Tensor Calculus
J.L. Synge and A.Schild (Dover Publication)
Solutions to exercises

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github.

Some notation conventions

$$\partial_r \equiv \tfrac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \quad \text{Christoffel symbol of the second kind}$$

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Curvature of space

1.1 p82 - Exercise

Explain why the surfaces of an ordinary cylinder and an ordinary cone are to be regarded as "flat" in the sense of our definition.

The reason is because those surfaces can be "unwrapped" like the figure below shows.

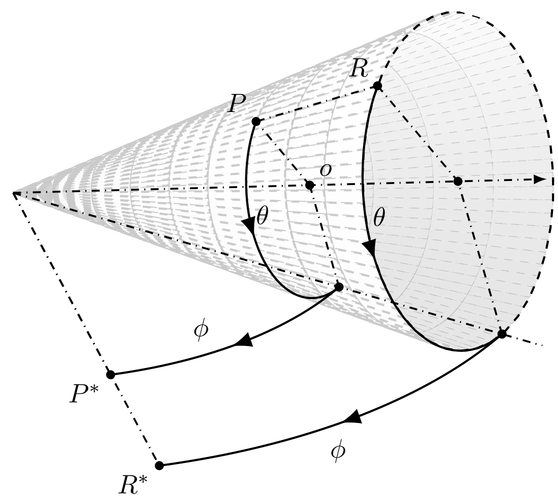

Figure 1.1: Unwrapping of a cone

For the cone, we can for each point P on the cone, lying on a distance h from the apex and making an angle θ, associate on a plane, a point P^* lying at the same distance h from the apex, taken as origin for the coordinate system, and making an angle $\phi = \theta \sin \alpha$ with α the angle of the cone. This pair of coordinates are polar coordinates with $r \in (-\infty, +\infty)$ and $\theta \in [0, 2k\pi)$. The same reasoning can be applied to a cylinder which is a cone with the apex at ∞. In that case the coordinate system becomes a Cartesian coordinate system.
As a continuous mapping exist from polar to orthogonal Cartesian coordinates both coordinate system can be written under the required form (3.101) and so can be called "flat".

♦

1.2 p83 - Exercise

What are the values of $R^{s}_{.rmn}$ in an Euclidean plane, the coordinates being rectangular Cartesians? Deduce the values of the components of this tensor for polar coordinates from its tensor character, or else by direct calculation.

See 2.64 page 139 (exercise 18).

♦

1.3 p86 - Exercise

Show that in a V_2 all the components of the covariant curvature tensor either vanish or are expressible in terms of R_{1212}.

We have (3.115) and (3.116)

$$\begin{cases} R_{rsmn} = -R_{srmn} \\ R_{rsmn} = -R_{rsnm} \\ R_{rsmn} = R_{mnrs} \\ R_{rsmn} + R_{rmns} + R_{rnsm} = 0 \end{cases} \tag{1}$$

It is clear from the two first identities that in pairs (rs) and (mn) both indices have to be different when the tensor is not 0. So we only have to consider R_{1212}, R_{1221}, R_{2112} and R_{2121}.
The two first identities gives us:

$$R_{1221} = -R_{1212} \tag{2}$$

$$R_{2112} = -R_{1212} \tag{3}$$

$$R_{2121} = -R_{2112} = R_{1212} \tag{4}$$

The third identity does give us any additional information. The fourth identity gives us only trivial statements:

$$R_{1212} + \underbrace{R_{1122}}_{=0} + \underbrace{R_{1221}}_{=-R_{1212}} = 0 \tag{5}$$

$$\underbrace{R_{1221}}_{=-R_{1212}} + R_{1212} + \underbrace{R_{1122}}_{=0} = 0 \tag{6}$$

$$\underbrace{R_{2112}}_{=-R_{1212}} + \underbrace{R_{2121}}_{=R_{1212}} + \underbrace{R_{2211}}_{=0} = 0 \tag{7}$$

$$\underbrace{R_{2121}}_{=R_{1212}} + \underbrace{R_{2211}}_{=0} + \underbrace{R_{2112}}_{=-R_{1212}} = 0 \tag{8}$$

Conclusion:
We get the identities (2), (3) and (4) in function of R_{1212} and all vanish if $R_{1212} = 0$

♦

1.4 p86-87 - clarification

The number of independent components of the covariant curvature tensor in a space of N dimensions is

$$\frac{1}{12}N^2\left(N^2-1\right)$$

We have (3.115) and (3.116)

$$\begin{cases} R_{rsmn} = -R_{srmn} \\ R_{rsmn} = -R_{rsnm} \\ R_{rsmn} = R_{mnrs} \\ R_{rsmn} + R_{rmns} + R_{rnsm} = 0 \end{cases} \tag{1}$$

It is clear from the two first identities that in the tuple (rs) and (mn) both indices have to be different when the component is not 0. So we only have to consider the component with the pair of tuples (r, s) and $m, n)$ with $r \neq s$ and $m \neq n$. For the tuple (r, s) we have N possibilities to draw an index for r but for s only $N-1$ indices remain as $r \neq s$. So for the tuple (r, s) we get $N(N-1)$ possibilities. But note by the first identity $R_{rsmn} = -R_{srmn}$ that we only have to consider the half of this quantity as once we have chosen a tuple (r, s) we also know the component for the tuple (s, r). So the total number of possibilities we have for (r, s) is $M = \frac{1}{2}N(N-1)$. The same yields for the tuple (mn). So, we get in total M^2 possibilities according to the two first identities.
The third identity $R_{rsmn} = R_{mnrs}$ puts an extra constraint on the number of possibilities as we have to subtract from M^2 the number of possibilities covered by this third identity. Note that, once we have chosen a tuple (rs) we have to exclude the tuple $(m, n) = (r, s)$ as the identity $R_{rsrs} = R_{rsrs}$ becomes trivial.. So for the first tuple we have M possibilities, but once chosen, only $M-1$ remain for the second tuple. So we get $M(M-1)$ possibilities. But, again we only have to take half of these possibilities as the identities $R_{rsmn} = R_{mnrs}$ and $R_{mnrs} = R_{rsmn}$ are equivalent.
So the total number of possibilities reduces to

$$M^2 - \frac{1}{2}M(M-1) \quad \text{with} \quad M = \frac{1}{2}N(N-1)$$

What about the fourth identity

$$R_{rsmn} + R_{rmns} + R_{rnsm} = 0$$

First we note that this identity implies that all indices are different as it becomes trivial in the other cases. This is a consequence of the first 3 identities. Indeed, we know already that

$$\begin{cases} r \neq s \\ m \neq n \\ (r, s) \neq (m, n) \end{cases} \tag{2}$$

Let's consider the following cases

$$\begin{cases} r=m & \to m\neq s\ m\neq n\ r\neq n & \to & R_{rsrn}+\underbrace{R_{rrns}}_{=0}+\underbrace{R_{rnsr}}_{=-R_{rnrs}=-R_{rsrn}}=0 \\ r=n & \to n\neq s\ m\neq n\ r\neq s & \to & R_{rsmr}+\underbrace{R_{rmrs}}_{=-R_{mrrs}=-R_{rsmr}}+\underbrace{R_{rrsm}}_{=0}=0 \\ s=m & \to m\neq r\ n\neq s\ r\neq s & \to & R_{rssn}+\underbrace{R_{rsns}}_{=-R_{rssn}}+\underbrace{R_{rnss}}_{=0}=0 \\ s=n & \to r\neq s\ m\neq s\ m\neq n & \to & R_{rsms}+\underbrace{R_{rmss}}_{=0}+\underbrace{R_{rssm}}_{=-R_{rsms}}=0 \end{cases} \tag{3}$$

So indeed, once two indices are equal, the fourth identity becomes trivial and does not put extra constraints to the number of possibilities. For the tuple (r,s,m,n) we have N possibilities to draw an index for r, for s only $N-1$, for m only $N-2$ and for n only $N-3$ indices remain as $r\neq s\neq m\neq n$. The maximum number of constraint generated by the fourth identity is thus

$$N(N-1)(N-2)(N-3)$$

But here again double counts occur. Indeed the fourth identity is true for the 6 tuples

$$(rsmn),(rmsn),(rmns),(rsnm),(rnsm),(rnms)$$

as first entry in the identity. The same reasoning is valid for the tuples $(n...)$, $(s...)$ $(m...)$.
So in total we get $6\times 4=24$ equivalent identities and the number of constraints generated by the fourth identity reduces to

$$\frac{1}{24}N(N-1)(N-2)(N-3)$$

Note that this number of constraints vanish for $N\leq 3$.
Putting it all together the number of independent components of R_{rsmn} becomes

$$\begin{aligned} \mho &= M^2-\frac{1}{2}M(M-1)-\frac{1}{24}N(N-1)(N-2)(N-3) && (4)\\ &= \frac{1}{2}M(M+1)-\frac{1}{24}N(N-1)(N-2)(N-3) && (5)\\ &= \frac{1}{8}N(N-1)\left(N(N-1)+2\right)-\frac{1}{24}N(N-1)(N-2)(N-3) && (6)\\ &= \frac{N}{24}\left(3N^2-6N^2+9N-6-N^3+3N^2+3N^2-9N-2N+6\right) && (7)\\ &= \frac{1}{12}N^2\left(N^2-1\right) && (8) \end{aligned}$$

