Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

Part II

Chapters V to VIII

by

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Relative tensors, ideas of volume, Green-Stokes theorems.

7.1 p241 - Exercise

If b_{rs} is an absolute tensor, show that the determinant $|b_{rs}|$ is a relative invariant of weight 2. What are the tensor characters of $|c^{rs}|$ and $|f^r_s|$?

As b_{rs} is an absolute tensor, we have

$$b'_{uv} = b_{rs} \frac{\partial x^r}{\partial x'^u} \frac{\partial x^s}{\partial x'^v} \tag{1}$$

Hence,

$$\left| b'_{uv} \right| = |b_{rs}| \left| \frac{\partial x^r}{\partial x'^u} \right| \left| \frac{\partial x^s}{\partial x'^v} \right| \tag{2}$$

and as $J = \left| \frac{\partial x^k}{\partial x'^s} \right|$ we get

$$\left| b'_{uv} \right| = J^2 \left| b_{rs} \right| \tag{3}$$

Conclusion, $|b_{rs}|$ is a relative invariant of weight 2.

◊

As c^{rs} is an absolute tensor, we have

$$c'^{uv} = c^{rs} \frac{\partial x'^u}{\partial x^r} \frac{\partial x'^v}{\partial x^s} \tag{4}$$

Hence,

$$\left| c'^{uv} \right| = |c^{rs}| \left| \frac{\partial x'^u}{\partial x^r} \right| \left| \frac{\partial x'^v}{\partial x^s} \right| \tag{5}$$

and as $J^{-1} = \left| \frac{\partial x'^s}{\partial x^k} \right|$ we get

$$\left| c'^{uv} \right| = J^{-2} \left| c^{rs} \right| \tag{6}$$

Conclusion, $|c^{rs}|$ is a relative invariant of weight -2.

◊

As f^r_s is an absolute tensor, we have

$$f'^u_v = f^r_s \frac{\partial x'^u}{\partial x^r} \frac{\partial x^s}{\partial x'^v} \tag{7}$$

Hence,

$$\left|f'^{u}_{v}\right| = |f^{r}_{s}| \left|\frac{\partial x'^{u}}{\partial x^{r}}\right| \left|\frac{\partial x^{s}}{\partial x'^{v}}\right| \tag{8}$$

and we get

$$\left|f'^{u}_{v}\right| = JJ^{-1} |f^{r}_{s}| \tag{9}$$

Conclusion, $|f^{r}_{s}|$ is an absolute invariant tensor .

♦

7.2 p242 - Exercise

Show that, in three dimensions, the only non-vanishing components of δ^{kl}_{rs} are

$$\delta^{23}_{23} = \delta^{32}_{32} = \delta^{31}_{31} = \delta^{13}_{13} = \delta^{12}_{12} = \delta^{21}_{21} = 1$$

$$\delta^{23}_{32} = \delta^{32}_{23} = \delta^{31}_{13} = \delta^{13}_{31} = \delta^{12}_{21} = \delta^{21}_{12} = -1$$

This is easily seen. If $(k,l),(r,s)$ are considered as sets, then $\delta^{kl}_{rs} \neq 0 \quad \Leftrightarrow \quad (k,l) \neq (r,s)$. And , $\delta^{kl}_{rs} = 1 \quad \Leftrightarrow \quad k = r \wedge l = s$ and on the opposite $\delta^{kl}_{rs} = -1 \quad \Leftrightarrow \quad k = s \wedge l = r$

♦

7.3 p243 - Exercise

Show that equations **5.231** and **6.128** can be written as follows:

$$M_{rs} = \delta^{kl}_{rs} z_k F_l$$

$$\omega_{rs} = \frac{1}{2}\delta^{kl}_{rs} v_{l,k}$$

$$(5.231) \quad M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r \tag{1}$$

In this expression $M_{rs} = 0$ when $r = s$, but this is also the case with δ^{kl}_{rs}.
In $M_{rs} = \delta^{kl}_{rs} z_k F_l$ we see that there is no contribution in the summation when $k = l$. The only contribution being those for which $k = r \wedge l = s$ (positive contribution) $\vee \quad k = s \wedge l = r$ (negative contribution) , hence

$$\delta^{kl}_{rs} z_k F_l \quad \Leftrightarrow \quad z_r F_s - z_s F_r$$

$$\lozenge$$

$$(6.128) \quad \omega_{rs} = \frac{1}{2}\left(v_{s,r} - v_{r,s}\right) \tag{2}$$

The same arguments of the previous case apply to this case (a way to see this is to represent symbolically, $z_r F_s$ and $v_{s,r}$ by T_{rs})

$$\blacklozenge$$

7.4 p243 - Exercise

If $T_{k_1 k_2 \ldots k_M}$ is completely skew-symmetric, determine

$$\delta^{k_1 k_2 \ldots k_M}_{s_1 s_2 \ldots s_M} T_{k_1 k_2 \ldots k_M}$$

$\delta^{k_1 k_2 \ldots k_M}_{s_1 s_2 \ldots s_M} T_{k_1 k_2 \ldots k_M}$ is a sum of $M!$ terms: the first of these is $T_{s_1 s_2 \ldots s_M}$; the other terms are obtained from it by permuting the subscripts and a minus sign is attached if the permutation is odd. Since $T_{s_1 s_2 \ldots s_M}$ is completely skew-symmetric, each of the $M!$ terms equals $+T_{s_1 s_2 \ldots s_M}$ Hence,

$$\delta^{k_1 k_2 \ldots k_M}_{s_1 s_2 \ldots s_M} T_{k_1 k_2 \ldots k_M} = M! \, T_{s_1 s_2 \ldots s_M}$$

♦

7.5 p245 - Exercise

Show that $\epsilon^{r_1 r_2 \ldots r_N} \epsilon_{r_1 r_2 \ldots r_N} = N!$.

First note that $sign(\epsilon^{r_1 r_2 \ldots r_N}) = sign(\epsilon_{r_1 r_2 \ldots r_N})$ so that each term in the summation is always +1.
There are N choices to chose from for r_1, $N-1$ for r_2 , etc. and only one for r_N. And so $\epsilon^{r_1 r_2 \ldots r_N} \epsilon_{r_1 r_2 \ldots r_N} = N!$

♦

7.6 p245 - Clarification to 7.113

$$\epsilon^{k_1 \ldots k_M r_1 \ldots r_{N-M}} \epsilon_{s_1 \ldots s_M r_1 \ldots r_{N-M}} = (N-M)!\ \delta^{k_1 \ldots k_M}_{s_1 \ldots s_M}$$

This can be seen as followed.

As the permutation $(r_1 \ldots r_{N-M})$ is the same for both covariant and contravariant permutation symbols, the product $\epsilon^{k_1 \ldots k_M r_1 \ldots r_{N-M}} \epsilon_{s_1 \ldots s_M r_1 \ldots r_{N-M}}$ for a fixed permutation $(r_1 \ldots r_{N-M})$ (i.e. no summation on repeated indexes) will be determined by $\delta^{k_1 \ldots k_M}_{s_1 \ldots s_M}$ .Indeed, the difference in "oddness" between $(k_1 \ldots k_M r_1 \ldots r_{N-M})$ and $(s_1 \ldots s_M r_1 \ldots r_{N-M})$ is only determined by the difference in "oddness" between $(k_1 \ldots k_M r_1)$ and $(s_1 \ldots s_M)$. So each term in the summation has the same contribution, i.e.; $\delta^{k_1 \ldots k_M}_{s_1 \ldots s_M}$.

