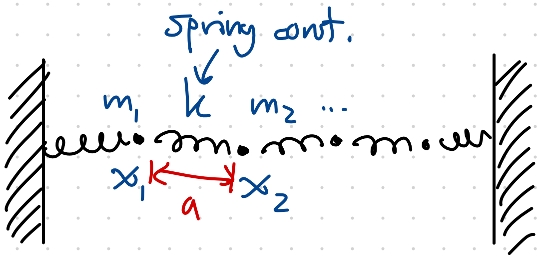
Field Quantization

Quantization of scalar fields



Consider a classical linear chain of N oscillators.

Go to the continuum limit $N \to \infty$

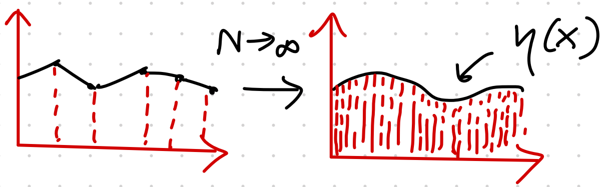
$a \to dx \quad \left(\sum a \to \int dx\right)$

Mass density $\mu = \frac{m}{a}$

Young modulus $Y = ka$

$$T = \sum_j \frac{1}{2} m \dot{x}_j^2 = \sum_j \frac{1}{2} a \frac{m}{a} \dot{x}_j^2 \to \int dx \frac{1}{2} \mu \left[\frac{\partial \eta(x,t)}{\partial t}\right]^2$$

Field variable $\eta(x,t)$



$$V = \sum_j \frac{1}{2} k (x_{j+1} - x_j)^2 = \sum_j \frac{1}{2} a (ka) \left(\frac{x_{j+1} - x_j}{a}\right)^2$$

$$\to \int dx \frac{1}{2} Y \left[\lim_{a \to 0} \frac{\eta(x_{j+1}) - \eta(x_j)}{a}\right]^2$$

$$= \int dx \frac{1}{2} Y \left[\frac{\partial \eta(x,t)}{\partial x}\right]^2$$

The equation of motion can be obtain, say, from the Lagrangian. But in continuum mechanics, we need Lagrangian density $\mathcal{L}$ and functional derivatives

$$\partial_t \underbrace{\left[\frac{\delta \mathcal{L}}{\delta(\partial_t \eta)}\right]}_{= \mu \frac{\partial \eta}{\partial t}} + \partial_x \underbrace{\left[\frac{\delta \mathcal{L}}{\delta(\partial_x \eta)}\right]}_{= -Y \frac{\partial \eta}{\partial x}} = - \underbrace{\frac{\delta \mathcal{L}}{\delta \eta}}_{= 0}$$

See Goldstein's classical mechanics, 3rd ed.

$$\boxed{\frac{\partial^2 \eta}{\partial t^2} - \underbrace{\frac{Y}{\mu}}_{v_p^2} \frac{\partial^2 \eta}{\partial x^2} = 0}$$ Wave equation

v_p^2 phase velocity

We can recover discrete degrees of freedom by confine the wave in a bounded region (putting in a boundary condition (B.C.)) and thinking about a discrete set of <u>modes</u>.

The choice of B.C. will not matter in the end once we extend the boundaries of the region to $\pm \infty$.

For normalization, will explain later

Normal mode expansion (Separation of variables)

$$\eta(x,t) = \sqrt{L} \sum_k q_k(t)\, u_k(x)$$

$$\xrightarrow[\text{the wave equation}]{\text{Plugging in}} \quad \frac{d^2 q_k}{dt^2} + \omega_k^2 q_k = 0, \quad \omega_k = v_k k$$

$$\sum_k u_k(x)\frac{d^2q}{dt^2} - \frac{Y}{\mu}\sum_k q_k(t)\frac{d^2u_k(x)}{dx^2} = 0$$

For each mode (labeled by k), the DE separates

$$\frac{1}{v_p^2}\frac{\ddot{q}_k}{q} = \frac{u''}{u} = -k^2 \leftarrow \text{Constant}$$

($\omega_k = v_k k$ Frequency of the oscillation)

$$\frac{d^2u_k}{dx^2} + k^2 u_k = 0 \qquad\qquad \frac{d^2q_k}{dt^2} + \omega_k^2 q_k = 0$$

Hard wall B.C.	Periodic B.C.
$u_k(x) = \sqrt{\frac{2}{L}}\sin(kx),\ k = \frac{n\pi}{L}$	$u_k(x) = \frac{e^{ikx}}{\sqrt{L}},\ k = \frac{2n\pi}{L}$

