

TENSOR CALCULUS  
J.L. SYNGE AND A.SCHILD (DOVER PUBLICATION)  
SOLUTIONS TO EXERCISES  
PART II  
CHAPTERS V TO VIII

by

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## Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github. An overview of the material covered in the book can be found in the separate document "Synge overview.pdf".

## Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma_{mn}^r \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

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# Applications to Classical Mechanics

## 5.1 p153 - Exercise

If  $\mu^\alpha$  are the contravariant components of a unit vector in a surface  $S$ , show that  $\mu^\alpha f_\alpha$  is the physical component of acceleration in the direction tangent to  $S$  defined by  $\mu^\alpha$ .

As we are in an Euclidean space we can interpret  $a_{mn}\mu^\alpha f^\alpha$  as  $|\mu||f|\cos\theta$  with  $\theta$  the angle between the two vectors. As  $|\mu| = 1$  we have

$$a_{mn}\mu^\alpha f^\alpha = \mu^\alpha f_\alpha \tag{1}$$

$$= |f|\cos\theta \tag{2}$$

which is the projection of the vector  $f$  on the unit vector  $\mu$ .



## 5.2 p154 - Clarification to 5.226.

$$5.226. \quad \mathbf{v} \frac{d\mathbf{v}}{ds} = \mathbf{0}, \quad \bar{\kappa} \mathbf{v}^2 = \mathbf{0}$$

Assuming that the particle is not at rest  $v \neq 0$ , and therefore  $\bar{\kappa} = 0$ . ***Since this implies that the curve is a geodesic...***

The assertion in bold is a direct consequence

$$2.513. \quad \frac{\delta \frac{dx^r}{ds}}{\delta s} = 0$$

As in **5.233** we have  $\frac{\delta \lambda^\alpha}{\delta s} = \frac{\delta \frac{dx^\alpha}{ds}}{\delta s} = 0$ , the considered curve follows the geodesic curve.



### 5.3 p155 - Exercise

Show that in relativity the force 4-vector  $X^r$  lies along the first normal of the trajectory in space-time. Express the first curvature in terms of the proper mass  $m$  of the particle and the magnitude  $X$  of  $X^r$ .

Let us recall the first Frenet formula **2.705** without forgetting that the metric form is not positive-definite,

$$\frac{\delta \lambda^r}{\delta s} = \kappa \nu^r, \quad \epsilon_{(1)} \nu_n \nu^n = 1$$

As **5.299**

$$m \frac{\delta \lambda^r}{\delta s} = X^r$$

it is clear that  $X^r = m \kappa \nu^r$  and is collinear with the first normal.

$$X^r = m \kappa \nu^r \tag{1}$$

$$\times \quad a_{mr} X^m \quad \Rightarrow \quad \underbrace{a_{mr} X^m X^r}_{=(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2} = m \kappa \underbrace{a_{mr} \nu^m \nu^r}_{=\epsilon_{(1)}} \tag{2}$$

$$\Rightarrow \quad \kappa = \epsilon_{(1)} \frac{(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2}{m}$$





## 5.4 p156 - Clarification

Interpretation of

$$\mathbf{5.231.} \quad M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r$$

What do the  $M_{rs}$  represent?

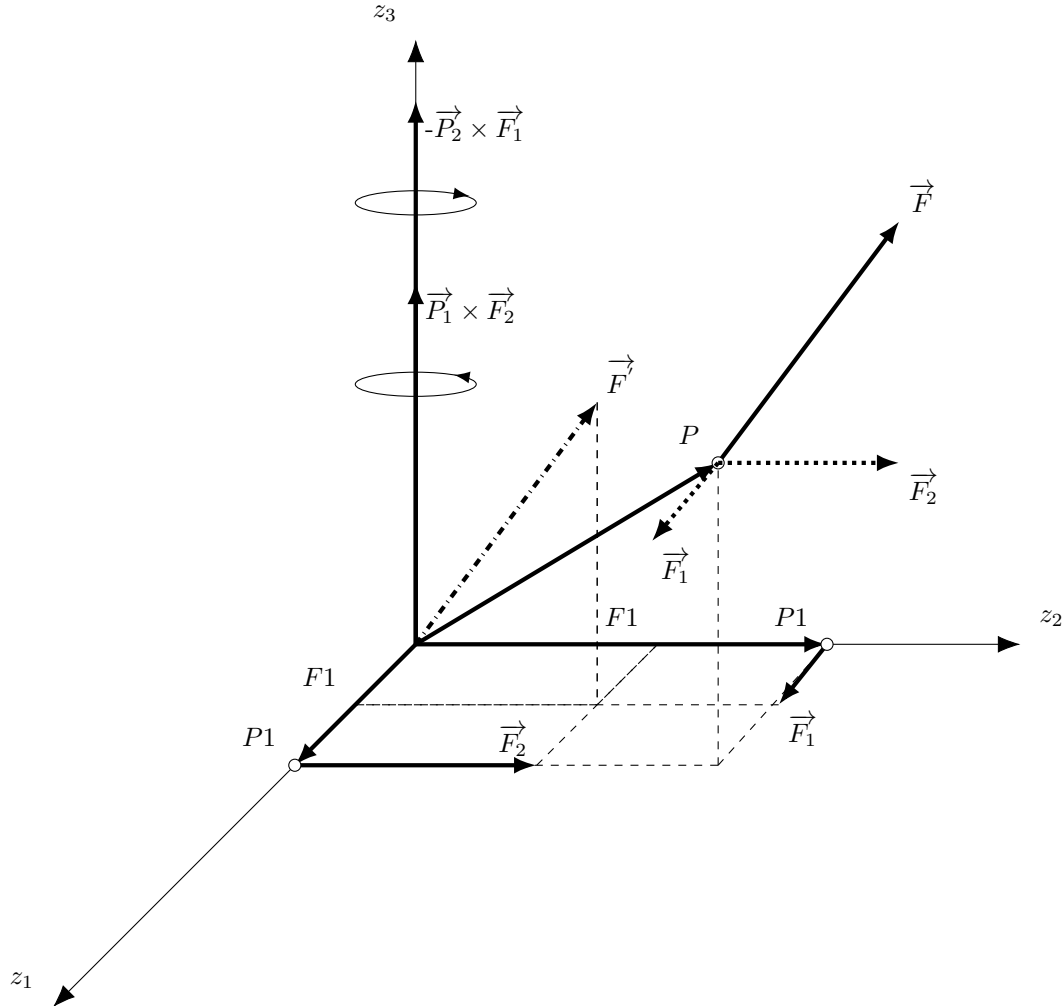


Figure 5.1: Interpretation of the tensor moment  $M_{12}$

Let's consider a mass point  $P$  on which a force  $\vec{F}$  is acting. The force has components  $(F_x, F_y, F_z)$  in the space  $V'_3$  (which is by the way not the space  $V_3$  of the considered mass point).

Let's investigate the element  $M_{12}$  of the *tensor moment*.

$P_1 F_2 \vec{e}_3$  is the vector product  $\vec{P}_1 \times \vec{F}_2$  and is as such the torque of the component  $F_2$  of  $\vec{F}$  acting on the mass point situated at  $P_1$ . The origin being fixed,  $\vec{F}_2$  tries to move  $P_1$ , clockwise along the  $z_3$  axis. The same is true for the component  $\vec{F}_1$  acting on the mass point situated at  $P_2$ , and is represented here by the vector  $-\vec{P}_2 \times \vec{F}_1$  ( $\vec{F}_1$  tries to move  $P_2$ , counter clockwise along the  $z_3$  axis). Hence,  $P_1 F_2 - P_2 F_1$  is the net force trying to move the point  $P$  along the  $z_3$  axis (i.e. in the plane

|| with the  $z_3 = 0$  plane).



## 5.5 p156 - Clarification

$$\mathbf{5.234.} \quad \frac{dh_r}{dt} = M_r$$

$$h_r = m\epsilon_{rmn}z_mv_n \tag{1}$$

$$\Rightarrow \quad \frac{dh_r}{dt} = m\epsilon_{rmn} \frac{dz_m}{dt} v_n + m\epsilon_{rmn} z_m \frac{dv_n}{dt} \tag{2}$$

$$= m \underbrace{\epsilon_{rmn} v_m v_n}_{=0} + \underbrace{\epsilon_{rmn} z_m F_n}_{=M_r} \tag{3}$$

$$= M_r \tag{4}$$



## 5.6 p158-159 - Clarification

$$\mathbf{5.313.} \quad \omega_{rs} = -\omega_{sr}$$

From 5.310 and the vector character of  $v_r$  and  $z_r$  (for transformations which do not change the origin), **it follows that  $\omega_{rs}$  is a Cartesian tensor of second order.**