There are M choices to chose from for r_1, $M-1$ for r_2 , etc. and only one for r_M. And so

$$\epsilon^{k_1 \ldots k_M r_1 \ldots r_{N-M}} \epsilon_{s_1 \ldots s_M r_1 \ldots r_{N-M}} = (N-M)!\ \delta^{k_1 \ldots k_M}_{s_1 \ldots s_M}$$

♦

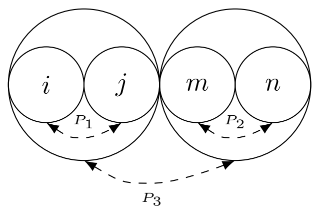
7.7 p247 - Exercise

If T_{rs} is an absolute skew-symmetric tensor in a $4-$space, show that

$$T_{14}T_{23} + T_{24}T_{31} + T_{34}T_{12}$$

is a tensor density

Be $P = T_{14}T_{23} + T_{24}T_{31} + T_{34}T_{12}$, we can write this as $P = \frac{1}{8}\delta^{ijmn}_{1234}T_{ij}T_{mn}$



(a)

Figure 7.1: Permutations

The factor $\frac{1}{8}$ is explained by te fact that are 2^3 possible permutations in the i, j, m, n indexes i.e. 2×2 for the permutations P_1 and P_2 and again 2 for the permutation P_3. Note that a single permutation P_1 or P_2 changes the sign of $T_{ij}T_{mn}$ but also changes the sign of δ^{ijmn}_{1234}, so the combined sign doesn't change. A double permutation P_1 and P_2 changes the sign of T_{ij} and T_{mn} resulting in a unchanged sign of $T_{ij}T_{mn}$ but also δ^{ijmn}_{1234} is unchanged because of the double permutation. Finally P_3 has no effect, nor on $T_{ij}T_{mn}$ nor on δ^{ijmn}_{1234}. So we have 8 repetitions for the same set i, j, m, n. So we have,

$$P = \frac{1}{8}\delta^{ijmn}_{1234}T_{ij}T_{mn} \tag{1}$$

$$= \frac{1}{8}\epsilon^{ijmn}T_{ij}T_{mn} \tag{2}$$

From this follows immediately as ϵ^{ijmn} is a relative tensor of weight 1 and T_{ij} an absolute tensor (i.e. a relative tensor of weight 0) that P is a relative tensor of eight 1 i.e. a density.

♦

7.8 p247 - Exercise

Show that, for rectangular Cartesian coordinates, the vorticity tensor and the vorticity vector of a fluid are duals (cf. **6.130**).

6.130:

$$\omega_r = \frac{1}{2}\epsilon_{rmn}\omega_{mn}, \qquad \omega_{mn} = \epsilon_{rmn}\omega_r \tag{1}$$

Put $\hat{T}^r = \omega_r$ and $T_{mn} = \omega_{mn}$ then the expressions in (1) can be expressed as (considering that the covariant an contravariant expressions are identical in rectangular Cartesian coordinates)

$$\hat{T}^r = \frac{1}{(3-2)!}\epsilon^{mnr}T_{mn}, \qquad T_{mn} = \epsilon_{rmn}\hat{T}^r \tag{2}$$

which are exactly the general definitions **7.121** and **7.122** (with $N = 3$ and $M = 2$) for dual tensors.

♦

7.9 p250 - Exercise

Show that

$$\eta_{r_1 \dots r_N} = \epsilon(a) a_{r_1 s_1} \dots a_{r_N s_N} \eta^{s_1 \dots s_N}$$

$$\eta^{r_1 \dots r_N} = \epsilon(a) a^{r_1 s_1} \dots a^{r_N s_N} \eta_{s_1 \dots s_N}$$

♦

7.10 p252 - Exercise

Using Riemannian coordinates, prove that **7.216** $\epsilon_{r_1 \ldots r_N|k} = \epsilon^{r_1 \ldots r_N}{}_{|k} = 0 \qquad \eta_{r_1 \ldots r_N|k} = \eta^{r_1 \ldots r_N}{}_{|k} = 0$

♦

7.11 p252 - Exercise

Prove that if T^n is a relative vector of weight W then,

$$\mathbf{7.220} \quad T^n_{\ |n} = (\epsilon(a)a)^{\frac{1}{2}(W-1)} \frac{\partial}{\partial x^n} \left[(\epsilon(a)a)^{\frac{1}{2}(1-W)} T^n \right]$$

♦

7.12 p255 - Clarification

$$\textbf{7.304} \quad \Delta^{k_1\dots k_M} = \delta^{k_1\dots k_M}_{s_1\dots s_M}\Delta_{(1)}x^{s_1}\dots\Delta_{(M)}x^{s_M}$$

Using **7.303** and noting that a permutation of columns in the matrix changes or not the sign of its determinant depending on the sign of the permutation, we can write

$$\Delta^{k_1\dots k_M} = \begin{vmatrix} \Delta_{(1)}x^{k_1} & \Delta_{(1)}x^{k_2} & \dots & \Delta_{(1)}x^{k_M} \\ \Delta_{(2)}x^{k_1} & \Delta_{(2)}x^{k_2} & \dots & \Delta_{(2)}x^{k_M} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{(M)}x^{k_1} & \Delta_{(M)}x^{k_2} & \dots & \Delta_{(M)}x^{k_M} \end{vmatrix} \tag{1}$$

$$= \epsilon^{k_1k_2\dots k_m} \begin{vmatrix} \Delta_{(1)}x^{1} & \Delta_{(1)}x^{2} & \dots & \Delta_{(1)}x^{M} \\ \Delta_{(2)}x^{1} & \Delta_{(2)}x^{k_2} & \dots & \Delta_{(2)}x^{k_M} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{(M)}x^{1} & \Delta_{(M)}x^{2} & \dots & \Delta_{(M)}x^{M} \end{vmatrix} \tag{2}$$

And using **4.313** : $\quad |A_{pq}| = \epsilon_{s_1s_2\dots s_M}A_{1s_1}A_{2s_2}\dots A_{Ms_M}$:

$$\Delta^{k_1\dots k_M} = \underbrace{\epsilon^{k_1k_2\dots k_m}\epsilon_{s_1s_2\dots s_M}}_{\text{see (7.114)}}\Delta_{(1)}x^{s_1}\Delta_{(2)}x^{s_2}\dots\Delta_{(M)}x^{s_M} \tag{3}$$

$$= \delta^{k_1\dots k_M}_{s_1\dots s_M}\Delta_{(1)}x^{s_1}\Delta_{(2)}x^{s_2}\dots\Delta_{(M)}x^{s_M} \tag{4}$$