Both are complete, orthonormal sets

$$\int dx\, u_k^*(x)\, u_{k'}(x) = \delta_{kk'}$$

$$\sum_k u_k^*(x)\, u_k(x') = \delta(x-x')$$

These look very similar to what we encountered when solving the Schrödinger eq., but remember that the waves here are classical!
They are not quantum mechanical wave functions!
So when we eventually quantize these waves, it would not be a second quantization. There is only first quantization.

$$\underset{[L]}{\eta(x,t)} = \sqrt{L} \sum_k q_k(t)\, \underset{[L]^{-1/2}}{u_k(x)} \sim \frac{e^{ikx}}{\sqrt{L}}$$

We put the $\sqrt{L}$ there so that q_k has the same unit as $\eta(x,t)$ (the unit of length).

$$T = \int dx\, \frac{1}{2}\mu \left(\frac{\partial\eta}{\partial t}\right)^2 = \sum_{kk'} \frac{1}{2} \overbrace{\mu L}^{M}\, \dot q_k \dot q_{k'} \overbrace{\int dx\, u_k^*(x) u_{k'}(x')}^{\delta_{kk'}}$$

$$= \sum_k \frac{1}{2} M \dot q_k^2$$

$$V = \int dx\, \frac{1}{2} Y \left(\frac{\partial\eta}{\partial x}\right)^2 = \sum_{kk'} \frac{1}{2} Y L\, q_k q_{k'} \int dx\, (\partial_x u_k)^* (\partial_x u_{k'})$$

(Integration by parts)

$$= -\sum_{kk'} \frac{1}{2} YL\, q_k q_{k'} \int dx\, (\partial_x^2 u_k)^* u_{k'}$$

For the mode function u_k that we have, $\partial_x^2 u_k = -k^2 u_k$

$$= +\sum_{kk'} \frac{1}{2} YL\, k^2 q_k q_{k'} \int dx\, u_k^* u_{k'}$$

$$\omega_k = k v_k = k\sqrt{\frac{Y}{\mu}} = k\sqrt{\frac{YL}{M}} \Rightarrow k^2 = \frac{M}{YL}\omega_k^2$$

$$= \sum_k \frac{1}{2} M \omega_k^2 q_k^2$$

$$\therefore H = T + V = \sum_k \left(\frac{1}{2} M \dot{q}_k^2 + \frac{1}{2} M \omega_k^2 q_k^2 \right)$$

Quantize

$$q \to \hat{q}, \quad p \to \hat{p}, \quad [q_k, p_{k'}] = i\hbar \delta_{kk'}$$

k vibrational modes $\Rightarrow$ k quantum harmonic oscillators

Quantization of E&M fields

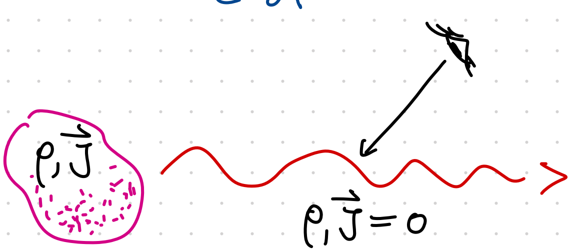
Quantization of E&M fields is more complicate than the procedure for scalar fields due to a few reasons: E&M fields are three-dim vector fields, and their six components $E_x, E_y, E_z, B_x, B_y, B_z$ are not all independent; they're related by Maxwell's equations.

Here we're only going to consider E&M radiations in free space. Thus, we begin with the source-free Maxwell's eqs.

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

(Gaussian units. There is no ε_0, μ_0. Moreover, $\vec{E}$ and $\vec{B}$ have the same unit.)



$\vec{E}$ and $\vec{B}$ fields are completely specified by the 4-potential

$A^\mu = (\phi, \vec{A})$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad * \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\nabla^2\phi = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} \quad \star$$

$$\nabla^2\vec{A} - \frac{1}{c}\frac{\partial^2\vec{A}}{\partial t^2} = -\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)$$

$\vec{E}$ and $\vec{B}$ fields are invariant under any gauge transformation

$$\phi \mapsto \phi + \frac{1}{c}\frac{\partial\chi}{\partial t}, \qquad \vec{A} \mapsto \vec{A} + \vec{\nabla}\chi$$

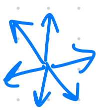
χ arbitrary scalar field

Helmholtz theorem

Any vector field $\vec{A}$ can be decomposed as

$$\vec{A} = \vec{A}_{\parallel} + \vec{A}_{\perp}$$

Longitudinal   Transversal



$\vec{\nabla}\times\vec{A}_{\parallel} = 0 \qquad \vec{\nabla}\cdot\vec{A}_{\perp} = 0$