Be

$$v_r = -\omega_{rn} z_n \quad (1)$$

Considering orthogonal transformation in a flat space  $z'_m = A_{mr} z_r + B_m$  with  $B_m = 0$  as we consider only transformations which do not change the origin. Differentiation with the parameter  $t$  gives

$$v'_m = A_{mr} v_r \quad (2)$$

$$= -\omega_{rn} A_{mr} z_n \quad (3)$$

$$(4)$$

But  $z'_q = A_{qr} z_r \Rightarrow A_{qn} z'_q = A_{qn} A_{qr} z_r \Rightarrow A_{qn} z'_q = z_n$  Hence

$$v'_m = -\omega_{rn} A_{mr} z_n \quad (5)$$

$$= -\underbrace{\omega_{rn} A_{mr} A_{qn}}_{\stackrel{\text{def}}{=} \omega'_{mq}} z'_q \quad (6)$$

$$v'_m = -\omega'_{mq} z'_q \quad (7)$$



## 5.7 p159 - Exercise

Show that if a rigid body rotates about the point  $z_r = b_r$  as fixed point, the velocity of a general point of the body is given by

$$v_r = -\omega_{rm} (z_m - b_m)$$

By 5.302.:

$$\left(z_m^{(1)} - z_m^{(2)}\right) \left(dz_m^{(1)} - dz_m^{(2)}\right) = 0 \quad (1)$$

At the fixed point we have  $z_m^{(2)} = b_m$  and  $dz_m^{(2)} = 0$ , hence

$$\left(z_m^{(1)} - b_m\right) \left(dz_m^{(1)}\right) = 0 \quad (2)$$

$$\Rightarrow z_m^{(1)} dz_m^{(1)} = b_m dz_m^{(1)} \quad (3)$$

As this is true for any point of the rigid mass, expanding (1) and using (3) we get when dividing by  $dt$

$$\left(z_m^{(2)} - b_m\right) v_m^{(1)} + \left(z_m^{(1)} - b_m\right) v_m^{(2)} = 0 \quad (4)$$

Taking twice the partial derivative  $\frac{\partial^2}{\partial z_p^{(1)} \partial z_q^{(1)}}$  we get

$$\left(z_m^{(2)} - b_m\right) \frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \quad (5)$$

As this is true for any arbitrary point in the rigid body we get

$$\frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \quad (6)$$

$$\Rightarrow v_m = K_{mr} z_r + B_m \quad (7)$$

At the fixed point we have

$$K_{mr} b_r + B_m = 0 \quad (8)$$

Plugging this in (7)

$$v_m = K_{mr} (z_r - b_m) \quad (9)$$

Putting  $K_{mr} = -\omega_{mr}$  gives us indeed the asked expression.



## 5.8 p161 - Clarification

$$\mathbf{5.325.} \quad \Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p$$

and hence, since  $\Omega_{np}$  is arbitrary,

$$\mathbf{5.326.} \quad \sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

To be complete the following step should be inserted

$$\Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p \quad (1)$$

$$\text{As } \Omega_{np} \text{ is skew-symmetric:} \quad -\Omega_{np} \sum (m f_p z_n) = -\Omega_{np} \sum F_p z_n \quad (2)$$

$$(1)+(2) \quad \Omega_{np} \sum m (f_n z_p - f_p z_n) = \Omega_{np} \sum (F_n z_p - F_p z_n) \quad (3)$$

and hence, since  $\Omega_{np}$  is arbitrary,

$$\mathbf{5.326.} \quad \sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$



## 5.9 p161 - Clarification

$$\begin{aligned} \mathbf{5.329.} \quad h_{np} &= \sum m (\omega_{nq} z_q z_p - \omega_{pq} z_q z_n) \\ &= J_{npqr} \omega_{rq} \end{aligned}$$

where

$$\mathbf{5.330.} \quad J_{npqr} = \sum m (\delta_{nr} z_q z_p - \delta_{pr} z_n z_q)$$

$$h_{np} = \sum m (\omega_{nq} z_q z_p - \omega_{pq} z_q z_n) \tag{1}$$

$$= \sum m (\omega_{rq} \delta_{rn} z_q z_p - \omega_{rq} \delta_{rp} z_q z_n) \tag{2}$$

$$= \omega_{rq} \sum m (\delta_{rn} z_q z_p - \delta_{rp} z_q z_n) \tag{3}$$

$$= J_{npqr} \omega_{rq} \tag{4}$$



## 5.10 p166 - Exercise

Deduce immediately from **5.420.** that the Coriolis force is perpendicular to the velocity.

$$G'_s = 2m\omega'_{sm}(S', S)v'_m(S') \quad (1)$$

$$\times v'_s(S') \quad : \quad G'_s v'_s(S') = m \left( \omega'_{sm}(S', S)v'_m(S')v'_s(S') + \omega'_{ms}(S', S)v'_m(S')v'_s(S') \right) \quad (2)$$

$$= 0 \quad \text{as } \omega'_{ms} \text{ is skew-symmetric} \quad (3)$$





## 5.11 p166 - Exercise

Show that if  $N = 3$  and  $\dot{\omega}'_r(S', S) = 0$ , then the centrifugal force may be written

$$\mathbf{5.422.} \quad C'_s = m\omega'_n(S', S)\omega'_n(S', S)z'_s - m\omega'_n(S', S)z'_n\omega'_s(S', S)$$

Deduce that  $C'_s$  is coplanar with the vectors  $\omega'_s(S', S)$  and  $z'_n$  and perpendicular to the former.

By **5.420.** with  $\dot{\omega}'_r(S', S) = 0$  and using **5.316.** ( $\omega'_{rs} = \epsilon_{rsn}\omega'_n$ )

$$C'_s = m\omega'_{sm}(S', S)\omega'_{nm}(S', S)z'_n \quad (1)$$

$$= m\epsilon_{smk}\omega'_k(S', S)\epsilon_{nmp}\omega'_p(S', S)z'_n \quad (2)$$

$$= m\epsilon_{msk}\epsilon_{mnp}\omega'_k(S', S)\omega'_p(S', S)z'_n \quad (3)$$

$$= m(\delta_{sn}\delta_{kp} - \delta_{sp}\delta_{kn})\omega'_k(S', S)\omega'_p(S', S)z'_n \quad (4)$$

$$= m\delta_{sn}\delta_{kp}\omega'_k(S', S)\omega'_p(S', S)z'_n - m\delta_{sp}\delta_{kn}\omega'_k(S', S)\omega'_p(S', S)z'_n \quad (5)$$

$$= m\omega'_p(S', S)\omega'_p(S', S)z'_s - m\omega'_n(S', S)\omega'_s(S', S)z'_n \quad (6)$$

To deduce that  $C'_s$  is coplanar with the vectors  $\omega'_s(S', S)$  and  $z'_n$  we calculate the mixed triple product

$$P = \epsilon_{spr}C'_s\omega'_p(S', S)z'_r \quad (7)$$

$$= m \underbrace{\epsilon_{spr}\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_p(S', S)z'_r}_{=0} - \underbrace{m\epsilon_{spr}\omega'_n(S', S)\omega'_s(S', S)z'_n\omega'_p(S', S)z'_r}_{=0} \quad (8)$$

$$= 0 \quad (9)$$

Both terms vanish: the first by the presence of the terms  $\epsilon_{spr}z'_s z'_r$  which cancel each other and for the second by the terms  $\epsilon_{spr}\omega'_s(S', S)\omega'_p(S', S)$ . As  $P = 0$ , the three vectors are coplanar.

We now calculate the inner product  $C'_s\omega'_s(S', S)$

$$P = m\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_s(S', S) - \underbrace{m\omega'_n(S', S)\omega'_s(S', S)z'_n\omega'_s(S', S)}_{\Leftrightarrow m\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_s(S', S)} \quad (10)$$

$$= 0 \quad (11)$$



## 5.12 p168 - Exercise

Taking  $N = 3$ , show that **5.424** may be reduced to the usual Euler equations:

$$I_{11} \frac{d\omega'_1(S', S)}{dt} - (I_{22} - I_{33}) \omega_2(S', S) \omega'_3(S', S) = M'_1$$

and two similar equations.

We first begin with an approach which leads to nothing. I probably made a reasoning error. I give here the whole calculation as this was interesting and also to, later, find my mistake. After this buggy solution, I will give a second version, which works. **5.424**:

$$M'_{ab} = J'_{abrq} \frac{d\omega'_{rq}(S', S)}{dt} + J'_{cdrq} (\delta_{ac} \delta_{du} \delta_{bv} + \delta_{bd} \delta_{cu} \delta_{av}) \omega'_{rq}(S', S) \omega'_{uv}(S', S) = \quad (1)$$

$$\times \epsilon_{sab}: \quad 2M'_s = \epsilon_{sab} J'_{abrq} \frac{d\omega'_{rq}(S', S)}{dt} + \epsilon_{sab} J'_{cdrq} (\delta_{ac} \delta_{du} \delta_{bv} + \delta_{bd} \delta_{cu} \delta_{av}) \omega'_{rq}(S', S) \omega'_{uv}(S', S) \quad (2)$$