♦

7.13 p255 - Exercise

Show that **7.305** may be written in the equivalent form

$$d\tau_{(M)}^{k_1\dots k_m} = \epsilon^{\beta_1\dots\beta_M} d_{(\beta_1)}x^{k_1}\dots d_{(\beta_M)}x^{k_M}$$

The determinant of a matrix and its transpose are equal.
Hence we can rewrite **7.305** $\delta^{k_1\dots k_M}_{s_1\dots s_M} d_{(1)}x^{s_1}\dots d_{(M)}x^{s_M}$ as

$$d\tau_{(M)}^{k_1\dots k_m} = \delta^{s_1\dots s_M}_{k_1\dots k_M} d_{(s_1)}x^1\dots d_{(s_M)}x^M \tag{1}$$

In order to be consistent with the notation we replace the s_i by α_i as the summation occurs along the constants $c^{(i)}$

$$d\tau_{(M)}^{k_1\dots k_m} = \delta^{\alpha_1\dots\alpha_M}_{k_1\dots k_M} d_{(\alpha_1)}x^1\dots d_{(\alpha_M)}x^M \tag{2}$$

Given the set $\{k_1, k_2, \dots, k_M\}$ we can represent the sequence $\{1, 2, \dots, M\}$ by $\{k_j, k_m, \dots, k_M, \dots k_n\}$ (imagine that $k_j = 1, k_m = 2, ...$ etc.). We rewrite (2) as

$$d\tau_{(M)}^{k_1k_2\dots k_m} = (\theta_\alpha)\delta^{\alpha_1\dots\alpha_M}_{12\dots M} d_{(\alpha_1)}x^{k_j} d_{(\alpha_2)}x^{k_m}\dots d_{(\alpha_M)}x^{k_n} \tag{3}$$

where

$$\theta_\alpha = \epsilon_{k_1k_2\dots k_M} \tag{4}$$

(a permutation in the lower indexes of the generalized Kronecker deltas symbol will invert the sign depending on the 'oddness' of the permutation). Let's rearrange the product $d_{(\alpha_1)}x^{k_j} d_{(\alpha_2)}x^{k_m}\dots d_{(\alpha_M)}x^{k_n}$ so that the indexes k_i are naturally ordered

$$d\tau_{(M)}^{k_1k_2\dots k_m} = (\theta_\alpha)\delta^{\alpha_1\dots\alpha_M}_{12\dots M} d_{(\alpha_r)}x^{k_1} d_{(\alpha_n)}x^{k_2}\dots d_{(\alpha_1)}x^{k_n}\dots d_{(\alpha_s)}x^{k_M} \tag{5}$$

and changing the order in the upper indexes of the general Kroneckers delta's:

$$d\tau_{(M)}^{k_1k_2\dots k_m} = (\theta_\alpha)(\theta_k)\delta^{\alpha_r\alpha_n\dots\alpha_s}_{12\dots M} d_{(\alpha_r)}x^{k_1} d_{(\alpha_n)}x^{k_2}\dots d_{(\alpha_1)}x^{k_n}\dots d_{(\alpha_s)}x^{k_M} \tag{6}$$

where $\theta_k = \pm 1$ depending on the 'oddness' of the permutation needed to go from $\{\alpha_1\dots\alpha_M\}$ to $\{\alpha_r\alpha_n\dots\alpha_s\}$.
As we can see in figure 7.2, it's no hard to see that

$$\theta_k = \theta_\alpha \tag{7}$$

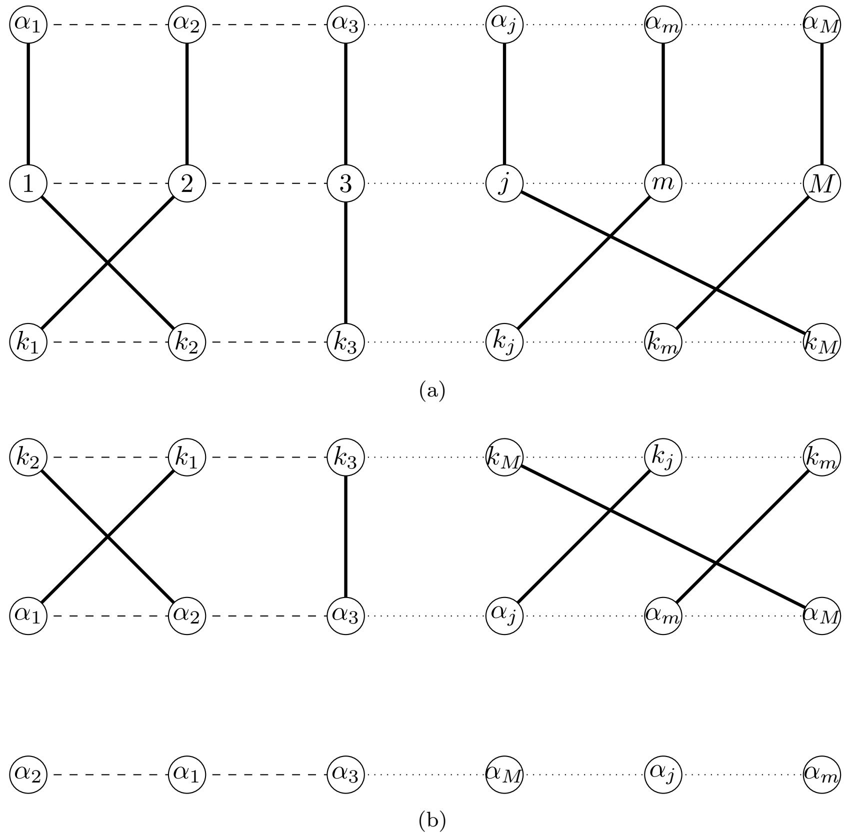


Figure 7.2: Permutations

Indeed suppose, as in the example (a) , $k_1 = 2, k_2 = 1, k_3 = 3, , \ldots, k_j = m, \ldots, k_m = j, \ldots$ etc., so we get a sequence $\{k_2, k_1, k_3, \ldots, k_m, \ldots k_j, \ldots\}$ as illustrated in (b). But to have - with this sequence - an equivalent expression of $d\tau_{(M)}^{k_1 k_2 \ldots k_m} = (\theta_\alpha)\delta_{12\ldots M}^{\alpha_1 \ldots \alpha_M} d_{(\alpha_1)}x^{k_j} d_{(\alpha_2)}x^{k_m} \ldots d_{(\alpha_M)}x^{k_n}$, we need to make an equivalent permutation so that α_r gets in the same position as k_r, resulting in a new sequence $\{\alpha_2, \alpha_1, \alpha_3, \ldots, \alpha_M, \ldots, \alpha_j, \alpha_m\}$.
The number of permutations to generate θ_α and θ_k are identical resulting in $\theta_\alpha \theta_k = 1$. So (6) can be rewritten (noting that the α_r are dummy indexes and that we are free to rename them so that $r = 1, n = 2, \ldots$)

$$d\tau_{(M)}^{k_1 k_2 \ldots k_m} = \delta_{12\ldots M}^{\beta_1 \beta_2 \ldots \beta_M} d_{(\beta_1)}x^{k_1} d_{(\beta_2)}x^{k_2} \ldots d_{(\beta_n)}x^{k_n} \ldots d_{(\beta_M)}x^{k_M} \tag{8}$$

