

OPT HW2

April 21, 2025

```
[1]: import numpy as np
import cvxpy as cp
import mosek
import matplotlib.pyplot as plt

from tqdm import tqdm
```

0.1 Q1

```
[2]: np.random.seed(42)

m = 100 # sample size
n = 200 # dimension

A = np.random.randn(m, n)
b = np.random.randn(m)

L = 2 * np.linalg.norm(A, 2)**2
t = 1.0 / L

#to compute optimal value f^*
def lasso(X, y, lambd):
    beta = cp.Variable(X.shape[1])
    objective = cp.Minimize(cp.sum_squares(y - X @ beta))
    constraints = [cp.norm1(beta) <= lambd]
    prob = cp.Problem(objective, constraints)
    prob.solve(solver=cp.MOSEK)
    return prob.value
```

0.2 (A) Projected Gradient-Descent

```
[3]: def proj_l1(v, lambd):
    z = cp.Variable(v.shape[0])
    objective = cp.Minimize(cp.sum_squares(z - v))
    constraints = [cp.norm1(z) <= lambd]
    prob = cp.Problem(objective, constraints)
    prob.solve(solver=cp.MOSEK)
```

```

    return z.value

def PGD_lasso(A, b, lambd, stepsize, iters):
    n = A.shape[1]
    x = np.zeros(n)
    f_vals = np.zeros(iters)

    for t in range(iters):
        grad = 2 * A.T @ (A @ x - b)
        x = proj_l1(x - stepsize * grad, lambd)
        f_vals[t] = np.linalg.norm(A @ x - b)**2

    return f_vals

```

0.3 (B) Frank-Wolfe

```
[4]: def lmo_l1(grad, lambd):
    n = grad.shape[0]
    s = cp.Variable(n)
    objective = cp.Minimize(grad @ s)
    constraints = [cp.norm1(s) <= lambd]
    prob = cp.Problem(objective, constraints)
    prob.solve(solver=cp.MOSEK)
    return s.value

def FW_lasso(A, b, lambd, iters):
    n = A.shape[1]
    x = np.zeros(n)
    f_vals = np.zeros(iters)

    for t in range(iters):
        grad = 2 * A.T @ (A @ x - b)
        s = lmo_l1(grad, lambd)
        stepsize = 2.0/(t+2.0)
        x = x + stepsize * (s - x)
        f_vals[t] = np.linalg.norm(A @ x - b)**2

    return f_vals
```

```
[5]: lambdas = [1, 10, 100]
iters = 500

f_star = {lam: lasso(A, b, lam) for lam in lambdas}
hist_FW = {lam: FW_lasso(A, b, lam, iters) for lam in lambdas}
hist_PGD = {lam: PGD_lasso(A, b, lam, t, iters) for lam in lambdas}

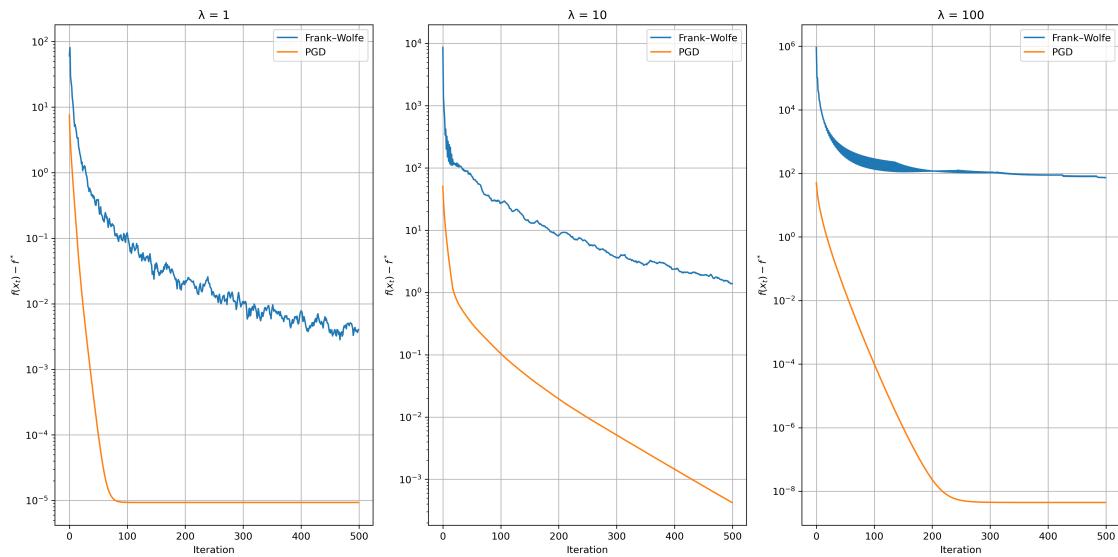
fig, axes = plt.subplots(1, 3, figsize=(16,8), dpi=300)
```

```

for ax, lam in zip(axes, lambdas):
    ax.plot(hist_FW[lam] - f_star[lam], label='Frank-Wolfe')
    ax.plot(hist_PGD[lam] - f_star[lam], label='PGD')
    ax.set_yscale('log')
    ax.set_title(f' = {lam}')
    ax.set_xlabel('Iteration')
    ax.set_ylabel(r'$f(x_t) - f^*$')
    ax.legend()
    ax.grid(True)

plt.tight_layout()
plt.show()

