Angular Momentum

First non-trivial example of classification of states according to their symmetries, in this case in particular according to how the quantum states transform under rotations in 3D.

$$\vec{L} = \vec{r} \times \vec{p}$$

Orbital angular momentum operator

$$\left.\begin{aligned} \hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{aligned}\right\} \quad \hat{L}_j = \epsilon_{jkl}\,\hat{r}_k\,\hat{p}_l$$

No ordering problem

$$[\hat{r}_j, \hat{p}_k] = i\hbar\,\delta_{jk}$$

Vector operator $\vec{L} = \hat{L}_x \hat{e}_x + \hat{L}_y \hat{e}_y + L_z \hat{e}_z$

$$\begin{aligned} [\hat{L}_x, \hat{L}_y] &= [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y,\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] \\ &= \hat{y}[\hat{p}_z, \hat{z}]\hat{p}_x + \hat{x}[\hat{z}, \hat{p}_z]\hat{p}_y \\ &= i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) = i\hbar\,\hat{L}_z \end{aligned}$$

$\therefore\ [\hat{L}_j, \hat{L}_k] = i\hbar\, \epsilon_{jkl} L_l$ same as Pauli matrices

$\left(\{\hat{L}_j\}_j\ \&\ \{\hat{\sigma}_j\}_j\right.$ are two reps of the same Lie algebra)

9 equations (3 independent)

To derive all of them in one go, we index notation

$$\vec{A} \times \vec{B} = \hat{e}_j\, \epsilon_{jkl} A_k B_l = \det\begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$$

$$\delta_{jk}\delta_{kl} = \delta_{jl}$$

$$\delta_{jk}\delta_{jk} = \delta_{jj} = \mathrm{Tr}\, \mathbb{1}_{d\times d} = d$$ (Dimension of the space)

$$\epsilon_{jkl}\epsilon_{jmn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$$

$$\epsilon_{jkl}\epsilon_{jkm} = \delta_{kk}\delta_{lm} - \delta_{km}\delta_{lm}$$

$$= d\delta_{lm} - \delta_{lm} = (d-1)\delta_{lm}$$

$$\hat{L}_j = \epsilon_{jkl}\hat{r}_k\hat{p}_l$$

$$[\hat{L}_j, \hat{L}_k] = \epsilon_{jlm}\epsilon_{knq}\underbrace{[\hat{r}_l\hat{p}_m, \hat{r}_n\hat{p}_q]}$$

$$\hat{r}_l[\hat{p}_m, \hat{r}_n]\hat{p}_q + \hat{r}_n[\hat{r}_l, \hat{p}_q]\hat{p}_m$$

$$= -i\hbar\,\delta_{mn}\hat{r}_l\hat{p}_q + i\hbar\,\delta_{lq}\hat{r}_n\hat{p}_m$$

$$= i\hbar\left(-\epsilon_{jlm}\epsilon_{kmq}\hat{r}_l\hat{p}_q + \epsilon_{jlm}\epsilon_{knl}\hat{r}_n\hat{p}_m\right)$$

$$m \quad l \quad l \quad n \quad m \quad n$$

$$= i\hbar\left(-\epsilon_{jml}\epsilon_{kln}\hat{r}_m\hat{p}_n + \epsilon_{jlm}\epsilon_{knl}\hat{r}_n\hat{p}_m\right)$$

Swapping introduces a minus sign twice

$$= i\hbar\underbrace{\epsilon_{jlm}\epsilon_{knl}}_{\delta_{mk}\delta_{jn} - \delta_{mn}\delta_{jk}}(\hat{r}_n\hat{p}_m - \hat{r}_m\hat{p}_n)$$

$$= i\hbar\left(\hat{r}_j\hat{p}_k - \cancel{\hat{r}_m\hat{p}_m} - \hat{r}_k\hat{p}_j + \cancel{\hat{r}_m\hat{r}_m}\right)$$

$$= i\hbar\,\epsilon_{jkl}\hat{L}_l$$

$\hat{L}_j \leftarrow$ Angular momentum operator in 3D

$\hat{J}_j \leftarrow$ Abstract angular momentum operator

$$[\hat{J}_j, \hat{J}_k] = i\epsilon_{jkl}\hat{J}_l$$

$$\vec{J}^2 := \hat{J}_x^2 + J_y^2 + J_z^2$$

$$[\vec{J}^2, \hat{J}_j] = [\hat{J}_k\hat{J}_k, \hat{J}_j]$$

$$= \hat{J}_k[\hat{J}_k, \hat{J}_j] + [\hat{J}_k, \hat{J}_j]\hat{J}_k$$

$$= i\hbar \underbrace{\epsilon_{kjl}}_{\text{Antisymmetric}} \underbrace{(\hat{J}_k\hat{J}_l + \hat{J}_l\hat{J}_k)}_{\text{Symmetric}} = 0$$

under interchange of j and k

$\vec{J}^2$ and $\hat{J}_z$ form an CSCO (Complete set of commuting observables)

