Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

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January 26, 2022

Remarks and warnings

Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

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Spaces and Tensors

1.1 p5-exercise

The parametric equations of a hypersurface in V_n are

$$
\begin{aligned}
x^1 &= a\cos(u^1) \\
x^2 &= a\sin(u^1)\cos(u^2) \\
x^3 &= a\sin(u^1)\sin(u^2)\cos(u^3) \\
&\vdots \\
x^{N-1} &= a\sin(u^1)\sin(u^2)\sin(u^3)\ldots\sin(u^{N-2})\cos(u^{N-1}) \\
x^N &= a\sin(u^1)\sin(u^2)\sin(u^3)\ldots\sin(u^{N-2})\sin(u^{N-1})
\end{aligned}
$$

where a is a constant. Find the single equation of the hyperspace in the form 1.103.

We have:

$$
\begin{aligned}
(x^N)^2 + (x^{N-1})^2 &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i)(\cos^2(u^{N-1}) + \sin^2(u^{N-1})) \\
&= a^2 \prod_{i=1}^{N-2} \sin^2(u^i) \\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i)\sin^2(u^{N-2}) \\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i)(1 - \cos^2(u^{N-2}) \\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - a^2 \prod_{i=1}^{N-3} \sin^2(u^i)\cos^2(u^{N-2}) \\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - (x^{N-2})^2
\end{aligned}
$$

giving

$$
(x^N)^2 + (x^{N-1})^2 + (x^{N-2})^2 = a^2 \prod_{i=1}^{N-3} \sin^2(u^i)
$$

In general, by recursion

$$
\sum_{i=0}^{k} (x^{N-i})^2 = a^2 \prod_{i=1}^{N-k-1} \sin^2(u^i) \quad (k \leq N-2)
$$

be k = N - 2 (N - k - 1 = 1) and in the left term put j = N - i (j goes from 2 to N), we get

$$\begin{aligned}\sum_{j=2}^{N}(x^j)^2 &= a^2\prod_{i=1}^{1}\sin^2(u^i)\\ &= a^2(1-\cos^2(u^1))\\ &= a^2-(x^1)^2\end{aligned}$$

and thus the equation of the hyperspace is given by

$$\sum_{j=1}^{N}(x^j)^2 - a^2 = 0$$

Determine whether the points $(\frac{1}{2}a, 0, 0, ...0)$, $(0, 0, ..., 0, 2a)$ lie on the same or opposite sides of the hyperspace.

For $(\frac{1}{2}a, 0, 0, ...0)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = -\frac{3a^2}{4} < 0$ and for $(0, 0, ..., 0, 2a)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = \frac{3a^2}{4} > 0$.
So the points lie on opposite sides of the hyperplane.

♦

1.2 p6-exercise

Let U_2 and W_2 be subspaces of V_N. Show that if N = 3 they will in general intersect in a curve; if N = 4 they will in general intersect in a finite number of points; and if $N > 4$ they will not in general intersect at all.

We have (see 1.102 page 5): $x^r = f^r(u^1, u^2, ..., u^M) \quad (r = 1, 2, ..., N)$
Case N=3:

For U_2 we have:

$$x^r = \phi^r(u^1, u^2) \quad (r = 1, 2, 3)$$

For W_2 we have:

$$x^r = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

The intersect of the two hyperplanes is given by the N equations:

$$\phi^r(u^1, u^2) = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

So we have 3 equations in 4 unknown u^1, u^2, v^1, v^2 and can choose (fix) one e.g. u^1 and solve the set of equations for u^2, v^1, v^2 giving

$$x^r = \theta^r(u^1) \quad (r = 1, 2, 3)$$

This is an equation of a curve in space (1 parameter equation)
Case N=4:
Using the same reasoning as with N=3, we get 4 equations for 4 unknown u^1, u^2, v^1, v^2.
Provided that the set of equation does not degenerate, these 4 equations will determine u^1, u^2, v^1, v^2 without any degree of freedom. So we get points as solutions. This solution does not to be unique, e.g. if the $\phi^r(u^1, u^2)$ are quadratic form, then the solutions

$$(u^1, u^2, v^1, v^2)$$

$$(-u^1, u^2, v^1, v^2)$$

$$(u^1, -u^2, v^1, v^2)$$

$$(-u^1, -u^2, v^1, v^2)$$

are possible.
Case N=5: There are more equations than variables. If the equations are not linear dependent, no solutions will be found.

♦

1.3 p8-exercise

Show that $(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = 3a_{rst}x^r x^s x^t$

$(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = a_{rst}x^r x^s x^t + a_{rts}x^r x^s x^t + a_{srt}x^r x^s x^t$ so by just renaming the dummy indices e.g. for the second term $r \mapsto s$, $s \mapsto t$ and $t \mapsto r$ we get the desired result.

♦

1.4 p8-exercise

If $\phi = a_{rs}x^r x^s$, show that

$$\frac{\partial \phi}{\partial x^r} = (a_{rs} + a_{sr})x^s \quad , \quad \frac{\partial^2 \phi}{\partial x^r \partial x^s} = a_{rs} + a_{sr}$$

Simplify these expressions in the case where $a^{rs} = a^{sr}$

We have

$$\begin{aligned}
\frac{\partial \phi}{\partial x^t} &= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs}\frac{\partial x^r}{\partial x^t} x^s + a_{rs}x^r \frac{\partial x^s}{\partial x^t} && (1)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs}\delta^r_t x^s + a_{rs}x^r \delta^s_t && (2)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts}x^s + a_{rt}x^r && (3)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts}x^s + a_{st}x^s \quad \text{(rename dummy variable in third term)} && (4)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + (a_{ts} + a_{st})x^s && (5)
\end{aligned}$$

Replace x^t by x^r, we get

$$\frac{\partial \phi}{\partial x^r} = \frac{\partial a_{rs}}{\partial x^r} x^r x^s + (a_{rs} + a_{sr})x^s \tag{6}$$

So the asked expression is only true if a_{rs} is not a function of the x^s. Assuming that a_{rs} is not a function of the x^s, take the partial derivative of (6) with respect to x^t, we get

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x^r \partial x^t} &= (a_{rs} + a_{sr})\frac{\partial x^s}{\partial x^t} && (7)\\
&= (a_{rs} + a_{sr})\delta^s_t && (8)\\
&= (a_{rt} + a_{tr}) && (9)
\end{aligned}$$

Replace x^t by x^s, and we get the proposed expression.

♦

1.5 p8-clarification on expression 1.210

$$\frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} = 0$$

From 1.209:

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s} + \frac{\partial x^r}{\partial x'^n}\frac{\partial^2 x'^n}{\partial x^p \partial x^s} = 0 \tag{1}$$

multiply (1) with $\frac{\partial x'^q}{\partial x^r}$

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} + \frac{\partial x^r}{\partial x'^n}\frac{\partial^2 x'^n}{\partial x^p \partial x^s}\frac{\partial x'^q}{\partial x^r} = 0 \tag{2}$$

$$\Leftrightarrow \frac{\partial x^r}{\partial x'^n}\frac{\partial^2 x'^n}{\partial x^p \partial x^s}\frac{\partial x'^q}{\partial x^r} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} = 0 \tag{3}$$

in the first term we get $$\frac{\partial x'^q}{\partial x^r}\frac{\partial x^r}{\partial x'^n} = \frac{\partial x'^q}{\partial x'^n} = \delta^q_n \tag{4}$$

(3) becomes

$$\frac{\partial^2 x'^n}{\partial x^p \partial x^s}\delta^q_n + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} = 0 \tag{5}$$

$$\Leftrightarrow \frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} = 0 \tag{6}$$

♦

1.6 p9-exercise

If A^r_s are the elements of a determinant A, and B^r_s the elements of a determinant B, show that the element of the product determinant is $A^r_n B^n_s$. Hence show that the product of the two jacobians

$$J = \left| \frac{\partial x^r}{\partial x^{,s}} \right|, \quad J^{'} = \left| \frac{\partial x^{,r}}{\partial x^s} \right|$$

is unity.

Remark: Some nitpick about the formulation: A^r_s are not the elements of a determinant A, but elements of the matrix A which gives $\det\{A\}$ provided that A is square (which is not explicitly mentioned.). The same remark for B and $A^r_n B^n_s$.
Be A^i_k the elements of matrix A and B^k_j the elements of matrix B and C = A.B the resulting matrix of the multiplication of A and B, then

$$C^i_j = A^i_k B^k_j$$

are the elements of matrix C. Now, put $A^i_k = \frac{\partial x^i}{\partial x^{,k}}$ and $B^k_j = \frac{\partial x^{,k}}{\partial x^j}$ then,

$$C^i_j = A^i_k B^k_j \tag{1}$$
$$= \frac{\partial x^i}{\partial x^{,k}} \frac{\partial x^{,k}}{\partial x^j} \tag{2}$$
$$= \delta^i_k \tag{3}$$

So $C = JJ^{,}$ becomes the unity matrix.

♦

1.7 p11-exercise

Show that a finite contravariant vector determines the ratios of the components of an infinitesimal displacement. (Consider the transformation of the equation $dx^r = \theta T^r$, where θ is an arbitrary factor which does not change under the transformation. Alternatively, show that the equations $T^r dx^s - T^s x^r = 0$ remain true when we transform the coordinates.)

Be T^q a contravariant vector.

$$T'^q = T^r \frac{\partial x'^q}{\partial x^r} \quad \text{(by definition)} \tag{1}$$

Be θ a small infinitesimal factor invariant for a coordinate transformation, define

$$dx^r = \theta T^r \tag{2}$$

$$\tag{3}$$

then

$$\frac{dx^r}{dx^s} = \frac{\theta T^r}{\theta T^s} \tag{4}$$

$$\Leftrightarrow T^s dx^r - T^r dx^s = 0 \tag{5}$$

Alternatively, multiply (5) with $\partial_{x^r} x'^q$, then

$$\frac{\partial x'^q}{\partial x^r} dx^r T^s - \frac{\partial x'^q}{\partial x^r} dx^s T^r = 0 \tag{6}$$

$$\Leftrightarrow \frac{\partial x'^q}{\partial x^r} dx^r T^s - dx^s T'^q = 0 \quad \text{(use (1) in the second term)} \tag{7}$$

$$\Leftrightarrow dx'^q T^s - dx^s T'^q = 0 \tag{8}$$

$$\tag{9}$$

Multiply (8) with $\partial_{x^s} x'^p$, then

$$dx'^q T^s \partial_{x^s} x'^p - dx^s T'^q \partial_{x^s} x'^p = 0 \tag{10}$$

$$\Leftrightarrow \quad T'^p dx^q - T'^q dx'^p = 0 \quad \text{(use (1) in the first term)} \tag{11}$$

and thus

$$\frac{dx'^q}{dx'^p} = \frac{T'^q}{T'^p}$$

♦

1.8 p12-exercise

Write down the equation of transformation, analogous to 1.305, of a contravariant tensor of the third order. Solve the equation so as to express the unprimed components in terms of the primed components.

Be

$$T^{,uvw} = T^{rst}\frac{\partial x^{,u}}{\partial x^r}\frac{\partial x^{,v}}{\partial x^s}\frac{\partial x^{,w}}{\partial x^t} \quad \text{(by definition)} \tag{1}$$

a contravariant vector.
Multiply (1) by $\frac{\partial x^n}{\partial x^{,u}}$

$$T^{,uvw}\frac{\partial x^n}{\partial x^{,u}} = T^{rst}\frac{\partial x^{,u}}{\partial x^r}\frac{\partial x^n}{\partial x^{,u}}\frac{\partial x^{,v}}{\partial x^s}\frac{\partial x^{,w}}{\partial x^t} \tag{2}$$

$$\Leftrightarrow T^{,uvw}\frac{\partial x^n}{\partial x^{,u}} = T^{rst}\delta^n_r\frac{\partial x^{,v}}{\partial x^s}\frac{\partial x^{,w}}{\partial x^t} \tag{3}$$

$$\Leftrightarrow T^{,uvw}\frac{\partial x^n}{\partial x^{,u}} = T^{nst}\frac{\partial x^{,v}}{\partial x^s}\frac{\partial x^{,w}}{\partial x^t} \tag{4}$$

Multiply (4) by $\frac{\partial x^m}{\partial x^{,v}}$

$$T^{,uvw}\frac{\partial x^n}{\partial x^{,u}}\frac{\partial x^m}{\partial x^{,v}} = T^{nst}\frac{\partial x^{,v}}{\partial x^s}\frac{\partial x^m}{\partial x^{,v}}\frac{\partial x^{,w}}{\partial x^t} \tag{5}$$

$$\Leftrightarrow \; T^{,uvw}\frac{\partial x^n}{\partial x^{,u}}\frac{\partial x^m}{\partial x^{,v}} = T^{nst}\delta^m_s\frac{\partial x^{,w}}{\partial x^t} \tag{6}$$

$$\Leftrightarrow \; T^{,uvw}\frac{\partial x^n}{\partial x^{,u}}\frac{\partial x^m}{\partial x^{,v}} = T^{nmt}\frac{\partial x^{,w}}{\partial x^t} \tag{7}$$

Multiply (7) by $\frac{\partial x^p}{\partial x^{,w}}$

$$T^{,uvw}\frac{\partial x^n}{\partial x^{,u}}\frac{\partial x^m}{\partial x^{,v}}\frac{\partial x^p}{\partial x^{,w}} = T^{nmt}\frac{\partial x^{,w}}{\partial x^t}\frac{\partial x^p}{\partial x^{,w}} \tag{8}$$

$$\Leftrightarrow \; T^{,uvw}\frac{\partial x^n}{\partial x^{,u}}\frac{\partial x^m}{\partial x^{,v}}\frac{\partial x^p}{\partial x^{,w}} = T^{nmt}\delta^p_t \tag{9}$$

$$\Leftrightarrow \; T^{,uvw}\frac{\partial x^n}{\partial x^{,u}}\frac{\partial x^m}{\partial x^{,v}}\frac{\partial x^p}{\partial x^{,w}} = T^{nmp} \tag{10}$$

Giving

$$T^{nmp} = T^{,uvw}\frac{\partial x^n}{\partial x^{,u}}\frac{\partial x^m}{\partial x^{,v}}\frac{\partial x^p}{\partial x^{,w}}$$

♦

1.9 p14-exercise

For a transformation from on set of rectangular Cartesian coordinates to another in Euclidean 3-space, show that the law of transformation of a contravariant vector is precisely the same as that of a covariant vector. Can this statements be extended to cover tensor of higher orders?

We have to prove that, given that,

$$T^{,i} = T^j \frac{\partial x^{,i}}{\partial x^j} \quad T^{,}_i = T_j \frac{\partial x^j}{\partial x^{,i}}$$

that also

$$T^{,i} = T^j \frac{\partial x^j}{\partial x^{,i}} \quad T^{,}_i = T_j \frac{\partial x^{,i}}{\partial x^j} \tag{1}$$

$$\Leftrightarrow \frac{\partial x^j}{\partial x^{,i}} = \frac{\partial x^{,i}}{\partial x^j} \tag{2}$$

Be

$$\hat{e^{,i}} = g^i_k \hat{e^k} \quad \text{and } \hat{e^i} = h^i_k \hat{e^{,k}} \tag{3}$$

the transformation rules from one set of (rectangular Cartesian) basis vectors to another set of (rectangular Cartesian) basis vectors. Then,

$$\langle \hat{e^{,i}}, \hat{e^{,j}} \rangle = \langle g^i_k \hat{e^k}, g^j_k \hat{e^k} \rangle \text{ and } \langle \hat{e^i}, \hat{e^j} \rangle = \langle h^i_k \hat{e^{,k}}, h^j_k \hat{e^{,k}} \rangle \tag{4}$$

$$\Leftrightarrow \delta^p_j = g^p_k g^j_k \text{ and } \delta^p_j = h^p_k h^j_k \tag{5}$$

$$\tag{6}$$

Be $\vec{v}$ a random vector in the Euclidean space,

$$\vec{v} = x^j \hat{e^j} = x^{,j} \hat{e^{,j}} \tag{7}$$

then

$$(3) \;\Rightarrow x^j \hat{e^j} = x^j h^j_k \hat{e^{,k}} \text{ and } x^{,j} \hat{e^{,j}} = x^{,j} g^j_k \hat{e^k} \tag{8}$$

$$\Rightarrow x^{,j} = x^m h^m_j \text{ and } x^m = x^{,j} g^j_m \tag{9}$$

$$\Rightarrow x^{,j} = x^{,i} g^i_m h^m_j \text{ and } x^m = x^k h^k_j g^j_m \tag{10}$$

$$\Rightarrow \delta^p_j = g^p_k h^k_j \text{ and } \delta^p_j = g^k_j h^p_k \tag{11}$$

$$(5) \;\Rightarrow g^p_k g^j_k = g^p_k h^k_j \text{ and } h^p_k h^j_k = g^k_j h^p_k \tag{12}$$

$$\Rightarrow g^j_k = h^k_j \text{ and } h^j_k = g^k_j \tag{13}$$

From (9)

$$x^j = x'^m g_j^m \text{ and } x'^k = x^n h_k^n \tag{14}$$

$$\Rightarrow \frac{\partial x'^k}{\partial x^j} = \frac{\partial x^n}{\partial x^j} h_k^n \text{ and } \frac{\partial x^j}{\partial x'^k} = \frac{\partial x'^m}{\partial x'^k} g_j^m \tag{15}$$

$$\Leftrightarrow \frac{\partial x'^k}{\partial x^j} = \delta_j^n h_k^n \text{ and } \frac{\partial x^j}{\partial x'^k} = \delta_k^m g_j^m \tag{16}$$

$$\Leftrightarrow \frac{\partial x'^k}{\partial x^j} = h_k^j \text{ and } \frac{\partial x^j}{\partial x'^k} = g_j^k \tag{17}$$

$$(13) \Rightarrow \frac{\partial x'^k}{\partial x^j} = \frac{\partial x^j}{\partial x'^k} \tag{18}$$

So (13) matches (2), proving the assertion.

Can this statements be extended to cover tensor of higher orders? Consider

$$T'^{i,j,\ldots,n} = T^{r,s,\ldots w} \frac{\partial x'^i}{\partial x^r} \frac{\partial x'^j}{\partial x^s} \cdots \frac{\partial x'^n}{\partial x^w} \text{ and } T^{r,s,\ldots w} = T'^{i,j,\ldots,n} \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} \cdots \frac{\partial x^w}{\partial x'^n}$$

Using the same reasoning as in (1) to (2) we need

$$\frac{\partial x'^i}{\partial x^r} \frac{\partial x'^j}{\partial x^s} \cdots \frac{\partial x'^n}{\partial x^w} = \frac{\partial x^r}{\partial x'^i} \frac{\partial x^s}{\partial x'^j} \cdots \frac{\partial x^w}{\partial x'^n}$$

As the conclusion (18) is independent of the order of the tensor, it is obvious that the above equality yields. Hence, the answer is YES.

♦

1.10 p16-exercise

In a space of 4 dimensions, the tensor A_{rst} is skew-symmetric in the last pair of suffixes. Show that only 24 of the 64 components may be chosen arbitrarily. If the further condition $A_{rst} + A_{str} + A_{trs} = 0$ is imposed, show that that only 20 components may be chosen arbitrarily.

We have, as A is skew-symmetric in the last pair of suffixes

$$A_{rst} = -A_{rts} \Rightarrow s = t\colon\ A_{rst} = 0$$

So, for each r (4 possible choices as N = 4) we have 4x4/2 - 4 = 6 degrees of freedom. [we have the term 4x4/2 as the tensor is (skew-)symmetric, e.g. once we choose element a_{12}, then a_{21} is also known. The term -4 takes into account the diagonal element which are 0 and thus cannot be chosen.] So, we have 4x6 = 24 degrees of freedom.
What about the supplementary constraint $A_{rst} + A_{str} + A_{trs} = 0$:
Consider the two possible excluding cases:

i) $r = s \neq t$ ($\Leftrightarrow r = t \neq s$)
This case gives - without the additional constraint (1) - 4x(4x3/2-4) = 8 degrees of freedom. Does the constraint (1) reduces this degree of freedom?
We have,

$$A_{rst} + A_{str} + A_{trs} = 0 \tag{1}$$

$$\Rightarrow \underbrace{A_{rrt} + A_{rtr}}_{=\,0\text{ (non-diagonal terms)}} + \underbrace{A_{trr}}_{=\,0\text{ (diagonal terms)}} = 0 \tag{2}$$

So, no additional constraints are added by (1) to the restriction i) and the DOF remains 8.

ii) $t \neq r \neq s \neq t$
This case means that we have to choose a set of 3 elements out of 4 elements without repetition. This a *variation* of 3 elements out of 4.

$$V_k^n = \frac{n!}{(n-k)!} \text{ giving } V_3^4 = \frac{4!}{(4-3)!} = 24$$

The constraint (1) gives us 24 equations but as $A_{rst} = -A_{rts}$ only 12 equations have to be considered. So, with the additional constraints the DOF becomes 24-12 = 12.
As i) and ii) are independent and excluding events we can add the DOF of both events and we get 8+12 = 20 DOF.

♦

1.11 p16-exercise

If A^{rs} is skew-symmetric and B_{rs} is symmetric, prove that $A^{rs}B_{rs} = 0$. Hence show that the quadratic form $a_{ij}x^ix^j$ is unchanged if a_{ij} is replaced by its symmetric part.

We can split the summation $A^{rs}B_{rs}$ in three subsummations:

$$A^{rs}B_{rs} = A^{rs}B_{rs}|_{r=s} \tag{1}$$
$$+ A^{rs}B_{rs}|_{r>s} \tag{2}$$
$$+ A^{rs}B_{rs}|_{r<s} \tag{3}$$

We have:
(1) $= 0$ as $A^{kk} = 0$ (skew-symmetric)
(2)+(3) $= A^{rs}B_{rs}|_{r>s} \ + A^{rs}B_{rs}|_{r<s}$
As $A^{rs} = -A^{sr}$ and $B^{rs} = B^{sr}$ we can write (2)+(3) as :

$$A^{rs}B_{rs}|_{r>s} \ + (-A^{sr})B_{sr}|_{r>s} = 0$$

So, $A^{rs}B_{rs} = 0$

Consider the quadratic form $\phi = a_{ij}x^ix^j$
Be $A_{ij} = (a_{ij})$ and $B_{ij} = (x^ix^j)$, then it is obvious that B_{ij} is symmetric and that $C_{ij} = -A_{ij}$ is the form where $-a_{ij}$ is replaced by its symmetric part (skew-symmetric). Hence $\phi = a_{ij}x^ix^j = a_{ij}b^{ij} = 0$ and so is $\phi = c_{ij}b^{ij} = 0$

♦

1.12 p18-exercise

What are the values (in a space of N dimensions) of the folllowing contractions formed from the Kronecker delta?

$$\delta^m_m, \delta^m_n \delta^n_m, \delta^m_n \delta^n_r \delta^r_m$$

$$\delta^m_m = N \tag{1}$$

$$\delta^m_n \delta^n_m = \delta^m_m = N \tag{2}$$

$$\delta^m_n \delta^n_r \delta^r_m = \delta^m_n \delta^n_m = \delta^m_m = N \tag{3}$$

♦

1.13 p19-exercise

If X^r, Y^r are arbitrary contravariant vectors and $a_{rs}X^rY^s$ is an invariant, then a_{rs} are the components of a covariant tensor of the second order.

We have to prove that

$$a'_{rs} = a_{ij}\frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^s} \text{ or } a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{1}$$

$a_{rs}X^rY^s$ is an invariant, means

$$a'_{rs}X'^rY'^s = a_{rs}X^rY^s \tag{2}$$

As X^r, Y^r are arbitrary contravariant vectors, we have

$$X'^r = X^i\frac{\partial x'^r}{\partial x^i} \text{ and } Y'^s = Y^j\frac{\partial x'^s}{\partial x^j} \tag{3}$$

(3) in (2) gives

$$a'_{rs}X^i\frac{\partial x'^r}{\partial x^i}Y^j\frac{\partial x'^s}{\partial x^j} = a_{rs}X^rY^s \tag{4}$$

$$\Leftrightarrow a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j}X^iY^j = a_{ij}X^iY^j \tag{5}$$

$$\Leftrightarrow (a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij})X^iY^j = 0 \tag{6}$$

As X^r, Y^r are arbitrary contravariant vectors, we conclude that

$$a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij} = 0 \tag{7}$$

$$\Leftrightarrow a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{8}$$

(8) = (1): OK

♦

1.14 p19-exercise

If X_{rs} is an arbitrary covariant tensor of the second order, and $A_r^{mn} X_{mn}$ is a covariant vector, then A_r^{mn} has the mixed tensor character indicated by the positions of its suffixes

We have to prove that

$$A_r^{,vw} = A_k^{mn} \frac{\partial x^k}{\partial x^{,r}} \frac{\partial x^{,v}}{\partial x^m} \frac{\partial x^{,w}}{\partial x^n} \tag{1}$$

We have

$$P_r = A_r^{mn} X_{mn} \tag{2}$$

is a covariant vector

$$\Rightarrow P_r^{,} = A_k^{mn} X_{mn} \frac{\partial x^k}{\partial x^{,r}} \tag{3}$$

but X_{mn} is a covariant tensor

$$\Rightarrow X_{mn} = X_{ps}^{,} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \tag{4}$$

So (4) in (3) gives

$$P_r^{,} = A_k^{mn} X_{ps}^{,} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}} \tag{5}$$

$$\Leftrightarrow P_r^{,} = \underbrace{A_k^{mn} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}}_{(*)} X_{ps}^{,} \tag{6}$$

Putting (*) as $A_r^{,ps} = A_k^{mn} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}$ we see that (6) has the form (2) and that $A_r^{,ps}$ obeys the rule of a mixed tensor (1).

♦

1.15 p21-exercise

If A_{rs} is a skew-symmetric covariant tensor, prove that B_{rst} defined as

$$B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$$

is a covariant tensor, and that it is skew-symmetric in all pairs of suffixes.

We have A_{rs} is a covariant tensor

$$A_{ij} = A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^i}\frac{\partial x^\beta}{\partial x^j} \tag{1}$$

$$\Rightarrow B_{rst} = \partial_r(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}) + \partial_s(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r}) + \partial_t(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}) \tag{2}$$

Note that

$$\partial_k(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}) = \partial_k(A_{\alpha\beta})\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \ A_{\alpha\beta}\partial_k(\frac{\partial x^\alpha}{\partial x^s})\frac{\partial x^\beta}{\partial x^t} + \ A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\partial_k(\frac{\partial x^\beta}{\partial x^t}) \tag{3}$$

$$\tag{4}$$

so,

$$\begin{aligned} B_{rst} &= \partial_r A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^s}\tfrac{\partial x^\beta}{\partial x^t} + \underbrace{A_{\alpha\beta}\partial_r\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}}_{*} + \underbrace{A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\partial_r\frac{\partial x^\beta}{\partial x^t}}_{**} \\ &+\partial_s A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^t}\tfrac{\partial x^\beta}{\partial x^r} + \underbrace{A_{\alpha\beta}\partial_s\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r}}_{***} + \underbrace{A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\partial_s\frac{\partial x^\beta}{\partial x^r}}_{*} \\ &+\partial_t A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^r}\tfrac{\partial x^\beta}{\partial x^s} + A_{\alpha\beta}\underbrace{\partial_t\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}}_{**} + \underbrace{A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\partial_t\frac{\partial x^\beta}{\partial x^s}}_{***} \end{aligned} \tag{5}$$

In (5) consider the two terms with (*)

$$T = A_{\alpha\beta}\partial_r\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\partial_s\frac{\partial x^\beta}{\partial x^r} \tag{6}$$

$$= A_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x^s\partial x^r}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial^2 x^\beta}{\partial x^r\partial x^s} \tag{7}$$

$$= A_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x^s\partial x^r}\frac{\partial x^\beta}{\partial x^t} + A_{\beta\alpha}\frac{\partial x^\beta}{\partial x^t}\frac{\partial^2 x^\alpha}{\partial x^r\partial x^s} \text{ (by renaming dummy variables)} \tag{8}$$

As $A_{ij} = -A_{ji}$ (skew-symmetric tensor), we get $T = 0$. The same yields for the (**) and (***) terms. So, B_{rst} reduces to

$$B_{rst} = \partial_r A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \partial_s A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r} + \partial_t A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s} \tag{9}$$

$$\Leftrightarrow \ B_{rst} = \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^r}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^s}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^t}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s} \tag{10}$$

By adequate renaming of the dummy variable in the 3 terms:

$$\begin{bmatrix} 1^{st} term \\ 2^{nd} term \\ 3^{rd} term \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma \to \alpha & \alpha \to \beta & \beta \to \gamma \\ \beta \to \alpha & \gamma \to \beta & \alpha \to \gamma \\ \alpha \to \alpha & \beta \to \beta & \gamma \to \gamma \end{bmatrix}$$

we get

$$B_{rst} = (\frac{\partial A_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial A_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma})\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}\frac{\partial x^\gamma}{\partial x^t} \tag{11}$$

$$\Leftrightarrow B_{rst} = (\underbrace{\partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta}}_{(****)})\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}\frac{\partial x^\gamma}{\partial x^t} \tag{12}$$

The expression (****) has exactly the required form $B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$ and is transformed (12) according the rules of a covariant tensor.

Let's prove now that it is skew-symmetric in all pairs of suffixes. We have to consider the following permutations

$$\begin{bmatrix} rst \\ rts \\ srt \\ str \\ trs \\ tsr \end{bmatrix}$$

E.g. *srt*

$$B_{rts} = \partial_r A_{ts} + \partial_t A_{sr} + \partial_s A_{rt} \tag{13}$$

$$= -\partial_r A_{st} - \partial_t A_{rs} - \partial_s A_{tr} \tag{14}$$

$$= -B_{rst} \tag{15}$$

The same calculations can be done for the other permutations.

♦

1.16 p23-exercise 1.

In a V_4 there are two 2-spaces with equations

$$x^r = f^r(u^1, u^2),\ x^r = g^r(u^3, u^4)$$

Prove that if these 2-spaces have a curve of intersection, then the determinal equation

$$\left|\frac{\partial x^r}{\partial u^s}\right| = 0$$

is satisfied along the curve.

Having a curve means that one of the parameters u^i can be freely chosen while the other 3 are determined by the chosen parameter.
We have,

$$\left|\frac{\partial x^r}{\partial u^s}\right| = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} & \frac{\partial x^1}{\partial u^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^4}{\partial u^1} & \frac{\partial x^4}{\partial u^2} & \frac{\partial x^4}{\partial u^3} & \frac{\partial x^4}{\partial u^4} \end{vmatrix} \tag{1}$$

Suppose we choose u^4 as parameter. This means $u^i = \phi^i(u^4)$ for i=1,2,3 and thus we can write

$$\frac{\partial x^i}{\partial u^4} = \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} + \frac{\partial x^i}{\partial u^4} \quad \text{with j=1,2,3} \quad \text{i = 1,2,3,4} \tag{2}$$

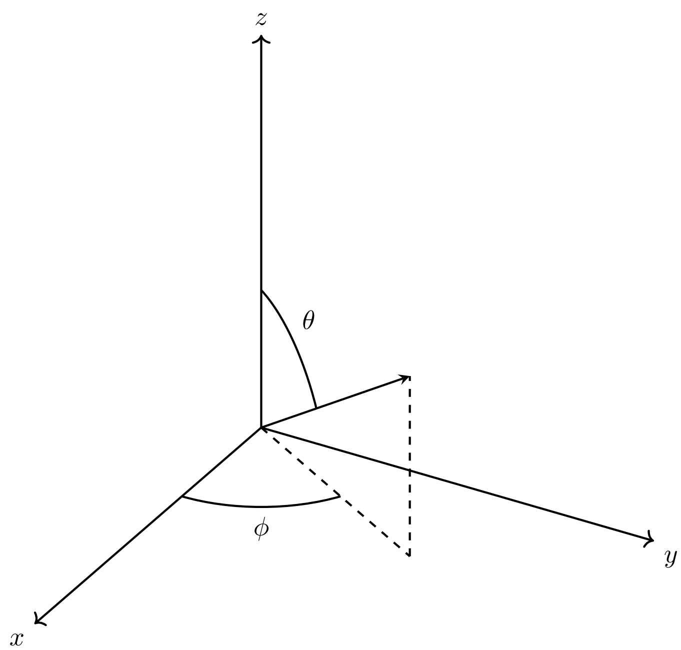
$$\Rightarrow \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} = 0 \tag{3}$$

This means that in (1) the three first columns a not linearly independent and thus have $\left|\frac{\partial x^r}{\partial u^s}\right| = 0$

♦

1.17 p23-exercise 2.

In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates x, y, z and spherical polar coordinates r,θ,ϕ.
Find the Jacobian of the transformation. Where is it zero or infinite?



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

Figure 1.1: Spherical coordinate system

Partial differentiating of (x,y,z) with respect to (r,θ,ϕ) gives the Jacobian

$$\begin{aligned} J &= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \qquad (1) \\ &= r^2\sin\theta(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi) \qquad (2) \\ &= r^2\sin\theta \qquad (3) \end{aligned}$$

J=0: for $r = 0$ or $\theta = 0$ $_{r\epsilon(-\infty,+\infty)}$ and $J \to \pm\infty$ or $\mp\infty$ for $r \to \pm\infty|_{\theta\neq 0}$. But what about the case $r \to \pm\infty|_{\theta\to 0}$? This case is not determined as long as no path is chosen in the (r, θ) configuration space.

♦

1.18 p23-exercise 3.

If X, Y, Z are the components of a contravariant vector for rectangular Cartesian coordinates in Euclidean 3-space, find it's components for spherical polar coordinates.

Be x^α the components of a contravariant vector in spherical polar coordinates and x^i it's components in rectangular Cartesian coordinates. As we have

$$\begin{aligned} x^\rho &= \sqrt{x^j x^j} \\ x^\theta &= \operatorname{atan} \tfrac{x^2}{x^1} \\ x^\phi &= \operatorname{asin} \tfrac{x^3}{\sqrt{x^j x^j}} \end{aligned} \qquad \text{and} \qquad A^\alpha = A^i \frac{\partial x^\alpha}{\partial x^i} \tag{1}$$

$$\Rightarrow [A^\alpha] = \left[A^i \frac{\partial x^\alpha}{\partial x^i} \right] = \begin{bmatrix} \frac{x^1}{\sqrt{x^j x^j}} & \frac{x^2}{\sqrt{x^j x^j}} & \frac{x^3}{\sqrt{x^j x^j}} \\ -\frac{x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} & 0 \\ -\frac{x^3 x^1}{(x^j x^j)\sqrt{(x^1)^2+(x^2)^2}} & -\frac{x^3 x^2}{(x^j x^j)\sqrt{(x^1)^2+(x^2)^2}} & \frac{\sqrt{(x^1)^2+(x^2)^2}}{(x^j x^j)} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} \tag{2}$$

♦

1.19 p23-exercise 4.

