Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

Part II

Chapters V to VIII

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Non-Riemannian spaces.

8.1 p309 - Exercise 10

In a projective space, *the coefficients of projective connection* P^r_{mn} are defined as follows

$$P^r_{mn} = \Gamma^r_{mn} - \frac{1}{N+1}\left(\delta^r_m \Gamma^p_{pn} + \delta^r_n \Gamma^p_{pm}\right)$$

Show that P^r_{mn} is invariant under projective transformations of Γ^r_{mn}. verify that $P^r_{nm} = P^r_{mn}$, $P^r_{rn} = 0$. Find the transformation properties of P^r_{mn} under changes of coordinate system. (T.Y. Thomas.)

Using (2)from the previous exercise, we have

$$P'^r_{mn} = \Gamma'^r_{mn} - \frac{1}{N+1}\left(\delta^r_m \Gamma'^p_{pn} + \delta^r_n \Gamma'^p_{pm}\right) \tag{1}$$

$$= \begin{cases} \Gamma^r_{mn} + \delta^r_n\psi_m + \delta^r_m\psi_n \\ \\ -\frac{1}{N+1}\delta^r_m\left(\Gamma^p_{pn} + \delta^p_n\psi_p + \delta^p_p\psi_n\right) \\ \\ -\frac{1}{N+1}\delta^r_n\left(\Gamma^p_{pm} + \delta^p_m\psi_p + \delta^p_p\psi_m\right) \end{cases} \tag{2}$$

$$\tag{3}$$

$$= \begin{cases} \Gamma^r_{mn} + \delta^r_n\psi_m + \delta^r_m\psi_n \\ \\ -\frac{1}{N+1}\delta^r_m\Gamma^p_{pn} - \frac{1}{N+1}\delta^r_m\psi_n - \frac{N}{N+1}\delta^r_m\psi_n \\ \\ -\frac{1}{N+1}\delta^r_n\Gamma^p_{pm} - \frac{1}{N-1}\delta^r_n\psi_m - \frac{N}{N+1}\delta^r_n\psi_m \end{cases} \tag{4}$$

$$= \begin{cases} \Gamma^r_{mn} - \frac{1}{N+1}\delta^r_n\Gamma^p_{pm} - \frac{1}{N-1}\delta^r_m\Gamma^p_{pn} \\ \\ +\delta^r_n\psi_m + \delta^r_m\psi_n - \frac{1}{N+1}\delta^r_n\psi_m - \frac{N}{N+1}\delta^r_n\psi_m \\ \\ -\frac{1}{N+1}\delta^r_m\psi_n - \frac{N}{N+1}\delta^r_m\psi_n \\ \end{cases} \tag{5}$$

$$\tag{6}$$

$$= \begin{cases} \underbrace{\Gamma^r_{mn} - \frac{1}{N+1}\left(\delta^r_m\Gamma^p_{pn} + \delta^r_n\Gamma^p_{pm}\right)}_{=P^r_{mn}} \\ \\ +\cancel{\frac{1}{N+1}\delta^r_n\psi_m} + \cancel{\frac{1}{N+1}\delta^r_m\psi_n} + \cancel{\frac{N}{N+1}\delta^r_n\psi_m} + \cancel{\frac{N}{N+1}\delta^r_m\psi_n} - \cancel{\frac{1}{N+1}\delta^r_n\psi_m} - \cancel{\frac{N}{N+1}\delta^r_n\psi_m} \\ \\ -\cancel{\frac{1}{N+1}\delta^r_m\psi_n} - \cancel{\frac{N}{N+1}\delta^r_m\psi_n} \end{cases} \tag{7}$$

$$= P^r_{mn} \tag{8}$$

proving the invariance of P^r_{mn} under projective transformations of the linear symmetric connections.