For example, $\vec{B}$ is divergence-less $\quad \vec{B} = \vec{B}_{\perp}, \quad \vec{B} = \vec{\nabla}\times\vec{A}_{\perp}$

$*$ implies that $\vec{E}_{\parallel} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}_{\parallel}}{\partial t}$

$\Rightarrow \vec{\nabla}\cdot\vec{E}_{\parallel} = -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}_{\parallel} = 0$ in free space ($\star$)

So $\vec{\nabla}\cdot\vec{E}_{\parallel}$ is just a constraint on $(\phi, \vec{A})$. What's important dynamically is

$$\boxed{\vec{B}_{\perp} = \vec{\nabla}\times\vec{A}_{\perp}, \qquad \vec{E}_{\perp} = -\frac{1}{c}\frac{\partial \vec{A}_{\perp}}{\partial t}}$$ an

The <u>Coulomb gauge</u> chooses $\boxed{\vec{\nabla}\cdot\vec{A} = 0 \Leftrightarrow \nabla^2\phi = 0}$.

In free space, $\vec{E} = \vec{E}_{\perp}$, so $\vec{A}_{\parallel} = 0$ further implies that $\vec{\nabla}\phi = 0$

(Coulomb gauge is not Lorentz invariant, but harder to quantize.)

Now we're ready to derive the equation of motion for $\vec{A}$ by noticing that the dynamics is in $\vec{\nabla}\times\vec{E}$ and $\vec{\nabla}\times\vec{B}$.

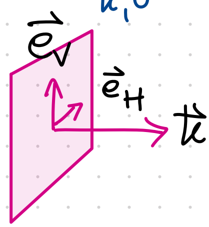
$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{A}}) - \nabla^2\vec{A} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0}$$

Uncoupled wave equations for three scalar fields A_x, A_y, A_z

$\rightarrow$ We know how to quantize!

Mode functions $\vec{u}_{\vec{k},\sigma}(\vec{r}) = \dfrac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}}\vec{\varepsilon}_{\vec{k},\sigma}$, $\quad \vec{k}\cdot\vec{\varepsilon}_{\vec{k},\sigma} = 0$



$$\vec{k} = \frac{2\pi n_x}{L}\hat{e}_x + \frac{2\pi n_y}{L}\hat{e}_y + \frac{2\pi n_z}{L}\hat{e}_z$$

- $[\vec{\nabla}\times(\vec{\nabla}\phi)]_\ell = \epsilon_{jk\ell}\,\partial_j\partial_k\phi$ but $\partial_j\partial_k = \partial_k\partial_j$ whereas $\epsilon_{jk\ell}$ is antisymmetric, so $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$. □

Moreover, $\vec{E}_{\parallel} = -\vec{\nabla}\phi - \dfrac{1}{c}\dfrac{\partial \vec{A}_{\parallel}}{\partial t}$

$$\vec{\nabla}\cdot\vec{E}_{\parallel} = -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}_{\parallel} = 0 \text{ in free space } (☆)$$

So $\vec{\nabla}\cdot\vec{E}_{\parallel}$ is just a constraint on $(\phi, \vec{A})$. What's important dynamically is

$$\vec{B}_{\perp} = \vec{\nabla}\times\vec{A}_{\perp}, \qquad \vec{E}_{\perp} = \frac{1}{c}\frac{\partial \vec{A}_{\perp}}{\partial t}$$

The Coulomb gauge chooses $\vec{\nabla}\cdot\vec{A} = 0 \Leftrightarrow \nabla^2\phi = 0$.

In free space, $\vec{E} = \vec{E}_{\perp}$, so $\vec{A}_{\parallel} = 0$ further implies that $\vec{\nabla}\phi = 0$

(Coulomb gauge is not Lorentz invariant, but harder to quantize.)

Now we're ready to derive the equation of motion for $\vec{A}$ by noticing that the dynamics is in $\vec{\nabla}\times\vec{E}$ and $\vec{\nabla}\times\vec{B}$.

