Introduction to field quantization and many-particle systems

Quantization of scalar fields

A one-dimensional scalar field can be obtained by taking the continuum limit of a linear chain of N oscillators. Denote the position and the mass of the jth oscillator in the chain by x_j and m_j, and suppose that the spring that connects the jth and the $(j+1)$th oscillators has a spring constant k_j and equilibrium length a.

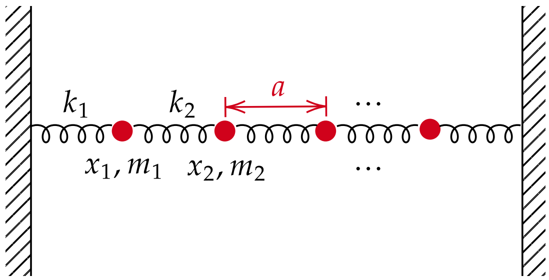

Figure 1. A linear chain of harmonic oscillators.

We can take the continuum limit $N \to \infty$ of the kinetic energy T and the potential energy V by changing $a \to dx$ and a sum to an integral; the discrete position $x_j(t)$ will become a continuous *field variable* $\eta(x,t)$.

$$T = \sum_j \frac{1}{2} m \dot{x}_j^2 = \sum_j \frac{1}{2} a \frac{m}{a} \dot{x}_j^2 \to \int \mathrm{d}x \frac{1}{2} \mu \left[\frac{\partial \eta(x,t)}{\partial t} \right]^2, \tag{0.1}$$

$$V = \sum_j \frac{1}{2} k (x_{j+1} - x_j)^2 = \sum_j \frac{1}{2} a (ka) \left(\frac{x_{j+1} - x_j}{a} \right)^2 \tag{0.2}$$

$$\to \int \mathrm{d}x \frac{1}{2} Y \left(\lim_{a \to \infty} \frac{\eta(x_{j+1,t} - \eta(x_j,t))}{a} \right)^2 \mathrm{d}x = \int dx \frac{1}{2} Y \left[\frac{\partial \eta(x,t)}{\partial x} \right]^2, \tag{0.3}$$

where $\mu = m/a$ is the mass density and $Y = ka$ is the Young modulus.

The Euler-Lagrange equation for a continuous field [7] which generalizes that for a point particle, is

$$\partial_t \left[\frac{\delta \mathcal{L}}{\delta \partial_t \eta} \right] + \partial_x \left[\frac{\delta \mathcal{L}}{\delta \partial_x \eta} \right] = -\frac{\delta \mathcal{L}}{\delta \eta}. \tag{0.4}$$

In our case, this reduces to

$$\mu \frac{\partial \eta}{\partial t} - Y \frac{\partial \eta}{\partial x} = 0, \tag{0.5}$$

yielding a PDE

$$\boxed{\frac{\partial^2 \eta}{\partial t^2} - \frac{Y}{\mu}\frac{\partial^2 \eta}{\partial x^2} = 0}, \tag{0.6}$$

whose solution in free space is a traveling wave with a phase velocity $v_p = \sqrt{Y/\mu}$. Here we have to put in some kind of boundary conditions at the two ends of the chain. This could be the hard-wall boundary condition ($\eta(0) = \eta(Na) = 0$) or the periodic boundary condition ($\eta(0) = \eta(Na)$). If we are interested in the free space solution, we are going to take the limit of the length $L = Na$ going to infinity, in which case we intuitively expect that the choice of boundary condition would not matter in the end.

A simple guess for the quantization procedure emerges if we expand a general solution in terms of separable solutions[1]

$$\eta(x,t) = \sum_k \sqrt{L}\, q_k(t) u_k(x). \tag{0.7}$$

Plugging this expansion into the wave equation 1.6 gives a consistency equation

$$\frac{1}{v_p^2}\frac{\ddot{q}}{q} = \frac{u''}{u} = -k^2, \tag{0.8}$$

where k^2 is some constant, yielding two ODEs

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}x^2} + k^2 u_k = 0, \qquad \frac{\mathrm{d}^2 q_k}{\mathrm{d}t^2} + \omega_k^2 q_k = 0. \tag{0.9}$$

The latter is the equation of motion for a simple harmonic oscilattor with an angular frequency $\omega_k \equiv v_k k$. The solutions for the hard-wall boundary condition and the periodic boundary condition are

$$u_k(x) = \frac{2}{L}\sin(kx), k = \frac{n\pi}{L}, \qquad u_k(x) = \frac{e^{ikx}}{L}, k = \frac{2n\pi}{L}. \tag{0.10}$$

By this choice of normalization, both are complete, orthonormal sets:[2]

$$\int \mathrm{d}x u_k^*(x) u_{k'}(x) = \delta_{kk'}, \tag{0.11}$$

$$\sum_k u_k^*(x) u_k(x') = \delta(x - x'). \tag{0.12}$$

Moreover, the form 1.7 means that q_k and η have the same unit of length.

