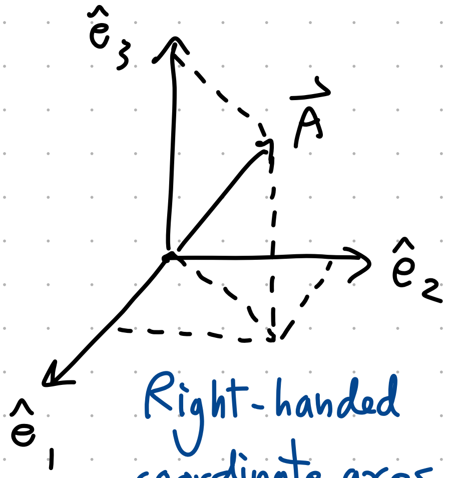


Right-handed coordinate axes

$$\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = \sum_{i=1}^{3} A_i \hat{e}_i$$

Implicitly using an orthonormal basis

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Dot product and cross product (See also Tutorial 24 Nov)

You can define "cross product" in arbitrary dim (exterior product), but it will be a vector only in 3 dim

$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3, \quad \vec{A} \times \vec{B} = \begin{pmatrix} A_2 B_3 - B_2 A_3 \\ A_3 B_1 - B_3 A_1 \\ A_1 B_2 - B_1 A_2 \end{pmatrix}$$

Einstein summation convention: we sum over a repeated index

$$\vec{A} = A_i \hat{e}_i \qquad \vec{A} \cdot \vec{B} = (A_i \hat{e}_i) \cdot (B_j \hat{e}_j) = A_i B_j \, \hat{e}_i \cdot \hat{e}_j = A_i B_j \delta_{ij}$$
$$= A_i B_i$$

dummy indices (หุ่นจำลอง)

Don't do this:

$$\vec{x} \cdot \vec{y} = (x_i \hat{e}_i) \cdot (y_i \cancel{\hat{e}_i})$$

Don't know what are included in which summation

Length $\sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_i A_i} = \sqrt{A_1^2 + A_2^2 + A_3^2}$

$\hat{e}_i \times \hat{e}_j = \hat{e}_k$ where i, j, k are cyclic

$$\begin{array}{ccc} & k & \\ \swarrow & & \nwarrow \\ i & \longrightarrow & j \end{array}$$

$(i,j,k) = (1,2,3)$ or $(3,1,2)$ or $(2,3,1)$

Antisymmetry

$$\hat{e}_i \times \hat{e}_j = -\hat{e}_j \times \hat{e}_i \quad (\text{In particular } \hat{e}_i \times \hat{e}_i = 0)$$

More generally, we can compactly define the cross product as

$$\vec{A} \times \vec{B} = \epsilon_{ijk} A_i B_j \hat{e}_k$$

Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) \text{ is cyclic} \\ -1 & (i,j,k) \text{ is anticyclic} \\ 0 & \text{otherwise} \end{cases}$$

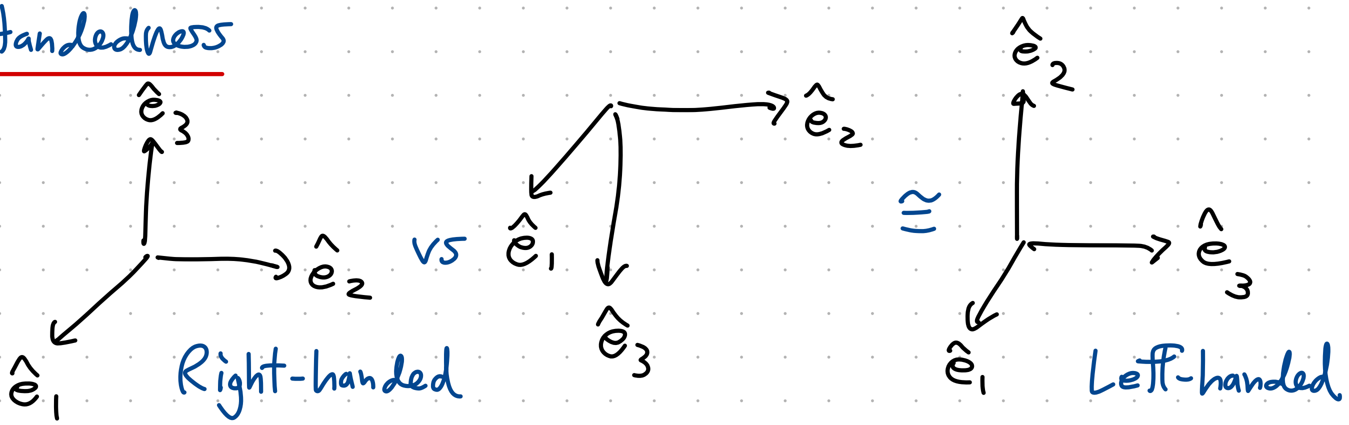
The kth component

$$(\vec{A} \times \vec{B})_k = \epsilon_{ijk} A_i B_j$$

Fixed index Dummy

Examples $\epsilon_{312} = 1$, $\epsilon_{321} = -1$
$\epsilon_{112} = 0$

Handedness



The difference is encapsulated in the triple product $\hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k)$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \epsilon_{ijk} A_i B_j C_k = \det\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

Proof (optional)

