M520 Non-Relativistic Quantum Mechanics	Ninnat Dangniam

Homework Assignment 2

DUE: Monday 24 July 2023 — 40 points

1. (5 points). Cohen-Tannoudji (C-T) H_{II} 3.

2. (5 points). C-T H_{II} 5.

3. (10 points). C-T H_{II} 10.

---

4. CSCO (10 points).

Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle, |u_4\rangle\}$ be an orthonormal basis in the four-dimensional Hilbert space of some quantum system. Consider the following four observables:

$$\begin{aligned}
\hat{A} &= \frac{1}{2}(|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| - |u_3\rangle\langle u_3| - |u_4\rangle\langle u_4|), \\
\widehat{B} &= \frac{1}{2}(|u_1\rangle\langle u_1| - |u_2\rangle\langle u_2| + |u_3\rangle\langle u_3| - |u_4\rangle\langle u_4|), \\
\widehat{C} &= \hat{A} + \widehat{B} = |u_1\rangle\langle u_1| - |u_4\rangle\langle u_4|, \\
\widehat{D} &= |u_1\rangle\langle u_1| + |u_2\rangle\langle u_3| + |u_3\rangle\langle u_2| + |u_4\rangle\langle u_4|.
\end{aligned}$$

The state of the system at $t = 0$ is given by

$$|\psi(0)\rangle = \frac{1}{2}(|u_1\rangle + |u_2\rangle + |u_3\rangle + |u_4\rangle).$$

(a) For each of the four operators, give eigenvectors and the corresponding eigenvalues. Identity two complete sets of commuting observables that have different sets of eigenvectors.

(b) For each of the four operators, give the possible results of a measurement of the operator at $t = 0$ and the probabilities for the various results.

(c) Suppose that the Hamiltonian is $H = \hbar\omega\widehat{C}$. Calculate $|\psi(t)\rangle$. If a measurement of $\widehat{C}$ is made at time t, what are the possible results of the measurement and their probabilities? If a measurement of $\widehat{D}$ is made at time t, what are the possible results of the measurement and their probabilities?

(d) Suppose that the Hamiltonian is $H = \hbar\omega\widehat{A}$. Repeat what you did for part **(c)**. That is, calculate $|\psi(t)\rangle$. If a measurement of $\widehat{C}$ is made at time t, what are the possible results of the measurement and their probabilities? If a measurement of $\widehat{D}$ is made at time t, what are the possible results of the measurement and their probabilities?

---

5. Commutator algebra (10 points). Throughout this problem, I will omit the hat on the linear operators $\hat{A}$ and $\widehat{B}$.

(a) If A and B both commute with their commutator $[A, B]$, prove by induction that

$$\begin{aligned}
[A, B^n] &= nB^{n-1}[A, B], \\
[A^n, B] &= nA^{n-1}[A, B].
\end{aligned}$$

(b) Define the operator function $f(\lambda) := e^{\lambda A} B e^{-\lambda A}$ and write out its Taylor expansion. Set $\lambda = 1$ to show that

$$e^{A} B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots.$$

(c) If $[A, B] = \phi B$, where ϕ is some complex number, show that

$$e^{A} B e^{-A} = e^{\phi} B.$$

(d) If A and B both commute with their commutator $[A, B]$, prove the Baker-Campbell-Hausdorf (BCH) identity,

$$e^{A+B} = e^{-[A,B]/2} e^{A} e^{B}.$$

Hint: Define the operator function $f(\lambda) := e^{\lambda(A+B)} e^{-\lambda B} e^{-\lambda A}$, establish the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = -\lambda [A, B] f,$$

and integrate the equation.