$\Diamond$

As we are dealing with linear symmetric connections and $\delta^r_m \Gamma^p_{pn} - \delta^r_n \Gamma^p_{pm}$ also is symmetric, we can conclude that P^r_{mn} is symmetric.
Also,

$$
\begin{aligned}
P^r_{rn} &= \Gamma^r_{rn} - \frac{1}{N+1}\left(\delta^r_r \Gamma^p_{pn} + \delta^r_n \Gamma^p_{pr}\right) && (9)\\
&= \Gamma^r_{rn} - \frac{1}{N+1}\left(N\Gamma^p_{pn} + \Gamma^p_{pn}\right) && (10)\\
&= \Gamma^r_{rn} - \Gamma^p_{pn} && (11)\\
&= 0 && (12)
\end{aligned}
$$

$\Diamond$

Let's perform a coordinate transformation, and use

$$(\mathbf{8.112}): \quad \Gamma^\rho_{\mu\nu} = \Gamma^r_{mn} X^\rho_r X^m_\mu X^n_\nu + X^r_{\mu\nu} X^\rho_r$$

from which we obtain

$$
\begin{aligned}
P^\rho_{\mu\nu} &= \Gamma^\rho_{\mu\nu} - \frac{1}{N+1}\left(\delta^\rho_\mu \Gamma^\tau_{\tau\nu} + \delta^\rho_\nu \Gamma^\tau_{\tau\mu}\right) && (13)\\
&= \begin{cases} \Gamma^r_{mn} X^\rho_r X^m_\mu X^n_\nu + X^r_{\mu\nu} X^\rho_r \\ -\frac{1}{N+1}\delta^\rho_\mu \left(\Gamma^r_{mn} \underbrace{X^\tau_r X^m_\tau}_{=\delta^m_r} X^n_\nu + \underbrace{X^r_{\tau\nu} X^\tau_r}_{=0}\right) \\ -\frac{1}{N+1}\delta^\rho_\nu \left(\Gamma^r_{mn} \underbrace{X^\tau_r X^m_\tau}_{=\delta^m_r} X^n_\mu + \underbrace{X^r_{\tau\mu} X^\tau_r}_{=0}\right) \end{cases} && (14)\\
&= \begin{cases} \Gamma^r_{mn} X^\rho_r X^m_\mu X^n_\nu + X^r_{\mu\nu} X^\rho_r \\ -\frac{\Gamma^r_{rn}}{N+1}\left(\delta^\rho_\mu X^n_\nu + \delta^\rho_\nu X^n_\mu\right) \end{cases} && (15)\\
(2.542): \quad &= \begin{cases} \Gamma^r_{mn} X^\rho_r X^m_\mu X^n_\nu + X^r_{\mu\nu} X^\rho_r \\ -\frac{\frac{\partial}{\partial x^r}\sqrt{a}}{\sqrt{a}(N+1)}\left(\delta^\rho_\mu X^n_\nu + \delta^\rho_\nu X^n_\mu\right) \end{cases} && (16)
\end{aligned}
$$

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8.2 p309 - Exercise 11

Show that the differential equation of a path can be written in the form

$$\lambda^s\left(\frac{d\lambda^r}{du}+P^r_{mn}\lambda^m\lambda^n\right)=\lambda^r\left(\frac{d\lambda^s}{du}+P^s_{mn}\lambda^m\lambda^n\right)$$

where P^r_{mn} are the coefficients of projective connection defined in Ex. 10. Deduce that no change in the Γ^r_{mn} other than a projective transformation leaves the P^r_{mn} invariant.

Starting from (**8.328**) we have the differential equation of a path

$$\lambda^s\left(\frac{d\lambda^r}{du}+\Gamma^r_{mn}\lambda^m\lambda^n\right)=\lambda^r\left(\frac{d\lambda^s}{du}+\Gamma^s_{mn}\lambda^m\lambda^n\right) \tag{1}$$

and using the definition of P^r_{mn} we check the following expression

$$\lambda^s\left(\frac{d\lambda^r}{du}+\left[\Gamma^r_{mn}-\frac{1}{N+1}\left(\delta^r_m\Gamma^p_{pn}+\delta^r_n\Gamma^p_{pm}\right)\right]\lambda^m\lambda^n\right)\stackrel{?}{=}\lambda^r\left(\frac{d\lambda^s}{du}+\left[\Gamma^s_{mn}-\frac{1}{N+1}\left(\delta^s_m\Gamma^p_{pn}+\delta^s_n\Gamma^p_{pm}\right)\right]\lambda^m\lambda^n\right) \tag{2}$$

$$\Rightarrow \qquad \Gamma^p_{pn}\lambda^r\lambda^n\lambda^s+\Gamma^p_{pm}\lambda^m\lambda^r\lambda^s\stackrel{?}{=}\Gamma^p_{pn}\lambda^s\lambda^n\lambda^r+\Gamma^p_{pm}\lambda^m\lambda^s\lambda^r \tag{3}$$

Obviously the last expression is true, proving the equivalence of the equation of a path.

