

Tensor Calculus
J.L. Synge and A.Schild (Dover Publication)
Solutions to exercises

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github.

Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma_{mn}^r \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

Contents

1	Curvature of space	4
1.1	p82 - Exercise	5
1.2	p83 - Exercise	6
1.3	p86 - Exercise	7
1.4	p86-87 - clarification	8
1.5	p87 - Exercise	10
1.6	p91 - Exercise	11
1.7	p93 - Exercise	13
1.8	p96 - Clarification	14

List of Figures

1.1	Unwrapping of a cone	5
1.2	Geodesics on a cylinder	12
1.3	Geometrical deduction of the geodesical deviation equation in an Euclidean space. .	13
1.4	Parallel transportation along a closed path	14
1.5	Relationship between parallel transportation along a closed path and the excess of the angle-sum over four right angles of a quadrilateral.	15
1.6	Relationship between parallel transportation along a closed path and the excess of the angle-sum over four right angles of a quadrilateral.	16

Curvature of space

1.1 p82 - Exercise

Explain why the surfaces of an ordinary cylinder and an ordinary cone are to be regarded as "flat" in the sense of our definition.

The reason is because those surfaces can be "unwrapped" like the figure below shows.

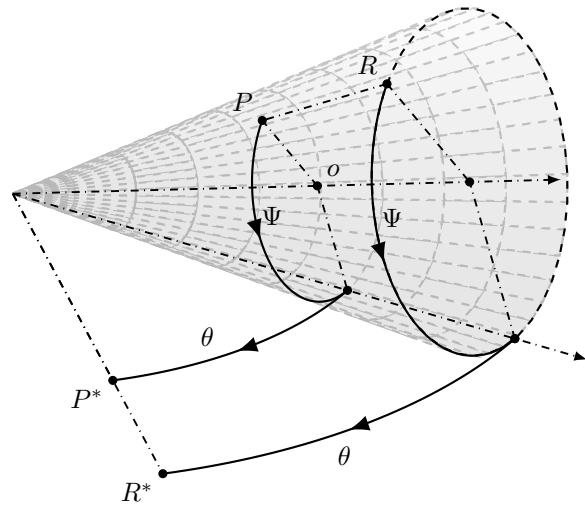


Figure 1.1: Unwrapping of a cone

For the cone, we can for each point P on the cone, lying on a distance h from the apex and making an angle Ψ , associate on a plane, a point P^* lying at the same distance h from the apex, taken as origin for the coordinate system, and making an angle $\theta = k\Psi$ with k a constant depending on the shape of the cone. This pair of coordinates are polar coordinates with $r \in (-\infty, +\infty)$ and $\theta \in [0, 2k\pi)$. The same reasoning can be applied to a cylinder which is a cone with the apex at ∞ . In that case the coordinate system becomes a Cartesian coordinate system.

As a continuous mapping exist from polar to orthogonal Cartesian coordinates both coordinate system can be written under the required form (3.101) and so can be called "flat".



1.2 p83 - Exercise

What are the values of R^s_{rmn} in an Euclidean plane, the coordinates being rectangular Cartesians? Deduce the values of the components of this tensor for polar coordinates from its tensor character, or else by direct calculation.

See 2.64 page 139 (exercise 18).



1.3 p86 - Exercise

Show that in a V_2 all the components of the covariant curvature tensor either vanish or are expressible in terms of R_{1212} .

We have (3.115) and (3.116)

$$\left\{ \begin{array}{l} R_{rsmn} = -R_{srmn} \\ R_{rsmn} = -R_{rsnm} \\ R_{rsmn} = R_{mnrs} \\ R_{rsmn} + R_{rmns} + R_{rns m} = 0 \end{array} \right. \quad (1)$$

It is clear from the two first identities that in pairs (rs) and (mn) both indices have to be different when the tensor is not 0. So we only have to consider R_{1212} , R_{1221} , R_{2112} and R_{2121} .