Using  $\omega'_{rq}(S', S) = \epsilon_{rqt} \omega'_t(S', S)$  and  $I_{st} = \frac{1}{2} J'_{abrq} \epsilon_{abs} \epsilon_{rqt}$

$$2M'_s = 2I_{st} \frac{d\omega'_t(S', S)}{dt} + \epsilon_{sab} \epsilon_{rqi} \epsilon_{uvj} J'_{cdrq} (\delta_{ac} \delta_{du} \delta_{bv} + \delta_{bd} \delta_{cu} \delta_{av}) \omega'_i(S', S) \omega'_j(S', S) \quad (3)$$

$$= \left\{ \begin{array}{l} 2I_{st} \frac{d\omega'_t(S', S)}{dt} \\ + \left( \epsilon_{sab} \epsilon_{rqi} \epsilon_{uvj} J'_{cdrq} \delta_{ac} \delta_{du} \delta_{bv} + \epsilon_{sab} \epsilon_{rqi} \epsilon_{uvj} J'_{cdrq} \delta_{bd} \delta_{cu} \delta_{av} \right) \omega'_i(S', S) \omega'_j(S', S) \end{array} \right. \quad (4)$$

$$= \left\{ \begin{array}{l} 2I_{st} \frac{d\omega'_t(S', S)}{dt} \\ + \left( \epsilon_{scb} \epsilon_{rqi} \epsilon_{dbj} J'_{cdrq} + \epsilon_{sad} \epsilon_{rqi} \epsilon_{caj} J'_{cdrq} \right) \omega'_i(S', S) \omega'_j(S', S) \end{array} \right. \quad (5)$$

$$= \left\{ \begin{array}{l} 2I_{st} \frac{d\omega'_t(S', S)}{dt} \\ + (\epsilon_{bcs} \epsilon_{bdj}) \epsilon_{rqi} J'_{cdrq} \omega'_i(S', S) \omega'_j(S', S) \end{array} \right. \quad (6)$$

$$= \left\{ \begin{array}{l} 2I_{st} \frac{d\omega'_t(S', S)}{dt} \\ + (\epsilon_{asd} \epsilon_{acj}) \epsilon_{rqi} J'_{cdrq} \omega'_i(S', S) \omega'_j(S', S) \\ + (\delta_{cd} \delta_{sj} - \delta_{cj} \delta_{sd}) \epsilon_{rqi} J'_{cdrq} \omega'_i(S', S) \omega'_j(S', S) \\ + (\delta_{sc} \delta_{dj} - \delta_{sj} \delta_{dc}) \epsilon_{rqi} J'_{cdrq} \omega'_i(S', S) \omega'_j(S', S) \end{array} \right. \quad (7)$$

$$= \begin{cases} 2I_{st} \frac{d\omega'_t(S', S)}{dt} \\ +\epsilon_{rqi} J'_{ccrq} \omega'_i(S', S) \omega'_s(S', S) \\ -\epsilon_{rqi} J'_{jsrq} \omega'_i(S', S) \omega'_j(S', S) \\ +\epsilon_{rqi} J'_{sjrq} \omega'_i(S', S) \omega'_j(S', S) \\ -\epsilon_{rqi} J'_{ccrq} \omega'_i(S', S) \omega'_s(S', S) \end{cases} \quad (8)$$

giving

$$2M'_s = \begin{cases} 2I_{st} \frac{d\omega'_t(S', S)}{dt} \\ +\epsilon_{rqi} J'_{sjrq} \omega'_i(S', S) \omega'_j(S', S) \\ -\epsilon_{rqi} J'_{jsrq} \omega'_i(S', S) \omega'_j(S', S) \end{cases} \quad (9)$$

For  $s = 1$ :

	$+\epsilon_{rqi} J_{1jrq} \omega_i \omega_j$	$-\epsilon_{rqi} J_{j1rq} \omega_i \omega_j$
$\epsilon_{123}$	$+J_{1112} \omega_3 \omega_1 + J_{1212} \omega_3 \omega_2 + J_{1312} \omega_3 \omega_3$	$-J_{1112} \omega_3 \omega_1 - J_{2112} \omega_3 \omega_2 - J_{3112} \omega_3 \omega_3$
$\epsilon_{132}$	$-J_{1113} \omega_2 \omega_1 - J_{1213} \omega_2 \omega_2 - J_{1313} \omega_2 \omega_3$	$+J_{1113} \omega_2 \omega_1 + J_{2113} \omega_2 \omega_2 + J_{3113} \omega_2 \omega_3$
$\epsilon_{213}$	$-J_{1121} \omega_3 \omega_1 - J_{1221} \omega_3 \omega_2 - J_{1321} \omega_3 \omega_3$	$+J_{1121} \omega_3 \omega_1 + J_{2121} \omega_3 \omega_2 + J_{3121} \omega_3 \omega_3$
$\epsilon_{231}$	$+J_{1123} \omega_1 \omega_1 + J_{1223} \omega_1 \omega_2 + J_{1323} \omega_1 \omega_3$	$-J_{1123} \omega_1 \omega_1 - J_{2123} \omega_1 \omega_2 - J_{3123} \omega_1 \omega_3$
$\epsilon_{321}$	$-J_{1132} \omega_1 \omega_1 - J_{1232} \omega_1 \omega_2 - J_{1332} \omega_1 \omega_3$	$+J_{1132} \omega_1 \omega_1 + J_{2132} \omega_1 \omega_2 + J_{3132} \omega_1 \omega_3$
$\epsilon_{312}$	$+J_{1131} \omega_2 \omega_1 + J_{1231} \omega_2 \omega_2 + J_{1331} \omega_2 \omega_3$	$-J_{1131} \omega_2 \omega_1 - J_{2131} \omega_2 \omega_2 - J_{3131} \omega_2 \omega_3$

Taking into account that  $J_{abcd} = 0$  for  $a \neq c \wedge b \neq d$

	$+\epsilon_{rqi} J_{1jrq} \omega_i \omega_j$	$-\epsilon_{rqi} J_{j1rq} \omega_i \omega_j$
$\epsilon_{123}$	$+J_{1112} \omega_3 \omega_1 + J_{1212} \omega_3 \omega_2 + J_{1312} \omega_3 \omega_3$	$-J_{1112} \omega_3 \omega_1$
$\epsilon_{132}$	$-J_{1113} \omega_2 \omega_1 - J_{1213} \omega_2 \omega_2 - J_{1313} \omega_2 \omega_3$	$+J_{1113} \omega_2 \omega_1$
$\epsilon_{213}$	$-J_{1121} \omega_3 \omega_1$	$+J_{1121} \omega_3 \omega_1 + J_{2121} \omega_3 \omega_2 + J_{3121} \omega_3 \omega_3$
$\epsilon_{231}$	$+J_{1323} \omega_1 \omega_3$	$-J_{2123} \omega_1 \omega_2$
$\epsilon_{321}$	$-J_{1232} \omega_1 \omega_2$	$+J_{3132} \omega_1 \omega_3$
$\epsilon_{312}$	$+J_{1131} \omega_2 \omega_1$	$-J_{1131} \omega_2 \omega_1 - J_{2131} \omega_2 \omega_2 - J_{3131} \omega_2 \omega_3$

Opposite sign terms vanish, giving

	$+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$	$-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$
$\epsilon_{123}$	$+J_{1212}\omega_3\omega_2 + J_{1312}\omega_3\omega_3$	
$\epsilon_{132}$	$-J_{1213}\omega_2\omega_2 - J_{1313}\omega_2\omega_3$	
$\epsilon_{213}$		$+J_{2121}\omega_3\omega_2 + J_{3121}\omega_3\omega_3$
$\epsilon_{231}$	$+J_{1323}\omega_1\omega_3$	$-J_{2123}\omega_1\omega_2$
$\epsilon_{321}$	$-J_{1232}\omega_1\omega_2$	$+J_{3132}\omega_1\omega_3$
$\epsilon_{312}$		$-J_{2131}\omega_2\omega_2 - J_{3131}\omega_2\omega_3$

Considering  $J_{abcd} = -J_{badc}$

	$+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$	$-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$
$\epsilon_{123}$	$+\cancel{J_{1212}}\omega_3\omega_2 + \cancel{J_{1312}}\omega_3\omega_3$	
$\epsilon_{132}$	$-\cancel{J_{1213}}\omega_2\omega_2 - \cancel{J_{1313}}\omega_2\omega_3$	
$\epsilon_{213}$		$+\cancel{J_{2121}}\omega_3\omega_2 + \cancel{J_{3121}}\omega_3\omega_3$
$\epsilon_{231}$	$+\cancel{J_{1323}}\omega_1\omega_3$	$-\cancel{J_{2123}}\omega_1\omega_2$
$\epsilon_{321}$	$-\cancel{J_{1232}}\omega_1\omega_2$	$+\cancel{J_{3132}}\omega_1\omega_3$
$\epsilon_{312}$		$-\cancel{J_{2131}}\omega_2\omega_2 - \cancel{J_{3131}}\omega_2\omega_3$

?? We get

$$m'_s = I_{st} \frac{d\omega'_t(S', S)}{dt}$$

?????