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{A}}) - \nabla^2\vec{A} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0}$$

$\sigma = \pm 1$ labels the helicity

$$\epsilon_{\sigma=+1} = \frac{\vec{e}_H + i\vec{e}_V}{\sqrt{2}}$$

$$\epsilon_{\sigma=-1} = \frac{\vec{e}_H - i\vec{e}_V}{\sqrt{2}}$$

Orthogonality (Inner product)

$$\int d^3x\, \vec{u}^*_{\vec{k},\sigma}(\vec{r}) \cdot \vec{u}_{\vec{k}',\sigma'}(\vec{r}) = \delta_{\vec{k},\vec{k}'}\delta_{\sigma,\sigma'}$$

Completeness (Outer product)

$$\left[\sum_{\vec{k},\sigma} u^*_{\vec{k},\sigma}(\vec{r}\,')\, u_{\vec{k},\sigma}(\vec{r})\right]_{jl} = \left[\delta_{jl}\,\delta(\vec{r}-\vec{r}\,')\right]_\perp$$

Project onto $\vec{A}_\perp$ to keep the transversality condition.

Might expect ||

$\delta_{jl}\,\delta(\vec{r}-\vec{r}\,')$

$$\vec{A}_\perp(\vec{r},t) = \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar c^2}{\omega_{\vec{k}} V}} \left[\alpha^{(0)}_{\vec{k},\lambda}\vec{\epsilon}_{\vec{k},\lambda} e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)} + \alpha^*_{\vec{k},\lambda}\vec{\epsilon}^{\,*}_{\vec{k},\lambda} e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}\right]$$

$$\vec{E}_\perp(\vec{r},t) = i\sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar\omega_{\vec{k}}}{V}} \left[\alpha_{\vec{k},\lambda}\vec{\epsilon}_{\vec{k},\lambda} e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)} - \alpha^*_{\vec{k},\lambda}\vec{\epsilon}^{\,*}_{\vec{k},\lambda} e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}\right]$$

$$\vec{B}_\perp(\vec{r},t) = i\sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar c^2}{\omega_{\vec{k}} V}} \left[\alpha_{\vec{k},\lambda}\,\vec{k}\times\vec{\epsilon}_{\vec{k},\lambda} e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)} - \alpha^*_{\vec{k},\lambda}\,\vec{k}\times\vec{\epsilon}^{\,*}_{\vec{k},\lambda} e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}\right]$$

Quantization

$$\alpha(0) \to \hat{a}(0), \quad \alpha^*(0) \to \hat{a}^\dagger(0), \quad [\hat{a}(0), \hat{a}^\dagger(0)] = 1$$

$$\hat{E}_\perp(\vec{r},t) = \hat{E}_\perp^{(+)}(\vec{r},t) + \text{H.C.} = \hat{E}_\perp^{(+)}(\vec{r},t) + \hat{E}_\perp^{(-)}(\vec{r},t)$$

(Hermitian conjugate)

Positive frequency component of the E-field

$$\boxed{\hat{E}_\perp^{(+)}(\vec{r},t) = i \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar\omega_{\vec{k}}}{V}}\, \hat{a}_{\vec{k},\lambda}(0)\, \vec{\epsilon}_{\vec{k},\lambda}\, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)}}$$

$$\mathcal{H} = \bigotimes_{k,\sigma} \mathcal{H}_{k,\sigma}$$

Vacuum state $|0\rangle \equiv \bigotimes_{k,\sigma} |0_{k,\sigma}\rangle$

$$\hat{E}_\perp^{(+)}(\vec{r},t)\,|0\rangle = 0$$

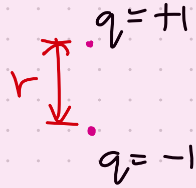
The magnetic field operator $\hat{B}$ can also be defined similarly. Crucially, $[\hat{E}(\vec{r},t), \hat{B}(\vec{r},t)] \neq 0$. Consequently, the uncertainty relation forbids simultaneous definite values of E- and B-field, even in the vacuum!

$\Rightarrow$ In quantum theory, darkness is never truly dark!

Atom-light interaction

Ex Dipole interaction

$$\hat{H}_{int} = -e\hat{r}\cdot\hat{E}(\vec{r},t)$$



Classical dipole

$$U_{int} = q\vec{r}\cdot\vec{E}$$

$$\hat{r} = \sum_n \sum_m \underbrace{\langle n|\hat{r}|m\rangle}\, |n\rangle\langle m|$$

Include terms like $\langle e|\hat{r}|g\rangle |e\rangle\langle g|$

$$\hat{H}_{int} = -\hat{d}\cdot\hat{E} = -\hat{d}\cdot(\hat{E}^{(+)} + \hat{E}^{(-)})$$

$$\approx -\langle e|\hat{d}|g\rangle |e\rangle\langle g| \hat{E}^{(+)} \quad \leftarrow \text{Absorption}$$

$$-\langle g|\hat{d}|e\rangle |g\rangle\langle e| \hat{E}^{(-)} \quad \leftarrow \text{Emission}$$

Rotating-wave approximation