Benders Decomposition Step-by-Step Solution

1 Problem Setup

Original Problem:

$$\begin{aligned} &\min z = -1.5x - 10y && (1)\\ &\text{s.t.} \quad 0 \le x \le 16 && (2)\\ &\qquad y - 1.5x \le 5 && (3)\\ &\qquad y - 0.5x \le 7.5 && (4)\\ &\qquad y + 0.5x \le 17.5 && (5)\\ &\qquad -y + x \le 10 && (6)\\ &\qquad y \ge 0 && (7) \end{aligned}$$

Treating x as the complicating variable, we decompose into:

- Master problem (in x and α)
- Subproblem (in y, given fixed x)

2 Iteration 1

2.1 First Master Problem

$$\begin{aligned} \min_{x,\alpha} \quad & z_M = -1.5x + \alpha && (8)\\ \text{s.t.} \quad & 0 \le x \le 16 && (9)\\ & \alpha \ge -100 \quad \text{(initial lower bound)} && (10) \end{aligned}$$

Solution: $x^{(1)} = 16$, $\alpha^{(1)} = -100$
Objective: $z_M^{(1)} = -1.5(16) - 100 = -124$

2.2 First Subproblem

Given $x = 16$:

$$\min_{y} \quad z = -10y \tag{11}$$

$$\text{s.t.} \quad y - 1.5(16) \leq 5 \quad \Rightarrow \quad y \leq 29 \tag{12}$$

$$y - 0.5(16) \leq 7.5 \quad \Rightarrow \quad y \leq 15.5 \tag{13}$$

$$y + 0.5(16) \leq 17.5 \quad \Rightarrow \quad y \leq 9.5 \quad \text{(active)} \tag{14}$$

$$-y + 16 \leq 10 \quad \Rightarrow \quad y \geq 6 \tag{15}$$

$$y \geq 0 \tag{16}$$

Solution: $y^{(1)} = 9.5$, $\lambda^{(1)} = 5$ (dual multiplier from constraint $y + 0.5x \leq 17.5$)

2.3 First Bounds Calculation

- **Upper bound:** $z_{\text{up}}^{(1)} = -1.5(16) + (-10)(9.5) = -24 - 95 = -119$
- **Lower bound:** $z_{\text{down}}^{(1)} = -1.5(16) + (-100) = -124$
- **Gap:** $|-119 - (-124)| = 5 > \varepsilon$

2.4 First Benders Cut

$$\alpha \geq -10(9.5) + 5(x - 16) = -95 + 5(x - 16)$$

3 Iteration 2

3.1 Second Master Problem

$$\min_{x,\alpha} \quad z_M = -1.5x + \alpha \tag{17}$$

$$\text{s.t.} \quad 0 \leq x \leq 16 \tag{18}$$

$$\alpha \geq -100 \tag{19}$$

$$\alpha \geq -95 + 5(x - 16) \quad \text{(Benders cut)} \tag{20}$$

Solution: $x^{(2)} = 6.25$, $\alpha^{(2)} = -143.75$
Objective: $z_M^{(2)} = -1.5(6.25) + (-143.75) = -153.125$

3.2 Second Subproblem

Given $x = 6.25$:

$$\min_{y} \quad z = -10y \tag{21}$$

$$\text{s.t.} \quad y \leq 14.375 \quad (\text{from } y - 1.5x \leq 5) \tag{22}$$

$$y \leq 10.625 \quad (\text{from } y - 0.5x \leq 7.5, \text{ active}) \tag{23}$$

$$y \leq 14.375 \quad (\text{from } y + 0.5x \leq 17.5) \tag{24}$$

$$y \geq -6.25 \quad (\text{from } -y + x \leq 10, \text{ not binding}) \tag{25}$$

$$y \geq 0 \tag{26}$$

Solution: $y^{(2)} = 10.625$, $\lambda^{(2)} = -5$ (dual multiplier)

3.3 Second Bounds Calculation

- **Upper bound:** $z_{\text{up}}^{(2)} = -1.5(6.25) + (-10)(10.625) = -9.375 - 106.25 = -115.625$
- **Lower bound:** $z_{\text{down}}^{(2)} = -1.5(6.25) + (-143.75) = -153.125$
- **Gap:** $|-115.625 - (-153.125)| = 37.5 > \varepsilon$

3.4 Second Benders Cut

$$\alpha \geq -10(10.625) + (-5)(x - 6.25) = -106.25 - 5(x - 6.25)$$

4 Iteration 3

4.1 Third Master Problem

$$\min_{x,\alpha} \quad z_M = -1.5x + \alpha \tag{27}$$

$$\text{s.t.} \quad 0 \leq x \leq 16 \tag{28}$$

$$\alpha \geq -100 \tag{29}$$

$$\alpha \geq -95 + 5(x - 16) \tag{30}$$

$$\alpha \geq -106.25 - 5(x - 6.25) \tag{31}$$

Solution: $x^{(3)} = 10$, $\alpha^{(3)} = -125$
Objective: $z_M^{(3)} = -1.5(10) + (-125) = -140$

4.2 Third Subproblem

Given $x = 10$:

$$\min_{y} \quad z = -10y \tag{32}$$

$$\text{s.t.} \quad y \leq 20 \quad (\text{from } y - 1.5x \leq 5) \tag{33}$$

$$y \leq 12.5 \quad (\text{from } y - 0.5x \leq 7.5, \text{ active}) \tag{34}$$

$$y \leq 12.5 \quad (\text{from } y + 0.5x \leq 17.5, \text{ active}) \tag{35}$$

$$y \geq 0 \quad (\text{from } -y + x \leq 10, \text{ not binding}) \tag{36}$$

$$y \geq 0 \tag{37}$$

Solution: $y^{(3)} = 12.5$, $\lambda^{(3)} = -5$

4.3 Third Bounds Calculation

- **Upper bound:** $z_{\text{up}}^{(3)} = -1.5(10) + (-10)(12.5) = -15 - 125 = -140$
- **Lower bound:** $z_{\text{down}}^{(3)} = -1.5(10) + (-125) = -140$
- **Gap:** $|-140 - (-140)| = 0 \leq \varepsilon$

5 Convergence

Optimal Solution:

- $x^* = 10$
- $y^* = 12.5$
- $z^* = -140$

The algorithm converged in 3 iterations, finding the same optimal solution as the direct method.

6 Summary Table

Iter	x	α	y	λ	z_{up}	z_{down}	Gap
1	16	-100	9.5	5	-119	-124	5
2	6.25	-143.75	10.625	-5	-115.625	-153.125	37.5
3	10	-125	12.5	-5	-140	-140	0

Table 1: Benders Decomposition Iteration Summary