♦

1.5 p87 - Exercise

Using the fact that the absolute derivative of the fundamental tensor vanishes, prove that 3.107 may be written

$$\frac{\delta^2 T_r}{\delta u \delta v} - \frac{\delta^2 T_r}{\delta v \delta u} = R_{rpmn} T^p \partial_u x^m \partial_v x^n$$

By 2.519 and 2.619 we have

$$\frac{\delta T_r}{\delta u} = \frac{\delta(a_{rk}T^k)}{\delta u} = \underbrace{\frac{\delta(a_{rk})}{\delta u}}_{=0} T^k + a_{rk}\frac{\delta(T^k)}{\delta u} \tag{1}$$

$$= a_{rk}T^k_{|n}\partial_u x^n \tag{2}$$

$$\Rightarrow \quad \frac{\delta^2 T_r}{\delta u \delta v} = \frac{\delta(a_{rk}T^k_{|n}\partial_u x^n)}{\delta v} \tag{3}$$

$$= \underbrace{\frac{\delta(a_{rk})}{\delta v}}_{=0} T^k_{|n}\partial_u x^n + a_{rk}\frac{\delta(T^k_{|n})}{\delta v}\partial_u x^n + a_{rk}T^k_{|n}\delta\frac{(\partial_u x^n)}{\delta v} \tag{4}$$

$$= a_{rk} \underbrace{\frac{\delta(T^k_{|n})}{\delta v}}_{=T^k_{|nm}\partial_v x^m} \partial_u x^n + a_{rk}T^k_{|n} \underbrace{\delta\frac{(\partial_u x^n)}{\delta v}}_{=(\partial_u x^n)_{|m}\partial_v x^m} \tag{5}$$

$$= a_{rk}T^k_{|nm}\partial_v x^m \partial_u x^n + a_{rk}T^k_{|n} \underbrace{(\partial_u x^n)_{|m}}_{=\partial_m(\partial_u x^n)+\Gamma^n_{pm}\partial_u x^p} \partial_v x^m \tag{6}$$

$$= a_{rk}T^k_{|nm}\partial_v x^m \partial_u x^n + a_{rk}T^k_{|n}\left(\underbrace{\partial_m(\partial_u x^n)}_{=\partial_u(\delta^n_m)=0} + \Gamma^n_{pm}\partial_u x^p\right)\partial_v x^m \tag{7}$$

$$= a_{rk}T^k_{|nm}\partial_v x^m \partial_u x^n + a_{rk}T^k_{|n}\Gamma^n_{pm}\partial_u x^p \partial_v x^m \tag{8}$$

Hence we have

$$\frac{\delta^2 T_r}{\delta v \delta u} = a_{rk}T^k_{|nm}\partial_u x^m \partial_v x^n + a_{rk}T^k_{|n}\Gamma^n_{pm}\partial_v x^p \partial_u x^m \tag{9}$$

$$= a_{rk}T^k_{|mn}\partial_u x^n \partial_v x^m + a_{rk}T^k_{|n}\Gamma^n_{mp}\partial_v x^m \partial_u x^p \tag{10}$$

$$\Rightarrow \quad \frac{\delta^2 T_r}{\delta u \delta v} - \frac{\delta^2 T_r}{\delta v \delta u} = a_{rk}T^k_{|nm}\partial_v x^m \partial_u x^n - a_{rk}T^k_{|mn}\partial_u x^n \partial_v x^m \tag{11}$$

$$= \left(a_{rk}T^k_{|nm} - a_{rk}T^k_{|mn}\right)\partial_u x^n \partial_v x^m \tag{12}$$

$$= \left(\underbrace{T_{r|nm} - T_{r|mn}}_{=-R_{rpmn}T^p}\right)\partial_u x^n \partial_v x^m \tag{13}$$

$$= -\underbrace{R_{rpmn}}_{=-R_{rpnm}} T^p \partial_u x^n \partial_v x^m = R_{rpmn}T^p \partial_u x^m \partial_v x^n \tag{14}$$

♦

1.6 p91 - Exercise

Would the study of geodesic deviation enable us to distinguish between a plane and a right circular cylinder?

For geodesic lines we have

$$\frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \tag{1}$$

$$\tag{2}$$

with the fundamental tensor for a cylinder (see exercise page 27)

$$(a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{3}$$

As no element of this tensor is a function of the coordinates, it is clear that all Christoffel symbols vanish and the geodesic curve are solutions of the simple system of 2^{nd} order differential equations

$$\frac{d^2x^r}{du^2} = 0 \tag{4}$$

Hence

$$x^r = \kappa u + \mu \tag{5}$$

$$\text{or} \quad x^r = \kappa^r u \tag{6}$$

by choosing the origin of the coordinates system with the initial condition position of the point. Choosing polar coordinates ϕ, z as coordinates, the distance one walks when following a geodesic is given by

$$s = \int_{u_0}^{u_1} \sqrt{(r^2(d\phi)^2 + (dz)^2)} \tag{7}$$

and if we take $u = \phi$ as independent a parameter, by (6) we get

$$s - s_0 = \int_{\psi_0}^{\psi_1} \sqrt{(r^2 + \kappa^2)} d\psi \equiv g(\psi - \psi_0) \tag{8}$$

$$\text{or} \quad s = \int_{\psi_0}^{\psi_1} \sqrt{(r^2 + \kappa^2)} d\psi \equiv g\psi \tag{9}$$

by introducing transformed coordinates $s^{'} = s - s_0$ and $\psi^{'} = \psi - \psi_0$.
and thus we get

