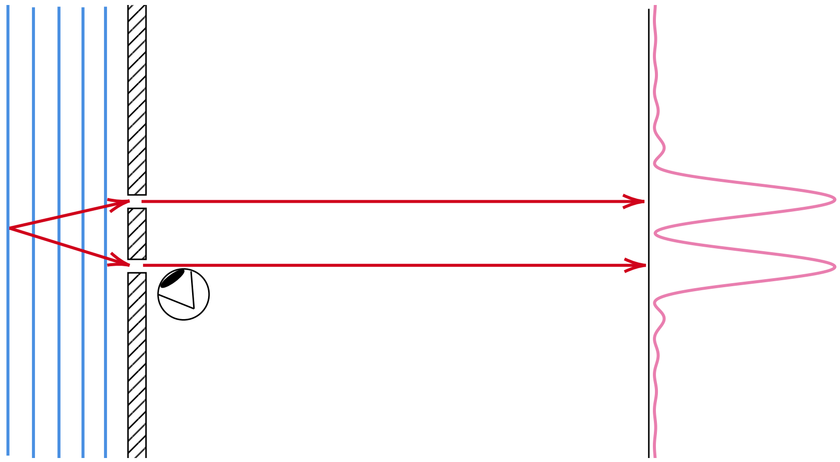

Figure 2. The neutron double-slit experiment with a camera placed next to the slits.

source that generates a monochromatic beam of light that is so attenuated that no more than one photon comes out at the same time[3]. If we send the beam through a 50-50 beamsplitter and place a detector at each of the output port (Fig. 3), there will be 50% chance to observe a click at the upper detector and a 50% chance to observe a click at the lower detector, but never both at the same time. Typically, a beamsplitter is made of a half-silvered mirror, which gives it

[3] *The electromagnetic wave cannot be identified as the wave function of a photon in the strict sense, because a photon cannot be localized and, consequently, there can be no spatial wave function of a photon. Though one can still talk about detecting a photon at some location.*

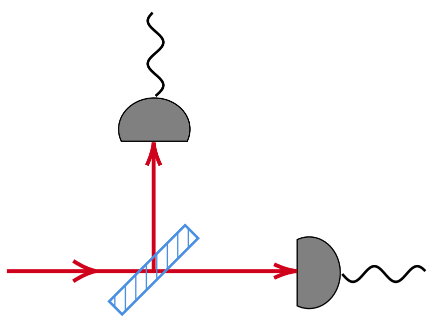

Figure 3. Light passing through a beamsplitter.

a special characteristic that, if a beam is incident on one side of the mirror, it splits into two equal-intensity beams with the same phase, but if the beam is incident on the other side, the reflected beam acquires a π-phase shift. (This can be derived from classical electromagnetism.)

A key to an interferometric experiment is that a wave is split into two parts, which are then recombined, possibly with a phase difference acquired along the two paths[4]. We can perform the same kind of experiments using beamsplitters and photodetectors, where the beamsplitters not only plays the role of the barrier in the double-slit experiment, but is also responsible for refocusing the two beams before they reach the photodetectors (4). This setup is called the *Mach-Zehnder interferometer*. An arbitrary phase difference between the two paths can be introduced by adjusting the relative length of the two arms. But for now let

[4] *The Stern-Gerlach spin recombination experiment follows the same principle.*

us consider the case in which the length of the two arms are balanced.