Now let us compute the Hamiltonian that governs the system.

$$T = \int \mathrm{d}x \frac{1}{2}\mu \left(\frac{\partial \eta}{\partial t}\right)^2 = \sum_{kk'} \frac{1}{2}\mu L\, \dot{q}_k \dot{q}_{k'} \underbrace{\int \mathrm{d}x u_k^*(x) u_{k'}(x)}_{\delta_{kk'}} \tag{0.13}$$

$$= \sum_k \frac{1}{2} M \dot{q}_k^2 \tag{0.14}$$

$$V = \int \mathrm{d}x \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x}\right)^2 = \sum_{kk'} \frac{1}{2} YL\, \dot{q}_k \dot{q}_{k'} \int \mathrm{d}x (\partial_x u_k)^* (\partial_x u_{k'}) \tag{0.15}$$

$$= -\sum_{kk'} \frac{1}{2} YL\, \dot{q}_k \dot{q}_{k'} \int \mathrm{d}x (\partial_x^2 u_k)^* (u_{k'}) \tag{0.16}$$

$$= \sum_{kk'} \frac{1}{2} YL\, k^2 \dot{q}_k \dot{q}_{k'} \underbrace{\int \mathrm{d}x u_k^*(x) u_{k'}(x)}_{\delta_{kk'}} \tag{0.17}$$

$$= \sum_k \frac{1}{2} M \omega_k^2 q_k^2, \tag{0.18}$$

where in the last line we have used the fact that $\omega_k = v_k k$, hence $k^2 = M\omega_k^2 / YL$.

$$\therefore H = T + V = \sum_k \left(\frac{1}{2} M \dot{q}_k^2 + \frac{1}{2} M \omega_k^2 q_k^2 \right) \tag{0.19}$$

Dirac quantization rule.

$$q \mapsto \hat{q}, \qquad p \mapsto \hat{p}, \qquad [\hat{q}_k, \hat{p}_{k'}] = i\hbar \delta_{kk'} \tag{0.20}$$

$$\boxed{k \text{ vibrational modes} \xrightarrow{\text{quantization}} k \text{ quantum harmonic oscillators}} \tag{0.21}$$

Quantization of electromagnetic radiations

Quantization of E&M fields is more complicate than the procedure we just outlined for scalar fields due to a few reasons: E&M fields are three-dimensional vector fields, and their six components $E_x, E_y, E_z, B_x, B_y, B_z$ are not all independent, as their relations are given by Maxwell's equations.

Here we consider only E&M radiations in free space, which must obey the source-free Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{0.22}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{0.23}$$

E and B fields are completely specified by the 4-potential $A^\mu = (\phi, \mathbf{A})$;

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{0.24}$$

$$\nabla \cdot \mathbf{E} = 0 \implies \nabla^2 \phi = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{A}, \tag{0.25}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times (\nabla \times \mathbf{A}) \implies \nabla^2 \mathbf{A} - \frac{1}{c}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) \tag{0.26}$$

$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

PROOF

$$[\nabla \times (\nabla \times \mathbf{A})]_j = \epsilon_{jkl}\partial_k(\nabla \times \mathbf{A})_l = \epsilon_{jkl}\epsilon_{lmn}\partial_k\partial_m A_n \tag{0.27}$$

$$= \delta_{jm}\delta_{kn}\partial_k\partial_m A_n - \delta_{jn}\delta_{km}\partial_k\partial_m A_n \tag{0.28}$$

$$= \partial_j\partial_k A_k - \partial_k^2 A_j \tag{0.29}$$

$$= [\nabla(\nabla \cdot \mathbf{A})]_j - (\nabla^2 \mathbf{A})_j \tag{0.30}$$

□

We will quantize A^μ since it has a fewer number of components. But to do so we have to deal with the gauge freedom. That is because E and B fields are invariant under any transformation of the form

$$\phi \mapsto \phi + \frac{1}{c}\frac{\partial \chi}{\partial t}, \qquad \mathbf{A} \mapsto \mathbf{A} + \nabla \chi, \tag{0.31}$$

