M520 Non-Relativistic Quantum Mechanics Ninnat Dangniam

Homework Assignment 6

DUE: Monday 3 October 2023 40 points

1. (10 points). C-T F_{VI} 5.

2. (10 points). C-T F_{VI} 8.

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3. Angular momentum of an isotropic oscillator (10 points).

The Hamiltonian of an isotropic, three-dimensional harmonic oscillator is given by

$$\widehat{H}(t) = \hbar\omega\left(\hat{a}_x^\dagger a_x + \hat{a}_y^\dagger a_y + \hat{a}_z^\dagger a_z + \frac{3}{2}\right) = \hbar\omega\left(\hat{a}_R^\dagger a_R + \hat{a}_L^\dagger a_L + \hat{a}_z^\dagger a_z + \frac{3}{2}\right), \tag{1}$$

where $\hat{a}_x$, $\hat{a}_y$, and $\hat{a}_z$ are annihilation operators for linear oscillators along the three Cartesian axes and

$$\hat{a}_R = \frac{\hat{a}_x - i\hat{a}_y}{\sqrt{2}}, \qquad \hat{a}_L = \frac{\hat{a}_x + i\hat{a}_y}{\sqrt{2}}, \tag{2}$$

are annihilation operators for right- and left-circular oscillators. The energy eigenstates are

$$|\chi_{n_R,n_L,n_z}\rangle = \frac{(\hat{a}_R^\dagger)^{n_R}(\hat{a}_L^\dagger)^{n_L}(\hat{a}_z^\dagger)^{n_z}}{\sqrt{n_R!n_L!n_z!}}\,|\chi_{0,0,0}\rangle\,. \tag{3}$$

One can show that the three angular momentum operators can be written in terms of the creation and annihilation operators as follow.

$$\hat{L}_x = i\hbar(\hat{a}_z^\dagger\hat{a}_y - \hat{a}_y^\dagger\hat{a}_z), \tag{4}$$
$$\hat{L}_y = i\hbar(\hat{a}_x^\dagger\hat{a}_z - \hat{a}_z^\dagger\hat{a}_x), \tag{5}$$
$$\hat{L}_z = i\hbar(\hat{a}_y^\dagger\hat{a}_x - \hat{a}_x^\dagger\hat{a}_y), \tag{6}$$

or, written compactly,

$$\hat{L}_j = \epsilon_{jkl}\hat{x}_k\hat{p}_l = i\hbar\epsilon_{jkl}\hat{a}_l^\dagger\hat{a}_k. \tag{7}$$

The raising and lowering operators can also be written in the form

$$\hat{L}_+ = \sqrt{2}\hbar(\hat{a}_z^\dagger\hat{a}_L - \hat{a}_R^\dagger\hat{a}_z), \tag{8}$$
$$\hat{L}_- = \sqrt{2}\hbar(\hat{a}_L^\dagger\hat{a}_z - \hat{a}_z^\dagger\hat{a}_R). \tag{9}$$

Consider, in this problem, putting two quanta of energy into the oscillators. That is, we are considering the six-dimensional subspace $\mathcal{H}_2$ spanned by the energy eigenstates with energy $7\hbar\omega/2$.

(a) Show that the energy eigenstates $|\chi_{n_R,n_L,n_z}\rangle$ in $\mathcal{H}_2$ are eigenstates of $\hat{L}_z$ and find their eigenvalues.

(b) In $\mathcal{H}_2$, find eigenstates and the corresponding eigenvalues of $\hat{\vec{L}}^2$ and $\hat{L}_z$ in terms of the states $|\chi_{n_R,n_L,n_z}\rangle$. Explain the physical meaning of the eigenvalues you assign to $\hat{\vec{L}}^2$.

4. Addition of three spin-1/2 particles (10 points). C-T G_X 5.

Hint. When adding an integral angular momentum l and a spin-1/2, you may use the formula

$$\left|L = l \pm \frac{1}{2}, M\right\rangle = \frac{1}{\sqrt{2l+1}}\left[\pm\sqrt{l \pm M + \frac{1}{2}}\left|l, \frac{1}{2}; M - \frac{1}{2}, +\right\rangle + \sqrt{l \mp M + \frac{1}{2}}\left|l, \frac{1}{2}; M + \frac{1}{2}, -\right\rangle\right],$$

from Complement A_X in C-T.