We can also choose $\{\vec{J}^2, \hat{J}_x\}$ or $\{\vec{J}^2, \hat{J}_y\}$ to be a CSCO, but once we include one of the $\hat{J}$'s, we cannot include other $\hat{J}$'s since they don't commute.

$$\vec{J}^2 |\beta, m\rangle = \beta\hbar^2 |\beta, m\rangle$$
$$\vec{J}_z |\beta, m\rangle = m\hbar |\beta, m\rangle$$

I follow Ballentine Section 7.2

Fact 1 $\boxed{\beta \geq m^2}$

- $$\underbrace{\langle \beta, m | \vec{J}^2 | \beta, m\rangle}_{\beta\hbar^2} = \langle \beta, m | \hat{J}_x^2 | \beta, m\rangle + \langle \beta, m | J_y^2 | \beta, m\rangle + \underbrace{\langle \beta, m | \hat{J}_z^2 | \beta, m\rangle}_{m\hbar^2} \quad \square$$

Define $\hat{J}_\pm := \hat{J}_x \pm i\hat{J}_y$

$$[\hat{J}_z, \hat{J}_\pm] = \pm\hbar \hat{J}_\pm$$
$$[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$$
$$\hat{J}_- \hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_x, \hat{J}_y]$$
$$= \vec{J}^2 - \hat{J}_z^2 - \hbar J_z \leftarrow$$

Simple action on $|\beta, m\rangle$

$$\hat{J}_z(\hat{J}_+|\beta,m\rangle) = ([\hat{J}_z,\hat{J}_+] + \hat{J}_+\hat{J}_z)|\beta,m\rangle$$

$$= \hbar(m+1)|\beta,m\rangle$$

$\Longrightarrow$ Either $\hat{J}_+|\beta,m\rangle$ is an eigenvector with eigenvalue $(m+1)\hbar$ or $\hat{J}_+|\beta,m\rangle = 0$

Suppose that we call the value of m for which the equation $\hat{J}_+|\beta,m\rangle = 0$ is true $m=j$

$$\hat{J}_+|\beta,j\rangle = 0$$

$$0 = \hat{J}_-\hat{J}_+|\beta,j\rangle = (\vec{J}^2 - \vec{J}_z^2 - \hbar\vec{J}_z)|\beta,j\rangle$$

$$= \hbar^2[\beta - j(j+1)]|\beta,j\rangle$$

$$\Longrightarrow \boxed{\beta = j(j+1)}$$

Since β is positive (Proof: $\langle\psi|\vec{J}^2|\psi\rangle$ is a sum of positive numbers $\langle\psi|\hat{J}_j^2|\psi\rangle = (\langle\psi|\hat{J}_j)(\hat{J}_j|\psi\rangle) \geq 0$) for a given β, we can solve for a unique j. Thus, we can label the eigenstates by j instead of β.

$$|j,m\rangle$$

Similarly, $\hat{J}_z(\hat{J}_-|j,m\rangle) = \hbar(m-1)(\hat{J}_-|j,m\rangle)$

$\Longrightarrow$ Either $\hat{J}_-|\beta,m\rangle$ is an eigenvector with eigenvalue $(m-1)\hbar$ or $\hat{J}_-|\beta,m\rangle = 0$

Call this lowest value of $m = k$

$$0 = \hat{J}_+\hat{J}_-|j,k\rangle = \hbar^2[j(j+1) - k(k-1)]|j,k\rangle$$

$$\Longrightarrow j(j+1) = k(k-1)$$

$$= (-k)(-k+1)$$

$$\Longrightarrow \quad k = -j$$

Thus, we have that $\boxed{-j \leqslant m \leqslant j}$

Now we don't know yet the range of values m or j can take, but we do know that an application of $\hat{J}_+$ (resp. $\hat{J}_-$) increases (resp. decreases) the value of m by 1. Therefore,

# of steps to hit the highest rung from m

1. $m + p = j$
2. $m - q = -j$

# of steps to hit the lowest rung from m

① − ②;

$$\frac{p+q}{2} = j$$

$\therefore$ $\boxed{j \text{ is either an integer or half-integer}}$

	$j=0$	$j=\frac{1}{2}$	$j=1$	$j=\frac{3}{2}$	$\cdots$
m	0	$1/2$ $-1/2$	1 0 -1	$3/2$ $1/2$ $-1/2$ $-3/2$	

