POLIMI
Graduate School of Management

Quantitative Finance

Asset Management

Portfolio Optimization Mean Variance

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Introduction of the 5 Stocks

Discover Financial Services

Discover Financial Services (DFS) is a company founded in 1960. The Company is both a banking holding and a financial holding. It operates through two segments: Direct Banking and Payment Services.

First Solar Inc.

First Solar Inc. is a company founded in 1999 that provides photovoltaic solar energy solutions. It operates through two segments: Components (design, production, and sale of solar modules) and Systems (development, construction, and maintenance of photovoltaic solar energy systems).

W.W. Grainger Inc.

W.W. Grainger Inc., founded in 1927, provides maintenance products for buildings in North America. It operates through three segments: branch distribution, laboratory and hospital equipment, and an integrated supply chain to optimize storage and distribution.

NiSource Inc.

NiSource Inc. is a holding in the energy market, founded in 1912, whose subsidiaries provide natural gas and electricity to customers in a geographic area that, through the Midwest, extends from the Gulf Coast to New England. The Company's main subsidiaries include Columbia Energy Group and Northern Indiana Public Service Company, operating in Indiana, Kentucky, Maryland, Ohio, Pennsylvania, and Virginia.

Verizon Communications Inc.

Verizon Communications Inc. is a holding founded in 1983. The Company, through its subsidiaries, provides communication, information, and entertainment products and services to consumers, businesses, and government entities. Its operating segments are Verizon Consumer Group and Verizon Business Group.

Properties	**Discover Financial Services**	**First Solar**	**W.W. Grainger**	**NiSource**	**Verizon Communications**
Foundation	1960	1999	1928	1987	1983
Market Cap	42.25 B USD	19.73 B USD	53.15 B USD	16.54 B USD	169.40 B USD
Employees	21000	6700	26000	7410	105400
Dividend Yield (ttm)	0.0161	N/A	0.0073	0.0299	0.0673
PE Ratio (ttm)	14.03	16.22	30.18	22.00	17.58
Basic EPS (ttm)	12,40 USD	11,67 USD	37,25 USD	1,68 USD	2,32 USD
Beta (1Y)	1.09	0.69	0.94	0.10	0.07
Market	NYSE	NASDAQ	NYSE	NYSE	NYSE
Industry	Finance, Rental, Leasing	Semiconductors	Wholesale Distributors	Gas Distributors	Wireless Telecommunications
Sector	Finance	Electronic Technology	Distribution Services	Utilities	Communications

Table 1: Table of stocks with inverted rows and columns

Estimates of $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$

Variance-Covariance Matrix $\boldsymbol{\Sigma}$

Before proceeding with the estimation of the mean return values of the assets $\boldsymbol{\mu}$ and the variance-covariance matrix $\boldsymbol{\Sigma}$, we verified that the provided data were consistent and free from discrepancies.

We then selected the data from the date "2007-07-31" since the data preceding that date contained *missing values*. The dataset we used consists of the following features:

- **Date**: Date of observation
- **MonthTotalReturn**: Monthly returns [%] including dividends
- **CompanyMarketCap**: Market Capitalization
- **PriceClose**: Closing price not including dividend values

For the calculation of the standard values of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ we used the values of **MonthTotalReturn** for each asset since they include dividend values. The choice to use these values, which incorporate dividends, allows us to be more faithful to the financial reality of the stocks under analysis.

We evaluated the possibility of using *log returns* to normalize the returns and to better resemble the theoretical case of normally distributed returns. The formula to transform the monthly total return (TR) to the log return (LR) is as follows:

$$LR = \ln(1 + TR) \tag{1}$$

where TR is the monthly total return and ln is the natural logarithm.
However, this idea was discarded since the improvement from the transformation is not consistent and, moreover, the interpretability of the values is less intuitive compared to the monthly total returns.

Therefore, we proceeded with the standard calculation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ for each asset using the values of **MonthTotalReturn** (not in %).

Constant Correlation Approach Σ_{cc}
The use of the **Constant Correlation Approach** is recommended when the analysis is extended to a large number of assets.
In this case the computational cost for calculating the matrix $\boldsymbol{\Sigma}$ becomes particularly high; an alternative is to replace the matrix with an approximated version, where the off-diagonal covariance values ($\boldsymbol{\sigma_{i,j}}$) are substituted with approximated values according to the relation $\boldsymbol{\sigma_{i,j}} = \sigma_i \sigma_j \rho$.

$$\rho = \frac{\frac{1}{n-1}\sum_{k=1}^{n}(r_{k,i}-\mu_i)(r_{k,j}-\mu_j)}{\sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(r_{k,i}-\mu_i)^2}\cdot\sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(r_{k,j}-\mu_j)^2}} \tag{2}$$

In conclusion, we decided not to use the constant correlation matrix since the number of assets considered does not entail a prohibitive computational expense.