In a space of three dimensions, how many different expressions are represented by the product $A^m_{np}B^{pq}_{rs}C^s_{tu}$? How many terms occur in each such expression, when written out explicitly?

As we have V_3 and considering that in $A^m_{np}B^{pq}_{rs}C^s_{tu}$ the six indices m, n, q, r, t, u are not dummy indices, we get 3^6 different expressions (first choose m: you have three choices, then n: also three choices giving 3x3 possibilities, etc for q, r, t, u).
For the second question, as in $A^m_{np}B^{pq}_{rs}$ there is only summation on over index (p) we get three terms for this part. As the summation with $A^m_{np}B^{pq}_{rs}$ and C^s_{tu} occurs only on one index also (s) we get 3x3 terms in the expression.

♦

1.20 p23-exercise 5.

If A is an invariant in V_n, are the second derivatives $\frac{\partial^2 A}{\partial x^r \partial x^s}$ the components of a tensor?

As A is invariant (note: different alphabets in the indices indicates different coordinate systems):

$$A(x^\rho) = A(x^i) \tag{1}$$

$$\Rightarrow \frac{\partial A(x^\rho)}{\partial x^i} = \frac{\partial A(x^j)}{\partial x^i} \tag{2}$$

To simplify the notation, we put $A(x^\rho) = A^{,}$ and $A(x^j) = A^{,}$ then (2) can be written as

$$\frac{\partial A^{,}}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^i} = \frac{\partial A}{\partial x^i} \tag{3}$$

Conclusion: $\frac{\partial A}{\partial x^i}$ is a covariant tensor.

Consider now $\frac{\partial A}{\partial x^i} = \frac{\partial A^{,}}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^i}$. Then,

$$\frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A^{,}}{\partial x^\rho \partial x^j}\frac{\partial x^\rho}{\partial x^i} + \frac{\partial A^{,}}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \tag{4}$$

$$\Leftrightarrow \frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A^{,}}{\partial x^\rho \partial x^\gamma}\frac{\partial x^\gamma}{\partial x^j}\frac{\partial x^\rho}{\partial x^i} + \frac{\partial A^{,}}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \tag{5}$$

The first term on the right side, behaves as covariant tensor but the presence of the second term makes that generally, $\frac{\partial^2 A}{\partial x^i \partial x^j}$ has not a tensor character. This is only when $\frac{\partial A^{,}}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} = 0$, which means that x^ρ, x^i are a linear map of each other.

♦

1.21 p23-exercise 6.

Suppose that in V_2 the components of a contravariant tensor field T^{mn} in a coordinate system x^r are

$$T^{11} = 1 \quad T^{12} = 0$$

$$T^{21} = 1 \quad T^{22} = 0$$

Find the components $T^{,mn}$ in a coordinate system $x^{,r}$, where

$$x^{,1} = (x^1)^2 \quad x^{,2} = (x^2)^2$$

Write down the values of these components in particular at the point $x^1 = 1, x^2 0 =$.

As we have a contravariant tensor field :

$$T^{,mn} = T^{ij} \frac{\partial x^{,m}}{\partial x^i} \frac{\partial x^{,n}}{\partial x^j} \tag{1}$$

$$\begin{array}{ll} x^{,1} = (x^1)^2 \Rightarrow & \frac{\partial x^{,1}}{\partial x^1} = 2x^1 \qquad \frac{\partial x^{,1}}{\partial x^2} = 0 \\ x^{,2} = (x^2)^2 \Rightarrow & \frac{\partial x^{,2}}{\partial x^1} = 0 \qquad \frac{\partial x^{,2}}{\partial x^2} = 2x^2 \end{array} \tag{2}$$

$$\tag{3}$$

$$\Rightarrow T^{,11} = 4(x^1)^2 + 4(x^2)^2 \tag{4}$$

$$\Rightarrow T^{,12} = T^{,21} = 0 \tag{5}$$

$$\Rightarrow T^{,22} = 4(x^1)^2 + 4(x^2)^2 \tag{6}$$

The components in at the point $x^1 = 1, x^2 = 0$ are

$$T^{,}(1,0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

♦

1.22 p24-exercise 7.

Given that if T_{mnrs} is a covariant tensor, and

$$T_{mnrs} + T_{mnsr} = 0$$

in a coordinate system x^p, establish directly that

$$T'_{mnrs} + T'_{mnsr} = 0$$

in any other coordinate system x,q.

Note: in the following, different alphabets in the indices indicates different coordinate systems. As we T_{mnrs} is a covariant tensor :

$$T_{\alpha\beta\gamma\delta} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} \tag{1}$$

$$\Rightarrow T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} + T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\delta}\frac{\partial x^s}{\partial x^\gamma} \tag{2}$$

Now, swap the dummy indices r and s in the second term on the right and as $T_{mnrs} = -T_{mnsr}$:

$$T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} + T_{mnsr}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^s}{\partial x^\delta}\frac{\partial x^r}{\partial x^\gamma} \tag{3}$$

$$= (T_{mnrs} + T_{mnsr})\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^s}{\partial x^\delta}\frac{\partial x^r}{\partial x^\gamma} \tag{4}$$

$$= 0 \tag{5}$$

♦

1.23 p24-exercise 8.

Prove that if A_r is a covariant vector, then $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ is a skew-symmetric covariant tensor of the second order (use the notation of 1.7).

Be $B_{rs} = \frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$.

i) B_{rs} is skew-symmetric: It is obvious that:

$$-B_{rs} = -\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r} = \frac{\partial A_s}{\partial x^r} - \frac{\partial A_r}{\partial x^s} \equiv B_{sr}$$

ii) B_{rs} is covariant:

Note: in the following, different alphabets in the indices indicates different coordinate systems.

Let

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta. \tag{1}$$

We know that $A_i = A_\gamma X^\gamma_i$ as A_i is covariant. Hence,

$$\partial_j A_i = \partial_j A_\gamma X^\gamma_i + A_\gamma \partial_j X^\gamma_i \tag{2}$$

$$= \partial_\alpha A_\gamma X^\alpha_j X^\gamma_i + A_\gamma \partial_j X^\gamma_i \tag{3}$$

Using (3), we compute the first term in (1)

$$\partial_s A_r X^r_\alpha X^s_\beta = \partial_\rho A_\gamma X^\rho_s X^\gamma_r X^r_\alpha X^s_\beta + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{4}$$

$$= \partial_\rho A_\gamma X^\rho_\beta X^\gamma_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{5}$$

$$= \partial_\rho A_\gamma \delta^\rho_\beta \delta^\gamma_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{6}$$

$$= \partial_\beta A_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{7}$$

In the same way, we get for the second term in (1)

$$\partial_r A_s X^s_\alpha X^r_\beta = \partial_\alpha A_\beta + A_\gamma \partial_r X^\gamma_s X^r_\alpha X^s_\beta \tag{8}$$

And thus,

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta = \partial_\beta A_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta - \partial_\alpha A_\beta - A_\gamma \partial_r X^\gamma_s X^r_\alpha X^s_\beta \tag{9}$$

$$\Rightarrow \partial_\beta A_\alpha - \partial_\alpha A_\beta = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta \tag{10}$$

So, i) and (10) proves that $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ is skew-symmetric tensor of the second order.

♦

1.24 p24-exercise 9.

Let $x^r, \overline{x}^r, y^r, \overline{y}^r$ be four systems of coordinates. Examine the tensor character of $\frac{\partial x^r}{\partial y^s}$ with respect to the following transformations:
i) A transformation $x^r = f^r(\overline{x}^1, \ldots, \overline{x}^N)$, with y^r unchanged;
ii) A transformation $y^r = g^r(\overline{y}^1, \ldots, \overline{y}^N)$, with x^r unchanged;

Note: in the following, different alphabets in the indices indicates different coordinate systems.

i) Let's compute the expression $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta}$. Obviously, the right side is an expression of a (possible) mixed tensor of the second order ($\frac{\partial x^r}{\partial y^s}$) under transformation from the (r) coordinate system to the (α) coordinate system. Then,

$$\begin{aligned} A(\alpha, \beta) &= \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta} && (1) \\ &= \frac{\partial x^\alpha}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} && (2) \\ &= \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} && (3) \end{aligned}$$

If we consider the $\overline{y}^r$ coordinate system as the y^ρ coordinate system and as $\overline{y}^r = y^r$ then $\frac{\partial y^\rho}{\partial y^s} = \delta^\rho_s$ and we get from (3)

$$\begin{aligned} A(\alpha, \beta) &= \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} && (4) \\ &= \frac{\partial x^\alpha}{\partial y^\rho}\delta^\rho_s\frac{\partial x^s}{\partial x^\beta} && (5) \\ &= \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial x^\rho}{\partial x^\beta} && (6) \\ &= \frac{\partial x^\alpha}{\partial y^\rho}\delta^\rho_\beta && (7) \\ &= \frac{\partial x^\alpha}{\partial y^\beta} && (8) \\ \text{(1) and (8)} \Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} &= \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta} && (9) \end{aligned}$$

So $A(r, s) = \frac{\partial x^r}{\partial y^s}$ is a mixed tensor of type A^r_s

ii) Let's compute the expression $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta}$. Obviously, the right side is an expression of a (possible) mixed tensor of the second order ($\frac{\partial x^r}{\partial y^s}$) under transformation from the (r) coordinate system to the (α) coordinate system. Then,

$$A(\alpha,\beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{10}$$

$$= \frac{\partial x^r}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{11}$$

$$= \frac{\partial x^r}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{12}$$

$$= \frac{\partial x^r}{\partial y^\rho}\delta^\rho_\beta\frac{\partial y^\alpha}{\partial y^r} \tag{13}$$

$$= \frac{\partial x^r}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{14}$$

$$= \frac{\partial x^r}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{15}$$

If we consider the $\overline{x}^r$ coordinate system as the x^σ coordinate system and as $\overline{x}^r = x^r$ then $\frac{\partial x^\sigma}{\partial x^r} = \delta^\sigma_r$ and we get from (15)

$$A(\alpha,\beta) = \frac{\partial x^r}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{16}$$

$$= \delta^r_\sigma\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{17}$$

$$= \frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^\sigma} \tag{18}$$

$$= \frac{\partial x^\sigma}{\partial y^\beta}\delta^\alpha_\sigma \tag{19}$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \tag{20}$$

$$\text{(10) and (19)} \;\Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{21}$$

So $A(r,s) = \frac{\partial x^r}{\partial y^s}$ is a mixed tensor of type A^r_s

♦

1.25 p24-exercise 10.

If x^r, y^r, z^r are three systems of coordinates, prove the follwoing rule for the multiplication of Jacobians.

$$\left|\frac{\partial x^m}{\partial y^n}\right|\left|\frac{\partial y^r}{\partial z^s}\right| = \left|\frac{\partial x^t}{\partial z^u}\right|$$

As we have

$$\frac{\partial x^t}{\partial z^u} = \frac{\partial x^t}{\partial y^k}\frac{\partial y^k}{\partial z^u} \tag{1}$$

$$\begin{bmatrix} \frac{\partial x^1}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial z^N} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial y^k}\frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial y^k}\frac{\partial y^k}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^k}\frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial y^k}\frac{\partial y^k}{\partial z^N} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^1} & \cdots & \frac{\partial x^N}{\partial y^k} \end{bmatrix} \begin{bmatrix} \frac{\partial y^1}{\partial z^1} & \cdots & \frac{\partial y^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial y^N}{\partial z^1} & \cdots & \frac{\partial y^N}{\partial z^N} \end{bmatrix} \tag{3}$$

$$\Rightarrow \left|\frac{\partial x^m}{\partial y^n}\right|\left|\frac{\partial y^r}{\partial z^s}\right| = \left|\frac{\partial x^t}{\partial z^u}\right| \tag{4}$$

♦

1.26 p24-exercise 11.

Prove that with respect to transformations

$$x^{,r} = C_{rs}x^s$$

where the coefficients are constants satisfying

$$C_{mr}C_{ms} = \delta^r_s$$

contravariant and covariant vectors have the same formula of transformation

$$A^{,r} = C_{rs}A^s,\ A_{,r} = C_{rs}A_s$$

i) $A^{,r} = C_{rs}A^s$

Be $A^{,r} = A^s \frac{\partial x^{,r}}{\partial x^s}$ and as $x^{,r} = C_{rs}x^s$ we have $\frac{\partial x^{,r}}{\partial x^s} = C_{rs}$. Hence,

$$A^{,r} = C_{rs}A^s$$

.

i) $A_{,r} = C_{rs}A_s$
Be $A_{,r} = A_s \frac{\partial x^s}{\partial x^{,r}}$ and as $x^{,r} = C_{rs}x^s$ we have

$$\frac{\partial x^{,r}}{\partial x^{,t}} = C_{rs}\frac{\partial x^s}{\partial x^{,t}} \tag{1}$$

$$\Rightarrow \delta^r_t = C_{rs}\frac{\partial x^s}{\partial x^{,t}} \tag{2}$$

Now, multiply (2) by C_{rq}. We get,

$$\delta^r_t C_{rq} = C_{rq}C_{rs}\frac{\partial x^s}{\partial x^{,t}} \tag{3}$$

$$C_{tq} = C_{rq}C_{rs}\frac{\partial x^s}{\partial x^{,t}} \tag{4}$$

$$\text{as } C_{mr}C_{ms} = \delta^r_s \Rightarrow \qquad C_{tq} = \delta^q_s\frac{\partial x^s}{\partial x^{,t}} \tag{5}$$

$$\Rightarrow C_{tq} = \frac{\partial x^q}{\partial x^{,t}} \text{ or } C_{rs} = \frac{\partial x^s}{\partial x^{,r}} \tag{6}$$

$$\text{as } A_{,r} = A_s\frac{\partial x^s}{\partial x^{,r}} \Rightarrow A_{,r} = C_{rs}\frac{\partial x^s}{\partial x^{,r}} \tag{7}$$

♦

1.27 p25-exercise 12.

Prove that

$$\frac{\partial ln\left|\frac{\partial x^m}{\partial y^n}\right|}{\partial x^r} = \frac{\partial^2 y^m}{\partial x^r \partial x^n}\frac{\partial x^n}{\partial y^m}$$

.

Lemma *Be A a square matrix $N \times N$; Be f a C^1 function $f : \mathbb{R}^{NxN} \to \mathbb{R}$. Define $A^{,}$ as $(A^{,}_{ij}) = \frac{df}{dA_{ij}}$. Then,*

$$(ln\,|A|)^{,} = (A^{-1})^T \;\; wih\; f = |A|$$

Proof:
By definition of the determinant, we have

$$|A| = A_{iK}C^i_K \qquad \textit{(no summation on K!)} \tag{1}$$

with $(C^i_K) = (-1)^{i+K}M^i_K$ being the cofactor of element A_{iK} and M^i_K the minor (N-1)x(N-1) matrix associated with the cofactor A_{K^i}. Be $C = (C_{ij})$ the NxN matrix formed with all possible cofactor elements C^i_j $(i, j = 1 \ldots, N)$.
We have

$$A^{-1} = \frac{C^T}{|A|} \tag{2}$$

$$\Rightarrow (A^{-1})^T = \frac{C}{|A|} \tag{3}$$

$$\textit{differentiating (1)} \;\Rightarrow \frac{\partial\,|A|}{\partial A_{mn}} = \frac{\partial A_{iK}}{\partial A_{mn}}C^i_K + A_{iK}\frac{\partial C^i_K}{\partial A_{mn}} \tag{4}$$

$$\textit{we have for } i = m \quad \begin{array}{ll} \frac{\partial A_{iK}}{\partial A_{mn}} = 1 & K = n \\ \frac{\partial A_{iK}}{\partial A_{mn}} = 0 & K \neq n \end{array} \tag{5}$$

Also, $\forall K : \frac{\partial C^i_K}{\partial A_{in}} = 0$ as by definition of the cofactor matrix, A_{ij} is not contained in C_{ij}.
Hence, (4) becomes

$$\frac{\partial\,|A|}{\partial A_{ij}} = C^i_j \tag{6}$$

$$\textit{But,} \quad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{\frac{\partial |A|}{\partial A_{ij}}}{|A|} \tag{7}$$

$$\textit{(6) and (7) gives} \quad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{C^i_j}{|A|} \tag{8}$$

$$\textit{(3) and (8) gives} \quad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{(A^{-1}_{ij})^T\,|A|}{|A|} = (A^{-1}_{ij})^T \tag{9}$$

$$\Rightarrow (ln\,|A|)^{,} = (A^{-1})^T \tag{10}$$

⋄

Now the main proof:
Let,

$$A \equiv [a_{mn}] = \left[\frac{\partial y^m}{\partial x^n}\right] \tag{11}$$

$$\Rightarrow \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial ln\,|A|}{\partial a_{mn}} \frac{\partial a_{mn}}{\partial x^r} \tag{12}$$

$$\text{from (10) we get} \quad \frac{\partial ln\,|A|}{\partial a_{mn}} = (A^{-1})^T_{mn} \tag{13}$$

$$\text{But A is a Jacobian, so } (A^{-1})_{mn} = \frac{\partial x^m}{\partial y^n} \tag{14}$$

$$\text{and thus } (A^{-1})^T_{mn} = \frac{\partial x^n}{\partial y^m} \tag{15}$$

$$\text{(13) can be written as } \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m} \frac{\partial a_{mn}}{\partial x^r} \tag{16}$$

$$\Rightarrow \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m} \frac{\partial^2 y^m}{\partial x^r \partial x^n} \tag{17}$$

♦

1.28 p25-exercise 13.

Consider the quantities $\frac{dx^r}{dt}$ for a particle moving in the plane. If x^r are the rectangular Cartesian coordinates, are these quantities the components of a contravariant or covariant vector with respect to rotation of the axes? Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?

Note: we suppose that by a rotation of the axes, the problem means a fixed rotation and not a rotation varying in time.

i) Be $v^r = \frac{dx^r}{dt}$ and consider v^α the same object but in another the coordinate system. A rotation of the axes implies the linear form

$$x^\alpha = R^\alpha_k x^k \quad \text{with } R^\alpha_k \neq R^\alpha_k(x^k) \tag{1}$$

$$\Rightarrow \quad \frac{\partial x^\alpha}{\partial x^r} = R^\alpha_k \delta^k_r \tag{2}$$

$$\Rightarrow \quad R^\alpha_r = \frac{\partial x^\alpha}{\partial x^r} \tag{3}$$

Consider $v^\alpha = \frac{dx^\alpha}{dt}$

$$v^\alpha = \frac{dx^\alpha}{dt} \tag{4}$$

$$(1) \;\Rightarrow\quad v^\alpha = R^\alpha_k \frac{dx^k}{dt} \tag{5}$$

$$\Rightarrow\quad v^\alpha = R^\alpha_k v^k \tag{6}$$

$$(3) \;\Rightarrow\quad v^\alpha = v^k \frac{\partial x^\alpha}{\partial x^r} \tag{7}$$

Conclusion: v^k is a contravariant vector.

ii) Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?
We know that

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^r} dx^r \tag{8}$$

$$\Rightarrow\quad \frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r}\frac{dx^r}{dt} \tag{9}$$

$$\Rightarrow\quad v^\alpha = v^r \frac{\partial x^\alpha}{\partial x^r} \tag{10}$$

So v^r is a contravariant vector in general. Note that this proof is more straightforward than the prove in i).

♦

1.29 p25-exercise 14.

Consider the question raised in No. 13 for the acceleration $\frac{d^2x^r}{dt^2}$.

From exercise 13. we know that

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r}\frac{dx^r}{dt} \tag{1}$$

$$\Rightarrow \quad \frac{d^2x^\alpha}{dt^2} = \frac{d^2x^r}{dt^2}\frac{\partial x^\alpha}{\partial x^r} + \frac{d\frac{\partial x^\alpha}{\partial x^r}}{dt}\frac{dx^r}{dt} \tag{2}$$

$$= \frac{d^2x^r}{dt^2}\frac{\partial x^\alpha}{\partial x^r} + \frac{\partial^2 x^\alpha}{\partial x^r \partial x^m}\frac{dx^m}{dt}\frac{dx^r}{dt} \tag{3}$$

The second term on the right does not vanish in general, hence $\frac{d^2x^r}{dt^2}$ has not a tensor character.

♦

1.30 p25-exercise 15.

It is well known that the equation of an ellipse may be written

$$ax^2 + 2hxy + by^2 = 1$$

What is the tensor character of a, h, b with respect to transformation to any Cartesian coordinates (rectangular or oblique) in the plane?

Consider the transformation from a (w, z) coordinate system to a (x, y) coordinate system. For the considered type of transformation we have

$$\begin{aligned} x &= \alpha w + \beta z \\ y &= \gamma w + \delta z \end{aligned} \tag{1}$$

$$\text{consider} \quad \begin{aligned} ax^2 + 2hxy + by^2 &= 1 \\ pw^2 + 2qwz + rz^2 &= 1 \end{aligned} \tag{2}$$

the two representations of the same ellipse in the respective coordinate systems. Plugging (1) in (2):

$$a\alpha^2 w^2 + a2\alpha\beta wz + \alpha\beta^2 z^2 + 2h\alpha\gamma w^2 + \beta\delta z^2 + 2h(\alpha\delta + \gamma\beta)wz + b\gamma^2 w^2 + 2b\gamma\delta wz + \delta^2 z^2 b = 1 \tag{3}$$

$$\tag{4}$$

Rearranging and equating the terms in w^2, wz, z^2 in (2) gives

$$p = a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \tag{5}$$

$$q = a\alpha\beta + h(\alpha\delta + \gamma\beta) + \gamma d \tag{6}$$

$$r = a\beta^2 + 2h\beta\delta + b\delta^2 \tag{7}$$

Consider the following objects

$$(A_{ij}) = \begin{bmatrix} a & h \\ h & b \end{bmatrix} \tag{8}$$

$$(A_{ij})' = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{21} \end{bmatrix} \tag{9}$$

$$\text{we calculate} \quad A'_{ij} = A_{km} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} \tag{10}$$

with $(x^{,1}, x^{,2}) = (w, z)$ and $(x^1, x^2) = (x, y)$. We have,

$$\frac{\partial x^1}{\partial x^{,1}} = \alpha, \frac{\partial x^1}{\partial x^{,2}} = \beta, \frac{\partial x^2}{\partial x^{,1}} = \gamma, \frac{\partial x^2}{\partial x^{,2}} = \delta \tag{11}$$

$$(10) \text{ and } (11) \quad \Rightarrow \quad \begin{array}{c} a^{,}_{11} = a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \\ a^{,}_{22} = a\beta^2 + 2h\delta\beta + b\delta^2 \\ a^{,}_{12} = a^{,}_{21} = a\alpha\beta + h(\alpha\delta + \gamma\beta) + b\gamma\delta \end{array} \tag{12}$$

Combining (5), (6), (7) and (12) we get

$$p = a^{,}_{11}, r = a^{,}_{22}, q = a^{,}_{12} = a^{,}_{21}$$

and so (9) becomes

$$(A_{ij})^{,} = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$$

Considering (10) we see that $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ transforms to $\begin{bmatrix} p & q \\ q & r \end{bmatrix}$ according the rules of a covariant tensor of order two.

♦

1.31 p25-exercise 16.

Matter is distributed in a plane and A, B, H are the moments and product of inertia with respect to rectangular aces $0xy$ in a plane. Examine the tensor character of the set of quantities A, B, H under rotation of the axes. What notation would you suggest for moments and product of inertia in order to exhibit the tensor character? What simple invariant can be formed from A, B, H ?

Consider the transformation from a (x^1, x^2) coordinate system to a (y^1, y^2) coordinate system. For the considered type of transformation we have

$$\begin{aligned} y^1 &= \alpha x^1 + \beta x^2 \\ y^2 &= \gamma x^1 + \delta x^2 \end{aligned} \tag{1}$$

$$\text{Be} \quad \begin{aligned} A &= \textstyle\sum_\rho m_\rho (x^{2,\rho})^2 \\ B &= \textstyle\sum_\rho m_\rho (x^{1,\rho})^2 \\ H &= \textstyle\sum_\rho m_\rho x^{1,\rho} x^{2,\rho} \end{aligned} \quad \text{and} \quad \begin{aligned} A^{,} &= \textstyle\sum_\rho m_\rho (y^{2,\rho})^2 \\ B^{,} &= \textstyle\sum_\rho m_\rho (y^{1,\rho})^2 \\ H^{,} &= \textstyle\sum_\rho m_\rho y^{1,\rho} y^{2,\rho} \end{aligned} \tag{2}$$

the moments and product of inertia, ρ being the index of summation over all the points with mass m_ρ.
For the sake of notational simplicty we consider only one point of mass as the linearity of A, B, H related to ρ ensures the validity of the nexts steps for all points in the plane.
From (1) and (2) we have:

$$\frac{A^{,}}{m_\rho} = \gamma^2 (x^1)^2 + 2\gamma\delta x^1 x^2 + \delta^2 (x^2)^2 \tag{3}$$

$$\frac{B^{,}}{m_\rho} = \alpha^2 (x^1)^2 + 2\alpha\beta x^1 x^2 + \beta^2 (x^2)^2 \tag{4}$$

$$\frac{H^{,}}{m_\rho} = \alpha\gamma (x^1)^2 + (\gamma\beta + \alpha\delta) x^1 x^2 + \beta\delta (x^2)^2 \tag{5}$$

$$\text{Note that} \quad \begin{aligned} \tfrac{\partial y^1}{\partial x^1} &= \alpha \quad & \tfrac{\partial y^1}{\partial x^2} &= \beta \\ \tfrac{\partial y^2}{\partial x^1} &= \gamma \quad & \tfrac{\partial y^2}{\partial x^2} &= \delta \end{aligned} \tag{6}$$

$$\text{(6) in (4):} \quad \frac{B^{,}}{m_\rho} = (x^1)^2 \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + 2(x^1)(x^2)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} \tag{7}$$

$$= (x^1)^2 \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + (x^1)(x^2)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)(x^1)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} \tag{8}$$

Repeating the same calculations for $\frac{A^{,}}{m_\rho}$ and $\frac{H^{,}}{m_\rho}$ gives:

$$\begin{aligned} \tfrac{A^{,}}{m_\rho} &= (x^1)^2 \tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + (x^1)(x^2)\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + (x^2)(x^1)\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + (x^2)^2 \tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \\ \tfrac{B^{,}}{m_\rho} &= (x^1)^2 \tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^1} + (x^1)(x^2)\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + (x^2)(x^1)\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + (x^2)^2 \tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^1}{\partial x^2} \\ \tfrac{H^{,}}{m_\rho} &= (x^1)^2 \tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + (x^1)(x^2)\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + (x^2)(x^1)\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + (x^2)^2 \tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \end{aligned} \tag{9}$$

Now, define

$$(K_{ij}) = \begin{bmatrix} (x^1)^2 & (x^1)(x^2) \\ \\ (x^2)(x^1) & (x^2)^2 \end{bmatrix} \qquad (K_{ij})^{,} = \begin{bmatrix} (y^1)^2 & (y^1)(y^2) \\ \\ (y^2)(y^1) & (y^2)^2 \end{bmatrix} \tag{10}$$

Then (9) can be written as

$$\begin{aligned}
\tfrac{A^{,}}{m_\rho} &= (y^1)^2 = K^{11}\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + K^{12}\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + K^{21}\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + K^{22}\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \\
\tfrac{B^{,}}{m_\rho} &= (y^2)^2 = K^{11}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^1} + K^{12}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + K^{21}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + K^{22}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^1}{\partial x^2} \qquad (11) \\
\tfrac{H^{,}}{m_\rho} &= (y^1)(y^2) = K^{11}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + K^{12}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + K^{21}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + K^{22}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^2}
\end{aligned}$$

Hence,

$$\begin{aligned}
K^{,11} &= K^{11}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^1} + K^{12}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + K^{21}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + K^{22}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^1}{\partial x^2} \\
K^{,22} &= K^{11}\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + K^{12}\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + K^{21}\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + K^{22}\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \qquad (12) \\
K^{,12} = K^{,21} &= K^{11}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + K^{12}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + K^{21}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + K^{22}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^2}
\end{aligned}$$

So the object $(K_{ij}) \equiv K^{ij}$ transforms according (12) like a contravariant second order tensor.

Now, consider $|K^{ij}|$, obviously $|K^{ij}| = (x^1)^2(x^2)^2 - (x^1)(x^2)(x^2)(x^1) = 0$, but so is also $|K^{,ij}|$.

$$\Rightarrow \quad |K^{ij}| \text{ is an invariant under the considered transformation}$$

.

♦

1.32 p25-exercise 17.

S_{nmr} is a skew-symmetric tensor in the first two indices. $-f_{mnr} + f_{nmr} = S_{mnr}$.

From exercise 13. we know that

$$-f_{mnr} + f_{nmr} = S_{mnr} \tag{1}$$

Swap the indices three times

$$\text{i)} \quad n \leftrightarrow r : (1) \Rightarrow -f_{mrn} + f_{rmn} = S_{mrn} \tag{2}$$

$$\Leftrightarrow \underbrace{f_{mnr}}_{*} + \underbrace{f_{rmn}}_{**} = -S_{rmn} \tag{3}$$

$$\text{ii)} \quad m \leftrightarrow r : (1) \Rightarrow -f_{rnm} + f_{nrm} = S_{rnm} \tag{4}$$

$$\Leftrightarrow \underbrace{f_{rmn}}_{**} + \underbrace{f_{nrm}}_{***} = -S_{nrm} \tag{5}$$

$$\text{iii)} \quad m \leftrightarrow n : (1) \Rightarrow -f_{nmr} + f_{mnr} = S_{nmr} \tag{6}$$

$$\Leftrightarrow \underbrace{f_{nrm}}_{***} + \underbrace{f_{mnr}}_{*} = -S_{mnr} \tag{7}$$

$$\text{(3) - (5) + (7):} \quad 2\underbrace{f_{mnr}}_{*} = -S_{rmn} + S_{nrm} - S_{mnr} \tag{8}$$

$$\Leftrightarrow f_{mnr} = \frac{-S_{rmn} + S_{nrm} - S_{mnr}}{2} \tag{9}$$

♦

Basic Operations in Riemannian Space

2.1 p27-exercise

Take polar coordinates r, θ in a plane. Draw the infinitesimal triangle with vertices (r, θ), $(r + dr, \theta)$, $(r, \theta + d\theta)$. Evaluate the square on the hypotenuse of this infinitesimal triangle, and so obtain the metric tensor for the plan for the coordinates(r, θ).

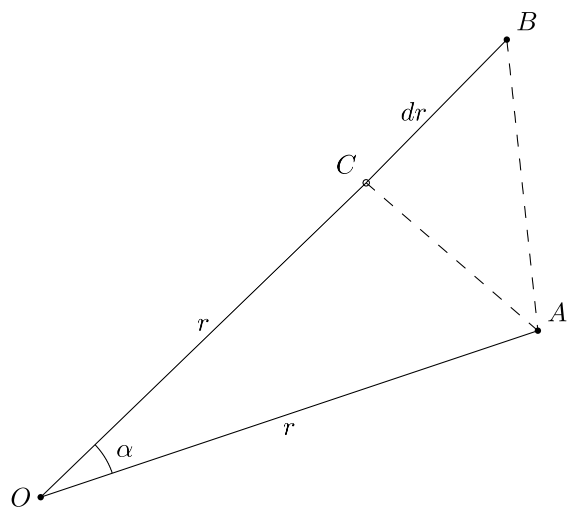


Figure 2.1: Metric tensor in polar coordinate system

$$
\begin{aligned}
ds^2 &= |AB|^2 && (1)\\
&= dr^2 + |CA|^2 && (2)\\
|CA| &= r\sin(d\theta) \approx r d\theta && (3)\\
\Rightarrow ds^2 &= dr^2 + r^2 d\theta^2 && (4)\\
\Rightarrow (a_{mn}) &= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} && (5)
\end{aligned}
$$

♦

2.2 p27-exercise

Show that if $x^1 = r, x^2 = \theta, x^3 = \phi$, in the usual notation of spherical polar coordinates, then

$$a_{11} = 1, a_{22} = r^2, a_{33} = r^2 \sin^2 \theta$$

and the other components vanish.

One can choose to start from $ds^2 = dx^2 + dy^2 + dz^2$ and then expanding the dx^i along (r, θ, ϕ) but this a rather tedious way. So we use a more geometrical way of deriving the metric
Consider an infinitesimal displacement of point E to J with $(dr, d\theta, d\phi)$.