Finally, using **7.114**

$$d\tau_{(M)}^{k_1 k_2 \ldots k_m} = \underbrace{\epsilon_{12\ldots M}}_{=1} \epsilon^{\beta_1 \beta_2 \ldots \beta_M} d_{(\beta_1)}x^{k_1} d_{(\beta_2)}x^{k_2} \ldots d_{(\beta_n)}x^{k_n} \ldots d_{(\beta_M)}x^{k_M} \tag{9}$$

$$= \epsilon^{\beta_1 \beta_2 \ldots \beta_M} d_{(\beta_1)}x^{k_1} d_{(\beta_2)}x^{k_2} \ldots d_{(\beta_n)}x^{k_n} \ldots d_{(\beta_M)}x^{k_M} \tag{10}$$

♦

7.14 p257 - Exercise

Let x^k be rectangular Cartesian coordinates in Euclidean 3−space. Introduce polar coordinates r, θ, ϕ and consider the surface of the sphere $r = a$. On this sphere form the infinitesimal 2−cell with corners (θ, ϕ), $(\theta + d\theta, \phi)$, $(\theta, \phi + d\phi)$, $(\theta + d\theta, \phi + d\phi)$. Determine the extension of this cell and interpret the rectangular components. In particular, show that the three independent components of the extension are (apart from the sign) equal to the areas obtained by normal projection of the cell onto the three rectangular planes. Does this interpretation remain valid if the sphere is replaces by some other surface?

We use **7.312**:

$$d\tau_{(2)}^{k_1 k_2} = \epsilon^{\alpha_1 \alpha_2} \frac{\partial x^{k_1}}{\partial y^{\alpha_1}} \frac{\partial x^{k_2}}{\partial y^{\alpha_2}} \left| d_{(\beta)} y^\gamma \right| \tag{1}$$

with $(y^1, y^2) = (\theta, \phi)$ giving if we take $f^{(i)} = c^{(i)}$ as $\theta = c^{(1)},\ \phi = c^{(2)}$:

$$\begin{aligned} \left| d_{(\beta)} y^\gamma \right| &= \begin{vmatrix} d_{(1)} y^1 & d_{(1)} y^2 \\ d_{(2)} y^1 & d_{(2)} y^2 \end{vmatrix} & (2) \\ &= \begin{vmatrix} d\theta & 0 \\ 0 & d\phi \end{vmatrix} & (3) \\ &= d\theta d\phi & (4) \end{aligned}$$

We also have

$$\begin{cases} x = a \sin\theta \cos\phi \\ y = a \sin\theta \sin\phi \\ z = a \cos\theta \end{cases} \tag{5}$$

$$\tag{6}$$

giving

$$\begin{cases} \frac{\partial x}{\partial \theta} = a \cos\theta \cos\phi \\ \frac{\partial x}{\partial \phi} = -a \sin\theta \sin\phi \\ \frac{\partial y}{\partial \theta} = a \cos\theta \sin\phi \\ \frac{\partial y}{\partial \phi} = a \sin\theta \cos\phi \\ \frac{\partial z}{\partial \theta} = -a \sin\theta \\ \frac{\partial z}{\partial \phi} = 0 \end{cases} \tag{7}$$

$$\tag{8}$$

and get

$$
\begin{aligned}
d\tau_{(2)}^{xy} &= \underbrace{\epsilon^{11}}_{=0}\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\theta}d\theta d\phi + \epsilon^{12}\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\phi}d\theta d\phi + \epsilon^{21}\frac{\partial x}{\partial\phi}\frac{\partial y}{\partial\theta}d\theta d\phi + \underbrace{\epsilon^{22}}_{=0}\frac{\partial x}{\partial\phi}\frac{\partial y}{\partial\phi}d\theta d\phi && (9)\\
&= a^2\cos\theta\cos\phi\sin\theta\cos\phi d\theta d\phi + a^2\sin\theta\sin\phi\cos\theta\sin\phi d\theta d\phi && (10)\\
&= a^2\cos\theta\sin\theta d\theta d\phi && (11)\\
d\tau_{(2)}^{yx} &= -a^2\cos\theta\sin\theta d\theta d\phi && (12)
\end{aligned}
$$

$$
\begin{aligned}
d\tau_{(2)}^{xz} &= \underbrace{\epsilon^{11}}_{=0}\frac{\partial x}{\partial\theta}\frac{\partial z}{\partial\theta}d\theta d\phi + \epsilon^{12}\frac{\partial x}{\partial\theta}\underbrace{\frac{\partial z}{\partial\phi}}_{=0}d\theta d\phi + \epsilon^{21}\frac{\partial x}{\partial\phi}\frac{\partial z}{\partial\theta}d\theta d\phi + \underbrace{\epsilon^{22}}_{=0}\frac{\partial x}{\partial\phi}\frac{\partial z}{\partial\phi}d\theta d\phi && (13)\\
&= -a^2\sin^2\theta\sin\phi d\theta d\phi && (14)\\
d\tau_{(2)}^{zx} &= a^2\sin^2\theta\sin\phi d\theta d\phi && (15)
\end{aligned}
$$

$$
\begin{aligned}
d\tau_{(2)}^{yz} &= \underbrace{\epsilon^{11}}_{=0}\frac{\partial y}{\partial\theta}\frac{\partial z}{\partial\theta}d\theta d\phi + \epsilon^{12}\frac{\partial y}{\partial\theta}\underbrace{\frac{\partial z}{\partial\phi}}_{=0}d\theta d\phi + \epsilon^{21}\frac{\partial y}{\partial\phi}\frac{\partial z}{\partial\theta}d\theta d\phi + \underbrace{\epsilon^{22}}_{=0}\frac{\partial y}{\partial\phi}\frac{\partial z}{\partial\phi}d\theta d\phi && (16)\\
&= a^2\sin^2\theta\cos\phi d\theta d\phi && (17)\\
d\tau_{(2)}^{zy} &= -a^2\sin^2\theta\cos\phi d\theta d\phi && (18)
\end{aligned}
$$

$$
\begin{aligned}
d\tau_{(2)}^{xx} &= \underbrace{\epsilon^{11}}_{=0}\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\theta}d\theta d\phi + \epsilon^{12}\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\phi}d\theta d\phi + \epsilon^{21}\frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\theta}d\theta d\phi + \underbrace{\epsilon^{22}}_{=0}\frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\phi}d\theta d\phi && (19)\\
&= 0 && (20)
\end{aligned}
$$

$$
\begin{aligned}
d\tau_{(2)}^{yy} &= \underbrace{\epsilon^{11}}_{=0}\frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\theta}d\theta d\phi + \epsilon^{12}\frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\phi}d\theta d\phi + \epsilon^{21}\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\theta}d\theta d\phi + \underbrace{\epsilon^{22}}_{=0}\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\phi}d\theta d\phi && (21)\\
&= 0 && (22)
\end{aligned}
$$

$$
\begin{aligned}
d\tau_{(2)}^{zz} &= \underbrace{\epsilon^{11}}_{=0}\frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\theta}d\theta d\phi + \epsilon^{12}\frac{\partial z}{\partial\theta}\underbrace{\frac{\partial z}{\partial\phi}}_{=0}d\theta d\phi + \epsilon^{21}\underbrace{\frac{\partial z}{\partial\phi}}_{=0}\frac{\partial z}{\partial\theta}d\theta d\phi + \underbrace{\epsilon^{22}}_{=0}\frac{\partial z}{\partial\phi}\frac{\partial z}{\partial\phi}d\theta d\phi && (23)\\
&= 0 && (24)
\end{aligned}
$$

