M520 Non-Relativistic Quantum Mechanics Ninnat Dangniam

Homework Assignment 4

DUE: Monday 21 August 2023 50+25 points

Potentially useful identities:

Simplified BCH identity: $e^{A+B} = e^A e^B e^{-[A,B]/2}$ if both A and B commute with $[A,B]$

Braiding identity: $e^B A e^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \frac{1}{3!}[B,[B,[B,A]]] + \cdots$

$$\cosh r = \sum_{\text{even}\, k\geq 0} \frac{r^k}{k!} = \frac{e^r + e^{-r}}{2}, \qquad \sinh r = \sum_{\text{odd}\, k\geq 1} \frac{r^k}{k!} = \frac{e^r - e^{-r}}{2}$$

1. Nuclear Magnetic Resonance (10+10 points).

Consider a spin-1/2 particle in a constant external field $\mathbf{B} = B_0 \hat{\mathbf{e}}_\mathbf{z}$ under the influence of some time-dependent Hamiltonian $\widehat{W}(t)$, leading to the total Hamiltonian

$$\widehat{H}(t) = \widehat{H}_0 + \widehat{W}(t); \tag{1}$$

$\widehat{H}_0 = -\boldsymbol{\mu}\cdot\mathbf{B} = \gamma B_0 \hat{S}_z = -\hbar\Omega\,\hat{\sigma}_z/2$, where $\Omega = \gamma B_0$ is the Larmor frequency. In order to study the dynamics governed by the time-dependent part, it is useful to move into the rotating frame of reference in which the spin no longer precesses about the z axis. Let $\widehat{U}_0 = \exp(i\Omega\hat{\sigma}_z t/2)$ be the free time-evolution operator, and define

$$|\psi_I(t)\rangle := \widehat{U}_0^\dagger(t,0)\,|\psi(t)\rangle\,, \qquad \widehat{H}_I(t) := \widehat{U}_0^\dagger(t,0)\widehat{W}\,\widehat{U}_0(t,0). \tag{2}$$

The subscript I indicates that we are in the *interaction picture* (also known as the *Dirac picture*) in which both states and operators evolve in time.

(a) Show that the Schrödinger equation in the interaction picture takes on the familiar form

$$i\hbar\frac{d}{dt}\,|\psi_I(t)\rangle = \widehat{H}_I(t)\,|\psi_I(t)\rangle\,. \tag{3}$$

(b) Show that

$$\begin{aligned}
\widehat{U}_0^\dagger\hat{\sigma}_z\widehat{U}_0 &= \hat{\sigma}_z,\\
\widehat{U}_0^\dagger\hat{\sigma}_x\widehat{U}_0 &= \cos(\Omega t)\hat{\sigma}_x + \sin(\Omega t)\hat{\sigma}_y,\\
\widehat{U}_0^\dagger\hat{\sigma}_y\widehat{U}_0 &= \cos(\Omega t)\hat{\sigma}_y - \sin(\Omega t)\hat{\sigma}_x.
\end{aligned} \tag{4}$$

Since any arbitrary $\widehat{W}(t)$ can be decomposed in the Pauli basis

$$\widehat{W}(t) = A_0\hat{\mathbb{1}} + A_x\hat{\sigma}_x + A_y\hat{\sigma}_y + A_z\hat{\sigma}_z, \tag{5}$$

we have that

$$\widehat{H}_I(t) = A_0\hat{\mathbb{1}} + \left[A_x\cos(\Omega t) - A_y\sin(\Omega t)\right]\hat{\sigma}_x + \left[A_x\sin(\Omega t) + A_y\cos(\Omega t)\right]\hat{\sigma}_y + A_z\hat{\sigma}_z. \tag{6}$$

If the Schrödinger-picture Hamiltonian is constant or varies slowly with time, the $\hat{\sigma}_x$ and $\hat{\sigma}_y$ terms in (6) will oscillate rapidly around zero, and thus the effect of these terms will be canceled out on average, leaving the net effect in the rotating frame being just an extra precession about the z axis.

(c) Now consider a specific oscillatory Hamiltonian

$$\widehat{W}(t) = \cos\big(\Omega' t - \phi\big)\hat{\sigma}_x - \sin\big(\Omega' t - \phi\big)\hat{\sigma}_y, \tag{7}$$

which could have been turned on by shining a circularly-polarized light onto the spin-1/2 particle. Use (6) to write down the interaction-picture Hamiltonian $\widehat{H}_I(t)$. You can ignore the identity component $A_0\hat{\mathbb{1}}$, since it only contributes an overall global phase. Discuss what happens in the rotating frame if $\Omega - \Omega'$ is close to zero.