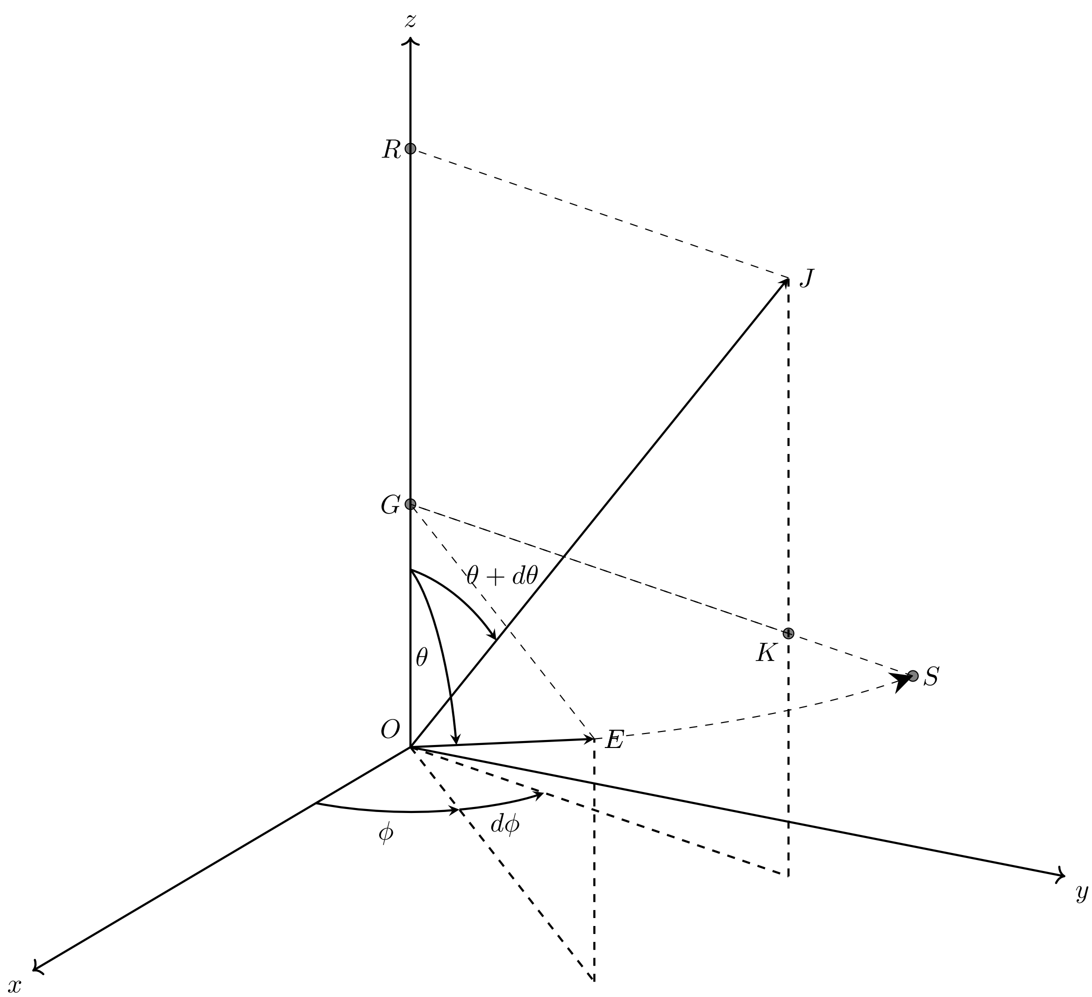


Figure 2.2: Metric tensor in spherical coordinate system

$$ds^2 = |EJ|^2 \tag{1}$$

As we use infinitesimal displacements we can assume that

$$|ES| \perp |GK| \perp |JK| \perp |ES|$$

. Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \tag{2}$$

We have the following relationships

$$\left.\begin{array}{c}
|GE| = |GS| = r\sin\theta \\
\\
|ES| = |GE|d\phi = r\sin\theta d\phi \\
\\
|GK| = |RJ| = (r+dr)\sin(\theta+d\theta) \\
= (r+dr)(\cos(\theta)\sin(d\theta) + \sin(\theta)\cos(d\theta)) \\
= (r+dr)(\cos(\theta)d\theta + \sin(\theta)) \\
= r\cos(\theta)d\theta + r\sin(\theta) + \sin(\theta)dr \\
\\
|OR| = (r+dr)\cos(\theta+d\theta) \\
= (r+dr)(\cos(\theta)\cos(d\theta) - \sin(\theta)\sin(d\theta)) \\
= (r+dr)(\cos(\theta) - \sin(\theta)d\theta) \\
= r\cos(\theta) - r\sin(\theta)d\theta + \cos(\theta)dr \\
\\
|OG| = r\cos(\theta) \\
\\
|JK| = |OR| - |OG| = \cos(\theta)dr - r\sin(\theta)d\theta \\
\\
|SK| = |GK| - |GS| = r\cos(\theta)d\theta + \sin(\theta)dr
\end{array}\right\} \tag{3}$$

$$\left.\begin{array}{c}
|ES|^2 = r^2\sin^2(\theta)d\phi^2 \\
|SK|^2 = r^2\cos^2(\theta)d\theta^2 + \sin^2(\theta)dr^2 + 2r\cos(\theta)\sin(\theta)drd\theta \\
|JK|^2 = \cos^2(\theta)dr^2 + r^2\sin^2(\theta)d\theta^2 - 2r\cos(\theta)\sin(\theta)drd\theta
\end{array}\right\} \tag{4}$$

Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \tag{5}$$

$$= \begin{cases} r^2 \sin^2(\theta) d\phi^2 \\ +r^2 \cos^2(\theta) d\theta^2 + \sin^2(\theta) dr^2 + 2r\cos(\theta)\sin(\theta) dr d\theta \\ +r^2 \sin^2(\theta) d\theta^2 + \cos^2(\theta) dr^2 - 2r\cos(\theta)\sin(\theta) dr d\theta \end{cases} \tag{6}$$

$$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \tag{7}$$

$$\Rightarrow (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{8}$$

♦

2.3 p27-exercise

Starting from 3.103, show that

$$a_{mn} = \frac{\partial y^1}{\partial x^m}\frac{\partial y^1}{\partial x^n} + \frac{\partial y^2}{\partial x^m}\frac{\partial y^2}{\partial x^n} + \frac{\partial y^3}{\partial x^m}\frac{\partial y^3}{\partial x^n}$$

and calculate the quantities for a sphere, taking as curvilinear coordinates on he sphere

$$x^1 = y^1, x^2 = y^2$$

We have

$$\begin{aligned}
(2.103) \quad & \Rightarrow y^1 = x^1, y^2 = x^2, y^3 = f^3(x^1, x^2) && (1)\\
\text{surface} = \text{sphere} \quad & \Rightarrow y^3 = \pm\sqrt{R^2 - (x^1)^2 - (x^2)^2} && (2)\\
ds^2 &= (dx^1)^2 + (dx^2)^2 + (dx^3)^2 && (3)\\
(1) \text{ and } (2) \quad & \Rightarrow \begin{cases} dy^1 = dx^1 \\ dy^2 = dx^2 \\ dy^3 = \pm\frac{1}{2}\frac{-2x^1dx^1 - 2x^2dx^2}{\sqrt{R^2-(x^1)^2-(x^2)^2}} \end{cases} && (4)\\
& \Rightarrow ds^2 = (dx^1)^2 + (dx^2)^2 + \frac{(x^1)^2(dx^1)^2 + (x^2)^2(dx^2)^2 + 2x^1x^2dx^1dx^2}{R^2 - (x^1)^2 - (x^2)^2} && (5)\\
& \Leftrightarrow ds^2 = \frac{(R^2 - (x^2)^2)(dx^1)^2 + (R^2 - (x^1)^2)(dx^2)^2 + 2x^1x^2dx^1dx^2}{R^2 - (x^1)^2 - (x^2)^2} && (6)\\
& \Rightarrow (a_{mn}) = \frac{1}{R^2 - (x^1)^2 - (x^2)^2}\begin{pmatrix} R^2 - (x^2)^2 & x^1x^2 \\ x^1x^2 & R^2 - (x^1)^2 \end{pmatrix} && (7)
\end{aligned}$$

♦

2.4 p30-clarification 2.202

$$a_{mr}\Delta^{ms} = a_{rm}\Delta^{sm} = \delta_r^s a$$

Case 1: $r = s$
We have, $a_{Rm}\Delta^{Rm}$ (no summation on R) is the definition of the determinant of A developed along the row R: OK.

Case 2: $r \neq s$
Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \tag{1}$$

and consider the matrix A'

$$A' = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{R1} & a_{R2} & \dots & a_{RN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{R1} & a_{R2} & \dots & a_{RN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \begin{array}{c} \vdots \\ \vdots \\ \leftarrow S^{th} \text{ row} \\ \vdots \\ \leftarrow R^{th} \text{ row} \\ \vdots \\ \vdots \end{array} \tag{2}$$

This matrix corresponds to the way $a_{Rm}\Delta^{Sm}$ is computed. Indeed with the factor a_{Rm} is not associated it's own cofactor Δ^{Rm} but the cofactor of the m^{th} column in row S. Replacing the S^{th} row with the row R and calculating it's determinant is the same as calculating $a_{Rm}\Delta^{Sm}$
But, $|A'| = 0$ as we have two identical rows. So, $a_{Rm}\Delta^{Sm} = 0$

Conclusion : The same reasoning can be applied when expanding the determinant along the columns instead of the rows we have indeed $a_{mr}\Delta^{ms} = a_{rm}\Delta^{sm} = \delta_r^s a$.

2.5 p31-exercise

Show that if $amn = 0$ for $m \neq n$, then

$$a^{11} = \frac{1}{a_{11}}, a^{22} = \frac{1}{a_{22}}, \ldots, a12 = 0, \ldots$$

We have to prove that:

$$a^{ij} = \begin{cases} \frac{1}{a_{ij}} & : i = j \\ 0 & : i \neq j \end{cases} \tag{1}$$

From 2.204:

$$a_{mR} a_{mS} = \delta_R^S \tag{2}$$

i) Be $R \neq S$

$$(2) \qquad \Rightarrow a_{mR} a_{mS} = 0 \tag{3}$$

$$\text{but} \qquad a_{mR} = 0 \quad \forall m \neq R \tag{4}$$

$$\Rightarrow a_{RR} a^{RS} = 0 \tag{5}$$

but $a_{RR} \neq 0$ (a_{RR} can't be 0 as the metric tensor would degenerate if $a_{mn} = 0 \quad \forall m \neq n$

$$\Rightarrow a^{Rs} = 0 \tag{6}$$

$$\tag{7}$$

$$\tag{8}$$

i) Be $R = S$

$$(2) \qquad \Rightarrow a_{mR} a_{mR} = 1 \tag{9}$$

$$\text{but} \qquad a_{mR} = 0 \quad \forall m \neq R \tag{10}$$

$$\Rightarrow a_{RR} a^{RR} = 1 \tag{11}$$

$$\Rightarrow a^{RR} = \frac{1}{a_{RR}} \tag{12}$$

♦

2.6 p31-exercise

Find the components of a^{mn} for spherical polar coordinates in Eulidean 3-space.

We have (see exercise page 27):

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \tag{1}$$

As $a_{mn} = 0 \quad \forall m \neq n$ we deduce (see previous exercise p31)

$$(a^{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta)} \end{pmatrix} \tag{2}$$

♦

2.7 p32-exercise

Find the mixed metric tensor $a_m^{\cdot n}$ obtained from a_{mn} by raising the second subscript

We have :

$$\begin{aligned} a_i^{\cdot j} &= a_{in}a^{nj} && (1)\\ &= a_{in}a^{jn} \quad a^{jn} \text{ is symmetric} && (2)\\ &= \delta_i^j \qquad \text{(see 2.205 pg. 30)} && (3)\\ &\Rightarrow a_i^{\cdot j} = \delta_i^j && (4) \end{aligned}$$

♦

2.8 p32-clarification 2.214

$$\frac{\partial a}{\partial a_{mn}} = a a^{mn}$$

Put $a_{MN} \equiv a_{mn}$. By definition, we have

$$a \equiv |a_{mn}| = a_{Mk}\Delta^{Mk} \quad \text{(develop determinant along row M)} \tag{1}$$

$$\Rightarrow \frac{\partial a}{\partial a_{mn}} = \frac{\partial a_{Mk}}{\partial a_{mn}}\Delta^{Mk} + a_{Mk}\frac{\partial \Delta^{Mk}}{\partial a_{mn}} \tag{2}$$

$$\text{but} \quad \frac{\partial a_{Mk}}{\partial a_{mn}} = \begin{cases} 1 & \text{if} \quad k = N \\ 0 & \text{if} \quad k \neq N \end{cases} \tag{3}$$

$$\text{and} \quad \frac{\partial \Delta^{Mk}}{\partial a_{mn}} = 0 \quad \forall k \text{ as } \Delta^{Mk} \text{does not contain the row with } a_{mn} \text{ as element.} \tag{4}$$

$$\text{(3) and (4)} \quad \Rightarrow \frac{\partial a}{\partial a_{mn}} = \Delta^{MN} \tag{5}$$

$$a^{mn} = \frac{\Delta^{mn}}{a} \quad \text{by definition (see 2.203 page 30)} \tag{6}$$

$$\Rightarrow \frac{\partial a}{\partial a_{mn}} = a a^{mn} \tag{7}$$

♦

2.9 p32-exercise

Prove that $a_{mn}a^{mn} = N$.

From 2.204, we have

$$a_{mr}a^{ms} = \delta_r^s \tag{1}$$

$$\text{Consider} \quad a_{mR}a^{mR} = 1 \tag{2}$$

$$\text{We can repeat (2) for R = 1,2,...,N} \quad \Rightarrow a_{mr}a^{mr} = N \tag{3}$$

$$\tag{4}$$

♦

2.10 p33-exercise

Show that in Euclidean 3-space with rectangular Cartesian coordinates, the definition 2.301 coincides with the usual definition of the magnitude of a vector.

The length of an arbitrary vector in Euclidean 3-space with rectangular Cartesian coordinates, is

$$ds^2 = (dy^1)^2 + (dy^2)^2(dy^2)^2 \tag{1}$$

From 2.301, it is obvious that the metrixc tensor can be expressed as,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

♦

2.11 p34-exercise

A curve in Euclidean 3-space has the equations

$$x^1 = a\cos(u), x^2 = a\sin(u), x^3 = bu$$

where x^1, x^2, x^3 are rectangular Cartesian coordinates,u is a parameter, and a, b are positive constants. Find the length of this curve between the point $u = 0$ and $u = 2\pi$.

The metric tensor has the following form,

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$\text{and (2.306)} \qquad s = \int_0^{2\pi} [\epsilon a_{mn} p^m p^n]^{\frac{1}{2}} du \tag{2}$$

with

$$p^1 = \frac{dx^1}{du} = -a\sin(u), \quad p^2 = \frac{dx^2}{du} = a\cos(u), \quad p^3 = \frac{dx^3}{du} = b \tag{3}$$

Hence (2) becomes

$$s = \int_0^{2\pi} \epsilon[a^2\sin^2(u) + a^2\cos^2(u) + b^2]^{\frac{1}{2}} du \tag{4}$$

$$= \int_0^{2\pi} \epsilon[a^2 + b^2]^{\frac{1}{2}} du \tag{5}$$

$$= [a^2 + b^2]^{\frac{1}{2}}\ u|_0^{2\pi} \tag{6}$$

$$= 2\pi[a^2 + b^2]^{\frac{1}{2}} \tag{7}$$

♦

2.12 p36-clarification 2.314

Going from 2.313 to 2.314 yields because both X^m and Y^m are unit vectors and by definition of the magnitude (see 2.301) both $a_{mn}X^mX^n$ and $a_{mn}Y^mY^n$ are 1 (also due to the fact that only a positive definite metric tensor is considered, $\epsilon = 1$).

♦

2.13 p37-exercise

Show that the small angle between unit vectors X^r and $X^r + dX^r$ (these increments being infenitesimal) is given by

$$\theta^2 = a_{mn} X^m X^n$$

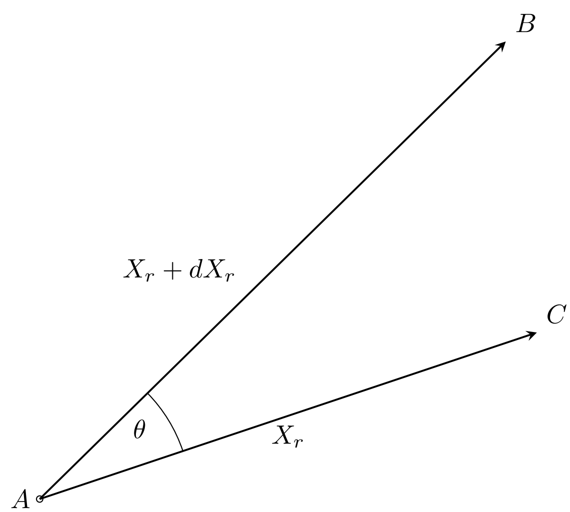


Figure 2.3: Small angle expression

By definition (2.302 page 33)

$$|BC|^2 = \epsilon a_{mn} dX^m dX^n \tag{1}$$

$$\tag{2}$$

We can drop $\epsilon = 1$ as the considered space is positive definite.
As θ is infinitesimal, we can state

$$|BC| \approx |AC|\theta \tag{3}$$

$$\text{and} \quad |AC| = X^r = 1 \quad \text{(as } X^r \text{ is a unit vector)} \tag{4}$$

$$\Rightarrow \theta^2 = a_{mn} dX^m dX^n \tag{5}$$

♦

2.14 p39-clarification 2.409

We clarify the integration by parts in the derivation of the general geodesic equation.

We have

$$\int d(A.B) = \int AdB + \int BdA \tag{1}$$

$$\Rightarrow \int AdB = \int BdA - \int d(A.B) \tag{2}$$

Now, substitute 2.407 in 2.406, we get

$$\frac{dL}{dv} = \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial p^r}{\partial v}du \tag{3}$$

$$= \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial \frac{\partial x^r}{\partial v}}{\partial u}du \tag{4}$$

$$= \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}d(\frac{\partial x^r}{\partial v}) \tag{5}$$

To integrate by parts the second term in (5) we put in (2)

$$A = \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} \quad \text{and } B = \frac{\partial x^r}{\partial v}$$

$$\int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}d(\frac{\partial x^r}{\partial v}) = \int AdB \tag{6}$$

$$= \int BdA - \int d(A.B) \tag{7}$$

$$= \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial x^r}{\partial v}\Big|_{u_0}^{u_1} - \int \frac{\partial x^r}{\partial v}d(\frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}) \tag{8}$$

$$= \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial x^r}{\partial v}\Big|_{u_0}^{u_1} - \int \frac{\partial x^r}{\partial v}\frac{\partial \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r})}{\partial u}du \tag{9}$$

Replacing (9) in (5) gives the formulea 2.409.

♦

2.15 p41-exercise

Prove the following identities:

$$[mn, r] = [nm, r], \quad [rm, n] + [rn, m] = \partial_r a_{mn}$$

$$[mn, r] = \frac{1}{2}(\partial_n a_{mr} + \partial_m a_{nr} - \partial_r a_{mn}) \tag{1}$$

$$= \frac{1}{2}(\partial_m a_{nr} + \partial_n a_{mr} - \partial_r a_{nm}) \tag{2}$$

$$= [nm, r] \tag{3}$$

and

$$[rm, n] + [rn, m] = \frac{1}{2}(\partial_r a_{mn} + \partial_m a_{rn} - \partial_n a_{rm} + \partial_n a_{rm} + \partial_r a_{mn} - \partial_m a_{rn}) \tag{4}$$

$$= \frac{1}{2}(\partial_r a_{mn} + \partial_r a_{mn}) \tag{5}$$

$$= \partial_r a_{mn} \tag{6}$$

♦

2.16 p42-exercise

Prove that

$$[mn, r] = a_{rs}\Gamma^s_{mn}$$

$$a_{rs}\Gamma^s_{mn} = a_{rs}a^{sk}[mn, k] \tag{1}$$
$$= \delta^k_r[mn, k] \tag{2}$$
$$= [mn, r] \tag{3}$$

♦

2.17 p42-clarification on 2.430

... This may proved without difficulty by starting with 2.427, in which λ is a known function of u , and defining s by the relation

$$s = \int_{u_0}^{u} (exp \int_{v_0}^{v} \lambda(w)dw)dv$$

u_0, v_0 being constants....

$$\text{see 2.428 :} \quad \lambda(u) = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{1}$$

$$\tag{2}$$

We suppose $u(s)$ continuous by parts with continuous inverse.

$$\Rightarrow \frac{ds}{du} = \frac{1}{\frac{du}{ds}} \tag{3}$$

$$\Rightarrow \frac{d^2s}{du^2} = \frac{d(\frac{1}{\frac{du}{ds}})}{ds}\frac{ds}{du} \tag{4}$$

$$= -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2}\frac{ds}{du} \tag{5}$$

$$\Rightarrow \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{6}$$

By definition (2.428)

$$\lambda(u) = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{7}$$

$$\text{hence by (6) and (7):} \quad \lambda(u) = \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} \tag{8}$$

$$\text{in (8) put} \quad y = \frac{ds}{du} \tag{9}$$

$$\text{and so} \quad \lambda(w) = \frac{y'}{y} \tag{10}$$

$$\Rightarrow \int \frac{y'}{y}dw = \int \lambda(w)dw \tag{11}$$

$$\Leftrightarrow \int d(lny) = \int \lambda(w)dw \tag{12}$$

$$\Rightarrow ln(y)|_{v_0}^{v} = \int_{v_0}^{v} \lambda(w)dw \tag{13}$$

$$\Rightarrow y = exp(\int_{v_0}^{v} \lambda(w)dw) + C \tag{14}$$

Taking into account (9), we get:

$$\frac{ds}{dv} = exp(\int_{v_0}^{v} \lambda(w)dw) + C \tag{15}$$

$$\Rightarrow s|_{u_0}^{u} = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv + Cu + B' \tag{16}$$

$$\Leftrightarrow s = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv + Cu + B \tag{17}$$

We show tha we have to put $C = 0$ and can drop the constant B. Remember by (8)

$$\lambda(u) = \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} \tag{18}$$

$$\text{by (17)} \quad \frac{ds}{du} = exp(\int_{u_0}^{u} \lambda(w)dw) + C \tag{19}$$

$$\text{and} \quad \frac{d^2s}{du^2} = \lambda(u)exp(\int_{u_0}^{u} \lambda(w)dw) \tag{20}$$

$$\text{hence by (18), (19) and (20):} \quad \lambda(u) = \frac{\lambda(u)exp(\int_{u_0}^{u} \lambda(w)dw)}{exp(\int_{u_0}^{u} \lambda(w)dw) + C} \tag{21}$$

So, whatever the constant B, the relation (18) is correct on the condition that C=0. So, indeed, we can choose the independent variable s as

$$s = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv$$

♦

2.18 p42-clarification on 2.430

After 2.430 it is stated:
"No matter what values these constants have, 2.424 is satisfied, and by adjusting the constant v_0, we can ensure that $a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \pm 1$ along C, so that s is actually the arc length."

We first prove that 2.4.24 is satisfied, no matter what values the constants take. We have

$$(2.430) \qquad s = \int_{u_0}^{u} (exp \int_{v_0}^{v} \lambda(w)dw)dv \tag{1}$$

$$\text{and } (2.427) \qquad \frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = \lambda\frac{dx^r}{du} \tag{2}$$

In (2) we can write the first term as

$$\frac{d^2x^r}{du^2} = \frac{d(\frac{dx^r}{du})}{ds}\frac{ds}{du} \tag{3}$$

$$\text{with} \quad \frac{d(\frac{dx^r}{du})}{ds} = \frac{d(\frac{dx^r}{ds}\frac{ds}{du})}{ds} = \frac{d^2x^r}{ds^2}\frac{ds}{du} + \frac{dx^r}{ds}\frac{d(\frac{ds}{du})}{ds} \tag{4}$$

Assuming the curve smooth, we have

$$\frac{d(\frac{ds}{du})}{ds} = \frac{d(\frac{1}{\frac{du}{ds}})}{ds} = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} = \lambda \tag{5}$$

Putting (4) and (5) in (3) we get

$$\frac{d^2x^r}{du^2} = \frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \lambda\frac{dx^r}{ds}\frac{ds}{du} \tag{6}$$

Plugging (6) in 2.427 gives:

$$\frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \lambda\frac{dx^r}{ds}\frac{ds}{du} + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \lambda\frac{dx^r}{ds}\frac{ds}{du} \tag{7}$$

$$\Leftrightarrow \frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = 0 \tag{8}$$

We can assume that $\frac{ds}{du}$ does not become 0 or $\pm\infty$ along the curve by choosing an adequate constant v_0. Indeed, from (2.430) we get

$$\frac{ds}{du} = exp(\int_{u_0}^{u} \lambda(w)dw) \tag{9}$$

$$= \frac{\phi(u)}{\phi(u_0)} \tag{10}$$

with $\phi(u_0) = e^{\theta(u_0)}$, $\theta(u)$ being the indefinite integral $\int \lambda(w)dw$.
So, it is sufficient to choose v_0 so that $\theta(u_0)$ does not become $\pm\infty$ to ensure that $\frac{ds}{du} \neq 0$ or $\neq \pm\infty$

along the curve and so we have from (8)

$$\frac{d^2x^r}{ds^2} + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = 0 \tag{11}$$

which is the definition (2.424) of a geodesic.

The same reasoning about $\frac{ds}{du} \neq 0$ can be made to prove that $a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \pm 1$ along C. Indeed, by definition (2.305):

$$ds = \left[\epsilon a_{mn}\frac{dx^m}{du}\frac{dx^n}{du}\right]^{\frac{1}{2}} \tag{12}$$

$$\text{equating with (9)} \quad \left[\epsilon a_{mn}\frac{dx^m}{du}\frac{dx^n}{du}\right]^{\frac{1}{2}} = exp(\int_{u_0}^{u}\lambda(w)dw) \tag{13}$$

$$\Rightarrow \quad \epsilon a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \left[exp(\int_{u_0}^{u}\lambda(w)dw)\right]^2 \tag{14}$$

$$\text{but} \quad \frac{ds}{du} = exp(\int_{u_0}^{u}\lambda(w)dw) \tag{15}$$

$$\text{and so, (9) becomes} \quad \epsilon a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \left(\frac{ds}{du}\right)^2 \tag{16}$$

$$\Rightarrow \quad a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \epsilon \tag{17}$$

♦

2.19 p43-clarification

$$\boxed{\lambda = \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + 2\Gamma^N_{\mu N}\frac{dx^\mu}{dx^N} + \Gamma^N_{NN}}$$

We start with 2.427 with $r = N$

$$\frac{d^2x^N}{dx^{N^2}} + \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} = \lambda\frac{dx^N}{dx^N} \tag{1}$$

$$\Rightarrow \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} = \lambda \quad \left(\text{as} \quad \frac{d^2x^N}{dx^{N^2}} = 0 \quad \frac{dx^N}{dx^N} = 1\right) \tag{2}$$

But (2) is only valid with the dummy indices μ and ν spanning the whole dimension $(1, 2, \ldots N)$, but by choice $\mu, \nu \quad \epsilon \quad (1, 2, \ldots, N-1)$. We have thus to add in the left term of (2) the cases

$$\begin{cases} \Gamma^N_{N\nu} & \nu = (1, 2, \ldots, N-1) \\ \\ \Gamma^N_{\mu N} & \mu = (1, 2, \ldots, N-1) \\ \\ \Gamma^N_{NN} \end{cases} \tag{3}$$

$$\text{(2) becomes} \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{N\nu}\frac{dx^N}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{\mu N}\frac{dx^\mu}{dx^N}\frac{dx^N}{dx^N} + \Gamma^N_{NN}\frac{dx^N}{dx^N}\frac{dx^N}{dx^N} = \lambda \tag{4}$$

$$\Rightarrow \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{N\nu}\frac{dx^\nu}{dx^N} + \Gamma^N_{\mu N}\frac{dx^\mu}{dx^N} + \Gamma^N_{NN} = \lambda \tag{5}$$

As $\Gamma^N_{\mu N}$ is symmetric on the lower indices and μ, ν being dummy indices:

$$\lambda = \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + 2\Gamma^N_{N\nu}\frac{dx^\nu}{dx^N} + \Gamma^N_{NN} \tag{6}$$

The other $N-1$ equation for $r = 1, \ldots, N-1$ can be deduced following the same reasoning.

♦

2.20 p45-clarification

$f_r \equiv a_{rm}\frac{d^2x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du}$ is covariant and $f^r \equiv \frac{d^2x^m}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du}$ is contravariant.

$$f_r \equiv a_{rm}\frac{d^2x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \tag{1}$$

$$\text{multiply with} \quad a^{sr} \quad \Rightarrow \quad f_r a^{sr} = a^{sr}a_{rm}\frac{d^2x^m}{du^2} + a^{sr}[mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \tag{2}$$

$$\Rightarrow \quad f^s = \delta^s_m\frac{d^2x^m}{du^2} + \underbrace{a^{sr}[mn,r]}_{\Gamma^s_{mn}}\frac{dx^m}{du}\frac{dx^n}{du} \tag{3}$$

$$\Rightarrow \quad f^s = \frac{d^2x^s}{du^2} + \Gamma^s_{mn}\frac{dx^m}{du}\frac{dx^n}{du} \tag{4}$$

By lifting the index of f_r we get a contravariant vector confirming that (4) is contravariant.

♦

2.21 p45-clarification

(2.443) and (2.444)

$$p^r \frac{\partial w}{\partial p^r} - w = C^t \quad \Rightarrow \quad w = C^t$$

By definition

$$w = a_{mn} p^m p^n \tag{1}$$

$$\Rightarrow \quad \frac{\partial w}{\partial p^r} = a_{mn} \left(\frac{\partial p^m}{\partial p^r} p^n + p^m \frac{\partial p^n}{\partial p^r} \right) \tag{2}$$

$$= a_{mn} (\delta^m_r p^n + p^m \delta^n_r) \tag{3}$$

$$= a_{rn} p^n + a_{mr} p^m \tag{4}$$

$$= 2a_{mr} p^m \quad \text{(as } a_{mn} \quad \text{is symmetric)} \tag{5}$$

$$(4) \quad \Rightarrow \quad p^r \frac{\partial w}{\partial p^r} = 2a_{mr} p^r p^m \tag{6}$$

$$= 2w \tag{7}$$

$$(2.443) \quad \Rightarrow \quad p^r \frac{\partial w}{\partial p^r} - w = 2w - w = w = C^t \tag{8}$$

$$\Rightarrow \quad w \equiv a_{mn} p^m p^n = C^t \tag{9}$$

♦

2.22 p47-exercise

The class of all parameters u, for which the equations of a geodesic null line assume the simple form 2.445, are obtained from any one such parameter by linear transformation

$$\overline{u} = au + b$$

a and b being constants.

The simple form 2.445 is :

$$\frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \tag{1}$$

$$\tag{2}$$

The general form of a geodesic is (2.447)

$$\frac{d^2x^r}{d\overline{u}^2} + \Gamma^r_{mn}\frac{dx^m}{d\overline{u}}\frac{dx^n}{d\overline{u}} = \lambda\frac{dx^r}{d\overline{u}} \tag{3}$$

$$\tag{4}$$

So (2.447) can only of the form (2.445) if $\lambda = 0$

$$\lambda = -\frac{\frac{d^2\overline{u}}{du^2}}{\left(\frac{d\overline{u}}{du}\right)^2} = 0 \tag{5}$$

$$\tag{6}$$

We can state that $\frac{d\overline{u}}{du} \neq 0$ as $\overline{u}$ can't be a constant (being a parameter of a curve). So,

$$\frac{d^2\overline{u}}{du^2} = 0 \tag{7}$$

$$\Rightarrow \quad \frac{d\overline{u}}{du} = a \tag{8}$$

$$\Rightarrow \quad \overline{u} = au + b \tag{9}$$

♦

2.23 p47-exercise

Consider a 3-space with coordinates x, y, z and a metric form $\Phi = (dx)^2 + (dy)^2 - (dz)^2$. prove that the geodesic null lines may be represented by the equations

$$x = au + a^{,} \quad y = bu + b^{,} \quad z = cu + c^{,}$$

where u is a parameter and $a, a^{,}, b, b^{,}, c, c^{,}$ are constants which are arbitrary except for the relation $a^2 + b^2 - c^2 = 0$.

Given is

$$\Phi = (dx)^2 + (dy)^2 - (dz)^2 \tag{1}$$

From the previous exercise we have already proven that x, y, z are of the form

$$x^i = q_i u + q_i^{,} \tag{2}$$

To be a null geodesic null line we need to have (2.448)

$$a_{mn} \frac{dx^m}{du} \frac{dx^n}{du} = 0 \tag{3}$$

$$\text{from (1) we deduce} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4}$$

$$(3) \quad \Rightarrow \quad (dx)^2 + (dy)^2 - (dz)^2 = 0 \tag{5}$$

$$(2) \quad \Rightarrow \quad (q_1)^2 + (q_2)^2 - (q_3)^2 = 0 \tag{6}$$

♦

2.24 p48-exercise

Prove that the Christoffel symbols of the first kind transform according the equation

$$[mn,r]' = [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^s}{\partial x'^r} + a_{pq}\frac{\partial x^p}{\partial x'^r}\frac{\partial^2 x^q}{\partial x'^m \partial x'^n}$$

From 2.438 page 45, we have

$$f_r \equiv a_{rm}\frac{d^2x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \quad \text{is covariant} \tag{1}$$

$$\Rightarrow \quad f'_r = f_s\frac{\partial x^s}{\partial x'^r} \tag{2}$$

$$\text{with} \quad f'_r = a'_{rm}\frac{d^2x'^m}{du^2} + [mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} \tag{3}$$

Combining (1), (2) and (3) gives

$$a'_{rm}\frac{d^2x'^m}{du^2} + [mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} = (a_{sm}\frac{d^2x^m}{du^2} + [mn,s]\frac{dx^m}{du}\frac{dx^n}{du})\frac{\partial x^s}{\partial x'^r} \tag{4}$$

We rewrite (4) as

$$[mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} = -\underbrace{a'_{rm}\frac{d^2x'^m}{du^2}}_{(*)} + \underbrace{a_{sm}\frac{d^2x^m}{du^2}\frac{\partial x^s}{\partial x'^r}}_{(**)} + \underbrace{[mn,s]\frac{dx^m}{du}\frac{dx^n}{du}\frac{\partial x^s}{\partial x'^r}}_{(***)} \tag{5}$$

$$(***) \Leftrightarrow [mn,s]\frac{\partial x^m}{\partial x'^p}\frac{dx'^p}{du}\frac{\partial x^n}{\partial x'^q}\frac{dx'^q}{du}\frac{\partial x^s}{\partial x'^r} \tag{6}$$

In (6) renaming the dummy indices m, n, p, q gives

$$(***) \Leftrightarrow [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{dx'^m}{du}\frac{\partial x^q}{\partial x'^n}\frac{dx'^n}{du}\frac{\partial x^s}{\partial x'^r} \tag{7}$$

$$\Leftrightarrow [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^s}{\partial x'^r}\left(\frac{dx'^m}{du}\frac{dx'^n}{du}\right) \tag{8}$$

Also,

$$(**) \Leftrightarrow a_{sm}\frac{d^2x^m}{du^2}\frac{\partial x^s}{\partial x'^r} \tag{9}$$

$$\text{As we have also} \quad \frac{d^2x^m}{du^2} = \frac{d(\frac{\partial x^m}{\partial x'^p}\frac{dx'^p}{du})}{du} \tag{10}$$

$$= \frac{\partial x^m}{\partial x'^p}\frac{d^2x'^p}{du^2} + \frac{dx'^p}{du}\frac{\partial^2 x^m}{\partial x'^p \partial x'^q}\frac{dx'^q}{du} \tag{11}$$

(11) and (9) gives by changing the dummy indices ($m \to t,\ p \to m,\ q \to n$)

$$(**) = \underbrace{a_{st} \frac{\partial x^t}{\partial x^{,p}} \frac{d^2 x^{,p}}{du^2} \frac{\partial x^s}{\partial x^{,r}}}_{(****)} + a_{pq} \frac{\partial^2 x^q}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^p}{\partial x^{,r}} \left(\frac{dx^{,m}}{du} \frac{dx^{,m}}{du} \right) \tag{12}$$

$$\text{with } (****) \quad = a_{st} \left(\frac{\partial x^t}{\partial x^{,m}} \frac{\partial x^s}{\partial x^{,r}} \right) \frac{d^2 x^{,m}}{du^2} \tag{13}$$

But a_{st} is a covariant tensor, so

$$a^{,}_{rm} = a_{st} \frac{\partial x^t}{\partial x^{,m}} \frac{\partial x^s}{\partial x^{,r}} \tag{14}$$

$$(13) \text{ becomes} \quad (****) = a^{,}_{rm} \frac{d^2 x^{,m}}{du^2} \tag{15}$$

$$\text{and from (5) we have} \quad (*) = -a^{,}_{rm} \frac{d^2 x^{,m}}{du^2} \tag{16}$$

$$\tag{17}$$

and both terms cancel each other in equation (5). So adding $(*), (**)$ and $(***)$ in (5) , we get

$$[mn, r]^{,} \frac{dx^{,m}}{du} \frac{dx^{,n}}{du} = a_{pq} \frac{\partial^2 x^q}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^p}{\partial x^{,r}} \left(\frac{dx^{,m}}{du} \frac{dx^{,m}}{du} \right) + [pq, s] \frac{\partial x^p}{\partial x^{,m}} \frac{\partial x^q}{\partial x^{,n}} \frac{\partial x^s}{\partial x^{,r}} \left(\frac{dx^{,m}}{du} \frac{dx^{,n}}{du} \right) \tag{18}$$

$$\Rightarrow [mn, r]^{,} = [pq, s] \frac{\partial x^p}{\partial x^{,m}} \frac{\partial x^q}{\partial x^{,n}} \frac{\partial x^s}{\partial x^{,r}} + a_{pq} \frac{\partial x^p}{\partial x^{,r}} \frac{\partial^2 x^q}{\partial x^{,m} \partial x^{,n}} \tag{19}$$

♦

2.25 p50-clarification 2.515

$$\frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du} S^r + T_r \frac{dS^r}{du} = \left(\frac{dT_r}{du} - \Gamma^m_{rn} T_m \frac{dx^n}{du}\right) S^r$$

with

$$\frac{\delta T_r}{\delta u} = \frac{dT_r}{du} - \Gamma^m_{rn} \frac{dx^n}{du} T_m$$

a covariant vector.

