Tensor Calculus
J.L. Synge and A.Schild (Dover Publication)
Solutions to exercises

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github.

Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \quad \text{Christoffel symbol of the second kind}$$

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Curvature of space

1.1 p109 - Exercise 6

For an orthogonal coordinates system in a V_2 we have

$$ds^2 = a_{11}\left(dx^1\right)^2 + a_{22}\left(dx^2\right)^2$$

Show that

$$\frac{1}{a}R_{1212} = -\frac{1}{2}\frac{1}{\sqrt{a}}\left[\partial_1\left(\frac{1}{\sqrt{a}}\partial_1 a_{22}\right) + \partial_2\left(\frac{1}{\sqrt{a}}\partial_2 a_{11}\right)\right]$$

We have

$$(a_{mn}) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \qquad (a^{mn}) = \frac{1}{a}\begin{pmatrix} a_{22} & 0 \\ 0 & a_{11} \end{pmatrix} \qquad a = a_{11}a_{22} \tag{1}$$

We have also

$$R = -\frac{2}{a}R_{1212} \tag{2}$$

$$R = a^{mn}R_{mn} \quad \Rightarrow \quad R = a^{11}R_{11} + a^{22}R_{22} \tag{3}$$

Looking at the pattern generated by equations (2) and (3) suggests that using these equations could lead to the proposed equation. Let's have a try ...

$$\begin{cases} \Gamma^1_{11} = \frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11} & \Gamma^1_{22} = -\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22} \\ \\ \Gamma^2_{11} = -\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11} & \Gamma^2_{22} = \frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22} \\ \\ \Gamma^1_{12} = \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} & \Gamma^2_{12} = \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} \end{cases} \tag{4}$$

$$\text{3.205.} \quad \Rightarrow \quad R_{rm} = \frac{1}{2}\partial_{rm}\log a - \frac{1}{2}\Gamma^p_{rm}\partial_p \log a - \partial_n \Gamma^n_{rm} + \Gamma^p_{rn}\Gamma^n_{pm} \tag{5}$$

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\Gamma^1_{11}\partial_1 \log a - \frac{1}{2}\Gamma^2_{11}\partial_2 \log a \\ -\partial_1\Gamma^1_{11} - \partial_2\Gamma^2_{11} + \\ \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^2_{21} \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a - \frac{1}{2}\Gamma^1_{22}\partial_1 \log a - \frac{1}{2}\Gamma^2_{22}\partial_2 \log a \\ -\partial_1\Gamma^1_{22} - \partial_2\Gamma^2_{22} + \\ \Gamma^1_{21}\Gamma^1_{12} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{21}\Gamma^1_{22} + \Gamma^2_{22}\Gamma^2_{22} \end{cases} \tag{6}$$

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}\partial_1 \log a - \frac{1}{2}(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11})\partial_2 \log a \\ -\partial_1(\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}) - \partial_2(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11}) + \\ \frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11} + \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11}(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11}) + \\ (-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11})\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} + \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22}\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a - \frac{1}{2}(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22})\partial_1 \log a - \frac{1}{2}\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}\partial_2 \log a \\ -\partial_1(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22}) - \partial_2(\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}) + \\ \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11}\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} + (-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22})\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} + \\ \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22}(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22}) + \frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22} \\ \end{cases} \tag{7}$$

Simplifying the notational burden by replacing a_{11} by γ and a_{22} by η:

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\partial_1\gamma\partial_1\log a + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\partial_2\gamma\partial_2\log a \\ -\frac{1}{2}\partial_1(\frac{1}{\gamma}\partial_1\gamma) + \frac{1}{2}\partial_2(\frac{1}{\eta}\partial_2\gamma) \\ +\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_1\gamma\partial_1\gamma - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\gamma \\ -\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\gamma + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_1\eta\partial_1\eta \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a + \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\partial_1\eta\partial_1\log a - \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\partial_2\eta\partial_2\log a \\ +\frac{1}{2}\partial_1(\frac{1}{\gamma}\partial_1\eta) - \frac{1}{2}\partial_2(\frac{1}{\eta}\partial_2\eta) \\ +\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_2\gamma\partial_2\gamma - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\eta\partial_1\eta \\ -\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\eta\partial_1\eta + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_2\eta\partial_2\eta \\ \end{cases} \tag{8}$$