1(d) is an optional bonus problem.

(d) We can use this phenomenon of *nuclear magnetic resonance* to generate an arbitrary rotation of the quantum state of the spin-1/2 particle. Suppose that Ω' exactly matches the Larmor frequency: $\Omega - \Omega' = 0$. Describe how one would generate an arbitrary Z rotation in the rotating frame using a time-sequence of Hamiltonians of the form (7) with varying values of ϕ.

The following matrix forms may be helpful:

$$\widehat{R}_x(\theta) = e^{-i\theta\hat{\sigma}_x/2} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{8}$$

$$\widehat{R}_y(\theta) = e^{-i\theta\hat{\sigma}_y/2} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{9}$$

$$\widehat{R}_z(\theta) = e^{-i\theta\hat{\sigma}_z/2} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}. \tag{10}$$

Magnetic Resonance Imaging (MRI) is a technique based on NMR that allows us to distinguish various kinds of tissues in the human body. In a typical MRI machine, the magnetic field's magnitude B_0 is a few Tesla, and radio-frequency pulses (with frequencies of the order of one to several hundred megahertz) are used to match the Larmor frequency. The actual mechanism responsible for MRI signals that one reads out is *decoherence*: non-unitary time evolutions that take pure quantum states to mixed quantum states.

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2. Anisotropic oscillator (10 points).

Consider a three-dimensional harmonic potential,

$$V(x,y,z) = \frac{m\omega^2}{2}\left[\left(1 + \frac{2\lambda}{3}\right)(x^2+y^2) + \left(1 - \frac{4\lambda}{3}\right)z^2\right], \tag{11}$$

where $0 \leq \lambda \leq 3/4$.

(a) What are the eigenstates of the Hamiltonian and the corresponding energy eigenvalues?

(b) Compute and discuss the energy, the parity, and the degree of degeneracy of the ground state and the first two excited states as λ varies.

3. Squeezed states (15 points).

We have seen from the lectures that, for a state $|\psi\rangle$ to be a minimum-uncertainty state, $|\psi\rangle$ must satisfy the equation

$$\left(\widehat{\Delta X} + \lambda \widehat{\Delta P}\right)|\psi\rangle = 0, \tag{12}$$

for a purely imaginary λ, where

$$\widehat{X} := \sqrt{\frac{m\omega}{\hbar}}\hat{x}, \qquad\qquad \widehat{P} := \frac{\hat{p}}{\sqrt{\hbar m\omega}}, \tag{13}$$

are the dimensionless position and momentum operators respectively. In the lectures, we have noted that $|\lambda| = \Delta X/\Delta P$ signifies the trade-off in the variances of $\widehat{X}$ and $\widehat{P}$. By choosing $\lambda = i$, (12) becomes precisely the condition that $|\psi\rangle$ is an eigenstate of the annihilation operator a. This exercise is about what happens if we choose other values of λ.

For simplicity, let us omit the zero-point energy and write

$$\widehat{H} = \hbar\omega\hat{a}^\dagger\hat{a}, \qquad\qquad \widehat{U}(t,0) = \exp\left(-\frac{i\widehat{H}t}{\hbar}\right) = \exp\left(-i\omega\hat{a}^\dagger\hat{a}t\right). \tag{14}$$

Consider λ to be a decaying exponential function.

$$\left(\hat{a}\cosh r + \hat{a}^\dagger\sinh r\right)|\psi_r\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(e^r\hat{x} + ie^{-r}\frac{\hat{p}}{m\omega}\right)|\psi_r\rangle = 0, \tag{15}$$

where r is a real number called the *squeeze parameter*.

(a) Evaluate the expectation values $\langle\hat{x}\rangle, \langle\hat{p}\rangle, \langle\hat{x}^2\rangle, \langle\hat{p}^2\rangle$, and $\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle$ with respect to the state $|\psi_r\rangle$.