It is given that S^r is a tensor propagated parallelly along the curve. The by (2.5212) we have

$$\frac{dS^r}{du} + \Gamma^r_{mn} S^m \frac{dx^n}{du} = 0 \tag{1}$$

$$\frac{dS^r}{du} = -\Gamma^r_{mn} S^m \frac{dx^n}{du} \tag{2}$$

$$\text{and} \quad \frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du} S^r + T_r \frac{dS^r}{du} \tag{3}$$

$$= \frac{dT_r}{du} S^r - \Gamma^r_{mn} S^m \frac{dx^n}{du} T_r \tag{4}$$

$$\tag{5}$$

Swap dummy indices r and m in the second term:

$$\frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du} S^r - \Gamma^m_{rn} S^r \frac{dx^n}{du} T_m \tag{6}$$

$$= \left(\frac{dT_r}{du} - \Gamma^m_{rn} \frac{dx^n}{du} T_m\right) S^r \tag{7}$$

$$\tag{8}$$

As $T_r S^r$ is an invariant and thus is also $\frac{d(T_r S^r)}{du}$ and as S^r can be chosen arbitrarily (as long it is a contravariant tensor), implies that

$$\frac{dT_r}{du} - \Gamma^m_{rn} \frac{dx^n}{du} T_m$$

is covariant and thus also

$$\frac{\delta T_r}{\delta u} = \frac{dT_r}{du} - \Gamma^m_{rn} \frac{dx^n}{du} T_m$$

♦

2.26 p50-clarification 2.516

$$\frac{\delta T_{rs}}{\delta u} \equiv \frac{dT_{rs}}{du} - \Gamma^m_{rn} T_{ms} \frac{dx^n}{du} - \Gamma^m_{sn} T_{rm} \frac{dx^n}{du}$$

is a covariant vector.

We build an invariant $T_{rs}S^rU^s$ with S^r and U^s arbitrary contravariant tensors. The we know that $\frac{d(T_{rs}S^rU^s)}{du}$ is also an invariant. We have

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = \frac{dT_{rs}}{du} S^rU^s + T_{rs}\frac{dS^r}{du}U^s + T_{rs}S^r\frac{dU^s}{du} \tag{1}$$

with with S^r and U^s propagated parallelly along the curve. Then,

$$\frac{dS^r}{du} = -\Gamma^r_{mn} S^m \frac{dx^n}{du} \tag{2}$$

$$\frac{dU^s}{du} = -\Gamma^s_{mn} Y^m \frac{dx^n}{du} \tag{3}$$

(2), (3) in (1) gives

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = \frac{dT_{rs}}{du} S^rU^s - T_{rs}U^s\Gamma^r_{mn}S^m\frac{dx^n}{du} - T_{rs}S^r\Gamma^s_{mn}U^m\frac{dx^n}{du} \tag{4}$$

Changing the dummy indices in the second and third term gives:

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = (\frac{dT_{rs}}{du} - \Gamma^m_{rn}T_{ms}\frac{dx^n}{du} - \Gamma^m_{sn}T_{rm}\frac{dx^n}{du})S^rU^s \tag{5}$$

As the left term is an invariant and S^r and U^s are arbitrary contravariant tensors, means that the expression in the brackets in the right part of the equation, is a covariant tensor.

♦

2.27 p51-exercise

Find the absolute derivative of T^r_{st}.

Define the invariant $I = D^r_{st} R^r S_s T_t$

$$I = D^r_{st} R^r S_s T_t \tag{1}$$
$$\Rightarrow A = \frac{dI}{du} = \frac{d\left(D^r_{st}\right)}{du} R^r S_s T_t + \ D^r_{st} S_s T_t \frac{d\left(R^r\right)}{du} + \ D^r_{st} R_r T_t \frac{d\left(S^s\right)}{du} + \ D^r_{st} R_r S_s \frac{d\left(T^t\right)}{du} \tag{2}$$

Reminder, performing a parallel propagation of a covariant and contravariant vector gives as equations

$$\frac{dV^v}{du} = -\Gamma^v_{mn} V^m \frac{dx^n}{du} \tag{3}$$
$$\frac{dW_w}{du} = +\Gamma^m_{wn} W^m \frac{dx^n}{du} \tag{4}$$

So (2) becomes:

$$A = \begin{cases} \frac{d\left(D^r_{st}\right)}{du} R^r S_s T_t \\ \\ -D^r_{st} S_s T_t \Gamma^r_{mn} R^m \frac{dx^n}{du} \\ \\ +D^r_{st} R_r T_t \Gamma^m_{sn} S^m \frac{dx^n}{du} \\ \\ +D^r_{st} R_r S_s \Gamma^m_{tn} T^m \frac{dx^n}{du} \end{cases} \tag{5}$$

In (5) apply the following renaming of dummy variables

$$\begin{cases} 2^{nd} line: & r \rightarrow m, m \rightarrow r \\ 3^{rd} line: & s \rightarrow m, m \rightarrow s \\ 4^{th} line: & t \rightarrow m, m \rightarrow t \end{cases}$$

and regrouping terms with $R^r S_s T_t$,(5) becomes then

$$A = \left[\frac{d\left(D^r_{st}\right)}{du} + (\ D^r_{mt} \Gamma^s_{mn} + \ D^r_{sm} \Gamma^t_{mn} - \ D^m_{st} \Gamma^m_{rn}) \frac{dx^n}{du}\right] R^r S_s T_t \tag{6}$$

But A is an invariant, so the expression in the square parenthesis is a tensor of the form T^r_{st} and we define the absolute derivative of T^r_{st} as:

$$\frac{\delta T^r_{st}}{\delta u} = \frac{d\left(T^r_{st}\right)}{du} + \Gamma^s_{mn} T^r_{mt} \frac{dx^n}{du} + \Gamma^t_{mn} T^r_{sm} \frac{dx^n}{du} - \Gamma^m_{rn} T^m_{st} \frac{dx^n}{du}$$

♦

2.28 p53-exercise

Prove that

$$\delta^r_{s|t} = 0, \quad a^{rs}_{|t} = 0$$

i)$\delta^t_{s|t} = 0$

$$(2.524) \text{ gives:} \quad \delta^r_{s|t} = \underbrace{\frac{\partial \delta^r_s}{\partial x^t}}_{=0} + \Gamma^r_{mt}\delta^m_s - \Gamma^m_{st}\delta^r_m \tag{1}$$

$$= \Gamma^r_{st} - \Gamma^r_{st} = 0 \tag{2}$$

ii)$a^{rs}_{|t} = 0$ We know that

$$\delta^r_{s|t} = a_{sk}a^{kr}|_t \tag{3}$$

$$\Rightarrow \delta^r_{s|t} = \frac{\partial a_{sk}}{\partial x^t}a^{kr} + a_{sk}\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a_{sk}a^{km} - \Gamma^m_{st}a_{mk}a^{kr} \tag{4}$$

Rearrange (4) and add $\Gamma^k_{mt}a_{ks}a^{mr}$ and subtract $\Gamma^m_{kt}a_{ms}a^{kr}$ (as $\Gamma^k_{mt}a_{ks}a^{mr} - \Gamma^m_{kt}a_{ms}a^{kr} = 0$)

$$\delta^r_{s|t} = (\frac{\partial a_{sk}}{\partial x^t} - \Gamma^m_{st}a_{mk} - \Gamma^m_{kt}a_{ms})a^{kr} + (\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a^{km} + \Gamma^k_{mt}a^{mr})a_{sk} \tag{5}$$

$$\text{but} \quad a_{sk|t} = (\frac{\partial a_{sk}}{\partial x^t} - \Gamma^m_{st}a_{mk} - \Gamma^m_{kt}a_{ms}) \tag{6}$$

$$\text{and as (2.526)} \quad a_{sk|t} = 0 \tag{7}$$

$$\text{(5) becomes} \quad \delta^r_{s|t} = \underbrace{(\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a^{km} + \Gamma^k_{mt}a^{mr})}_{a^{kr}_{|t}} a_{sk} \tag{8}$$

$$= a^{kr}_{|t}a_{sk} \tag{9}$$

$$= 0 \quad \text{as} \quad \delta^r_{s|t} = 0 \quad \text{(see first part of this exercise)} \tag{10}$$

As all a_{ks} can't be zero and as we didn't choose any special Riemannian space, we can conclude from $a^{kr}_{|t}a_{sk} = 0$ that

$$a^{rs}_{|t} = 0$$

♦

2.29 p54-exercise

Prove that

$$\frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s}$$

$$\frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{d\lambda^n}{ds} \tag{1}$$

By definition of the absolute derivative, we have:

$$\frac{\delta\lambda^n}{\delta s} = \frac{d\lambda^n}{ds} + \Gamma^n_{pk}\lambda^p\frac{dx^k}{ds} \tag{2}$$

$$\text{(2) in (1)} \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m(\frac{\delta\lambda^n}{\delta s} - \Gamma^n_{pk}\lambda^p\frac{dx^k}{ds}) \tag{3}$$

$$= \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2a_{mn}\lambda^m\Gamma^n_{pk}\lambda^p\frac{dx^k}{ds} \tag{4}$$

$$\text{we have} \quad \Gamma^n_{pk} = a^{ns}[pk,s] \tag{5}$$

$$\text{so,} \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2\underbrace{a_{mn}a^{ns}}_{=\delta^s_m}[pk,s]\lambda^m\lambda^p\frac{dx^k}{ds} \tag{6}$$

$$= \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2[pk,m]\lambda^m\lambda^p\frac{dx^k}{ds} \tag{7}$$

$$\text{but} \quad 2[pk,m]\lambda^m\lambda^p = [pk,m]\lambda^m\lambda^p + [pk,m]\lambda^m\lambda^p \tag{8}$$

$$= [pk,m]\lambda^m\lambda^p + [mk,p]\lambda^m\lambda^p \tag{9}$$

$$\text{we have also} \quad \begin{cases} [pk,m] = \frac{1}{2}(\partial_k a_{pm} + \partial_p a_{km} - \partial_m a_{pk}) \\ [mk,p] = \frac{1}{2}(\partial_k a_{pm} + \partial_m a_{pk} - \partial_p a_{mk}) \end{cases} \tag{10}$$

$$\Rightarrow 2[pk,m] = \partial_k a_{pm} \tag{11}$$

$$\text{so} \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n - \underbrace{\partial_k a_{pm}\frac{dx^k}{ds}}_{=\frac{da_{mn}}{ds}}\lambda^m\lambda^p + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} \tag{12}$$

$$\Rightarrow \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} \tag{13}$$

♦

2.30 p54-exercise

Prove that

$$(T^r S_s)_{|n} = T^r_{|n} S_s + T^r S_{|n}$$

$$\begin{aligned}
(T^r S_s)_{|n} &= \partial_n (T^r S_s) + \Gamma^r_{nm} T^m S_s - \Gamma^m_{sn} T^r S_m && (1)\\
&= \partial_n (T^r) S_s + T^r \partial_n (S_s) + S_s \Gamma^r_{nm} T^m - T^r \Gamma^m_{sn} S_m && (2)\\
&= T^r \underbrace{(\partial_n (S_s) - \Gamma^m_{sn} S_m)}_{S_{s|n}} + S_s \underbrace{(\partial_n (T^r) + \Gamma^r_{nm} T^m)}_{T^r_{|n}} && (3)\\
&= T^r S_{s|n} + S_s T^r_{|n} && (4)
\end{aligned}$$

♦

2.31 p57-exercise

Compute the Christoffel symbols in 2.540 directly from the definitions 2.421 and 2.422. Check that all Christoffels symbols not shown explicitly in 2.540 vanish.

Easy but very tedious, not reproduced yet, later perhaps

♦

2.32 p57-exercise

Show that for the spherical polar metric 2.532, we have $ln\sqrt{a} = 2ln(x^1) + ln(sin(x^2))$ and

$$\textbf{2.544} \quad \Gamma^n_{1n} = \frac{2}{x^1}, \quad \Gamma^n_{2n} = \cot(x^2), \quad \Gamma^n_{3n} = 0$$

The spherical polar metric 2.532 is,

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1\sin(x^2))^2 \end{pmatrix} \tag{1}$$

$$\Rightarrow |a_{mn}| = \left[(x^1)^2\sin(x^2)\right]^2 \tag{2}$$

$$\Rightarrow ln(\sqrt{|a_{mn}|}) = 2ln(x^1) + ln(\sin(x^2)) \tag{3}$$

$$\Rightarrow \quad \begin{cases} \Gamma^n_{1n} = \partial_1(ln(\sqrt{a})) = \frac{2}{x^1} \\ \\ \Gamma^n_{2n} = \partial_2(ln(\sqrt{a})) = \frac{\cos(x^2)}{\sin(x^2)} = \cot(x^2) \\ \\ \Gamma^n_{3n} = 0 \end{cases} \tag{4}$$

♦

2.33 p58-exercise

Show that for the spherical polar metric

$$\textbf{2.546} \quad T^n_{\ |n} = \frac{1}{r^2}\partial_r(r^2\ T^1) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\ T^2) + \partial_\phi T^3$$

Obtain a similar expression for the "Laplacian" ΔV of an invariant V defined as

$$\textbf{2.547} \quad \Delta V = (a^{mn}\partial_m V)_{|n}$$

We have

$$\textbf{(2.545)} \quad T^n_{\ |n} = \frac{1}{\sqrt{a}}\partial_n(\sqrt{a}\ T^n) \tag{1}$$

$$\text{and from the previous exercise p.58:} \quad \sqrt{a} = (x^1)^2\sin\big(x^2\big) \tag{2}$$

$$\Rightarrow \quad T^n_{\ |n} = \frac{1}{\sqrt{a}}\left(\sin\big(x^2\big)\partial_1[(x^1)^2T^1] + (x^1)^2\partial_2[\sin\big(x^2\big)T^2] + (x^1)^2\sin\big(x^2\big)\partial_3T^3\right) \tag{3}$$

$$= \frac{1}{x^1}\partial_1[(x^1)^2T^1] + \frac{1}{\sin(x^2)}\partial_2[\sin\big(x^2\big)T^2] + \partial_3T^3 \tag{4}$$

Replace in (4) $\quad x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$

$$T^n_{\ |n} = \frac{1}{r^2}\partial_r[r^2\ T^r] + \frac{1}{\sin\theta}\partial_\theta[\sin\theta\ T^\theta] + \partial_\phi T^\phi \tag{5}$$

Let's calculate the Laplacian.

$$\Delta V = (a^{mn}\partial_m V)_{|n} \tag{6}$$

$$\text{be} \qquad G^n = a^{mn}\partial_m V \tag{7}$$

$$\text{then (see exercise p.32)} \qquad \begin{cases} G^1 = \partial_r V \\ G^2 = \frac{1}{r^2}\partial_\theta V \\ G^1 = \frac{1}{r^2\sin^2\theta}\partial_\phi V \end{cases} \tag{8}$$

and by the previous result of this exercise

$$\Delta V = G^n_{|n} = \frac{1}{r^2}\partial_r[r^2\ G^1] + \frac{1}{\sin\theta}\partial_\theta[\sin\theta\ G^2] + \partial_\phi G^3 \tag{9}$$

$$= \frac{1}{r^2}\partial_r[r^2\partial_r V] + \frac{1}{r^2\sin\theta}\partial_\theta[\sin\theta\ \partial_\theta V] + \frac{1}{r^2\sin^2\theta}\partial^2_\phi V \tag{10}$$

♦

2.34 p60 - clarification for 2.609

$$A^r_{.mns} = -\partial_s \Gamma^r_{mn} + 2\Gamma^r_{sp}\Gamma^p_{mn}$$

We have (2.608):

$$\frac{d^2x^r}{ds^2} = -\Gamma^r_{mn}p^m p^n \tag{1}$$

$$\Rightarrow \quad \frac{d^3x^r}{ds^3} = -\frac{d\Gamma^r_{mn}}{ds}p^m p^n - \Gamma^r_{mn}\left(p^m \frac{dp^n}{ds} + p^n \frac{dp^m}{ds}\right) \tag{2}$$

$$= -\partial_k \Gamma^r_{mn} \underbrace{\frac{dx^k}{du}}_{=p^k} p^m p^n - \Gamma^r_{mn}\left(p^m \frac{dp^n}{ds} + p^n \frac{dp^m}{ds}\right) \tag{3}$$

$$\text{as} \quad \frac{dp^g}{ds} = \frac{d^2x^g}{ds^2} = -\Gamma^g_{ik}p^i p^k \quad \Rightarrow \quad = -\partial_k \Gamma^r_{mn} \underbrace{\frac{dx^k}{ds}}_{=p^k} p^m p^n + \Gamma^r_{mn}p^m \Gamma^n_{ik}p^i p^k + \Gamma^r_{mn}p^n \Gamma^m_{ki}p^k p^i \tag{4}$$

In the second and third terms, rename the dummy indices : $m \leftrightarrow k, n \leftrightarrow i$ and $m \leftrightarrow i, n \leftrightarrow k$.
So (4) becomes

$$\frac{d^3x^r}{ds^3} = -\partial_k \Gamma^r_{mn}p^m p^n p^k + \Gamma^r_{ki}p^k \Gamma^i_{nm}p^n p^m + \Gamma^r_{ik}p^k \Gamma^i_{nm}p^n p^m \tag{5}$$

$$= \left(-\partial_k \Gamma^r_{mn} + \Gamma^r_{ik}\Gamma^i_{mn} + \Gamma^r_{ik}\Gamma^i_{nm}\right) p^m p^n p^k \tag{6}$$

$$= \left(-\partial_k \Gamma^r_{mn} + 2\Gamma^r_{ik}\Gamma^i_{mn}\right) p^m p^n p^k \tag{7}$$

♦

2.35 p62-exercise

Prove that if a pair of vectors are unit orthogonal vectors at a point on a curve, and if they are both propagated parallelly along the curve, then they remain unit orthogonal vectors along the curve.

Given is, a pair of vectors U^m and V^m which are unit orthogonal vectors at a point on a curve. So,

$$\begin{array}{rl} \text{U is a unit vector (2.302)} & a_{mn}U^mU^n = \epsilon \\ \text{V is a unit vector (2.302)} & a_{mn}V^mV^n = \epsilon \\ \text{U, V are orthogonal (2.317)} & a_{mn}U^mV^n = 0 \end{array} \tag{1}$$

at one point on the curve.
We have to prove that the above properties are valid along the curve (i.e. $\forall$ points on the curve) provided that the vectors are propagated // along the curve, which means

$$\textbf{(2.512)} \qquad \frac{\delta U^r}{\delta u} = \frac{dU^r}{du} + \Gamma^r_{mn}U^m\frac{dx^n}{du} = 0 \tag{2}$$

for both vectors U, V. Thus,

$$\frac{dU^r}{du} = -\Gamma^r_{mn}U^m\frac{dx^n}{du} \tag{3}$$

i) Consider the magnitude M at a random point on the curve

$$M = a_{mn}U^mU^n \tag{4}$$

$$\Rightarrow \qquad \frac{dM}{ds} = \frac{da_{mn}}{du}U^mU^n + 2a_{mn}U^m\frac{dU^n}{du} \tag{5}$$

Obviously M and $\frac{dM}{ds}$ are invariants. Also, we can choose at any point on the curve a Riemannian coordinate system (RCS) for which the Christoffel symbols vanish at that point. Hence, $\frac{dU^r}{du} = 0$ at that point and the second term in the right part of (5) vanish. (5) becomes then,

$$\frac{dM}{ds} = \frac{\partial a'_{mn}}{\partial x'^k}\frac{dx'^k}{ds}U'^mU'^n \tag{6}$$

We also know **(2.425. page 41)** that $[km,n]' + [kn,m]' = \frac{\partial a'_{mn}}{\partial x'^k}$. But in the chosen coordinate system, $[km,n] = 0$ at the origin of this coordinate system. So by (6) we get $\frac{dM}{ds} = 0$.
So the magnitude is constant along the curve and as we know that a certain point $M = 1$:

U,V are unit vectors along the curve

ii) Consider now the angle between the vectors U, V. Be $A = \cos\ \theta$.By definition

$$A = a_{mn} U^m V^n \tag{7}$$

$$\Rightarrow \quad \frac{dA}{ds} = \frac{da_{mn}}{du} U^m V^n + a_{mn}(V^m \frac{dU^n}{du} + U^m \frac{dV^n}{du}) \tag{8}$$

We follow the same reasoning as in i) and so

$$\frac{dA}{ds} = 0$$

So, the angle is constant and we know is is $\frac{\pi}{2}$ at a certain point. So,:

U,V are orthogonal along the curve

♦

2.36 p62-exercise

Given that λ^r is a unit vector field, prove that

$$\lambda^r{}_{|s}\lambda_r = 0 \quad \text{and} \quad \lambda^r\lambda_{r|s} = 0$$

Is the relation $\lambda^r{}_{|s}\lambda_s = 0$ true for a general unit vector field?

To simplify the calculation, we choose a random element in the unit vector field and use at that point a Riemannian coordinate system (RCS). So, we have

$$\lambda^r{}_{|s} = \partial_s\lambda^r \quad \text{and} \quad \lambda_{r|s} = \partial_s\lambda_r \tag{1}$$

$$\text{as we have a unit vector fields:} \qquad a_{mn}\lambda^m\lambda^n = 1 \tag{2}$$

$$\Leftrightarrow \lambda_n\lambda^n = 1 \qquad \text{(by lowering the index m)} \tag{3}$$

$$\Rightarrow \qquad \lambda^n\partial_s\lambda_n + \lambda_n\partial_s\lambda^n = 0 \tag{4}$$

We prove that

$$\lambda^n\partial_s\lambda_n = \lambda_n\partial_s\lambda^n \quad \forall \text{ vector fields}$$

We have the trivial identity

$$\partial_s(\lambda^r\lambda_r) = \partial_s(\lambda^r\lambda_r) \tag{5}$$

$$\Leftrightarrow \qquad \partial_s(a^{rm}\lambda_m\lambda_r) = \partial_s(a_{rm}\lambda^m\lambda^r) \tag{6}$$

Lemma *: $\partial_s a^{rm} = 0$ in a Riemannian coordinate system (i.e. at the origin)*

We have

$$a^{rm}a_{ms} = \delta^r_s \tag{7}$$

$$\Rightarrow a_{ms}\partial_k a^{rm} + a^{rm}\partial_k a_{ms} = 0 \tag{8}$$

$$\textit{we know (2.618)} \qquad \partial_r a_{mn} = 0 \qquad \textit{at the origin of a RCS} \tag{9}$$

$$\textit{so (8) becomes} \qquad a_{ms}\partial_k a^{rm} = 0 \tag{10}$$

$$\textit{multiply (10) by} \quad a^{ns} \quad \Rightarrow \underbrace{a^{ns}a_{ms}}_{=\delta^n_m}\partial_k a^{rm} = 0 \tag{11}$$

$$\Rightarrow \qquad \partial_k a^{rn} = 0 \tag{12}$$

◇

Now, expanding (6) and using **2.618** and the lemma:

$$a^{rm}\lambda_m\partial_s\lambda_r + a^{rm}\lambda_r\partial_s\lambda_m = a_{rm}\lambda^m\partial_s\lambda^r + a_{rm}\lambda^r\partial_s\lambda^m \tag{13}$$

$$\text{renaming dummy indices:} \qquad a^{rm}\lambda_r\partial_s\lambda_m = a_{rm}\lambda^r\partial_s\lambda^m \tag{14}$$

$$\Rightarrow \qquad \lambda^m\partial_s\lambda_m = \lambda_m\partial_s\lambda^m \tag{15}$$

Considering (5) and (15) we conclude:

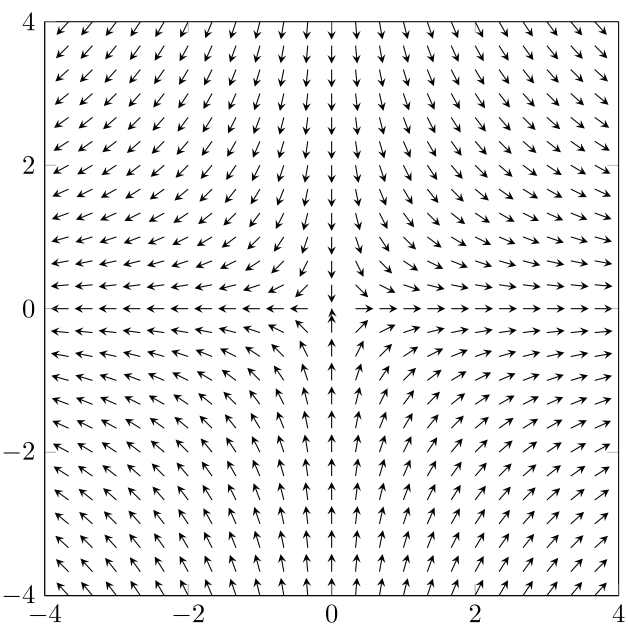
$$\lambda^n\partial_s\lambda_n = \lambda_n\partial_s\lambda^n = 0 \tag{16}$$

and as $\partial_s\lambda_n = \lambda_{n|s}$ and $\partial_s\lambda^n = \lambda^n{}_{|s}$ at the origin of the considered coordinate system, we have:

$$\lambda^m\lambda_{n|s} = \lambda_m\lambda^n{}_{|s} = 0 \tag{17}$$

◇

Is the relation $\lambda^r_{|s}\lambda_s = 0$ true for a general unit vector field?
The answer is NO. Let's consider the following unit vector field in a Cartesian Coordinate system:



$$V : \mathbb{R}^2_* \to \mathbb{R}^2 | V(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

Figure 2.4: Vector field for which $\lambda^r_{|s}\lambda_s = 0$ does not hold

Put $r = \sqrt{x^2+y^2}$, we get (as we have a Cartesian Coordinate system, the Christoffel symbols vanish and the covariant components of the vectors are equal to their contravariant part):

$$\begin{cases} V^1 = V_1 = +\frac{x}{r} \\ V^2 = V_2 = -\frac{y}{r} \end{cases} \tag{18}$$

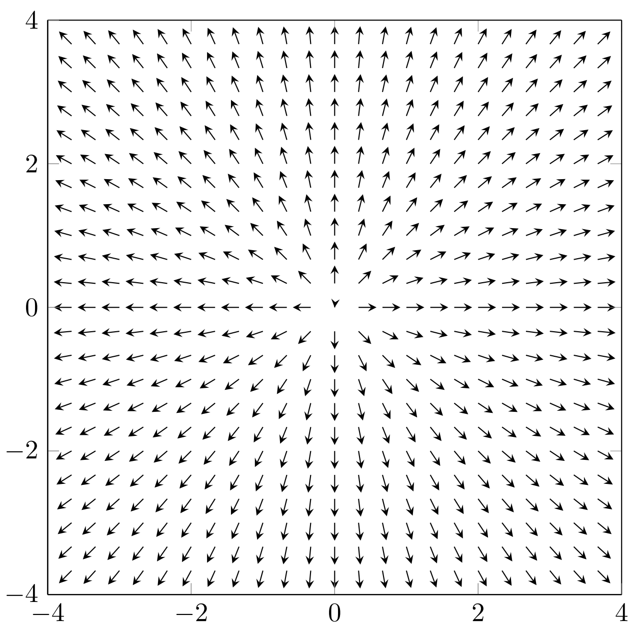
$$\begin{cases} V^1_{\ |1} = V_{1|1} = \frac{y^2}{r^3} \quad V^1_{\ |2} = V_{1|2} = -\frac{xy}{r^3} \\ A\ V^2_{\ |1} = V_{2|1} = \frac{xy}{r^3} \quad V^2_{\ |2} = V_{2|2} = -\frac{x^2}{r^3} \end{cases} \tag{19}$$

$$\Rightarrow \quad \begin{cases} V^1_{\ |s} V_s = V^1_{\ |1} V_1 + V^1_{\ |2} V_2 = \frac{y^2}{r^3}\frac{x}{r} + (-\frac{xy}{r^3})(-\frac{y}{r}) = \frac{xy^2}{r^4} \neq 0 \\ V^2_{\ |s} V_s = V^2_{\ |1} V_1 + V^2_{\ |2} V_2 = \frac{xy}{r^3}\frac{x}{r} + (-\frac{y}{r})(-\frac{y}{r}) = \frac{x^2 y}{r^4} \neq 0 \end{cases} \tag{20}$$

Just as a check, we calculate $V^r_{\ |s} V_r$ which should be zero:

$$\Rightarrow \quad \begin{cases} V^s_{\ |1} V_s = V^1_{\ |1} V_1 + V^2_{\ |1} V_2 = (+\frac{y^2}{r^3})\frac{x}{r} + (+\frac{xy}{r^3})(-\frac{y}{r}) = 0 \\ V^s_{\ |2} V_s = V^1_{\ |2} V_1 + V^2_{\ |2} V_2 = (-\frac{xy}{r^3})\frac{x}{r} + (-\frac{y}{r})(-\frac{x^2}{r^3}) = 0 \end{cases} \tag{21}$$

Now, let's consider another unit vector field in a Cartesian Coordinate system:



$$V : \mathbb{R}^2_* \to \mathbb{R}^2 | V(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

Figure 2.5: Vector field for which $\lambda^r_{\ |s}\lambda_s = 0$ hold

$$\begin{cases} V^1 = V_1 = +\frac{x}{r} \\ V^2 = V_2 = +\frac{y}{r} \end{cases} \tag{22}$$

$$\begin{cases} V^1_{|1} = V_{1|1} = \frac{y^2}{r^3} & V^1_{|2} = V_{1|2} = -\frac{xy}{r^3} \\ V^2_{|1} = V_{2|1} = -\frac{xy}{r^3} & V^2_{|2} = V_{2|2} = +\frac{x^2}{r^3} \end{cases} \tag{23}$$

$$\Rightarrow \quad \begin{cases} V^1_{|s}V_s = V^1_{|1}V_1 + V^1_{|2}V_2 = (+\frac{y^2}{r^3})\frac{x}{r} + (-\frac{xy}{r^3})(+\frac{y}{r}) = 0 \\ V^2_{|s}V_s = V^2_{|1}V_1 + V^2_{|2}V_2 = (-\frac{xy}{r^3})\frac{x}{r} + (+\frac{y}{r})(+\frac{y}{r}) = 0 \end{cases} \tag{24}$$

Just as a check, we calculate $V^r_{|s}V_r$ which should be zero:

$$\Rightarrow \quad \begin{cases} V^s_{|1}V_s = V^1_{|1}V_1 + V^2_{|1}V_2 = (+\frac{y^2}{r^3})(+\frac{x}{r}) + (-\frac{xy}{r^3})(+\frac{y}{r}) = 0 \\ V^s_{|2}V_s = V^1_{|2}V_1 + V^2_{|2}V_2 = (-\frac{xy}{r^3})(+\frac{x}{r}) + (+\frac{y}{r}) + \frac{x^2}{r^3}) = 0 \end{cases} \tag{25}$$

So, in the second example the relationship $\lambda^r_{|s}\lambda_s = 0$ holds. Question (to investigate further and later) : does the fact that in the first case $\boldsymbol{\nabla} \times \overline{V} \neq 0$ and in the second case $\boldsymbol{\nabla} \times \overline{V} = 0$, means that there is some relation with this expression?