◇

Let's try another approach. Start with **5.332**:  $\frac{d}{dt}(I_{st}\omega_t) = M_s$

$$\frac{d}{dt}(I_{st}(S', S)\omega_t(S', S)) = M_s(S', S) \quad (10)$$

Cf. **5.408**.

$$\omega'_u(S', S) = A_{uq}\omega_q(S', S) \quad (11)$$

$$\times A_{ut} \rightarrow A_{ut}\omega'_u(S', S) = A_{ut}A_{uq}\omega_q(S', S) \quad (12)$$

$$= \omega_t(S', S) \quad (13)$$

$$\omega_t(S', S) = A_{ut}\omega'_u(S', S) \quad (14)$$

$$(10) \Rightarrow M_s(S', S) = \frac{d}{dt}(I_{st}(S', S)A_{ut}\omega'_u(S', S)) \quad (15)$$

$$\times A_{ps} \Rightarrow M'_p(S', S) = A_{ps}\frac{d}{dt}(I_{st}(S', S)A_{ut}\omega'_u(S', S)) \quad (16)$$

$$I_{st}(S', S) = A_{as}A_{bt}I'_{ab}(S', S) \quad (17)$$

$$(16) \Rightarrow M'_p(S', S) = A_{ps}\frac{d}{dt}(A_{as}A_{bt}I'_{ab}(S', S)A_{ut}\omega'_u(S', S)) \quad (18)$$

$$= A_{ps}\frac{d}{dt}(A_{as}I'_{ak}(S', S)\omega'_k(S', S)) \quad (19)$$

As we transformed  $I_{st}(S', S)$  to a coordinate system fixed to the body we have that the elements of  $I'_{ab}(S', S)$  are constants.

Hence,

$$M'_p(S', S) = I'_{ak}(S', S)A_{ps}\frac{d}{dt}(A_{as}\omega'_k(S', S)) \quad (20)$$

$$= I'_{ak}(S', S)A_{ps}(\dot{A}_{as}\omega'_k(S', S) + A_{as}\dot{\omega}'_k(S', S)) \quad (21)$$

$$= I'_{ak}(S', S)A_{ps}A_{as}\dot{\omega}'_k(S', S) + I'_{ak}(S', S)A_{ps}\dot{A}_{as}\omega'_k(S', S) \quad (22)$$

$$= I'_{pk}(S', S)\dot{\omega}'_k(S', S) + I'_{ak}(S', S)A_{ps}\dot{A}_{as}\omega'_k(S', S) \quad (23)$$

$$\mathbf{5.408.} \Rightarrow A_{ps}\dot{A}_{as} = \omega'_{ap}(S', S) \quad (24)$$

$$(23) \Rightarrow M'_p(S', S) = I'_{pk}(S', S)\dot{\omega}'_k(S', S) + I'_{ak}(S', S)\omega'_{ap}(S', S)\omega'_k(S', S) \quad (25)$$

Let's now calculate the last expression for  $p = 1$

$$M'_1(S', S) = I'_{1k}(S', S)\dot{\omega}'_k(S', S) + I'_{ak}(S', S)\omega'_{a1}(S', S)\omega'_k(S', S) \quad (26)$$

As we want an arbitrary, fixed to the body of course, coordinate system, it is possible to chose one so that the  $I'_{kj}(S', S) = 0$  for  $k \neq j$  i.e.  $I'_{kj}(S', S)$  is diagonal. This is possible because  $I'_{kj}(S', S)$  is symmetric (the finite-dimensional spectral theorem says that any symmetric matrix whose entries are real can be diagonalized by an orthogonal matrix).

We get, noticing that  $\omega'_{ab}(S', S)$  is skew-symmetric and hence  $\omega'_{11}(S', S) = 0$  :

$$M'_1(S', S) = I'_{11}(S', S)\dot{\omega}'_1(S', S) + I'_{22}(S', S)\omega'_{21}(S', S)\omega'_2(S', S) + I'_{33}(S', S)\omega'_{31}(S', S)\omega'_3(S', S) \quad (27)$$

Using **5.317**:  $\omega'_{21}(S', S) = -\omega'_3(S', S)$  and  $\omega'_{31}(S', S) = \omega'_2(S', S)$  we get the asked expression

$$M'_1(S', S) = I'_{11}(S', S)\dot{\omega}'_1(S', S) - \left(I'_{22}(S', S) - I'_{33}(S', S)\right)\omega'_2(S', S)\omega'_3(S', S) \quad (28)$$



### 5.13 p169 - Exercise

Assign convenient generalized coordinates for the three systems (a), (b), and (c) mentioned at the beginning of this section, and calculate the kinematical metric form in each case

(a) **a particle on a surface** ( $N = 2$ )

No need here for fancy general coordinates: the  $V_2$  coordinate system in the plane is the metric form of choice. Indeed  $|v|^2 = a_{mn}v_m v_n$  and for a  $V_2$

$$ds^2 = \left( a_{11} (v^1)^2 + 2a_{12} v^1 v^2 + a_{22} (v^2)^2 \right) dt^2$$

and if the space is Euclidean and the plane smooth, we can choose an orthogonal system where  $a_{12}$  will vanish.

(b) **a rigid body which can turn about a fixed point, as in the preceding section** ( $N = 3$ )

For a rigid body we can choose a coordinate system  $S'$  fixed to the body to describe the geometry of the rigid body. The kinetic energy referenced to a 'non-moving' (abuse of language) coordinate system  $S$  is

$$T = \frac{1}{2} \sum \rho v'_n(S) v'_n(S) \quad (\text{summation over all masses in the rigid body}) \quad (1)$$

We know by **5.409**:  $v'_n(S) = v'_n(S') + \omega'_{mn}(S', S) z'_m$ . As the  $v'_n(S')$  are fixed, we have  $v'_n(S') = 0$  giving

$$T = \frac{1}{2} \sum \rho z'_m z'_k \omega'_{mn}(S', S) \omega'_{kn}(S', S) \quad (2)$$

Note in (2) that we bring  $\omega'_{mn}(S', S)$  out of the summation as this expression is the same for all masses in the body.

$$\omega_{mn}(S', S) = \epsilon_{mnt} \omega'_t(S', S) \quad (3)$$

$$\Rightarrow T = \frac{1}{2} \sum \rho \epsilon_{mnt} \epsilon_{kns} z'_m z'_k \omega'_t(S', S) \omega'_s(S', S) \quad (4)$$

$$= \frac{1}{2} \sum \rho (\delta_{mk} \delta_{ts} - \delta_{ms} \delta_{kt}) z'_m z'_k \omega'_t(S', S) \omega'_s(S', S) \quad (5)$$

$$= \frac{1}{2} \sum \rho \left( z'_m z'_m \omega'_t(S', S) \omega'_t(S', S) - z'_s z'_t \omega'_t(S', S) \omega'_s(S', S) \right) \quad (6)$$

$$= \frac{1}{2} \sum \rho \left( \delta_{st} z'_m z'_m \omega'_s(S', S) \omega'_t(S', S) - z'_s z'_t \omega'_t(S', S) \omega'_s(S', S) \right) \quad (7)$$

$$= \frac{1}{2} \sum \rho \left( \delta_{st} z'_m z'_m - z'_s z'_t \right) \omega'_s(S', S) \omega'_t(S', S) \quad (8)$$

By **5.335**. we have  $I_{st} = \delta_{st} \sum \rho z_m z_m - \sum \rho z_s z_t$  and so (8) can be written as

$$T = \frac{1}{2} I_{st} \omega'_s(S', S) \omega'_t(S', S) \quad (9)$$

So we can choose the three angles  $\Omega'_s(S', S)$  with  $(\omega'_s(S', S) = \frac{d\Omega'_s(S', S)}{dt})$  as generalized coordinates and define

$$ds^2 = I_{st} d\Omega'_s(S', S) d\Omega'_t(S', S)$$

with

$$a_{mn} = I_{mn}$$

having constants as elements. Some check on consistency of the metric tensor defined by (14):

**Positive definite ?** : Yes, as  $T$  is positive by construction.

**Symmetric ?** : Yes, as  $a_{mn} = I_{km}$  and  $I_{km}$  is symmetric.

(c) **a chain of six rods smoothly hinged together, with one end fixed and all moving on a smooth plane** ( $N = 6$ )

To simplify the notation we will assume that the mass  $m_k$  of each rod (with length  $L_k$ ) is concentrated at it's endpoint .

First we note that the velocity of a rod is composed of two vectors, one (labelled as  $\bar{v}_k$ ) generated by its own rotation relative to the previous rod and the other (labelled as  $\bar{v}_{k-1}$ ) generated by the velocity of the endpoint of the rod to which it is attached (see.fig. 5.2).