Noting that $\partial_{ii}\log a = \partial_i\left(\frac{1}{a_{11}}\partial_i a_{11}\right) + \partial_i\left(\frac{1}{a_{22}}\partial_i a_{22}\right)$ and $\partial_i \log a = \frac{1}{a_{11}}\partial_i a_{11} + \frac{1}{a_{22}}\partial_i a_{22} \quad (i = 1, 2)$, we get:

$$
\left|\begin{array}{c|c}
2R_{11} = & 2R_{22} = \\
\underbrace{\partial_1\left(\frac{1}{\gamma}\partial_1\gamma\right)}_{*} + \partial_1\left(\tfrac{1}{\eta}\partial_1\eta\right) & \partial_2\left(\tfrac{1}{\gamma}\partial_2\gamma\right) + \underbrace{\partial_2\left(\frac{1}{\eta}\partial_2\eta\right)}_{*} \\
\underbrace{-\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_1\gamma)^2}_{-} - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_1\gamma\partial_1\eta & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}\partial_1\gamma\partial_1\eta + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_1\eta)^2 \\
+\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_2\gamma)^2 + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta & -\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta - \underbrace{\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_2\eta)^2}_{-} \\
-\underbrace{\partial_1(\frac{1}{\gamma}\partial_1\gamma)}_{*} + \partial_2(\tfrac{1}{\eta}\partial_2\gamma) & +\partial_1(\tfrac{1}{\gamma}\partial_1\eta) - \underbrace{\partial_2(\frac{1}{\eta}\partial_2\eta)}_{*} \\
+\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_1\gamma)^2}_{-} - \underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2}_{+} & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}(\partial_2\gamma)^2 - \underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2}_{+} \\
-\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2}_{+} + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}(\partial_1\eta)^2 & -\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2}_{+} + \underbrace{\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_2\eta)^2}_{-}
\end{array}\right| \tag{9}
$$

$$
\Rightarrow \left|\begin{array}{c|c}
2R_{11} = & 2R_{22} = \\
\partial_1\left(\tfrac{1}{\eta}\partial_1\eta\right) + \partial_2(\tfrac{1}{\eta}\partial_2\gamma) & \partial_1(\tfrac{1}{\gamma}\partial_1\eta) + \partial_2\left(\tfrac{1}{\gamma}\partial_2\gamma\right) \\
+\tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}(\partial_1\eta)^2 - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_2\gamma)^2 & -\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_1\eta)^2 + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}(\partial_2\gamma)^2 \\
-\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_1\gamma\partial_1\eta + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}\partial_1\gamma\partial_1\eta - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta
\end{array}\right| \tag{10}
$$

Be $R = \frac{1}{\gamma}R_{11} + \frac{1}{\eta}R_{22}$, all first order derivatives vanish and we get,

$$
\frac{1}{\gamma}R_{11} + \frac{1}{\eta}R_{22} = \frac{1}{2}\left[\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)\right] + \frac{1}{2}\left[\frac{1}{\gamma}\partial_2(\frac{1}{\gamma}\partial_2\gamma) + \frac{1}{\eta}\partial_2(\frac{1}{\eta}\partial_2\gamma)\right] \tag{11}
$$

We further simplify this expression. Considering the symmetry of (11) we only explicit the calculations for the first terms in ∂_1.

$$\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta) = \frac{1}{\eta}\partial_1(\frac{1}{\sqrt{\gamma}}\frac{1}{\sqrt{\gamma}}\frac{\sqrt{\eta}}{\sqrt{\eta}}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\sqrt{\eta}}\frac{1}{\sqrt{\eta}}\frac{\sqrt{\gamma}}{\sqrt{\gamma}}\partial_1\eta) \tag{12}$$

$$= \frac{1}{\eta}\partial_1\left[\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}\frac{1}{\sqrt{a}}\partial_1\eta\right] + \frac{1}{\gamma}\partial_1\left[\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{a}}\partial_1\eta\right] \tag{13}$$

$$= \begin{cases} \underbrace{\frac{1}{\eta}\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}}_{=\frac{1}{\sqrt{a}}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1\eta\right] + \underbrace{\frac{1}{\gamma}\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}}_{=\frac{1}{\sqrt{a}}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1\eta\right] \\ \\ +\frac{1}{\sqrt{a}}\partial_1\eta\underbrace{\left[\frac{1}{\eta}\partial_1\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}} + \frac{1}{\gamma}\partial_1\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}\right]}_{=0} \end{cases} \tag{14}$$

$$= 2\frac{1}{\sqrt{a}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1 a_{22}\right] \tag{15}$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)\right] = \frac{1}{\sqrt{a}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1 a_{22}\right] \tag{16}$$

Using (16) and the same calculations for the terms in ∂_2 and using (2) and (3) we get

$$\frac{1}{a}R_{1212} = -\frac{1}{2}\frac{1}{\sqrt{a}}\left[\partial_1\left(\frac{1}{\sqrt{a}}\partial_1 a_{22}\right) + \partial_2\left(\frac{1}{\sqrt{a}}\partial_2 a_{11}\right)\right]$$

♦

1.2 p109 - Exercise 7

Suppose that in a V_3 the metric is :

$$ds^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2$$

whre h_1, h_2, h_3 are functions of the three coordinates. Calculate the curvature tensor in terms of the $h^{'}s$ and their derivatives. Check your result by nothing that the curvature tensor will vanish if h_1 is a function of x^1 only, h_2 a function of x^2 only, and h_3 a function of x^3 only.

