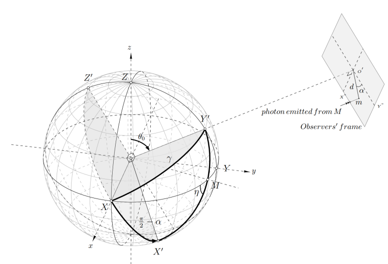
LuminetCpp: The maths behind the code.
C++ version of the Python code by bgmeulem
//https://github.com/bgmeulem/Luminet/



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Status: DRAFT

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$$\left\{\frac{1}{r^2}\left(\frac{dr}{d\phi}\right)\right\}^2+\frac{1}{r^2}\left(1-\frac{2M}{r}\right)=\frac{1}{b^2}$$

from

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+sin^2\theta d\phi^2\right)$$

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2. Image of a bare black hole

2.1 Derivation of

$$\left\{\frac{1}{r^2}\left(\frac{dr}{d\phi}\right)\right\}^2+\frac{1}{r^2}\left(1-\frac{2M}{r}\right)=\frac{1}{b^2}$$

from

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+sin^2\theta d\phi^2\right)$$

The metric form Φ is (considering the invariance of a rotation along the θ generating axis)

$$\Phi=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2d\phi^2 \tag{1}$$

For geodesic null lines ($\Phi=0$) we have (λ is an affine parameter):

$$\begin{cases} \frac{d^2x^\sigma}{d\lambda^2}+\Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=0 \\ \\ g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=0 \end{cases} \tag{2}$$

For the considered metric form (modified Schwarzschild), the connection functions are:

$$\Gamma^t_{\ rt}=-\frac{M}{r\left(2M-r\right)} \tag{3}$$

$$\Gamma^r_{\ tt}=-\frac{M\left(2M-r\right)}{r^3} \tag{4}$$

$$\Gamma^r_{\ rr}=\frac{M}{r\left(2M-r\right)} \tag{5}$$

$$\Gamma^r_{\ \phi\phi}=2M-r \tag{6}$$

$$\Gamma^\phi_{\ \phi r}=\frac{1}{r} \tag{7}$$

all the others being zero.
We could try to plug in these connections into (2) but this would give us a set of coupled 2^{nd} order differential equations. This is certainly difficult (doable?) to solve analytically.
Using the inherent symmetries in the Schwarschild metric we can use Killing vectors to find conserved quantities under certain coordinate transformation. There are two of them:

i) Conservation of energy E:

$$K^\mu = (\partial_t)^\mu = (-1,0,0)$$

ii) Conservation of magnitude of angular momentum L:

$$R^\mu = (\partial_\phi)^\mu = (0,0,1)$$

Lowering the index, gives

$$K_\mu = g_{\mu\sigma} K^\sigma = \left(\left(1-\frac{2M}{r}\right),0,0\right)$$

and

$$R_\mu = g_{\mu\sigma} R^\sigma = \left(0,0,r^2\right)$$

giving

$$E = K_\mu \frac{dx^\mu}{d\lambda} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \tag{8}$$

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \tag{9}$$

Putting (1) and $\Phi = 0$ (light like geodesic) in another form:

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{10}$$

and plugging (8) and (9) in (10) gives:

$$-\left(1-\frac{2M}{r}\right)^{-1} E^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} = 0$$

or

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)$$

Using (9) with $\left(\frac{d\phi}{d\lambda}\right)^2 = \frac{L^2}{r^4}$:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\left(\frac{dr}{d\lambda}\right)^2}{\left(\frac{d\phi}{d\lambda}\right)^2} = \frac{E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}{\frac{L^2}{r^4}}$$

rearranging:

$$\left\{\frac{1}{r^2}\frac{dr}{d\phi}\right\}^2 + \frac{1}{r^2}\left(1-\frac{2M}{r}\right) = \frac{E^2}{L^2}$$

And putting $\frac{E^2}{L^2} = \frac{1}{b^2}$ gives finally equation (**2**).

♦

2.2 Calculating the roots of $G(u)$ (equation (3)).

We have in (**3**):

$$G(u) = u^3 - \frac{u^2}{2M} + \frac{1}{2Mb^2}$$

If we assume that the trajectory of a ray reaching the observers' plaque, has a minimum distance to the black hole (the perastion) than, equation (**3**) has the right form as it represents the (squared) derivative of the trajectory and thus the point where $G(u)$ is zero, represent extrema of $u(\phi)$. Suppose that $r = P$ is a minimum then we can write:

$$\begin{aligned} G(u) &= \left(u - \frac{1}{P}\right)\left(u^2 + cu + d\right) \\ &= u^3 + \left(c - \frac{1}{P}\right)u^2 + \left(d - c\frac{1}{P}\right)u - \frac{d}{P} \end{aligned}$$

and get

$$\begin{cases} c - \frac{1}{P} = -\frac{1}{2M} \\ d - c\frac{1}{P} = 0 \\ \frac{d}{P} = -\frac{1}{2Mb^2} \end{cases}$$
$$\Rightarrow \begin{cases} c = \frac{2M-P}{2MP} \\ d = \frac{2M-P}{2MP^2} \\ \frac{2M-P}{2MP^3} = -\frac{1}{2Mb^2} \end{cases}$$