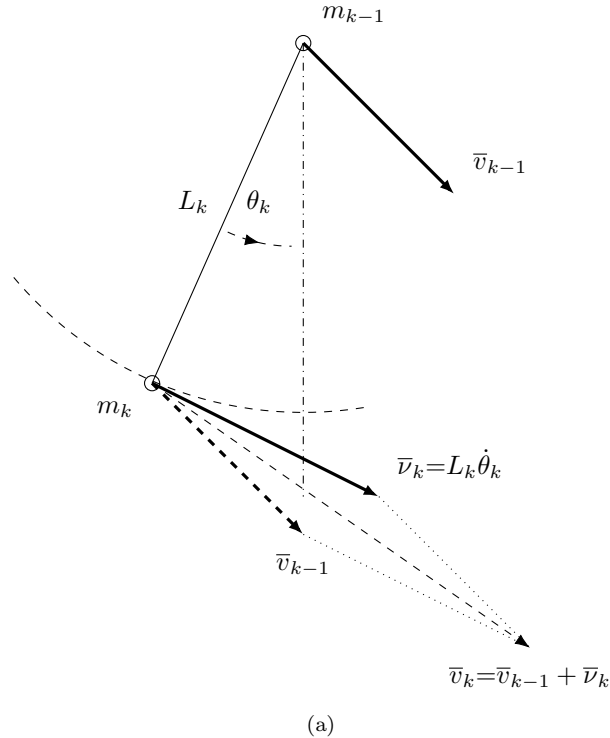


Figure 5.2: Composition of absolute and relative velocities of a chain of rods



If we take Cartesian coordinates it is easy to see that rod (1) will have components

$$\left( L_1 \dot{\theta}_1 \cos \theta, L_1 \dot{\theta}_1 \sin \theta_1 \right)$$

rod (2)

$$\left( L_1 \dot{\theta}_1 \cos \theta_1 + L_2 \dot{\theta}_2 \cos \theta_2, L_1 \dot{\theta}_1 \sin \theta_1 + L_2 \dot{\theta}_2 \sin \theta_2 \right)$$

$\vdots$

rod (k)

$$\left( \sum_{i=1}^k L_i \dot{\theta}_i \cos \theta_i, \sum_{i=1}^k L_i \dot{\theta}_i \sin \theta_i \right)$$

and so

$$\left( v^{(k)} \right)^2 = \left( \sum_{i=1}^k L_i \dot{\theta}_i \cos \theta_i \right)^2 + \left( \sum_{i=1}^k L_i \dot{\theta}_i \sin \theta_i \right)^2 \quad (10)$$

$$= \sum_{i=1}^k \left( L_i \dot{\theta}_i \right)^2 + 2 \sum_{i=1}^k \sum_{j=1}^{k-i} \left( L_i L_{i+j} \dot{\theta}_i \dot{\theta}_{i+j} \cos (\theta_i - \theta_{i+j}) \right) \quad (11)$$

So the kinetic energy of one rod and the total kinetic energy of the system are

$$T^{(k)} = \frac{1}{2} m_k \left[ \sum_{i=1}^k \left( L_i \dot{\theta}_i \right)^2 + 2 \sum_{i=1}^k \sum_{j=1}^{k-i} \left( L_i L_{i+j} \dot{\theta}_i \dot{\theta}_{i+j} \cos (\theta_i - \theta_{i+j}) \right) \right] \quad (12)$$

$$T = \sum_{k=1}^N T^{(k)} \quad (13)$$

For  $N = 6$  we get

rod	$T^{(k)}$
1	$\frac{1}{2} m_1 \left[ \left( L_1 \dot{\theta}_1 \right)^2 \right]$
2	$\frac{1}{2} m_2 \left[ \left( L_1 \dot{\theta}_1 \right)^2 + \left( L_2 \dot{\theta}_2 \right)^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) \right]$
3	$\frac{1}{2} m_3 \left[ \left( L_1 \dot{\theta}_1 \right)^2 + \left( L_2 \dot{\theta}_2 \right)^2 + \left( L_3 \dot{\theta}_3 \right)^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) + 2 L_1 L_3 \dot{\theta}_1 \dot{\theta}_3 \cos (\theta_1 - \theta_3) + \dots \right]$
4	$\frac{1}{2} m_4 \left[ \left( L_1 \dot{\theta}_1 \right)^2 + \left( L_2 \dot{\theta}_2 \right)^2 + \left( L_3 \dot{\theta}_3 \right)^2 + \left( L_4 \dot{\theta}_4 \right)^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) + 2 L_1 L_3 \dot{\theta}_1 \dot{\theta}_3 \cos (\theta_1 - \theta_3) + \dots \right]$
5	$\frac{1}{2} m_5 \left[ \left( L_1 \dot{\theta}_1 \right)^2 + \left( L_2 \dot{\theta}_2 \right)^2 + \left( L_3 \dot{\theta}_3 \right)^2 + \left( L_4 \dot{\theta}_4 \right)^2 + \left( L_5 \dot{\theta}_5 \right)^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) + \dots \right]$
6	$\frac{1}{2} m_6 \left[ \left( L_1 \dot{\theta}_1 \right)^2 + \left( L_2 \dot{\theta}_2 \right)^2 + \left( L_3 \dot{\theta}_3 \right)^2 + \left( L_4 \dot{\theta}_4 \right)^2 + \left( L_5 \dot{\theta}_5 \right)^2 + \left( L_6 \dot{\theta}_6 \right)^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) + \dots \right]$

Giving for  $T$

$$2T = \left\{ \begin{array}{l} (m_1 + m_2 + m_3 + m_4 + m_5 + m_6) \left( L_1 \dot{\theta}_1 \right)^2 \\ + (m_2 + m_3 + m_4 + m_5 + m_6) \left( L_2 \dot{\theta}_2 \right)^2 \\ + (m_3 + m_4 + m_5 + m_6) \left( L_3 \dot{\theta}_3 \right)^2 \\ + (m_4 + m_5 + m_6) \left( L_4 \dot{\theta}_4 \right)^2 \\ + (m_5 + m_6) \left( L_5 \dot{\theta}_5 \right)^2 \\ + (m_6) \left( L_6 \dot{\theta}_6 \right)^2 \\ + 2(m_2 + m_3 + m_4 + m_5 + m_6) L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ + 2(m_3 + m_4 + m_5 + m_6) L_1 L_3 \dot{\theta}_1 \dot{\theta}_3 \cos(\theta_1 - \theta_3) \\ + 2(m_3 + m_4 + m_5 + m_6) L_2 L_3 \dot{\theta}_2 \dot{\theta}_3 \cos(\theta_2 - \theta_3) \\ + 2(m_4 + m_5 + m_6) L_1 L_4 \dot{\theta}_1 \dot{\theta}_4 \cos(\theta_1 - \theta_4) \\ + 2(m_4 + m_5 + m_6) L_2 L_4 \dot{\theta}_2 \dot{\theta}_4 \cos(\theta_2 - \theta_4) \\ + 2(m_4 + m_5 + m_6) L_3 L_4 \dot{\theta}_3 \dot{\theta}_4 \cos(\theta_3 - \theta_4) \\ + 2(m_5 + m_6) L_1 L_5 \dot{\theta}_1 \dot{\theta}_5 \cos(\theta_1 - \theta_5) \\ + 2(m_5 + m_6) L_2 L_5 \dot{\theta}_2 \dot{\theta}_5 \cos(\theta_2 - \theta_5) \\ + 2(m_5 + m_6) L_3 L_5 \dot{\theta}_3 \dot{\theta}_5 \cos(\theta_3 - \theta_5) \\ + 2(m_5 + m_6) L_4 L_5 \dot{\theta}_4 \dot{\theta}_5 \cos(\theta_4 - \theta_5) \\ + 2(m_6) L_1 L_6 \dot{\theta}_1 \dot{\theta}_6 \cos(\theta_1 - \theta_6) \\ + 2(m_6) L_2 L_6 \dot{\theta}_2 \dot{\theta}_6 \cos(\theta_2 - \theta_6) \\ + 2(m_6) L_3 L_6 \dot{\theta}_3 \dot{\theta}_6 \cos(\theta_3 - \theta_6) \\ + 2(m_6) L_4 L_6 \dot{\theta}_4 \dot{\theta}_6 \cos(\theta_4 - \theta_6) \\ + 2(m_6) L_5 L_6 \dot{\theta}_5 \dot{\theta}_6 \cos(\theta_5 - \theta_6) \end{array} \right. \quad (14)$$

We define as general coordinates the angles  $\theta^i$  and express  $ds^2$  as

$$ds^2 = 2T dt^2$$

and see that  $ds^2$  is of the required form

$$ds^2 = a_{mn} d\theta^m d\theta^n$$

The metric tensor  $a_{mn}$  contains elements depending on the  $\theta_k$  chosen as general coordinates of the system and is a good candidate as metric tensor. Some check on consistency of the metric tensor defined by (8):

**Positive definite ?** : Yes, as  $T$  is positive by definition

**Symmetric ?** : Yes, as the non-diagonal term  $a_{ij}$  contains  $\cos(\theta_i - \theta_j) = \cos(\theta_j - \theta_i)$

**Number of elements** : the metric tensor  $a_{mn}$  for  $N = 6$  should contain 6 diagonal elements and  $\frac{6 \times 6 - 6}{2} = 15$  independent non-diagonal elements. Checking (8), one can find that the numbers yield.



## 5.14 p174 - Exercise

Establish the general result

$$v \frac{dv}{ds} = X_r \lambda^r, \quad \kappa v^2 = X_r \nu^r$$

Deduce that, if no forces act on the system, the trajectory is a geodesic in configuration space and the magnitude of the velocity is constant.

In configuration space  $f_r = X_r$ . Hence by **5, 515**

$$X^r = v \frac{dv}{ds} \lambda^r + \kappa v^2 \nu^r \quad (1)$$

$$\Rightarrow \quad X^r \lambda_r = X_r \lambda^r = v \frac{dv}{ds} \quad \text{as } \lambda^r \perp \nu^r \quad (2)$$

$$\text{and} \quad X^r \nu_r = X_r \nu^r = \kappa v^2 \quad \text{as } \lambda^r \perp \nu^r \quad (3)$$

$$(4)$$

The trajectory is a geodesic if  $\kappa = 0$  which is the case as  $X_r = 0$  and

$$v \frac{dv}{ds} = 0 \Rightarrow \frac{dv}{ds} = 0 \Rightarrow v = C^t$$



## 5.15 p174 - Clarification

It is easy to see that the lines of force are the orthogonal trajectories of the equipotential surface  $V = C^t$

Consider a curve given by  $x^r = x^r(u)$ .