Shrinkage Correlation Approach $\Sigma_{shrinkage}$
The **Shrinkage Correlation Approach** matrix combines the values of the standard theoretical matrix $\boldsymbol{\Sigma}$ with a target matrix, which could be the constant correlation matrix $\boldsymbol{\Sigma_{cc}}$.
This allows the $\boldsymbol{\Sigma_{shrinkage}}$ matrix to have greater adaptability to the examined case compared to the $\boldsymbol{\Sigma_{cc}}$ matrix and also provides greater robustness for the estimates that use it.

$$\Sigma_{\text{shrinkage}} = \lambda\Sigma_{\text{target}} + (1-\lambda)\Sigma_{\text{cc}} \tag{3}$$

Where $\lambda \in [0,1]$ determines the intensity of the shrinkage applied to the variance-covariance matrix $\boldsymbol{\Sigma}$.

Also in this case, given the simplicity of the calculations under study, the number of observations available, and the difficulty of correctly fine-tuning the parameter $\boldsymbol{\lambda}$, we preferred to opt for the use of the standard matrix $\boldsymbol{\Sigma}$.

$$\Sigma = \begin{bmatrix} 0.0101 & 0.0037 & 0.0019 & 0.0030 & 0.0012 \\ 0.0037 & 0.0053 & 0.0013 & 0.0026 & 0.0012 \\ 0.0019 & 0.0013 & 0.0030 & 0.0017 & 0.0013 \\ 0.0030 & 0.0026 & 0.0017 & 0.0299 & 0.0014 \\ 0.0012 & 0.0012 & 0.0013 & 0.0014 & 0.0027 \end{bmatrix}, \quad \Sigma_{\text{cc}} = \begin{bmatrix} 0.0101 & 0.0021 & 0.0016 & 0.0051 & 0.0015 \\ 0.0021 & 0.0053 & 0.0012 & 0.0037 & 0.0011 \\ 0.0016 & 0.0012 & 0.0030 & 0.0028 & 0.0008 \\ 0.0051 & 0.0037 & 0.0028 & 0.0299 & 0.0026 \\ 0.0015 & 0.0011 & 0.0008 & 0.0026 & 0.0027 \end{bmatrix}$$

$$\Sigma_{\text{shrinkage}} = \begin{bmatrix} 0.0101 & 0.0032 & 0.0018 & 0.0037 & 0.0013 \\ 0.0032 & 0.0053 & 0.0013 & 0.0030 & 0.0012 \\ 0.0018 & 0.0013 & 0.0030 & 0.0020 & 0.0011 \\ 0.0037 & 0.0030 & 0.0020 & 0.0299 & 0.0018 \\ 0.0013 & 0.0012 & 0.0011 & 0.0018 & 0.0027 \end{bmatrix}$$

Mean Returns μ
Exponential Mean μ_{exp}
The calculation of the exponential mean differs from that of the simple mean by the use of a parameter $\boldsymbol{\gamma}$ which weights the contribution of the historical series relative to the time t at which it is evaluated.
This results in a weighted mean over the temporal values of the contributions, giving more emphasis to values closer to the end of the observation period.

$$\mu_i = \frac{\sum_{t=1}^{N} R_{i,t}e^{-\gamma(N-t)}}{\sum_{j=1}^{N} e^{-\gamma(N-j)}} \tag{4}$$

Also here $\gamma \in [0,1]$.

The use of the exponential mean μ_{exp} is recommended in the case of short time series, where recent changes are considered more important. Since the series available to us cover an extended time period, we preferred to use the classic mean of returns μ.

However, it is interesting to note that, when comparing the two mean returns, the exponential return updates the average return showing appreciation if $\boldsymbol{\mu_{exp}} > \boldsymbol{\mu}$; depreciation if $\boldsymbol{\mu_{exp}} < \boldsymbol{\mu}$.

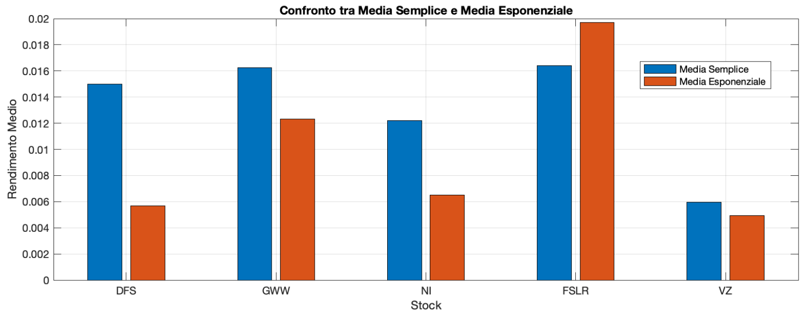

Figure 1: Exponential and classical returns

Efficient Portfolio Frontier

In the context of Modern Portfolio Theory, Markowitz's contribution was to create a reference framework for the optimized management of portfolio weights $\mathbf{w}$, so as to minimize the variance of the entire portfolio Σ.
The basic assumptions for solving the Markowitz problem are:

- Quadratic utility function: $u(x) = x - \frac{a}{2}x^2$
- Normally distributed returns: $R \sim (\mu, \sigma^2)$

Under these assumptions, the formulation of the Markowitz problem consists of minimizing the risk value $\boldsymbol{\sigma_p^2}$ for a given target portfolio return $\boldsymbol{\mu_p}$.