$$z = ms^{'} \tag{10}$$

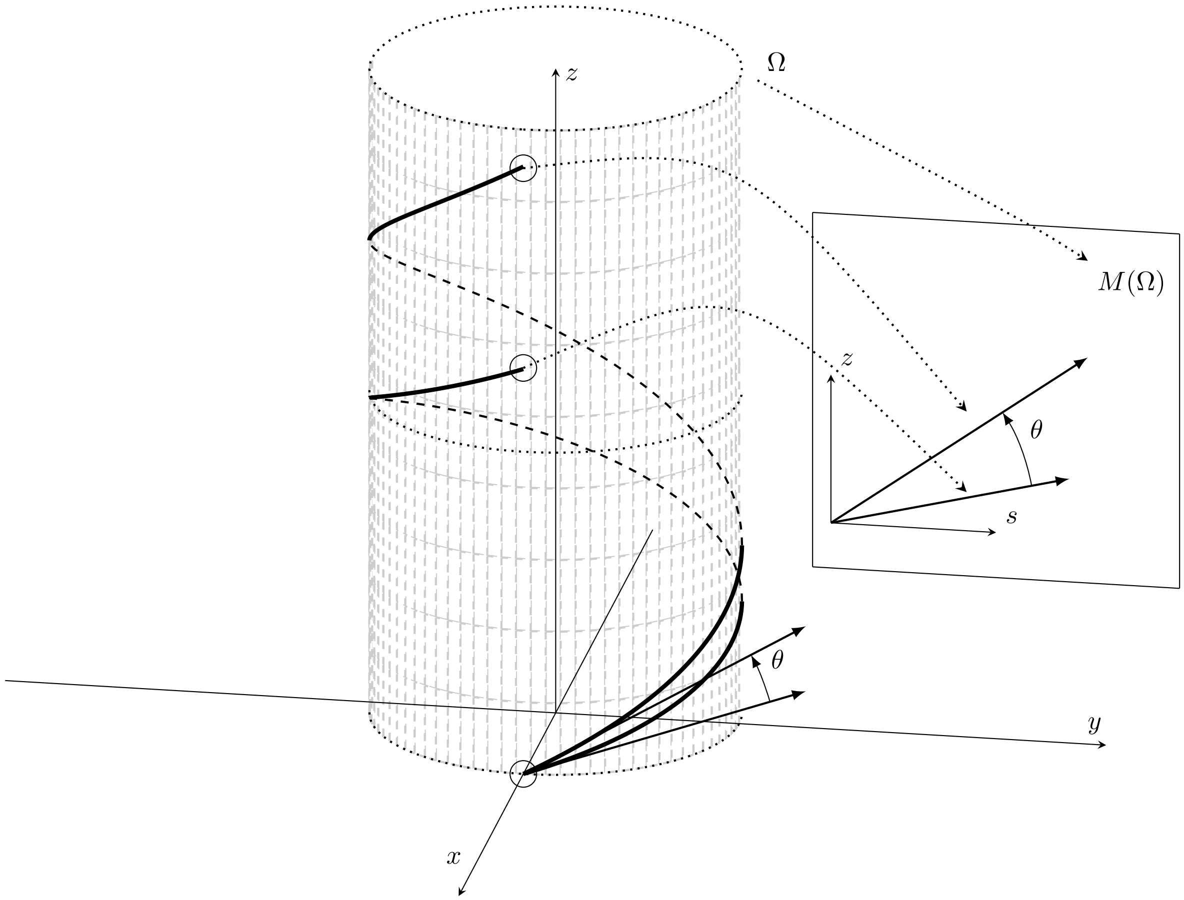

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Figure 1.2: Geodesics on a cylinder

The above figure illustrates what an observer living in the manifold Ω sees when walking along geodesics on the cylinder. He only can measure the distance $s^{'}$ and the displacement along z and by (10) can only draw a chart like the one seen on the right of the cylinder. A "flatlander" living in the mapped manifold $M(\Omega)$ would see the same chart when walking along geodesics in his plane.

Conclusion: No, studying the geodesic deviation on a right circular cylinder does not enable us to say on which surface we are.

♦

1.7 p93 - Exercise

For rectangular cartesians in Euclidean 3-space, show that the general solution of 3.311 is $\eta^r = A^r s + B^r$, where $A^{,}$ B^r are constants. verify this by elementary geometry.

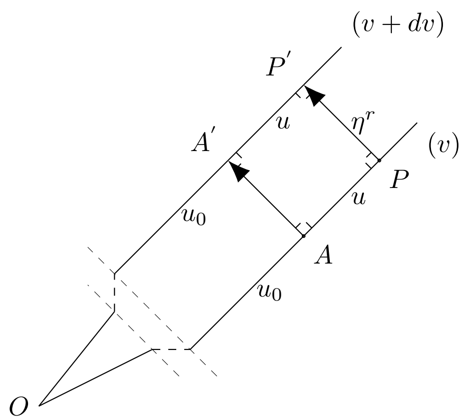
We have equation 3.111

$$\frac{\delta^2\eta^r}{\delta s^2} + R^r_{.smn} p^s \eta^m p^n = 0 \tag{1}$$

From exercise on page 83 we know that $R^r_{.smn} = 0$ in an Euclidean space. Also, in such spaces, the Christoffels symbols vanish and equation (1) reduces to $\frac{d^2\eta}{ds^2} = 0$. And so,

$$\eta^r = A^r s + B^r$$

This is also easily deducted from a geometrical point of view. In an Euclidean space, the geodesics are straight lines. For an infinitesimal change in the geodesic family parameter v, we can assume that a vector, going perpendicular from 1 point from one geodesic with parameter v to another infinitesimal close geodesic with parameter $v + dv$, will also be perpendicular on this geodesic. This situation is depicted in fig. 1.3(a). We conclude that $\overrightarrow{AA'} \parallel \overrightarrow{PP'}$. This can also be deducted from Thales theorem (see fig. 1.3 (b)).



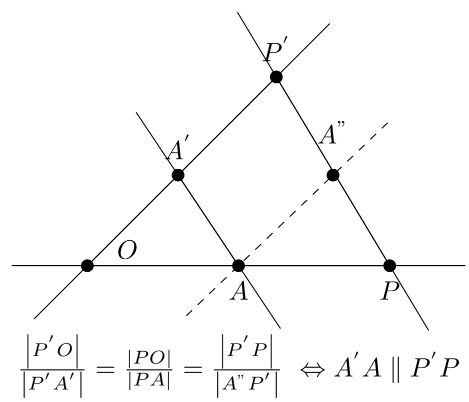


Figure 1.3: Geometrical deduction of the geodesical deviation equation in an Euclidean space.

Hence, we than can say that,

$$\frac{\left|AA'\right|}{u_0} = \frac{\left|PP'\right|}{u}$$

or,

$$\eta^r = A^r u + B^r$$

as the reference point A can be chosen arbitrarily on the line AP.

♦

1.8 p96 - Clarification

> ... But under parallel propagation along a geodesic, a vector makes a constant angle with the geodesic; following the vector round the small quadrilateral, it is easy to see that the angle through which the vector has turned on completion of the circuit is E, the excess of the angle-sum of four right angles...

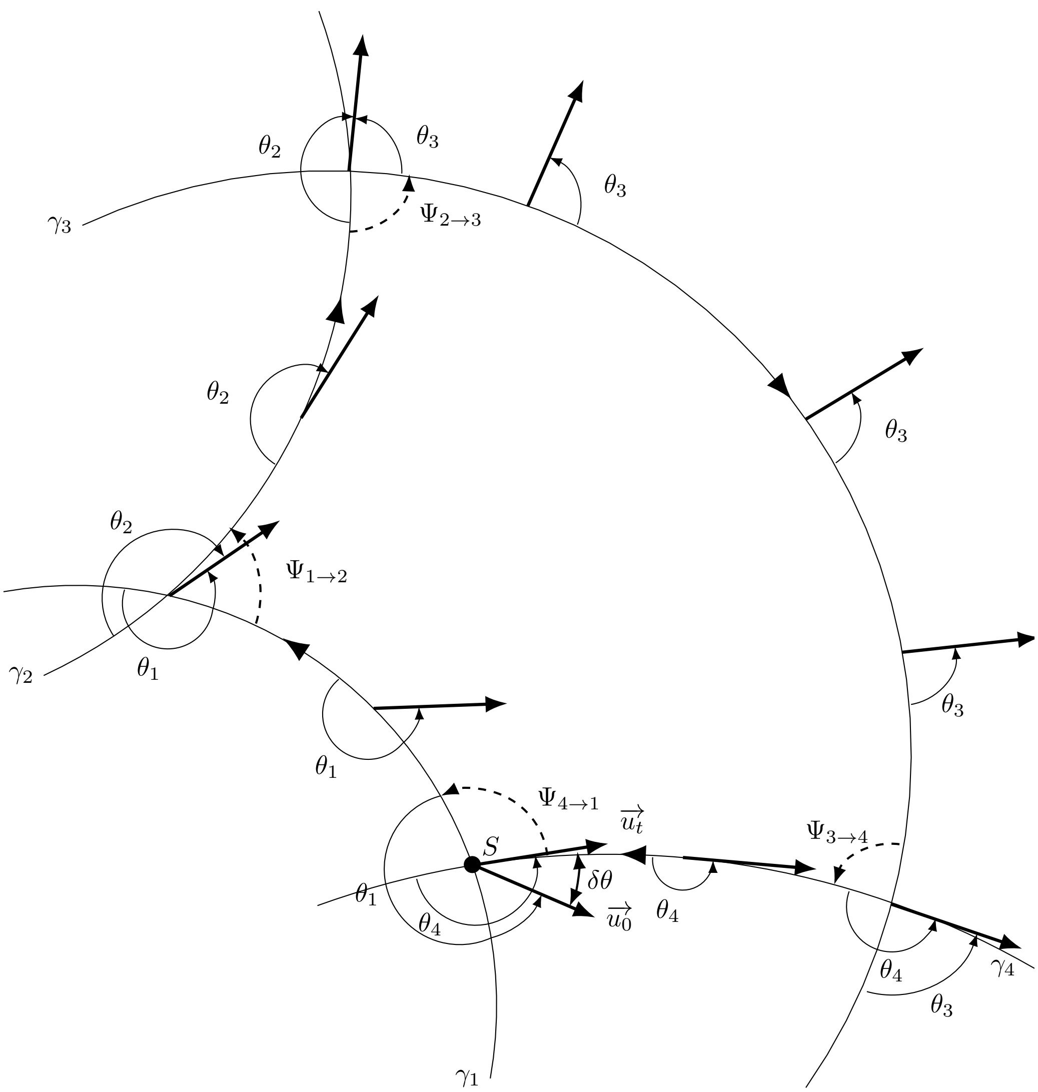


Figure 1.4: Parallel transportation along a closed path

Consider 4 geodesics $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ close to each other so that they form a small quadrilateral. At each intersection they form an angle $\Psi_{i\to i+1}$. A vector $\overrightarrow{u_0}$ is transported parallely along the path starting at the intersection S of γ_1, γ_4 and ends as vector $\overrightarrow{u_t}$ at the same point S. In general $\overrightarrow{u_0} \neq \overrightarrow{u_t}$ and will differ by a small angle $\delta\theta$. Let's investigate the relationship between $\delta\theta$ and the $\Psi_{i\to i+1}$.