From **3.115**. and **3.115**. we get for the non vanishing components of the covariant curvature tensor(6 independent components to calculate):

$$R_{1212} = \begin{cases} -R_{1221} \\ -R_{2112} \\ R_{2121} \end{cases} \quad R_{2323} = \begin{cases} -R_{2332} \\ -R_{3223} \\ R_{3232} \end{cases} \quad R_{1313} = \begin{cases} -R_{1331} \\ -R_{3113} \\ R_{3131} \end{cases} \tag{1}$$

$$R_{1213} = \begin{cases} -R_{1231} \\ R_{1312} \\ -R_{1321} \\ -R_{2113} \\ R_{2131} \\ -R_{3112} \\ R_{3121} \end{cases} \quad R_{1223} = \begin{cases} -R_{1232} \\ -R_{2123} \\ R_{2132} \\ R_{2312} \\ -R_{2321} \\ R_{3212} \\ -R_{3221} \end{cases} \quad R_{1323} = \begin{cases} -R_{1332} \\ R_{2313} \\ -R_{2331} \\ -R_{3123} \\ R_{3132} \\ -R_{3213} \\ R_{3231} \end{cases} \tag{2}$$

The metrix tensors:

$$(a_{mn}) = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \quad (a_{mn}) = \begin{pmatrix} \frac{1}{h_1^2} & 0 & 0 \\ 0 & \frac{1}{h_2^2} & 0 \\ 0 & 0 & \frac{1}{h_3^2} \end{pmatrix} \tag{3}$$

The Christoffel symbols:

$$\begin{array}{lll} [11,1] = h_1\partial_1 h_1 & [11,2] = -h_1\partial_2 h_1 & [11,3] = -h_1\partial_3 h_1 \\ [12,1] = h_1\partial_2 h_1 & [12,2] = h_2\partial_1 h_2 & [12,3] = 0 \\ [22,1] = -h_2\partial_1 h_2 & [22,2] = h_2\partial_2 h_2 & [22,3] = -h_2\partial_3 h_2 \\ [23,1] = 0 & [23,2] = h_2\partial_3 h_2 & [23,3] = h_3\partial_2 h_3 \\ [33,1] = -h_3\partial_1 h_3 & [33,2] = -h_3\partial_2 h_3 & [33,3] = -h_3\partial_3 h_3 \\ [31,1] = h_1\partial_3 h_1 & [31,2] = 0 & [31,3] = h_3\partial_1 h_3 \end{array} \tag{4}$$

$$\begin{array}{lll}
\Gamma^1_{11} = \frac{1}{h_1}\partial_1 h_1 & \Gamma^2_{11} = -\frac{h_1}{h_2^2}\partial_2 h_1 & \Gamma^3_{11} = -\frac{h_1}{h_3^2}\partial_3 h_1 \\
\Gamma^1_{12} = \frac{1}{h_1}\partial_2 h_1 & \Gamma^2_{12} = \frac{1}{h_2}\partial_1 h_2 & \Gamma^3_{12} = 0 \\
\Gamma^1_{22} = -\frac{h_2}{h_1^2}\partial_1 h_2 & \Gamma^2_{22} = \frac{1}{h_2}\partial_2 h_2 & \Gamma^3_{22} = -\frac{h_2}{h_3^2}\partial_3 h_2 \\
\Gamma^1_{23} = 0 & \Gamma^2_{23} = \frac{1}{h_2}\partial_3 h_2 & \Gamma^3_{23} = \frac{1}{h_3}\partial_2 h_3 \\
\Gamma^1_{33} = -\frac{h_3}{h_1^2}\partial_1 h_3 & \Gamma^2_{33} = -\frac{h_3}{h_2^2}\partial_2 h_3 & \Gamma^3_{33} = \frac{1}{h_3}\partial_3 h_3 \\
\Gamma^1_{31} = \frac{1}{h_1}\partial_3 h_1 & \Gamma^2_{31} = 0 & \Gamma^3_{31} = \frac{1}{h_3}\partial_1 h_3
\end{array} \tag{5}$$

We use 3.113.

$$R_{rsmn} = \partial_m[sn, r] - \partial_n[sm, r] + \Gamma^p_{sm}[rn, p] - \Gamma^p_{sn}[rm, p]$$

Note that we only have to perform the full calculation for two curvature tensors e.g. R_{1212} and $R_{1213}*$ as the others can be retrieved by using adequate indices renaming and use of the identities 3.115.

