Dot product
$x, y \in \mathbb{R}^n$:
$x.y = x_1 y_1 + \cdots + x_n y_n$

Inner product Space

An inner product space is a vector space V equipped with an innner product on V.

Inner product

$< u, v >: V \times V \to \mathbb{F}$

Norm

$\|v\| = \sqrt{< v, v >}$

- $\|v\| = 0 \Leftrightarrow v = 0$
- $\|\lambda v\| = |\lambda| \, \|v\|$, $\forall \lambda \in \mathbb{F}$

Properties

- $\forall \ fixed u \in V : < v, u >$ is a linear map from $V \ to \mathbb{F}$
- $< 0, u >= 0, \ \forall u \in V$
- $< u, 0 >= 0, \ \forall u \in V$
- $< u, v + w >=< u, v > + < u, w > \ \ \forall u, v \in V$
- $< u, \lambda v >= \overline{\lambda} < u, v >, \ \forall \lambda \ in \mathbb{F} \ and \ u, v \in V$

Positivity

$< v, v > \geq 0, \forall v \in V$

Definiteness

$< v, v >= 0 \Leftrightarrow v = 0$

Additive in the first slot

$< u + v, w >=< u, w > + < v, w >, \ \forall u, v, w \in V$

Homogeneous in first slot

$< \lambda u, v >= \lambda < u, v >, \ \forall \lambda \in \mathbb{F}, \forall u, v \in V$

Conjugate Symmetry

$< u, v >= \overline{< v, u >}, \ \forall u, v \in V$

Orthogonal decomposition

Be $u, v \in V : \ v \neq 0$

then,

Set $c = \frac{<u,v>}{\|v\|^2}$ and $w = u - \frac{<u,v>}{\|v\|^2} v$

$< w, v >= 0$ and $u = cv + w$

Orthogonality and 0

- 0 is orthogonal to every vector in V
- 0 is the only vector in V that is orhogonal to itself.

Orthogonality

$u, v \in V :< u, v >= 0$

Cauchy-Schwarz Inequality

Be $u, v \in V$, Then,

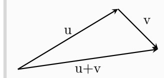
$| < u, v > | \leq \|u\| \, \|v\|$

Pythogorean Theorem

If $u, v \in V$ are orthogonal vectors, then,

$\|u + v\|^2 = \|u\|^2 + \|v\|^2$

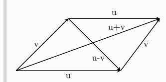
Triangle Inequality



Be $u, v \in V : \ v \neq 0$ then,

$\|u + v\| \leq \|u\| + \|v\|$

Parallelogram Equality



Be $u, v \in V : \ v \neq 0$

then,

$\|u + v\|^2 + \|u - v\|^2 = 2 \left(\|u\|^2 + \|v\|^2 \right)$

Orthonormal bases

$e_1 \ldots e_m \in V$

is orthonormal if

$$< e_j, e_k >= \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Orthonormal basis of V

is an orthonormal list of vectors of V that is also a basis of V.

Orthonormal list is linearly independent.

Orthonormal list of length $dim \ V$ **is an orthonormal basis of** V

Be $e_1 \ldots e_m$ an orthonormal list in V

$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \ldots |a_m|^2$

$\forall a_1 \ldots a_m \in \mathbb{F}$