♦

2.37 p64-clarification 2.625

2.625	$\frac{dx^r}{dx^N} = \frac{X^r}{X^N}$

Be C a surface defined by the function $F(x^1, \ldots, x^{N-1}) = C$ and $c_\perp$ the curve intersecting the surface C perpendicularly at a point p.
Along the curve at that point p we have

$$\text{as} \quad \begin{cases} \frac{dx^r}{ds} & \text{is the tangent vector along} \quad c_\perp \\ X^r = a^{rn}\frac{\partial F}{\partial x^n} & \text{is orthogonal on the surface (2.623)} \quad C \end{cases} \tag{1}$$

$$\Rightarrow \quad \frac{dx^r}{ds} = kX^r \tag{2}$$

So, $\frac{dx^r}{ds}$ is proportional to X^r (as the curve intersects the surface orthogonally). This means that also all the components (coordinates) of both quantities are proportional. And so,

$$\frac{\frac{dx^r}{ds}}{\frac{dx^N}{ds}} = \frac{kX^r}{kX^N} \tag{3}$$

$$\Rightarrow \quad \frac{dx^r}{dx^N} = \frac{X^r}{X^N} \tag{4}$$

♦

2.38 p65-exercise

Deduce from **2.629** that

$$a^{N\rho} = 0 \qquad a^{NN} = \frac{1}{a_{NN}}$$

We have (see 2.629):

$$a_{N\rho} = 0 \tag{1}$$

$$\text{and also} \qquad a_{Nm}a^{ms} = \delta^s_N \tag{2}$$

In (2) split the m index in the subspace and the remaining coordinate N

$$a_{N\rho}a^{\rho s} + a_{NN}a^{Ns} = \delta^s_N \tag{3}$$

$$\text{as} \quad a_{N\rho} = 0 \Rightarrow \qquad a_{NN}a^{Ns} = \delta^s_N \tag{4}$$

Case 1: $s \neq N$

$$a_{NN}a^{Ns} = 0 \quad \Leftrightarrow \quad a_{NN}a^{N\rho} = 0 \qquad (\text{as} \quad s \neq N) \tag{5}$$

$$\text{as we suppose} \quad a_{NN} \neq 0 \quad \Rightarrow \quad a^{N\rho} = 0 \tag{6}$$

Case 2: $s = N$

$$a_{NN}a^{NN} = 1 \tag{7}$$

$$\Rightarrow \quad a^{NN} = \frac{1}{a_{NN}} \tag{8}$$

♦

2.39 p69-clarification on 2.645

In 2.645 we have

$$T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\alpha\beta} T_N$$

and

$$T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N$$

Indeed,

$$\begin{aligned}
&& T^N &= a^{mN} T_m && (1)\\
&& &= a^{\alpha N} T_\alpha + a^{NN} T_N && (2)\\
&\text{but (2.631)} & a^{\alpha N} &= 0 && (3)\\
&\Rightarrow & T^N &= a^{NN} T_N && (4)\\
\text{as} \quad a^{NN} = \frac{1}{a_{NN}} &\Rightarrow & T_{\alpha|\beta} &= T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N && (5)
\end{aligned}$$

♦

2.40 p69-exercise

Show that

$$\textbf{2.648} \qquad T^{\alpha}_{\ |\beta} = T^{\alpha}_{\ ||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N$$

$$\textbf{2.649} \qquad T^{N}_{\ |\alpha} = \partial_\alpha T^N - \frac{1}{2a_{NN}}\partial_N a_{\alpha\mu}T^\mu + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$$

$$\textbf{2.650} \qquad T^{\alpha}_{\ |N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$$

i) $T^{\alpha}_{\ |\beta} = T^{\alpha}_{\ ||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N$

$$\textbf{(2.520)} \quad \Rightarrow \quad T^{\alpha}_{\ |\beta} = \partial_\beta T^\alpha + \Gamma^{\alpha}_{m\beta}T^m \qquad (m = 1, \ldots, N) \tag{1}$$

$$\Leftrightarrow \quad T^{\alpha}_{\ |\beta} = \underbrace{\partial_\beta T^\alpha + \Gamma^{\alpha}_{\mu\beta}T^\mu}_{T^{\alpha}_{\ ||\beta}} + \Gamma^{\alpha}_{N\beta}T^N \tag{2}$$

$$\textbf{(2.639)} \quad \Gamma^{\alpha}_{N\beta} = \frac{1}{2}a^{\alpha\mu}\partial_N a^{\mu\beta} \tag{3}$$

$$\text{(2) and (3):} \quad T^{\alpha}_{\ |\beta} = T^{\alpha}_{\ ||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N \tag{4}$$

$\diamond$

Remark: We also use **(2.639)** for the two other identities.

ii) $T^{N}_{\ |\alpha} = \partial_\alpha T^N - \frac{1}{2a_{NN}}\partial_N a_{\alpha\mu}T^\mu + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$

$$\textbf{(2.520)} \quad \Rightarrow \quad T^{N}_{\ |\alpha} = \partial_\alpha T^N + \Gamma^{N}_{m\alpha}T^m \qquad (m = 1, \ldots, N) \tag{5}$$

$$\Leftrightarrow \quad T^{N}_{\ |\alpha} = \partial_\beta T^\alpha + \underbrace{\Gamma^{N}_{\sigma\alpha}}_{-\frac{1}{2a_{NN}}\partial_N a_{\sigma\alpha}} T^\sigma + \underbrace{\Gamma^{N}_{N\alpha}}_{\frac{1}{2a_{NN}}\partial_\alpha a_{NN}} T^N \tag{6}$$

$$\Rightarrow \quad T^{N}_{\ |\alpha} = \partial_\beta T^\alpha - \frac{1}{2a_{NN}}\partial_N a_{\sigma\alpha}T^\sigma + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N \tag{7}$$

iii) $T^{\alpha}_{\ |N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$

$$\textbf{(2.520)} \quad \Rightarrow \quad T^{\alpha}_{\ |N} = \partial_N T^\alpha + \Gamma^{\alpha}_{mN}T^m \qquad (m = 1, \ldots, N) \tag{8}$$

$$\Leftrightarrow \quad T^{\alpha}_{\ |N} = \partial_N T^\alpha + \underbrace{\Gamma^{\alpha}_{\sigma N}}_{\frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}} T^\sigma + \underbrace{\Gamma^{\alpha}_{NN}}_{-\frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}} T^N \tag{9}$$

$$\Rightarrow \quad T^{\alpha}_{\ |N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N \tag{10}$$

$\blacklozenge$

2.41 p71-exercise

Write down equation 2.643 tot 2.650 for the special cae of a geodesic normal coordinate system.

$$
\begin{aligned}
\textbf{(2.643)} \qquad T_{\alpha||\beta} &= \partial_\beta T_\alpha - \Gamma^\gamma_{\alpha\beta} T_\gamma \qquad \text{(does not change)} && (1)\\
\textbf{(2.644)} \qquad T_{\alpha|\beta} &= T_{\alpha||\beta} - \underbrace{\Gamma^N_{\alpha\beta}}_{=\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}} T_N && (2)\\
&= T_{\alpha||\beta} + \frac{1}{2}\epsilon \partial_N a_{\alpha\beta} T_N && (3)\\
\textbf{(2.645)} \qquad T_{\alpha|\beta} &= T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N \qquad \text{(does not change)} && (4)\\
\textbf{(2.646)} \qquad T_{N|\alpha} &= \partial_\alpha T_N - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N && (5)\\
&= \partial_\alpha T_N - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu && (6)\\
\textbf{(2.647)} \qquad T_{\alpha|N} &= \partial_N T_\alpha - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N && (7)\\
&= \partial_N T_\alpha - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu && (8)\\
\textbf{(2.648)} \qquad T^\alpha_{\ |\beta} &= T^\alpha_{\ ||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta} T_N \qquad \text{(does not change)} && (9)\\
\textbf{(2.649)} \qquad T^N_{\ |\alpha} &= \partial_\alpha T^N - \frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\frac{1}{a_{NN}}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N && (10)\\
&= \partial_\alpha T^N - \frac{1}{2}\epsilon \partial_N a_{\mu\alpha} T^\mu && (11)\\
\textbf{(2.650)} \qquad T^\alpha_{\ |N} &= \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma} T^\sigma - \frac{1}{2}a^{\alpha\mu}\underbrace{\partial_\mu a_{NN}}_{=0} T^N && (12)\\
&= \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma} T^\sigma && (13)
\end{aligned}
$$

To investigate: note (3) and (4) which suggest that $\epsilon T_N = T^N$. Prove formally?

♦

2.42 p73-Clarification 2.706

... Let us now define a unit vector $\lambda^r_{(2)}$ and a positive invariant $\kappa_{(2)}$ by the equation

$$\begin{cases} \frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \epsilon\epsilon_{(1)}\kappa_{(1)}\lambda^r \\ \epsilon_{(2)}\lambda^n_{(2)}\lambda_{(2)n} = 1 \end{cases} \tag{1}$$

We can state that $\kappa_{(2)}$ is an invariant but one has to check whether the expression (1) implies that $\kappa_{(2)}$ is indeed invariant.
What we know is that $\lambda^r, \frac{\delta\lambda^r}{\delta s}, \frac{\delta\lambda^r_{(1)}}{\delta s}, \lambda^r_{(2)}$ are contravariant vectors. Also $\kappa_{(1)}$ is an invariant as $\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ and the magnitude of $\frac{\delta\lambda^r}{\delta s}$ does not depend on the coordinate system. So,

$$(1) \times \lambda_{(2)r} \quad \Rightarrow \quad \frac{\delta\lambda^r_{(1)}}{\delta s}\lambda_{(2)r} = \kappa_{(2)}\lambda^r_{(2)}\lambda_{(2)r} - \epsilon\epsilon_{(1)}\kappa_{(1)}\lambda^r\lambda_{(2)r} \tag{2}$$

$$\Rightarrow \quad \kappa_{(2)}\underbrace{\lambda^r_{(2)}\lambda_{(2)r}}_{\text{invariant}} = \underbrace{\frac{\delta\lambda^r_{(1)}}{\delta s}\lambda_{(2)r}}_{\text{invariant}} + \underbrace{\epsilon\epsilon_{(1)}}_{\text{invariant}}\underbrace{\kappa_{(1)}}_{\text{invariant}}\underbrace{\lambda^r\lambda_{(2)r}}_{\text{invariant}} \tag{3}$$

$$\Rightarrow \quad \kappa_{(2)} = \text{invariant} \tag{4}$$

♦

2.43 p74-Clarification 2.710

2.710 $$\begin{cases} \frac{\delta\lambda^r_{(M-1)}}{\delta s} = \kappa_{(M)}\lambda^r_{(M)} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\lambda^r_{(M-2)} \\ \epsilon_{(M-1)}\lambda^n_{(M-1)}\lambda_{(M-1)n} = 1 \qquad \text{(M=1,2,...,N)} \end{cases} \tag{1}$$

... It is easily proved by mathematical induction that the whole sequence of vectors defined by 2.710 are perpendicular to the tangent and to one another ...

We already know from 2.703 to 2.709 that $\lambda^r, \lambda^r_{(1)}, \lambda^r_{(2)}, \lambda^r_{(3)}$, satisfying equations (1), are all mutually perpendicular. Let us assume that the orthogonality for the set $\{\lambda^r_{(k)} : k = 0, 1, 2, 3, ..., M-1\}$ has been verified. We prove by induction that then, $\lambda^r_{(M)}$ will be orthogonal to all elements of the set.

i) Consider the set $\{\lambda^r_{(k)} : k = 0, 1, 2, 3, ..., M-3\}$ where we already know that $\lambda^r_{(k)}$ are mutually perpendicular and also $\lambda^r_{(k)} \perp \lambda^r_{(M-1)}$, $\lambda^r_{(k)} \perp \lambda^r_{(M-2)}$ and $\lambda^r_{(M-1)} \perp \lambda^r_{(M-1)} \quad \forall k$.

$$(1) \times \lambda_{(k)r} \quad \Rightarrow \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(k)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(k)r} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\lambda^r_{(M-2)}\lambda_{(k)r}}_{=0} \tag{2}$$

$$\text{We have} \quad \lambda_{(k)r}\lambda^r_{(M-1)} = 0 \tag{3}$$

$$\Rightarrow \quad \frac{\delta\lambda_{(k)r}\lambda^r_{(M-1)}}{\delta s} = \lambda^r_{(M-1)}\frac{\delta\lambda_{(k)r}}{\delta s} + \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = 0 \tag{4}$$

$$\Rightarrow \quad \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = -\lambda^r_{(M-1)}\frac{\delta\lambda_{(k)r}}{\delta s} \tag{5}$$

$$\text{We have} \quad \frac{\delta\lambda_{(k)r}}{\delta s} = \kappa_{(k+1)}\lambda_{(k+1)r} - \epsilon_{(k)}\epsilon_{(k-1)}\kappa_{(k)}\lambda_{(k-1)r} \tag{6}$$

$$(5) \text{ and } (6) \Rightarrow \quad \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = -\kappa_{(k+1)}\underbrace{\lambda_{(k+1)r}\lambda^r_{(M-1)}}_{=0} - \epsilon_{(k)}\epsilon_{(k-1)}\kappa_{(k)}\underbrace{\lambda_{(k-1)r}\lambda^r_{(M-1)}}_{=0} \tag{7}$$

$$\text{From (2)} \Rightarrow \quad \kappa_{(M)}\lambda^r_{(M)}\lambda_{(k)r} = 0 \tag{8}$$

$$\Rightarrow \quad \lambda^r_{(M)} \perp \lambda_{(k)r} \qquad \forall k = 0, 1, 2, 3, ..., M-3 \tag{9}$$

ii) Consider the case $k = M - 1$

$$(1)\times\lambda_{(M-1)r} \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-1)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-1)r} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\lambda^r_{(M-2)}\lambda_{(M-1)r}}_{=0} \tag{10}$$

$$\text{from (2.530):} \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-1)r} = \frac{1}{2}\underbrace{\frac{\delta\lambda_{(M-1)r}\lambda^r_{(M-1)}}{\delta s}}_{=0 \text{ as } \lambda_{(M-1)r}\lambda^r_{(M-1)}=\epsilon_{(M-1)}} \tag{11}$$

$$\Rightarrow \quad \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-1)r} = 0 \tag{12}$$

$$\Rightarrow \quad \lambda_{(M-1)r} \perp \lambda_{(M)r} \tag{13}$$

iii) Consider the case $k = M-2$

$$(1)\times\lambda_{(M-2)r} \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-2)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\lambda^r_{(M-2)}\lambda_{(M-2)r}}_{=\epsilon_{(M-2)}} \tag{14}$$

$$\Rightarrow \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-2)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} - \epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\epsilon_{(M-2)}\epsilon_{(M-2)}}_{=1} \tag{15}$$

$$\Rightarrow \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-2)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} - \epsilon_{(M-1)}\kappa_{(M-1)} \tag{16}$$

$$\text{We have} \quad \lambda^r_{(M-1)}\lambda_{(M-2)r} = 0 \tag{17}$$

$$\Rightarrow \quad \lambda_{(M-2)r}\frac{\delta\lambda^r_{(M-1)}}{\delta s} = -\lambda^r_{(M-1)}\frac{\delta\lambda_{(M-2)r}}{\delta s} \tag{18}$$

$$\text{We have also} \quad \frac{\delta\lambda_{(M-2)r}}{\delta s} = \kappa_{(M-1)}\lambda_{(M-1)r} - \epsilon_{(M-3)}\epsilon_{(M-2)}\kappa_{(M-2)}\lambda_{(M-3)r} \tag{19}$$

$$(19)\times\lambda^r_{(M-1)} \text{ and (18):} \quad \lambda_{(M-2)r}\frac{\delta\lambda^r_{(M-1)}}{\delta s} = -\kappa_{(M-1)}\underbrace{\lambda^r_{(M-1)}\lambda_{(M-1)r}}_{=\epsilon_{(M-1)}} - \epsilon_{(M-3)}\epsilon_{(M-2)}\kappa_{(M-2)}\underbrace{\lambda^r_{(M-3)}\lambda_{(M-1)r}}_{=0} \tag{20}$$

$$\Rightarrow \quad \lambda_{(M-2)r}\frac{\delta\lambda^r_{(M-1)}}{\delta s} = -\kappa_{(M-1)}\epsilon_{(M-1)} \tag{21}$$

$$\text{(16) and (21):} \quad -\kappa_{(M-1)}\epsilon_{(M-1)} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} - \epsilon_{(M-1)}\kappa_{(M-1)} \tag{22}$$

$$\Rightarrow \quad \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} = 0 \tag{23}$$

$$\Rightarrow \quad \lambda_{(M-2)r} \perp \lambda_{(M)r} \tag{24}$$

With, i), ii), iii) all possible cases are covered which makes the proof complete.

♦

2.44 p75-Clarification 2.714

2.714 $(\kappa_{(1)})^2 = \epsilon_{(1)} a_{mn} \frac{\delta\lambda^m}{\delta s} \frac{\delta\lambda^n}{\delta s}, \qquad \epsilon_{(1)} = \pm 1$

$$
\begin{aligned}
& & \frac{\delta\lambda^n}{\delta s} &= \kappa_{(1)} \lambda^n_{(1)} && (1)\\
(1) \times (1) \quad &\Rightarrow & \frac{\delta\lambda^m}{\delta s} \frac{\delta\lambda^n}{\delta s} &= (\kappa_{(1)})^2 \lambda^m_{(1)} \lambda^n_{(1)} && (2)\\
(2) \times a_{mn} \Rightarrow & & a_{mn} \frac{\delta\lambda^m}{\delta s} \frac{\delta\lambda^n}{\delta s} &= a_{mn} (\kappa_{(1)})^2 \lambda^m_{(1)} \lambda^n_{(1)} && (3)\\
& & &= (\kappa_{(1)})^2 \underbrace{\lambda_{(1)m} \lambda^n_{(1)}}_{=\epsilon_{(1)}} && (4)\\
& & &= (\kappa_{(1)})^2 && (5)
\end{aligned}
$$

♦

2.45 p75-exercise

For positive definite metric forms, write out explicitly the Frenet formulae for the case N=2, 3 and 4.

The general Frenet formulae are

$$\begin{cases} \frac{\delta \lambda^r_{(M-1)}}{\delta s} = \kappa_{(M)}\lambda^r_{(M)} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\lambda^r_{(M-2)} \\ \epsilon_{(M-1)}\lambda^n_{(M-1)}\lambda_{(M-1)n} = 1 \qquad \text{(M=1,2,...,N)} \end{cases} \tag{1}$$

As Φ is positive definite, we have $\epsilon_{(k)} = 1 \quad \forall k$

N=2	N=3	N=4
$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(3)}}{\delta s} = \kappa_{(4)}\lambda^r_{(4)} - \kappa_{(3)}\lambda^r_{(2)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$ $\lambda^n_{(3)}\lambda_{(3)n} = 1$

Taking into account that $\kappa_{(N)} = 0$ for a space V_N, we get,

N=2	N=3	N=4
$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = -\kappa_{(1)}\lambda^r$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = -\kappa_{(2)}\lambda^r_{(1)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(3)}}{\delta s} = -\kappa_{(3)}\lambda^r_{(2)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$ $\lambda^n_{(3)}\lambda_{(3)n} = 1$

♦

2.46 p76-exercise

In an Euclidean space V_N, the fundamental form is given as $\Phi = dx^n dx^n$. Show that a curve which has $\kappa_{(2)} = 0$ and $\kappa_{(1)} =$ constant satisfies equations of the form

$$x^r = A^r \cos \kappa_{(1)} s + B^r \sin \kappa_{(1)} s + C^r$$

where A^r, B^r, C^r are constants satisfying

$$A^r A^R = B^r B^r = \frac{1}{\kappa_{(1)}^2}, \quad A^r B^r = 0$$

so that A^r and B^r are vectors of equal magnitude and perpendicular to one another. (This curve is a circle in the N-space)

$$\text{What we know} \qquad \Phi = dx^n dx^n \tag{1}$$

$$\Rightarrow \qquad (a_{mn}) = (\delta_n^m) \tag{2}$$

$$\text{and given} \qquad \kappa_{(1)} = \text{constant} \quad \kappa_{(2)} = 0 \quad \epsilon_{(1)} = \epsilon_{(2)}, \cdots = 1 \tag{3}$$

$$\text{we have (2.705)} \qquad \frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \qquad \text{with} \quad \lambda^r = \frac{dx^r}{ds} \tag{4}$$

$$\text{but} \quad (a_{mn}) = (\delta_n^m) \quad \Rightarrow \qquad \frac{\delta \lambda^r}{\delta s} = \frac{d\lambda^r}{ds} \tag{5}$$

$$\text{(4) and (5)} \quad \Rightarrow \qquad \frac{d\lambda^r}{ds} = \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \tag{6}$$

$$\text{also} \qquad \frac{\delta \lambda_{(1)}^r}{\delta s} = \underbrace{\kappa_{(1)}}_{=0} \frac{\delta \lambda_{(1)}^r}{\delta s} - \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \tag{7}$$

Hence we get the following set of equations

$$\begin{cases} \frac{dx^r}{ds} = \lambda^r \\ \frac{d\lambda^r}{ds} = \kappa_{(1)}\lambda^r_{(1)} \\ \frac{d\lambda^r_{(1)}}{ds} = -\kappa_{(1)}\lambda^r_{(1)} \\ \kappa_{(1)} = \kappa \quad (=\text{constant}) \\ \kappa_{(2)} = 0 \\ \lambda^n\lambda_n = 1 \\ \lambda^n_{(1)}\lambda_{(1)n} = 1 \end{cases} \tag{8}$$

$$(8) \quad \Rightarrow \quad \frac{d^2\lambda^r_{(1)}}{ds^2} + \kappa^2\lambda^r_{(1)} = 0 \tag{9}$$

Solving the ODE (9). Put $e^{rs} = \lambda^k_{(1)}$

$$(9): \qquad r^2 + \kappa^2 = 0 \tag{10}$$

$$\Rightarrow \qquad r = \pm\imath\kappa \tag{11}$$

$$\text{Hence, a general solution of (9) is of the form:} \qquad \lambda^r_{(1)} = p^r e^{\imath\kappa s} + q^r e^{-\imath\kappa s} \tag{12}$$

$$\text{put} \quad p^r + q^r = A^{,r} \quad \text{and} \quad p^r - q^r = B^{,r} \tag{13}$$

$$\Leftrightarrow \qquad p^r = \frac{A^{,r} + B^{,r}}{2} \text{and} \quad p^r = \frac{A^{,r} - B^{,r}}{2} \tag{14}$$

$$\text{(12) can then be written as} \quad \lambda^r_{(1)} = A^{,r}\frac{e^{\imath\kappa s} + e^{-\imath\kappa s}}{2} + B^{,r}\frac{e^{\imath\kappa s} - e^{-\imath\kappa s}}{2} \tag{15}$$

$$\text{or} \quad \lambda^r_{(1)} = A^{,r}\cos\kappa s + B^{,r}\sin\kappa s \tag{16}$$

$$\text{We have (8)} \qquad \lambda^r = -\,\kappa_{(1)}\frac{d\lambda^r_{(1)}}{ds} \tag{17}$$

$$\frac{d(16)}{ds} \quad \text{and (17)} \Rightarrow \qquad \lambda^r = A^{,r}\sin\kappa s - B^{,r}\cos\kappa s \tag{18}$$

$$\text{as} \quad \lambda^r = \frac{dx^r}{ds} \quad \text{with (18)} \quad \Rightarrow \qquad x^r = -\frac{A^{,r}}{\kappa}\cos\kappa s - \frac{B^{,r}}{\kappa}\sin\kappa s + C^r \tag{19}$$

$$\tag{20}$$

Replace $-\frac{A^{,r}}{\kappa}$ with A^r and $-\frac{B^{,r}}{\kappa}$ with B^r, we get then the following set of equations,

$$\begin{cases} x^r = A^r \cos\kappa s + B^{,r} \sin\kappa s + C^r \\ \lambda^r = -\kappa A^r \sin\kappa s + \kappa B^r \cos\kappa s \\ \lambda^r_{(1)} = -\kappa A^r \cos\kappa s - \kappa B^r \sin\kappa s \end{cases} \tag{21}$$

with the following constraints $\qquad \lambda^n \lambda_n = 1 \qquad \lambda^n_{(1)} \lambda_{(1)n} = 1 \qquad (22)$

$$\lambda^n \lambda_n = 1 \quad \Rightarrow \kappa^2 A^r A^r \sin^2\kappa s + \kappa^2 B^r B^r \cos^2\kappa s - 2\kappa^2 A^r B^r \sin\kappa s \cos\kappa s = 1 \tag{23}$$

$$\text{or} \qquad A^r A^r \sin^2\kappa s + B^r B^r \cos^2\kappa s - 2A^r B^r \sin\kappa s \cos\kappa s = \frac{1}{\kappa^2} \tag{24}$$

$$\lambda^n_{(1)} \lambda_{(1)n} = 1 \quad \Rightarrow \kappa^2 A^r A^r \cos^2\kappa s + \kappa^2 B^r B^r \sin^2\kappa s + 2\kappa^2 A^r B^r \sin\kappa s \cos\kappa s = 1 \tag{25}$$

$$\text{or} \qquad A^r A^r \cos^2\kappa s + B^r B^r \sin^2\kappa s + 2A^r B^r \sin\kappa s \cos\kappa s = \frac{1}{\kappa^2} \tag{26}$$

$$\text{Choose} \quad \kappa s = \frac{\pi}{2} \quad \text{and} \quad \kappa s = 0 \tag{27}$$

$$\Rightarrow \qquad A^r A^r = \frac{1}{\kappa^2} \quad \text{and} \quad B^r B^r = \frac{1}{\kappa^2} \tag{28}$$

Morover considering (26)-(24) and (28) $\quad \Rightarrow \qquad 2A^r B^r \sin\kappa s \cos\kappa s = 0 \quad \forall \kappa s \qquad (29)$

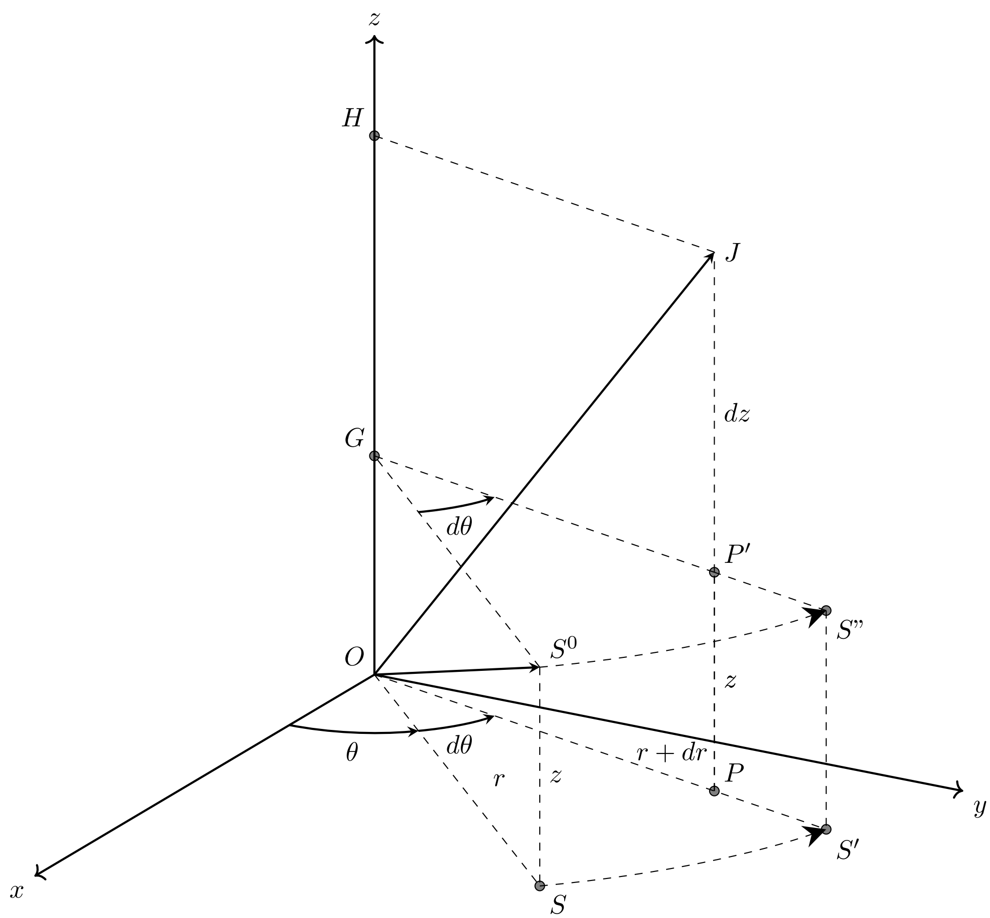
$$\Rightarrow \qquad A^r B^r = 0 \tag{30}$$

Note: when deriving expressions (23) and (26) we use the fact that $(a_{mn}) = (a^{mn}) = (\delta^m_n)$

♦

2.47 p78-exercise 1

For cylindrical coordinates in Euclidean 3-space, write down the metric form by inspection of a diagram showing a general infinitesimal displacement, and calculate all the Christoffel symbols of both kinds.



From the figure we may (assuming an infinitesimal displacement), we may approximate $\left|\overrightarrow{SS'}\right|$ with the arclength $rd\theta$ and assume $\left|\overrightarrow{SS''}\right| \perp \left|\overrightarrow{GS'}\right|$) Hence, the infinitesimal displacement from S

$$
\begin{aligned}
ds^2 &= \left|\overrightarrow{SS'}\right|^2 + \left|\overrightarrow{S''P'}\right|^2 + \left|\overrightarrow{P'J}\right|^2 && (1)\\
&= dr^2 + ((r+dr)d\theta)^2 + dz^2 && (2)\\
&= dr^2 + r^2 d\theta^2 + dz^2 && (3)
\end{aligned}
$$

$$
\text{Hence} \qquad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (4)
$$

Note that all $a_{mn} = 0 \quad \forall m \neq n$. So,

$$\begin{cases} [r\ r, r] = [\theta\ \theta, \theta] = [z\ z, z] = 0 \\ [r\ \theta, r] = [r\ r, \theta] = [r\ r, z] = 0 \\ [r\ z, \theta] = [z\ \theta, r] = [z\ \theta, z] = 0 \end{cases} \tag{5}$$

$$\text{But:} \qquad [\theta\ \theta, r] = -r \quad \text{and} \quad [r\ \theta, \theta] = r \tag{6}$$

$$\text{Hence} \qquad \begin{cases} \Gamma^m_{nk} = 0 \quad \forall \quad (nk) \neq (r,\theta), (\theta,\theta) \\ \\ \Gamma^\theta_{r\theta} = \frac{1}{r} \quad \text{and} \quad \Gamma^r_{\theta\theta} = -r \end{cases} \tag{7}$$

♦

2.48 p78-exercise 2

If a_{rs} and b_{rs} are covariant tensors, show that the roots of the determinant equation

$$|Xa_{rs} - b_{rs}| = 0$$

are invariants.

$$\begin{aligned}
&\text{Be} && c_{rs} = Xa_{rs} - b_{rs} && (1)\\
&\text{Given} && a_{rs} = a'_{mk}\frac{\partial x'^m}{\partial x^r}\frac{\partial x'^k}{\partial x^s} && (2)\\
&\text{and} && b_{rs} = b'_{mk}\frac{\partial x'^m}{\partial x^r}\frac{\partial x'^k}{\partial x^s} && (3)\\
&(1) \quad \Rightarrow && c_{rs} = \underbrace{(Xa'_{mk} - b'_{mk})}_{=c'_{km}}\frac{\partial x'^m}{\partial x^r}\frac{\partial x'^k}{\partial x^s} && (4)\\
& && = c'_{km}\frac{\partial x'^m}{\partial x^r}\frac{\partial x'^k}{\partial x^s} && (5)\\
&\text{Be} && J = \left|\frac{\partial x'^m}{\partial x^r}\right| = \left|\frac{\partial x'^k}{\partial x^s}\right| && (6)\\
&\text{In (5) put} && d_{kr} = c'_{km}\frac{\partial x'^m}{\partial x^r} && (7)\\
&\Rightarrow && c_{rs} = d'_{kr}\frac{\partial x'^k}{\partial x^s} && (8)\\
&\text{or in matrix form} && C = D^T J \quad \text{with} \quad D = C'J && (9)\\
&\Rightarrow && |C| = \left|(C'J)^T J\right| && (10)\\
&\Leftrightarrow && |C| = |C'|\,|J|\,|J| && (11)
\end{aligned}$$

As $|J| \neq 0$ (J is the Jacobian of the transformation, and thus can't be zero), then

$$|C| = 0 \Rightarrow |C'| = 0$$

.

So, the root of $|C| = 0$ is also a root of $|C'| = 0$ and is as a consequence, invariant.

