07.30.25

Last time

- Symmetries & conservation laws

... in classical mechanics

Every conserved quantity $\Longleftrightarrow$ Symmetry (Noether)

Canonical momentum $\quad p_j \equiv \dfrac{\partial L}{\partial \dot q_j} \qquad \dfrac{\partial p_j}{\partial t} = 0 \Leftrightarrow \dfrac{\partial}{\partial t}\left(\dfrac{\partial L}{\partial \dot q_j}\right) - \dfrac{\partial L}{\partial q_j} = 0$

$$\frac{\partial p_j}{\partial t} = 0 \Leftrightarrow \frac{\partial H}{\partial q_j} = 0$$

- $\dfrac{d\langle \hat A \rangle}{dt}$ or $\dfrac{d\hat A_H(t)}{dt} = \dfrac{1}{i\hbar}[\hat A, \hat H] + \dfrac{\partial \hat A}{\partial t} \quad \longleftrightarrow \quad \{A, H\}_{P.B.}$

Heisenberg E.O.M.

$\langle \psi(0)| \hat U^\dagger(t,0)\, \hat A\, \hat U(t,0) |\psi(0)\rangle$

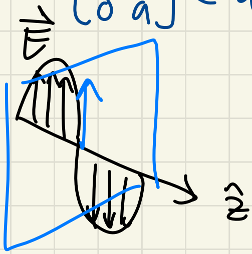
Demonstrate how the postulates/axioms of QM work in a simple toy system

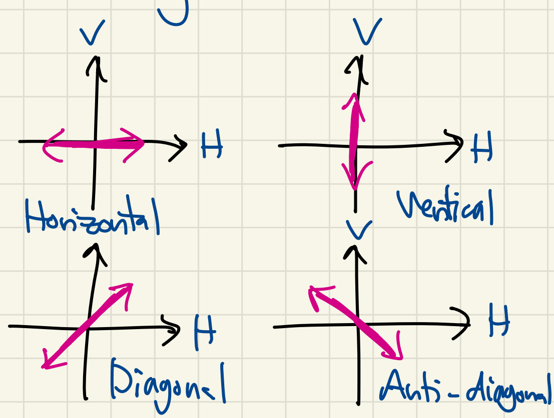
Two-level systems

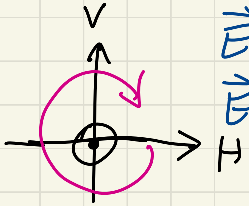
Two-level system/qubit has many peculiar/special feature that doesn't to generalize to higher dimension $\leftarrow$ Be careful!

Ex C.S.C.O. $\quad \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a\mathbb{1} \leftarrow$ Boring

① Polarization

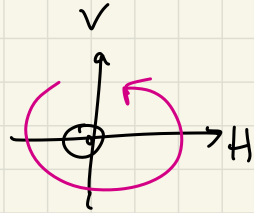




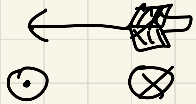


Left-circular

$$\vec{E}_x \propto \cos(kz-\omega t-\phi_x)\,\hat{e}_x$$
$$\vec{E}_y \propto \cos(kz-\omega t-\phi_y)\,\hat{e}_y$$
$$\delta = \phi_y - \phi_x$$

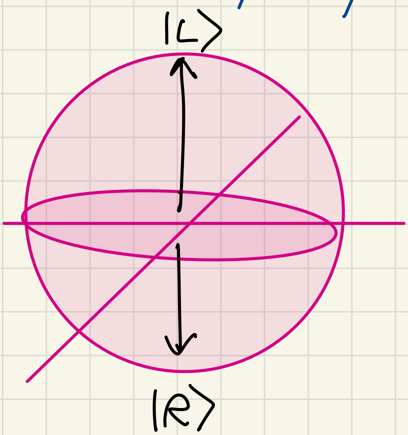


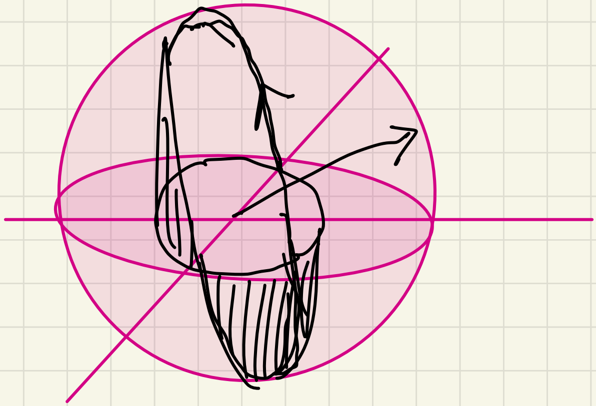
Right-circular



Polarization vector (2D complex) ⟷ Ellipse ⟷ Poincaré sphere (3D real)