- j labels an $(2j+1)$-dimensional subspace $\mathcal{H}_j$
- m labels a specific element of the ONB $|j,m\rangle$ for $\mathcal{H}_j$

Normalization of $\hat{J}_\pm |j,m\rangle$

$$\| \hat{J}_+ |j,m\rangle \|^2 = \langle j,m| \hat{J}_- \hat{J}_+ |j,m\rangle$$

$$= \hbar^2 [j(j+1) - m(m-1)] |j,m\rangle$$

$$\Rightarrow \quad \hat{J}_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\;|j,m+1\rangle$$
$$= \hbar\sqrt{(j+m+1)(j-m)}\;|j,m+1\rangle$$

Similarly

$$\hat{J}_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\;|j,m-1\rangle$$
$$= \hbar\sqrt{(j-m+1)(j+m)}\;|j,m-1\rangle$$

Example

① $j=\frac{1}{2}$ $\quad \hat{J}_+|\frac{1}{2},\frac{1}{2}\rangle = \hbar\sqrt{\frac{3}{4}-\frac{3}{4}}\;|\frac{1}{2},\frac{1}{2}\rangle = 0$

$$\hat{J}_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{3}{4}+\tfrac{1}{4}}\;|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

Ordered basis $\{|\frac{1}{2},\frac{1}{2}\rangle, |\frac{1}{2},-\frac{1}{2}\rangle\}$

$$\hat{J}_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad \hat{J}_- = \hat{J}_+^\dagger = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$$\hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_x$$

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i} = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_y$$

$\hat{J}_z$ just has $m\hbar$ in the diagonal: $\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_z$

② $j = 1$

$$\hat{J}_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

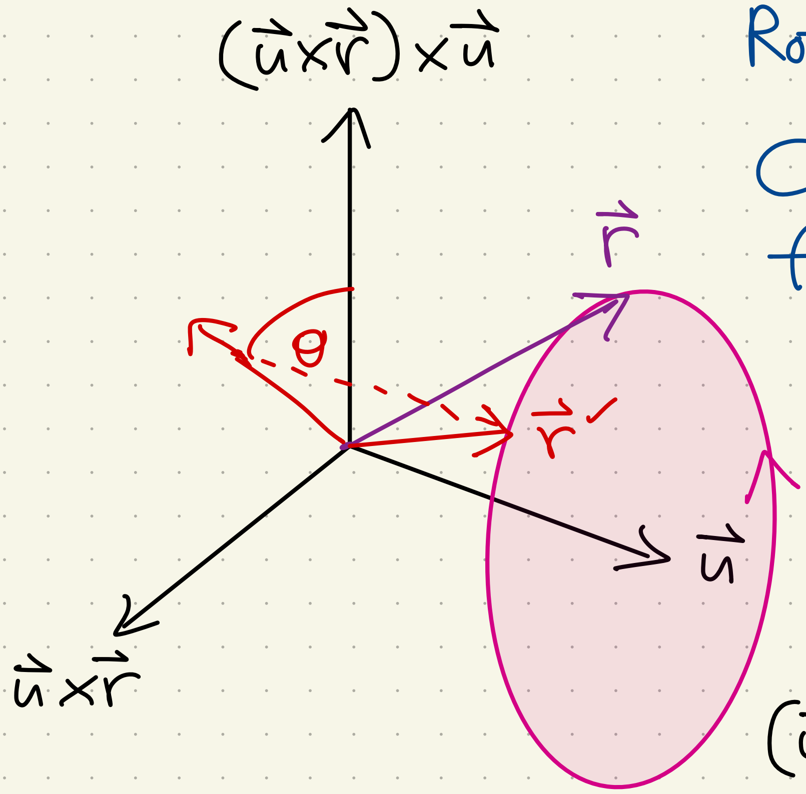
$$\hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{J}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{array}{ll|c|cc|ccc|cccc}
 & j = & 0; & \frac{1}{2} & ; & & 1 & ; & & \frac{3}{2} & & \\
 & m = & 0; & \frac{1}{2}, & -\frac{1}{2}; & 1, & 0, & -1; & \frac{3}{2}, & \frac{1}{2}, & -\frac{1}{2}, & -\frac{3}{2} \\
\hline
j' = 0,\ m' = & 0 & 0 & & & & & & & & & \\
\hline
j' = \frac{1}{2},\ m' = & \frac{1}{2} & & 0 & 1 & & & & & & & \\
 & -\frac{1}{2} & & 0 & 0 & & & & & & & \\
\hline
j' = 1,\ m' = & 1 & & & & 0 & \sqrt{2} & 0 & & & & \\
 & 0 & & & & 0 & 0 & \sqrt{2} & & & & \\
 & -1 & & & & 0 & 0 & 0 & & & & \\
\hline
j' = \frac{3}{2},\ m' = & \frac{3}{2} & & & & & & & 0 & \sqrt{3} & 0 & 0 \\
 & \frac{1}{2} & & & & & & & 0 & 0 & \sqrt{4} & 0 \\
 & -\frac{1}{2} & & & & & & & 0 & 0 & 0 & \sqrt{3} \\
 & -\frac{3}{2} & & & & & & & 0 & 0 & 0 & 0
\end{array} \qquad (7.16)$$

