M520 Non-Relativistic Quantum Mechanics Ninnat Dangniam

Homework Assignment 1

DUE: Monday 10 July 2023 — 40 points

1. Estimation of atomic parameters (5 points).

Use the uncertainty relation to estimate the ground-state energy of a particle of mass m in a three-dimensional potential well

$$V(r) = -\frac{\alpha}{r^s},$$

where $\alpha, s > 0$ are real and positive, but otherwise arbitrary numbers.

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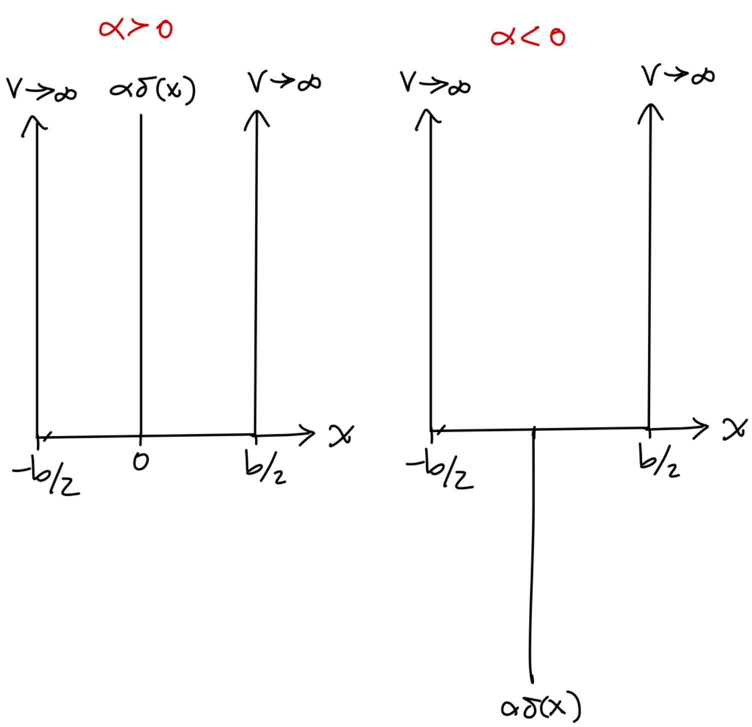
2. An infinite square well with a δ function (15 points).

A particle of mass m is confined in an infinite square well of width b, which has a δ function well *or* barrier in the middle.

$$V(x) = \begin{cases} \alpha\delta(x), & |x| < b/2, \\ \infty, & |x| > b/2, \end{cases}$$

with $\alpha < 0$ for a well and $\alpha > 0$ for a barrier. You don't need to normalize the wave functions in this problem.



(a) Find the energy of the **odd** bound states, and make a rough plot of the first odd wave function $\psi(x)$ for a typical value of α.

(b) For a **barrier** ($\alpha > 0$), derive a transcendental equation that determines the energies of the **even** bound states. Indicate how to obtain the energies of the even states by a graphical method. Discuss how the wave function and energy of the lowest even state change as α increases from 0 to ∞. (**Hint**: Assume a wave function of the form $\psi(x) = A\sin(k|x| - \varphi)$.)

(c) Repeat what you did for part **(b)** but for a **well** ($\alpha < 0$). That is, derive a transcendental equation that determines the energies of the **even** bound states. Indicate how to obtain the energies of the even states by a graphical method. You will have to consider carefully what happens to the lowest-energy even state when its energy goes to zero. Discuss how the wave function and energy of the lowest two even states change as α decreases from 0 to $-\infty$.

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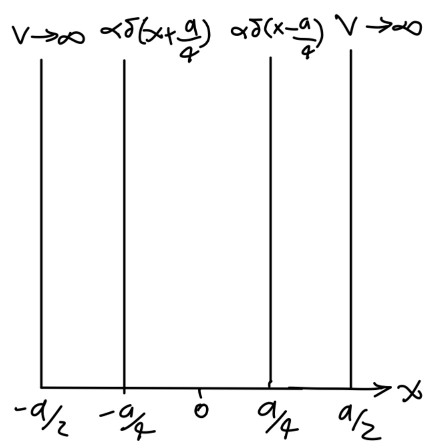
3. An infinite square well with two δ functions (10 points).

A particle of mass m is confined in an infinite square well of width a, which has δ function barriers with equal strength α at $x = \pm a/4$.

$$V(x) = \begin{cases} \alpha\delta(x + a/4) + \alpha\delta(x - a/4), & |x| < a/2, \\ \infty, & |x| > a/2, \end{cases}$$

with $\alpha \geq 0$. You don't need to normalize the wave functions in this problem.



(a). Without actually solving the problem, plot and discuss how the ground-state wave function changes as α increases from 0 to ∞.

(b). Find the ground-state wave function $\psi(x)$ and the ground energy E as functions of α. **Hint**: The ground state has an even wave function, which can be chosen to be

$$\psi(x) = \begin{cases} \cos(kx)m & |x| < a/4, \\ B\cos(k|x| - \phi), & a/4 < |x| < a/2. \end{cases}$$

Your answer should be in the form of a graph that allows you to determine k, ϕ and B as functions of α.

(c). Check that the ground-state wave function you found in part **(b)** has the behavior you guessed in part **(a)**.

4. Finite double wells (10 points).

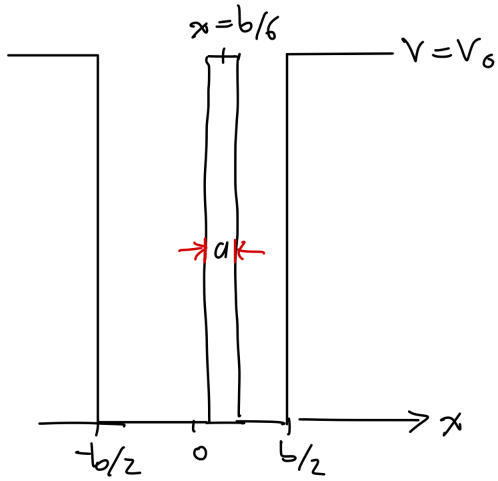
A particle of mass m experiences the following potential $V(x)$ shown below. The potential is a square well of depth V_0 and width b_0, with a barrier of the same height V_0 and width a centered at $x_0 = b/6$.

$$V(x) = \begin{cases} V_0, & x < -\frac{b}{2}, x > \frac{b}{2} \text{ and } x_0 - \frac{a}{2} < x < x_0 + \frac{a}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

Assume a very deep well:

$$\frac{\sqrt{2mV_0}}{\hbar} b \gg 1,$$

and that $a \leq b/2$.



(a). Estimate the number of bound states confined inside the wells.

(b). Assume that the barrier is very narrow such that

$$\frac{2mV_0 ab}{\hbar} = \left(\frac{\sqrt{2mV_0}}{\hbar} a\right)\left(\frac{\sqrt{2mV_0}}{\hbar} b\right) \ll 1.$$

Estimate the energies of the lowest five bound states. Explain the logic of your approximation.

(c). Assuming that $a = b/3$, estimate the energies of the lowest five bound states. Explain the logic of your approximation.