♦

2.49 p78-exercise 3

Is the form $dx^2 + 3dxdy + 4dy^2 + dz^2$ positive definite?

$$\Phi = dx^2 + 3dxdy + 4dy^2 + dz^2 \qquad (1)$$

$$\text{Put (1) in the form} \qquad \Phi = X^2 + 3XY + 4Y^2 + Z^2 \qquad (2)$$

$$\text{Z has only a positive contribution: so put} \quad Z = 0 \quad \Rightarrow \quad \Phi = X^2 + 3XY + 4Y^2 \qquad (3)$$

$$\text{(3) can only be zero or negative if} \quad XY < 0 \quad \text{:put} \quad Y = -aX \quad (a > 0) \qquad (4)$$

$$\Rightarrow \Phi = X^2 - 3aX^2 + 4a^2X^2 \qquad (5)$$

$$\text{The roots of (5) are} \quad a_{1,2} = \frac{3 \pm \sqrt{9-16}}{8} \qquad (6)$$

So, by (6) we can't get a $a \in \mathbb{R}_*$, so that (1) can be 0 or negative. Hence,

$$\text{The form} \quad \Phi \quad \text{is positive definite}$$

♦

2.50 p78-exercise 4

If X^r, Y^r are unit vectors inclined at an angle θ, prove that

$$\sin^2\theta = (a_{rm}a_{sn} - a_{rs}a_{mn})X^r Y^s X^m Y^n$$

$X^r Y^s$ are unit vectors. So,

$$a_{rm}X^r X^m = 1 \quad \text{and} \quad a_{sn}Y^s Y^n = 1 \tag{1}$$

$$\text{We have} \qquad \sin^2\theta = 1 - \cos^2\theta \tag{2}$$

$$\text{and (2.312)} \qquad \cos\theta = a_{mn}X^m Y^n \tag{3}$$

$$\Rightarrow \qquad \sin^2\theta = a_{rm}X^r X^m a_{sn}Y^s Y^n - a_{mn}X^m Y^n a_{rs}X^r Y^s \tag{4}$$

$$= (a_{rm}a_{sn} - a_{mn}a_{rs})X^r Y^s X^m Y^n \tag{5}$$

♦

2.51 p78-exercise 5

Show that, if θ is the angle between the normals to the surfaces $x^1 = C^{st}, x^2 = C^{st}$, then

$$\cos\theta = \frac{a^{12}}{\sqrt{a^{11}a^{22}}}$$

$$\text{be} \qquad \phi'_1(x^1, x^2, \ldots, x^N) = C^{st} \quad \phi'_2(x^1, x^2, \ldots, x^N) = C^{st} \tag{1}$$

the two equations representing S_1, S_2 (see page 63). We can rewrite (1) as:

$$x^1 = \phi'_1(x^1, x^2, \ldots, x^N) = C^{st} \tag{2}$$

$$x^2 = \phi'_2(x^1, x^2, \ldots, x^N) = C^{st} \tag{3}$$

$$\text{From (2.622) we know that} \quad X^m = a^{mn}\partial_n\phi_1,\ Y^m = a^{mn}\partial_n\phi_2 \text{ are } \perp \text{ vectors to the surfaces } \phi_1, \phi_2 \tag{4}$$

$$\text{We know also} \qquad |X^m|^2 = a^{mk}X^m X^k \tag{5}$$

$$= a_{mk}a^{mn}\partial_n\phi_1 a^{kp}\partial_p\phi_1 \tag{6}$$

$$= \delta^k_k a^{kp}\partial_n\phi_1\partial_p\phi_1 \tag{7}$$

$$= a^{np}\partial_n\phi_1\partial_p\phi_1 \tag{8}$$

$$\text{as } \phi_1 = x^1 = C^{st} \ \Rightarrow \ = a^{np}\delta^1_n\delta^1_p \tag{9}$$

$$= \epsilon a^{11} \tag{10}$$

$$\text{Analog, we have} \qquad |Y^m|^2 = \epsilon a^{22} \tag{11}$$

$$\text{By definition:} \qquad \cos\theta = \frac{a_{mn}X^m Y^n}{|X^r|\,|Y^s|} \tag{12}$$

$$\text{and} \qquad a_{mn}X^m Y^n = a_{mn}a^{mk}\partial_k\phi_1\ a^{np}\partial_p\phi_2 \tag{13}$$

$$= \delta^k_n a^{np}\partial_k\phi_1\ \partial_p\phi_2 \tag{14}$$

$$= a^{kp}\partial_k\phi_1\ \partial_p\phi_2 \tag{15}$$

$$\text{as } \phi_1 = x^1 = C^{st},\ \phi_2 = x^2 = C^{st} \ \Rightarrow \ = a^{kp}\delta^1_k\delta^2_p \tag{16}$$

$$= a^{12} \tag{17}$$

$$\text{So (12) becomes with (10), (11) and (17)} \qquad \cos\theta = \frac{a_{12}}{\sqrt{\epsilon a^{11}\epsilon a^{22}}} \tag{18}$$

$$= \frac{a_{12}}{\sqrt{a^{11}a^{22}}} \tag{19}$$

♦

2.52 p78-exercise 6

Let $x^1,\ x^2,\ x^3$ be rectangular Cartesian coordinates in Euclidean 3-space, and let $x^1,\ x^2$ be taken as coordinates on a surface $x^3 = f(x^1,\ x^2)$. Show that the Christoffel symbols of the second kind for the surface are

$$\Gamma^r_{mn} = \frac{f_r f_{mn}}{1 + f_n f_p}$$

the suffixes taking the values $1,\ 2$ and the subscripts indicating partial derivatives.

We have (rectangular Cartesian coordinates in Euclidean 3-space)

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1}$$

with
$$x^3 = f(x^1,\ x^2) \tag{2}$$

and thus
$$dx^3 = \partial_1 f\ dx^1 + \partial_2 f\ dx^2 \tag{3}$$

$\Rightarrow$
$$ds^2 = (1 + (\partial_1 f)^2)(dx^1)^2 + (1 + (\partial_2 f)^2)(dx^2)^2 + 2\partial_1 f\ \partial_2 f\ dx^1 dx^2 \tag{4}$$

put
$$\begin{cases} f_1 = \partial_1 f \\ f_2 = \partial_2 f \\ f_{11} = \partial_{11} f \\ f_{22} = \partial_{22} f \\ f_{12} = f_{21} = \partial_{12} f \end{cases} \tag{5}$$

from (4)
$$(a_{mn}) = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix} \tag{6}$$

$\Rightarrow$
$$|a_{mn}| = (1 + f_1^2)(1 + f_2^2) - (f_1 f_2)^2 \tag{7}$$

$$= 1 + f_1^2 + f_2^2 \tag{8}$$

also
$$(a^{mn}) = \frac{1}{1 + f_1^2 + f_2^2} \begin{pmatrix} 1 + f_2^2 & -f_1 f_2 \\ -f_1 f_2 & 1 + f_1^2 \end{pmatrix} \tag{9}$$

$$\textbf{Calculating the Christoffels symbols:} \quad [mn,\ k] = \frac{1}{2}(\partial_m a_{nk} + \partial_n a_{mk} - \partial_k a_{mn}) \tag{10}$$

$$\Rightarrow \qquad \begin{cases} [11,\ 1] = f_1 f_{11} \\ [11,\ 2] = f_2 f_{11} \\ [12,\ 1] = f_1 f_{12} \\ [12,\ 2] = f_2 f_{21} \\ [22,\ 2] = f_2 f_{22} \end{cases} \tag{11}$$

$$\text{From (11) we can see that the general form is:} \quad [mn, s] = f_{mn} f_s \tag{12}$$

$$\textbf{Calculating the Christoffels symbols:} \quad \Gamma^r_{mn} = a_{rs}[mn, s] \tag{13}$$

$$\text{(13) with (12):} \quad \Gamma^r_{mn} = a_{rs} f_{mn} f_s = f_{mn}(a^{rs} f_s) \tag{14}$$

$$\text{put } \Delta = \frac{1}{1 + f_1^2 + f_2^2} = \frac{1}{1 + f_p f_p} \tag{15}$$

$$\Rightarrow \qquad \begin{cases} \Gamma^1_{mn} & = (a_{11} f_1 + a_{12} f_2) f_{mn} \\ & = \Delta(f_1 + f_2^2 f_1 - f_1 f_2^2) f_{mn} \\ & = \Delta f_1 f_{mn} \\ & \\ \Gamma^2_{mn} & = (a_{21} f_1 + a_{22} f_2) f_{mn} \\ & = \Delta(-f_1^2 f_2 + f_2 + f_2 f_1^2) f_{mn} \\ & = \Delta f_2 f_{mn} \end{cases} \tag{16}$$

$$\Rightarrow \quad \Gamma^r_{mn} = \frac{f_r f_{mn}}{1 + f_p f_p}$$

♦

2.53 p78-exercise 7

Write down the differential equations of the geodesics on a sphere, using colatitude θ and the azimuth ϕ as coordinates. Integrate the differential equations and obtain a finite equation

$$A \sin\theta \cos\phi + B \sin\theta \sin\phi + C \cos\theta = 0$$

where A, B, C are arbitrary constants.

We will use two different approaches to determine the relation and finally use a geometrical reasoning allowing us to avoid solving the ODE's resulting from the above mentioned approaches.

We will first find the solution, starting from the variational principle , defining a geodesic. In spherical coordinates we have (see exercise page 27) $ds^2 = dr^2 + r^2 \sin^2(\theta) d\phi^2 + r^2 d\theta^2$.
As $r = R = C^{st}$ this reduces to

$$ds^2 = R^2 \sin^2(\theta) d\phi^2 + R^2 d\theta^2 \tag{1}$$

So the length of a curve on the sphere from a point P_1 to another point P_2 , the curve being determined by $\theta = \theta(u) \quad \phi = \phi(u)$ is:

$$L = R \int_{P_1}^{P_2} = \sqrt{\sin^2(\theta) d\phi^2 + d\theta^2} du \tag{2}$$

$$\text{Be} \quad \theta = u \quad \phi = \phi(\theta) \tag{3}$$

$$\Rightarrow \quad L = R \int_{\theta_1}^{\theta_2} = \sqrt{1 + \sin^2(\theta) (\frac{d\phi}{d\theta})^2} d\theta \tag{4}$$

Applying the variational principle on L and using the Euler-Langrange equations:

$$\frac{d\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}{d\theta} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \text{with} \quad \mathcal{L} = \sqrt{1+\sin^2(\theta)(\dot{\phi})^2} \tag{5}$$

$$\text{as} \quad \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{6}$$

$$\text{(5) becomes:} \quad \frac{d\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}{d\theta} = 0 \tag{7}$$

$$\Leftrightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = C \quad (= \text{constant}) \tag{8}$$

$$\text{with} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial\sqrt{1+\sin^2\theta\ \dot{\phi}^2}}{\partial\dot{\phi}} = \frac{\sin^2\theta\ \dot{\phi}}{\sqrt{1+\sin^2\theta\ \dot{\phi}^2}} \tag{9}$$

$$\text{(9) and (10):} \quad C^2 = \frac{\sin^2\theta\ \dot{\phi}}{\sqrt{1+\sin^2\theta\ \dot{\phi}^2}} \tag{10}$$

$$\text{or} \quad \dot{\phi} = \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} \tag{11}$$

Solving the ODE (12). Put $u = \cot\theta \quad \Rightarrow \quad du = -\csc^2 d\theta = -\frac{1}{\sin^2\theta}d\theta$. So,

$$\phi = -C\int \frac{\sin\theta}{\sqrt{\sin^2\theta - C^2}}du \tag{12}$$

$$= -C\int \frac{du}{\sqrt{1-\frac{C^2}{\sin^2\theta}}} \tag{13}$$

$$= -C\int \frac{du}{\sqrt{1-C^2-C^2\cot^2\theta}} \tag{14}$$

$$= -C\int \frac{du}{\sqrt{1-C^2-C^2u^2}} \tag{15}$$

$$\text{be} \quad a = \frac{\sqrt{1-C^2}}{C} \tag{16}$$

$$\text{(15) becomes} \quad \phi = -\int \frac{1}{\sqrt{a^2-u^2}}du \tag{17}$$

$$\text{put} \quad u = av \tag{18}$$

$$\text{(18) becomes} \quad \phi = -\int \frac{1}{\sqrt{1-v^2}}dv \tag{19}$$

$$= -\arccos v + C^{st} \tag{20}$$

$$= -\arccos\frac{u}{a} + \phi_0 \tag{21}$$

$$\text{or:} \quad \frac{u}{a} = \cos(\phi-\phi_0) \ \text{(by choosing an adequate } \phi_0\text{)} \tag{22}$$

$$\text{so :} \quad \cot\theta = a\cos(\phi-\phi_0) \tag{23}$$

$$\text{expanding } \cos(\phi-\phi_0) \text{ gives:} \quad \frac{\cos\theta}{\sin\theta} = A\cos\phi + B\sin\phi \tag{24}$$

$$\text{or:} \quad A\cos\phi\sin\theta + B\sin\phi\sin\theta - \cos\theta = 0 \tag{25}$$

◇

Finding the geodesics from the tensorial formula's.
Note: For ease of notation we put $R = 1$ without losing any general solutions.

$$\text{from (1) we get:} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{26}$$

$$\text{and} \quad (a^{mn}) = \begin{pmatrix} \frac{1}{\sin^2\theta} & 0 \\ 0 & 1 \end{pmatrix} \tag{27}$$

$$\text{hence:} \quad \begin{cases} \Gamma^1_{11} = 0 & \Gamma^2_{11} = 0 \\ \Gamma^1_{12} = 0 & \Gamma^2_{12} = \cot\theta \\ \Gamma^1_{22} = -\cos\theta\sin\theta & \Gamma^2_{22} = 0 \end{cases} \tag{28}$$

Finding the geodesics from the tensorial formula's, implies solving 2^{nd} order ODE's. In order to find the simpliest form to solve , we write down three possible forms of the geodesic equations:

$$\text{arc-length s as independent variable} \quad \begin{cases} (a) \quad \frac{d^2\phi}{ds^2} + 2\cot\theta \frac{d\phi}{ds}\frac{d\theta}{ds} = 0 \\ \\ (b) \quad \frac{d^2\theta}{ds^2} - \sin\theta\cos\theta(\frac{d\phi}{ds})^2 = 0 \\ \\ (c) \quad (\frac{d\theta}{ds})^2 + \sin^2\theta(\frac{d\phi}{ds})^2 = 1 \end{cases} \tag{29}$$

$$\theta \text{ as independent variable} \quad \begin{cases} \lambda = -\sin\theta\cos\theta(\frac{d\phi}{d\theta})^2 \\ \\ \frac{d^2\phi}{d\theta^2} = \lambda\frac{d\phi}{d\theta} - 2\cot\theta\frac{d\phi}{d\theta} \end{cases} \tag{30}$$

$$\Rightarrow \quad \frac{d^2\phi}{d\theta^2} = -\sin\theta\cos\theta(\frac{d\phi}{d\theta})^3 - 2\cot\theta\frac{d\phi}{d\theta} \tag{31}$$

$$\phi \text{ as independent variable} \quad \begin{cases} \lambda = 2\cot\theta\frac{d\theta}{d\phi} \\ \\ \frac{d^2\theta}{d\phi^2} = \lambda\frac{d\theta}{d\phi} + \sin\theta\cos\theta(\frac{d\theta}{d\phi})^2 \end{cases} \tag{32}$$

$$\Rightarrow \quad \frac{d^2\theta}{d\phi^2} = 2\cot\theta(\frac{d\theta}{d\phi})^2 + \sin\theta\cos\theta(\frac{d\theta}{d\phi})^2 \tag{33}$$

$$\tag{34}$$

Inspection shows hat the expression (32) and (34) are quite complicated while using (30b) and (30c) we can get an expression of the form

$$\ddot{\theta} - \sin\theta\cos\theta\left(\frac{1-\dot{\theta}^2}{\sin^2\theta}\right) = 0 \tag{35}$$

$$\text{or:} \quad \ddot{\theta} - \cot\theta\left(1-\dot{\theta}^2\right) = 0 \tag{36}$$

$$\text{Put} \quad u(\theta) = \dot{\theta} \quad \Rightarrow \quad \ddot{\theta} = \dot{u}u \tag{37}$$

$$\text{(37) can be written as:} \quad \dot{u}u - \cot\theta\left(1-u^2\right) = 0 \tag{38}$$

$$\text{or} \quad \frac{\dot{u}u}{(1-u^2)} = \cot\theta \tag{39}$$

$$\text{or} \quad \frac{u}{(1-u^2)}du = \frac{\cos\theta}{\sin\theta}d\theta \tag{40}$$

$$\text{or} \quad -\frac{1}{2}\frac{1}{(1-u^2)}d(1-u^2) = \frac{1}{\sin\theta}d(\sin\theta) \tag{41}$$

$$\text{hence} \quad -d(\log\left(\sqrt{1-u^2}\right)) = d(\log(\sin\theta)) \tag{42}$$

$$\Rightarrow \quad d(\log\sqrt{1-u^2} + \log(\sin\theta)) = 0 \tag{43}$$

$$\Leftrightarrow \quad d(\log\left(\sqrt{1-u^2}\sin\theta\right)) = 0 \tag{44}$$

$$\Rightarrow \quad (1-\dot{\theta}^2)\sin^2\theta = C^2 \tag{45}$$

$$\Rightarrow \quad \dot{\theta}^2 = 1 - \frac{C^2}{\sin^2\theta} \tag{46}$$

$$\text{we have (30c)} \quad \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta = 1 \tag{47}$$

$$\text{and so (47):} \quad 1 - \frac{C^2}{\sin^2\theta} + \dot{\phi}^2\sin^2\theta = 1 \tag{48}$$

$$\Rightarrow \quad \dot{\phi}^2 = \frac{C^2}{\sin^4\theta} \tag{49}$$

$$\Rightarrow \quad \dot{\phi} = \frac{C}{\sin^2\theta} \tag{50}$$

$$\text{we have} \quad \dot{\phi} = \frac{d\phi}{d\theta}\dot{\theta} \tag{51}$$

$$\text{so} \quad d\phi = \frac{C}{\sin^2\theta\sqrt{1-\frac{C^2}{\sin^2\theta}}}d\theta \tag{52}$$

$$\text{so} \quad d\phi = \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}}d\theta \tag{53}$$

Note that expression (54) is exactly the expression (12) we found by applying directly the variational principle to find the general expression. So applying steps (13) to (26) gives us the same expression.

◇

Instead of solving the ODE's (46) and (53) we can use geometrical considerations to get the asked expression.
Due to the invariance of a sphere regarding rotation of the axes, we can choose a reference axis system XYZ (from which θ, ϕ are measured) so that at $s = 0$ of the geodesic, corresponds the point $r = 1,\ \theta = 0$. As from (45), $(1 - \dot{\theta}^2)\sin^2\theta\Big|_{s=0} = C^2$ follows that $C = 0$ (because $\theta|_{s=0} = 0$). So we get $(1 - \dot{\theta}^2)\sin^2\theta = 0\ \forall\ \theta : \Rightarrow\ \dot{\theta} = 1$ and $\theta = s$. Then from (37) $\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta = 1$ follows that $\dot{\phi} = 0$ and thus $\phi = C^{st}$. Again, considering symmetry we can choose the axis system so that $\phi = 0$. The set of equations $\theta = s,\ \phi = 0$ represents a circle on the sphere generated by the intersection of the sphere with the XZ plane. Again, considering symmetry, we can conclude that any circle on the sphere generated by the intersection a plane going through the origin of the axis system, is also a geodesic curve. So, be $\widehat{n} = (A, B, C)$ the normal vector defining a plane going through the origin and $\widehat{p} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ a point on the sphere, then the intersection of the plane and the sphere is given by

$$\langle\widehat{n}|\widehat{p}\rangle = A\cos\phi\sin\theta + B\sin\phi\sin\theta + C\cos\theta = 0$$

♦

2.54 p79-exercise 8

Find in integrated form the geodesic null lines in a V_3 for which the metric form

$$(dx^1)^2 - R^2[(dx^2)^2 + (dx^3)^2]$$

R being a function of x^1 only.

We have,

$$\Phi = (dx^1)^2 - R^2[(dx^2)^2 + (dx^3)^2] \tag{1}$$

$$\text{Hence,}\quad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -R^2 & 0 \\ 0 & 0 & -R^2 \end{pmatrix} \tag{2}$$

$$\text{and,}\quad (a^{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{R^2} & 0 \\ 0 & 0 & -\frac{1}{R^2} \end{pmatrix} \tag{3}$$

$$\text{Hence,}\quad \begin{cases} [22,1] = R\,\partial_1 R & [12,2] = -R\,\partial_1 R \\ [33,1] = R\,\partial_1 R & [13,3] = -R\,\partial_1 R \end{cases} \tag{4}$$

$$\text{and,}\quad \begin{cases} \Gamma^1_{22} = R\,\partial_1 R & \Gamma^2_{12} = \frac{1}{R}\partial_1 R \\ \Gamma^1_{33} = R\,\partial_1 R & \Gamma^3_{13} = \frac{1}{R}\partial_1 R \end{cases} \tag{5}$$

$$\text{with all other } [mn,s] \text{ and } \Gamma^s_{mn} \text{ being zero.} \tag{6}$$

The equations of null geodesics give:

$$\begin{cases} \frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \\ a_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \end{cases} \tag{7}$$

$\boldsymbol{x^r = x^1}$ gives:

$$\frac{d^2x^1}{du^2} + R\partial_1 R(\frac{dx^2}{du})^2 + R\partial_1 R(\frac{dx^3}{du})^2 = 0 \tag{8}$$

Put $u = x^1$

$$\text{from (8):}\quad R\,\partial_1 R\left[(\frac{dx^2}{du})^2 + (\frac{dx^3}{du})^2\right] = 0 \tag{9}$$

If, $R = R(x^1) \neq C^{st}$, the form (9) can only be zero if $(\frac{dx^2}{du})^2 + (\frac{dx^3}{du})^2 = 0$ and thus $\frac{dx^2}{du} = \frac{dx^3}{du} = 0$ and hence $x^2,\ x^3$ are constant.
Conclusion: the null geodesics are the bundle of rays parallel with the x^1 axis with vector equation $\widehat{p} = (s, A, B), s \in (-\infty, +\infty)$ and A, B arbitrary constants.

♦

2.55 p79-exercise 9

Show that, for normal coordinate system, the Christoffel symbols

$$[\rho N,\ \sigma],\ [\rho\sigma,\ N],\ [\rho N,\ N],\ [NN,\ N]$$

$$\Gamma^{\rho}_{N\sigma},\ \Gamma^{N}_{\rho\sigma},\ \Gamma^{\rho}_{NN},\ \Gamma^{N}_{N\rho},\ \Gamma^{N}_{NN}$$

have tensor character with respect to the transformation of the coordinates $x^1, \ldots, x^{N-1}$

We know that $a_{\rho\sigma} = a'_{mn}\partial_\rho x'^m \partial_\sigma x'^n$ with $\rho, \sigma = 1, \ldots, N-1$ and $m, n = 1, \ldots, N$. We have also $x^N = x'^N$.

$[\rho N,\ \sigma]$

$$\begin{aligned}
[\rho N,\ \sigma] &= \frac{1}{2}\partial_N a_{\rho\sigma} \quad \text{see (2.639)} && (1)\\
&= \frac{1}{2}\partial_N (a'_{mn}\partial_\rho x'^m \partial_\sigma x'^n) && (2)\\
&= \begin{cases} \frac{1}{2}(\partial_N a'_{mn}\partial_\rho x'^m \partial_\sigma x'^n \\ +a'_{mn}\partial_\sigma x'^n \partial_{N\rho} x'^m \\ +a'_{mn}\partial_\rho x'^m \partial_{N\sigma} x'^n) \end{cases} && (3)\\
\text{We have} \quad \partial_N x'^m = \delta^m_N \Rightarrow\ & \partial_{N\rho} x'^m = \partial_{N\sigma} x'^n = 0 && (4)\\
\Rightarrow \quad [\rho N,\ \sigma] &= \frac{1}{2}\partial_N a'_{mn}\partial_\rho x'^m \partial_\sigma x'^n && (5)\\
&= \begin{cases} \frac{1}{2}(\partial_N a'_{\alpha\beta}\partial_\rho x'^\alpha \partial_\sigma x'^\beta \\ +\partial_N a'_{NN}\underbrace{\partial_\rho x'^N}_{=0}\underbrace{\partial_\sigma x'^N}_{=0} \\ +\partial_N a'_{\alpha N}\partial_\rho x'^\alpha \underbrace{\partial_\sigma x'^N}_{=0} \\ +\partial_N a'_{\beta N}\partial_\rho x'^\alpha \underbrace{\partial_\sigma x'^N}_{=0}) \end{cases} && (6)\\
\Rightarrow \quad [\rho N,\ \sigma] &= \underbrace{\frac{1}{2}\partial_N a'_{\alpha\beta}}_{=[\alpha N,\ \beta]'}\partial_\rho x'^\alpha \partial_\sigma x'^\beta && (7)\\
&= [\alpha N,\ \beta]'\partial_\rho x'^\alpha \partial_\sigma x'^\beta && (8)
\end{aligned}$$

This confirms the tensor character of $[\rho N,\ \sigma]$

◇

$[\rho\sigma,\ N]$ this follows immediately from the previous and considering $[\rho\sigma,\ N] = -[\rho N,\ \sigma]$ see(2.639)

◇

$[\boldsymbol{\rho N},\ \boldsymbol{N}]$

We prove the case for $[\boldsymbol{NN},\ \boldsymbol{\rho}]$ as $[\rho N,\ N] = -[NN,\ \rho]$

$$[NN,\ \rho] = \frac{1}{2}\partial_\rho a_{NN} \quad \text{see } (2.639) \tag{9}$$

$$= \frac{1}{2}\partial_\rho (a'_{mn}\partial_N x'^m \partial_N x'^n) \tag{10}$$

$$= \begin{cases} \frac{1}{2}(\partial_\rho a'_{mn}\partial_N x'^m \partial_N x'^n \\ \\ +a'_{mn}\partial_\rho x'^n \underbrace{\partial_{N\rho} x'^m}_{=0} \\ \\ +a'_{mn}\partial_\rho x'^m \underbrace{\partial_{N\rho} x'^n}_{=0}) \end{cases} \tag{11}$$

$$= \frac{1}{2}\partial_\rho a'_{mn}\partial_N x'^m \partial_N x'^n \tag{12}$$

$$= \begin{cases} \frac{1}{2}(\partial_\rho a'_{\alpha\beta} \underbrace{\partial_N x'^\alpha}_{=0} \underbrace{\partial_N x'^\beta}_{=0} \\ \\ +\partial_\rho a'_{NN} \underbrace{\partial_N x'^N}_{=1} \underbrace{\partial_N x'^N}_{=1} \\ \\ +\partial_\rho a'_{\alpha N} \underbrace{\partial_N x'^\alpha}_{=0} \underbrace{\partial_N x'^N}_{=1} \\ \\ +\partial_\rho a'_{\beta N} \underbrace{\partial_N x'^\alpha}_{=0} \underbrace{\partial_N x'^N}_{=1}) \end{cases} \tag{13}$$

$$\Rightarrow \quad [NN,\ \rho] = \frac{1}{2}\partial_\rho a'_{NN} \tag{14}$$

$$\Leftrightarrow \quad [NN,\ \rho] = \underbrace{\frac{1}{2}\partial_\alpha a'_{NN}}_{=[NN,\ \alpha]'} \partial_\rho x'^\alpha \tag{15}$$

$$\Rightarrow \quad [NN,\ \rho] = [NN,\ \alpha]' \partial_\rho x'^\alpha \tag{16}$$

This confirms the tensor character of $[NN,\ \rho]$ and consequently of $[\rho N,\ N]$

◇

$[NN,\ N]$

$$
\begin{aligned}
[NN,\ N] &= \frac{1}{2}\partial_N a_{NN} \quad \text{see (2.639)} && (17)\\
&= \frac{1}{2}\partial_N (a'_{mn}\partial_N x'^m \partial_N x'^n) && (18)\\
&= \begin{cases} \frac{1}{2}(\partial_N a'_{mn}\underbrace{\partial_N x'^m}_{\delta^m_N}\underbrace{\partial_N x'^n}_{\delta^n_N} \\ \\ +a'_{mn}\partial_N x'^n \underbrace{\partial_{NN} x'^m}_{=0} \\ \\ +a'_{mn}\partial_N x'^m \underbrace{\partial_{NN} x'^n}_{=0}) \end{cases} && (19)\\
&= \frac{1}{2}\partial_N a'_{mn}\delta^m_N \delta^n_N && (20)\\
&= \underbrace{\frac{1}{2}\partial_N a'_{NN}}_{=[NN,\ N]'} && (21)\\
\Rightarrow \quad [NN,\ N] &= [NN,\ N]' && (22)
\end{aligned}
$$

This confirms the tensor character of $[NN,\ N]$ as an invariant under transformation of the coordinates $x^1, \ldots, x^{N-1}$

$\diamond$

$$\Gamma^\rho_{N\sigma},\ \Gamma^N_{\rho\sigma},\ \Gamma^\rho_{NN},\ \Gamma^N_{N\rho},\ \Gamma^N_{NN}$$

For the Christoffel symbols of the second kind we use:

$$
\begin{aligned}
\Gamma^r_{st} &= a^{rk}[st,\ k] && (23)\\
a^{rk} &= a'^{mn}\frac{\partial x^r}{\partial x'^m}\frac{\partial x^k}{\partial x'^n} && (24)\\
\text{(2.631) page 65:} \qquad a^{N\rho} &= 0 && (25)
\end{aligned}
$$

$\Gamma^{\rho}_{N\sigma}$

$$\Gamma^{\rho}_{N\sigma} = a^{\rho k}[N\sigma,\ k] \tag{26}$$
$$= a^{\rho\tau}[N\sigma,\ \tau] + \underbrace{a^{\rho N}}_{=0}[N\sigma,\ N] \tag{27}$$
$$(24):\quad = a^{,mn}\frac{\partial x^{\rho}}{\partial x^{,m}}\frac{\partial x^{\tau}}{\partial x^{,n}}[N\sigma,\ \tau] \tag{28}$$
$$\text{also (see previous results):}\quad [N\sigma,\ \tau] = [N\mu,\ \nu]^{,}\partial_{\sigma}x^{,\mu}\partial_{\tau}x^{,\nu} \tag{29}$$

And so,

$$\Gamma^{\rho}_{N\sigma} = a^{,mn}\frac{\partial x^{\rho}}{\partial x^{,m}}\frac{\partial x^{\tau}}{\partial x^{,n}}[N\mu,\ \nu]^{,}\partial_{\sigma}x^{,\mu}\partial_{\tau}x^{,\nu} \tag{30}$$
$$= a^{,mn}\frac{\partial x^{\rho}}{\partial x^{,m}}\underbrace{\frac{\partial x^{,\nu}}{\partial x^{,n}}}_{=\delta^{\nu}_{n}}[N\mu,\ \nu]^{,}\partial_{\sigma}x^{,\mu} \tag{31}$$
$$= a^{,\theta\nu}[N\mu,\ \nu]^{,}\frac{\partial x^{\rho}}{\partial x^{,\theta}}\frac{\partial x^{,\mu}}{\partial x^{\sigma}} + \underbrace{a^{,N\nu}}_{=0}[N\mu,\ \nu]^{,}\frac{\partial x^{\rho}}{\partial x^{,N}}\frac{\partial x^{,\mu}}{\partial x^{\sigma}} \tag{32}$$
$$= \underbrace{a^{,\theta\nu}[N\mu,\ \nu]^{,}}_{=\Gamma^{,\theta}_{N\mu}}\frac{\partial x^{\rho}}{\partial x^{,\theta}}\frac{\partial x^{,\mu}}{\partial x^{\sigma}} \tag{33}$$
$$\Rightarrow\quad \Gamma^{\rho}_{N\sigma} = \Gamma^{,\theta}_{N\mu}\frac{\partial x^{\rho}}{\partial x^{,\theta}}\frac{\partial x^{,\mu}}{\partial x^{\sigma}} \tag{34}$$

So, $\Gamma^{\rho}_{N\sigma}$ is a 2^{nd} order mixed tensor (contravariant in ρ, covariant in σ.)

⋄

$\Gamma^{N}_{\rho\sigma}$

$$\Gamma^{N}_{\rho\sigma} = a^{Nk}[\rho\sigma,\ k] \tag{35}$$
$$= \underbrace{a^{N\tau}}_{=0}[\rho\sigma,\ \tau] + a^{NN}[\rho\sigma,\ N] \tag{36}$$
$$(8):\quad = a^{,NN}[\alpha\beta,\ N]^{,}\partial_{\rho}x^{,\alpha}\partial_{\sigma}x^{,\beta} \tag{37}$$
$$\text{considering :}\quad a^{,N\tau} = 0 \quad \text{we can write this as:} \tag{38}$$
$$\Gamma^{N}_{\rho\sigma} = a^{,N\tau}[\alpha\beta,\ \tau]^{,}\partial_{\rho}x^{,\alpha}\partial_{\sigma}x^{,\beta} + a^{,NN}[\alpha\beta,\ N]^{,}\partial_{\rho}x^{,\alpha}\partial_{\sigma}x^{,\beta} \tag{39}$$
$$= \underbrace{a^{,Nk}[\alpha\beta,\ k]^{,}}_{=\Gamma^{,N}_{\alpha\beta}}\partial_{\rho}x^{,\alpha}\partial_{\sigma}x^{,\beta} \tag{40}$$
$$\Rightarrow\quad \Gamma^{N}_{\rho\sigma} = \Gamma^{,N}_{\alpha\beta}\partial_{\rho}x^{,\alpha}\partial_{\sigma}x^{,\beta} \tag{41}$$

So, $\Gamma^{N}_{\rho\sigma}$ is a 2^{nd} order mixed tensor (covariant in both indices)

⋄

$\boldsymbol{\Gamma^{\rho}_{NN}}$

$$
\begin{aligned}
\Gamma^{\rho}_{NN} &= a^{\rho k}[NN,\ k] && (42)\\
&= a^{\rho\tau}[NN,\ \tau] + \underbrace{a^{\rho N}}_{=0}[NN,\ N] && (43)\\
(16){:}\quad &= a^{,mn}\frac{\partial x^{\rho}}{\partial x^{,m}}\frac{\partial x^{\tau}}{\partial x^{,n}}[NN,\ \alpha]^{,}\partial_{\tau}x^{,\alpha} && (44)\\
&= a^{,mn}\frac{\partial x^{\rho}}{\partial x^{,m}}\underbrace{\frac{\partial x^{\alpha}}{\partial x^{,n}}}_{=\delta^{\alpha}_{n}}[NN,\ \alpha]^{,} && (45)\\
&= a^{,\tau\alpha}[NN,\ \alpha]^{,}\frac{\partial x^{\rho}}{\partial x^{,\tau}} + \underbrace{a^{,N\alpha}}_{=0}[NN,\ \alpha]^{,}\frac{\partial x^{\rho}}{\partial x^{,N}} && (46)\\
&= \underbrace{a^{,\tau\alpha}[NN,\ \alpha]^{,}}_{=\Gamma^{,\tau}_{NN}}\frac{\partial x^{\rho}}{\partial x^{,\tau}} && (47)\\
\Rightarrow\quad \Gamma^{\rho}_{NN} &= \Gamma^{,\tau}_{NN}\frac{\partial x^{\rho}}{\partial x^{,\tau}} && (48)
\end{aligned}
$$

So, Γ^{ρ}_{NN} is a 1^{st} order contravariant tensor.

◇

$\boldsymbol{\Gamma^{N}_{N\rho}}$

$$
\begin{aligned}
\Gamma^{N}_{N\rho} &= a^{Nk}[N\rho,\ k] && (49)\\
&= a^{NN}[N\rho,\ N] + \underbrace{a^{N\tau}}_{=0}[N\rho,\ \tau] && (50)\\
(16){:}\quad &= a^{,NN}[N\alpha,\ N]^{,}\partial_{\rho}x^{,\alpha} && (51)\\
\text{considering}:\quad a^{,N\tau} &= 0 \quad \text{we can write this as:} && (52)\\
&= a^{,N\tau}[N\alpha,\ \tau]^{,}\partial_{\rho}x^{,\alpha} + a^{,NN}[N\alpha,\ N]^{,}\partial_{\rho}x^{,\alpha} && (53)\\
&= \underbrace{a^{,Nk}[N\alpha,\ k]^{,}}_{=\Gamma^{,N}_{N\alpha}}\partial_{\rho}x^{,\alpha} && (54)\\
\Rightarrow\quad \Gamma^{N}_{N\rho} &= \Gamma^{,N}_{N\alpha}\partial_{\rho}x^{,\alpha} && (55)
\end{aligned}
$$

So, $\Gamma^{N}_{N\rho}$ is a 1^{st} order covariant tensor.