Along that line we have  $V = V(x^r(u))$ . Take  $u = s$  as parameter and let's impose that  $V(s) = C^t$ .

We have  $\frac{dV}{ds} = \frac{\partial V}{\partial x^r} \frac{dx^r}{ds} = \frac{\partial V}{\partial x^r} \lambda^r = 0$  with  $\lambda^r = \frac{dx^r}{ds}$  the tangent vector along that curve.

But  $X_r = \frac{\partial V}{\partial x^r}$ .

So,  $X_r \lambda^r = 0$  and as  $X_r$  is collinear with  $dx^r$  (the infinitesimal line element of the line of force) we have  $dx_n \lambda^n = a_{mn} dx^m \lambda^n = 0$  proving the perpendicularity of both curves.



## 5.16 p181 and p182 - Clarification Figures 13., 14. and 15.

There are several ways to perform a map of the configuration space of a rigid body with fixed point.

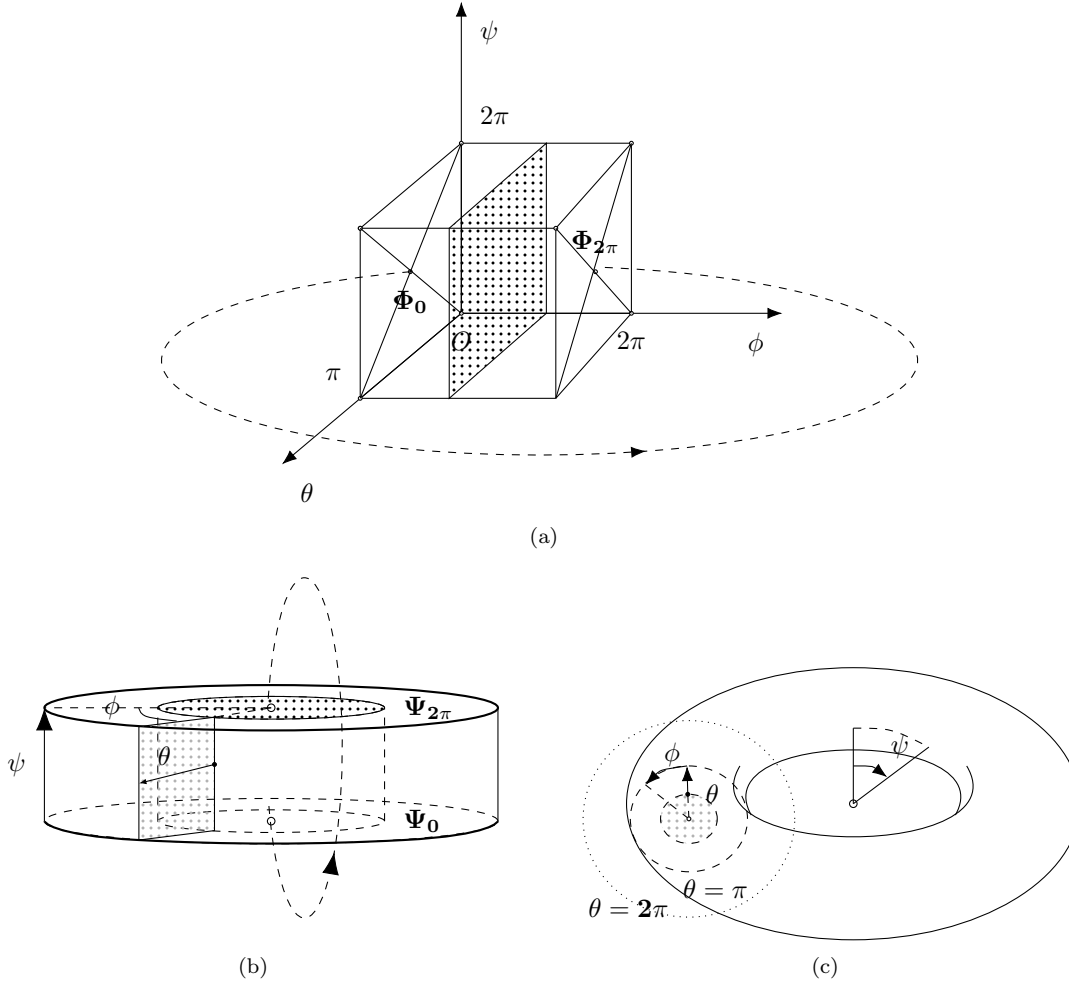


Figure 5.3: Map of the configuration space of a rigid body with fixed point.

Consider figure 5.2(a). We can stretch like an accordion the cuboid along the  $\phi$  axis and bent it so that the planes  $\phi = 0$  and  $\phi = 2\pi$  join. We get (b), a torus with square sections. The dimension  $\phi$  is dealt with as a point  $P(\theta, \phi, \psi)$  in the configuration space returns to the same point when varying  $\phi$  to  $\phi + 2k\pi$ .

We can apply the same procedure of stretching and bending for the  $\psi$  dimension so that the planes  $\Psi = 0$  and  $\Psi = 2\pi$  join. We get (c), a torus-like object.

The only dimension left is  $\theta$  which our multi-dimensional crippled mind can't find a way to reshape this pseudo-torus so that when varying  $\theta$  we can come back to the same point as started.



## 5.17 p183-5.561 - Clarification

The kinetic energy is

$$\mathbf{5.561.} \quad T = \frac{1}{2}I \left( \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos \theta \right)$$

We first determine the general form of the kinetic energy for a rigid body rotating around a fixed point. From **5, 310** we have

$$v_r = -\omega_{rm} z_m = -\epsilon_{rst} \omega_s z_t \quad (1)$$

$$T = \frac{1}{2} \sum m v_r v_r \quad (2)$$

$$\Rightarrow T = \frac{1}{2} \sum m \epsilon_{rst} \omega_s z_t \epsilon_{ruv} \omega_u z_v \quad (3)$$

$$= \frac{1}{2} \sum m (\delta_{su} \delta_{tv} \omega_s \omega_u z_t z_v - \delta_{sv} \delta_{tu} \omega_s \omega_u z_t z_v) \quad (4)$$

For the case  $N = 3$  we get from (4):

$$T = \frac{1}{2} \sum m [\omega_1^2 (z_2^2 + z_3^2) + \omega_2^2 (z_1^2 + z_3^2) + \omega_3^2 (z_1^2 + z_2^2) - 2\omega_1 \omega_2 z_1 z_2 - 2\omega_1 \omega_3 z_1 z_3 - 2\omega_2 \omega_3 z_2 z_3] \quad (5)$$

Using the result from **5.336** this can be written as

$$T = \frac{1}{2} [I_{11} \omega_1^2 + I_{22} \omega_2^2 + I_{33} \omega_3^2 + 2I_{12} \omega_1 \omega_2 + 2I_{13} \omega_1 \omega_3 + 2I_{23} \omega_2 \omega_3] \quad (6)$$

Considering that the matrix  $I_{ij}$  is symmetric, one can always find an appropriate basis so that the matrix becomes diagonal. Hence (6) can be simplified to

$$T = \frac{1}{2} [I_{11} \omega_1^2 + I_{22} \omega_2^2 + I_{33} \omega_3^2] \quad (7)$$

Of course the  $\omega_i$  in (7) are not the Euler angles and we have to express the  $\omega_i$  as functions of the Euler angles.

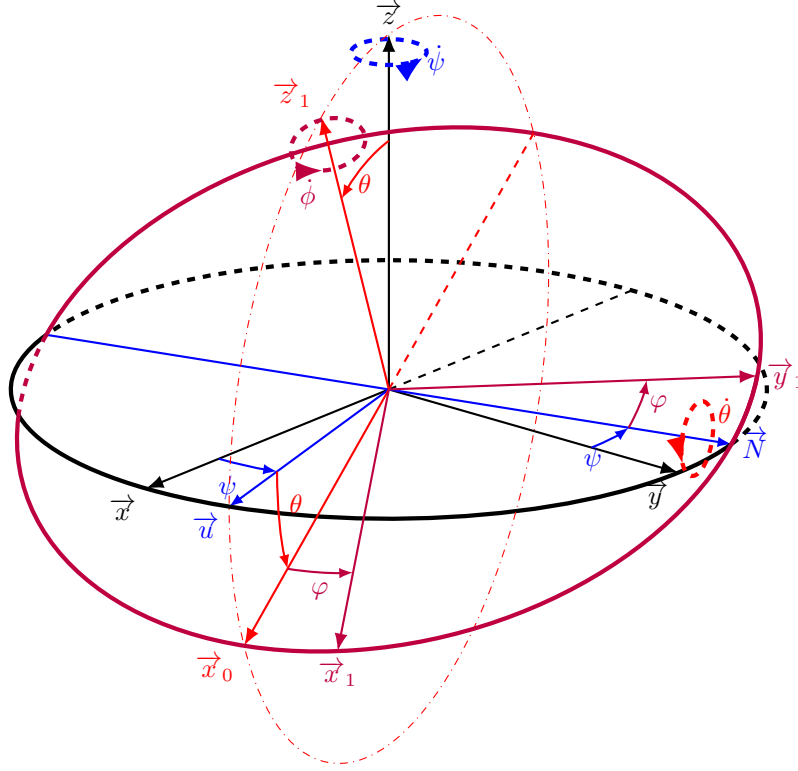


Figure 5.4: Euler angles

Consider the Euler angles as in figure 5.5. The resulting angular velocity of the rigid body can be expressed as

$$\vec{\omega} = \dot{\psi} \vec{z} + \dot{\theta} \vec{N} + \dot{\phi} \vec{z}_1 \quad (8)$$

The projection of  $\vec{\omega}$  on the basis  $\vec{x}_1, \vec{y}_1, \vec{z}_1$  (which we choose fixed to the rigid body) will then coincide with the  $\omega_i$ .