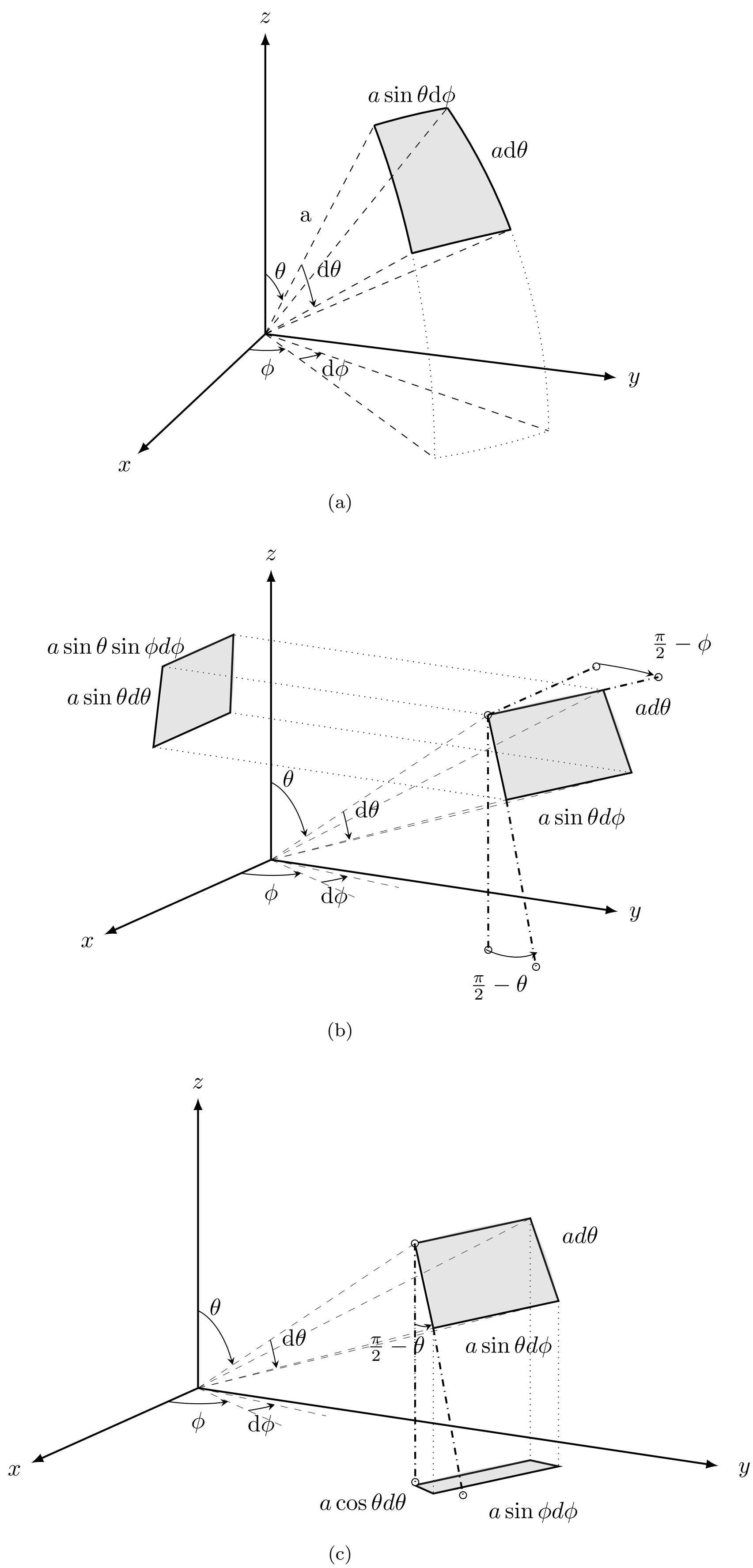


Figure 7.3: Projections of extensions

The quantities $d\tau_{(2)}^{xy}$, $d\tau_{(2)}^{xz}$, $d\tau_{(2)}^{yz}$ are the projections of the extension on the respective Cartesian coordinates planes as can be seen in figure 7.3 where figure (a) depicts the extension (area $= a^2 \sin\theta d\theta d\phi$) when choosing θ, ϕ as the parameters y^k, while figure (b) represents the projection of this extension on the $xz-$plane and figure (c) represents the projection of this extension on the $xy-$plane.

◊

Does this interpretation remain valid if the sphere is replaces by some other surface? The answer is no. For a two-space in Cartesian coordinates system and with surface with parameters (u, v), equation (2) reduces to

$$d\tau_{(2)}^{xy} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) dudv \tag{25}$$

So $d\tau_{(2)}^{xy} = 0$ if $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 0$. Consider the disk defined by the following parametric function

$$S : \left\{\mathbb{R}^2 \to \mathbb{R}^3 : \; S(u,v) = \left(\frac{u}{\sqrt{u^2+v^2+C}}, \; \frac{v}{\sqrt{u^2+v^2+C}}, \; 1\right)\right\}$$

(the constant C is there just to avoid the undefinedness of the surface for $(u, v) = (0, 0)$).
It is easy to see that $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 0$, yet the surface is parallel with the $xy-$plane, which implies that the projection on the $xy-$plane of an elementary cell on S will not have a zero area as can be seen in the figure hereunder.

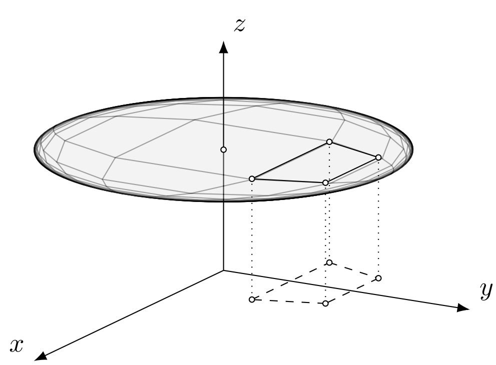


Figure 7.4: A disk defined as $S : \left\{\mathbb{R}^2 \to \mathbb{R}^3 : \; S(u,v) = \left(\frac{u}{\sqrt{u^2+v^2+C}}, \; \frac{v}{\sqrt{u^2+v^2+C}}, \; 1\right)\right\}$

Conclusion: The interpretation of $d\tau_{(2)}^{k_1 k_2}$ as the projection of a cell on a axis-plane, does not hold for every surface.