where χ could be any scalar function. Recall the statement of Helmholtz theorem that any vector field $\mathbf{A}$ can be decomposed as a sum

$$\mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp; \tag{0.32}$$

$A_\parallel$ is the longitudinal (curl-less) component, $\nabla \times \mathbf{A}_\parallel = 0$, and $A_\perp$ is the transversal (divergence-less) component $\nabla \cdot \mathbf{A}_\perp = 0$. The condition that E is divergence-less in free space 1.25 is just a constraint on $\mathbf{A}_\parallel$ and ϕ that plays no dynamical role:

$$\nabla \cdot \mathbf{E}_\parallel = -\nabla^2 \phi - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{A}_\parallel, \tag{0.33}$$

What is important are

$$\boxed{\mathbf{B}_\perp = \nabla \times \mathbf{A}_\perp, \qquad \mathbf{E}_\perp = \frac{1}{c}\frac{\partial \mathbf{A}_\perp}{\partial t}.} \tag{0.34}$$

The *Coulomb gauge* chooses $\mathbf{A} = \mathbf{A}_\perp \iff \nabla \cdot \mathbf{A} = 0 \iff \nabla^2 \phi = 0$. This choice of gauge implies the wave equation

$$\boxed{\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0.} \tag{0.35}$$

Crucially, these are *uncoupled* equations for three scalar fields A_x, A_y, A_z, which we know how to quantize!

Let us put the field inside a cubic box of volume $V = L \times L \times L$. The mode functions now have an additional index $\sigma = \pm 1$ for the helicity of the radiation.

$$\mathbf{u}_{\mathbf{k},\sigma}(\mathbf{r}) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}\boldsymbol{\epsilon}_{\mathbf{k},\sigma}, \tag{0.36}$$

where $\mathbf{k}\cdot\boldsymbol{\epsilon}_{\mathbf{k},\sigma} = 0$ and

$$\mathbf{k} = \frac{2\pi n_x}{L}\hat{\mathbf{e}}_x + \frac{2\pi n_y}{L}\hat{\mathbf{e}}_y + \frac{2\pi n_z}{L}\hat{\mathbf{e}}_z \tag{0.37}$$

We give the rest of the results without derivation.

$$\mathbf{A}_\perp(\mathbf{r},t) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{2\pi\hbar c^2}{\omega_\mathbf{k} V}}\left[\alpha_{\mathbf{k},\sigma}(0)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_\mathbf{k}t)}\boldsymbol{\epsilon}_{\mathbf{k},\sigma} + \alpha^*_{\mathbf{k},\sigma}(0)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_\mathbf{k}t)}\boldsymbol{\epsilon}^*_{\mathbf{k},\sigma}\right] \tag{0.38}$$

$$\mathbf{E}_\perp(\mathbf{r},t) = i\sum_{\mathbf{k},\sigma}\sqrt{\frac{2\pi\hbar\omega_\mathbf{k}}{V}}\left[\alpha_{\mathbf{k},\sigma}(0)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_\mathbf{k}t)}\boldsymbol{\epsilon}_{\mathbf{k},\sigma} - \alpha^*_{\mathbf{k},\sigma}(0)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_\mathbf{k}t)}\boldsymbol{\epsilon}^*_{\mathbf{k},\sigma}\right] \tag{0.39}$$

$$\mathbf{B}_\perp(\mathbf{r},t) = i\sum_{\mathbf{k},\sigma}\sqrt{\frac{2\pi\hbar c^2}{\omega_\mathbf{k} V}}\left[\alpha_{\mathbf{k},\sigma}(0)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_\mathbf{k}t)}\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k},\sigma} + \alpha^*_{\mathbf{k},\sigma}(0)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_\mathbf{k}t)}\mathbf{k}\times\boldsymbol{\epsilon}^*_{\mathbf{k},\sigma}\right] \tag{0.40}$$

Dirac quantization rule.

$$\alpha(0) \mapsto \hat{a}(0), \qquad \alpha^*(0) \mapsto \hat{a}^\dagger(0), \qquad [\hat{a}_{\mathbf{k},\sigma}, \hat{a}^\dagger_{\mathbf{k}',\sigma'}] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \tag{0.41}$$

$$\boxed{k \text{ E\&M radiation modes} \xrightarrow{\text{quantization}} 2k \text{ quantum harmonic oscillators}} \tag{0.42}$$