The two first identities gives us:

$$R_{1221} = -R_{1212} \quad (2)$$

$$R_{2112} = -R_{1212} \quad (3)$$

$$R_{2121} = -R_{2112} = R_{1212} \quad (4)$$

The third identity does give us any additional information. The fourth identity gives us only trivial statements:

$$R_{1212} + \underbrace{R_{1122}}_{=0} + \underbrace{R_{1221}}_{=-R_{1212}} = 0 \quad (5)$$

$$\underbrace{R_{1221}}_{=-R_{1212}} + R_{1212} + \underbrace{R_{1122}}_{=0} = 0 \quad (6)$$

$$\underbrace{R_{2112}}_{=-R_{1212}} + \underbrace{R_{2121}}_{=R_{1212}} + \underbrace{R_{2211}}_{=0} = 0 \quad (7)$$

$$\underbrace{R_{2121}}_{=R_{1212}} + \underbrace{R_{2211}}_{=0} + \underbrace{R_{2112}}_{=-R_{1212}} = 0 \quad (8)$$

Conclusion:

We get the identities (2), (3) and (4) in function of R_{1212} and all vanish if $R_{1212} = 0$



1.4 p86-87 - clarification

The number of independent components of the covariant curvature tensor in a space of N dimensions is

$$\frac{1}{12}N^2(N^2 - 1)$$

We have (3.115) and (3.116)

$$\begin{cases} R_{rsmn} = -R_{srnm} \\ R_{rsmn} = -R_{rsnm} \\ R_{rsmn} = R_{mnrs} \\ R_{rsmn} + R_{rmns} + R_{rns m} = 0 \end{cases} \quad (1)$$

It is clear from the two first identities that in the tuple (rs) and (mn) both indices have to be different when the component is not 0. So we only have to consider the component with the pair of tuples (r, s) and m, n with $r \neq s$ and $m \neq n$. For the tuple (r, s) we have N possibilities to draw an index for r but for s only $N - 1$ indices remain as $r \neq s$. So for the tuple (r, s) we get $N(N - 1)$ possibilities. But note by the first identity $R_{rsmn} = -R_{srnm}$ that we only have to consider the half of this quantity as once we have chosen a tuple (r, s) we also know the component for the tuple (s, r) . So the total number of possibilities we have for (r, s) is $M = \frac{1}{2}N(N - 1)$. The same yields for the tuple (mn) . So, we get in total M^2 possibilities according to the two first identities.

The third identity $R_{rsmn} = R_{mnrs}$ puts an extra constraint on the number of possibilities as we have to subtract from M^2 the number of possibilities covered by this third identity. Note that, once we have chosen a tuple (rs) we have to exclude the tuple $(m, n) = (r, s)$ as the identity $R_{rsrs} = R_{rsrs}$ becomes trivial.. So for the first tuple we have M possibilities, but once chosen, only $M - 1$ remain for the second tuple. So we get $M(M - 1)$ possibilities. But, again we only have to take half of these possibilities as the identities $R_{rsmn} = R_{mnrs}$ and $R_{mnrs} = R_{rsmn}$ are equivalent.

So the total number of possibilities reduces to

$$M^2 - \frac{1}{2}M(M - 1) \quad \text{with} \quad M = \frac{1}{2}N(N - 1)$$

What about the fourth identity

$$R_{rsmn} + R_{rmns} + R_{rns m} = 0$$

First we note that this identity implies that all indices are different as it becomes trivial in the other cases. This is a consequence of the first 3 identities. Indeed, we know already that

$$\begin{cases} r \neq s \\ m \neq n \\ (r, s) \neq (m, n) \end{cases} \quad (2)$$

Let's consider the following cases

$$\left\{ \begin{array}{l} r = m \rightarrow m \neq s \ m \neq n \ r \neq n \rightarrow R_{rsrn} + \underbrace{R_{rrns}}_{=0} + \underbrace{R_{rnsr}}_{=-R_{rnrs}=-R_{rsrn}} = 0 \\ r = n \rightarrow n \neq s \ m \neq n \ r \neq s \rightarrow R_{rsmr} + \underbrace{R_{rmrs}}_{=-R_{mrrs}=-R_{rsmr}} + \underbrace{R_{rrsm}}_{=0} = 0 \\ s = m \rightarrow m \neq r \ n \neq s \ r \neq s \rightarrow R_{rssn} + \underbrace{R_{rsns}}_{=-R_{rssn}} + \underbrace{R_{rnsr}}_{=0} = 0 \\ s = n \rightarrow r \neq s \ m \neq s \ m \neq n \rightarrow R_{rsmr} + \underbrace{R_{rmss}}_{=0} + \underbrace{R_{rsmr}}_{=-R_{rsmr}} = 0 \end{array} \right. \quad (3)$$

So indeed, once two indices are equal, the fourth identity becomes trivial and does not put extra constraints to the number of possibilities. For the tuple (r, s, m, n) we have N possibilities to draw an index for r , for s only $N - 1$, for m only $N - 2$ and for n only $N - 3$ indices remain as $r \neq s \neq m \neq n$. The maximum number of constraint generated by the fourth identity is thus

$$N(N - 1)(N - 2)(N - 3)$$

But here again double counts occur. Indeed the fourth identity is true for the 6 tuples

$$(rsmn), (rmsn), (rmns), (rsnm), (rnsr), (rnms)$$

as first entry in the identity. The same reasoning is valid for the tuples $(n...), (s...) (m...)$.