From the last expression we get already an important relation:

$$b^2 = \frac{P^3}{P - 2M} \tag{1}$$

which is what equation (**5**) should be.[1].

From the other equation we obtain the following quadratic expression

$$F(u) = u^2 + \frac{2M - P}{2MP}u + \frac{2M - P}{2MP^2}$$

which will generate the two other roots of $G(u)$. As only the roots of it, are interesting we rewrite it as

$$F(u) = 2MP^2u^2 + (2M - P)Pu + 2M - P$$

[1]there is a typo in the paper as equation (**5**) needs to be read as $b = \sqrt{\frac{P^3}{P-2M}}$

The two roots are

$$\begin{aligned} u_{1,2} &= \frac{(P-2M)P \pm \sqrt{8MP^2(P-2M)+(P-2M)^2P^2}}{4MP^2} \\ &= \frac{(P-2M) \pm \sqrt{8M(P-2M)+(P-2M)^2}}{4MP} \end{aligned}$$

Put $Q \equiv \sqrt{8M(P-2M)+(P-2M)^2}$ or

$$\begin{aligned} Q^2 &= 8M(P-2M)+(P-2M)^2 \\ &= (P-2M)(8M+P-2M) \\ &= (P-2M)(P+6M) \end{aligned}$$

Thus, we get the three roots of $G(u)$

$$u_1 = -\frac{Q-P+2M}{4MP}, \quad u_2 = \frac{1}{P}, \quad u_3 = \frac{Q+P-2M}{4MP} \tag{2}$$

as in equation (**4**) in the paper. Also from (1) we have

$$b = \sqrt{\frac{P^3}{P-2M}}$$

♦

2.3 Derivation of $\phi_\infty = 2\left(\frac{P}{Q}\right)^{\frac{1}{2}} \int_{\zeta_\infty}^{\frac{\pi}{2}} \left(1 - k^2 \sin^2 x\right)^{-\frac{1}{2}} dx$

We have in (**3**): $\left(\frac{du}{d\phi}\right)^2 = 2MG(u)$ with $G(u) = u^3 - \frac{u^2}{2M} + \frac{1}{2Mb^2}$. We rewrite this as

$$\left(\frac{d\phi}{du}\right)^2 = \frac{1}{2MG(u)}$$

As we are only interested in the value of the impact parameter b at infinity, we have $\lim_{r\to\infty} b \Leftrightarrow \lim_{u\to 0_+} b$ and for symmetry reasons, only have to integrate (**3**) from $\frac{1}{P}$ to 0 or

$$\phi_\infty = \frac{1}{\sqrt{2M}} \int_0^{\frac{1}{P}} \frac{dt}{\sqrt{G(t)}} \tag{1}$$

with (see (2) in the preceding section)

$$G(t) = t^3 - \frac{1}{2M}t^2 + \frac{P-2M}{2MP^3} = (t-u_1)(t-u_2)(t-u_3) \tag{2}$$

Reduction of the elliptic integral to Legendre form:

Let's put $E(z) = \sqrt{2M}\frac{1}{\sqrt{2M}} \int_z^{u_2} \frac{dt}{\sqrt{G(t)}}$. Also be, $u_1 < z < u_2 < u_3$,

$$E(z) = \int_z^{u_2} \frac{dt}{\sqrt{(u_1-t)(u_2-t)(t-u_3)}},$$

and set

$$\varphi := \arcsin\left(\sqrt{\frac{z-u_1}{u_2-u_1}}\right) \in \left(0, \frac{\pi}{2}\right),$$

$$k := \sqrt{\frac{u_2-u_1}{u_3-u_1}} \in (0,1).$$

Consider the transformation given by the substitution $\sqrt{\frac{t-u_1}{u_2-u_1}} = x$. Observing that

$$\sqrt{\frac{t-u_1}{u_2-u_1}} = x \Rightarrow t = u_1 + (u_2-u_1)x^2 \Rightarrow \frac{dt}{dx} = 2(u_2-u_1)x,$$