In quantum optics, say, the electric field operator is usually written as the sum of its *positive*- and *negative*-frequency components:

$$\widehat{\mathbf{E}}_\perp(\mathbf{r},t) = \widehat{\mathbf{E}}^{(+)}_\perp(\mathbf{r},t) + \text{H.C.} = \mathbf{E}^{(+)}_\perp(\mathbf{r},t) + \mathbf{E}^{(-)}_\perp(\mathbf{r},t), \tag{0.43}$$

$$\boxed{\widehat{\mathbf{E}}^{(+)}_\perp(\mathbf{r},t) \equiv i\sum_{\mathbf{k},\sigma}\sqrt{\frac{2\pi\hbar\omega_\mathbf{k}}{V}}\;\hat{a}_{\mathbf{k},\sigma}(0)\;e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_\mathbf{k}t)}\boldsymbol{\epsilon}_{\mathbf{k},\sigma}} \tag{0.44}$$

because the positive-frequency parts are responsible for the absorption of light by an atom, while the negative-frequency parts are responsible for the emission of light by an atom.

The state space of traversal E&M radiation modes is the tensor product of Hilbert spaces of the simple harmonic oscillator, each for a particular mode and a particular polarization direction:

$$\mathcal{H} = \bigotimes_{\mathbf{k},\sigma} \mathcal{H}_{\mathbf{k},\sigma}. \tag{0.45}$$

The complete, orthogonal set that spans the total Hilbert space can be built from the vacuum state

$$|0\rangle \equiv \bigotimes_{\mathbf{k},\sigma} |0_{\mathbf{k},\sigma}\rangle\,, \qquad\qquad \widehat{\mathbf{E}}_{\perp}^{(+)}\,|0\rangle = 0. \tag{0.46}$$

It is important to note that $[\widehat{\mathbf{E}}(\mathbf{r},t), \widehat{\mathbf{B}}(\mathbf{r},t)] \neq 0$ in general. Consequently, the uncertainty relation forbids a state which possesses simultaneous definite values of the E- and B-field, even in the vacuum! Therefore, we come to a striking conclusion that, in the quantum world, *even nothingness is not completely dark*.

Tensor product

The quantum state of a joint system AB lives in the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$. The tensor product space can be defined without relying on specific bases on $\mathcal{H}_A$ and $\mathcal{H}_B$, but for the sake of simplicity, we will define the tensor product via ONBs.

Suppose that $\{|e_j\rangle\}_j$ and $\{|f_k\rangle\}_k$ are ONBs for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. (Their dimensions may not be the same.) $\mathcal{H}_A \otimes \mathcal{H}_B$ is simply the span (formal linear combinations) of elements in the Cartesian product $\{|e_j\rangle\}_j \times \{|f_k\rangle\}_k$, denoting the elements by $|e_j\rangle \otimes |f_k\rangle$.

A vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ is said to be a **product state** if it can be written as a simple product $|\psi\rangle \otimes |\phi\rangle$ of some $|\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$. Otherwise, a vector is said to represent an **entangled state**. If needed, subscripts may be added to indicate which vector belongs to which Hilbert space, for example, $|\psi\rangle_A \otimes |\phi\rangle_B$. It is common to omit the tensor product symbol $\otimes$ and write a product state simply as $|\psi\rangle\,|\phi\rangle$, or even $|\psi\phi\rangle$ when no confusion may arise.

Scalar multiplication

$$\lambda(|\psi\rangle \otimes |\phi\rangle) = (\lambda\,|\psi\rangle) \otimes |\phi\rangle = \lambda(|\psi\rangle \otimes (\lambda\,|\phi\rangle) \tag{0.47}$$

Vector addition

$$(|\psi_1\rangle + |\psi_2\rangle) \otimes |\phi\rangle = |\psi_1\rangle \otimes |\phi\rangle + |\psi_2\rangle \otimes |\phi\rangle \tag{0.48}$$
$$|\psi\rangle \otimes (|\phi_1\rangle + |\phi_2\rangle) = |\psi\rangle \otimes |\phi_1\rangle + |\psi\rangle \otimes |\phi_2\rangle \tag{0.49}$$

Linear combinations of product vectors with no common factor (entangled states) are genuinely new objects in $\mathcal{H}_A \otimes \mathcal{H}_B$.

Compare these to the rules 1.50 and 1.51 for the direct sum of vector spaces in the box below.