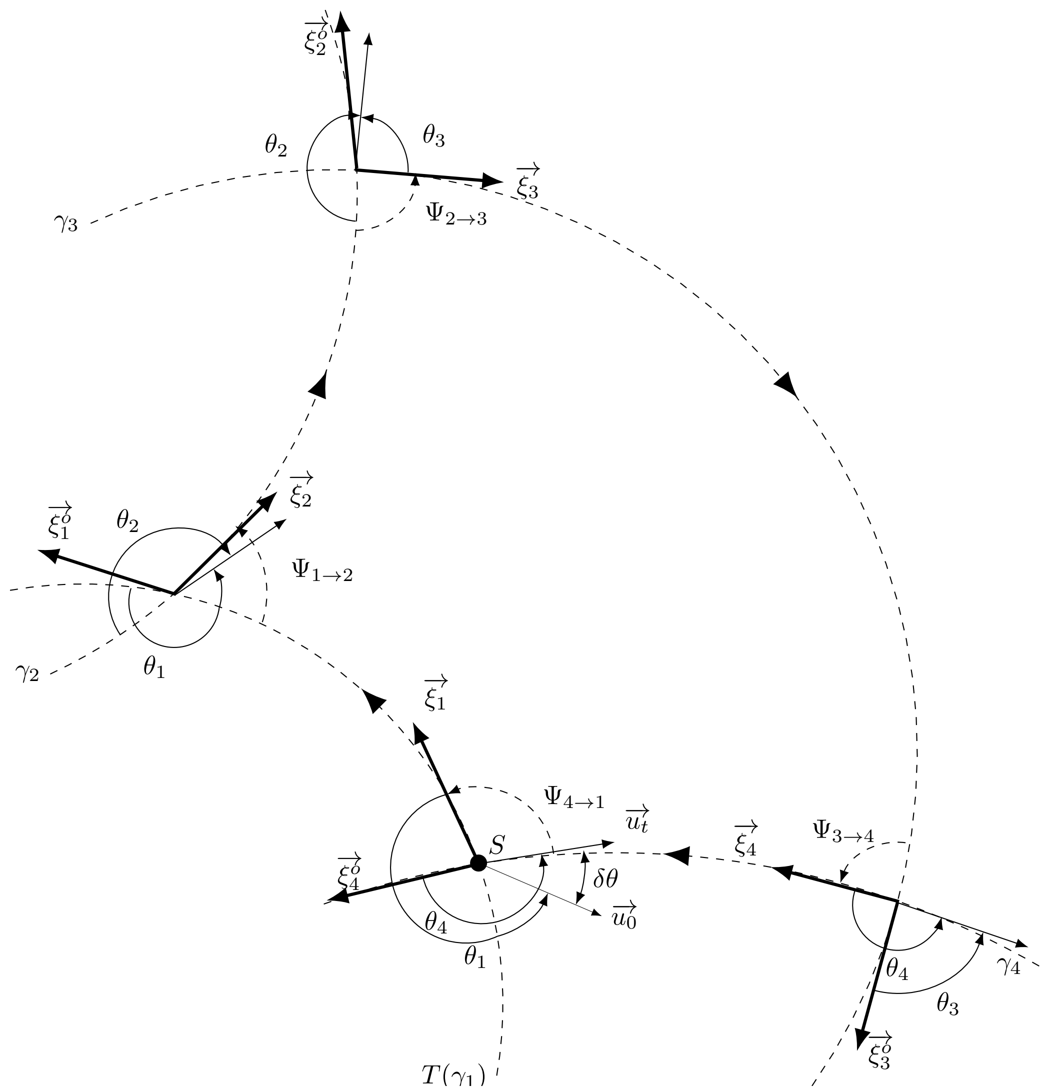


Figure 1.5: Relationship between parallel transportation along a closed path and the excess of the angle-sum over four right angles of a quadilateral.

Let ξ_i and ξ_i^o $(i = 1, 2, 3, 4)$ be respectively, tangent unit vectors to the geodesics at the beginning and at the end of the intersections of the geodesics. Be $\widehat{\xi}_i$ and $\widehat{\xi}_i^o$ $(i = 1, 2, 3, 4)$ the angles of these vectors relative to an arbitrary reference vector and be $\widehat{\tau}_i$ and $\widehat{\tau}_i^o$ $(i = 0, 2, 3, 4)$ the angles of the transported vector (relative to this arbitrary reference vector) at the intersections of the geodesics. Then,

$$\begin{cases} \widehat{\tau}_0 = \widehat{\xi}_1 + \theta_1 & \\ \widehat{\tau}_1 = \widehat{\xi}_1^o + \theta_1 & \widehat{\tau}_1 = \widehat{\xi}_2 + \theta_2 \\ \widehat{\tau}_2 = \widehat{\xi}_2^o + \theta_1 & \widehat{\tau}_2 = \widehat{\xi}_3 + \theta_3 \\ \widehat{\tau}_3 = \widehat{\xi}_3^o + \theta_3 & \widehat{\tau}_3 = \widehat{\xi}_4 + \theta_4 \\ \widehat{\tau}_4 = \widehat{\xi}_4^o + \theta_4 & \end{cases} \tag{1}$$

We have also

$$\begin{cases} \widehat{\xi}_2 - \widehat{\xi}_1^o = \Psi_{1\to 2} \\ \widehat{\xi}_3 - \widehat{\xi}_2^o = \Psi_{2\to 3} \\ \widehat{\xi}_4 - \widehat{\xi}_3^o = \Psi_{3\to 4} \\ \widehat{\xi}_1 - \widehat{\xi}_4^o = \Psi_{4\to 1} \\ \widehat{\tau}_4 - \widehat{\tau}_0 = \delta\theta \end{cases} \tag{2}$$

Combining (1) and (2)

$$\begin{cases} \Psi_{1\to 2} = \theta_1 - \theta_2 \\ \Psi_{2\to 3} = \theta_2 - \theta_3 \\ \Psi_{3\to 4} = \theta_3 - \theta_4 \\ \Psi_{4\to 1} = \theta_4 - \theta_1 - \delta\theta \end{cases} \tag{3}$$

and so

$$\delta\theta = -(\Psi_{1\to 2} + \Psi_{2\to 3} + \Psi_{3\to 4} + \Psi_{4\to 1}) \tag{4}$$

Note that these relationships are valid on the (curved) V_2 manifold. In order to go further we map the quadrilateral, on the manifold, on it's tangent plane (see fig. 1.6 (a) hereunder) - supposing that the quadrilateral is infinitesimally small and that we can find a conformal map from γ to $T(\gamma)$.

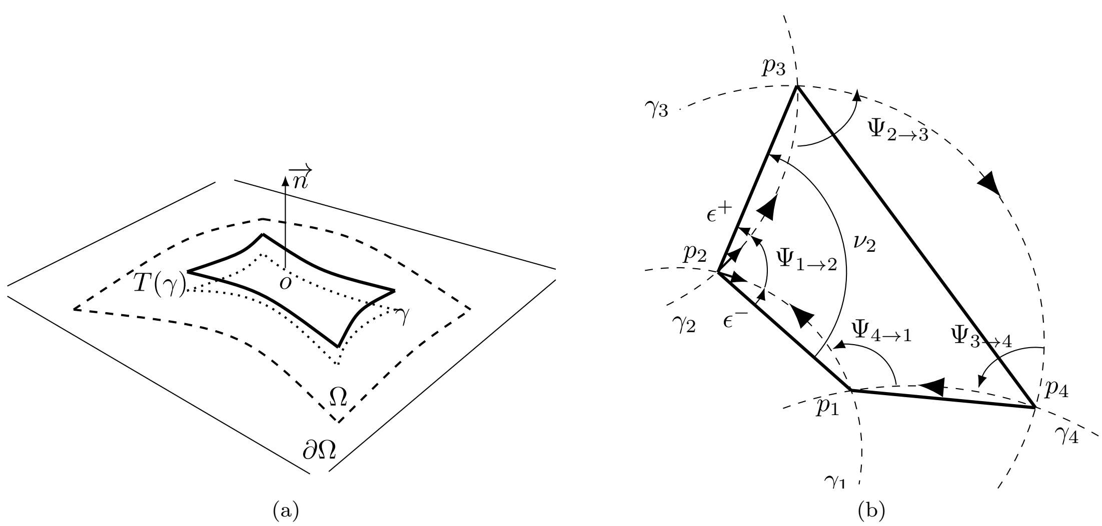


Figure 1.6: Relationship between parallel transportation along a closed path and the excess of the angle-sum over four right angles of a quadrilateral.