Determinant in N dimension

$$\det M = \sum_{\sigma \in S_N} \left(\operatorname{sgn}\sigma \prod_{i=1}^{N} M_{i,\sigma(i)} \right)$$

Permutation group of N objects

$$|S_N| = N!$$

$$3 \text{ dim} \Rightarrow \operatorname{sgn}\sigma = \begin{cases} 1 & \text{cyclic} \\ -1 & \text{anticyclic} \end{cases}$$

$$|S_3| = 3! = 6$$

and specialize the notation to $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$ (I can write this because $\det M^T = \det M$)

$\sigma = (i,j,k)$ means $\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ i & j & k \end{matrix}$

σ	$\prod M_{i,\sigma(i)}$	$\Rightarrow$ New notation	$\operatorname{sgn}\sigma$
$(1,2,3)$	$M_{11}M_{22}M_{33}$	$A_1B_2C_3$	$+1$
$(1,3,2)$	$M_{11}M_{23}M_{32}$	$A_1B_3C_2$	-1
	and so on $\vdots$		

$$\epsilon_{ijk} A_i B_j C_k = \vec{A}\cdot(\vec{B}\times\vec{C})$$

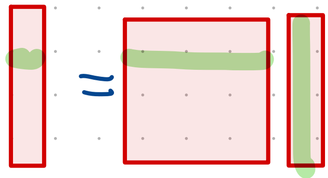
Linear transformations

Matrix $M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$

$$\left(\begin{array}{l}\text{Matrix multiplication} \\ (MN)_{ij} = M_{ik} N_{kj}\end{array}\right)$$

Multiplication with a vector

ith entry →



Let $\vec{B} = M\vec{A}$, $B_i = M_{ij} A_j$

If $\vec{C} = M\vec{e}_i$

$C_j = M_{ji} \Leftrightarrow \vec{C} = \vec{M}_i$, the ith column of M.

So if M maps the set of coordinate axes to a new one, $\det M$ determines whether M changes the handedness of the axes.

$$\hat{e}_1' = M\hat{e}_1$$
$$\hat{e}_2' = M\hat{e}_2$$
$$\hat{e}_3' = M\hat{e}_3$$

$$\hat{e}_1' \cdot (\hat{e}_2' \times \hat{e}_3') = \underbrace{\vec{m}_1 \cdot (\vec{m}_2 \times \vec{m}_3)}_{\text{(columns)}} = \det M$$

Transformations that preserve the dot product (lengths, angles)

$$\vec{A}\cdot\vec{B} \overset{?}{=} (O\vec{A})\cdot(O\vec{B}) = (O\vec{A})_i (O\vec{B})_i = O_{ij} A_j O_{ik} B_k$$
$$= A_j B_k O_{ij} O_{ik} = A_j B_k \underbrace{O^T_{ji} O_{ik}}_{\text{Want } \delta_{jk}}$$

Want $\delta_{jk} \Leftrightarrow \boxed{O^T O = \mathbb{1}}$

The definition of an orthogonal transformation

$$\left(\begin{array}{c}\text{Equivalently} \\ OO^T = \mathbb{1}\end{array}\right)$$

Can be proven using only $\hat{e}_i$

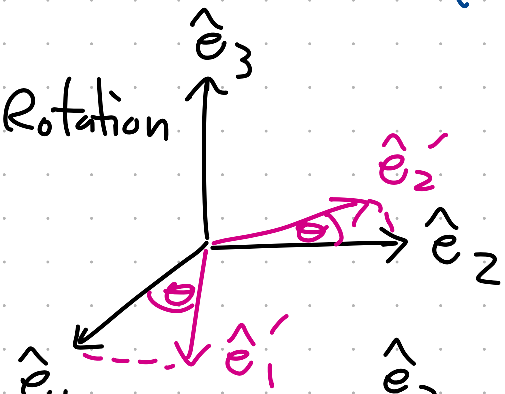
$$\delta_{ij} = \hat{e}_i \cdot \hat{e}_j \overset{?}{=} (O\hat{e}_i)\cdot(O\hat{e}_j)$$
$$= (O\hat{e}_i)_k (O\hat{e}_j)_k$$
$$= O_{ki} O_{kj} = O^T_{ik} O_{kj}$$
$$\Leftrightarrow O^T O = \mathbb{1}$$

What is the det of an orthogonal transform. ?

$$1 = \det \mathbb{1} = \det(O^T O) = \det(O^T)\det(O)$$
$$= \det(O)\det(O) = (\det O)^2$$

$$\Rightarrow \det O = \begin{cases} +1 & \text{Preserves the handedness} \\ -1 & \text{Switches ——— " ———} \end{cases}$$

Rotation

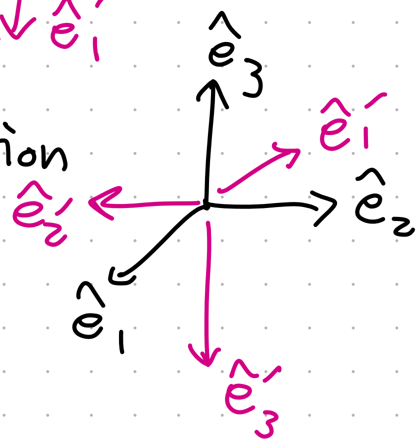


$$\hat{e}_1' = \cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2$$
$$\hat{e}_2' = -\sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_2$$

$$O = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det O = 1$$

Reflection



$$O = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \det O = -1$$

Let us agree that $\det(\vec{A}\vec{B}\vec{C})$ means the det of a 3×3 matrix with columns $\vec{A}$, $\vec{B}$, and $\vec{C}$.