Direct sum of vector spaces

In contrast, A subset $S \subset V$ of a vector space that is also a vector space is called a **subspace**. A vector space U is said to be a **direct sum** $V \oplus W$ if every $u \in U$ can be decomposed uniquely as a sum of $v \in V$ and $w \in W$. Equivalently, any $u \in U$ can be specified by a unique pair $(v, w) \in V \times W$ such that

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \tag{0.50}$$
$$a(v, w) = (av, aw). \tag{0.51}$$

Inner product.

$$(\langle\eta| \otimes \langle\xi|)(|\psi\rangle \otimes |\phi\rangle) = \langle\eta|\psi\rangle \langle\xi|\phi\rangle \tag{0.52}$$

Partial inner product.

$${}_A\langle\eta| \left(|\psi\rangle_A \otimes |\phi\rangle_B\right) = \langle\eta|\psi\rangle\, |\phi\rangle_B \tag{0.53}$$

Linear operators.

$$\widehat{A} \otimes \widehat{B}(|\psi\rangle \otimes |\phi\rangle) = (\widehat{A}\,|\psi\rangle) \otimes (\widehat{B}\,|\phi\rangle) \tag{0.54}$$
$$(\widehat{A} \otimes \widehat{B})(\widehat{C} \otimes \widehat{D}) = (\widehat{A}\widehat{C}) \otimes (\widehat{B}\widehat{D}) \tag{0.55}$$

Thus, the trace factorizes.

$$\begin{aligned} \mathrm{tr}\left(\widehat{A} \otimes \widehat{B}\right) &= \sum_{jk} \langle e_j| \langle f_k|\, \widehat{A} \otimes \widehat{B}\, |e_j\rangle\, |f_k\rangle && (0.56) \\ &= \sum_{jk} \langle e_j|\widehat{A}|e_j\rangle \langle f_k|\widehat{B}|f_k\rangle && (0.57) \\ &= \mathrm{tr}\left(\widehat{A}\right) \otimes \mathrm{tr}\left(\widehat{B}\right) && (0.58) \end{aligned}$$

Local operations always commute.

$$[\widehat{A} \otimes \widehat{\mathbb{1}}, \widehat{\mathbb{1}} \otimes \widehat{B}] = \widehat{A} \otimes \widehat{B} - \widehat{A} \otimes \widehat{B} = 0 \tag{0.59}$$

If one wants to represent a tensor product operator in a matrix form, one needs to fix the ordering of the bases of $\mathcal{H}_A \otimes \mathcal{H}_B$.[3] When the lexicographic ordering is chosen, the matrix form has the form of the Kronecker product, which only shown here for the 2-by-2 case:

$$\widehat{A}\widehat{B} \longleftrightarrow \left(\begin{array}{c:c} A_{00}B & A_{01}B \\ \hdashline A_{10}B & A_{11}B \end{array}\right). \tag{0.60}$$

The concept of tensor product is typically introduced to students in a highly unintuitive setting of quantum theory, and as a result, the idea may appear alien at first. However, the tensor product is already present in ordinary probability theory as a self-evident means of combining the probabilities of two independent random variables.

$$\begin{pmatrix} p \\ 1-p \end{pmatrix} \otimes \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} pq \\ p(1-q) \\ (1-p)q \\ (1-p)(1-q) \end{pmatrix} \tag{0.61}$$

In ordinary probability theory, a state that cannot be written as a product state $|p\rangle \otimes |q\rangle$ is a **correlated state**. Entanglement is a form of correlation, but we will see in Section 1.5.2 that it can be stronger than any classical correlation.

Quantum nonlocality

EPR argument

A philosophical troubling aspect of quantum theory is that one cannot in general think of an act of measuring as revealing a pre-existing value of the measured property. When the system is in an eigenstate of an observable $\hat{A}$ with an eigenvalue λ, subsequent measurements of $\hat{A}$ do not alter the state, hence there is a tendency to think of the value λ as pre-existed. Einstein, Padolsky, and Rosen (EPR) devised a clever argument using quantum correlation and the principle of relativity to argue that values of incompatible (non-commuting) observables can simultaneously pre-exist [8].