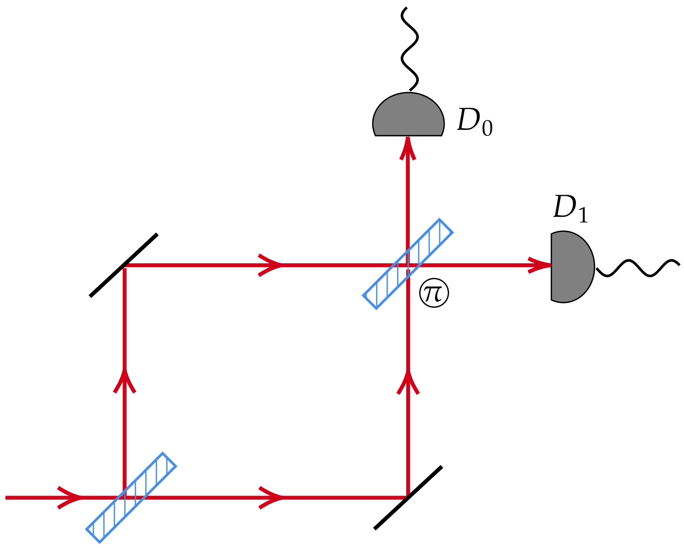

Figure 4. The balanced Mach-Zehnder interferometer. π indicates the side from which the reflected beam acquires a π-phase shift.

Classically, each beamsplitter acts like a perfectly random coin flip. So one might expect that the probability p_0 that detector D_0 registers a click is

$$p_0 = \overbrace{\frac{1}{2} \times \frac{1}{2}}^{\substack{\text{The probability that} \\ \text{the photon reaches } D_0 \\ \text{from the upper arm}}} + \overbrace{\frac{1}{2} \times \frac{1}{2}}^{\substack{\text{The probability that} \\ \text{the photon reaches } D_0 \\ \text{from the lower arm}}} = \frac{1}{2}. \tag{1.6}$$

Ditto for the probability p_1 that detector D_1 clicks.

Quantum mechanically, the contributions to the probability of a click at detector D_0 come from the reflection of the photon that traveled along the upper arm, associated to the amplitude $\psi_{U,R} = 1/\sqrt{2} \times 1/\sqrt{2} = 1/2$, [5] and the transmission of the photon that traveled along the lower arm, associated to the amplitude $\psi_{L,T} = 1/\sqrt{2} \times 1/\sqrt{2} = 1/2$. So the interference term is positive

[5] *The $\sqrt{2}$'s come from the normalization $|\psi_U|^2 + |\psi_L|^2 = 1$ and $|\psi_R|^2 + |\psi_T|^2$ at each beamsplitter.*

$$2\text{Re}(\psi^*_{U,R}\psi_{L,T}) = \frac{1}{2}, \tag{1.7}$$

and a click at detector D_0 is *certain* to happen. In contrast, the contributions to a click at detector D_1,

$$\psi_{U,T} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}, \tag{1.8}$$

$$\psi_{L,R} = \frac{1}{\sqrt{2}} \times \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}, \tag{1.9}$$

destructively interfere

$$2\text{Re}(\psi^*_{U,T}\psi_{L,R}) = -\frac{1}{2}, \tag{1.10}$$

and *no photon* is incident on detector D_1.

SECTION 1.2
Feynman rules

At the beginning of his lectures in quantum mechanics, Richard Feynman quickly gets to the double-slit interference phenomenon after the following remark [1].

> We choose to examine a phenomenon which is impossible, *absolutely* impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the *only* mystery.

Feynman's claim that the phenomena is impossible to explain classically can be disputed [2, 3]. Nevertheless, it is still true that this phenomenon illustrates the rules at the heart of quantum mechanics, sometimes called the *Feynman rules* (for probability amplitudes), which we now discuss in greater generality.

Suppose that there are N different processes that lead to the same final event, and associated to each process is a probability amplitude ψ_j, $j = 1, 2, \ldots, N$. How we compute the probability of the final event depends on whether the physical arrangement allows the processes to be distinguished.

1. If the N processes are **indistinguishable**, we sum up the amplitudes before taking the absolute square.

$$p = \left| \sum_{j=1}^{N} \psi_j \right|^2 \tag{1.11}$$

2. If the N processes are **distinguishable**, we take the absolute square of each amplitude before summing them up.

$$p = \sum_{j=1}^{N} \left| \psi_j \right|^2 \tag{1.12}$$

In the double-slit experiment, the two alternatives of a neutron passing through slit #1 or #2 are clearly distinct physical processes, but if we do not interact with the neutron at the slits, we have no way to know, even in principle, which slit the neutron has passed through. In this case, the two alternatives are said to be indistinguishable. The same reasoning applies in the case of the balanced Mach-Zehnder interferemoeter.