```



0.4 Q3

```

[6]: def objective(x):
       return x**2 + 1

def constraint(x):
    return (x - 2) * (x - 4)

def lagrangian(x, lam):
    return objective(x) + lam * constraint(x)

x = np.linspace(1, 5, 400)
plt.figure(figsize=(10, 6), dpi=300)
plt.plot(x, objective(x), label=r'$f(x)=x^2+1$', color='blue', linewidth=2)

```

```

plt.fill_between(x, objective(x), where=(x >= 2) & (x <= 4), color='lightgray', alpha=0.5, label='Feasible set [2,4]')

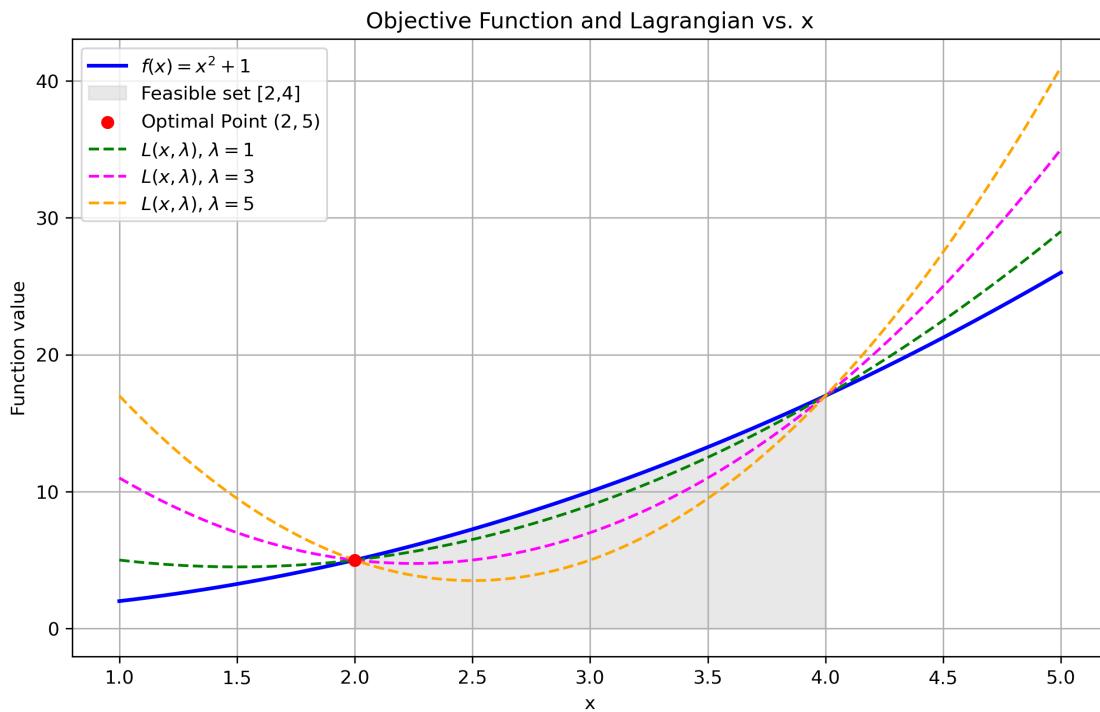
opt_x = 2
opt_val = objective(opt_x)
plt.scatter([opt_x], [opt_val], color='red', zorder=5, label=r'Optimal Point $(2,5)$')

lambdas = [1, 3, 5]
colors = ['green', 'magenta', 'orange']
for lam, color in zip(lambdas, colors):