we find after applying the substitution that $E(z)$ can be rewritten as

$$\begin{aligned}
E(z) &= \int_z^{u_2} \frac{dt}{\sqrt{(u_3-t)(u_2-t)(t-u_1)}} \\
&= \int_{\sqrt{\frac{z-u_1}{u_2-u_1}}}^{1} \frac{2(u_2-u_1)\,x dx}{\sqrt{[u_3-u_1-(u_2-u_1)x^2][u_2-u_1-(u_2-u_1)x^2](u_2-u_1)x^2}} \\
&= \frac{2}{\sqrt{u_3-u_1}} \int_{\sqrt{\frac{z-u_1}{u_2-u_1}}}^{1} \frac{dx}{\sqrt{(1-x^2)\left[1-\left(\frac{u_2-u_1}{u_3-u_1}\right)x^2\right]}} \\
&= \frac{2}{\sqrt{u_3-u_1}} \int_{\sin(\varphi)}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \\
&= \frac{2}{\sqrt{u_3-u_1}} \int_{\varphi}^{\frac{\pi}{2}} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} \qquad (x=\sin(\tau))
\end{aligned} \tag{3}$$

Let's compute now the parameters $\frac{2}{\sqrt{u_3-u_1}}$, k^2, φ:

$$\begin{aligned}
u_3-u_1 &= \frac{Q+P-2M}{4MP} + \frac{Q-P+2M}{4MP} \\
&= \frac{1}{2M}\frac{Q}{P}
\end{aligned}$$

giving

$$\frac{2}{\sqrt{u_3-u_1}} = 2\sqrt{2M}\sqrt{\frac{P}{Q}} \tag{4}$$

Also,

$$\begin{aligned}
k^2 &= \frac{u_2-u_1}{u_3-u_1} \\
&= \frac{\frac{1}{P}+\frac{Q-P+2M}{4MP}}{\frac{Q+P-2M}{4MP}+\frac{Q-P+2M}{4MP}} \\
&= \frac{\frac{4M+Q-P+2M}{4MP}}{\frac{Q}{2MP}} \\
&= \frac{Q-P+6M}{2Q}
\end{aligned} \tag{5}$$

And finally,

$$\begin{aligned}
\sin^2\varphi_{|z=0} &= \frac{0-u_1}{u_2-u_1} \\
&= \frac{\frac{Q-P+2M}{4MP}}{\frac{1}{P}+\frac{Q-P+2M}{4MP}} \\
&= \frac{\frac{Q-P+2M}{4MP}}{\frac{4M+Q-P+2M}{4MP}} \\
&= \frac{Q-P+2M}{Q-P+6M}
\end{aligned} \tag{6}$$

Putting this all together and replacing φ by ζ_∞ as the lower integration limit, we get:

$$\begin{aligned}
\phi_\infty &= \frac{E(0)}{\sqrt{2M}} \\
&= \frac{1}{\sqrt{2M}} 2\sqrt{2M}\sqrt{\frac{P}{Q}} \int_{\zeta_\infty}^{\frac{\pi}{2}} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} \\
&= 2\sqrt{\frac{P}{Q}} \int_{\zeta_\infty}^{0} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} + 2\sqrt{\frac{P}{Q}} \int_{0}^{\frac{\pi}{2}} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} \\
&= 2\sqrt{\frac{P}{Q}} \int_{0}^{\frac{\pi}{2}} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} - 2\sqrt{\frac{P}{Q}} \int_{0}^{\zeta_\infty} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} \\
&= 2\sqrt{\frac{P}{Q}} \left[K(k) - F(\zeta_\infty, k)\right],
\end{aligned}$$

where here the incomplete and complete elliptic integrals of the first kind are defined with the following argument convention:

$$F(\zeta_\infty, k) := \int_0^{\zeta_\infty} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} \qquad \theta \in \mathbb{R} \wedge k \in (-1,1),$$

$$K(k) := F\left(\frac{\pi}{2}, k\right) \qquad k \in (-1,1).$$

and

$$\begin{cases} k = \sqrt{\frac{Q-P+6M}{2Q}} \\ \zeta_\infty = \arcsin\sqrt{\frac{Q-P+2M}{Q-P+6M}} \end{cases}$$