♦

7.15 p263 - Exercise

Using polar coordinates in Euclidean 3−space find the volume of an infinitesimal cell whose edges are tangent to the coordinate curves. Obtain the volume of a sphere by integration.

For polar spherical coordinates we have

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{1}$$

giving

$$|a_{mn}| = r^4 \sin^2\theta \tag{2}$$

and using as parameters the $x^k \equiv (r, \theta, \phi)$ as parameters for the parametric surface we get

$$\left|d_{(s)}x^k\right| = \begin{vmatrix} dr & 0 & 0 \\ 0 & d\theta & 0 \\ 0 & 0 & d\phi \end{vmatrix} \tag{3}$$

$$= drd\theta d\phi \tag{4}$$

Using **7.405**:

$$dv_{(N)}^2 = \epsilon(a)\,|a_{mn}|\,\left|d_{(s)}x^k\right|^2 \tag{5}$$

$$= r^4 \sin^2\theta\,(drd\theta d\phi)^2 \tag{6}$$

getting for the volume of a sphere with radius R:

$$V = 8\int_0^R \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} dv_{(N)} \tag{7}$$

$$= 8\int_0^R \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r^2 \sin\theta drd\theta d\phi \tag{8}$$

$$= \frac{4}{3}\pi R^3 \tag{9}$$

♦

7.16 p265 - Exercise

In the relativistic theory of finite, expanding universe, the following line element is adopted:

$$ds^2 = R^2 \left[dr^2 + \sin^2 r \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right] - dt^2$$

where $R = R(t)$ is a function of the "time" t. the ranges of the coordinates may be taken to be $0 \leq r \leq \pi,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi < 2\pi,\ -\infty < t < +\infty$.
Find the total volume of "space", i.e., of the surface $t =$ constant, and show that it varies with the "time" t as $R^3(t)$.

For the considered metric, we have

$$(a_{mn}) = \begin{pmatrix} R^2 & 0 & 0 & 0 \\ 0 & R^2 \sin^2 r & 0 & 0 \\ 0 & 0 & R^2 \sin^2 r \sin^2 \theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

Using as parameters the $x^k \equiv (r, \theta, \phi)$ as parameters for the parametric surface and using **7.409** : $b_{\alpha\beta} = a_{ks} \frac{\partial x^k}{\partial y^\alpha} \frac{\partial x^s}{\partial y^\beta}$, we get for the 3−space $t =$ constant

$$(b_{mn}) = \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2 \sin^2 r & 0 \\ 0 & 0 & R^2 \sin^2 r \sin^2 \theta \end{pmatrix} \tag{2}$$

giving

$$|b_{mn}| = R^6 \sin^4 r \sin^2 \theta \tag{3}$$

Using **7.413**:

$$\begin{aligned} dv^2_{(M)} &= \frac{\epsilon(b)}{M!} a_{k_1 s_1} \ldots a_{k_M s_M} d\tau^{k_1 \ldots k_M}_{(M)} d\tau^{s_1 \ldots s_M}_{(M)} && (4) \\ &= -\frac{1}{6} a_{k_1 s_1} \ldots a_{k_M s_M} d\tau^{k_1 \ldots k_M}_{(M)} d\tau^{s_1 \ldots s_M}_{(M)} && (5) \\ &= -\frac{6}{6} a_{11} a_{22} a_{33} \left(\underbrace{d\tau^{123}_{(M)}}_{= dr d\theta d\phi} \right)^2 && (6) \\ &= -R^6 \sin^4 r \sin^2 \theta \left(dr d\theta d\phi\right)^2 && (7) \\ \Rightarrow \quad dv_{(M)} &= R^3 \sin^2 r \sin \theta dr d\theta d\phi && (8) \end{aligned}$$

getting for the volume of "space" with "radius" R:

$$
\begin{aligned}
V &= \int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} R^3 \sin^2 r \sin\theta dr d\theta d\phi && (9)\\
&= 2R^3\pi \int_0^R \sin^2 r dr \underbrace{\int_0^{\pi} \sin\theta d\theta}_{=-cos\theta|_0^{\pi}} && (10)\\
&= 4R^3\pi \underbrace{\int_0^R \sin^2 r dr}_{=\frac{1}{2}\left(\theta-\frac{1}{2}\sin(2r)\right)\big|_0^{\pi}} && (11)\\
&= 2\pi^2 R^3 && (12)
\end{aligned}
$$

(the integral in (11) can be found by substituting $\sin^2 r = 1-\cos^2 r$ and using the cosine sum of angles rule $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ with $\alpha=\beta=r$).

$\diamondsuit$

In order to try to understand a little bit better what manifold is represented by the metric, let's use the trick to embed it in an higher dimensional space with en Euclidean metric.
Therefore let's use the following map:

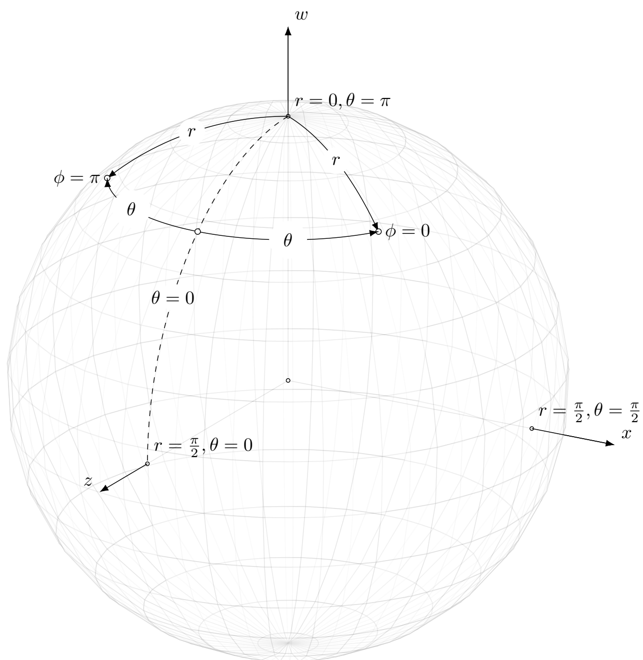
$$
\begin{cases}
x = R\sin r\sin\theta\cos\phi\\
y = R\sin r\sin\theta\sin\phi\\
z = R\sin r\cos\theta\\
w = R\cos r
\end{cases}
\tag{13}
$$