So in total we get $6 \times 4 = 24$ equivalent identities and the number of constraints generated by the fourth identity reduces to

$$\frac{1}{24}N(N - 1)(N - 2)(N - 3)$$

Note that this number of constraints vanish for $N \leq 3$.

Putting it all together the number of independent components of R_{rsmn} becomes

$$\mathcal{U} = M^2 - \frac{1}{2}M(M - 1) - \frac{1}{24}N(N - 1)(N - 2)(N - 3) \quad (4)$$

$$= \frac{1}{2}M(M + 1) - \frac{1}{24}N(N - 1)(N - 2)(N - 3) \quad (5)$$

$$= \frac{1}{8}N(N - 1)(N(N - 1) + 2) - \frac{1}{24}N(N - 1)(N - 2)(N - 3) \quad (6)$$

$$= \frac{N}{24}(3N^2 - 6N^2 + 9N - 6 - N^3 + 3N^2 + 3N^2 - 9N - 2N + 6) \quad (7)$$

$$= \frac{1}{12}N^2(N^2 - 1) \quad (8)$$



1.5 p87 - Exercise

Using the fact that the absolute derivative of the fundamental tensor vanishes, prove that 3.107 may be written

$$\frac{\delta^2 T_r}{\delta u \delta v} - \frac{\delta^2 T_r}{\delta v \delta u} = R_{r p m n} T^p \partial_u x^m \partial_v x^n$$

By 2.519 and 2.619 we have

$$\frac{\delta T_r}{\delta u} = \frac{\delta(a_{rk} T^k)}{\delta u} = \underbrace{\frac{\delta(a_{rk})}{\delta u}}_{=0} T^k + a_{rk} \frac{\delta(T^k)}{\delta u} \quad (1)$$

$$= a_{rk} T_{|n}^k \partial_u x^n \quad (2)$$

$$\Rightarrow \frac{\delta^2 T_r}{\delta u \delta v} = \frac{\delta(a_{rk} T_{|n}^k \partial_u x^n)}{\delta v} \quad (3)$$

$$= \underbrace{\frac{\delta(a_{rk})}{\delta v}}_{=0} T_{|n}^k \partial_u x^n + a_{rk} \frac{\delta(T_{|n}^k)}{\delta v} \partial_u x^n + a_{rk} T_{|n}^k \delta \left(\frac{\partial_u x^n}{\delta v} \right) \quad (4)$$

$$= a_{rk} \underbrace{\frac{\delta(T_{|n}^k)}{\delta v}}_{=T_{|nm}^k \partial_v x^m} \partial_u x^n + a_{rk} T_{|n}^k \underbrace{\delta \left(\frac{\partial_u x^n}{\delta v} \right)}_{=(\partial_u x^n)_{|m} \partial_v x^m} \quad (5)$$

$$= a_{rk} T_{|nm}^k \partial_v x^m \partial_u x^n + a_{rk} T_{|n}^k \underbrace{(\partial_u x^n)_{|m}}_{=\partial_m(\partial_u x^n) + \Gamma_{pm}^n \partial_u x^p} \partial_v x^m \quad (6)$$

$$= a_{rk} T_{|nm}^k \partial_v x^m \partial_u x^n + a_{rk} T_{|n}^k \left(\underbrace{\partial_m(\partial_u x^n) + \Gamma_{pm}^n \partial_u x^p}_{=\partial_u(\delta_m^n)=0} \right) \partial_v x^m \quad (7)$$

$$= a_{rk} T_{|nm}^k \partial_v x^m \partial_u x^n + a_{rk} T_{|n}^k \Gamma_{pm}^n \partial_u x^p \partial_v x^m \quad (8)$$