$$R_{1212} = -h_2\partial^2_{11}(h_2) - h_1\partial^2_{22}(h_1) + \frac{h_2}{h_1}\partial_1 h_1\partial_1 h_2 + \frac{h_1}{h_2}\partial_2 h_1\partial_2 h_2 - \frac{h_1 h_2}{h_3^2}\partial_3 h_1\partial_3 h_2 \tag{6}$$

$$R_{2323} = -h_3\partial^2_{22}(h_3) - h_2\partial^2_{33}(h_2) + \frac{h_3}{h_2}\partial_2 h_2\partial_2 h_3 + \frac{h_2}{h_3}\partial_3 h_2\partial_3 h_3 - \frac{h_2 h_3}{h_1^2}\partial_1 h_2\partial_1 h_3 \tag{7}$$

$$R_{1313} = -h_3\partial^2_{11}(h_3) - h_1\partial^2_{33}(h_1) + \frac{h_3}{h_1}\partial_1 h_1\partial_1 h_3 + \frac{h_1}{h_3}\partial_3 h_1\partial_3 h_3 - \frac{h_1 h_3}{h_2^2}\partial_2 h_1\partial_2 h_3 \tag{8}$$

$$R_{1213} = -h_1\partial^2_{32}(h_1) + \frac{h_1}{h_3}\partial_2 h_3\partial_3 h_1 + \frac{h_1}{h_2}\partial_2 h_1\partial_3 h_2 \tag{9}$$

$$R_{1223} = h_2\partial^2_{31}(h_2) - \frac{h_2}{h_1}\partial_1 h_2\partial_3 h_1 - \frac{h_2}{h_3}\partial_3 h_2\partial_1 h_3 \tag{10}$$

$$R_{1323} = -h_3\partial^2_{21}(h_3) + \frac{h_3}{h_1}\partial_1 h_3\partial_3 h_1 + \frac{h_3}{h_2}\partial_2 h_3\partial_1 h_2 \tag{11}$$

And, indeed, all curvature tensors vanish when the h_i are only a function of the indices' dimension.

♦

1.3 p109 - Exercise 8

In relativity we encounter the metric form

$$\Phi = e^{\alpha} + e^{x^1}\left[\left(dx^2\right)^2 + \sin^2 x^2 \left(dx^3\right)^2\right] - e^{\gamma}\left(dx^4\right)^2$$

where α and γ are functions of x^1 and x^4 only.
Show that the complete set of non-zero components of the Einstein tensor (see equation (3.214)) for the form given above are as follows

$$\begin{aligned}
G^1_{.1} &= e^{-\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_1\right) + e^{-x^1} \\
G^2_{.2} &= e^{\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_1\gamma_1\right) \\
&\quad + e^{\gamma}\left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4\right) \\
G^3_{.3} &= G^2_{.2} \\
G^4_{.4} &= e^{-\alpha}\left(-\frac{3}{4} - \frac{1}{2}\alpha_1\right) + e^{-x^1} \\
e^{\alpha}G^4_{.1} &= -e^{\gamma}G^4_{.1} = -\frac{1}{2}\alpha_4
\end{aligned}$$

The subscript on α and γ indicate partial derivatives with respect to x^1 and x^4.

We have

$$(a_{mn}) = \begin{pmatrix} e^{\alpha} & 0 & 0 & 0 \\ 0 & e^{x^1} & 0 & 0 \\ 0 & 0 & e^{x^1}\sin^2 x^2 & 0 \\ 0 & 0 & 0 & -e^{\gamma} \end{pmatrix} \quad (a^{mn}) = \begin{pmatrix} e^{-\alpha} & 0 & 0 & 0 \\ 0 & e^{-x^1} & 0 & 0 \\ 0 & 0 & \frac{e^{-x^1}}{\sin^2 x^2} & 0 \\ 0 & 0 & 0 & -e^{-\gamma} \end{pmatrix} \tag{1}$$

And will use the following definitions:

$$G^n_{.t} = R^n_{.t} - \frac{1}{2}\delta^n_t R \tag{2}$$

$$R^n_{.t} = a^{nk}R_{kt} \tag{3}$$

$$R_{kt} = a^{sn}R_{sktn} \tag{4}$$

$$R = a^{kt}R_{kt} \tag{5}$$

Considering that the non-diagonal components of a_{mn} vanish and as $R_{sktn} = 0$ when $s = k$ or $t = n$, we can write :