◊

As

$$\lambda^s\left(\frac{d\lambda^r}{du}+P^r_{mn}\lambda^m\lambda^n\right)=\lambda^r\left(\frac{d\lambda^s}{du}+P^s_{mn}\lambda^m\lambda^n\right)$$

describes a path and is equivalent to (**8.328**) we can follow the exact same reasoning from (**8.328**) on, to (**8.332**) where

$$\begin{aligned}
B^r_{mn} &= P'^r_{mn}-P^r_{mn} && (4)\\
&= \Gamma'^r_{mn}-\frac{1}{N+1}\left(\delta^r_m\Gamma'^p_{pn}+\delta^r_n\Gamma'^p_{pm}\right)-\Gamma^r_{mn}+\frac{1}{N+1}\left(\delta^r_m\Gamma^p_{pn}+\delta^r_n\Gamma^p_{pm}\right) && (5)\\
&= \underbrace{\Gamma'^r_{mn}-\Gamma^r_{mn}}_{=A^r_{mn}}-\frac{1}{N+1}\left[\delta^r_m\Gamma'^p_{pn}+\delta^r_n\Gamma'^p_{pm}-\delta^r_m\Gamma^p_{pn}-\delta^r_n\Gamma^p_{pm}\right] && (6)\\
&= A^r_{mn}-\frac{1}{N+1}\left[\delta^r_m\left(\Gamma'^p_{pn}-\Gamma^p_{pn}\right)+\delta^r_n\left(\Gamma'^p_{pm}-\Gamma^p_{pm}\right)\right] && (7)
\end{aligned}$$

As we want P^r_{mn} to be invariant under projective transformations, we have $B^r_{mn}=0$ and hence (7) can be written as

$$A^r_{mn}=\frac{1}{N+1}\left[\delta^r_m\left(\Gamma'^p_{pn}-\Gamma^p_{pn}\right)+\delta^r_n\left(\Gamma'^p_{pm}-\Gamma^p_{pm}\right)\right] \tag{8}$$

From this we see that A^r_{mn} must of the form

$$A^r_{mn} = \delta^r_n \phi_m + \delta^r_n \phi_n$$

with $\phi_k = \frac{1}{N+1}\left(\Gamma'^{p}_{pk} - \Gamma^p_{pk}\right)$, which leads to the type of transformations as defined in (**8.337**).

♦

8.3 p310 - Exercise 12

Defining

$$P^s_{.rmn} = P^s_{rn,m} - P^s_{rm,n} + P^p_{rn}P^s_{pm} - P^p_{rm}P^s_{pn}$$

$$P_{rm} = P^s_{.rms}$$

where P^r_{mn} are the coefficients of projective connection of Ex. 10, show that

$$P^s_{.smn} = 0, \quad P_{rm} = -P^s_{rm,s} + P^p_{rs}P^s_{pm}$$

Prove that

$$W^s_{.rmn} = P^s_{.rmn} + \frac{1}{N-1}\left(\delta^s_m P_{rn} - \delta^s_n P_{rm}\right)$$

where $W^s_{.rmn}$ is the projective curvature tensor.

$$P^s_{.smn} = P^s_{sn,m} - P^s_{sm,n} + \underbrace{P^p_{sn}P^s_{pm} - P^p_{sm}P^s_{pn}}_{=0} \tag{1}$$

from which we see that $P^s_{.smn} = -P^s_{.snm}$. From the definition of $P^s_{.rmn}$ it is easy to see that this quantity is symmetric in the last two suffixes. Hence we can conclude from $P^s_{.smn} = -P^s_{.snm}$ and $P^s_{.smn} = P^s_{.snm}$ that $P^s_{.smn} = 0$.

♢

$$P_{rm} = P^s_{.rms} \tag{2}$$
$$= \underbrace{P^s_{rs,m}}_{=0} - P^s_{rm,s} + P^p_{rs}P^s_{pm} - P^p_{rm}\underbrace{P^s_{ps}}_{=0} \tag{3}$$
$$= -P^s_{rm,s} + P^p_{rs}P^s_{pm} \tag{4}$$

♢

The last assignment requires about 5 pages of tedious and boring basic algebraic and suffix manipulations. There was no added value to transcript this in Latex.