♦

3. Image of a clothed black hole

3.1 General setting

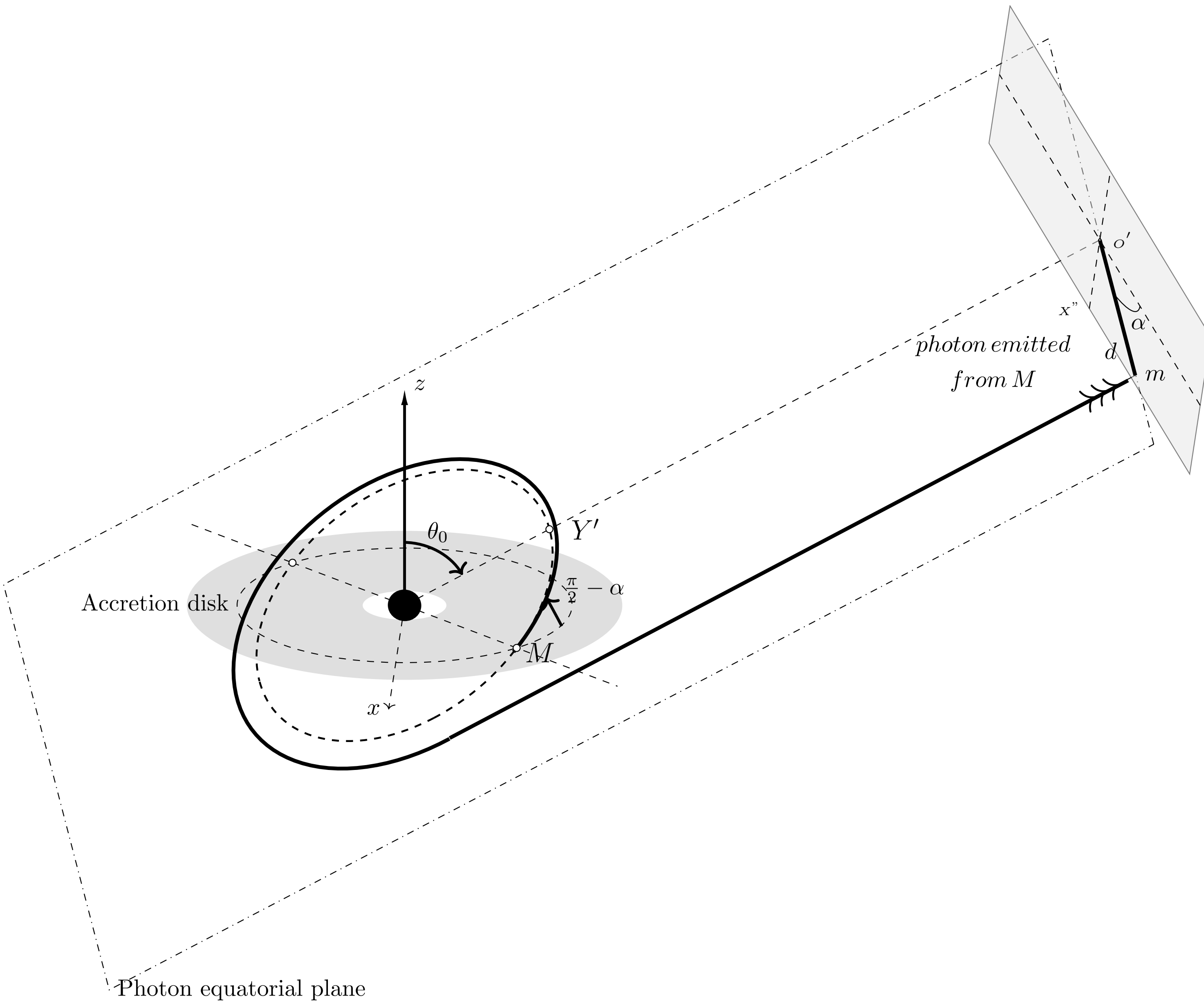

Figure 3.1: General setting.