Said another way, processes are indistinguishable if there is no information stored anywhere in the universe that can tell them apart. With this understanding in mind, we can now clarify what we actually meant when we said that quantum mechanics describes the behavior of a closed system. "Closed" here does not mean the absence of an energy or mass exchange with the surroundings; it means the absence of *communication* with the outside world. Thus, it is more accurate to say that quantum mechanics apply to systems that are *informationally isolated*. We are not going to discuss about quantum information in this course, but we can already see here that the notion of information is already baked into the formalism of quantum mechanics at a fundamental level. In my opinion, Rolf Landauer's slogan "information is physical" is at its most profound in the quantum world.

SECTION 1.3
Quantum reality

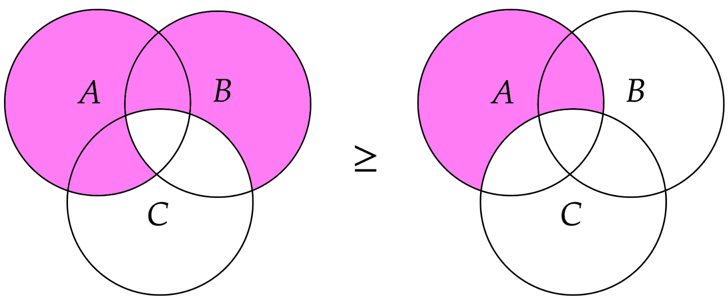

Figure 5. The Venn diagram showing that $\Pr(A,\overline{B}) + \Pr(B,\overline{C}) \geq \Pr(A,\overline{C})$.

Suppose that there are three binary random variables A, B, and C. Consider the inequality

$$\Pr(A,\overline{B}) + \Pr(B,\overline{C}) \geq \Pr(A,\overline{C}). \tag{1.13}$$

Why should we care about this inequality? Well, I will tell you in a minute. But first let us verify that it is true. The event $A \wedge \overline{B}$ can be broken up into two cases: either $A \wedge \overline{B} \wedge C$ or $A \wedge \overline{B} \wedge \overline{C}$ is true. Ditto for the event $B \wedge \overline{C}$ and $A \wedge \overline{C}$. So from (1.13) we have that

$$\Pr(A,\overline{B},C) + \cancel{\Pr(A,\overline{B},\overline{C})} + \cancel{\Pr(A,B,\overline{C})} + \Pr(\overline{A},B,\overline{C}) \geq \cancel{\Pr(A,B,\overline{C})} + \cancel{\Pr(A,\overline{B},\overline{C})} \tag{1.14}$$

$$\Pr(A,\overline{B},C) + \Pr(\overline{A},B,\overline{C}) \geq 0, \tag{1.15}$$

which is a truism. This inequality is simply a product of our common sense that probabilities cannot be negative.

However, John Stuart Bell pointed out that if we take these random variables to be the outcomes of some appropriate spin measurements of distanced quantum particles, this inequality would be violated. Numerous tests of (a variant of) this *Bell's inequality* have been performed, and the resulting violation of Bell's inequality, in agreement with the prediction of quantum mechanics, have been confirmed to an excellent degree.

The conclusion that one should draw from this, as it turns out, is not so much that probabilities can be negative; they can't. The crux of the issue is that, in the quantum world, there are good reasons to believe that there are no pre-existing values of A, B, and C waiting to be revealed by a measurement. Instead the act of a measurement itself participates in *creating* a value of the physical property. This worldview disturbed Einstein greatly, as he asked Abraham Pais "Do you really believe the moon is not there when you are not looking at it?" Today, for better or worse, evidences have forced us to accept this quantum reality.

We will look at Bell's inequality and the quantum prediction in more details later in this course.

References

[1] Richard Feynman, Robert Leighton, and Matthew Sands, *The Feynman Lectures on Physics*, Vol. 3, Addison-Wesley, Reading, Mass. (1965).

[2] Joseph Renes, *Quantum Information Theory: Concepts and Methods*, De Gruyter Oldenbourg (2022).

[3] Lorenzo Catani, Matthew Leifer, David Schmid, and Robert Spekkens, *Why interference phenomena do not capture the essence of quantum theory*, arXiv:2111.13727