Matrix representation of $\hat{J}_+$ in $\mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{\frac{3}{2}} \oplus \cdots$ where the subscript indicates the j eigenvalue.

(Ballentine p. 164)



Rotation of $\vec{r}$ about $\vec{u}$

Construct orthogonal axes from $\vec{u}$, $\vec{u}\times\vec{r}$, $(\vec{u}\times\vec{r})\times\vec{u}$

Can check that

$$(\vec{u}\times\vec{r})\times\vec{u} = \vec{r} - (\vec{u}\cdot\vec{r})\,\vec{u}$$

$$R_{\vec{u}}(\theta)\,\vec{r} = (\vec{u}\cdot\vec{r})\,\vec{u} + \cos\theta\,(\vec{u}\times\vec{r})\times\vec{u} + \sin\theta\,\vec{u}\times\vec{r}$$

$$\|$$

$$R_{\vec{u}}(\theta)\,x_j\hat{e}_j = x_j\left[R_{\vec{u}}(\theta)\,\hat{e}_j\right]$$

$$= x_j\,\hat{e}_k\,\underbrace{\left[\hat{e}_k R_{\vec{u}}(\theta)\,\hat{e}_j\right]}_{\text{Matrix element}}$$

Insert the identity

Infinitesimal rotation $d\theta$

$$R_{\vec{u}}(d\theta)\,\vec{r} = \vec{r} + d\theta\,\vec{u}\times\vec{r}$$

Compare

$$\boxed{\left[R_{\vec{u}}(d\theta)\right]_{jk} = \delta_{jk} + d\theta\,\epsilon_{jkl}\,u_l}$$

What are generators of rotations?

$$R_{\vec{u}}(d\theta)\psi(\vec{r}) = \langle R_{\vec{n}}(-d\theta)\vec{r}|\psi\rangle$$

$$= \langle \vec{r} - d\theta\, \vec{u}\times\vec{r}|\psi\rangle$$

Special case $\vec{u} = \hat{e}_z$

$\Rightarrow$ $\hat{e}_z \times \vec{r} = \epsilon_{3jk} x_j \hat{e}_k = x\hat{e}_y - y\hat{e}_x$

$$\therefore \langle\vec{r}|R_{\hat{e}_z}(d\theta)|\psi\rangle = \langle\vec{r} - d\theta(x\hat{e}_y - y\hat{e}_x)|\psi\rangle$$

$$= \psi(x + y\,d\theta, y - x\,d\theta, z)$$

$$= \psi(x,y,z) + \left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right)d\theta$$

$$= \langle\vec{r}|\left[\hat{\mathbb{1}} - \frac{i}{\hbar}\hat{L}_z(d\theta)\right]|\psi\rangle$$

To obtain the commutation relation, proceed as in the case of translations. Generally, suppose that we have vector operators $\vec{\hat{V}}$

$$\hat{R}^\dagger(d\theta)\,\vec{\hat{V}}\,\hat{R}(d\theta) = \vec{\hat{V}} + d\theta\,\vec{u}\times\vec{\hat{V}}$$

$$= \left(\mathbb{1} + \frac{i}{\hbar}d\theta\,\hat{J}_j\right)\vec{\hat{V}}\left(\mathbb{1} - \frac{i}{\hbar}d\theta\,\hat{J}_j\right)$$

$$= \vec{V} + \frac{i}{\hbar}d\theta\left(\hat{J}_j\vec{\hat{V}} - \vec{\hat{V}}\hat{J}_j\right)$$

compare

$$\therefore \quad \boxed{[\hat{J}_j, \hat{V}_k] = i\hbar\,\epsilon_{jkl}\hat{V}_l}$$

Thus, the angular momentum commutation relation can be understood as a consequence of the $\hat{J}_j$'s being vector operators.