We determine the components of  $\vec{z}, \vec{N}, \vec{z}_1$  with  $\vec{x}_1, \vec{y}_1, \vec{z}_1$  as basis.

We have

$$\begin{cases} \vec{N} = \cos \phi \vec{y}_1 + \sin \phi \vec{x}_1 \\ \vec{z} = \cos \theta \vec{z}_1 - \sin \theta \vec{x}_0 \\ \vec{x}_0 = \cos \phi \vec{x}_1 - \sin \phi \vec{y}_1 \end{cases} \quad (9)$$

$$\Rightarrow \begin{cases} \vec{N} = \cos \phi \vec{y}_1 + \sin \phi \vec{x}_1 \\ \vec{z} = \cos \theta \vec{z}_1 - \sin \theta \cos \phi \vec{x}_1 + \sin \theta \sin \phi \vec{y}_1 \end{cases} \quad (10)$$

Hence,

$$\vec{\omega} = \dot{\psi} \cos \theta \vec{z}_1 - \dot{\psi} \sin \theta \cos \phi \vec{x}_1 + \dot{\psi} \sin \theta \sin \phi \vec{y}_1 + \dot{\theta} \cos \phi \vec{y}_1 + \dot{\theta} \sin \phi \vec{x}_1 + \dot{\phi} \vec{z}_1 \quad (11)$$

giving

$$\begin{cases} \omega_1 = \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi \\ \omega_2 = \dot{\psi} \sin \theta \sin \phi + \dot{\theta} \cos \phi \\ \omega_3 = \dot{\psi} \cos \theta + \dot{\phi} \end{cases} \quad (12)$$

In the case considered  $I_{11} = I_{22} = I_{33} = I$ . Plugging (12) in (7) gives indeed

$$T = \frac{1}{2} I \left( \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos \theta \right)$$





## 5.18 p186 - Exercise 1

If a vector at the point with coordinates  $(1, 1, 1)$  in Euclidean 3-space has components  $(3, -1, 2)$ , find the contravariant, covariant and physical components in spherical polar coordinates.

The tensor  $T_n$  to consider is  $(3, -1, 2) - (1, 1, 1) = (2, -2, 1)$ .

The Jacobian matrix for the transformation  $z^n \rightarrow x^k$ , evaluated at the point  $(1, 1, 1)$  is

$$J_{(1,1,1)} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (2)$$

$$\Rightarrow \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'n} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \quad (4)$$

We have the metric tensor evaluated at  $(1, 1, 1)$

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (5)$$

$$\Rightarrow \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \quad (6)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \quad (7)$$

And the physical components

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \quad (8)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \quad (9)$$

Another way to find the physical components is to project orthogonally the tensor on the unit vectors of a local Cartesian coordinate system, oriented along the unit vectors  $\bar{e}_r, \bar{e}_\theta, \bar{e}_\phi$  corresponding to the vector  $P(1, 1, 1)$  with modulus  $|P| = \sqrt{3}$ . We have for the tensor  $T_n(2, -2, 1)$  with modulus  $|T_n| = 3$  as component along  $\bar{e}_r$ :

$$|T_n| \cos \alpha = |T_n| \frac{\langle T_n, P \rangle}{|T_n| |P|} \quad (10)$$

$$= |T_n| \frac{2 - 2 + 1}{|T_n| |P|} \quad (11)$$

$$= \frac{1}{\sqrt{3}} \quad (12)$$

For the component along  $\bar{e}_\theta$  we first have to determine the vector  $\bar{e}_\theta$ . As first equation we have the

orthogonality condition with  $\bar{e}_r$  and putting  $\bar{e}_\theta = (a, b, c)$ , get  $\langle \bar{e}_r, \bar{e}_\theta \rangle = a + b + c = 0$ . As  $\bar{e}_\theta$  lies in the plane  $(1, 1, 0) - (0, 0, 0) - (0, 0, 1)$  we can put  $a = b$  and get  $\bar{e}_\theta = \frac{1}{\sqrt{6}}(1, 1, -2)$  and get for the tensor  $T_n(2, -2, 1)$  as component along  $\bar{e}_\theta$ :

$$|T_n| \cos \beta = |T_n| \frac{\langle T_n, \bar{e}_\theta \rangle}{|T_n|} \quad (13)$$

$$= |T_n| \frac{2 - 2 - 2}{|T_n| \sqrt{6}} \quad (14)$$

$$= -\frac{\sqrt{2}}{\sqrt{3}} \quad (15)$$

For the component along  $\bar{e}_\phi$  we first have to determine the vector  $\bar{e}_\phi$ . As first equation we have the orthogonality condition with the pair  $\bar{e}_r, \bar{e}_\theta$  and get  $\bar{e}_\phi = \bar{e}_r \times \bar{e}_\theta = \frac{1}{\sqrt{3}\sqrt{6}}(-3, 3, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ . For the tensor  $T_n(2, -2, 1)$  as component along  $\bar{e}_\phi$ :

$$|T_n| \cos \gamma = |T_n| \frac{\langle T_n, \bar{e}_\phi \rangle}{|T_n|} \quad (16)$$

$$= |T_n| \frac{-2 - 2}{|T_n| \sqrt{2}} \quad (17)$$

$$= -\frac{4}{\sqrt{2}} \quad (18)$$

$$= -2\sqrt{2} \quad (19)$$

giving

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} 1 \\ \frac{\sqrt{3}}{\sqrt{2}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \quad (20)$$

as in (9).



## 5.19 p186 - Exercise 2

In cylindrical coordinates  $(r, \phi, z)$  in Euclidean 3-space, a vector field is such that the vector at each point points along the parametric line of  $\phi$ , in the sense of  $\phi$  increasing, and its magnitude is  $kr$ , where  $k$  is a constant. Find the contravariant, covariant and physical components of this vector field.

We can work backwards, with the physical components as starting point. Indeed, at a point  $P(r, \phi, z)$  the tensor of this vector field will have  $(0, kr, 0)$  as physical components in the cylindrical coordinates  $(r, \phi, z)$  system.

We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

Giving

$$\begin{cases} X_1 = h_1 X_1^{phys.} = 0 \\ X_2 = h_2 X_2^{phys.} = kr^2 \\ X_3 = h_3 X_3^{phys.} = 0 \end{cases} \quad (2)$$

and

$$\begin{cases} X^1 = \frac{X_1^{phys.}}{h_1} = 0 \\ X^2 = \frac{X_2^{phys.}}{h_2} = k \\ X^3 = \frac{X_3^{phys.}}{h_3} = 0 \end{cases} \quad (3)$$



## 5.20 p186 - Exercise 3

Find the physical components of velocity and acceleration along the parametric lines of cylindrical coordinates in terms of the and their derivatives with respect to time.

We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

and the contravariant velocities

$$\begin{cases} v^1 = \frac{dr}{dt} \\ v^2 = \frac{d\phi}{dt} \\ v^3 = \frac{dz}{dt} \end{cases} \quad (2)$$

giving by  $v_K^{phys.} = h_K v^K$

$$\begin{cases} v_r = \frac{dr}{dt} \\ v_\phi = r \frac{d\phi}{dt} \\ v_z = \frac{dz}{dt} \end{cases} \quad (3)$$

For the acceleration using  $f^r = \frac{\delta v^r}{\delta t}$  and the Christoffel symbols being

$$\begin{cases} \Gamma_{nk}^m = 0 \quad \forall \quad (nk) \neq (r, \theta), (\theta, \theta) \\ \Gamma_{r\theta}^\theta = \frac{1}{r} \quad \text{and} \quad \Gamma_{\theta\theta}^r = -r \end{cases} \quad (4)$$

we have

$$\left\{ \begin{array}{l} f^1 = \frac{dv^1}{dt} - \underbrace{r v^2 \frac{dx^2}{dt}}_{=(v^2)^2} \\ f^2 = \frac{dv^2}{dt} + \underbrace{\frac{1}{r} v^1 \frac{dx^2}{dt} + \frac{1}{r} v^2 \frac{dx^2}{dt}}_{=\frac{2}{r} v^1 v^2} \\ f^3 = \frac{dv^3}{dt} \end{array} \right. \quad (5)$$

giving by  $f_K^{phys.} = h_K f^K$

$$\left\{ \begin{array}{l} f_r = \frac{dv^1}{dt} - r (v^2)^2 \\ f_{phi} = r \frac{dv^2}{dt} + r \frac{2}{r} v^1 v^2 \\ f_z = \frac{dv^3}{dt} \end{array} \right. \quad (6)$$

$$\Rightarrow \left\{ \begin{array}{l} f_r = \frac{d^2 r}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \\ f_{phi} = r \frac{d^2 \phi}{dt^2} + 2 \frac{dr}{dt} \frac{d\phi}{dt} \\ f_z = \frac{d^2 z}{dt^2} \end{array} \right. \quad (7)$$

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## 5.21 p186 - Exercise 4

A particle moves on a sphere under the action of gravity. Find the contravariant and covariant components of the force, using colatitude and azimuth, and write down the equation of motion.