Let's look at the point p_2 on $\partial\Omega$. We have $\nu_2 = \epsilon^- + \Psi_{1\to 2} + \epsilon^+ = \Psi_{1\to 2} + \epsilon$. In general,

$$\sum_{i=1}^{4} \nu_i = 2\pi \tag{5}$$

$$\underbrace{\sum_{i=1}^{4} \Psi_{i\to i+1}}_{=-\delta\theta} + \sum_{i=1}^{4} \epsilon_i = \sum_{i=1}^{4} \frac{\pi}{2} \tag{6}$$

$$\Rightarrow \quad -\delta\theta = \sum_{i=1}^{4} (\frac{\pi}{2} - \epsilon_i) \tag{7}$$

Calling $\frac{\pi}{2} - \epsilon_i$ the excess, we get the assertion made.

♦

1.9 p102 - Clarification

Still using the same notation and Fig. 7, we have at B the three vectors $(T^r)_1,\ (T^r)_2,\ Y_r$ ... it follows that

3.516. $(\Delta T^r)_A(Y_r)_{A'2} = -(\Delta T^r)_B(Y_r)_B$

First we note that for an invariant "propagated parallely" along a curve we have

$$\frac{\delta(T^rY_r)}{\delta u} = \frac{\delta T^r}{\delta u}Y_r + T^r\frac{\delta Y_r}{\delta u} = 0$$

In fig. 1.7 we use the following convention:

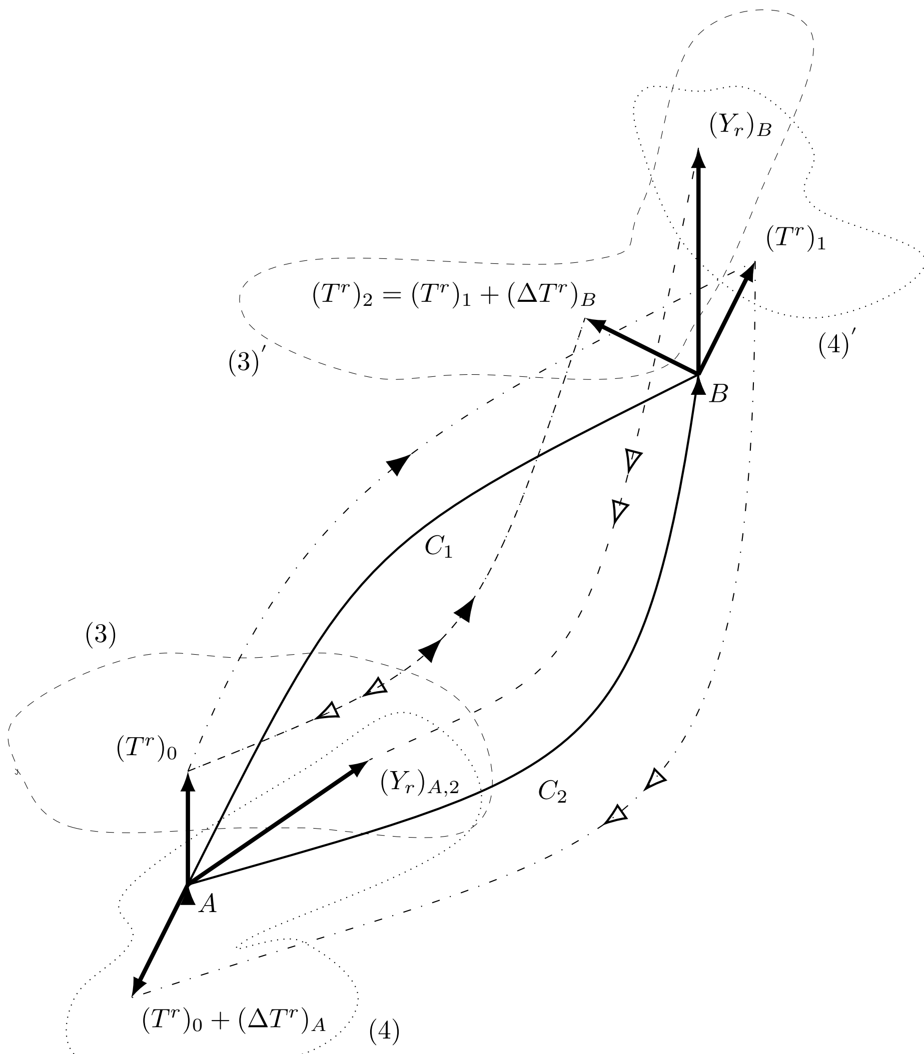


Figure 1.7: Parallel transportation along a closed path

- a one black arrowed line means a forward propagation from A to B along C_1

- a double black arrowed line means a forward propagation from A to B along C_2
- a double open arrowed line means a backward propagation from B to A along C_2

In order to find the angular displacement of the vector $(T^r)_0$ when propagated parallely from A to B and back to A along different paths, we follow the dash dotted line in Fig. 1.7. Following that path, we end with the vector $(T^r)_0 + (\Delta T^r)_A)$ in A and have the vector $(T^r)_1$ as intermediate forward propagation from A to B along C_1, that vector being transported backwards to A along C_2.

Note that for a vector the forward and backward propagation along the same curve is a null operation:

$$(T^r)_0 \underset{\substack{A \to B \\ C_2}}{\to} (T^r)_2 \underset{\substack{B \to A \\ C_2}}{\to} (T^r)_0$$

Also, we have in Fig.1.7.

$$(T^r)_0 \underset{\substack{A \to B \\ C_1}}{\to} (T^r)_1 \underset{\substack{B \to A \\ C_2}}{\to} (T^r)_0 + (\Delta T^r)_A$$

$$(Y_r)_B \underset{\substack{B \to A \\ C_2}}{\to} (Y_r)_{A,2}$$

At A we form the following invariants

$$\begin{cases} ((T^r)_0 + (\Delta T^r)_A)\ (Y_r)_{A,2} \\ (T^r)_0\ (Y_r)_{A,2} \end{cases} \tag{1}$$

and at B

$$\begin{cases} (T^r)_1\ (Y_r)_B \\ (T^r)_2\ (Y_r)_B = ((T^r)_1 + (\Delta T^r)_B)\ (Y_r)_B \end{cases} \tag{2}$$

Due to the null effect of parallel propagation on invariants, we get

$$(T^r)_1\ (Y_r)_B = ((T^r)_0 + (\Delta T^r)_A)\ (Y_r)_{A,2} \tag{3}$$

$$((T^r)_1 + (\Delta T^r)_B)\ (Y_r)_B = (T^r)_0\ (Y_r)_{A,2} \tag{4}$$

$$\text{(3)-(4)} \quad \Rightarrow \quad -(\Delta T^r)_B)\ (Y_r)_B = (\Delta T^r)_A)\ (Y_r)_{A,2} \tag{5}$$

♦

1.10 p105 - Clarification

3.521.
$$\begin{aligned}\frac{dI}{dv} &= \int_{u_1}^{u_2} \partial_v \left(T_n \partial_u x^n\right) du \\ &= \int_{u_1}^{u_2} \frac{\delta T_n}{\delta v} \partial_u x^n du + \int_{u_1}^{u_2} T_n \frac{\delta \left(\partial_u x^n\right)}{\delta v} du.\end{aligned}$$

Now $\frac{\delta T_n}{\delta v} = 0$, since T_r is propagated along *all* curves in V_n.

To better understand this last statement recall that from 3.515, we have

$$\begin{aligned}\left(\Delta T^r\right)_B \left(Y_r\right)_B &= \int \int Y_r R^r_{.pmn} T^p \partial_u x^m \partial_v x^n du dv \\ &= 0 \quad \text{as} \qquad R^r_{.pmn} = 0\end{aligned}$$

As $\left(Y_r\right)_B$ is arbitrary we have $\left(\Delta T^r\right)_B = 0$. Consider fig. 1.8 below.