Example

$$[\hat{L}_j, \hat{x}_k] = [\epsilon_{jlm}\hat{x}_l\hat{p}_m, \hat{x}_k]$$
$$= \epsilon_{jlm}\hat{x}_l[\hat{p}_m, \hat{x}_k]$$
$$= -i\hbar\,\epsilon_{jlk}\hat{x}_l$$
$$= i\hbar\,\epsilon_{jkl}\hat{x}_l$$

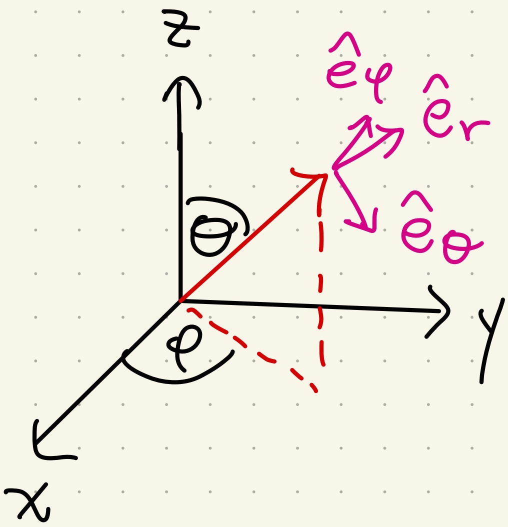
Orbital angular momentum

We will now show that the Hilbert space of wave functions $\psi(\vec{r})$ over $\mathbb{R}^3$ decomposes into a direct sum of all irreps of SO(3) with integral values of j.

(C-T VI.D pp. 663–664)

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\hat{L}_\pm = \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$



ℓ and m must be integral

$$\frac{\hbar}{i}\frac{\partial}{\partial\varphi}Y_\ell^m(\theta,\varphi) = m\hbar Y_\ell^m(\theta,\varphi)$$

Assuming separable solutions

$$Y_\ell^m(\theta,\varphi) = F_\ell^m(\theta)e^{im\varphi}$$

Then the continuity at $\varphi = 0$ and $\varphi = 2\pi$ implies that m is an integer $\Rightarrow$ ℓ is also an integer

All integral values of l appear

$$0 = \hat{L}_+ Y_l^l(\theta,\varphi) = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) Y_l^l(\theta,\varphi)$$

$$= \hbar e^{i\varphi}\left\{\left[\frac{dF_l^l(\theta)}{d\theta}\right] e^{il\varphi} + i\cot\theta\, F_l^l(\theta)\frac{de^{il\varphi}}{d\varphi}\right\}$$

$$\Longleftrightarrow \left(\frac{d}{d\theta} - l\cot\theta\right) F_l^l(\theta) = 0$$

$$dF = l\cot\theta\, d\theta\, F$$

$$\frac{dF}{F} = l\frac{d(\sin\theta)}{\sin\theta}$$

$$d(\ln F) = l\, d[\ln(\sin\theta)]$$

$$\ln F = l\ln(\sin\theta) + C$$

$$F \propto e^{l\ln\sin\theta} = (\sin\theta)^l$$

Clearly, for each l, there exists a unique

$$Y_l^l(\theta,\varphi) \propto (\sin\theta)^l e^{il\varphi}$$

The rest of Y_l^m can be obtained from applying $\hat{L}_-$ to Y_l^l. (C-T Complement A_{VI})

Orthonormality

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, Y_{\ell'}^{m'*}(\theta,\varphi) \, Y_\ell^m(\theta,\varphi) = \delta_{\ell'\ell}\delta_{m'm}$$

Completeness (Closure relation)

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\theta',\varphi') \, Y_\ell^m(\theta,\varphi) = \frac{\delta(\theta-\theta')}{\sin\theta} \delta(\varphi-\varphi')$$

3D delta function

$$\delta(x-x')\delta(y-y')\delta(z-z') = \frac{\delta(r-r')}{r} \frac{\delta(\theta-\theta')}{r\sin\theta} \delta(\varphi-\varphi')$$

Parity

Under reflection $\begin{cases} \theta \mapsto \pi-\theta \\ \varphi \mapsto \pi+\varphi \end{cases}$

$$Y_\ell^m(\pi-\theta,\pi+\varphi) = (-1)^\ell \, Y_\ell^m(\theta,\varphi)$$

$$Y_\ell^{m*}(\theta,\varphi) = (-1)^m \, Y_\ell^{-m}(\theta,\varphi)$$