We determine first the physical components of the force.

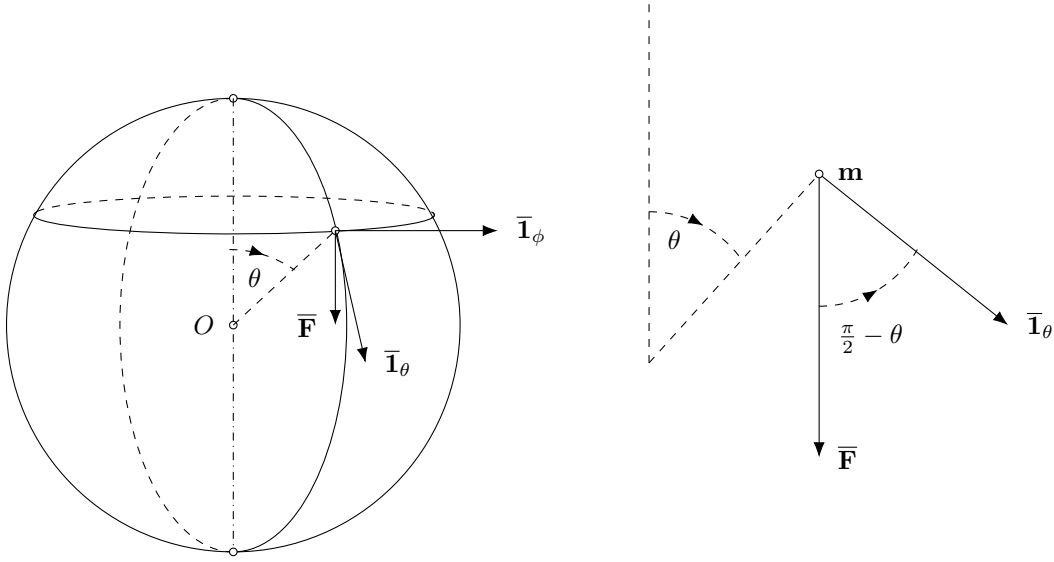


Figure 5.5: Physical components of the gravitational force tensor acting on a mass  $\mathbf{m}$  on a sphere

We note first that the unit vector  $\bar{l}_\phi$  is perpendicular to the plane formed by the vectors  $\bar{l}_\theta, \bar{F}$  and so the force has no components projected on this vector. The vector  $\bar{F}$  is parallel with the axis of reference of the sphere with radius  $R$  and so the physical components become

$$\Rightarrow \begin{cases} F_\phi^{phys} = 0 \\ F_\theta^{phys} = mg \sin \theta \end{cases} \quad (1)$$

$$\Rightarrow \begin{cases} F_\phi = 0 & F_\phi = 0 \\ F^\theta = \frac{1}{R} mg \sin \theta & F_\theta = R mg \sin \theta \end{cases} \quad (2)$$

We use equation 5.212.

$$\begin{cases} \frac{d}{dt} \frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ T = \frac{1}{2} m a_{pq} \dot{x}^p \dot{x}^q, \quad \dot{x}^s = \frac{dx^s}{dt} \end{cases} \quad (3)$$

with for our case

$$T = \frac{1}{2}mR^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) \quad (4)$$

and get the set of equation of motion (the second column gives the dimensional analysis as a check for consistency)

$$\left\{ \begin{array}{l} \frac{\ddot{\phi}}{\dot{\phi}} = -2 \cot \theta \dot{\theta} \quad : \quad \frac{[T]^{-2}}{[T]^{-1}} \cong [T]^{-1} \\ \ddot{\theta} - \left( \dot{\phi} \right)^2 \sin \theta \cos \theta = \frac{g}{R} \sin \theta \quad : \quad [T]^{-2} + ([T]^{-1})^2 \cong \frac{[L][T]^{-2}}{[L]} \end{array} \right. \quad (5)$$

Let's check the special case when  $\dot{\phi} = 0$ .

The first equation can be rewritten and gives of course  $\phi = C$  while the second equation becomes

$$\ddot{\theta} = \frac{g}{R} \sin \theta$$

which is similar to the equation of the simple gravity pendulum.





## 5.22 p186 - Exercise 5

Consider the motion of a particle on a smooth torus under no forces except normal reaction. The geometrical line element may be written

$$ds^2 = (a - b \cos \theta)^2 d\phi^2 + b^2 d\theta^2$$

where  $\phi$  is an azimuthal angle and  $\theta$  an angular displacement from the equatorial plane. Show that the path of a particle satisfies the following two differential equations in which  $h$  is a constant

$$(a) \quad (a - b \cos \theta)^2 \frac{d\phi}{ds} = h$$

$$(b) \quad b^2 \left( \frac{d\theta}{d\phi} \right)^2 = \frac{(a - b \cos \theta)^4}{h^2} - (a - b \cos \theta)^2$$

We use equation **5.212**.

$$\begin{cases} \frac{d}{dt} \frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ T = \frac{1}{2} m a_{pq} \dot{x}^p \dot{x}^q, \quad \dot{x}^s = \frac{dx^s}{dt} \end{cases} \quad (1)$$

with for our case

$$T = \frac{1}{2} m \left( b^2 \dot{\theta}^2 + (a - b \cos \theta)^2 \dot{\phi}^2 \right) \quad (2)$$

giving

$$\begin{cases} \frac{\partial T}{\partial \dot{\phi}} = m (a - b \cos \theta)^2 \dot{\phi} & \frac{\partial T}{\partial \phi} = 0 \\ \frac{\partial T}{\partial \dot{\theta}} = m b^2 \dot{\theta} & \frac{\partial T}{\partial \theta} = m b (a - b \cos \theta) \dot{\phi}^2 \sin \theta \end{cases} \quad (3)$$

$$\Rightarrow \begin{cases} (a - b \cos \theta)^2 \ddot{\phi} + 2b (a - b \cos \theta) \dot{\theta} \dot{\phi} \sin \theta = 0 \\ b^2 \ddot{\theta} - b (a - b \cos \theta) \dot{\phi}^2 \sin \theta = 0 \end{cases} \quad (4)$$

$$\Rightarrow \begin{cases} (a - b \cos \theta) \ddot{\phi} = -2b \dot{\theta} \dot{\phi} \sin \theta \\ b^2 \ddot{\theta} - b (a - b \cos \theta) \dot{\phi}^2 \sin \theta = 0 \end{cases} \quad (5)$$

In the first equation, put  $y \equiv \dot{\phi}$  giving for the first equation:

$$\frac{dy}{y} = -2b \frac{\sin \theta d\theta}{(a - b \cos \theta)} \quad (6)$$

$$\Leftrightarrow \frac{dy}{y} = -2 \frac{d(a - b \cos \theta)}{(a - b \cos \theta)} \quad (7)$$

$$\Rightarrow \log y = -2 \log(a - b \cos \theta) + \log C \quad (8)$$

$$\Rightarrow \dot{\phi} = C (a - b \cos \theta)^{-2} \quad (9)$$

Note that  $\dot{\phi}$  is a time derivative. But as we are on a geodesic, **5.226.** stands and so  $v$  is constant as  $\frac{dv}{ds} = 0$ . Using  $v = \frac{ds}{dt}$ , (9) can be written as

$$(a - b \cos \theta)^2 \frac{d\phi}{dt} = C \quad (10)$$

$$\Leftrightarrow (a - b \cos \theta)^2 \frac{d\phi}{ds} \underbrace{\frac{ds}{dt}}_{=v} = C \quad (11)$$

$$\Leftrightarrow (a - b \cos \theta)^2 \frac{d\phi}{ds} = h \quad \text{with } h = \frac{C}{v} \quad (12)$$

We don't use the second equation in (5) but the line element equation instead

$$ds^2 = (a - b \cos \theta)^2 d\phi^2 + b^2 d\theta^2 \quad (13)$$

$$\Rightarrow \left( \frac{ds}{d\phi} \right)^2 = (a - b \cos \theta)^2 + b^2 \left( \frac{d\theta}{d\phi} \right)^2 \quad (14)$$

$$\Rightarrow b^2 \left( \frac{d\theta}{d\phi} \right)^2 = \left( \frac{d\phi}{ds} \right)^{-2} - (a - b \cos \theta)^2 \quad (15)$$

$$(12) \quad : \quad b^2 \left( \frac{d\theta}{d\phi} \right)^2 = \frac{(a - b \cos \theta)^4}{h^2} - (a - b \cos \theta)^2 \quad (16)$$