$$\begin{pmatrix} R_{11} \\ R_{22} \\ R_{33} \\ R_{44} \end{pmatrix} = \begin{pmatrix} 0 & R_{2112} & R_{3113} & R_{4114} \\ R_{1221} & 0 & R_{3223} & R_{4224} \\ R_{1331} & R_{2332} & 0 & R_{4334} \\ R_{1441} & R_{2442} & R_{3443} & 0 \end{pmatrix} \begin{pmatrix} a^{11} \\ a^{22} \\ a^{33} \\ a^{44} \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} R_{12} \\ R_{13} \\ R_{14} \\ R_{23} \\ R_{24} \\ R_{34} \end{pmatrix} = \begin{pmatrix} 0 & 0 & R_{3123} & R_{4124} \\ 0 & R_{2132} & 0 & R_{4134} \\ 0 & R_{2142} & R_{3143} & 0 \\ R_{1231} & 0 & 0 & R_{4234} \\ R_{1241} & 0 & R_{3243} & 0 \\ R_{1341} & R_{2342} & 0 & 0 \end{pmatrix} \begin{pmatrix} a^{11} \\ a^{22} \\ a^{33} \\ a^{44} \end{pmatrix} \tag{7}$$

$$\tag{8}$$

The Christoffel symbols of the first kind are:

$$\begin{array}{llll}
[11,1] = \frac{1}{2}\alpha_1 e^{\alpha} & [11,2] = 0 & [11,3] = 0 & [11,4] = -\frac{1}{2}\alpha_4 e^{\alpha} \\
[12,1] = 0 & [12,2] = \frac{1}{2}e^{x^1} & [12,3] = 0 & [12,4] = 0 \\
[13,1] = 0 & [13,2] = 0 & [13,3] = \frac{1}{2}e^{x^1}\sin^2 x^2 & [13,4] = 0 \\
[14,1] = \frac{1}{2}\alpha_4 e^{\alpha} & [14,2] = 0 & [14,3] = 0 & [14,4] = -\frac{1}{2}\gamma_1 e^{\gamma} \\
[22,1] = -\frac{1}{2}e^{x^1} & [22,2] = 0 & [22,3] = 0 & [22,4] = 0 \\
[23,1] = 0 & [23,2] = 0 & [23,3] = \frac{1}{2}e^{x^1}\sin 2x^2 & [23,4] = 0 \\
[24,1] = 0 & [24,2] = 0 & [24,3] = 0 & [24,4] = 0 \\
[33,1] = -\frac{1}{2}e^{x^1} sin^2 x^2 & [33,2] = -\frac{1}{2}e^{x^1}\sin 2x^2 & [33,3] = 0 & [33,4] = 0 \\
[34,1] = 0 & [34,2] = 0 & [34,3] = 0 & [34,4] = 0 \\
[44,1] = \frac{1}{2}\gamma_1 e^{\gamma} & [44,2] = 0 & [44,3] = 0 & [44,4] = -\frac{1}{2}\gamma_4 e^{\gamma}
\end{array} \tag{9}$$

We use 3.114. and considering that $a_{mn} = a^{mn} = 0$ for $m \neq n$:

$$R_{rsmn} = \begin{cases} \frac{1}{2}\left(\partial^2_{sm} a_{rn} + \partial^2_{rn} a_{sm}\right) \\ +\frac{1}{e^{\alpha}}\left([rn,1][sm,1] - [rm,1][sn,1]\right) \\ +\frac{1}{e^{x^1}}\left([rn,2][sm,2] - [rm,2][sn,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([rn,3][sm,3] - [rm,3][sn,3]\right) \\ -\frac{1}{e^{\gamma}}\left([rn,4][sm,4] - [rm,4][sn,4]\right) \end{cases}$$

Giving:

$$R_{2112} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{22} + \partial_{22}^2 a_{11}\right) \\ +\frac{1}{e^\alpha}\left([22,1][11,1] - [21,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][11,2] - [21,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][11,3] - [21,3][12,3]\right) \\ -\frac{1}{e^\gamma}\left([22,4][11,4] - [21,4][12,4]\right) \end{cases} \tag{10}$$

$$\tag{11}$$

$$R_{3113} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{33} + \partial_{33}^2 a_{11}\right) \\ +\frac{1}{e^\alpha}\left([33,1][11,1] - [31,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][11,2] - [31,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][11,3] - [31,3][13,3]\right) \\ -\frac{1}{e^\gamma}\left([33,4][11,4] - [31,4][13,4]\right) \end{cases} \tag{12}$$

$$\tag{13}$$

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{44} + \partial_{44}^2 a_{11}\right) \\ +\frac{1}{e^\alpha}\left([44,1][11,1] - [41,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][11,2] - [41,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][11,3] - [41,3][14,3]\right) \\ -\frac{1}{e^\gamma}\left([44,4][11,4] - [41,4][14,4]\right) \end{cases} \tag{14}$$