Portfolio	Expected Return (μ)	Variance (σ^2)
Risk-free Asset	0.25%	0
Minimum Variance Portfolio	1.0%	0.0019
Tangent Portfolio	1.62%	0.0035

Table 2: Expected return and variance.

Stock	w_T	w_{MVP}
Discover Financial Services	0.0128	0.0044
Grainger Inc.	0.5317	0.1625
Nisource	0.6726	0.3639
First Solar	0.0476	0.0091
Verizon	-0.2648	0.4601

Table 3: Composition of the tangent portfolio and the minimum variance portfolio for the specific stocks.

In accordance with theory, since the minimum variance portfolio has a return higher than that of the risk-free asset, the tangency point between the maximum Sharpe ratio line and the efficient Markowitz frontier of the risky stocks lies on the efficient frontier.

By imposing further constraints on the optimization problem proposed above; in particular **v1: the sum of the first two stocks is 50%**, **v2: a minimum portfolio weight of 10% for each stock**.
The tangent portfolios, calculated for each of the three frontiers, have been computed using the definition of the tangent portfolio. It represents the portfolio, belonging to the frontier, that maximizes the Sharpe ratio.

We note, from the graph and the table, that for the tangent portfolios in the unconstrained case and under constraint **v1** the results are similar.
The additional constraint **v1** does not alter the portfolio performance.
The same cannot be said for the case **v2**; in this case, in accordance with theory, limiting the portfolio's freedom to take short positions leads to the possibility of "excluding" stocks with poor or even negative performance.

With reference to the target return $\mu_{target} = 0.5\%$ it is observed that the calculated frontiers, both with and without additional constraints, yield portfolios as follows:

- In the case of no additional constraints, the portfolio $\mathbf{w}_1$ has the lowest risk exposure. This is consistent with theory, as the problem provides the greatest flexibility for varying the portfolio weights.

- The portfolio subject to **v2** for low return values deviates from the previous case, generating a portfolio that for the same expected return $\mu_{target} = 0.5\%$ carries higher risk. This behavior appears to stem from the high returns (related to the stocks under consideration) of the two stocks constrained in positions 1 and 2. These stocks, in order to be rebalanced and achieve a return of $\mu_{target} = 0.5\%$, require strong short positions on the other stocks. So much so that the frontier, for equal risk exposure, suggests a portfolio with a higher return.
- Finally, in the case of **v2** there is no combination of portfolio weights that permits reaching the desired return. The minimum admissible portfolio return μ_p with the imposed values is $(1 - n_{stock} lb)\mu_{riskfree} + \sum \mu * lb = 3.41\%$.

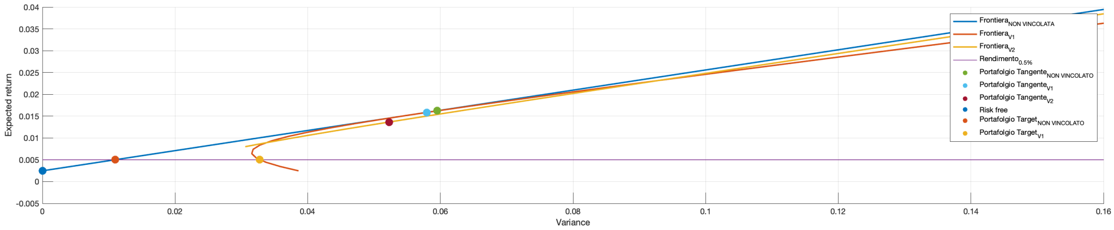

Figure 2: Constrained frontiers and notable portfolios

Stock	**No Constraint (w)**	**Constraint 1 (w)**
Discover Financial Services	0.0024	0.1328
Grainger Inc.	0.0979	0.3672
Nisource	0.1238	-0.3317
First Solar	0.0088	-0.0441
Verizon	-0.0488	-0.1022
Sum of Weights	0.1841	0.0219

Composition of the tangent portfolios in the cases of no constraints and under **v1**. For each stock, the absolute weights represent the allocation of wealth in the portfolio relative to the risk-free asset.

Alpha and Beta CAPM

The CAPM is a model for determining the equilibrium price of a financial asset. It focuses on the fact that the investor must consider systematic risk—that is, the risk associated with overall market movements that cannot be eliminated through diversification. In our case, we note that the stocks most exposed to market fluctuations are Discover Financial Services, First Solar, and Grainger Inc., for which a $\beta > 1$ and a positive excess return α are observed. Verizon should be sold because it has a negative excess return, whereas with NiSource it is advisable to remain neutral since it shows an excess return close to zero.

Stock	α	β	Significance (α) ($p < 0.05$)
Discover Financial Services	0.0162	1.4206	1
Grainger Inc.	0.0108	0.9936	1
Nisource	0.0013	0.6095	0
First Solar	0.0164	1.4829	0
Verizon	-0.0074	0.4824	1

Table 4: CAPM α and β values for each stock and their statistical significance.