Hence we have

$$\frac{\delta^2 T_r}{\delta v \delta u} = a_{rk} T_{|nm}^k \partial_u x^m \partial_v x^n + a_{rk} T_{|n}^k \Gamma_{pm}^n \partial_v x^p \partial_u x^m \quad (9)$$

$$= a_{rk} T_{|mn}^k \partial_u x^n \partial_v x^m + a_{rk} T_{|n}^k \Gamma_{mp}^n \partial_v x^m \partial_u x^p \quad (10)$$

$$\Rightarrow \frac{\delta^2 T_r}{\delta u \delta v} - \frac{\delta^2 T_r}{\delta v \delta u} = a_{rk} T_{|nm}^k \partial_v x^m \partial_u x^n - a_{rk} T_{|mn}^k \partial_u x^n \partial_v x^m \quad (11)$$

$$= (a_{rk} T_{|nm}^k - a_{rk} T_{|mn}^k) \partial_u x^n \partial_v x^m \quad (12)$$

$$= \left(\underbrace{T_{r|nm} - T_{r|mn}}_{=-R_{rpmn} T^p} \right) \partial_u x^n \partial_v x^m \quad (13)$$

$$= - \underbrace{R_{rpmn}}_{=-R_{rpnm}} T^p \partial_u x^n \partial_v x^m = R_{rpnm} T^p \partial_u x^m \partial_v x^n \quad (14)$$



1.6 p91 - Exercise

Would the study of geodesic deviation enable us to distinguish between a plane and a right circular cylinder?

For geodesic lines we have

$$\frac{d^2 x^r}{du^2} + \Gamma_{mn}^r \frac{dx^m}{du} \frac{dx^n}{du} = 0 \quad (1)$$

(2)

with the fundamental tensor for a cylinder (see exercise page 27)

$$(a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad (3)$$

As no element of this tensor is a function of the coordinates, it is clear that all Christoffel symbols vanish and the geodesic curve are solutions of the simple system of 2^{nd} order differential equations

$$\frac{d^2 x^r}{du^2} = 0 \quad (4)$$

Hence

$$x^r = \kappa u + \mu \quad (5)$$

$$\text{or } x^r = \kappa^r u \quad (6)$$

by choosing the origin of the coordinates system with the initial condition position of the point. Choosing polar coordinates ϕ, z as coordinates, the distance one walks when following a geodesic is given by

$$s = \int_{u_0}^{u_1} \sqrt{(r^2(d\phi)^2 + (dz)^2)} \quad (7)$$

and if we take $u = \phi$ as independent a parameter, by (6) we get

$$s - s_0 = \int_{\psi_0}^{\psi_1} \sqrt{(r^2 + \kappa^2)} d\psi \equiv g(\psi - \psi_0) \quad (8)$$

$$\text{or } s = \int_{\psi_0}^{\psi_1} \sqrt{(r^2 + \kappa^2)} d\psi \equiv g\psi \quad (9)$$

by introducing transformed coordinates $s' = s - s_0$ and $\psi' = \psi - \psi_0$. and thus we get