then it's easy to see that

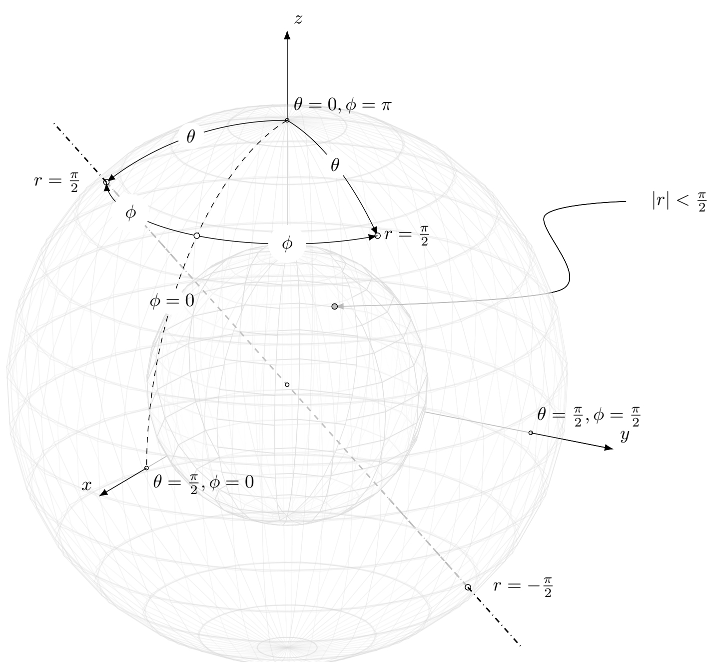
$$
\begin{aligned}
ds^2_{(4)} &= dx^2+dy^2+dz^2+dw^2 && (14)\\
&= R^2\left[dr^2+\sin^2 r\left(d\theta^2+\sin^2\theta d\phi^2\right)\right] && (15)
\end{aligned}
$$

which is exactly the hypersurface we sought. Now let's see what happens when we keep $\phi=0$ or π. Then $y=0$ and in fact we take a slice of the $4-space$ along the xzw subspace, and can visualize this $3-$space as a sphere as represented by figure 7.5(a).
We can also keep r at a certain value. We get also a sphere as represented in figure 7.5(b). Here the biggest possible sphere corresponds to $r=(2k+1)\frac{\pi}{2},\quad k=\{\dots,-1,0,1,\dots\}$ all other spheres having a smaller radius. When r tends to $r=k\pi\quad k=\{\dots,-1,0,1,\dots\}$ the sphere shrinks to a point with coordinates $(0,0,0,\pm R)$.
Finally, keeping $\theta=k\pi,\quad k=\{\dots,-1,0,1,\dots\}$ we get a circle of radius R situated in the xw plane.

Remember that what we do here is just take 'slices' of a space. compare it to an Euclidean $3-$space with a sphere: taking a slice e.g. parallel to the xy plane will give us a circle, a point or an empty set while taking a one-dimensional slice along a line will give us 2, 1 or 0 point(s).



(a)



(b)

Figure 7.5: Manifold with $ds^2 = R^2\left[dr^2 + \sin^2 r\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$ metric, embedded in a 4− Euclidean space

♦

7.17 p268 - Clarification

Since $\frac{\partial T_k}{\partial x^s}$ differs from the tensor $\frac{1}{2}\left(T_{k,s} - T_{s,k}\right)$ by an expression symmetric in $k,\ s$ and $d\tau^{ks}_{(2)}$ is an absolute tensor, skew-symmetric in these suffixes, it follows that the integrand on the left is an invariant.

Let's express **7.504** in curvilinear coordinates, equipped with metric

$$\int_{R_2} T_{k|s} d\tau^{ks}_{(2)} = \int_{R_1} T_r d\tau^{r}_{(1)} \tag{1}$$

Obviously we have

$$T_{k|s} = \frac{1}{2}\left(T_{k|s} - T_{s|k}\right) + \frac{1}{2}\left(T_{k|s} + T_{s|k}\right) \tag{2}$$

Let's put $A_{ks} = \frac{1}{2}\left(T_{k|s} - T_{s|k}\right)$ and $B_{ks} = \frac{1}{2}\left(T_{k|s} + T_{s|k}\right)$.
A_{ks} is skew-symmetric, while B_{ks} is symmetric. Note also that A_{ks} reduces to $A_{ks} = \frac{1}{2}\left(T_{k,s} - T_{s,k}\right)$ as the Christoffel symbols vanish in this expression, leaving only the partial differentials and hence, independent of a metric. So the integrand on the left side becomes

$$T_{k|s} d\tau^{ks}_{(2)} = A_{ks} d\tau^{ks}_{(2)} + B_{ks} d\tau^{ks}_{(2)} \tag{3}$$

$$\Rightarrow \quad T_{k|s} d\tau^{ks}_{(2)} = A_{ks} d\tau^{ks}_{(2)} - B_{ks} d\tau^{sk}_{(2)} \tag{4}$$

$$= A_{ks} d\tau^{ks}_{(2)} - B_{sk} d\tau^{sk}_{(2)} \tag{5}$$

$$(3)+(5) \quad \Rightarrow \quad T_{k|s} d\tau^{ks}_{(2)} = A_{ks} d\tau^{ks}_{(2)} \tag{6}$$

We note that both A_{ks} and $d\tau^{ks}_{(2)}$ are tensors, independent of the metric defined, and thus $A_{ks} d\tau^{ks}_{(2)}$ is an invariant of order 0.

♦

7.18 p274 - Exercise

> Due to the skew-symmetry of $d\tau_{(M)}^{k_1\ldots k_M}$, the integrand on the left-hand side is also an invariant. This may be proved in a few lines, preferably with the use of the compressed notation of 1.7; the proof is left as an exercise for the reader.

We will use the indices i, j for the transformed tensor components and p, q for the original components. We have

$$
\begin{aligned}
T'_{i_1\ldots i_{M-1}} &= T_{p_1\ldots p_{M-1}} X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}} && (1)\\
\Rightarrow \quad T'_{i_1\ldots i_{M-1},i_M} &= T_{p_1\ldots p_{M-1},i_M} X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}} + T_{p_1\ldots p_{M-1}} \left(X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}}\right)_{,i_M} && (2)\\
&= T_{p_1\ldots p_{M-1},p_M} X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}} X_{i_M}^{p_M} + T_{p_1\ldots p_{M-1}} \left(X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}}\right)_{,i_M} && (3)
\end{aligned}
$$

and

$$
d\tau'^{i_1\ldots i_M}_{(M)} = d\tau^{s_1\ldots s_M}_{(M)} X^{i_1}_{s_1} \ldots X^{i_M}_{s_M} \qquad (4)
$$

This gives

$$
\begin{aligned}
T'_{i_1\ldots i_{M-1},i_M} d\tau'^{i_1\ldots i_M}_{(M)} &= \begin{cases} T_{p_1\ldots p_{M-1},p_M} X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}} X_{i_M}^{p_M} d\tau^{s_1\ldots s_M}_{(M)} X^{i_1}_{s_1} \ldots X^{i_M}_{s_M} \\ + \underbrace{T_{p_1\ldots p_{M-1}} \left(X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}}\right)_{,i_M} d\tau^{s_1\ldots s_M}_{(M)} X^{i_1}_{s_1} \ldots X^{i_M}_{s_M}}_{=0} \end{cases} && (5)\\
&= T_{p_1\ldots p_{M-1},p_M} X_{s_1}^{p_1} \ldots X_{s_{M-1}}^{p_{M-1}} X_{s_M}^{p_M} d\tau^{s_1\ldots s_M}_{(M)} && (6)\\
&= T_{p_1\ldots p_{M-1},p_M} \delta_{s_1}^{p_1} \ldots \delta_{s_{M-1}}^{p_{M-1}} \delta_{s_M}^{p_M} d\tau^{s_1\ldots s_M}_{(M)} && (7)\\
&= T_{s_1\ldots s_{M-1},s_M} d\tau^{s_1\ldots s_M}_{(M)} && (8)
\end{aligned}
$$