♦

3.2 The coordinate system (enhanced):

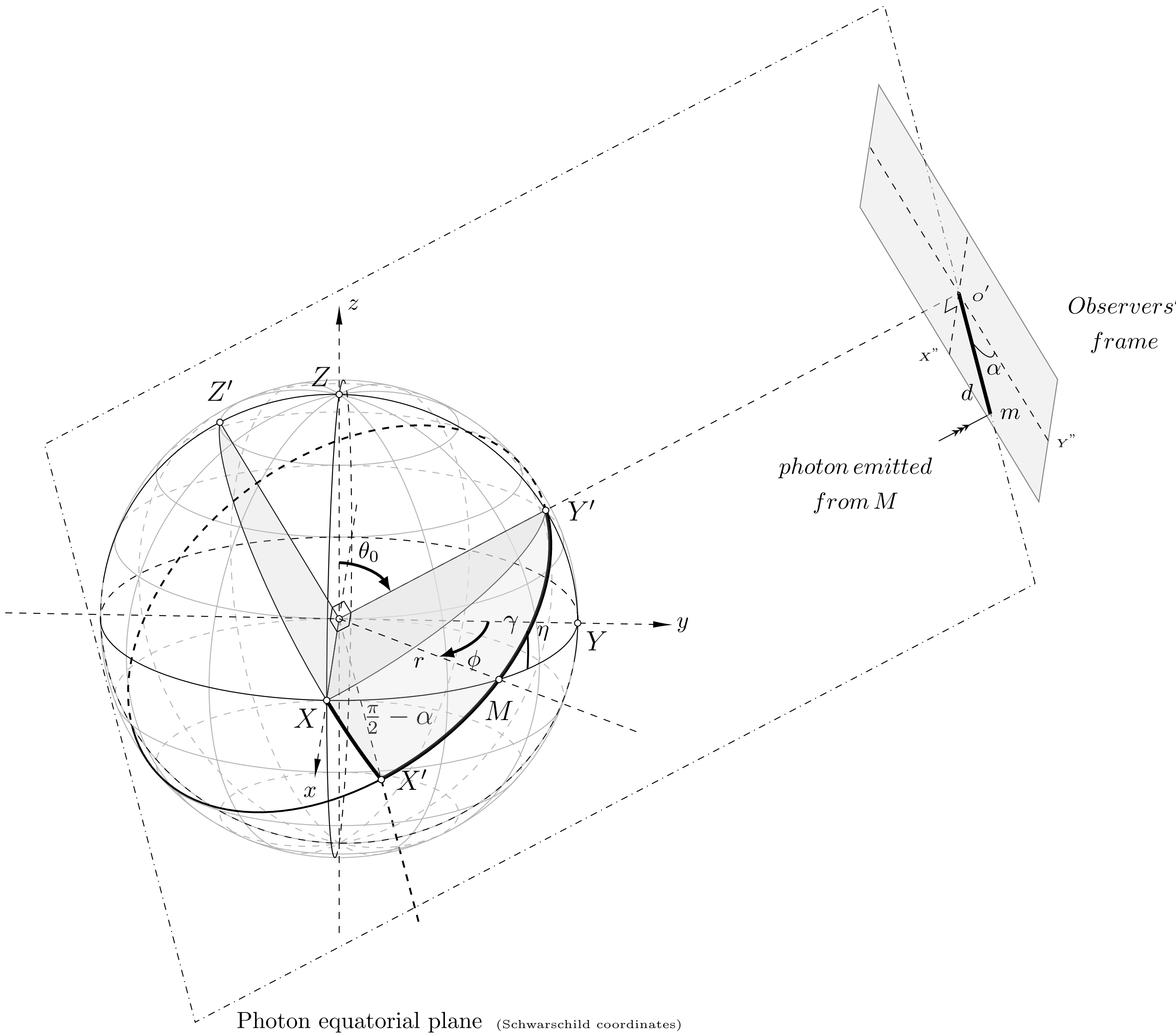

Figure 3.2: The coordinate system (see text).

♦

3.3 Angles relationships: equations (9) and (10)

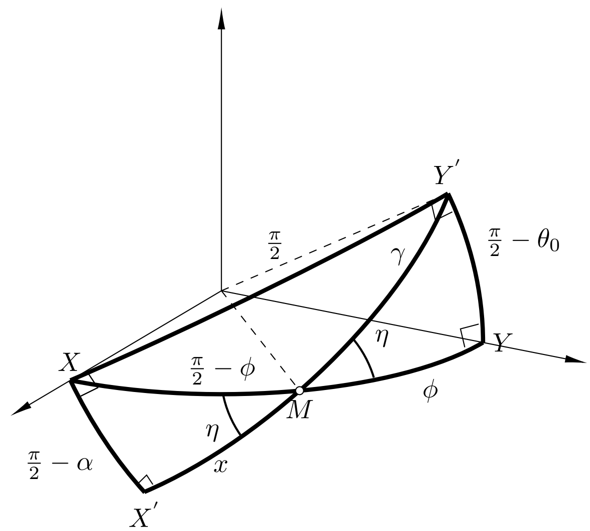

Figure 3.3: Different spheric triangles used to find a relation between ϕ, θ_0, α.

Resolving the different spherical triangles in order to have a relation between ϕ, θ_0, α.
We have (using the basis spherical triangles identities $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin B}$ and $\cos a = \cos b \cos c + \sin b \sin c \cos A$)

i) $\Delta Y_2' M Y$

$$\frac{\cos\theta_0}{\sin\eta} = \frac{\sin\gamma}{\sin\frac{\pi}{2}} = \frac{\sin\phi}{\sin Y_2}$$
$$\cos\left(\frac{\pi}{2}-\theta_0\right) = \cos\gamma\cos\phi + \sin\gamma\sin\phi\cos\eta$$
$$\cos\gamma = \cos\left(\frac{\pi}{2}-\theta_0\right)\cos\phi + \sin\left(\frac{\pi}{2}-\theta_0\right)\sin\phi\cos\frac{\pi}{2}$$
$$\cos\phi = \cos\gamma\cos\left(\frac{\pi}{2}-\theta_0\right) + \sin\gamma\sin\left(\frac{\pi}{2}-\theta_0\right)\cos Y_2'$$