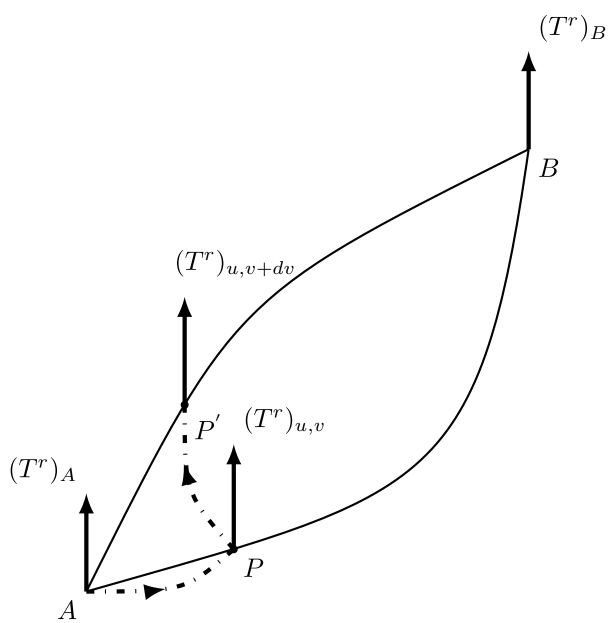


Figure 1.8: Parallel transportation along a path in a space with zero curvature tensor

Consider the path $A \rightarrow P \rightarrow P'$, P being situated at the parametric coordinates (u, v) and P' at $(u, v+dv)$. For this path we have also $\left(\Delta T^r\right)_{P,P'} = 0$. So $(T^r)_{u,v} = (T^r)_A$ and $(T^r)_{u,v+dv} = (T^r)_{u,v}$ and thus $\frac{\delta T_r}{\delta v} = 0$.

♦

1.11 p108 - Exercise 1

Taking polar coordinates on a sphere of radius a, calculate the curvature tensor, the Ricci tensor, and the curvature invariant.

We have

$$\Phi = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2 \tag{1}$$

We only have to calculate R_{1212} (see exercise page 86).

$$\begin{aligned} R_{\theta\phi\theta\phi} &= \partial_\theta \underbrace{[\phi\phi,\theta]}_{=-a^2\sin\theta\cos\theta} - \partial_\phi \underbrace{[\phi\theta,\theta]}_{=0} + \underbrace{\Gamma^\theta_{\phi\theta}}_{=0}[\theta\phi,\theta] + \underbrace{\Gamma^\phi_{\phi\theta}[\theta\phi,\phi]}_{=a^2\cos^2\theta} - \underbrace{\Gamma^\theta_{\phi\phi}}_{=0}[\theta\theta,\theta] - \underbrace{\Gamma^\phi_{\phi\phi}}_{=0}[\theta\theta,\phi] && (2) \\ &= a^2\sin^2\theta && (3) \end{aligned}$$

$$3.208.: \quad \frac{R_{11}}{a_{11}} = \frac{R_{22}}{a_{22}} = -\frac{R_{\theta\phi\theta\phi}}{det(a_{mn})} \quad \Rightarrow \quad \begin{cases} R_{11} = -1 \\ R_{12} = 0 \\ R_{22} = -\sin^2\theta \end{cases} \tag{4}$$

$$3.210.: \quad R = -\frac{2}{det(a_{mn})} R_{\theta\phi\theta\phi} \quad \Rightarrow \quad R = -\frac{2}{a^2} \tag{5}$$

♦

1.12 p108 - Exercise 2

Take as manifold V_2 the surface of an ordinary right circular cone, and consider one of the circular sections. A vector in V_2 is propagated parallely round this circle. Show that its direction is changed on completion of the circuit. can you reconcile this result with the fact that V_2 is flat?

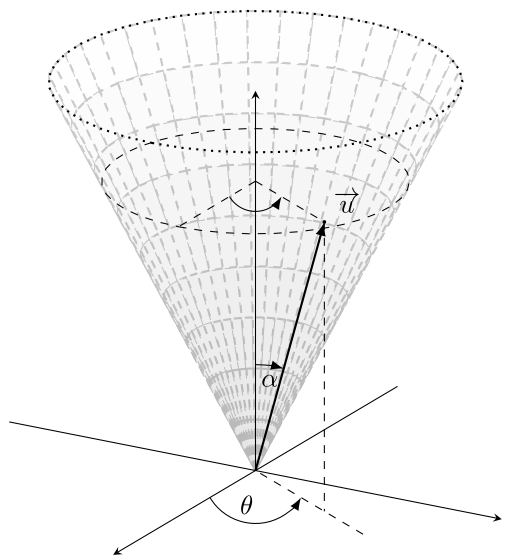


Figure 1.9: Intrinsic coordinates on a cone

We take as coordinate system (u, θ), embedded in the manifold, u being the distance of the generator of the considered point to the apex of the cone and θ the angle with a arbitrary vector laying in a plane perpendicular to the axis of the cone.
It is not hard to see that the fundamental form for this manifold is

$$\Phi = du^2 + \underbrace{k}_{= \sin^2 \alpha} u^2 d\theta^2$$

We have

$$(a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & ku^2 \end{pmatrix} \quad (a^{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{ku^2} \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} [mn, u] \\ [mn, \theta] \end{pmatrix} = \begin{pmatrix} 0 & 0 & -ku \\ ku & 0 & 0 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} \Gamma^u_{mn} \\ \Gamma^\theta_{mn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -ku \\ 0 & \frac{1}{u} & 0 \end{pmatrix} \tag{3}$$

Let's calculate the curvature tensor. From the exercise on page 86 we know that for a V_2 all components of the curvature tensor can be expressed in terms of R_{1212}. We have

$$R_{u\theta u\theta} = \left\{ \begin{array}{l} \frac{1}{2}\left(\partial_{\theta u}a_{u\theta} + \partial_{u\theta}a_{\theta u} - \partial_{\theta\theta}a_{uu} - \partial_{uu}a_{\theta\theta}\right) \\ +a^{pq}\left([u\theta,p][\theta u,q] - [uu,p][\theta\theta,q]\right) \end{array} \right. = \left\{ \begin{array}{l} -k \\ +a^{uu}\left(\underbrace{[u\theta,u][\theta u,u] - [uu,u][\theta\theta,u]}_{=0}\right) \\ +\underbrace{a^{u\theta}}_{=0}\left([u\theta,u][\theta u,\theta] - [uu,u][\theta\theta,\theta]\right) \\ +\underbrace{a^{\theta u}}_{=0}\left([u\theta,\theta][\theta u,u] - [uu,\theta][\theta\theta,u]\right) \\ +\underbrace{a^{\theta\theta}}_{=\frac{1}{ku^2}}\left(\underbrace{[u\theta,\theta][\theta u,\theta]}_{=k^2u^2} - \underbrace{[uu,\theta][\theta\theta,\theta]}_{=0}\right) \end{array} \right. \tag{4}$$

So indeed all components of the curvature tensor vanish and hence V_2 is flat.

Let's now calculate the parallel transportation of a vector T^r along a circle somewhere on the cone. Taking θ as the parameter of the curve, the equation of the curve is $(u = u_0,\ \theta) \quad \theta \in\ [0, 2\pi)$. We have for parallel transportation along that curve $\frac{\delta T^r}{\delta\theta} = 0$ and get

$$\left\{ \begin{array}{l} \frac{dT^u}{d\theta} + \Gamma^u_{\theta\theta}T^\theta\frac{d\theta}{d\theta} = 0 \\ \\ \frac{dT^\theta}{d\theta} + \Gamma^\theta_{u\theta}T^u\frac{d\theta}{d\theta} + \Gamma^\theta_{\theta u}T^\theta\underbrace{\frac{du}{d\theta}}_{=0} = 0 \quad \left(\frac{du}{d\theta} = 0 \quad \text{as} \quad u = C^{st}\right) \end{array} \right. \tag{5}$$

$$\Rightarrow \left\{ \begin{array}{l} \dot{T^u} - ku_0T^\theta = 0 \\ \\ \dot{T}^\theta + \frac{1}{u_0}T^u = 0 \end{array} \right. \tag{6}$$

$$\Rightarrow \left\{ \begin{array}{l} \frac{\ddot{T}^u}{T^u} = -k \\ \\ \dot{T}^\theta = -\frac{T^u}{u_0} \end{array} \right. \tag{7}$$