Under the reflection

$$\left.\begin{aligned} \vec{A} &\mapsto \vec{A}' = -\vec{A} \\ \vec{B} &\mapsto \vec{B}' = -\vec{B} \\ \vec{C} &\mapsto \vec{C}' = -\vec{C} \end{aligned}\right\}$$

(1) $\det(\vec{A}'\vec{B}'\vec{C}') = \det[(-\vec{A})(-\vec{B})(-\vec{C})] = -\det(\vec{A}\vec{B}\vec{C})$

$\Rightarrow$ Pseudoscalars change signs under reflections.

(2) Let $\vec{A}\times\vec{B} = \vec{D}$, then $\vec{A}'\times\vec{B}' = (-\vec{A})\times(-\vec{B}) = \vec{D}$

Pseudovectors remain unchanged under reflections

(A product of two vectors act like a scalar in the same way the dot product is a scalar)

Examples

Scalars	$\vec{A}\cdot\vec{B}$
Pseudoscalars	$\vec{A}\cdot(\vec{B}\times\vec{C})$
Vectors	$\vec{A}$, $\vec{A}\times(\vec{B}\times\vec{C})$
Pseudovectors	$\vec{A}\times\vec{B}$

$$\vec{F}_{magnetic} = q\underbrace{\vec{v}}_{\text{Vector}}\times \overbrace{\vec{B}}^{\text{Pseudo vector}}$$

because $\vec{B}$ is a cross product

$\vec{B} = \vec{\nabla}\times\vec{A}$ (Curl of a vector potential)

or $d\vec{B}(\vec{r}) = \frac{I}{c}d\vec{\ell}\times\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$

from Biot-Savart

It is worth emphasizing that these notions are only defined with respect to a given class of transformations

(Here, they are orthogonal transformations in 3 dimensions)

As an example, unitary transformations are privilaged in QM

$U^\dagger U = UU^\dagger = \mathbb{1}$ where $U^\dagger = (U^T)^*$ ← complex conjugation

Tensors

Now we will look at matrices (and more generally multi-index objects) from the point of view of their transformation properties.

Outer product $\vec{A}\vec{B}^T = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}(B_1 B_2 B_3) = \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_3B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{pmatrix}$

(Also called dyadics and can be written as $\vec{A}\otimes\vec{B}^T$ or $|A\rangle\langle B|$ for QM aficionado) or $T_{ij} = A_iB_j$ Example of a rank-2 tensor

Recall $A'_i = O_{ij} A_j$

$$T'_{ij} = O_{ik} O_{jl} A_k B_l = O_{ik} O_{jl} T_{kl}$$ ← want to have j as the last index

$$= O_{ik} T_{kl} (O^T)_{lj} = (O T O^T)_{ij}$$

$$\Leftrightarrow T' = O T O^T = O T O^{-1}$$

So it transforms under O likes an ordinary matrix.
But the transformation is reducible.

vector $\mapsto$ vector$'$ mixes all component = irreducible
scalar $\mapsto$ scalar$'$ —— " —— (trivially)

You will show in the homework a special case of the fact that any rank-2 tensor can be decompose into 3 irreducible parts:

The scalar part: $T^{(0)}_{ij} = \frac{\operatorname{Tr} T}{3} \delta_{ij}$

The vector part: $T^{(1)}_{ij} = \frac{T_{ij} - T_{ji}}{2}$

The rest: $T^{(2)}_{ij} = \frac{T_{ij} + T_{ji}}{2} - \frac{\operatorname{Tr} T}{3} \delta_{ij}$

where the trace $\operatorname{Tr} T = T_{ii} = T_{11} + T_{22} + T_{33}$

More generally, the components of a rank-n tensor transform as

$$T'_{i_1, i_2, \ldots, i_n} = O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_n j_n} T_{j_1, j_2, \ldots, j_n}$$

A contraction by setting a pair of indices to be the same turns a rank n tensor to a rank $(n-2)$ tensor

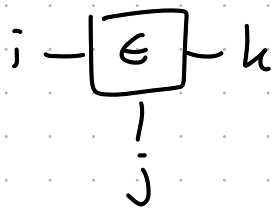
Example: Rank-2 $\quad T_{ij} \mapsto T_{ii} = \mathrm{Tr}\, T \quad$ scalar (rank-0)

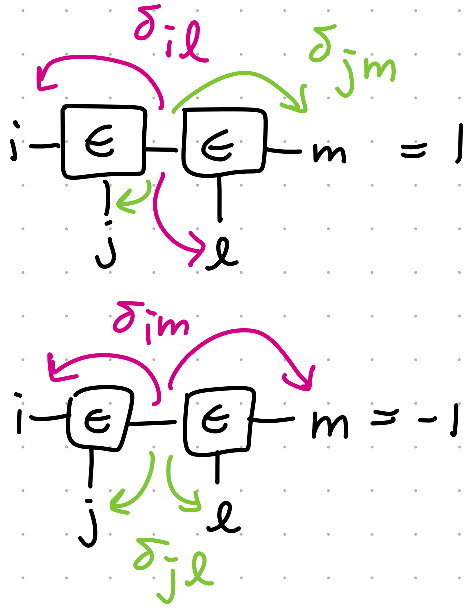
Rank-3 $\quad T_{ijk} = A_i B_j C_k \mapsto T_{iik} = A_i B_i C_k$

$$T \mapsto (\vec{A} \cdot \vec{B}) \vec{C} \quad \text{vector}$$

How to remember the product-of-Levi-Civita identity (Personal note)