    plt.plot(x, lagrangian(x, lam), label=r'$L(x,\lambda)$, $\lambda=' + str(lam), color=color, linestyle='--')

plt.title('Objective Function and Lagrangian vs. x')
plt.xlabel('x')
plt.ylabel('Function value')
plt.legend()
plt.grid(True)

plt.show()

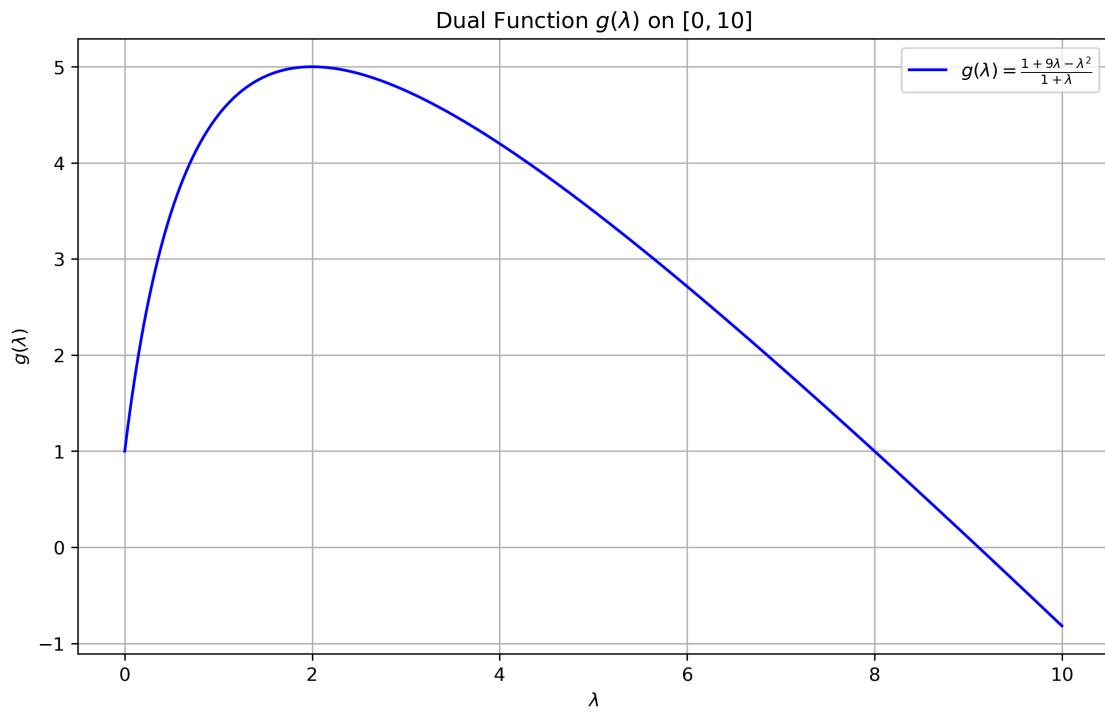
```



```
[7]: def g(lam):
    return (1 + 9*lam - lam**2) / (1 + lam)

lam_values = np.linspace(0, 10, 400)
g_values = g(lam_values)

plt.figure(figsize=(10, 6), dpi=300)
plt.plot(lam_values, g_values, label=r'$g(\lambda)=\frac{1+9\lambda-\lambda^2}{1+\lambda}$', color='blue')
plt.xlabel(r'$\lambda$')
plt.ylabel(r'$g(\lambda)$')
plt.title('Dual Function $g(\lambda)$ on $[0,10]$')
plt.legend()
plt.grid(True)
plt.show()
```