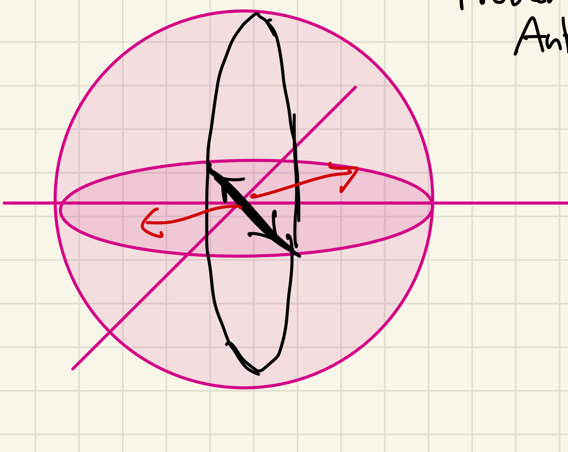
$$\tilde{E}_x\hat{e}_x + \tilde{E}_y e^{i\delta}\hat{e}_y$$

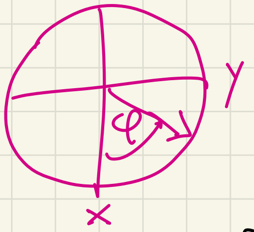


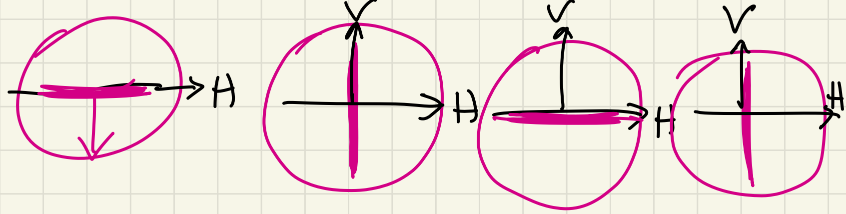


Problem !?

Antipodal points are associated to the same polarization states

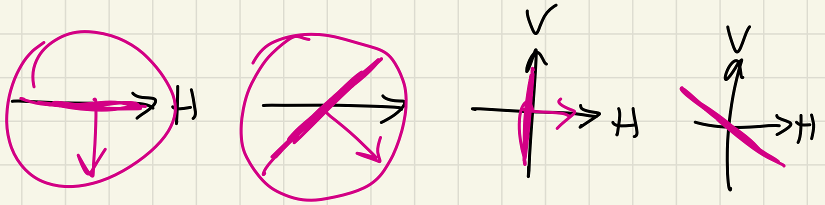






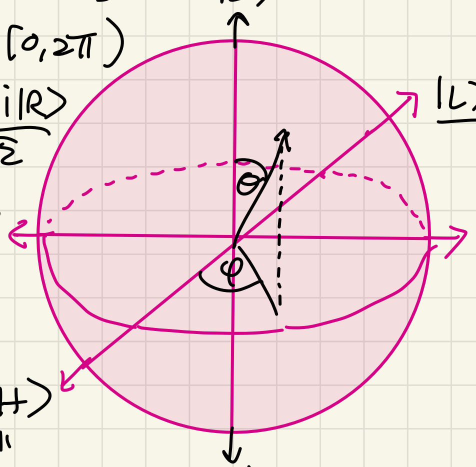
$[0, \pi]$ $\varphi = 0$ $\varphi = \frac{\pi}{2}$ $\varphi = \pi$ $\varphi = 3\pi/2$

$\delta \to \frac{\delta}{2}$ $[0, \pi]$



$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi)$



$|L\rangle$

$|R\rangle$

$\frac{|L\rangle - |R\rangle}{\sqrt{2}} = |V\rangle$

$|D\rangle = \frac{|L\rangle + i|R\rangle}{\sqrt{2}}$

$\frac{|L\rangle - i|R\rangle}{\sqrt{2}} = |A\rangle$

$|H\rangle = \frac{|L\rangle + |R\rangle}{\sqrt{2}}$

$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|L\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|R\rangle$

Antipodal points are orthogonal (in Hilbert space)

$\langle L|R\rangle = 0$

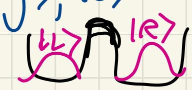
Two different geometries!