◇

$\boldsymbol{\Gamma^{N}_{NN}}$

$$\begin{aligned}
\Gamma^{N}_{NN} &= a^{Nk}[NN,\ k] && (56)\\
&= a^{NN}[NN,\ N] + \underbrace{a^{N\tau}}_{=0}[NN,\ \tau] && (57)\\
&= a^{,NN}[NN,\ N] && (58)\\
(22){:}\quad &= a^{,NN}[NN,\ N]^{,} && (59)\\
\Rightarrow \quad \Gamma^{N}_{NN} &= \Gamma^{,N}_{NN} && (60)
\end{aligned}$$

So, Γ^{N}_{NN} is an invariant.

♦

2.56 p79-exercise 10

If θ, ϕ are colatitude and azimuth on a sphere, and we take

$$x^1 = \theta\cos\phi, \; x^2 = \theta\sin\phi$$

Calculate the Christoffels symbols for the coordinate system x^1, x^2 and show that they vanish at the point $\theta = 0$

We have, page 48 (2.507) :
$$\Gamma'^{r}_{mn} = \Gamma^{s}_{pq}\frac{\partial x'^{r}}{\partial x^{s}}\frac{\partial x^{p}}{\partial x'^{m}}\frac{\partial x^{q}}{\partial x'^{n}} + \frac{\partial x'^{r}}{\partial x^{s}}\frac{\partial^2 x^{s}}{\partial x'^{m}\partial x'^{n}} \tag{1}$$

The calculations are really basic but lengthy and only train your skills in basic calculus. So I will only calculate Γ'^{1}_{11}.
We know that in a spherical coordinate system all Γ'^{s}_{pq} vanish except for $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^{\phi}_{\phi\theta} = \cot\theta$.

$$\boldsymbol{\Gamma'^{1}_{11} = \Gamma^{\theta}_{\phi\phi}\partial_\theta x^1\partial_1\phi\partial_1\phi + 2\Gamma^{\phi}_{\phi\theta}\partial_\phi x^1\partial_1\theta\partial_1\phi + \partial_\theta x^1\partial^2_{11}\theta + \partial_\phi x^1\partial^2_{11}\phi} \tag{2}$$

$$\text{we have}\quad \begin{cases} x^1 = \theta\cos\phi \\ x^2 = \theta\sin\phi \end{cases} \Rightarrow \begin{cases} \theta = \sqrt{(x^1)^2 + (x^2)^2} \\ \phi = \arctan\frac{x^2}{x^1} \end{cases} \tag{3}$$

$$\text{so}\begin{cases} \partial_1\theta = \frac{x^1}{\theta} = \cos\phi & \partial_1\phi = -\frac{x^2}{\theta^2} = -\frac{\sin\phi}{\theta} \\ \\ \partial_\theta x^1 = \cos\phi & \partial_\phi x^1 = -\theta\sin\phi \\ \\ \partial^2_{11}\theta = \frac{\theta^2-(x^1)^2}{\theta^3} = \frac{\sin^2\phi}{\theta} & \partial^2_{11}\phi = 2\frac{x^1x^2}{\theta^4} = 2\frac{\cos\phi\sin\phi}{\theta^2} \end{cases} \tag{4}$$

$$\Gamma'^{1}_{11} = \Gamma^{\theta}_{\phi\phi}\cos\phi\frac{\sin^2\phi}{\theta^2} + 2\Gamma^{\phi}_{\phi\theta}(-\theta\sin\phi)\cos\phi(-\frac{\sin\phi}{\theta}) + \cos\phi\frac{\sin^2\phi}{\theta} + (-\theta\sin\phi)(2\frac{\cos\phi\sin\phi}{\theta^2}) \tag{5}$$

$$= -\sin\theta\cos\theta\frac{\cos\phi\sin^2\phi}{\theta^2} + 2\cot\theta\cos\phi\sin^2\phi + \frac{\cos\phi\sin^2\phi}{\theta} - 2\frac{\cos\phi\sin^2\phi}{\theta} \tag{6}$$

$$= (2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2})\cos\phi\sin^2\phi \tag{7}$$

◇

Does $\mathbf{\Gamma^{;1}_{11}}$ vanish for $\theta \to 0$?
The problematic term in (7) is $2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2}$ for which $\lim_{\theta\to 0} = \pm\infty \mp \infty \mp \infty$ is undefined.

$$\begin{aligned}
\text{Consider} \quad L_+ &= \lim_{\theta\to 0_+} 2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} && (8)\\
\text{We have} \quad & \sin\theta, \theta \geq 0 \quad \sin\theta \leq \theta \quad \tfrac{1}{\theta} \leq \tfrac{1}{\sin\theta} \quad 0 \geq \cos\theta \leq 1 && (9)\\
\text{so} \quad L_+ &= \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\sin\theta} - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} && (10)\\
&\geq \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\theta} - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} && (11)\\
&\geq \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\theta} - \frac{1}{\theta} - \frac{\cancel{\theta}\cos\theta}{\theta^{\cancel{2}}} && (12)\\
&\geq \lim_{\theta\to 0_+} \frac{\cos\theta - 1}{\theta} && (13)\\
\text{(l'Hospitale rule)} \Rightarrow \quad L_+ &\geq 0 && (14)\\
\text{also} \quad L_+ &= \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\sin\theta} - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} && (15)\\
&\leq \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\sin\theta} - \frac{\cos\theta}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} && (16)\\
&\leq \lim_{\theta\to 0_+} \cos\theta\left(\frac{2}{\sin\theta} - \frac{1}{\theta} - \frac{\sin\theta}{\theta^2}\right) && (17)\\
\frac{\sin\theta}{\theta} \leq 1 \quad \Rightarrow \quad &\leq \lim_{\theta\to 0_+} \cos\theta\left(\frac{2}{\sin\theta} - \frac{1}{\theta} - \frac{1}{\theta}\right) && (18)\\
\frac{1}{\sin\theta} \geq \frac{1}{\theta} \quad \Rightarrow \quad &\leq \lim_{\theta\to 0_+} \cos\theta\left(\frac{2}{\theta} - \frac{1}{\theta} - \frac{1}{\theta}\right) && (19)\\
&\leq 0 && (20)\\
\text{(14) and ((20)} \quad &\Rightarrow L_+ = 0 && (21)
\end{aligned}$$

Considering that

$$L_- = \lim_{\theta\to 0_-} 2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2}$$

is equivalent to

$$L_+^\alpha = \lim_{\alpha\to 0_+} -(2\cot\alpha - \frac{1}{\alpha} - \frac{\sin\alpha\cos\alpha}{\alpha^2})$$

(substitute $\theta = -\alpha$) we conclude that

$$L_- = -L_+ = 0$$

. And so

$$\Gamma^{;1}_{11}\Big|_{\theta=0} = (2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2})\cos\phi\sin^2\phi\Big|_{\theta=0} = 0$$

♦

2.57 p79-exercise 11

If vectors T^r and S_r undergo parallel propagation along a curve, show that $T^n S_n$ is constant along the curve.

Parallel propagation of T^r and S_r along a curve means

$$\frac{\delta T^r}{\delta u} \equiv \frac{dT^r}{du} + \Gamma^r_{mn} T^m \frac{dx^n}{du} = 0 \tag{1}$$

$$\frac{\delta S_r}{\delta u} \equiv \frac{dS_r}{du} - \Gamma^m_{rn} S_m \frac{dx^n}{du} = 0 \tag{2}$$

$$\text{Hence} \quad \frac{dT^r S_r}{du} = S_r \left(-\Gamma^r_{mn} T^m \frac{dx^n}{du} \right) + T^r \left(\Gamma^m_{rn} S_m \frac{dx^n}{du} \right) \tag{3}$$

$$= \frac{dx^n}{du} \left(\Gamma^m_{rn} S_m T^r - \Gamma^m_{rn} T^r S_m \right) \tag{4}$$

$$= 0 \tag{5}$$

$$\Rightarrow \qquad T^r S_r = C^{st} \tag{6}$$

♦

2.58 p79-exercise 12

Deduce from 2.201 that the determinant $a = |a_{mn}|$ transforms according to

$$a' = aJ^2 \quad , \quad J = \left| \frac{\partial x^r}{\partial x'^s} \right|$$

$$a'_{rs} = a_{mk} \frac{\partial x^m}{\partial x'^r} \frac{\partial x^k}{\partial x'^s} \tag{1}$$

$$\text{Be} \qquad (J_{mr}) = \left(\frac{\partial x^m}{\partial x'^r} \right) \tag{2}$$

$$\text{In (1) put} \qquad c_{kr} = a_{mk} \frac{\partial x^m}{\partial x'^r} = a_{km} \frac{\partial x^m}{\partial x'^r} \tag{3}$$

$$\Rightarrow \qquad a'_{rs} = c_{kr} \frac{\partial x^k}{\partial x'^s} \tag{4}$$

$$\text{or in matrix form} \qquad A' = C^T J \quad \text{with} \quad C = AJ \tag{5}$$

$$\Rightarrow \qquad |A'| = \left| (AJ)^T J \right| \tag{6}$$

$$\Leftrightarrow \qquad |A'| = \left| J^T A^T J \right| \tag{7}$$

$$\Leftrightarrow \qquad |A'| = |A|\,|J|\,|J| \tag{8}$$

$$\Leftrightarrow \qquad |A'| = |A|\,|J|^2 \tag{9}$$

♦

2.59 p79-exercise 13

Using local Cartesians and applying the result of the previous exercise (N° 12), prove that, if the metric form is positive-definite, then the determinant $a = |a_{mn}|$ is always positive.

Using local Cartesian coordinates we have:

$$\Phi = \epsilon_i (dy^i)^2 \quad \text{with} \quad \Phi > 0 \tag{1}$$

$$\text{so} \quad |a_{mn}| = \begin{vmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & \epsilon_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \epsilon_N \end{vmatrix} = \prod_{i=1}^{N} \epsilon_i > 0 \tag{2}$$

$$\tag{3}$$

Going from (a_{mn}) to any arbitrary coordinate system, we have (see exercise 12 page 79)

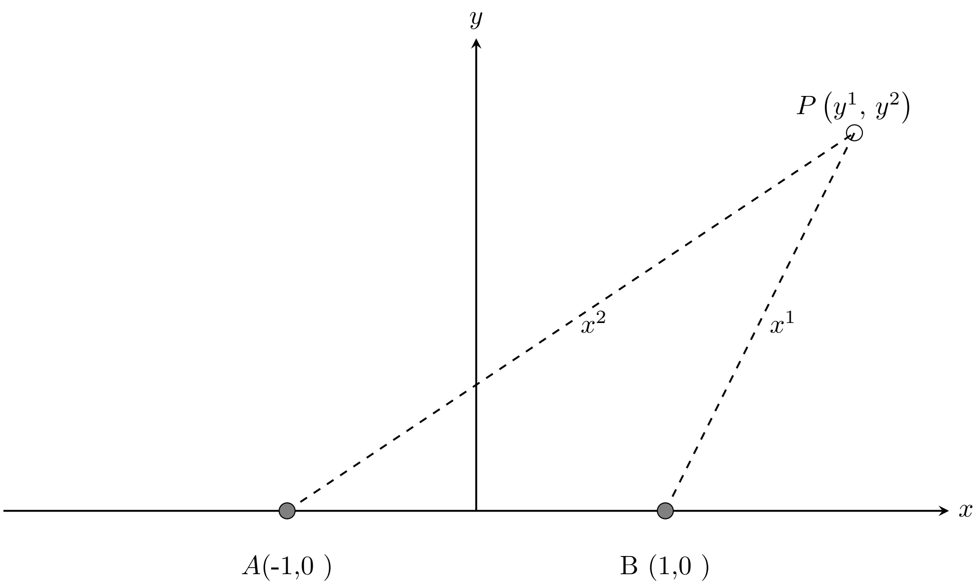
$$|a'_{mn}| = \underbrace{|a_{mn}|}_{>0} \underbrace{J^2}_{>0} \tag{4}$$

$$\Rightarrow \quad |a'_{mn}| > 0 \tag{5}$$

♦

2.60 p79-exercise 14

In a plane, let $x^1,\ x^2$ be the distances of a general point from the point with rectangular coordinates $(1,0)$, $(-1,0)$, respectively. (These are bipolar coordinates.) Find the line element for these coordinates, and find the conjugate tensor a^{mn}.



$$(x^1)^2 = \left(y^1-1\right)^2 + \left(y^2\right)^2 \tag{1}$$

$$(x^2)^2 = \left(y^1+1\right)^2 + \left(y^2\right)^2 \tag{2}$$

$$\partial_1(1) \quad \Rightarrow \quad x^1 = \left(y^1-1\right)\partial_1 y^1 + \left(y^2\right)\partial_1 y^2 \tag{3}$$

$$\partial_2(2) \quad \Rightarrow \quad x^2 = \left(y^1+1\right)\partial_2 y^1 + \left(y^2\right)\partial_2 y^2 \tag{4}$$

$$\partial_2(1) \quad \Rightarrow \quad 0 = \left(y^1-1\right)\partial_2 y^1 + \left(y^2\right)\partial_2 y^2 \tag{5}$$

$$\partial_1(2) \quad \Rightarrow \quad 0 = \left(y^1+1\right)\partial_1 y^1 + \left(y^2\right)\partial_1 y^2 \tag{6}$$

$$\Rightarrow \quad \begin{cases} \text{(3)-(6):} \quad \partial_1 y^1 = -\frac{x^1}{2} \\ \\ \text{(4)-(5):} \quad \partial_2 y^1 = \frac{x^2}{2} \\ \\ \text{(6):} \quad \partial_1 y^2 = \frac{y^1+1}{2y^2}x^1 \\ \\ \text{(5):} \quad \partial_2 y^2 = -\frac{y^1-1}{2y^2}x^2 \end{cases} \tag{7}$$

$$\text{Hence,} \quad J = \left(\frac{\partial y^m}{\partial x^n}\right) = \begin{pmatrix} -\frac{x^1}{2} & \frac{x^2}{2} \\ \frac{y^1+1}{2y^2}x^1 & -\frac{y^1-1}{2y^2}x^2 \end{pmatrix} \tag{8}$$

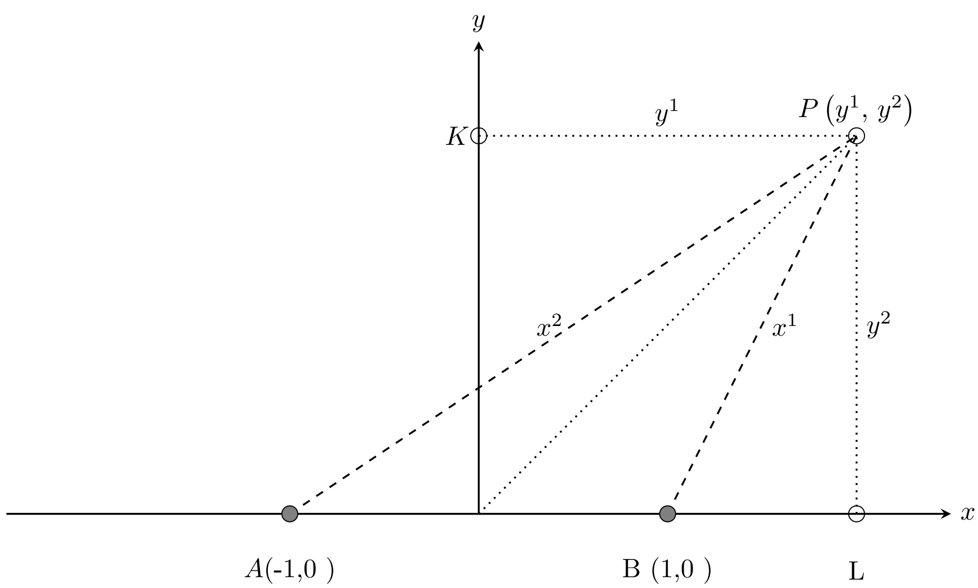
Be A the metric tensor in Cartesian coordinate system. Then, going to a arbitrary coordinate system gives a metric tensor according to

$$A' = J^T A J = J^T J \quad \text{as} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

$$\Rightarrow \quad A' = \begin{pmatrix} -\frac{x^1}{2} & \frac{y^1+1}{2y^2}x^1 \\ \frac{x^2}{2} & -\frac{y^1-1}{2y^2}x^2 \end{pmatrix} \begin{pmatrix} -\frac{x^1}{2} & \frac{x^2}{2} \\ \frac{y^1+1}{2y^2}x^1 & -\frac{y^1-1}{2y^2}x^2 \end{pmatrix} \tag{10}$$

$$\Rightarrow \quad A' = \frac{x^1 x^2}{4(y^2)^2} \begin{pmatrix} x^1 x^2 & -\left[(y^1)^2 + (y^2)^2 - 1)\right] \\ -\left[(y^1)^2 + (y^2)^2 - 1)\right] & x^1 x^2 \end{pmatrix} \tag{11}$$

We use plane geometry to express expression (11) as a function of only (x^1, x^2). For the triangle APB in the figure below, we have



$$|OP|^2 = \frac{(x^1)^2 + (x^2)^2}{2} - \underbrace{\frac{|AB|^2}{4}}_{=1} \tag{12}$$

$$\Rightarrow \quad (y^1)^2 + (y^2)^2 = \frac{(x^1)^2 + (x^2)^2}{2} - 1 \tag{13}$$

The area K of the triangle can be expressed in two ways

$$K = \frac{1}{2} y^2 \, |AB| = y^2 \tag{14}$$

$$\text{and} \quad K = \sqrt{s(s - x^1)(s - x^2)(s - 2)} \tag{15}$$

$$\text{with} \quad s = \frac{1}{2}(x^1 + x^2 + 2) \tag{16}$$

$$\text{hence} \quad (y^2)^2 = \frac{1}{16}(x^1 + x^2 + 2)(x^1 + 2)(x^2 + 2)(x^1 + x^2) \tag{17}$$

So we get from (11):

$$A^{,} = \frac{4x^1x^2}{(x^1 + x^2 + 2)(x^1 + 2)(x^2 + 2)(x^1 + x^2)} \begin{pmatrix} x^1x^2 & 2 - \frac{(x^1)^2 + (x^2)^2}{2} \\ 2 - \frac{(x^1)^2 + (x^2)^2}{2} & x^1x^2 \end{pmatrix} \tag{18}$$

For the conjugate metric tensor we start from expression (11) and invert the metric tensor

$$|A^{,}| = \left[\frac{x^1x^2}{4(y^2)^2}\right]^2 \left[(x^1x^2)^2 - \left((y^1)^2 + (y^2)^2 - 1\right)^2\right] \tag{19}$$

$$\text{hence} \quad (a^{,mn}) = \frac{1}{|A^{,}|} \frac{x^1x^2}{4(y^2)^2} \begin{pmatrix} x^1x^2 & \left[(y^1)^2 + (y^2)^2 - 1\right] \\ \left[(y^1)^2 + (y^2)^2 - 1\right] & x^1x^2 \end{pmatrix} \tag{20}$$

$$= \frac{4(y^2)^2 \begin{pmatrix} x^1x^2 & \left[(y^1)^2 + (y^2)^2 - 1\right] \\ \left[(y^1)^2 + (y^2)^2 - 1\right] & x^1x^2 \end{pmatrix}}{x^1x^2 \left[(x^1x^2)^2 - ((y^1)^2 + (y^2)^2 - 1)^2\right]} \tag{21}$$

$$\text{with} \quad x^1x^2 = \sqrt{[(y^1 - 1)^2 + (y^2)^2]\,[(y^1 + 1)^2 + (y^2)^2]} \tag{22}$$

♦

2.61 p79-exercise 15

Given $\Phi = a_{mn}dx^m dx^n$, with $a_{11} = a_{22} = 0$ but $a_{12} \neq 0$, show that Φ may be written in the form

$$\Phi = \epsilon\Psi_1^2 - \epsilon\Psi_2^2 + \Phi_2$$

where Φ_2 is a homogeneous quadratic form in $dx^3, dx^4, \ldots, dx^N$, $\epsilon = \pm 1$, and where

$$\Psi_1 = \frac{1}{\sqrt{(2\epsilon a_{12})}}\left[a_{12}(dx^1 + dx^2) + (a_{13} + a_{23})dx^3 + \cdots + (a_{1N} + a_{2N})dx^N\right]$$

$$\Psi_2 = \frac{1}{\sqrt{(2\epsilon a_{12})}}\left[a_{12}(-dx^1 + dx^2) + (a_{13} - a_{23})dx^3 + \cdots + (a_{1N} - a_{2N})dx^N\right]$$

Using local Cartesian coordinates we have: Be

$$\Phi = a_{mn}dx^m dx^n \tag{1}$$
$$\tag{2}$$

and consider the following sequences of terms:

$$\Psi_1 = b_{11}dx^1 + b_{12}dx^2 + b_{13}dx^3 + \cdots + b_{1N}dx^N \tag{3}$$
$$\Psi_2 = b_{21}dx^1 + b_{22}dx^2 + b_{23}dx^3 + \cdots + b_{2N}dx^N \tag{4}$$

The expressions (3) and (4) contain all the terms of Φ with $(dx^1)^2, (dx^2)^2$ and dx^1dx^2. So, Φ can be expressed as

$$\Phi = \Psi_1^2 - \Psi_2^2 + \Phi_2 \tag{5}$$

With Φ_2 being a homogeneous form containing only terms in $dx^i dx^j \quad i, j > 2$.
We can express $\Psi_1^2 - \Psi_2^2$ as

$$\Psi_1^2 - \Psi_2^2 = \begin{cases} (b_{11}^2 - b_{21}^2)(dx^1)^2 + (b_{12}^2 - b_{22}^2)(dx^2)^2 + (b_{13}^2 - b_{23}^2)(dx^3)^2 + \cdots + (b_{1N}^2 - b_{2N}^2)(dx^N)^2 + \\ 2(b_{11}b_{12} - b_{21}b_{22})dx^1dx^2 + 2(b_{11}b_{13} - b_{21}b_{23})dx^1dx^3 + \cdots + 2(b_{11}b_{1N} - b_{21}b_{2N})dx^1dx^N + \\ 2(b_{12}b_{13} - b_{22}b_{23})dx^2dx^3 + 2(b_{12}b_{14} - b_{22}b_{24})dx^2dx^4 + \cdots + 2(b_{12}b_{1N} - b_{21}b_{2N})dx^2dx^N + \\ + \ldots 2(b_{1j}b_{1k} - b_{2j}b_{2k})dx^kdx^j + \ldots \quad j > 2,\ k \neq j \end{cases} \tag{6}$$

Equating the terms in (1) and (6) and taking into account (=given) $a_{11} = a_{22} = 0$ and $a_{12} \neq 0$

$$\begin{cases} b_{11}^2 - b_{21}^2 = 0 \\ b_{12}^2 - b_{22}^2 = 0 \\ 2\left(b_{11}b_{12} - b_{21}b_{22}\right) = a_{12} + a_{21} \end{cases} \tag{7}$$

$$\Rightarrow \quad \begin{cases} b_{11}^2 = b_{21}^2 \\ b_{12}^2 = b_{22}^2 \\ \left(b_{11}b_{12} - b_{21}b_{22}\right) = a_{12} \end{cases} \tag{8}$$

$$\Rightarrow \quad b_{11} = \pm b_{21} \quad b_{22} = \pm b_{12} \tag{9}$$

$$\Rightarrow \quad \begin{cases} \pm b_{11}b_{22} \mp b_{11}b_{22} = a_{12} \\ \text{or} \\ \pm b_{12}b_{21} \mp b_{12}b_{21} = a_{12} \end{cases} \tag{10}$$

$$\tag{11}$$

As $a_{12} \neq 0$, the signs in the above expressions must be the same. Hence,

$$\begin{cases} 2b_{11}b_{22} = \pm a_{12} \\ 2b_{12}b_{21} = \pm a_{12} \end{cases} \tag{12}$$

(12) can be satisfied with an infinite combination of $b_{11}, b_{12}, b_{21}, b_{22}$. Choose,

$$b_{11} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{22} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{12} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{21} = -\sqrt{\frac{\epsilon a_{12}}{2}} \tag{13}$$

$$\text{with} \quad \epsilon = \pm 1 \quad \text{so that} \quad \epsilon a_{12} \geq 0 \tag{14}$$

Put $\xi = \sqrt{\frac{\epsilon a_{12}}{2}}$. Then, (3) and (4) can be expressed as

$$\Psi_1 = \xi dx^1 + \xi dx^2 + b_{13}dx^3 + \cdots + b_{1N}dx^1dx^N \tag{15}$$

$$\Psi_2 = -\xi dx^1 + \xi dx^2 + b_{23}dx^3 + \cdots + b_{2N}dx^1dx^N \tag{16}$$

What are the $b_{\cdot j} \quad (j > 2)$?
From (6), e.g. for $j = 3$, identifying the terms in dx^1dx^3 and dx^2dx^3 we see that

$$\begin{cases} \xi b_{13} + \xi b_{23} = a_{13} \\ \xi b_{13} - \xi b_{23} = a_{23} \end{cases} \tag{17}$$

$$\Rightarrow \quad \begin{cases} b_{13} = \frac{a_{13}+a_{23}}{2\xi} \\ b_{23} = \frac{a_{13}-a_{23}}{2\xi} \end{cases} \tag{18}$$

$$\text{or, in general} \quad \begin{cases} b_{1j} = \frac{a_{1j}+a_{2j}}{2\xi} \\ \\ b_{2j} = \frac{a_{1j}-a_{2j}}{2\xi} \end{cases} \quad (j > 2) \tag{19}$$

Bring $\frac{1}{\xi}$ out in Ψ_1, Ψ_2. This gives

$$\Psi_1 = \frac{1}{\xi}\left(\xi^2(dx^1 + dx^2) + \frac{a_{13} + a_{23}}{2}dx^3 + \cdots + \frac{a_{1N} + a_{2N}}{2}dx^N\right) \tag{20}$$

$$\Psi_2 = \frac{1}{\xi}\left(\xi^2(-dx^1 + dx^2) + \frac{a_{13} - a_{23}}{2}dx^3 + \cdots + \frac{a_{1N} - a_{2N}}{2}dx^N\right) \tag{21}$$

$$\Rightarrow \quad \begin{cases} \Psi_1 = \frac{\sqrt{2}}{\sqrt{\epsilon a_{12}}}\left(\frac{\epsilon a_{12}}{2}(dx^1 + dx^2) + \frac{a_{13}+a_{23}}{2}dx^3 + \cdots + \frac{a_{1N}+a_{2N}}{2}dx^N\right) \\ \\ \Psi_2 = \frac{\sqrt{2}}{\epsilon\sqrt{\epsilon a_{12}}}\left(\frac{\epsilon a_{12}}{2}(-dx^1 + dx^2) + \frac{a_{13}-a_{23}}{2}dx^3 + \cdots + \frac{a_{1N}-a_{2N}}{2}dx^N\right) \end{cases} \tag{22}$$

$$\Leftrightarrow \quad \begin{cases} \Psi_1 = \frac{1}{\sqrt{2\epsilon a_{12}}}\left(\epsilon a_{12}(dx^1 + dx^2) + (a_{13} + a_{23})dx^3 + \cdots + (a_{1N} + a_{2N})dx^N\right) \\ \\ \Psi_2 = \frac{1}{\sqrt{2\epsilon a_{12}}}\left(\epsilon a_{12}(-dx^1 + dx^2) + (a_{13} - a_{23})dx^3 + \cdots + (a_{1N} - a_{2N})dx^N\right) \end{cases} \tag{23}$$

What about the factor $\epsilon = \pm 1$ in the terms $\epsilon a_{12}(dx^1 + dx^2)$ and $\epsilon a_{12}(-dx^1 + dx^2)$? Consider the alternate form

$$\begin{cases} \Psi'_1 = \frac{1}{\sqrt{2\epsilon a_{12}}}\left(a_{12}(dx^1 + dx^2) + (a_{13} + a_{23})dx^3 + \cdots + (a_{1N} + a_{2N})dx^N\right) \\ \\ \Psi'_2 = \frac{1}{\sqrt{2\epsilon a_{12}}}\left(a_{12}(-dx^1 + dx^2) + (a_{13} - a_{23})dx^3 + \cdots + (a_{1N} - a_{2N})dx^N\right) \end{cases} \tag{24}$$

Let's have a look at the terms in $dx^i dx^j$ in $\Psi'^2_1 - \Psi'^2_2$

$$
\begin{aligned}
x_{11}(dx^1)^2 + 2x_{12}dx^1dx^2 + x_{22}(dx^2)^2 &= \frac{1}{2\epsilon a_{12}}\left((a_{12})^2(dx^1+dx^2)^2 - (a_{12})^2(-dx^1+dx^2)^2\right) && (25)\\
&= \frac{\epsilon a_{12}}{2}\left((dx^1+dx^2)^2 - (-dx^1+dx^2)^2\right) && (26)\\
&= \frac{\epsilon a_{12}}{2}\left((dx^1)^2 + 2dx^1dx^2 + (dx^2)^2 - (dx^1)^2 + 2dx^1dx^2 - (dx^2)^2\right) && (27)\\
&= 2\epsilon a_{12}dx^1dx^2 && (28)\\
2x_{13}dx^1dx^3 &= \frac{1}{2\epsilon a_{12}}(2a_{12}(a_{13}+a_{23})dx^1dx^3 - 2a_{12}(a_{13}-a_{23})dx^1dx^3) && (29)\\
&= \frac{1}{2\epsilon a_{12}}(4a_{12}a_{23})dx^1dx^3 && (30)\\
&= 2\epsilon a_{23}dx^1dx^3 && (31)\\
2x_{23}dx^2dx^3 &= \frac{1}{2\epsilon a_{12}}(2a_{12}(a_{13}+a_{23})dx^1dx^3 - 2a_{12}(a_{13}-a_{23})dx^1dx^3) && (32)\\
&= \frac{1}{2\epsilon a_{12}}(4a_{12}a_{23})dx^1dx^3 && (33)\\
&= 2\epsilon a_{23}dx^1dx^3 && (34)\\
2x_{34}dx^3dx^4 &= \frac{1}{2\epsilon a_{12}}\begin{pmatrix} 2(a_{13}+a_{23})(a_{14}+a_{24} \\ -2(a_{13}-a_{23})(a_{14}-a_{24}) \end{pmatrix} dx^3dx^4 && (35)\\
&= \frac{2\epsilon}{a_{12}}(a_{23}a_{14}+a_{13}a_{24}) && (36)
\end{aligned}
$$

We rewrite now the metric form Φ as

$$\Phi = \epsilon\Psi_1'^2 - \epsilon\Psi_2'^2 + \Phi_2$$

From (28), (31) and (34) we see that the terms in dx^1, dx^2 in $\Phi = \epsilon\Psi_1'^2 - \epsilon\Psi_2'^2 + \Phi_2$ correspond to the expected metric form and that the other terms in $dx^j, dx^k,\ j,k \neq 1,2$ can be corrected in the remaining term Φ_2.

2.62 p80-exercise 16

Find the null geodesics of a 4-space with line element

$$ds^2 = \epsilon\gamma\left(dx^2 + dy^2 + dz^2 - dt^2\right)$$

where γ is an arbitrary function of $x,\ y,\ z,\ t$.

We have:

$$(a_{mn}) = \gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (a^{mn}) = \frac{1}{\gamma} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

$$\tag{2}$$

Be $(x^1, x^2, x^3, x4) \equiv (x, y, z, t)$.
The general conditions for a null geodesic are

$$\begin{cases} \frac{d^2x^r}{dv^2} + \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = 0 \\ \\ a_{mn}dx^m dx^n = 0 \end{cases} \tag{3}$$

When calculating $[mn, r] = \frac{1}{2}\left(\partial_n a_{mr} + \partial_m a_{nr} - \partial_r a_{mn}\right)$ we note that (a_{mn}) is a diagonal matrix, ans so is also (a^{mn}). Hence Γ^r_{mn} will contain only one term:

$$\Gamma^r_{mn} = a^{RR}[mn, R] \tag{4}$$

$$\Rightarrow \begin{cases} i) & m, n \neq R \ \wedge\ m \neq n & : & [mn, R] = 0 \\ \\ ii) & m \neq n = R \ \vee\ n \neq m = R & : & [mn, R] = \frac{1}{2}\partial_m a_{RR} \\ \\ iii) & m = n \neq R & : & [mn, R] = -\frac{1}{2}\partial_R a_{mn} \\ \\ iv) & m = n = R & : & [mn, R] = \frac{1}{2}\partial_R a_{RR} \\ \end{cases} \tag{5}$$

and for the Γ^r_{mn}:

$$\Rightarrow \left\{ \begin{array}{lll} i) & m,n \neq R \ \wedge \ m \neq n & : \ \Gamma^R_{mn} = 0 \\ & & \\ ii) & m \neq n = R \ \vee \ n \neq m = R & : \ \Gamma^R_{nR} = \frac{1}{2\gamma}\partial_n \gamma \\ & & \\ iii) & m = n \neq R = \ 1,2,3 & : \ \Gamma^R_{kk} = -\frac{1}{2\gamma}\partial_R \gamma \\ & m = n \neq R = 4 & : \ \Gamma^4_{kk} = \frac{1}{2\gamma}\partial_R \gamma \\ & & \\ iv) & m = n = R & : \ \Gamma^R_{RR} = \frac{1}{2\gamma}\partial_R \gamma \end{array} \right. \tag{6}$$

Let's compute

$$A_r \equiv \frac{d^2x^r}{dv^2} + \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} \tag{7}$$

and take $v = x^4$, so $x^1, \ x^2, \ x^3 = f(x^4)$. (In the following $d_k x^j \equiv \frac{dx^j}{d(x^k)}$, $d^2_k x^j \equiv \frac{d^2x^j}{d(x^k)^2}$)

$$\left\{ \begin{array}{ll} A_1 = & d_4^2 x^1 + \Gamma^1_{mn} d_4 x^m d_4 x^n \\ A_2 = & d_4^2 x^2 + \Gamma^2_{mn} d_4 x^m d_4 x^n \\ A_3 = & d_4^2 x^3 + \Gamma^3_{mn} d_4 x^m d_4 x^n \\ A_4 = & \Gamma^4_{mn} d_4 x^m d_4 x^n \end{array} \right. \tag{8}$$

and get from (8) and (6):

$$A_1 - d_4^2 x^1 = \left\{ \begin{array}{l} \frac{1}{\gamma}\partial_2\gamma \ d_4x^1 d_4x^2 + \frac{1}{\gamma}\partial_3\gamma \ d_4x^1 d_4x^3 + \frac{1}{\gamma}\partial_4\gamma \ d_4x^1 d_4x^4 \\ \\ + \quad \underbrace{\frac{1}{2\gamma}\partial_1\gamma(d_4x^1)^2}_{= \left\{ \begin{array}{l} \frac{1}{\gamma}\partial_1\gamma(d_4x^1)(d_4x^1) \\ -\frac{1}{2\gamma}\partial_1\gamma(d_4x^1)^2 \end{array} \right.} \quad -\frac{1}{2\gamma}\partial_1\gamma(d_4x^2)^2 - \frac{1}{2\gamma}\partial_1\gamma(d_4x^3)^2 + \frac{1}{2\gamma}\partial_1\gamma(d_4x^4)^2 \end{array} \right. \tag{9}$$

$$= \left\{ \begin{array}{l} \frac{1}{\gamma}\partial_1\gamma d_4x^1 d_4x^1 + \frac{1}{\gamma}\partial_2\gamma \ d_4x^1 d_4x^2 + \frac{1}{\gamma}\partial_3\gamma \ d_4x^1 d_4x^3 + \frac{1}{\gamma}\partial_4\gamma \ d_4x^1 d_4x^4 \\ \\ -\frac{1}{2\gamma}\partial_1\gamma(d_4x^1)^2 - \frac{1}{2\gamma}\partial_1\gamma(d_4x^2)^2 - \frac{1}{2\gamma}\partial_1\gamma(d_4x^3)^2 + \frac{1}{2\gamma}\partial_1\gamma(d_4x^4)^2 \end{array} \right. \tag{10}$$

$$\tag{11}$$

$$= \left\{ \begin{array}{l} \frac{1}{\gamma}d_4x^1\left(\partial_1\gamma d_4x^1 + \partial_2\gamma \ d_4x^2 + \partial_3\gamma \ d_4x^3 + \partial_4\gamma \ d_4x^4\right) \\ \\ -\frac{1}{2\gamma}\partial_1\gamma\left(\underbrace{(d_4x^1)^2 + (d_4x^2)^2 + (d_4x^3)^2 - (d_4x^4)^2}_{=a_{mn}dx^m dx^n = 0}\right) \end{array} \right. \tag{12}$$

$$\Rightarrow \quad A_1 = d_4^2 x^1 + \frac{1}{\gamma} d_4 x^1 < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} > \tag{13}$$

with $\boldsymbol{\nabla}\gamma = (\partial_x\gamma, \ \partial_y\gamma, \ \partial_z\gamma, \ \partial_t\gamma)$ and $\partial_4\overline{x} = (\partial_t x, \ \partial_t y, \ \partial_t z, \ \partial_t t)$.