$$\tag{15}$$

$$R_{3223} = \begin{cases} \frac{1}{2}\left(\partial_{22}^2 a_{33} + \partial_{33}^2 a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][22,1] - [32,1][23,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][22,2] - [32,2][23,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][22,3] - [32,3][23,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][22,4] - [32,4][23,4]\right) \end{cases} \tag{16}$$

$$\tag{17}$$

$$R_{4224} = \begin{cases} \frac{1}{2}\left(\partial_{22}^2 a_{44} + \partial_{44}^2 a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][22,1] - [42,1][24,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][22,2] - [42,2][24,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][22,3] - [42,3][24,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][22,4] - [42,4][24,4]\right) \end{cases} \tag{18}$$

$$\tag{19}$$

$$R_{4334} = \begin{cases} \frac{1}{2}\left(\partial_{33}^2 a_{44} + \partial_{44}^2 a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][33,1] - [43,1][34,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][33,2] - [43,2][34,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][33,3] - [43,3][34,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][33,4] - [43,4][34,4]\right) \end{cases} \tag{20}$$

$$\tag{21}$$

$$R_{3123} = \begin{cases} \frac{1}{2}\left(\partial^2_{12} a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][12,1]-[32,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][12,2]-[32,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][12,3]-[32,3][13,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][12,4]-[32,4][13,4]\right) \end{cases} \tag{22}$$

(23)

$$R_{4124} = \begin{cases} \frac{1}{2}\left(\partial^2_{12} a_{44}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][12,1]-[42,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][12,2]-[42,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][12,3]-[42,3][14,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][12,4]-[42,4][14,4]\right) \end{cases} \tag{24}$$

(25)

$$R_{2132} = \begin{cases} \frac{1}{2}\left(\partial^2_{13} a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][13,1]-[23,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][13,2]-[23,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][13,3]-[23,3][12,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][13,4]-[23,4][12,4]\right) \end{cases} \tag{26}$$

(27)

$$R_{4134} = \begin{cases} \frac{1}{2}\left(\partial^2_{13} a_{44}\right) \\ +\frac{1}{e^\alpha}\left([44,1][13,1] - [43,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][13,2] - [43,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][13,3] - [43,3][14,3]\right) \\ -\frac{1}{e^\gamma}\left([44,4][13,4] - [43,4][14,4]\right) \end{cases} \tag{28}$$

(29)

$$R_{2142} = \begin{cases} \frac{1}{2}\left(\partial^2_{14} a_{22}\right) \\ +\frac{1}{e^\alpha}\left([22,1][14,1] - [24,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][14,2] - [24,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][14,3] - [24,3][12,3]\right) \\ -\frac{1}{e^\gamma}\left([22,4][14,4] - [24,4][12,4]\right) \end{cases} \tag{30}$$

(31)

$$R_{3143} = \begin{cases} \frac{1}{2}\left(\partial^2_{14} a_{33}\right) \\ +\frac{1}{e^\alpha}\left([33,1][14,1] - [34,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][14,2] - [34,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][14,3] - [34,3][13,3]\right) \\ -\frac{1}{e^\gamma}\left([33,4][14,4] - [34,4][13,4]\right) \end{cases} \tag{32}$$

(33)

$$R_{1231} = \begin{cases} \frac{1}{2}\left(\partial^2_{23} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][23,1]-[13,1][21,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][23,2]-[13,2][21,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][23,3]-[13,3][21,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][23,4]-[13,4][21,4]\right) \end{cases} \tag{34}$$

(35)

$$R_{4234} = \begin{cases} \frac{1}{2}\left(\partial^2_{23} a_{44}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][23,1]-[43,1][24,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][23,2]-[43,2][24,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][23,3]-[43,3][24,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][23,4]-[43,4][24,4]\right) \end{cases} \tag{36}$$

(37)

$$R_{1241} = \begin{cases} \frac{1}{2}\left(\partial^2_{24} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][24,1]-[14,1][21,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][24,2]-[14,2][21,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][24,3]-[14,3][21,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][24,4]-[14,4][21,4]\right) \end{cases} \tag{38}$$

(39)

$$R_{3243} = \begin{cases} \frac{1}{2}\left(\partial^2_{24} a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][24,1]-[34,1][23,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][24,2]-[34,2][23,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][24,3]-[34,3][23,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][24,4]-[34,4][23,4]\right) \end{cases} \tag{40}$$

(41)

$$R_{1341} = \begin{cases} \frac{1}{2}\left(\partial^2_{34} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][34,1]-[14,1][31,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][34,2]-[14,2][31,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][34,3]-[14,3][31,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][34,4]-[14,4][31,4]\right) \end{cases} \tag{42}$$

(43)

$$R_{2342} = \begin{cases} \frac{1}{2}\left(\partial^2_{34} a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][34,1]-[24,1][32,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][34,2]-[24,2][32,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][34,3]-[24,3][32,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][34,4]-[24,4][32,4]\right) \end{cases} \tag{44}$$