$$z = m s' \quad (10)$$

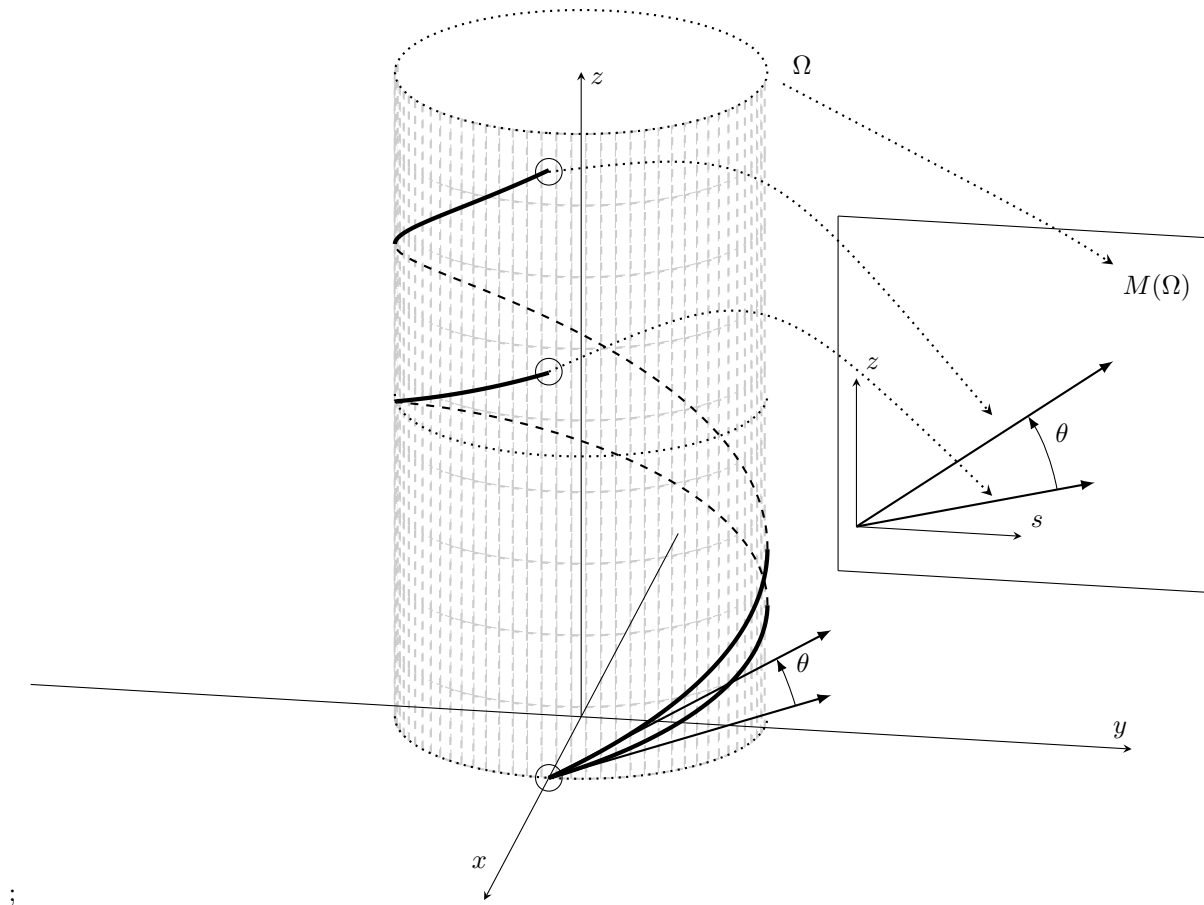


Figure 1.2: Geodesics on a cylinder

The above figure illustrates what an observer living in the manifold Ω sees when walking along geodesics on the cylinder. He only can measure the distance s' and the displacement along z and by (10) can only draw a chart like the one seen on the right of the cylinder. A "flatlander" living in the mapped manifold $M(\Omega)$ would see the same chart when walking along geodesics in his plane.

Conclusion: No, studying the geodesic deviation on a right circular cylinder does not enable us to say on which surface we are.



1.7 p93 - Exercise

For rectangular cartesians in Euclidean 3-space, show that the general solution of 3.311 is $\eta^r = A^r s + B^r$, where A^r, B^r are constants. verify this by elementary geometry.

We have equation 3.111

$$\frac{\delta^2 \eta^r}{\delta s^2} + R^r_{.smn} p^s \eta^m p^n = 0 \quad (1)$$

From exercise on page 83 we know that $R^r_{.smn} = 0$ in an Euclidean space. Also, in such spaces, the Christoffels symbols vanish and equation (1) reduces to $\frac{d^2 \eta}{ds^2} = 0$. And so,

$$\eta^r = A^r s + B^r$$

This is also easily deduced from a geometrical point of view. In an Euclidean space, the geodesics are straight lines. For an infinitesimal change in the geodesic family parameter v , we can assume that a vector, going perpendicular from 1 point from one geodesic with parameter v to another infinitesimal close geodesic with parameter $v + dv$, will also be perpendicular on this geodesic. This situation is depicted in figure (a). We conclude that $\overrightarrow{AA'} \parallel \overrightarrow{PP'}$. This can also be deduced from Thales theorem (see fig. (b)).

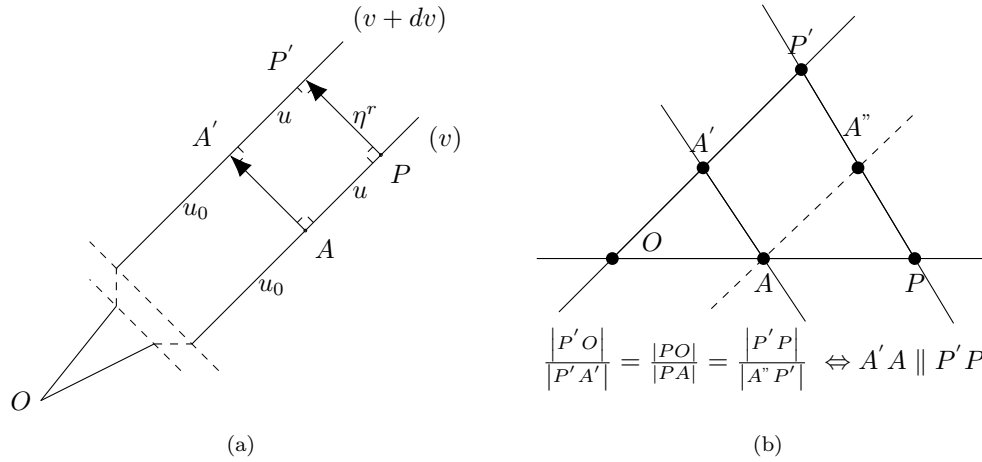


Figure 1.3: Geometrical deduction of the geodesical deviation equation in an Euclidean space.