0.5 Q8

0.6 (i) Gradient of Least Square

The gradient is $2X^\top(X\beta - y)$

0.7 (ii) Proximal Gradient Descent

Hard code the group indices and group omegas.

```
[8]: group_indices = [
    [0],
    [1, 2, 3],
    [4, 5, 6],
    [7, 8],
    [9],
    [10, 11],
    [12],
    [13],
    [14, 15, 16]
]

group_omegas = [
    0.0,                      # no penalty on intercept
    np.sqrt(3),
    np.sqrt(3),
    np.sqrt(2),
    1.0,
    np.sqrt(2),
    1.0,
    1.0,
    np.sqrt(3)
]

def proximal_operator(x, t, lamda):
    x = np.asarray(x)
    prox = np.zeros_like(x)
    for idxs, omega in zip(group_indices, group_omegas):
        x_j = x[idxs]
        norm_j = np.linalg.norm(x_j, 2)
        c = (1 - (t * lamda * omega) / norm_j) if norm_j > 0 else 0
        if c < 0:
            c = 0
        prox[idxs] = c * x_j
    return prox
```

```
[9]: def proximal_group_lasso(X, y, lambd, t, iters):
    p = X.shape[1]
    beta = np.ones(p)
    f_vals = np.zeros(iters)

    for k in tqdm(range(iters), desc="proximal"):
        grad = 2 * X.T @ (X @ beta - y)
        beta = proximal_operator(beta - t * grad, t, lambd)
```

```

    obj = np.linalg.norm(X @ beta - y)**2
    for idxs, omega in zip(group_indices, group_omegas):
        obj += lambd * omega * np.linalg.norm(beta[idxs])
    f_vals[k] = obj

    return beta, f_vals

```

```
[10]: def accelerated_proximal_group_lasso(X, y, lambd, t, iters):
    p = X.shape[1]
    beta_prev      = np.ones(p)
    beta_prev_prev = np.ones(p)
    f_vals         = np.zeros(iters)

    for k in tqdm(range(1, iters+1), desc="acc-proximal"):
        if k == 1:
            v = beta_prev
        else:
            v = beta_prev + (k-2)/(k+1) * (beta_prev - beta_prev_prev)

        grad = 2 * X.T @ (X @ v - y)
        beta = proximal_operator(v - t * grad, t, lambd)

        obj = np.linalg.norm(X @ beta - y)**2
        for idxs, omega in zip(group_indices, group_omegas):
            obj += lambd * omega * np.linalg.norm(beta[idxs])
        f_vals[k-1] = obj

        beta_prev_prev = beta_prev
        beta_prev      = beta

    return beta_prev, f_vals
```

```
[11]: import pandas as pd

X = pd.read_csv("X.csv").values
ones = np.ones((X.shape[0], 1))
X = np.hstack([ones, X])
y = pd.read_csv("y.csv").values
y = y.reshape(-1)

print("X:", X.shape, "y:", y.shape)
```

X: (189, 17) y: (189,)

```
[12]: t = 0.002
lambd = 4
iters = 1000
```

```
beta_prox, obj_prox = proximal_group_lasso(X, y, lambd, t, iters)
beta_acc, obj_acc = accelerated_proximal_group_lasso(X, y, lambd, t, iters)
```

```
proximal: 100%| 1000/1000 [00:00<00:00, 19719.16it/s]
acc-proximal: 100%| 1000/1000 [00:00<00:00, 17380.10it/s]
```

```
[13]: print(obj_prox[-1], obj_acc[-1])
```

```
84.6952082766263 84.69520827662632
```

We use CVXPY to find the optimal value.

```
[14]: def group_lasso(X, y, lambd):
    p = X.shape[1]
    beta = cp.Variable(p)
    loss = cp.sum_squares(X @ beta - y)

    penalty = 0
    for idxs, omega in zip(group_indices, group_omegas):
        penalty += omega * cp.norm(beta[idxs], 2)

    objective = loss + lambd * penalty
    problem = cp.Problem(cp.Minimize(objective))
    problem.solve(solver=cp.MOSEK)

    return beta.value, problem.value
```