Levi-Civita



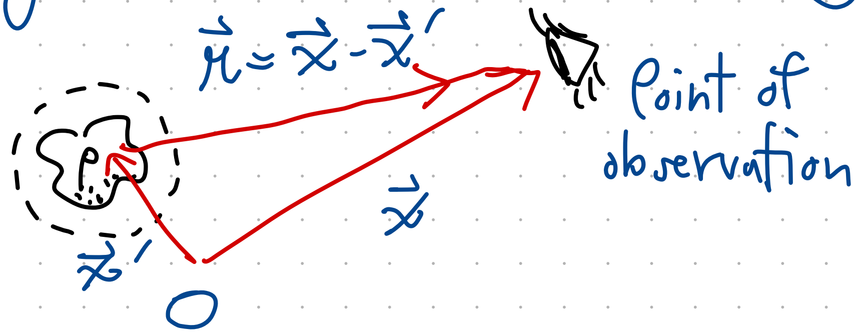


$$\Rightarrow \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Can we this graphical notation to contract any two indices

Ultimately, it's a contraction of two special 3-tensors. This kind of diagrammatical notations is used frequently in many-body quantum physics (and to some extent in QI/HEP)

Multipole moments: approximate, far field solutions for localized charge distributions without conducting boundaries



Let $r := |\vec{x}|$ (I will do multipole expantion in Cartesian coordinates first, so for now I will keep the vector as $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.)

$\vec{x} = r\hat{r}$, $\delta := \frac{r'}{r} \ll 1$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{r^2 - 2\vec{x}\cdot\vec{x}' + r'^2}} = \frac{1}{r}\frac{1}{\sqrt{1 - 2\frac{\vec{x}\cdot\vec{x}'}{r^2} + \left(\frac{r'}{r}\right)^2}}$$

$$(1+\epsilon)^n = 1 + n\epsilon + \frac{n(n-1)}{2}\epsilon^2 + O(\epsilon^3)$$

$$= \frac{1}{r}\left(1 - 2\hat{r}\cdot\hat{r}'\delta + \delta^2\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 + \hat{r}\cdot\hat{r}'\delta - \frac{\delta^2}{2} + \frac{3}{8}\left(-2\hat{r}\cdot\hat{r}'\delta + \delta^2\right)^2 + \dots\right)$$

$(\ )^2$

$$= \frac{1}{r}\left(1 + \hat{r}\cdot\hat{r}'\delta + \left(\frac{3}{2}(\hat{r}\cdot\hat{r}')^2 - \frac{1}{2}\right)\delta^2 + O(\delta^3)\right)$$

$$= \frac{1}{r} + \frac{\hat{r}\cdot\vec{x}'}{r^2} + \frac{1}{2}\frac{\hat{r}\cdot\left(3\vec{x}'\vec{x}' - \mathbb{1}r'^2\right)\cdot\hat{r}}{r^3} + \dots$$

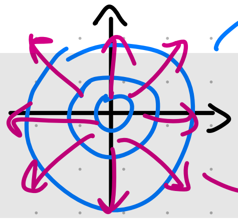
primed coordinates

Properties of the source alone, not the observation point

$$\Phi(\vec{x}) = \frac{q}{r} + \frac{\hat{r}\cdot\vec{p}}{r^2} + \frac{1}{2}\frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{r^3} + \dots$$

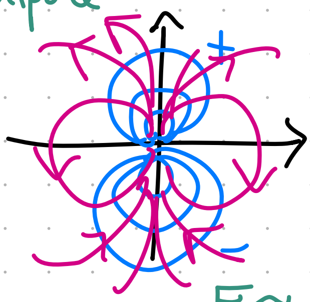
Multipole moments

Monopole $q = \int d^3x' \rho(\vec{x}')$



Dipole $\vec{p} = \int d^3x' \vec{x}' \rho(\vec{x}') = q\langle \vec{x}' \rangle$

Pure dipole $p_z = \frac{q\cos\theta}{r^2}$



$E \sim \frac{1}{r^3}$

If there are positive (resp. negative) charges with distribution N_+ (resp. N_-)

$$\vec{p} = q_+ \langle \vec{x}' \rangle_{N_+} - q_- \langle \vec{x}' \rangle_{N_-}$$

Special case $|q_+| = |q_-| \Rightarrow \vec{p} = q\underbrace{\left(\langle \vec{x}' \rangle_{N_+} - \langle \vec{x}' \rangle_{N_-}\right)}_{d \quad [Q][L]}$

$d \updownarrow \begin{matrix} \cdot +q \\ \cdot -q \end{matrix}$ No monopole.
Model of polarization in dielectric materials.

Quadrupole $Q_{ij} \quad 3\int d^3x' \underbrace{\left(x_i' x_j' - \frac{r'^2}{3}\delta_{ij}\right)}_{\text{Symmetric, traceless}} \rho(\vec{x}')$

Does this look familiar to you?

