LP with Complicating Constraints

1. Problem Data

$$c = (3, 5, 4, 2, 1, 6, 2.5, 3.5), \quad A = \begin{bmatrix} 2 & 1 & 1 & 3 & 2 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 1 & 3 & 2 \end{bmatrix}, \quad b = (40, 32, 27)$$

Blocks:

$B_1 = \{1, 2\},\ B_2 = \{3, 4\},\ B_3 = \{5, 6\},\ B_4 = \{7, 8\}, \quad f = (5, 6, 4, 7), \quad 0 \le x_j \le \bar{u}_j = (5, 5, 6, 6, 4, 4, 7, 7).$

2. Iteration 1

Duals from master:

$$\lambda^{(1)} = (10^6, -10^6, -10^6), \quad \sigma^{(1)} = 4.70000605 \times 10^7.$$

Reduced costs $\bar{c} = c - A^\top \lambda^{(1)}$:

$\bar{c} = (-9.99997{\times}10^5,\ 2.000005{\times}10^6,\ 4.000004{\times}10^6,\ -1.999998{\times}10^6,\ 1,\ 2.000006{\times}10^6,\ 4.0000025{\times}10^6,\ 1.0000035{\times}10^6).$

Block solutions (fill smallest $\bar{c}_j$ until $\sum x_j = f_k$):

$$x^{(1)} = (5, 0, 0, 6, 4, 0, 0, 7).$$

$$v^{(1)} = \bar{c}^\top x^{(1)} = -9.9999445 \times 10^6, \quad z^{(1)} = c^\top x^{(1)} = 55.5.$$

$$r^{(1)} = Ax^{(1)} = (50, 22, 18).$$

Since $v^{(1)} < \sigma^{(1)}$, add new column.

3. Iteration 2

$$\lambda^{(2)} = (-0.263158, 10^6, -10^6), \quad \sigma^{(2)} = -3.999931 \times 10^6.$$

Reduced costs:

$\bar{c} = (-9.9999647{\times}10^5,\ -9.9999474{\times}10^5,\ -9.9999574{\times}10^5,\ -9.9999721{\times}10^5,\ 1.5263,\ -9.9999374{\times}10^5,\ 1.0000028{\times}10^6$

Block solutions:

$$x^{(2)} = (5, 0, 0, 6, 0, 4, 7, 0).$$

$$v^{(2)} = -7.999921 \times 10^6, \quad z^{(2)} = 68.5, \quad r^{(2)} = (39, 33, 25).$$

Since $v^{(2)} < \sigma^{(2)}$, add column.

4. Iteration 3

$$\lambda^{(3)} = (0, 2.789474, -2.526316), \quad \sigma^{(3)} = 39.605263.$$

Reduced costs:

$$\bar{c} = (0.210526,\ 1.947368,\ 0.684210,\ -0.789474,\ 0.736842,\ 2.947368,\ 4.5,\ 5.763158).$$

Block solutions:

$$x^{(3)} = (5, 0, 0, 6, 4, 0, 7, 0).$$

$$v^{(3)} = 30.763158, \quad z^{(3)} = 48.5, \quad r^{(3)} = (43, 29, 25).$$

Since $v^{(3)} < \sigma^{(3)}$, add column.

5. Iteration 4 (Convergence)

$$\lambda^{(4)} = (0, 5, -4), \quad \sigma^{(4)} = 3.5.$$

Reduced costs:

$$\bar{c} = (-2, -1, -3, -3, 0, 0, 4.5, 6.5).$$

Block solutions:

$$x^{(4)} = (5, 0, 6, 0, 4, 0, 7, 0).$$

$$v^{(4)} = (-2) \times 5 + (-3) \times 6 + 0 \times 4 + 4.5 \times 7 = 3.5 = \sigma^{(4)}.$$

Convergence reached since $v^{(4)} = \sigma^{(4)}$.

6. Optimal Solution

Recovered from master convex combination:

$$x^\star = (5, 0, 1, 5, 3, 1, 7, 0), \quad z^\star = 55.5.$$

Verification:

$$Ax^\star = (40, 32, 27) = b.$$

7. Direct LP Verification

Solving full LP directly:

$$x_{\text{full}} = x^\star, \quad z_{\text{full}} = 55.5.$$

$$\max |x_{\text{full}} - x^\star| = 0, \quad z_{\text{full}} - z^\star = 0.$$

8. Iteration Summary

Iter	$\lambda^\top$	σ	v	z	$r^\top$
1	$(10^6, -10^6, -10^6)$	4.7×10^7	-9.9999×10^6	55.5	$(50, 22, 18)$
2	$(-0.26, 10^6, -10^6)$	-4.0×10^6	-7.9999×10^6	68.5	$(39, 33, 25)$
3	$(0, 2.789, -2.526)$	39.605	30.763	48.5	$(43, 29, 25)$
4	$(0, 5, -4)$	3.5	3.5	60.5	$(31, 41, 37)$