(45)

Considering that $[mn,q]=0$ for $m \neq n \neq q \neq m$ and replacing the remaining Christoffels symbols:

$$R_{2112} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{22} + \partial^2_{22} a_{11}\right) \\ -\frac{1}{4}\alpha_1 e^{x^1} \end{cases} \tag{46}$$

(47)

$$R_{3113} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{33} + \partial^2_{33} a_{11}\right) \\ \\ -\frac{1}{4}\left(1+\alpha_1\right) e^{x^1} sin^2 x^2 \end{cases} \tag{48}$$

(49)

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{44} + \partial^2_{44} a_{11}\right) \\ \\ +\frac{1}{4}\left(\alpha_1\gamma_1 e^{\gamma} - \alpha_4^2 e^{\alpha}\right) + \frac{1}{4}\left(\alpha_4\gamma_4 e^{\alpha} - \gamma_1^2 e^{\gamma}\right) \end{cases} \tag{50}$$

(51)

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{44} + \partial^2_{44} a_{11}\right) \\ \\ +\frac{1}{4}\left(\alpha_1\gamma_1 e^{\gamma} - \alpha_4^2 e^{\alpha}\right) - \frac{1}{4}\left(\alpha_4\gamma_4 e^{\alpha} - \gamma_1^2 e^{\gamma}\right) \end{cases} \tag{52}$$

(53)

$$R_{3223} = \begin{cases} \frac{1}{2}\left(\partial^2_{22} a_{33} + \partial^2_{33} a_{22}\right) \\ \\ +\frac{1}{4}\frac{e^{x^1}}{e^{2\alpha}} sin^2 x^2 - e^{x^1}\cos^2 x^2 \end{cases} \tag{54}$$

(55)

$$R_{4224} = \begin{cases} \frac{1}{2}\left(\partial^2_{22} a_{44} + \partial^2_{44} a_{22}\right) \\ \\ -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} \end{cases} \tag{56}$$

(57)

$$R_{4334} = \begin{cases} \frac{1}{2}\left(\partial^2_{33} a_{44} + \partial^2_{44} a_{33}\right) \\ \\ -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} sin^2 x^2 \end{cases} \tag{58}$$

(59)

$$R_{3123} = \frac{1}{2}\left(\partial^2_{12} a_{33}\right) = 0 \tag{60}$$
$$R_{4124} = \frac{1}{2}\left(\partial^2_{12} a_{44}\right) = 0 \tag{61}$$
$$R_{2132} = \frac{1}{2}\left(\partial^2_{13} a_{22}\right) = 0 \tag{62}$$
$$R_{4134} = \frac{1}{2}\left(\partial^2_{13} a_{44}\right) = 0 \tag{63}$$
$$R_{2142} = e^{-\alpha}\left([22,1][14,1]\right) \tag{64}$$
$$R_{3143} = e^{-\alpha}\left([33,1][14,1]\right) \tag{65}$$
$$R_{1231} = \frac{1}{2}\left(\partial^2_{23} a_{11}\right) = 0 \tag{66}$$
$$R_{4234} = \frac{1}{2}\left(\partial^2_{23} a_{44}\right) = 0 \tag{67}$$
$$R_{1241} = \frac{1}{2}\left(\partial^2_{24} a_{11}\right) = 0 \tag{68}$$
$$R_{3243} = \frac{1}{2}\left(\partial^2_{24} a_{33}\right) = 0 \tag{69}$$
$$R_{1341} = \frac{1}{2}\left(\partial^2_{34} a_{11}\right) = 0 \tag{70}$$
$$R_{2342} = \frac{1}{2}\left(\partial^2_{34} a_{22}\right) = 0 \tag{71}$$

Giving:

$$R_{2112} = \frac{1}{4}\left(1-\alpha_1\right) e^{x^1} \tag{72}$$
$$R_{3113} = \frac{1}{4}\left(1-\alpha_1\right) e^{x^1} \sin^2 x^2 \tag{73}$$
$$R_{4114} = \begin{cases} \frac{1}{2}e^{\alpha}\left(\alpha_{44} + \frac{1}{2}\alpha_4^2 - \frac{1}{2}\alpha_4\gamma_4\right) \\ -\frac{1}{2}e^{\gamma}\left(\gamma_{11} + \frac{1}{2}\gamma_1^2 - \frac{1}{2}\alpha_1\gamma_1\right) \end{cases} \tag{74}$$
$$R_{3223} = \left(\frac{1}{4}\frac{e^{x^1}}{e^{\alpha}} - 1\right) e^{x^1} \sin^2 x^2 \tag{75}$$
$$R_{4224} = -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} \tag{76}$$
$$R_{4334} = -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} sin^2 x^2 \tag{77}$$
$$R_{2142} = -\frac{1}{4}\alpha_4 e^{x^1} \tag{78}$$
$$R_{3143} = -\frac{1}{4}\alpha_4 e^{x^1} \sin^2 x^2 \tag{79}$$