The term $T_{p_1\ldots p_{M-1}} \left(X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}}\right)_{,i_M} d\tau^{s_1\ldots s_M}_{(M)} X^{i_1}_{s_1} \ldots X^{i_M}_{s_M}$ in (5) is zero. Indeed, rewriting this as :

$$
T_{p_1\ldots p_{M-1}} \left(X_{i_1}^{p_1} \ldots X_{i_{M-1}}^{p_{M-1}}\right)_{,p_M} X^{p_M}_{i_M} d\tau^{s_1\ldots s_M}_{(M)} X^{i_1}_{s_1} \ldots X^{i_M}_{s_M} \qquad (9)
$$

The terms in brackets are zero as they are of the form

$$
\cdots + X_{i_1}^{p_1} \ldots \underbrace{\frac{\partial^2 x^{p_m}}{\partial x^{p_M} \partial x'^{i_m}}}_{=\frac{\partial \delta^{p_m}_{p_M}}{\partial x'^{i_m}}=0} \ldots X_{i_{M-1}}^{p_{M-1}} + \ldots \qquad (10)
$$

From (8) we conclude that the integrand is indeed an invariant.

♦

7.19 p275 - Exercise

The skew-symmetric part of a tensor $T_{k_1 \ldots k_M}$ is defined as

$$T_{[k_1 \ldots k_M]} = (M!)^{-1} \, \delta^{s_1 \ldots s_M}_{k_1 \ldots k_M} T_{s_1 \ldots s_M}$$

Show that the left-hand side of equation **7.525** is unchanged if $T_{k_1 \ldots k_M}$ is replace by its skew-symmetric part. Show that the same is true for the right-hand side.

The integrand of the left-hand side of **7.525** is

$$T_{k_1 \ldots k_{M-1}, k_M} d\tau^{k_1 \ldots k_M}_{(M)} \tag{1}$$

replacing the tensor with it's skew-symmetric part gives

$$\begin{aligned}
T_{[k_1 \ldots k_{M-1}], k_M} d\tau^{k_1 \ldots k_M}_{(M)} &= \frac{1}{(M-1)!} \delta^{s_1 \ldots s_{M-1}}_{k_1 \ldots k_{M-1}} T_{s_1 \ldots s_{M-1}, k_M} d\tau^{k_1 \ldots k_M}_{(M)} && (2)\\
&= \frac{1}{(M-1)!} \delta^{s_1 \ldots s_{M-1}}_{k_1 \ldots k_{M-1}} \delta^{s_M}_{k_M} T_{s_1 \ldots s_{M-1}, s_M} d\tau^{k_1 \ldots k_M}_{(M)} && (3)\\
&= \frac{1}{(M-1)!} \delta^{s_1 \ldots s_{M-1}}_{k_1 \ldots k_{M-1}} T_{s_1 \ldots s_{M-1}, s_M} d\tau^{k_1 \ldots k_{M-1} s_M}_{(M)} && (4)\\
&= T_{s_1 \ldots s_{M-1}, s_M} \frac{1}{(M-1)!} \left(\delta^{s_1 \ldots s_{M-1}}_{k_1 \ldots k_{M-1}} d\tau^{k_1 \ldots k_{M-1} s_M}_{(M)} \right) && (5)\\
& && (6)
\end{aligned}$$

The terms in the brackets can be reduced to $(M-1)! d\tau^{s_1 \ldots s_{M-1} s_M}_{(M)}$. Indeed, the number of choices in choosing the $k_1, \ldots k_{M-1}$ are restricted to $M-1$ choices, due to the skew-symmetry of $d\tau^{k_1 \ldots k_{M-1} s_M}_{(M)}$ (choosing a $k_i = s_M$ would result in a zero value for $d\tau^{k_1 \ldots k_{M-1} s_M}_{(M)}$). So, in total there will be $(M-1)!$ terms and due to the skew-symmetry of $d\tau^{k_1 \ldots k_{M-1} s_M}_{(M)}$ a change in sign of $\delta^{s_1 \ldots s_{M-1}}_{k_1 \ldots k_{M-1}}$ will also result in a change of sign in $d\tau^{k_1 \ldots k_{M-1} s_M}_{(M)}$. Hence, we get

$$\begin{aligned}
T_{[k_1 \ldots k_{M-1}], k_M} d\tau^{k_1 \ldots k_M}_{(M)} &= T_{s_1 \ldots s_{M-1}, s_M} \frac{1}{(M-1)!} (M-1)! d\tau^{s_1 \ldots s_{M-1} s_M}_{(M)} && (7)\\
&= T_{s_1 \ldots s_{M-1}, s_M} d\tau^{s_1 \ldots s_{M-1} s_M}_{(M)} && (8)
\end{aligned}$$

◊

The integrand of the right-hand side of **7.525** is

$$T_{k_1 \ldots k_{M-1}} d\tau^{k_1 \ldots k_{M-1}}_{(M-1)} \tag{9}$$

replacing the tensor with it's skew-symmetric part gives

$$T_{[k_1\ldots k_{M-1}]}d\tau_{(M-1)}^{k_1\ldots k_{M-1}} = \frac{1}{(M-1)!}\delta_{k_1\ldots k_{M-1}}^{s_1\ldots s_{M-1}}T_{s_1\ldots s_{M-1}}d\tau_{(M-1)}^{k_1\ldots k_{M-1}} \tag{10}$$

$$= T_{s_1\ldots s_{M-1}}\frac{1}{(M-1)!}\left(\delta_{k_1\ldots k_{M-1}}^{s_1\ldots s_{M-1}}d\tau_{(M-1)}^{k_1\ldots k_{M-1}}\right) \tag{11}$$

The terms in the brackets can be reduced to $(M-1)!d\tau_{(M-1)}^{s_1\ldots s_{M-1}}$. Indeed, the number of choices in choosing the $k_1,\ldots k_{M-1}$ is $M-1$, giving $(M-1)!$ terms and due to the skew-symmetry of $d\tau_{(M-1)}^{k_1\ldots k_{M-1}}$ a change in sign of $\delta_{k_1\ldots k_{M-1}}^{s_1\ldots s_{M-1}}$ will also result in a change of sign in $d\tau_{(M)}^{k_1\ldots k_{M-1}}$. Hence, we get

$$T_{[k_1\ldots k_{M-1}]}d\tau_{(M-1)}^{k_1\ldots k_{M-1}} = T_{s_1\ldots s_{M-1}}\frac{1}{(M-1)!}(M-1)!d\tau_{(M-1)}^{s_1\ldots s_{M-1}} \tag{12}$$

$$== T_{s_1\ldots s_{M-1}}d\tau_{(M-1)}^{s_1\ldots s_{M-1}} \tag{13}$$

♦