ii) $\Delta X' M X_2$

$$\frac{\cos\alpha}{\sin\eta} = \frac{\sin x}{\sin X_2} = \frac{\cos\phi}{\sin X'}$$
$$\cos\left(\frac{\pi}{2}-\alpha\right) = \cos\left(\frac{\pi}{2}-\phi\right)\cos x + \sin\left(\frac{\pi}{2}-\phi\right)\sin x\cos\eta$$
$$\cos\left(\frac{\pi}{2}-\phi\right) = \cos\left(\frac{\pi}{2}-\alpha\right)\cos x + \sin\left(\frac{\pi}{2}-\alpha\right)\sin x\cos X'$$
$$\cos x = \cos\left(\frac{\pi}{2}-\alpha\right)\cos\left(\frac{\pi}{2}-\phi\right) + \sin\left(\frac{\pi}{2}-\alpha\right)\sin\left(\frac{\pi}{2}-\phi\right)\cos X_1$$

iii) $\Delta Y X_1 X'$

$$
\begin{aligned}
\frac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{2}} &= \frac{\cos\theta_0}{\sin X_1} = \frac{\sin\frac{\pi}{2}}{\sin Y'} \\
\cos(\frac{\pi}{2}-\theta_0) &= \cos\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\sin\frac{\pi}{2}\cos X_1 \\
\cos\frac{\pi}{2} &= \cos(\frac{\pi}{2}-\theta_0)\cos\frac{\pi}{2} + \sin(\frac{\pi}{2}-\theta_0)\sin\frac{\pi}{2}\cos\frac{\pi}{2} \\
\cos\frac{\pi}{2} &= \cos\frac{\pi}{2}\cos(\frac{\pi}{2}-\theta_0) + \sin\frac{\pi}{2}\sin(\frac{\pi}{2}-\theta_0)\cos Y'
\end{aligned}
$$

iv) $\Delta X Y_1' X'$

$$
\begin{aligned}
\frac{\cos\alpha}{\sin Y_1'} &= \frac{\sin\frac{\pi}{2}}{\sin X'} = \frac{\sin(x+\gamma)}{\sin\frac{\pi}{2}} \\
\cos(\frac{\pi}{2}-\alpha) &= \cos\frac{\pi}{2}\cos(x+\gamma) + \sin\frac{\pi}{2}\sin(x+\gamma)\cos Y_1' \\
\cos\frac{\pi}{2} &= \cos(x+\gamma)\cos(\frac{\pi}{2}-\alpha) + \sin(x+\gamma)\sin(\frac{\pi}{2}-\alpha)\cos X' \\
\cos(x+\gamma) &= \cos\frac{\pi}{2}\cos(\frac{\pi}{2}-\alpha) + \sin\frac{\pi}{2}\sin(\frac{\pi}{2}-\alpha)\cos\frac{\pi}{2}
\end{aligned}
$$

Simplifying:

$$
\begin{aligned}
\frac{\cos\theta_0}{\sin\eta} &= \sin\gamma = \frac{\sin\phi}{\sin Y_2} \\
\frac{\cos\alpha}{\sin\eta} &= \frac{\sin x}{\sin X_2} = \frac{\cos\phi}{\sin X'} \\
\cos\theta_0 &= \sin X_1 \\
Y' &= \frac{\pi}{2} \\
\frac{\cos\alpha}{\sin Y_1'} &= \frac{1}{\sin X'} = \sin(x+\gamma) \\
\sin\theta_0 &= \cos\gamma\cos\phi + \sin\gamma\sin\phi\cos\eta \\
\cos\gamma &= \sin\theta_0\cos\phi \\
\cos\phi &= \cos\gamma\sin\theta_0 + \sin\gamma\cos\theta_0\cos Y_2' \\
\sin\alpha &= \sin\phi\cos x + \cos\phi\sin x\cos\eta \\
\sin\phi &= \sin\alpha\cos x + \cos\alpha\sin x\cos X' \\
\cos x &= \sin\alpha\sin\phi + \cos\alpha\cos\phi\cos X_1 \\
\sin\theta_0 &= \cos X_1 \\
\sin\alpha &= \sin(x+\gamma)\cos Y_1' \\
0 &= \cos(x+\gamma)\sin\alpha + \sin(x+\gamma)\cos\alpha\cos X' \\
0 &= \cos(x+\gamma)
\end{aligned}
$$

and get (discarding all other equation which are not relevant for the problem):