From the first equation in (11) we deduce that a solution can be of the form
$T^u = p^{'}\left(e^{(a\theta+b^{'})} + e^{-(a\theta+b^{'})}\right)$. Substituting in (12) we see that $a^2 = -k \rightarrow a = \pm i\sqrt{k}$. Replacing

$b^{'}$ by ib the solution for the system of differential equations (12) becomes

$$
\begin{aligned}
T^u &= p^{'}\left(e^{i(\sqrt{k}\theta+b)} + e^{-i(\sqrt{k}\theta+b)}\right) && (8)\\
&= p\cos\left(\sqrt{k}\theta+b\right) && (9)\\
\Leftrightarrow \quad &= C_1\sin\sqrt{k}\theta + C_2\cos\sqrt{k}\theta && (10)
\end{aligned}
$$

$$
\text{(15) in (12) gives:} \quad \begin{cases} T^u = C_1\sin\sqrt{k}\theta + C_2\cos\sqrt{k}\theta \\ \\ T^\theta = \frac{1}{\sqrt{k}u_0}\left(C_1\cos\sqrt{k}\theta - C_2\sin\sqrt{k}\theta\right) \end{cases} \tag{11}
$$

$$
\text{with} \quad \begin{cases} C_1 = \sqrt{k}u_0\ T^\theta\big|_{\theta=0} \\ \\ C_2 = T^u\big|_{\theta=0} \end{cases} \tag{12}
$$

$$
\Rightarrow \quad \begin{cases} T^u = \sqrt{k}u_0T_0^\theta\sin\sqrt{k}\theta + T_0^u\cos\sqrt{k}\theta \\ \\ T^\theta = T_0^\theta\cos\sqrt{k}\theta - \frac{T_0^u}{\sqrt{k}u_0}\sin\sqrt{k}\theta \end{cases} \tag{13}
$$

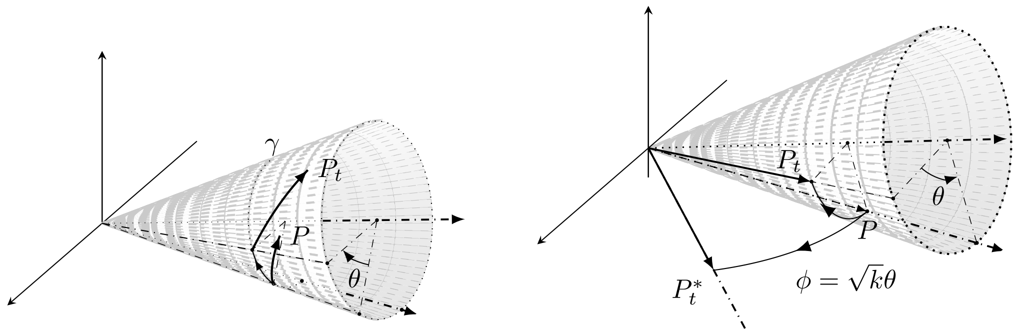
Let's now compute the angle ϕ between the starting vector T_0^r and the vector T^r parallely transported over an angle θ on the circle. We have (see **(2.301.)** and **(2.312.)**):

$$
\begin{cases} \left|T_0^r\right|^2 = \left(T_0^u\right)^2 + ku_0^2\left(T_0^\theta\right)^2 \\ \\ \left|T^r\right|^2 = \left(T_0^u\right)^2 + ku_0^2\left(T_0^\theta\right)^2 \\ \\ \cos\phi = \frac{T_0^uT^u + ku_0^2T_0^\theta T^\theta}{\left(T_0^u\right)^2 + ku_0^2\left(T_0^\theta\right)^2} \end{cases} \tag{14}
$$

The last equation in (19) becomes

$$
\begin{aligned}
&\cos\phi = \cos\sqrt{k}\theta && (15)\\
\Rightarrow \quad &\phi = \sqrt{k}\theta + 2m\pi \qquad (m = 0, \pm1, \pm2, \dots) && (16)
\end{aligned}
$$

So for $\theta = 2\pi$ the angle between the starting vector and the transported vector is not $2m\pi$.



(a) (b)

Figure 1.10: Relationship between parallel transportation along a circle of a cone and unwrapping a cone .

Fig. 1.10 illustrates the analogy between (a): the $\parallel$ transportation of a vector P along a circular curve γ on the cone over an angle θ giving a vector P_t making an angle $\phi = \sqrt{k}\theta$ with the starting vector, and

(b): the result of "unwrapping" a cone. Be two vectors $\overrightarrow{OP}$ and $\overrightarrow{OP_t}$, P and P_t being two points placed at a distance $r\theta$ along a circle with radius r. Placing the vector $\overrightarrow{OP}$ in the XY-plane and unwrapping the cone over an angle θ will map the vector $\overrightarrow{OP_t}$ in the XY-plane to a vector $\overrightarrow{OP_t^*}$ making an angle $\phi = \sqrt{k}\theta$ with the vector $\overrightarrow{OP}$.

The transported vector will only coincide with the initial vector for $\sqrt{k} = \sin\alpha = \frac{1}{n} \quad (n = 1, 2, \dots)$.

Fig. 1.11 illustrates this in the $X - Y$-plane, for n=3. Only after a transportation over three periods, will the transported vector coincide with the initial vector. The dotted area represented the unwrapped cone.

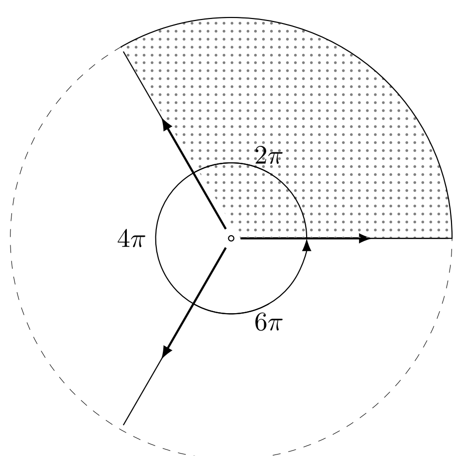


Figure 1.11: Parallel transportation along a circle of a cone with $\sin\alpha = \frac{1}{3}$.

♦

1.13 p109 - Exercise 3 and 4

Exercise3.

> Consider the equations
> $$\left(R_{mn}-\theta a_{mn}\right)X^n=0$$
> where R_{mn} is the Ricci tensor in a V_N $(N>2)$, θ an invariant, and X^n a vector. Show that, if these equations are to be consistent, θ must have one of a certain set of N values, and that the vectors X^n corresponding to different values of θ are perpendicular to one another. (The directions of these vectors are called the *Ricci principal directions*).

$$\left(R_{mn}-\theta a_{mn}\right)X^n=0 \tag{1}$$

$$(1)\times\left(a^{mp}\right)\quad\Rightarrow\quad a^{mp}R_{mn}X^n-\theta\underbrace{a^{mp}a_{mn}}_{=\delta^p_n}X^n=0 \tag{2}$$

$$\Rightarrow\quad a^{mp}R_{mn}X^n-\theta X^p=0 \tag{3}$$

$$\tag{4}$$

Define $T^p_n=a^{mp}R_{mn}$, then (3) can be written in matrix form with $\mathbf{T}\equiv\left(T^p_n\right)$, $\mathbf{X}\equiv\left(X^p\right)$ and $\mathbf{I}\equiv\left(\delta^i_j\right)$

$$\left(\mathbf{T}-\theta\mathbf{I}\right)\mathbf{X}=0 \tag{5}$$

This is an eigenvector equation with $\mathbf{T}$ being Hermitian i.e. $\mathbf{T}^\dagger=\mathbf{T}$. Indeed, obviously $\overline{\mathbf{T}}=\mathbf{T}$ and

$$\mathbf{T}^T=\left(\mathbf{AR}\right)^T \tag{6}$$

$$=\mathbf{R}^T\mathbf{A}^T \tag{7}$$

$$\Leftrightarrow\left(T^j_i\right)=\left(R_{kj}\right)^T\left(a^{ik}\right)^T \tag{8}$$

$$=\left(R_{jk}\right)\left(a^{ki}\right) \tag{9}$$

as both $R_{jk},\ a^{ki}$ are symmetric we have

$$\left(T^j_i\right)=\left(R_{kj}\right)\left(a^{ik}\right) \tag{10}$$

$$=\left(T^i_j\right) \tag{11}$$

$$\Rightarrow\qquad\mathbf{T}^\dagger=\mathbf{T} \tag{12}$$

This means that the N roots of $det\left(\mathbf{T}-\theta\mathbf{I_n}\right)=0$, which is a necessary condition to have equation (5) consistent, are real. Hence θ will take N values, being the eigenvalues of the transformation matrix $\mathbf{T}$. If all eigenvalues have multiplicity one, then the N eigenvectors in (5) corresponding to the N eigenvalues will be orthogonal to each other. But, can eigenvalues with algebraic multiplicity $m>1$

occur? The answer is yes. Let's rewrite $P(\theta) = det\,(\mathbf{T} - \theta \mathbf{I_n})$ as $\theta^N + q_i \theta^{n-i} \quad (i = N-1, N-2, \dots, 1)$ with q_i functions of the Ricci tensor components. The condition for eigenvalues with algebraic multiplicity $m > 1$ to occur is that the determinant of the Sylvester matrix of the two following polynomial should be zero.