Hence, we than can say that,

$$\frac{|AA'|}{u_0} = \frac{|PP'|}{u}$$

or,

$$\eta^r = A^r u + B^r$$

as the reference point A can be chosen arbitrarily on the line AP .



1.8 p96 - Clarification

... But under parallel propagation along a geodesic, a vector makes a constant angle with the geodesic; following the vector round the small quadrilateral, it is easy to see that the angle through which the vector has turned on completion of the circuit is E , the excess of the angle-sum of four right angles...

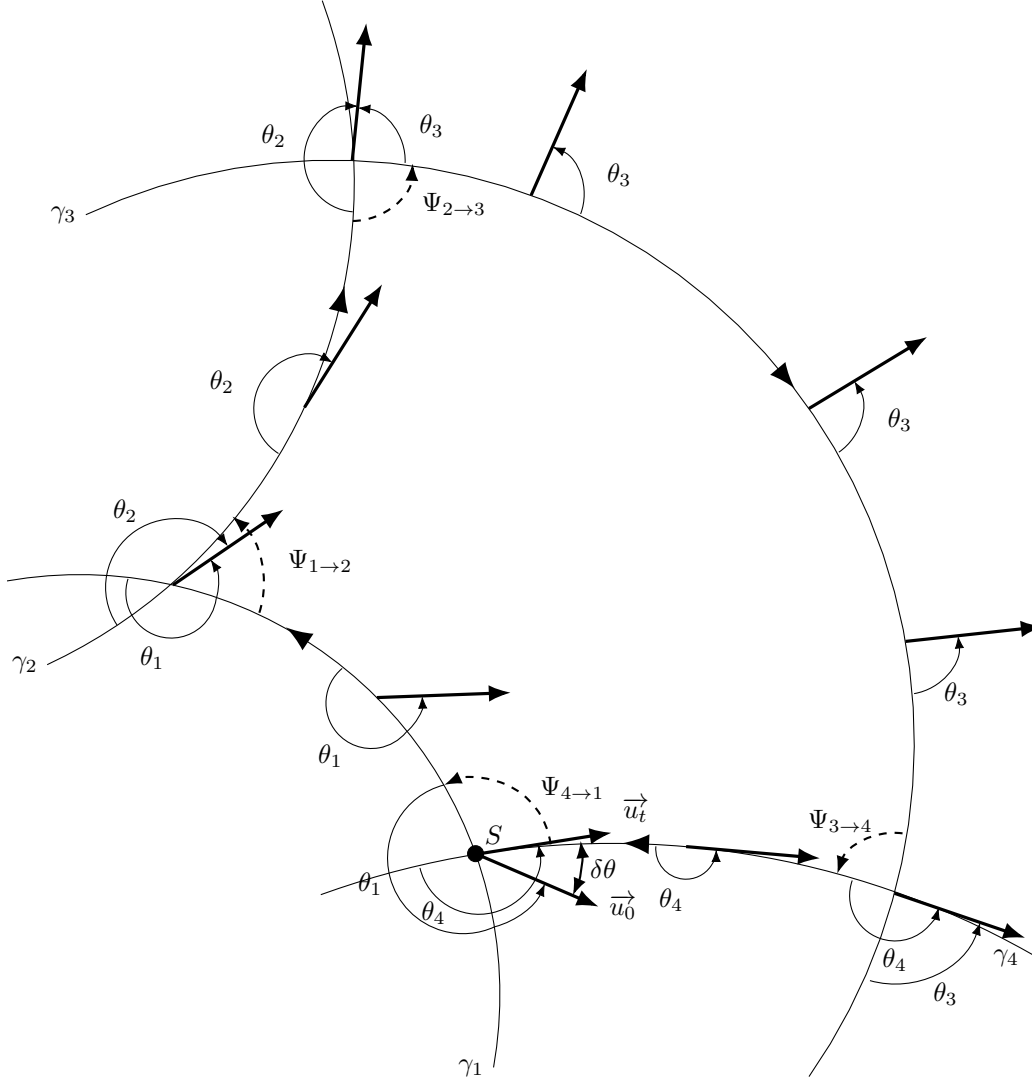


Figure 1.4: Parallel transportation along a closed path

Consider 4 geodesics $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ close to each other so that they form a small quadrilateral. At each intersection they form an angle $\Psi_{i \rightarrow i+1}$. A vector \vec{u}_0 is transported parallelly along the path starting at the intersection S of γ_1, γ_4 and ends as vector \vec{u}_t at the same point S . In general $\vec{u}_0 \neq \vec{u}_t$ and will differ by a small angle $\delta\theta$. Let's investigate the relationship between $\delta\theta$ and the $\Psi_{i \rightarrow i+1}$.