So for the geodesic with get for the first coordinate

$$d_4^2 x^1 + \frac{1}{\gamma} d_4 x^1 < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} >= 0$$

Doing analogous calculations give the following set of equations

$$d_4^2 x^k + \frac{1}{\gamma} d_4 x^k < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} >= 0 \tag{14}$$

Note that for $k = 4$ we have as $d_4 x^4 = 1$ and $d_4^2 x^4 = 0$:

$$\frac{1}{\gamma} < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} >= 0 \tag{15}$$

$$\Rightarrow \quad < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} >= 0 \tag{16}$$

Note that this means that $\boldsymbol{\nabla}\gamma$ and $\partial_4 \overline{x}$ are orthogonal 4-vectors.
So the set of equations in (14) reduce to the following set of 2^{nd} order differential equations:

$$\left\{ \begin{array}{l} x" = 0 \\ y" = 0 \\ z" = 0 \\ t = t \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = x_1 t + x_0 \\ y = y_1 t + y_0 \\ z = z_1 t + z_0 \\ t = t \end{array} \right. \tag{17}$$

This is the equations of 4-space cone, which was expected as the metric is localy a Minkowksi-like metric.

♦

2.63 p80-exercise 17

In a space V_N the metric tensor is a_{mn}. Show that the null geodesics are unchanged if the metric tensor is changed to $b_{mn} = \gamma a_{mn}$, γ being a function of the coordinates.

Null geodesics are determined by the following set of equations

$$\begin{cases} \frac{d^2x^r}{dv^2} + \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = 0 \\ \\ a_{mn}dx^m dx^n = 0 \end{cases} \tag{1}$$

$$\text{We have} \quad \Gamma^r_{mn} = a^{rs}[mn, s] \tag{2}$$

$$= \frac{1}{\gamma}a^{rs}\gamma[mn, s] \tag{3}$$

$$\text{We have also} \quad b^{rs} = \frac{1}{\gamma}a^{rs} \tag{4}$$

$$\text{Indeed} \quad a^{rs}a_{rs} = \delta^r_t \tag{5}$$

$$\Rightarrow \quad \frac{1}{\gamma}a^{rs}\gamma a_{rs} = \delta^r_t \tag{6}$$

$$\Rightarrow \quad \frac{1}{\gamma}a^{rs}b_{rs} = \delta^r_t \tag{7}$$

$$\Rightarrow \quad \frac{1}{\gamma}a^{rs} = b_{rs} \tag{8}$$

$$\text{So (3) becomes} \quad \Gamma^r_{mn} = b^{rs}\gamma[mn, s] \tag{9}$$

Be $[mn, s]'$ the Christoffel symbol associated with the metric $b_{mn} = \gamma a_{mn}$. Then with

$$[mn, s]' = \frac{1}{2}\left(\partial_n\gamma a_{ms} + \partial_m\gamma a_{ns} - \partial_s\gamma a_{mn}\right)$$

we get

$$[mn, s]' = \gamma[mn, s] + \frac{1}{2}\left(a_{ms}\partial_n\gamma + a_{ns}\partial_m\gamma - a_{mn}\partial_s\gamma\right) \tag{10}$$

Substitute $\gamma[mn,s]$ from (10) in (9) we get

$$\Gamma^r_{mn} = \underbrace{b^{rs}\gamma[mn,s]}_{=\Gamma'^r_{mn}} - \underbrace{\frac{1}{2}b^{rs}\left(a_{ms}\partial_n\gamma + a_{ns}\partial_m\gamma - a_{mn}\partial_s\gamma\right)}_{:=Q} \tag{11}$$

$$\text{with} \quad Q = \frac{1}{2}b^{rs}\left(a_{ms}\partial_n\gamma + a_{ns}\partial_m\gamma - a_{mn}\partial_s\gamma\right) \tag{12}$$

$$= \frac{1}{2}\gamma\left(\underbrace{a^{rs}a_{ms}}_{=\delta^r_m}\partial_n\gamma + \underbrace{a^{rs}a_{ns}}_{=\delta^r_n}\partial_m\gamma - a^{rs}a_{mn}\partial_s\gamma\right) \tag{13}$$

$$Q \times \frac{dx^m}{dv}\frac{dx^n}{dv} = \frac{1}{2}\gamma\left(\underbrace{\delta^r_m\frac{dx^m}{dv}}_{=\frac{dx^r}{dv}}\underbrace{\partial_n\gamma\frac{dx^n}{dv}}_{=\frac{d\gamma}{dv}} + \underbrace{\delta^r_n\frac{dx^n}{dv}}_{=\frac{dx^r}{dv}}\underbrace{\partial_m\gamma\frac{dx^m}{dv}}_{=\frac{d\gamma}{dv}} - a^{rs}\partial_s\gamma\underbrace{a_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv}}_{=0 \text{ by}(1)}\right) \tag{14}$$

$$= \frac{1}{2}\gamma\left(\frac{dx^r}{dv}\frac{d\gamma}{dv} + \frac{dx^r}{dv}\frac{d\gamma}{dv}\right) \tag{15}$$

$$= \left(\gamma\frac{d\gamma}{dv}\right)\frac{dx^r}{dv} \tag{16}$$

So from (11) and (16), (1) can be expressed as

$$\frac{d^2x^r}{dv^2} + \Gamma'^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = \left(\gamma\frac{d\gamma}{dv}\right)\frac{dx^r}{dv} \tag{17}$$

By (2.449) page 46 we see that the vector $\frac{d^2x^r}{dv^2} + \Gamma'^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv}$ is collinear to $\frac{dx^r}{dv}$. Also $b_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = 0$. And hence (17) determines the geodesic null lines. So both expressions (1) and (17) are equivalent to determine the same geodesic null lines in the space V_N equipped with the metric a_{mn} or b_{mn}

♦

2.64 p80-exercise 18

Are the relations

$$T_{|rs} = T_{|sr}$$

$$T_{r|sk} = T_{r|ks}$$

true (a) in curvilinear coordinates in Euclidean space, (b) in a general Riemannian space?

$$\boldsymbol{T_{|rs} \stackrel{?}{=} T_{|sr}}$$

We have $T|_r = \partial_r T$ (see (2.528) page 53)

$$\begin{array}{ccc} \boldsymbol{T_{|rs}} & & \boldsymbol{T_{|sr}} \\ & \leftrightarrow & \\ T|_{rs} = \partial^2_{rs} T - \Gamma^m_{rs} T|_m & & T|_{sr} = \partial^2_{sr} T - \Gamma^m_{sr} T|_m \end{array} \tag{1}$$

As $\partial^2_{rs} = \partial^2_{sr}$ and $\Gamma^m_{rs} = \Gamma^m_{sr}$ we can conclude that $\boldsymbol{T_{|rs} = T_{|sr}}$ in both cases (a) and (b).

$\diamond$

$$\boldsymbol{T_{r|sk} \stackrel{?}{=} T_{r|ks}}$$

$$\begin{array}{lcl} \boldsymbol{T_{r|sk}} & & \boldsymbol{T_{r|sk}} \\ A_{rs} := T_{r|s} = \partial_s T_r - \Gamma^m_{rs} T_m & & B_{rk} := T_{r|k} = \partial_k T_r - \Gamma^m_{rk} T_m \\ A_{rs|k} = \partial_k A_{rs} - \Gamma^m_{rk} A_{ms} - \Gamma^m_{sk} A_{rm} & & B_{rk|s} = \partial_s B_{rk} - \Gamma^m_{rs} B_{mk} - \Gamma^m_{ks} B_{rm} \\ A_{ms} = \partial_s T_m - \Gamma^n_{ms} T_n & & B_{mk} = \partial_k T_m - \Gamma^n_{mk} T_n \\ A_{rm} = \partial_m T_r - \Gamma^n_{rm} T_n & \leftrightarrow & B_{rm} = \partial_m T_r - \Gamma^n_{rm} T_n \\ A_{rs|k} = \begin{cases} \underbrace{\partial^2_{sk} T_r}_{*} - T_m \partial_k \Gamma^m_{rs} - \underbrace{\Gamma^m_{rs} \partial_k T_m}_{**} \\ - \underbrace{\Gamma^m_{rk} \partial_s T_m}_{***} + \Gamma^m_{rk} \Gamma^n_{ms} T_n \\ - \underbrace{\Gamma^m_{sk} \partial_m T_r}_{****} + \underbrace{\Gamma^m_{sk} \Gamma^n_{rm} T_n}_{*****} \end{cases} & & B_{rk|s} = \begin{cases} \underbrace{\partial^2_{ks} T_r}_{*} - T_m \partial_s \Gamma^m_{rk} - \underbrace{\Gamma^m_{rk} \partial_s T_m}_{***} \\ - \underbrace{\Gamma^m_{rs} \partial_k T_m}_{**} + \Gamma^m_{rs} \Gamma^n_{mk} T_n \\ - \underbrace{\Gamma^m_{ks} \partial_m T_r}_{****} + \underbrace{\Gamma^m_{ks} \Gamma^n_{rm} T_n}_{*****} \end{cases} \end{array} \tag{2}$$

$$\Rightarrow \quad A_{rs|k} - B_{rk|s} = T_m \left(\underbrace{\partial_s \Gamma^m_{rk} - \partial_k \Gamma^m_{rs} + \Gamma^n_{rk}\Gamma^m_{ns} - \Gamma^n_{rs}\Gamma^m_{nk}}_{:= R^s.rmn} \right) \tag{3}$$

So, $T_{r|sk} = T_{r|ks}$ only if $T_m R^s_{.rmn} = 0$ and as T_m is an arbitrary tensor, $R^s_{.rmn}$ must vanish for $T_{r|sk} = T_{r|ks}$.

Note:
Although Γ^n_{rk} is not a tensor, the quantity $R^s_{.rmn}$ is. Indeed, as both $A_{rs|k} - B_{rk|s}$ and T_m have the tensor character, this implies that $R^s_{.rmn}$ is a tensor. Now, for an Euclidean space equipped with Cartesian coordinates all Γ^n_{rk} are constant and vanish. So, $R^s_{.rmn} = 0$. Let's consider a change of coordinate system from Cartesian to curvilinear coordinate system. Then, by the tensor character of $R'^a_{.bcd}$ we have,

$$R'^a_{.bcd} = R^s_{.rmn} \partial_s x'^a \partial_{('b)} x^r \partial_{('c)} x^m \partial_{('d)} x^n \tag{4}$$

but $R^s_{.rmn} = 0$ in the Cartesian coordinate system and so is $R'^a_{.bcd}$

Conclusion:
In a general Riemannian space $T_{r|sk} \neq T_{r|ks}$ but $T_{r|sk} = T_{r|ks}$ in a curvilinear Euclidean space.

♦

2.65 p80-exercise 19

Consider a V_N with indefinite metric form. For all points P lying on the cone of geodesic null lines drawn from O, the definition 2.611 for Riemannian coordinates apparently breaks down. Revise the definition of Riemannian coordinates so as to include such points.

For geodesic null lines we have (2.445 page 46)

$$\begin{cases} \frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \\ a_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \end{cases} \tag{1}$$

or (2.448 page 46)

$$\begin{cases} \frac{d^2x^r}{dv^2} + \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = \lambda(v)\frac{dx^r}{dv} \\ a_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = 0 \end{cases} \tag{2}$$

where by suitable choice of the parameter v , $\lambda(v)$ can be made any preassigned function of v.

$$(2) \quad \Rightarrow \quad \frac{d^2x^r}{dv^2} = \lambda\frac{dx^r}{dv} - \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} \tag{3}$$

$$\Rightarrow \quad \frac{d^3x^r}{dv^3} = \frac{d\lambda}{dv}\frac{dx^r}{dv} + \lambda\frac{d^2x^r}{dv^2} + A^r_{.mns}\frac{dx^m}{dv}\frac{dx^n}{dv}\frac{dx^s}{dv} \tag{4}$$

$$\text{with} \quad \Rightarrow \quad A^r_{.mns} = -\partial_s\Gamma^r_{mn} + 2\Gamma^r_{sp}\Gamma^p_{mn} \tag{5}$$

Expanding x^r in a Taylor series around a point $O(a^r)$ we get (for ease of notation we put $p^r \coloneqq \frac{dx^r}{dv}$)

$$x^r = a^r + vp^r + \frac{1}{2}v^2\lambda p^r - \frac{1}{2}v^2\Gamma^r_{mn}p^mp^n + \frac{1}{6}v^3\frac{d\lambda}{dv}p^r + \frac{1}{6}v^3\lambda\frac{dp^r}{dv} + \frac{1}{6}v^3A^r_{.mns}p^mp^np^s + \dots \tag{6}$$

$$= a^r + \left(v + \frac{1}{2}v^2\lambda + \frac{1}{6}v^3\frac{d\lambda}{dv}\right)p^r - \frac{1}{2}v^2\Gamma^r_{mn}p^mp^n + \frac{1}{6}v^3\lambda\frac{dp^r}{dv} + \frac{1}{6}v^3A^r_{.mns}p^mp^np^s + \dots \tag{7}$$

Put $x'^r \coloneqq v\xi(v)p^r$ with $\xi(v) = \left(1 + \frac{1}{2}v\lambda + \frac{1}{6}v^2\frac{d\lambda}{dv}\right)$. Hence (7) becomes

$$x^r = a^r + x'^r + \frac{\lambda v^3}{6}\left(\frac{p'^r}{v\xi} - \frac{\xi + v\xi'}{v^2\xi^2}x'^r\right) - \frac{\Gamma^r_{mn}}{2\xi^2}x'^mx'^n + \frac{A^r_{.mns}}{6\xi^3}x'^mx'^nx'^s + \dots \tag{8}$$

$$= a^r + \tau(v)x'^r + \frac{\lambda v^2}{6\xi}p'^r - \frac{\Gamma^r_{mn}}{2\xi^2}x'^mx'^n + \frac{A^r_{.mns}}{6\xi^3}x'^mx'^nx'^s + \dots \tag{9}$$

$$\text{with} \qquad \tau(v) \coloneqq 1 - \frac{\lambda v}{6\xi^2}(\xi + v\xi') \tag{10}$$

Is the Jacobian non-zero?

$$\frac{\partial x^r}{\partial x'^q} = \tau\delta^r_q + \frac{\lambda v^2}{6\xi}\frac{d\delta^r_q}{dv} - \frac{\Gamma^r_{mq}}{\xi^2}x'^m + \frac{A^r_{.mnq}}{2\xi^3}x'^mx'^n + \dots \tag{11}$$

In the infinitesimal neighbourhood of O we have

$$\begin{cases} v \to 0 \\ \xi \to 1 \\ \xi^{'} \to \frac{\lambda}{2} \\ x^{'m} \to 0 \\ \tau \to 1 \end{cases} \tag{12}$$

$$\tag{13}$$

so that the Jacobian determinant becomes

$$\left| \frac{\partial x^r}{\partial x'^q} \right| = \left| \tau \delta^r_q \right| = \tau^N = 1 \tag{14}$$

Conclusion:

So in order to define a Riemannian coordinates system which is still valid on the null geodesics, it is sufficient to define the Riemannian coordinates around O as

$$x^{'r} \coloneqq v \xi p^r$$

with

$$\xi(v) = \left(1 + \frac{\lambda}{2} v + \frac{1}{6} \frac{d\lambda}{dv} v^2 \right)$$

and λ, by suitable choice of v, being any pre-defined function of v.

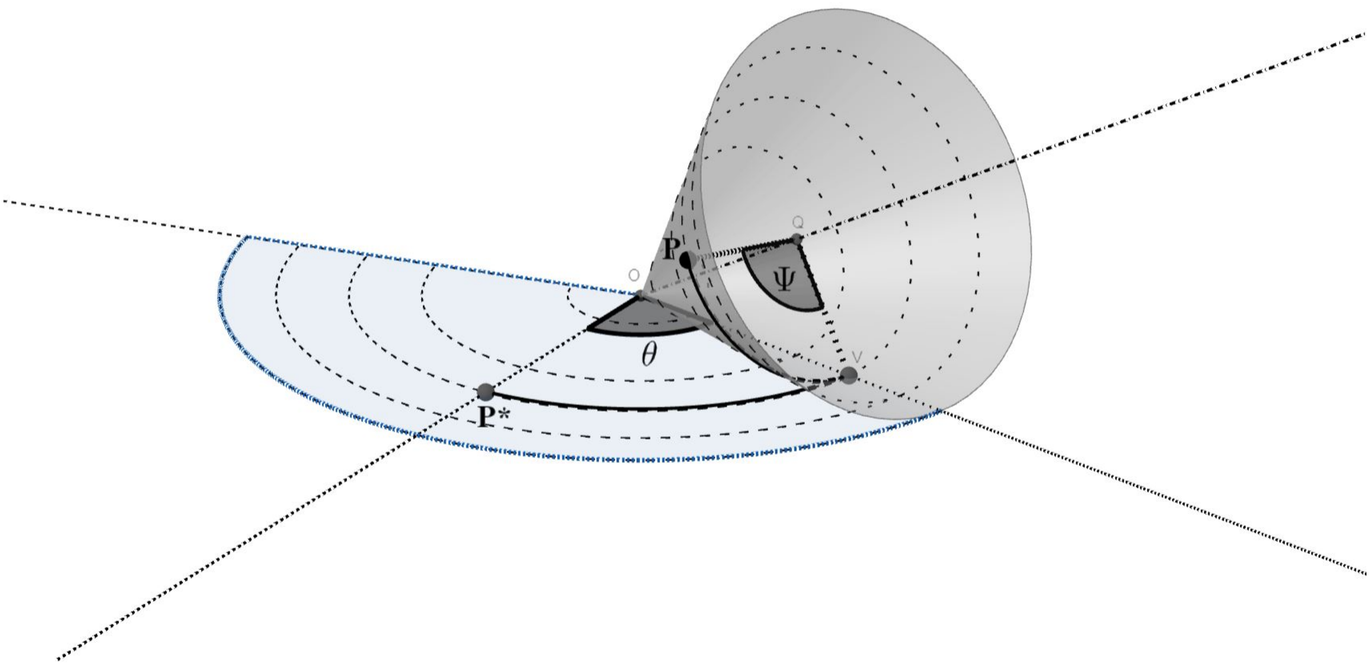
♦

Curvature of space

3.1 p82 - Exercise

Explain why the surfaces of an ordinary cylinder and an ordinary cone are to be regarded as "flat" in the sense of our definition.

The reason is because those surfaces can be "unwrapped" like the figure below shows.



For the cone, we can for each point P on the cone, lying on a distance r from the apex and making an angle Ψ, associate on a plane, a point P^* lying at the same distance r from the apex, taken as origin for the coordinate system, and making an angle $\theta = k\Psi$ with k a constant depending on the shape of the cone. This pair of coordinates are polar coordinates with $r \in (-\infty, +\infty)$ and $\theta \in [0, 2k\pi)$. The same reasoning can be applied to a cylinder which is a cone with the apex at ∞. In that case the coordinate system becomes a Cartesian coordinate system.
As a continuous mapping exist from polar to orthogonal Cartesian coordinates both coordinate system can be written under the required form (3.101) and so can be called "flat".

♦

3.2 p83 - Exercise

What are the values of $R^{s}_{.rmn}$ in an Euclidean plane, the coordinates being rectangular Cartesians? Deduce the values of the components of this tensor for polar coordinates from its tensor character, or else by direct calculation.

See 2.64 page 139 (exercise 18).

♦

3.3 p86 - Exercise

Show that in a V_2 all the components of the covariant curvature tensor either vanish or are expressible in terms of R_{1212}.

We have (3.115) and (3.116)

$$\begin{cases} R_{rsmn} = -R_{srmn} \\ R_{rsmn} = -R_{rsnm} \\ R_{rsmn} = R_{mnrs} \\ R_{rsmn} + R_{rmns} + R_{rnsm} = 0 \end{cases} \tag{1}$$

It is clear from the two first identities that in pairs (rs) and (mn) both indices have to be different when the tensor is not 0. So we only have to consider R_{1212}, R_{1221}, R_{2112} and R_{2121}.
The two first identities gives us:

$$R_{1221} = -R_{1212} \tag{2}$$

$$R_{2112} = -R_{1212} \tag{3}$$

$$R_{2121} = -R_{2112} = R_{1212} \tag{4}$$

The third identity does give us any additional information. The fourth identity gives us only trivial statements:

$$R_{1212} + \underbrace{R_{1122}}_{=0} + \underbrace{R_{1221}}_{=-R_{1212}} = 0 \tag{5}$$

$$\underbrace{R_{1221}}_{=-R_{1212}} + R_{1212} + \underbrace{R_{1122}}_{=0} = 0 \tag{6}$$

$$\underbrace{R_{2112}}_{=-R_{1212}} + \underbrace{R_{2121}}_{=R_{1212}} + \underbrace{R_{2211}}_{=0} = 0 \tag{7}$$

$$\underbrace{R_{2121}}_{=R_{1212}} + \underbrace{R_{2211}}_{=0} + \underbrace{R_{2112}}_{=-R_{1212}} = 0 \tag{8}$$

Conclusion:
We get the identities (2), (3) and (4) in function of R_{1212} and all vanish if $R_{1212} = 0$

♦

3.4 p86-87 - clarification

The number of independent components of the covariant curvature tensor in a space of N dimensions is

$$\frac{1}{12}N^2\left(N^2-1\right)$$

We have (3.115) and (3.116)

$$\begin{cases} R_{rsmn} = -R_{srmn} \\ R_{rsmn} = -R_{rsnm} \\ R_{rsmn} = R_{mnrs} \\ R_{rsmn} + R_{rmns} + R_{rnsm} = 0 \end{cases} \tag{1}$$

It is clear from the two first identities that in the tuple (rs) and (mn) both indices have to be different when the component is not 0. So we only have to consider the component with the pair of tuples (r,s) and $m,n)$ with $r \neq s$ and $m \neq n$. For the tuple (r,s) we have N possibilities to draw an index for r but for s only $N-1$ indices remain as $r \neq s$. So for the tuple (r,s) we get $N(N-1)$ possibilities. But note by the first identity $R_{rsmn} = -R_{srmn}$ that we only have to consider the half of this quantity as once we have chosen a tuple (r,s) we also know the component for the tuple (s,r). So the total number of possibilities we have for (r,s) is $M = \frac{1}{2}N(N-1)$. The same yields for the tuple (mn). So, we get in total M^2 possibilities according to the two first identities.
The third identity $R_{rsmn} = R_{mnrs}$ puts an extra constraint on the number of possibilities as we have to subtract from M^2 the number of possibilities covered by this third identity. Note that, once we have chosen a tuple (rs) we have to exclude the tuple $(m,n) = (r,s)$ as the identity $R_{rsrs} = R_{rsrs}$ becomes trivial.. So for the first tuple we have M possibilities, but once chosen, only $M-1$ remain for the second tuple. So we get $M(M-1)$ possibilities. But, again we only have to take half of these possibilities as the identities $R_{rsmn} = R_{mnrs}$ and $R_{mnrs} = R_{rsmn}$ are equivalent.
So the total number of possibilities reduces to

$$M^2 - \frac{1}{2}M(M-1) \quad \text{with} \quad M = \frac{1}{2}N(N-1)$$

What about the fourth identity

$$R_{rsmn} + R_{rmns} + R_{rnsm} = 0$$

First we note that this identity implies that all indices are different as it becomes trivial in the other cases. This is a consequence of the first 3 identities. Indeed, we know already that

$$\begin{cases} r \neq s \\ m \neq n \\ (r,s) \neq (m,n) \end{cases} \tag{2}$$

Let's consider the following cases

$$\begin{cases} r = m & \to m \neq s \; m \neq n \; r \neq n & \to & R_{rsrn} + \underbrace{R_{rrns}}_{=0} + \underbrace{R_{rnsr}}_{=-R_{rnrs}=-R_{rsrn}} = 0 \\ r = n & \to n \neq s \; m \neq n \; r \neq s & \to & R_{rsmr} + \underbrace{R_{rmrs}}_{=-R_{mrrs}=-R_{rsmr}} + \underbrace{R_{rrsm}}_{=0} = 0 \\ s = m & \to m \neq r \; n \neq s \; r \neq s & \to & R_{rssn} + \underbrace{R_{rsns}}_{=-R_{rssn}} + \underbrace{R_{rnss}}_{=0} = 0 \\ s = n & \to r \neq s \; m \neq s \; m \neq n & \to & R_{rsms} + \underbrace{R_{rmss}}_{=0} + \underbrace{R_{rssm}}_{=-R_{rsms}} = 0 \end{cases} \tag{3}$$

So indeed, once two indices are equal, the fourth identity becomes trivial and does not put extra constraints to the number of possibilities. For the tuple (r, s, m, n) we have N possibilities to draw an index for r, for s only $N-1$, for m only $N-2$ and for n only $N-3$ indices remain as $r \neq s \neq m \neq n$. The maximum number of constraint generated by the fourth identity is thus

$$N(N-1)(N-2)(N-3)$$

But here again double counts occur. Indeed the fourth identity is true for the 6 tuples

$$(rsmn), (rmsn), (rmns), (rsnm), (rnsm), (rnms)$$

as first entry in the identity. The same reasoning is valid for the tuples $(n...)$, $(s...)$ $(m...)$.
So in total we get $6 \times 4 = 24$ equivalent identities and the number of constraints generated by the fourth identity reduces to

$$\frac{1}{24} N(N-1)(N-2)(N-3)$$

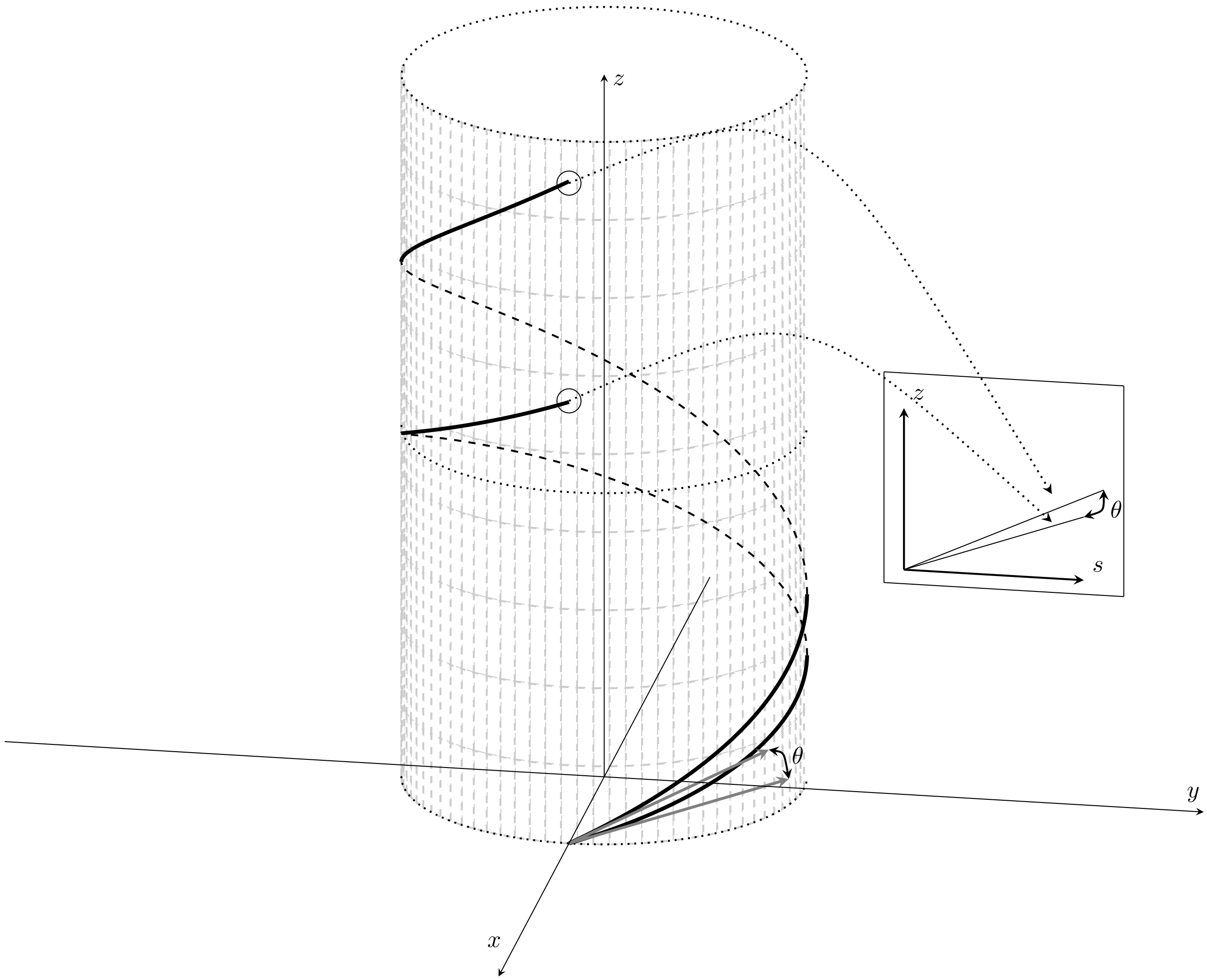
Note that this number of constraints vanish for $N \leq 3$.
Putting it all together the number of independent components of R_{rsmn} becomes

$$\begin{aligned} \mho &= M^2 - \frac{1}{2}M(M-1) - \frac{1}{24}N(N-1)(N-2)(N-3) && (4) \\ &= \frac{1}{2}M(M+1) - \frac{1}{24}N(N-1)(N-2)(N-3) && (5) \\ &= \frac{1}{8}N(N-1)\left(N(N-1)+2\right) - \frac{1}{24}N(N-1)(N-2)(N-3) && (6) \\ &= \frac{N}{24}\left(3N^2 - 6N^2 + 9N - 6 - N^3 + 3N^2 + 3N^2 - 9N - 2N + 6\right) && (7) \\ &= \frac{1}{12}N^2\left(N^2 - 1\right) && (8) \end{aligned}$$

♦

3.5 p82 - Exercise

Cylindar



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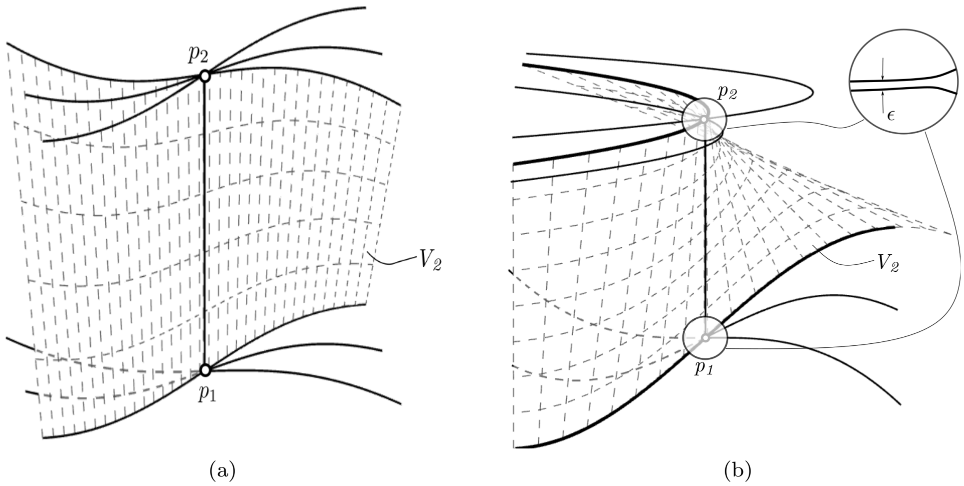
♦

3.6 p83 - clarification

V_2 from fammily of geodesics



♦