Recall that in HW 1.2, you have shown similar properties for the irreducible 3-tensor

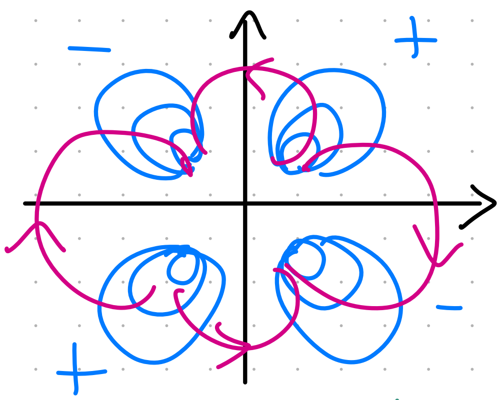
$$x_i x_j x_k - \frac{1}{5}(x_i \delta_{jk} + x_k \delta_{ij} + x_j \delta_{ki}) r^2$$

A rank-ℓ Cartesian tensor contains 3^ℓ components but they are not all independent. Picking $i = 1,2,3$ for each of the ℓ indices

- A symmetric tensor has $\binom{\ell+3-1}{3-1} = \frac{(\ell+2)(\ell+1)}{2}$ independent components
- A tensor has $\binom{\ell}{2} = \frac{\ell(\ell-1)}{2}$ traces. Choosing 2 indices to contract

- Therefore, a traceless symmetric tensor has

$$\frac{(\ell+2)(\ell+1)}{2} - \frac{\ell(\ell-1)}{2} = 2\ell+1 \text{ independent components}$$



Pure quadrupole

A nonzero quadrupole moment tells us about the anisotropy of the charge/mass distribution e.g. the Earth has a gravitational quadrupole moment because it is not a perfect sphere. (It is oblate – a little flat at the poles.)

Examples ① Spherically symmetric charge distribution

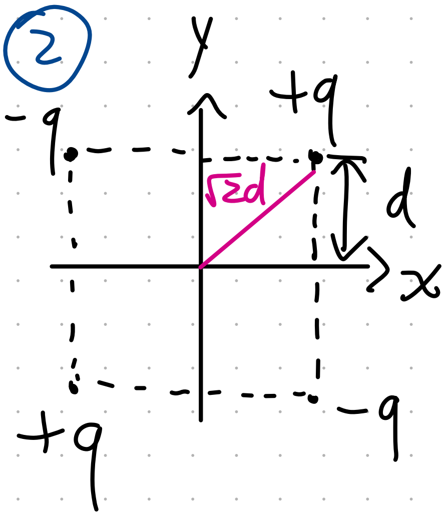
$$Q_{xx} = 3q\int d^3x\left(x^2 - \frac{r^2}{3}\right)\rho(\vec{x})$$

$$= 3q\left(\langle x^2\rangle - \frac{\langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle}{3}\right)$$

Spherical symmetry $\Rightarrow \langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle \Rightarrow Q_{xx} = 0.$

Turns out all $Q_{ij} = 0.$

②



No net charge ⇒ No monopole moment

$$\vec{p} = q\Big(\underbrace{\langle\vec{x}\rangle_{N_+}}_{\begin{pmatrix}0\\0\end{pmatrix}} - \underbrace{\langle\vec{x}\rangle_{N_-}}_{\begin{pmatrix}0\\0\end{pmatrix}}\Big) = 0 \quad \text{No dipole}$$

In contrast, a gravitational monopole is always nonzero because there is no negative mass.

$$Q_{ij} = \sum_{\alpha} \left(3 x_i^{(\alpha)} x_j^{(\alpha)} - (r^{(\alpha)})^2 \delta_{ij} \right) q^{(\alpha)}$$

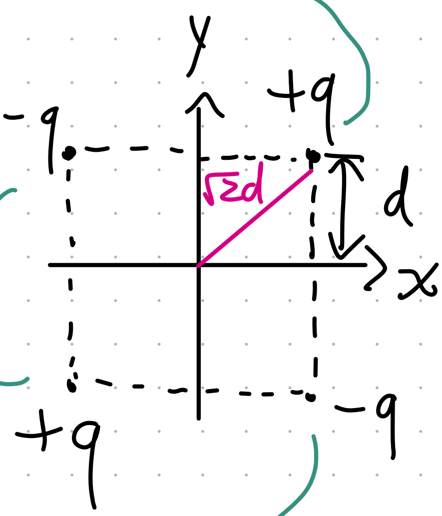
$2d^2$ for this charge config.

α: charge label

$Q_{xz} = Q_{yz} = 0$ because the charges lie in the $z=0$ plane.

x^2

$$\begin{aligned} Q_{xx} &= q\left[3d^2 - 2d^2\right] \\ &\quad + q\left[3(-d)^2 - 2d^2\right] \\ &\quad - q\left[3d^2 - 2d^2\right] \\ &\quad - q\left[3(-d)^2 - 2d^2\right] \\ &= 2qd^2 - 2qd^2 = 0 \end{aligned}$$



Interchanging x and y makes no difference $\Rightarrow Q_{yy} = 0$

Traceless condition $\Rightarrow Q_{zz} = 0$

The only component left is

$$\begin{aligned} Q_{xy} &= q\left[3d\cdot d - 2d^2\right] \\ &\quad + q\left[3(-d)(-d) - 2d^2\right] \\ &\quad - q\left[3d(-d) - 2d^2\right] \\ &\quad - q\left[3(-d)d - 2d^2\right] \\ &= 2qd^2 - 2q(-3d^2 - 2d^2) \\ &= 2qd^2 + 10qd^2 \\ &= 12qd^2 \end{aligned}$$

$$\overleftrightarrow{Q} = \begin{pmatrix} 0 & 12qd^2 & 0 \\ 12qd^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have just found that the first few multipole moments are symmetric and traceless by inspection. But now let us derive these properties once and for all by writing down the general expression for the Taylor expansion.