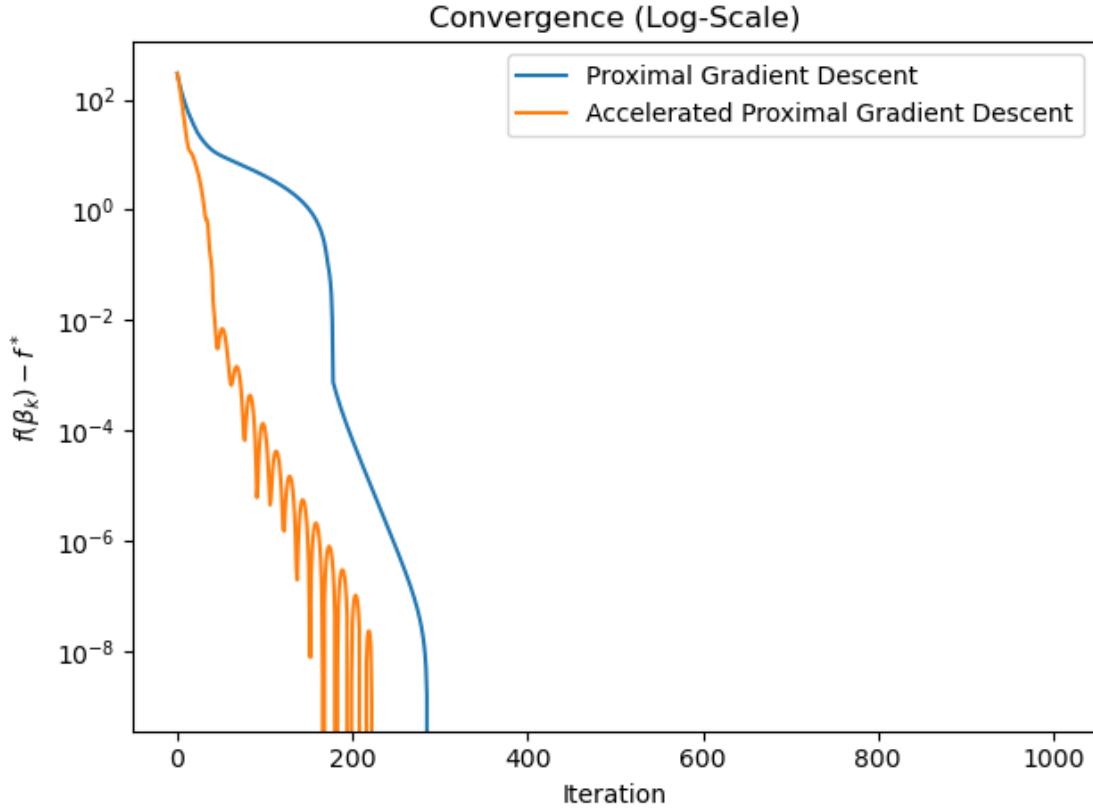
```
[15]: beta_opt, obj_opt = group_lasso(X, y, lambd)
print(obj_opt)
```

```
84.69520831035122
```

0.8 (ii) Convergence of Proximal Gradient Descent

```
[16]: iters = np.arange(len(obj_prox))

plt.figure()
plt.plot(iters, obj_prox - obj_opt, label='Proximal Gradient Descent')
plt.plot(iters, obj_acc - obj_opt, label='Accelerated Proximal Gradient Descent')
plt.yscale('log')
plt.xlabel('Iteration')
plt.ylabel(r'$f(\beta_k) - f^*$')
plt.title('Convergence (Log-Scale)')
plt.legend()
plt.tight_layout()
plt.show()
```



0.9 (iii) Coefficients

```
[17]: print(np.round(beta_prox,3))
```

```
[ 3.018e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00 -0.000e+00
  0.000e+00  3.030e-01 -5.400e-02 -2.890e-01 -2.250e-01  3.400e-02
 -2.750e-01 -4.580e-01  2.300e-02  3.000e-03 -9.000e-03]
```

```
[18]: print(np.round(beta_acc,3))
```

```
[ 3.018e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00 -0.000e+00
  0.000e+00  3.030e-01 -5.400e-02 -2.890e-01 -2.250e-01  3.400e-02
 -2.750e-01 -4.580e-01  2.300e-02  3.000e-03 -9.000e-03]
```

The coefficients are the same, and the first two groups (excluding the intercept) are not selected, which is mother's age and mother's weight.

0.10 (iv) Lasso

We let each coefficients be a singleton group.

```
[19]: p = 17
group_indices = [[j] for j in range(p)]
group_omegas = [1.0] * p
```

```
[20]: L = 2 * np.linalg.norm(X, 2)**2
t = 1.0 / L
```

```
[21]: beta_lasso, obj_lasso = accelerated_proximal_group_lasso(X, y, 0.35, t, 10000)
acc-proximal: 100%| 10000/10000 [00:00<00:00, 11874.59it/s]
```

```
[22]: print(np.round(beta_lasso,4))
```

```
[ 3.0412 -0.       1.3962  0.7315  1.6749  0.       1.1948  0.3013 -0.1331
 -0.2817 -0.2979  0.1847 -0.5335 -0.4744  0.0856  0.0141 -0.1491]
```

Comparing the solutions, when we run lasso, it happens that it selected coefficients from a group that were not selected by group lasso, for example, the first coefficient from the group of age.