The singlet state,

$$\left|\Psi^-\right\rangle \equiv \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}, \tag{0.62}$$

is an eigenstate shared by $\widehat{Z}\widehat{Z}$ and $\widehat{X}\widehat{X}$ with both eigenvalues -1. (Verify that they commute.) Now, while Alice cannot make a local Z and X measurements at the same time, if she chooses to measure one, say $\widehat{Z}_A$ and find the value $z_A = \pm 1$, then the spin of Bob's particle would need to have the opposite value to satisfy $z_A z_B = -1$. Since the spins are anti-correlated in every direction in the singlet stat, the same conclusion follows if Alice were to measure $\widehat{X}_A$. But, EPR argued, the act of measurement by Alice over here cannot effect the state of Bob's particle over there. Thus, the fact that Alice could have chosen to measure either observable and inferred z_B or x_B without disturbing Bob's particle means that those values already existed before the measurement.

$$\begin{pmatrix}\textbf{Entanglement}\\ \text{Anti-correlation}\\ \text{in the singlet state}\end{pmatrix} + \begin{pmatrix}\textbf{Relativity}\\ \text{The choice of}\\ \text{measurement on}\\ \text{A cannot have an}\\ \text{influence on B}\end{pmatrix} \Longrightarrow \begin{pmatrix}\textbf{Quantum theory}\\ \textbf{is incomplete}\end{pmatrix}$$

0.0.1 Bell's inequality

The hope that quantum theory can be completed with *hidden variables* that behave entirely classically is dashed by an experimental violation of Bell's inequality. Suppose that we have binary variables a, b, and c, each takes value either 0 or 1. Observe that

$$|b - c| \equiv 1 - bc. \tag{0.63}$$

In particular, the value of the expression is either 0 or 2, and multiplying the quantity $b - c$ inside the absolute value by another binary variable does not change the equality. Thus, we have that the *values* of any three binary variables have to obey the relation

$$|ab - ac| = 1 - bc. \tag{0.64}$$

Notice that behind this trivial inequality lies again the philosophical assumption that the values a, b, and c pre-exist and can be measured without disturbing one another. Thankfully, based on the principle of relativity, if, say, a and b are obtained from measurements of distant parties, then the discovery of one of them should not effect the the other value.

What would happen if we think of a, b and c as values obtained from making quantum spin measurements? To apply 1.64 to the quantum setting, we need to modify it a little bit. Quantum theory does not predict the outcome of any single measurement with certainty in general; it predicts the average value. So we should instead write[4]

$$|\langle ab\rangle - \langle ac\rangle| \le 1 - \langle bc\rangle\,. \tag{0.65}$$

This is the original Bell's inequality derived in John Bell's 1964 paper [9].

$$\begin{pmatrix}\textbf{Classical}\\ \textbf{correlation}\end{pmatrix} + \begin{pmatrix}\textbf{Relativity}\\ \text{The choice of}\\ \text{measurement on}\\ \text{A cannot have an}\\ \text{influence on B}\end{pmatrix} \Longrightarrow \big(\,\textbf{Bell's inequality}\,\big)$$

We can challenge Bell's inequality in the following scenario in quantum mechanics: in each measurement round, distribute a singlet state to Alice and Bob and let Alice makes a choice to measure either an observable $(\mathbf{a}\cdot\hat{\boldsymbol{\sigma}})\otimes\widehat{\mathbb{1}}$ or $(-\mathbf{c}\cdot\hat{\boldsymbol{\sigma}})\otimes\widehat{\mathbb{1}}$, and let Bob makes a choice to measure either $\widehat{\mathbb{1}}\otimes(\mathbf{b}\cdot\hat{\boldsymbol{\sigma}})$ or $\widehat{\mathbb{1}}\otimes(\mathbf{c}\cdot\hat{\boldsymbol{\sigma}})$. The singlet state ensures that the value c measured by both Alice and Bob coincides. One can compute

$$\langle \Psi^- | (\mathbf{a} \cdot \hat{\boldsymbol{\sigma}})(\mathbf{b} \cdot \hat{\boldsymbol{\sigma}}) | \Psi^- \rangle = -\mathbf{a} \cdot \mathbf{b}. \tag{0.66}$$

Plugging this into 1.65 gives

$$|\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}| + \mathbf{b} \cdot \mathbf{c} \leq 1. \tag{0.67}$$

By choosing the three unit vectors as in Figure 2, the inequality is violated,

$$|\cos 45^\circ| + \cos 45^\circ = \sqrt{2} \geq 1. \tag{0.68}$$

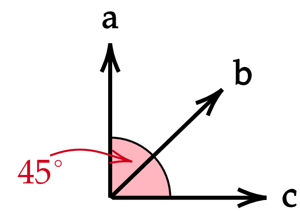

Figure 2. Three measurement angles that leads to a violation of Bell's inequality.

References

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[2] Albert Einstein, Boris Padolsky, and Nathan Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Physical Review **47**, 777 (1935).

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