ONB

② Spin-1/2 $|\uparrow\rangle, |\downarrow\rangle$

$|+\rangle, |-\rangle \equiv |\pm_z\rangle$, $|\pm_x\rangle$, $|\pm_y\rangle$, $|0\rangle, |1\rangle$ (QI), $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$ (QI)

③ Atomic energy levels $|g\rangle, |e\rangle$

④ Double-well potential 

* No tunneling

Test our axioms

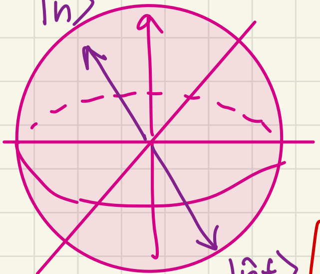
① States $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|L\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|R\rangle$

Normalized ket

② Measurements

Outcome $\longleftrightarrow$ Orthogonal Projector $\{|\hat{n}\rangle, |\hat{n}^\perp\rangle\}$

$|\hat{n}\rangle$



$$|\hat{n}(\theta,\varphi)\rangle = \cos\left(\frac{\theta}{2}\right)|L\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|R\rangle$$

$$\alpha|L\rangle + \beta|R\rangle \overset{\text{orthogonal}}{\longleftrightarrow} \beta^*|L\rangle - \alpha^*|R\rangle$$

$|\hat{n}^\perp\rangle$ $\begin{bmatrix}\beta^* \\ -\alpha^*\end{bmatrix}^\dagger \neq \begin{bmatrix}\beta^* \\ -\alpha^*\end{bmatrix} \cdot \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = [\beta \ \ -\alpha]\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \beta\alpha - \alpha\beta = 0$

$\begin{bmatrix}\beta^* \\ -\alpha^*\end{bmatrix}^\dagger = [(\beta^*)^* \ \ (-\alpha)^{**}]$

$[\beta'' \ \ -\alpha]$

$$|\hat{n}^\perp(\theta,\varphi)\rangle = \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}|L\rangle - \cos\left(\frac{\theta}{2}\right)|R\rangle$$

$$\boxed{\begin{aligned}\theta &\mapsto \pi - \theta \\ \varphi &\mapsto \pi + \varphi\end{aligned}}$$

Measurement $\longleftrightarrow$ $|\hat{n}\rangle\langle\hat{n}|$, $|\hat{n}^\perp\rangle\langle\hat{n}^\perp|$

$$|\hat{n}\rangle\langle\hat{n}| = \frac{1}{2}\begin{pmatrix}1+\cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & 1-\cos\theta\end{pmatrix} \longleftarrow \begin{aligned} x &= \sin\theta\cos\varphi \\ y &= \sin\theta\sin\varphi \\ z &= \cos\theta \end{aligned}$$

$$= \frac{1}{2}\begin{pmatrix}1+z & x-iy \\ x+iy & 1-z\end{pmatrix}$$

$$= \frac{1}{2}\Big[\mathbb{1} + \overbrace{z\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} + x\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} + y\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}}^{\text{Pauli matrices}}\Big]$$

Shorthand

$$\longrightarrow \frac{\mathbb{1} + \hat{n}\cdot\vec{\sigma}}{2}$$

$$\vec{\sigma} = \begin{pmatrix}\hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z\end{pmatrix} \qquad \hat{n} = \begin{pmatrix}x \\ y \\ z\end{pmatrix}$$

$$(|\phi\rangle, |\psi\rangle) = \langle\phi|\psi\rangle$$

$|-\hat{n}\rangle\langle-\hat{n}|$
$$|\hat{n}^{\perp}\rangle\langle\hat{n}^{\perp}| = \frac{\mathbb{1} - \hat{n}\cdot\vec{\sigma}}{2}$$

$|a_j\rangle$

$$\hat{A} \overset{?}{\doteq} \sum_j a_j |a_j\rangle\langle a_j|$$

Spin obrervable in any direction $\hat{n}$

$$\hat{\sigma}_n = |\hat{n}\rangle\langle\hat{n}| - |-\hat{n}\rangle\langle-\hat{n}| = \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & \cos\theta \end{bmatrix} = \hat{n}\cdot\vec{\sigma}$$

Uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$$

$$\hat{X} \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Focus on $\hat{X}, \hat{Y}, \hat{Z}$ Algebra

$$\hat{Y} \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{Z} \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties

① $\hat{\sigma}_j^{\dagger} = \hat{\sigma}_j$

② $\hat{\sigma}_j^2 = \hat{\mathbb{1}} \Rightarrow \hat{\sigma}_j = \hat{\sigma}_j^{-1} \Rightarrow \hat{\sigma}_j$ is unitary

generalize to higher dim

→ Hermitian ⇒ Gell-mann

→ Unitary ⇒ Displacement

③ $\hat{\sigma}_1\hat{\sigma}_2 = i\hat{\sigma}_3$

$\hat{\sigma}_2\hat{\sigma}_3 = i\hat{\sigma}_1$

$\hat{\sigma}_3\hat{\sigma}_1 = i\hat{\sigma}_2 \qquad (\hat{\sigma}_3\hat{\sigma}_1)^{\dagger} = (i\hat{\sigma}_2)^{\dagger}$