$$\frac{\cos\theta_0}{\sin\eta} = \sin\gamma \tag{1}$$

$$\frac{\cos\alpha}{\sin\eta} = \cos\phi \tag{2}$$

$$\cos\gamma = \sin\theta_0\cos\phi \tag{3}$$

From (1) and (2) we get [1]

$$\sin\gamma = \frac{\cos\phi\cos\theta_0}{\cos\alpha} \tag{4}$$

and combining this with (3) we get (squaring both equations and add them up)

$$\cos\alpha = \frac{\cos\phi\cos\theta_0}{\sqrt{1-\sin^2\theta_0\cos^2\phi}}$$

Equation (10) in the paper follows from equations (3) and (4) by elimination of ϕ.

♦

[1]In the paper, in equation (9) we see $\cos\alpha = \frac{\cot\phi\cos\theta_0}{\sin\gamma}$, which is wrong ($\cot\phi$ should be $\cos\phi$) as otherwise $\cos\alpha$ could become greater than 1 for certain values of ϕ.

3.4 Derivation of equation (11), (12), (13).

As can be seen in figure 3.2. the photon will reach the plaque (at infinity) after having spanned the angle γ. Hence,

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{2M}} \int_0^{\frac{1}{r}} \frac{dt}{\sqrt{G(t)}} \\
&= \underbrace{\frac{1}{\sqrt{2M}} \int_0^{\frac{1}{P}} \frac{dt}{\sqrt{G(t)}}}_{2\sqrt{\frac{P}{Q}}[K(k)-F(\zeta_\infty,k)]} + \frac{1}{\sqrt{2M}} \int_{\frac{1}{P}}^{\frac{1}{r}} \frac{dt}{\sqrt{G(t)}} \\
&= 2\sqrt{\frac{P}{Q}}\left[K(k) - F(\zeta_\infty, k)\right] - \frac{1}{\sqrt{2M}} \int_{\frac{1}{r}}^{\frac{1}{P}} \frac{dt}{\sqrt{G(t)}} \qquad (1)
\end{aligned}$$

Let's recall equation (3) in section 2.3.:

$$E(z) = 2\sqrt{2M}\sqrt{\frac{P}{Q}} \int_{\varphi}^{\frac{\pi}{2}} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} \qquad (x = \sin(\tau)) \qquad (2)$$

The upper integration limit $\frac{\pi}{2}$ corresponds to the $\frac{1}{P}$ spherical radial distance, but here for calculating $\varphi = \arcsin\left(\sqrt{\frac{z-u_1}{u_2-u_1}}\right) \in \left(0, \frac{\pi}{2}\right)$ instead of taking $z = 0$ as we did previously, we have to take $z = \frac{1}{r}$ which gives

$$\begin{aligned}
\sin^2\varphi_{|z=\frac{1}{r}} &= \frac{\frac{1}{r} - u_1}{u_2 - u_1} \\
&= \frac{\frac{1}{r} + \frac{Q-P+2M}{4MP}}{\frac{1}{P} + \frac{Q-P+2M}{4MP}} \\
&= \frac{Q - P + 2M + \frac{4MP}{r}}{Q - P + 6M} \qquad (3)
\end{aligned}$$

Let's put

$$\sin^2\zeta_r = \frac{Q - P + 2M + \frac{4MP}{r}}{Q - P + 6M} \qquad (4)$$

(2) becomes

$$
\begin{aligned}
E(z) &= 2\sqrt{2M}\sqrt{\frac{P}{Q}}\int_{\zeta_r}^{\frac{\pi}{2}} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} \\
&= 2\sqrt{2M}\sqrt{\frac{P}{Q}}\int_{\zeta_r}^{0} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} + 2\sqrt{2M}\sqrt{\frac{P}{Q}}\int_{0}^{\frac{\pi}{2}} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} \\
&= -2\sqrt{2M}\sqrt{\frac{P}{Q}}\int_{0}^{\zeta_r} \frac{d\tau}{\sqrt{1-k^2\sin^2(\tau)}} + 2\sqrt{2M}\sqrt{\frac{P}{Q}}K(k) \\
&= -2\sqrt{2M}\sqrt{\frac{P}{Q}}F(\zeta_r,k) + 2\sqrt{2M}\sqrt{\frac{P}{Q}}K(k) \qquad (5)
\end{aligned}
$$