(b) Squeezed states can be obtained by applying the unitary *squeeze operator*

$$\hat{S}(r,\theta) := \exp\left[\frac{r}{2}\left(\hat{a}^2 e^{-2i\theta} - (\hat{a}^\dagger)^2 e^{2i\theta}\right)\right] \tag{16}$$

to the oscillator's ground states $|0\rangle$:

$$|\varphi_{r,\theta}\rangle := \hat{S}(r,\theta)\,|0\rangle\,. \tag{17}$$

Show that

$$\hat{S}(r,\theta)\,\hat{a}\,\hat{S}^\dagger(r,\theta) = \hat{a}\cosh r + \hat{a}^\dagger e^{2i\theta}\sinh r. \tag{18}$$

(c) Show that the state $|\psi_r\rangle$ can be taken to be a squeezed state $|\varphi_{r,0}\rangle$.

(d) Suppose that the initial state of the oscillator is $|\psi(0)\rangle = |\varphi_{r,0}\rangle$. Show that the state

$$|\psi(t)\rangle = \hat{U}(t,0)\,|\psi(0)\rangle\,, \tag{19}$$

at an arbitrary time t remains a squeezed state $\left|\varphi_{r(t),\theta(t)}\right\rangle$ and determine the degree of squeezing $r(t)$ and the squeezing angle $\theta(t)$ as functions of t. Show that at time $t = \pi/(2\omega)$, the sign of the squeeze parameter reverses: $r \mapsto -r$.

(e) Find the wave function $\psi_r(x) = \langle x|\psi_r\rangle$ up to an irrelevant global phase.

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4. Oscillator driven by a time-dependent force. (15+15 points)

Consider a harmonic oscillator acted on by a generalized force $f(t)$, leading to the Hamiltonian with an explicit time dependence

$$\widehat{H}(t) = \widehat{H}_0 + i\hbar\left[\hat{a}^\dagger f(t) - \hat{a} f^*(t)\right]. \tag{20}$$

Solving the operator Schrödinger equation

$$i\hbar\frac{d\widehat{U}(t,0)}{dt} = \widehat{H}(t)\widehat{U}(t,0), \tag{21}$$

yields the solution

$$\widehat{U}(t,0) = e^{i\delta(t)}\widehat{U}_0(t,0)D(\hat{a},\alpha(t)), \tag{22}$$

where $\alpha(t)$ and $\delta(t)$ are functions to be determined, $D(\hat{a},\alpha(t)) = \exp\left[\alpha(t)\hat{a}^\dagger - \alpha^*(t)\hat{a}\right]$ is the displacement operator, and $\widehat{U}_0(t,0) = \exp\left(-i\omega\hat{a}^\dagger\hat{a}t\right)$ is the free time-evolution operator.

(a) Show that $D(\hat{a},\alpha(t))$ satisfies the following differential equation:

$$\frac{dD(\hat{a},\alpha(t))}{dt} = \left[-\frac{1}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) + (\dot{\alpha}\hat{a}^\dagger - \dot{\alpha}^*\hat{a})\right] D(\hat{a},\alpha(t)). \tag{23}$$

Hint: Use the Baker-Campbell-Hausdorff identity to write the displacement operator in the normal order first, that is, putting $\hat{a}^\dagger$ to the left of $\hat{a}$, and then differentiate.

(b) Use the result of **(a)** to show that the Schrödinger equation implies two ODEs for $\alpha(t)$ and $\delta(t)$, the solutions of which are

$$\alpha(t) = \int_0^t dt' f(t')e^{i\omega t'}, \qquad \delta(t) = \frac{i}{2}\int_0^t dt'(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*). \tag{24}$$

(c) Suppose that the oscillator is initially in the ground state $|\psi(0)\rangle = |0\rangle$. Find the expectation values $\langle\hat{a}\rangle, \langle\hat{a}^\dagger\rangle, \langle\hat{a}^\dagger\hat{a}\rangle$, and $\langle\hat{a}^2\rangle$ in terms of $\alpha(t)$ and $\delta(t)$. Use these expectation values to calculate the expectation values and variances of $\hat{x}$ and $\hat{p}$.

(d) Derive the Heisenberg equations of motion for $\hat{a}, \hat{a}^\dagger, \hat{x}$ and $\hat{p}$.

4(e) and 4(f) are optional bonus problems.

(e) Solve the resulting Heisenberg equations of motion for $\hat{a}$ and $\hat{a}^\dagger$ with appropriate initial conditions at $t = 0$. You can employ any technique at your disposal (for example, Green's functions).

(f) Repeat **(c)** using the form of the Heisenberg operators obtained in **(e)**. That is, show that the expectation values calculated in the Schrödinger picture and the Heisenberg picture are the same.