$$\begin{cases} P(\theta) = \theta^N + q_i \theta^{n-i} \quad (i = N-1, N-2, \dots, 1) \\ \frac{dP(\theta)}{d\theta} = N\theta^{N-1} + (n-i) q_i \theta^{n-i-1} \quad (i = N-1, N-2, \dots, 1) \end{cases} \tag{13}$$

The associated Sylvester matrix with these two polynomials will be of the form

$$S\left(P(\theta), \frac{dP(\theta)}{d\theta}\right) = \begin{pmatrix} 1 & q_{N-1} & \dots & q_1 & 0 & 0 & \dots & 0 \\ 0 & 1 & q_{N-1} & \dots & q_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & q_{N-1} & \dots & q_2 & q_1 \\ N & (N-1)q_{N-1} & \dots & q_1 & 0 & 0 & \dots & 0 \\ 0 & N & (N-1)q_{N-1} & \dots & q_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & N & (N-1)q_{N-1} & \dots & 2q_2 & q_1 \end{pmatrix}$$

If the determinant of this matrix is not zero, then there will be no algebraic multiplicity. In the other case, one has to check whether in the eigenspace, related to the eigenvalues with algebraic multiplicity $m > 1$, m linear independent eigenvectors can be found.

Exercise 4.

What becomes of the Ricci principal directions (see above) if $N = 2$?

From **3.208**. we have

$$\frac{R_{11}}{a_{11}} = \frac{R_{12}}{a_{12}} = \frac{R_{22}}{a_{22}} = -\frac{R_{1212}}{a} \tag{14}$$

$$\Rightarrow \begin{cases} T_1^1 = a^{11}R_{11} + a^{12}R_{21} \\ T_2^1 = a^{11}R_{12} + a^{12}R_{22} \\ T_2^2 = a^{21}R_{12} + a^{22}R_{22} \end{cases} \tag{15}$$

$$\text{put } K = -\frac{R_{1212}}{a} \quad \Rightarrow \begin{cases} T_1^1 = Ka^{11}a_{11} + Ka^{12}a_{12} \\ T_2^1 = Ka^{11}a_{12} + Ka^{12}a_{22} \\ T_2^2 = Ka^{21}a_{12} + Ka^{22}a_{22} \end{cases} \tag{16}$$

$$\Rightarrow \begin{cases} T_1^1 = K\delta_1^1 = K \\ T_2^1 = K\delta_2^1 = 0 \\ T_2^2 = K\delta_2^2 = K \end{cases} \tag{17}$$

hence the characteristic equation $det\,(\mathbf{T} - \theta \mathbf{I_n}) = 0$ becomes $(K - \theta)^2 = 0$. So only one value of θ exists as $\theta = K$. Equation (5) becomes $\mathbf{0X} = 0$. So we can chose any pair of linear independent vectors as eigenvectors and can make them perpendicular to one another.

♦

1.14 p109 - Exercise 5

Suppose that two spaces V_N, $V_N^{'}$ have metric tensors a_{mn}, $a_{mn}^{'}$ such that $a_{mn}^{'} = k\ a_{mn}$, where k is a constant. Write down the relations between the curvature tensors, the Ricci tensors, and the curvature invariants of the two spaces.

We have

$$ds^2 = a_{mn} dx^m dx^n \tag{1}$$

$$ds^{'2} = a_{mn}^{'} dx^{'m} dx^{'n} \tag{2}$$

But let's be careful: there is no reason to assume that $dx^m = dx^{'m}$. Let's embed the two spaces in a space V_{N+1}. If an observer in that space sees two dispacements ds^2 and $ds^{'2}$ which for him have the same magnitude, we have

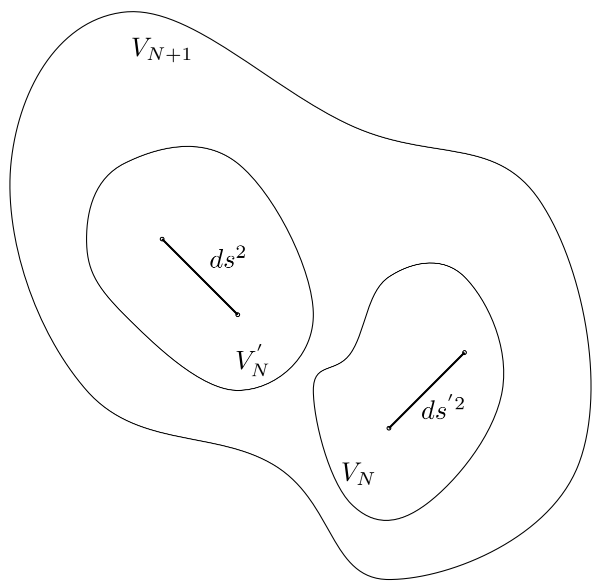


Figure 1.12: Embedded V_N spaces

$$ds^{'2} = ds^2 \tag{3}$$

$$\Rightarrow \qquad a_{mn}^{'} dx^{'m} dx^{'n} = a_{mn} dx^m dx^n \tag{4}$$

$$\Rightarrow \qquad k a_{mn} dx^{'m} dx^{'n} = a_{mn} dx^m dx^n \tag{5}$$

$$\Rightarrow \qquad dx^{'m} = \frac{1}{\sqrt{k}} dx^m \tag{6}$$

We have also

$$a^{'}_{mk}a^{'kn} = \delta^n_m \tag{7}$$

$$\Rightarrow \qquad ka_{mp}a^{'pn} = \delta^n_m \tag{8}$$

$$\Rightarrow \qquad a^{'pn} = \frac{1}{k}a^{pn} \tag{9}$$

And get the following relations

$$\begin{cases} [mn,r]^{'} = \sqrt{k}^3[mn,r] \\ \\ \Gamma^{'r}_{.mn} = \sqrt{k}\Gamma^r_{.mn} \end{cases} \tag{10}$$

$$\Rightarrow \qquad R^{'s}_{.rmn} = \frac{\partial\Gamma^{'s}_{.rn}}{\partial x^{'m}} + \dots \tag{11}$$

$$\Rightarrow \qquad R^{'s}_{.rmn} = \frac{\sqrt{k}\partial\Gamma^s_{.rn}}{\partial\left(\frac{x^m}{\sqrt{k}}\right)} + \dots \tag{12}$$

$$\Rightarrow \qquad R^{'s}_{.rmn} = kR^s_{.rmn} \tag{13}$$

$$\times\ a^{'ks} \qquad \Rightarrow \qquad a^{'ks}R^{'s}_{.rmn} = ka^{'ks}R^s_{.rmn} \tag{14}$$

$$\Rightarrow \qquad R^{'}_{krmn} = k\frac{1}{k}a^{ks}R^s_{.rmn} \tag{15}$$

$$\Rightarrow \qquad R^{'}_{krmn} = R_{krmn} \tag{16}$$

$$R_{rm} = R^n_{.rmn} \qquad \Rightarrow \qquad R^{'}_{\ mn} = kR_{rmn} \tag{17}$$

$$R = a^{mn}R_{mn} \qquad \Rightarrow \qquad R^{'} = a^{'mn}R^{'}_{mn} \tag{18}$$

$$\Rightarrow \qquad R^{'} = \frac{1}{k}a^{mn}kR_{mn} \tag{19}$$

$$\Rightarrow \qquad R^{'} = R \tag{20}$$

Summary

$$R^{'s}_{.rmn} = kR^s_{.rmn} \tag{21}$$

$$R^{'}_{krmn} = R_{krmn} \tag{22}$$

$$R^{'}_{mn} = kR_{rmn} \tag{23}$$

$$R^{'} = R \tag{24}$$

♦