Recall in 1D:

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_a + \frac{1}{2!}(x-a)^2\left.\frac{\partial^2 f}{\partial x^2}\right|_a + \cdots$$

Similarly in 3D:

$$f(\vec{x}) = f(\vec{a}) + (\vec{x}-\vec{a})\cdot\vec{\nabla}_{\vec{x}} f\Big|_{\vec{a}} + \cdots$$

$$f(\vec{x}+\vec{a}) = f(\vec{x}) + \vec{a}\cdot\vec{\nabla}_{\vec{x}+\vec{a}} f\Big|_{\vec{x}} \leftarrow a_i \partial_i f\Big|_{\vec{x}}$$

$$+ a_i a_j \partial_i \partial_j f\Big|_{\vec{x}}$$

Now the function that we want to Taylor expand is nothing but the Green function $G(\vec{x}-\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + (-\vec{x}')\cdot\vec{\nabla}_{\vec{r}}\frac{1}{|\vec{r}|}\Big|_{\vec{x}} + \frac{1}{2!}x'_i x'_j \partial_i\partial_j \frac{1}{|\vec{r}|}\Big|_{\vec{x}} + \cdots$$

$\frac{1}{r}$

$$= \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\underbrace{x'_{i_1}x'_{i_2}\cdots x'_{i_\ell}}_{\substack{P^{(\ell)} \\ \text{reducible} \\ \text{rank-}\ell\text{ tensor}}}\underbrace{\partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}\frac{1}{r}\Big|_{\vec{x}}}_{\substack{T^{(\ell)} \\ \text{irreducible} \\ \text{symmetric, traceless}}}$$

Why?

$$T_{i_1 i_2 \cdots i_\ell} = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}\frac{1}{\mathscr{r}}\Big|_{\vec{x}}$$ is obviously symmetric under interchanging any two indices.

$$\mathrm{Tr}_{jk}\left(T_{i_1\cdots j\cdots k\cdots i_\ell}\right) = T_{i_1\cdots j\cdots j\cdots i_\ell} = \partial_{i_1}\cdots\partial_{i_\ell}\underbrace{\nabla^2\frac{1}{\mathscr{r}}\Big|_{\vec{x}}}_{4\pi\delta(\vec{x}-\vec{x}')|_{\vec{x}}}$$

evaluated at the observation point $\vec{x}$ where there is no charge

$\Rightarrow \mathrm{Tr}(T) = 0$

This is the reason T is called a harmonic tensor

It satisfies the Laplace eq.

$\nabla^2 T = 0.$

Let us evaluate T explicitly.

$$T_{i_1 i_2 \cdots i_\ell} = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}\frac{1}{\mathscr{r}}\Big|_{\vec{x}}$$

Each derivative adds 2 to the power of $\mathscr{r}$:

$$\left(\partial_{\vec{\mathscr{r}}}\frac{1}{\mathscr{r}}\right)_i\Big|_{\vec{x}} = \left(\vec{\nabla}_{x-\vec{x}}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}\right)_i\Big|_{\vec{x}}$$

$$= -\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}\cdot 2(x_i-x_i')\Big|_{\vec{x}}$$

$$= -x_i/r^3$$

$$\Rightarrow \boxed{\partial_{i_1}\cdots\partial_{i_\ell}\frac{1}{\mathscr{r}}\Big|_{\vec{x}} = (-1)^\ell(2\ell-1)!!\,\frac{[x_{i_1}x_{i_2}\cdots x_{i_\ell}]^{(\ell)}}{r^{2\ell+1}}}$$

Plugging this expression for T into the multipole expansion gives

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{\cancel{(-1)^l}}{l!} \underbrace{x'_{i_1} x'_{i_2} \cdots x'_{i_l}}_{\rho^{(l)}} \underbrace{\cancel{(-1)^l}(2l-1)!! \frac{[x_{i_1} \cdots x_{i_l}]^{(l)}}{r^{2l+1}}}_{T^{(l)}}$$

Recall from HW 1.2 that a contraction with an irreducible tensor picks out only the irreducible part, therefore without loss of generality we can write

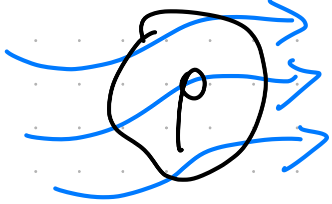
$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \underbrace{(2l-1)!! [x'_{i_1} \cdots x'_{i_l}]^{(l)}}_{\text{Primed}} \frac{1}{l!} \underbrace{\frac{[x_{i_1} \cdots x_{i_l}]^{(l)}}{r^{2l+1}}}_{\text{Unprimed}}$$

$\downarrow$

$$\boxed{\begin{array}{l}\text{Multipole moment} \\ Q^{(l)}_{i_1 i_2 \cdots i_l} = (2l-1)!! \int d^3x' \rho(\vec{x}') [x'_{i_1} x'_{i_2} \cdots x'_{i_l}]^{(l)}\end{array}}$$

This concludes the proof that we can always define multipole moments that are symmetric and traceless.