As all other curvature components vanish, we get

$$\begin{pmatrix} R_{11} \\ \\ R_{22} \\ \\ R_{33} \\ \\ R_{44} \end{pmatrix} = P \begin{pmatrix} e^{-\alpha} \\ \\ e^{-x^1} \\ \\ \frac{e^{-x^1}}{\sin^2 x^2} \\ \\ -e^{-\gamma} \end{pmatrix} \tag{80}$$

With

$$P = \begin{pmatrix} 0 & \frac{1}{4}\left(1-\alpha_1\right)e^{x^1} & \frac{1}{4}\left(1-\alpha_1\right)e^{x^1}\sin^2 x^2 & \begin{matrix} \frac{1}{2}e^{\alpha}\left(\alpha_{44}+\frac{1}{2}\alpha_4^2-\frac{1}{2}\alpha_4\gamma_4\right) \\ -\frac{1}{2}e^{\gamma}\left(\gamma_{11}+\frac{1}{2}\gamma_1^2-\frac{1}{2}\alpha_1\gamma_1\right) \end{matrix} \\ \\ . & 0 & \left(\frac{1}{4}\frac{e^{x^1}}{e^{\alpha}}-1\right)e^{x^1}\sin^2 x^2 & -\frac{1}{4}\gamma_1\frac{e^{\gamma}e^{x^1}}{e^{\alpha}} \\ \\ . & . & 0 & -\frac{1}{4}\gamma_1\frac{e^{\gamma}e^{x^1}}{e^{\alpha}}sin^2x^2 \\ \\ . & . & . & 0 \end{pmatrix} \tag{81}$$

and

$$\begin{pmatrix} R_{12} \\ R_{13} \\ R_{14} \\ R_{23} \\ R_{24} \\ R_{34} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4}\alpha_4 e^{x^1} & -\frac{1}{4}\alpha_4 e^{x^1}\sin^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-\alpha} \\ \\ e^{-x^1} \\ \\ \frac{e^{-x^1}}{\sin^2 x^2} \\ \\ -e^{-\gamma} \end{pmatrix} \tag{82}$$

Finally we can compute the Einstein tensor:

$$R = a^{11}R_{11} + a^{22}R_{22} + a^{33}R_{33} + a^{44}R_{44} \tag{83}$$

$$= \begin{cases} e^{-\alpha}\left(\gamma_{11}+\frac{1}{2}\gamma_1^2-\frac{1}{2}\alpha_1\gamma_1\right) \\ \\ e^{-\gamma}\left(\alpha_{44}+\frac{1}{2}\alpha_4^2-\frac{1}{2}\alpha_4\gamma_4\right) \\ \\ +e^{-\alpha}\left(\frac{3}{2}+\gamma_1-\alpha_1\right) \\ \\ -2e^{-x^1} \end{cases} \tag{84}$$

$$G^{1}_{.1} = e^{-\alpha} R_{11} - \frac{1}{2} R \tag{85}$$
$$= e^{-\alpha} \left(-\frac{1}{4} - \frac{1}{2}\gamma_1 \right) + e^{-x^1} \tag{86}$$

$$G^{2}_{.2} = e^{-x^1} R_{22} - \frac{1}{2} R \tag{87}$$
$$= \begin{cases} e^{-\alpha} \left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1\gamma_1 + \frac{1}{4}\alpha_1 \right) \\ \\ +e^{-\gamma} \left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4 \right) \end{cases} \tag{88}$$

$$G^{3}_{.3} = \frac{e^{-x^1}}{\sin^2 x^2} R_{33} - \frac{1}{2} R \tag{89}$$
$$= \begin{cases} e^{-\alpha} \left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1\gamma_1 + \frac{1}{4}\alpha_1 \right) \\ \\ +e^{-\gamma} \left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4 \right) \end{cases} \tag{90}$$

$$G^{4}_{.4} = -e^{-\gamma} R_{44} - \frac{1}{2} R \tag{91}$$
$$= e^{-\alpha} \left(-\frac{3}{4} + \frac{1}{2}\alpha_1 \right) + e^{-x^1} \tag{92}$$

$$G^{1}_{.4} = e^{-\alpha} R_{14} \tag{93}$$
$$= -\frac{1}{2} e^{-\alpha} \alpha_4 \tag{94}$$

$$G^{4}_{.1} = -e^{-\gamma} R_{14} \tag{95}$$
$$= \frac{1}{2} e^{-\gamma} \alpha_4 \tag{96}$$

♦