Substituting this in (1):

$$
\begin{aligned}
\gamma &= 2\sqrt{\frac{P}{Q}}\left[K(k) - F(\zeta_\infty,k)\right] - \frac{1}{\sqrt{2M}}2\sqrt{2M}\sqrt{\frac{P}{Q}}K(k) + \frac{1}{\sqrt{2M}}2\sqrt{2M}\sqrt{\frac{P}{Q}}F(\zeta_r,k) \\
&= 2\sqrt{\frac{P}{Q}}\left[K(k) - F(\zeta_\infty,k)\right] - 2\sqrt{\frac{P}{Q}}K(k) + 2\sqrt{\frac{P}{Q}}F(\zeta_r,k) \\
&= 2\sqrt{\frac{P}{Q}}\left[F(\zeta_r,k) - F(\zeta_\infty,k)\right] \qquad (6)
\end{aligned}
$$

The only term which depends explicitly on r is $F(\zeta_r,k)$. We rewrite (5) as

$$
F(\zeta_r,k) = \frac{1}{2}\gamma\sqrt{\frac{Q}{P}} + F(\zeta_\infty,k) \qquad (7)
$$

Let $\mathrm{sn}(u,k)$ be the usual Jacobian elliptic function with $\sin\zeta_r = \mathrm{sn}(F(\zeta_r,k))$. We get

$$
\sin\zeta_r = \mathrm{sn}\left(\frac{1}{2}\gamma\sqrt{\frac{Q}{P}} + F(\zeta_\infty,k)\right)
$$

From which:

$$
\begin{aligned}
& \sin^2\zeta_r = \mathrm{sn}^2\left(\frac{1}{2}\gamma\sqrt{\frac{Q}{P}} + F(\zeta_\infty,k)\right) \\
(4) \Rightarrow\quad & \frac{Q-P+2M+\frac{4MP}{r}}{Q-P+6M} = \mathrm{sn}^2\left(\frac{1}{2}\gamma\sqrt{\frac{Q}{P}} + F(\zeta_\infty,k)\right) \\
\Rightarrow\quad & \frac{1}{r} = -\frac{Q-P+2M}{4MP} + \frac{Q-P+6M}{4MP}\mathrm{sn}^2\left(\frac{1}{2}\gamma\sqrt{\frac{Q}{P}} + F(\zeta_\infty,k)\right) \qquad (8)
\end{aligned}
$$

Note that in the paper the coefficient $\frac{Q}{P}$ is wrongly written as $\frac{P}{Q}$

♦

The derivative of the Jacobi elliptic function $\mathrm{sn}(u, k)$ with respect to the parameter k is given by:

$$\frac{d}{dk}\mathrm{sn}(u, k) = \mathrm{cn}(u, k)\,\mathrm{dn}(u, k)$$

Where $\mathrm{cn}(u, k)$ and $\mathrm{dn}(u, k)$ are the Jacobi elliptic functions associated with $\mathrm{sn}(u, k)$.

The derivative of the Jacobi elliptic function $\mathrm{sn}(u, k)$ with respect to the variable u is given by:

$$\frac{d}{du}\mathrm{sn}(u, k) = \mathrm{cn}(u, k)\,\mathrm{dn}(u, k)$$

Where $\mathrm{cn}(u, k)$ and $\mathrm{dn}(u, k)$ are the Jacobi elliptic functions associated with $\mathrm{sn}(u, k)$.

The complete elliptic integral of the first kind, denoted as $K(k)$, is a function of the modulus k. Since it's not dependent on the variable u, its derivative with respect to u is zero:

$$\frac{d}{du}K(k) = 0$$

The derivative of the complete elliptic integral of the first kind, $(K(k))$, with respect to the parameter (k) is given by:

$$[\frac{dK(k)}{dk} = \frac{E(k)}{k(1 - k^2)}]$$

Where $(E(k))$ is the complete elliptic integral of the second kind.

The derivative of the incomplete elliptic integral of the first kind, $(F(p, k))$, with respect to the parameter (k) is not expressible in simple elementary functions. However, it can be represented in terms of the elliptic integrals of the first and second kind and the complete elliptic integral of the first kind.

The derivative is given by:

$$[\frac{dF(p, k)}{dk} = \frac{\Pi(n, k) - K(k)}{2k(1 - k^2)}]$$

Where $(n = 1 - p), (K(k))$ is the complete elliptic integral of the first kind, and $(\Pi(n, k))$ is the complete elliptic integral of the third kind with modulus (n).

The derivative of the incomplete elliptic integral of the first kind, $(F(p, k))$, with respect to the parameter (p) is not expressible in simple elementary functions. However, it can be represented in terms of the elliptic integrals of the first and second kind and the complete elliptic integral of the first kind.

The derivative is given by:

$$[\frac{dF(p, k)}{dp} = \frac{K(k') - E(k')}{2p(1 - p)(1 - k^2)}]$$

Where $(k' = \sqrt{1 - k^2})$,$(K(k))$ is the complete elliptic integral of the first kind, and $(E(k))$ is the complete elliptic integral of the second kind.