Multipole expansion of the energy in an external field



Here we will see the characteristic of how each multipole interact with an external field.

Expand the external potential around the "center of charge" x_0:

$$\nabla^2 \phi = 0$$

symmetric traceless again!

$$\phi(\vec{x}) = \phi(0) + \vec{x}\cdot\vec{\nabla}\phi\Big|_{x_0} + \cdots + \frac{1}{\ell!} x_{i_1}\cdots x_{i_\ell}\partial_{i_1}\cdots\partial_{i_\ell}\phi\Big|_{x_0} + \cdots$$

$$U = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\int d^3x'\rho(\vec{x}')\,[x_{i_1}\cdots x_{i_\ell}]^{(\ell)}\partial_{i_1}\cdots\partial_{i_\ell}\phi\Big|_{x_0}$$

$$= \sum_{\ell=0}^{\infty}\frac{1}{\ell!(2\ell-1)!!}Q^{(\ell)}_{i_1\cdots i_\ell}\partial_{i_1}\cdots\partial_{i_\ell}\phi\Big|_{x_0}$$

$$= Q^{(0)}\phi(x_0) + Q^{(1)}_i\partial_i\phi\Big|_{x_0} + \frac{1}{6}Q^{(2)}_{ij}\partial_i\partial_j\phi\Big|_{x_0} + \cdots$$

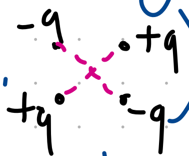
The charge interacts with the potential $q\phi(x_0)$

The dipole interacts with the electric field $-p_iE_i$

The quadrupole interacts with the gradient of the field

and so on...

Spherical representation of the multipole expansion

Cartesian multipole expansion is convenient when we have charge distribution with Cartesian symmetry (e.g. 

) but not when we have cylindrical or spherical symmetry.

The spherical representation is also more natural if you notice that what we have done in the Cartesian case is Taylor expanding $\frac{1}{|\vec{r}-\vec{r}'|}$ which has spherical symmetry around the point $\vec{r}=\vec{r}'$.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r}\underbrace{\left(1-2\hat{r}\cdot\hat{r}'\delta+\delta^2\right)^{-1/2}}_{\text{Azimuthally symmetric function on a sphere}} =$$

Linear combination of eigenvectors of $\hat{L}_z = -i\frac{\partial}{\partial\phi}$ corresponding to the eigenvalue 0.

$\delta = \frac{r'}{r}$

generates rotations
$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$P_\ell(\hat{r}\cdot\hat{r}')$, $\ell = 0,1,2,\dots,\infty$

$2\ell+1$ independent components of Cartesian multipole moments	$\Longleftrightarrow$	$2\ell+1$ dimension of of SO(3) irreps ("integer-spin")

Generating function

$$\left(1-2\hat{r}\cdot\hat{r}'\delta+\delta^2\right)^{-1/2}=\sum_{\ell=0}^{\infty}\delta^\ell P_\ell(\hat{r}\cdot\hat{r}')$$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|}=\sum_{\ell=0}^{\infty}\frac{(r')^\ell}{r^{\ell+1}}P_\ell(\hat{r}\cdot\hat{r}')$$

$$\phi(\vec{x})=\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\int d^3x'(r')^\ell P_\ell(\hat{r}\cdot\hat{r}')$$

Griffiths stops here (Section 3.4). However, notice that we have not separate the source variables (primed) and the variables for the observation point (unprimed). Hence we can't define the multipole moments as properties of the source yet.

(We could also have written $P_\ell(\cos\theta)$ where $\hat{r}\cdot\hat{r}'=\cos\theta$ and mistaken θ to be the angular coordinate for the source and defined $Q^{(\ell)}$ correspondingly. But something would clearly not be right as $Q^{(\ell)}$ in this case would just be one number, not $2\ell+1$.)

To separate the two sets of variables, we use the addition theorem: $P_\ell(\hat{r}\cdot\hat{r}') = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi')$

(This theorem actually has a simple and elegant interpretation by thinking of spherical harmonics as matrix elements of irreps of $SO(3)$, the so-called "Wigner D-matrices")

$$\Rightarrow \boxed{\phi(\vec{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, q_{\ell m}\, \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}}$$

(Eq. 4.1–4.3 of Jackson)

where $\boxed{q_{\ell m} = \int d^3x' (r')^\ell \rho(\vec{x}')\, Y^*_{\ell m}(\theta',\phi')}$

are the spherical multipole moments. Since there are $2\ell+1$ $Y_{\ell m}$'s for a given ℓ, there are $2\ell+1$ of the $q_{\ell m}$'s as it should be.

The $Y_{\ell m}(\theta',\phi')$ only reduces to $P_\ell(\cos\theta')$ if the charge distribution has azimuthal symmetry. If we did not use the addition theorem, we have to write out each $P_\ell(\hat{r}\cdot\hat{r}')$ and factor out the unprimed variables by hand (which is what Griffiths does for the first two moments).