$\hat{\sigma}_1\hat{\sigma}_3 = -i\hat{\sigma}_2$

Levi-Civita

Einstein summation convention

$$\hat{\sigma}_j\hat{\sigma}_k = \delta_{jk}\hat{\mathbb{1}} + i\sum_{l}\epsilon_{jkl}\sigma_l$$

$$\epsilon_{jkl} = \begin{cases} +1 & (jkl) = (123) \text{ \& cyclic} \\ -1 & (jkl) = (213) \\ 0 & \end{cases}$$

$$[\hat{\sigma}_j, \hat{\sigma}_k] = 2i\epsilon_{jkl}\hat{\sigma}_l$$

$$\{\hat{\sigma}_j, \hat{\sigma}_k\} \equiv \hat{\sigma}_j\hat{\sigma}_k + \hat{\sigma}_k\hat{\sigma}_j = 2\delta_{jk}\hat{\mathbb{1}}$$

Uncertainty relation

$$\Delta\sigma_j\Delta\sigma_k \geq \frac{1}{2}|\langle[\hat{\sigma}_j,\hat{\sigma}_k]\rangle| = |\langle\epsilon_{jkl}\sigma_l\rangle|$$

$$\Delta\hat{X}\Delta\hat{Y} \geq |\langle\hat{Z}\rangle| \leftarrow \text{state-dependent}$$

Expectation values

$$\langle \hat{n} | \hat{\sigma}_j | \hat{n} \rangle = \mathrm{Tr}\left[|\hat{n}\rangle\langle\hat{n}| \hat{\sigma}_j \right] = \mathrm{Tr}\left[\frac{\hat{\mathbb{1}} + \hat{n}\cdot\vec{\hat{\sigma}}}{2} \hat{\sigma}_j \right]$$

$$= \underbrace{\cancel{\tfrac{1}{2}\mathrm{Tr}(\hat{\mathbb{1}}\hat{\sigma}_j)}}_{0} + \frac{1}{2}\mathrm{Tr}\left[\sum_k n_k \hat{\sigma}_k \hat{\sigma}_j\right]$$

④ $\mathrm{Tr}\,\sigma_j = 0$

$$= \frac{1}{2}\sum_k n_k \, \mathrm{Tr}(\hat{\sigma}_k \hat{\sigma}_j) = n_j$$

Nonzero only if $j=k$

Implication

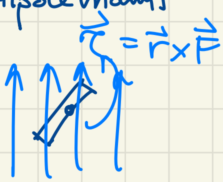
$$\begin{pmatrix} \langle \hat{\sigma}_x \rangle \\ \langle \hat{\sigma}_y \rangle \\ \langle \hat{\sigma}_z \rangle \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \hat{n}$$ Quantum Tomography

$$\hat{S} = \frac{\hbar}{2}\vec{\hat{\sigma}}$$

Dynamics $\hat{H} = \vec{\mu}\cdot\vec{B} = -\gamma\overbrace{\hbar\frac{\vec{\hat{\sigma}}}{2}}\cdot\vec{B} \Rightarrow \hat{U}(t,0) = e^{-i\hat{H}t/\hbar}$

$$= e^{i\gamma\vec{B}\cdot\vec{\hat{\sigma}}/2}$$

Magnetic dipole moment



HW C-T

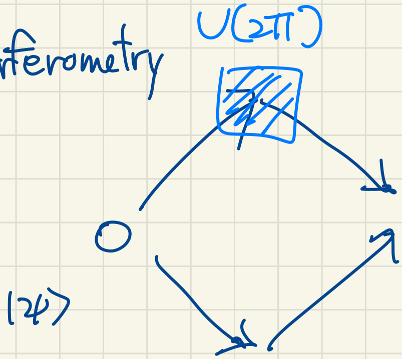
$$\boxed{\hat{U}(t,0) = \cos\left(\frac{\theta}{2}\right)\hat{\mathbb{1}} - i\sin\left(\frac{\theta}{2}\right)\hat{n}\cdot\vec{\hat{\sigma}}}$$

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle$$

$$|\psi(t)\rangle = \underbrace{e^{-iE_1 t/\hbar}|e_1\rangle + e^{-iE_2 t/\hbar}|e_2\rangle}_{\text{stationary states}}$$ Elementary way to compute the state

Interesting feature $\theta = 2\pi \Rightarrow \hat{U} = \underbrace{\cancel{\cos\pi}}_{-1}\hat{\mathbb{1}} - i\underbrace{\cancel{\sin\pi}}_{0}\,\hat{n}\cdot\vec{\hat{\sigma}} = -\mathbb{1}$

Interferometry



$|\uparrow\rangle \mapsto -|\uparrow\rangle$

$$\frac{|U\rangle + |L\rangle}{\sqrt{2}} \rightarrow \frac{-|U\rangle + |L\rangle}{\sqrt{2}}$$