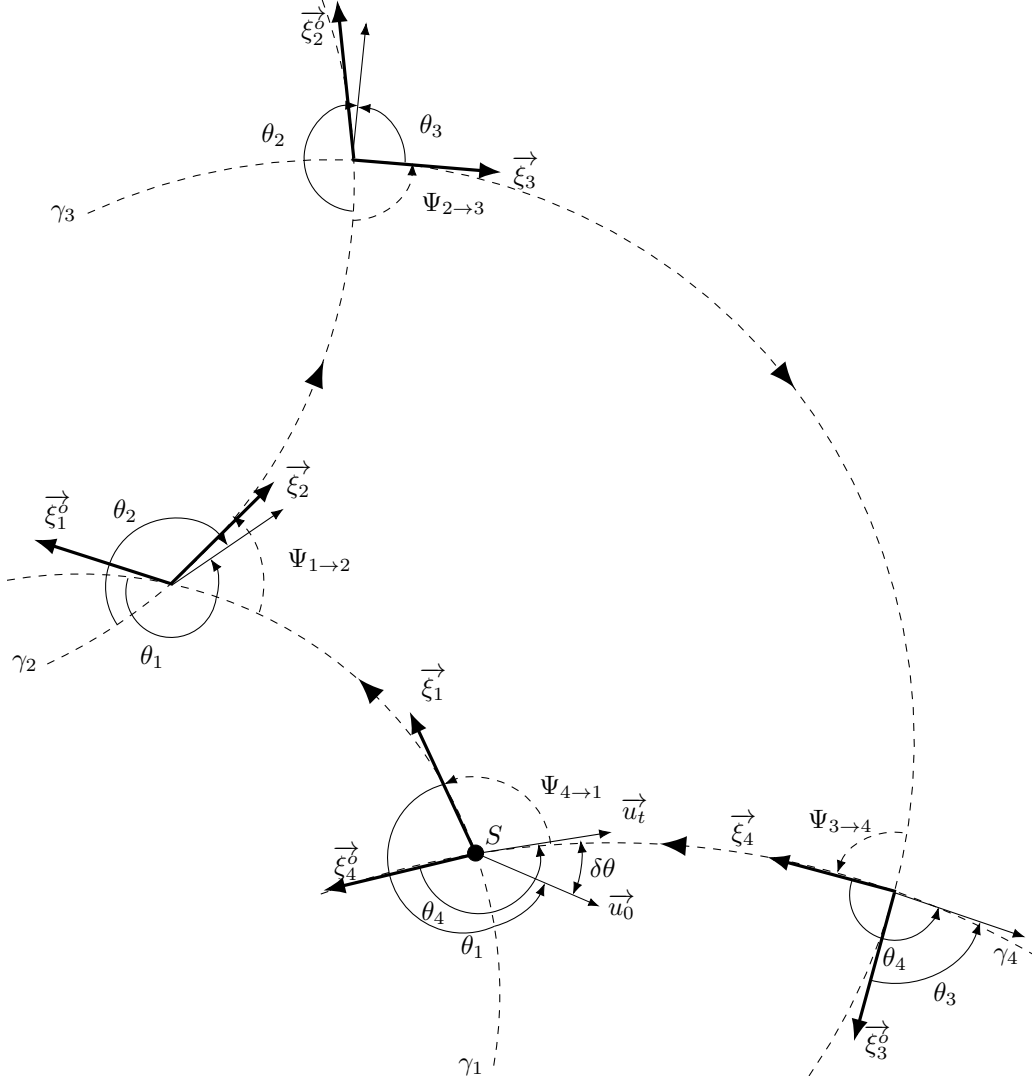


Figure 1.5: Relationship between parallel transportation along a closed path and the excess of the angle-sum over four right angles of a quadrilateral.