Components of $q_{\ell m}$ (Jackson p.146)

$m \backslash \ell$	0	1	2
0	$\frac{1}{\sqrt{4\pi}} q$	$\sqrt{\frac{3}{4\pi}}\, p_z$	$\frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33}$
1	–	$-\sqrt{\frac{3}{8\pi}}(p_x - i p_y)$	$-\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23})$
2	–	–	$\frac{1}{2}\sqrt{\frac{15}{2\pi}}(Q_{11} - 2iQ_{12} - Q_{22})$

Case of azimuthal symmetry

$q_{\ell m} = 0$ if $m \neq 0$

$$\phi(\vec{x}) = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} q_{\ell 0} \underbrace{Y_{\ell 0}(\theta)}_{\sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)} = \sum_{\ell=0}^{\infty} Q^{(\ell)} \frac{P_\ell(\cos\theta)}{r^{\ell+1}}$$

$$\boxed{Q^{(\ell)} = \int d^3x' (r')^\ell P_\ell(\cos\theta')}$$

A scalar for each ℓ. Not a (higher-rank) tensor anymore.

Relation to Cartesian multipoles

http://www.physics.sfsu.edu/~lea/courses/oldgrad/grad/sphermult.pdf

$l=0$ is trivial because it is only a constant function.

$$q_{00} = \int d^3x' \, r'^0 \, Y^*_{00} \, \rho(\vec{x}') = \frac{1}{\sqrt{4\pi}} \int d^3x' \, \rho(\vec{x}') = \frac{q}{\sqrt{4\pi}}$$

$l=1$ $m=0$

$$q_{10} = \int d^3x' \, r' \underbrace{\sqrt{\frac{3}{4\pi}} \cos\theta'}_{Y^*_{10}} \rho(\vec{x}') = \sqrt{\frac{3}{4\pi}} \int d^3x' \, z' \rho(\vec{x}')$$

$$= \sqrt{\frac{3}{4\pi}} \, p_z$$

$m=1$

$$Y^*_{11} = (-1)^l \, Y_{1,-1}$$

$$q_{11} = \int d^3x' \, r' \underbrace{\left(-\sqrt{\frac{3}{8\pi}} \sin\theta' e^{-i\phi'}\right)}_{Y^*_{11}} \rho(\vec{x}')$$

$$= -\sqrt{\frac{3}{8\pi}} \int d^3x' \underbrace{r' \sin\theta' (\cos\phi' - i\sin\phi')}_{x'-iy'} \rho(\vec{x}')$$

$$= -\sqrt{\frac{3}{8\pi}} (p_x - i p_y)$$

Similar for higher order multipoles. For example, $l=2$, $m=0$

$$q_{20} = \int d^3x' (r')^2 \frac{1}{2} \sqrt{\frac{5}{4\pi}} (3\cos^2\theta' - 1) \, \rho(\vec{x}')$$

$$= \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int d^3x' \left[3(z')^2 - (r')^2\right] \rho(\vec{x}') = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \, Q_{33}$$

What is curious is that, since the $Y_{\ell m}$'s are complex, the multipole moments are complex numbers!

How can we understand this? Spherical harmonics are simulataneous eigenfunctions of L^2 and L_z

L^2: Tells you the SO(3)-irrep label ℓ

L_z: Tells you the SO(2) (z rotations)-irrep label m

z as a function is invariant under $L_z = -i\frac{\partial}{\partial\phi} \Rightarrow m=0$

There is no more eigenfunction of L_z in 3D ... unless you allow complex vectors.

$$x \pm iy = r\sin\theta(\cos\phi \pm i\sin\phi) = r\sin\theta\, e^{\pm i\phi} \Rightarrow m = \pm 1$$

$$\left(-i\frac{\partial}{\partial\phi} e^{\pm i\phi} = \mp i^2 e^{\pm i\phi} = \pm e^{\pm i\phi}\right)$$

$$Y_{\ell m}(\vec{x}) = \sum_{m_1, m_2, m_3} (x+iy)^{m_1}(x-iy)^{m_2} z^{m_3} C_{m_1, m_2, m_3}$$

where $\begin{cases} \ell = m_1 + m_2 + m_3 \\ m = m_1 - m_2 \end{cases}$

Examples

$\ell = 2 \quad (x \pm iy)^2 = r^2 \sin^2\theta\, e^{\pm i2\phi}$

Summary

$$\phi(\vec{x}) = \sum_{\ell} \frac{1}{\ell!} Q^{(\ell)}_{i_1 i_2 \cdots i_\ell} \frac{[x_{i_1} \cdots x_{i_\ell}]^{(\ell)}}{r^{2\ell+1}}$$

$$= \sum_{\ell m} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}}$$

Same quantity, two different "representations"

Cartesian $Q^{(\ell)}_{i_1 \cdots i_\ell} = (2\ell-1)!! \int d^3x' \rho(\vec{x}') [x'_{i_1} \cdots x'_{i_\ell}]^{(\ell)}$

Spherical $q_{\ell m} = \int d^3x' (r')^\ell \rho(\vec{x}') Y^*_{\ell m}(\theta', \phi')$

SO(3)-irreps as $\begin{cases} \text{functions on the unit sphere} \\ \text{tensors} \end{cases}$

Physics: Approximation of the potential due to a localized charge distribution in the power of $1/r$.

Each multipole contributes to the energy (of the charge distribution in an external field) in a different way.

$$U = Q^{(0)} \phi(x_0) + Q^{(1)}_i \partial_i \phi \Big|_{x_0} + \frac{1}{6} Q^{(2)}_{ij} \partial_i \partial_j \phi \Big|_{x_0} + \cdots$$