Let ξ_i and ξ_i^o ($i = 1, 2, 3, 4$) be respectively, tangent unit vectors to the geodesics at the beginning and at the end of the intersections of the geodesics. Be $\hat{\xi}_i$ and $\hat{\xi}_i^o$ ($i = 1, 2, 3, 4$) the angles of these vectors relative to an arbitrary reference vector and Be $\hat{\tau}_i$ and $\hat{\tau}_i^o$ ($i = 0, 2, 3, 4$) the angles of the transported vector (relative to this arbitrary reference vector) at the intersections of the geodesics. Then,

$$\begin{cases} \hat{\tau}_0 = \hat{\xi}_1 + \theta_1 \\ \hat{\tau}_1 = \hat{\xi}_1^o + \theta_1 & \hat{\tau}_1 = \hat{\xi}_2 + \theta_2 \\ \hat{\tau}_2 = \hat{\xi}_2^o + \theta_1 & \hat{\tau}_2 = \hat{\xi}_3 + \theta_3 \\ \hat{\tau}_3 = \hat{\xi}_3^o + \theta_3 & \hat{\tau}_3 = \hat{\xi}_4 + \theta_4 \\ \hat{\tau}_4 = \hat{\xi}_4^o + \theta_4 \end{cases} \quad (1)$$

We have also

$$\begin{cases} \widehat{\xi}_2 - \widehat{\xi}_1^o = \Psi_{1 \rightarrow 2} \\ \widehat{\xi}_3 - \widehat{\xi}_2^o = \Psi_{2 \rightarrow 3} \\ \widehat{\xi}_4 - \widehat{\xi}_3^o = \Psi_{3 \rightarrow 4} \\ \widehat{\xi}_1 - \widehat{\xi}_4^o = \Psi_{4 \rightarrow 1} \\ \widehat{\tau}_4 - \widehat{\tau}_0 = \delta\theta \end{cases} \quad (2)$$

Combining (1) and (2)

$$\begin{cases} \Psi_{1 \rightarrow 2} = \theta_1 - \theta_2 \\ \Psi_{2 \rightarrow 3} = \theta_2 - \theta_3 \\ \Psi_{3 \rightarrow 4} = \theta_3 - \theta_4 \\ \Psi_{4 \rightarrow 1} = \theta_4 - \theta_1 - \delta\theta \end{cases} \quad (3)$$

and so

$$\delta\theta = -(\Psi_{1 \rightarrow 2} + \Psi_{2 \rightarrow 3} + \Psi_{3 \rightarrow 4} + \Psi_{4 \rightarrow 1}) \quad (4)$$

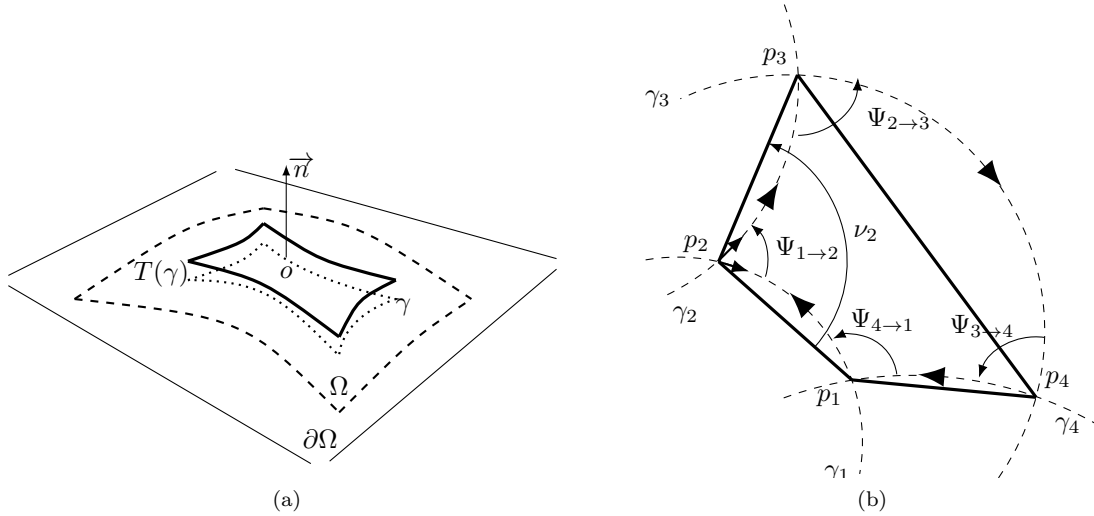


Figure 1.6: Relationship between parallel transportation along a closed path and the excess of the angle-sum over four right angles of a quadrilateral.

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