♦

8.4 p310 - Exercise 13

Show that

$$W^s_{.smn} = 0, \quad W^s_{.rsn} = 0, \quad W^s_{.rms} = 0,$$
$$W^s_{.rmn} = -W^s_{.rnm}, \quad W^s_{.rmn} + W^s_{.mnr} + W^s_{.nrm} = 0,$$

$$\begin{aligned} W^s_{.smn} &= \underbrace{P^s_{.smn}}_{=0} + \frac{1}{N-1}\left(\underbrace{\delta^s_m P_{sn} - \delta^s_n P_{sm}}_{=P_{mn}-P_{nm}=0}\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} W^s_{.rsn} &= P^s_{.rsn} + \frac{1}{N-1}\left(\delta^s_s P_{rn} - \delta^s_n P_{rs}\right) \\ &= P^s_{.rsn} + \underbrace{P_{rn}}_{=P^s_{.rns}} \end{aligned}$$

but $P^s_{.rmn}$ is skew-symmetric in the last two suffixes, so

$$W^s_{.rsn} = 0$$

The same reasoning holds for

$$W^s_{.rms} = 0$$

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$$\begin{aligned} W^s_{.rmn} + W^s_{.mnr} + W^s_{.nrm} &= \begin{cases} \frac{1}{N-1}\left(\delta^s_m P_{rn} - \delta^s_n P_{rm} + \delta^s_n P_{mr} - \delta^s_r P_{mn} + \delta^s_r P_{nm} - \delta^s_m P_{nr}\right) \\ \quad +P^s_{.rn,m} - P^s_{.rm,n} + P^p_{.rn} P^s_{.pm} - P^p_{.rm} P^s_{.pn} \\ \quad +P^s_{.mr,n} - P^s_{.mn,r} + P^p_{.mr} P^s_{.pm} - P^p_{.mn} P^s_{.pr} \\ \quad +P^s_{.nm,r} - P^s_{.nr,m} + P^p_{.nm} P^s_{.pr} - P^p_{.nr} P^s_{.pm} \end{cases} \\ &= 0 \end{aligned}$$

$$\blacklozenge$$

8.5 p310 - Exercise 14

In a space with linear connection, we say that the *directions* of two vectors, X^r at a point A and Y^r at a point B, are parallel with respect to a curve C which joins A and B if the vector obtained by parallel propagation of X^r along C from A to B is a multiple of Y^r. prove that the most general change of linear connection which preserves parallelism of directions (with respect to all curves) is given by

$$\Gamma'^r_{mn} = \Gamma^r_{mn} + \delta^r_m \psi_n$$

where ψ_n is an arbitrary vector. If Γ^r_{mn} are the coefficients of a symmetric connection, show that Γ'^r_{mn} are semi-symmetric(cf. Exercise 4).

Let's propagate parallely the vector X^r along the same curve C but with two different linear connections Γ^r_{mn} and Γ'^r_{mn} . So we have

$$\begin{cases} X^r{}_{,n} + \Gamma^r_{mn} X^m = 0 & (1) \\ \\ X^r_{,mn} + \Gamma'^r_{mn} X^m = 0 & (2) \end{cases}$$

Let's evaluate these expressions at the point B, requiring in both cases that at this point $X^r = \lambda_{(1)} Y^r$ for Γ^r_{mn} and $X^r = \lambda_{(2)} Y^r$ for Γ'^r_{mn} where Y^r is a single valued vector at this point. We get,

$$\begin{cases} \left(\lambda_{(1)} Y^r\right)_{,n} + \Gamma^r_{mn} \lambda_{(1)} Y^m = 0 \\ \\ \left(\lambda_{(2)} Y^r\right)_{,n} + \Gamma'^r_{mn} \lambda_{(2)} Y^m = 0 \end{cases} \tag{3}$$

As Y^r is fixed, the partial derivatives can be reduced to $\left(\lambda_{(.)}\right)_{,n} Y^r + 2^{nd}$ order terms and we rewrite (3) as

$$\begin{cases} \left(\Gamma^r_{mn} \lambda_{(1)} + \delta^r_m \left(\lambda_{(1)}\right)_{,n}\right) Y^m = 0 \\ \\ \left(\Gamma'^r_{mn} \lambda_{(2)} + \delta^r_m \left(\lambda_{(2)}\right)_{,n}\right) Y^m = 0 \end{cases} \tag{4}$$

$$\Leftrightarrow \qquad \begin{cases} \left(\Gamma^r_{mn} + \delta^r_m \frac{\left(\lambda_{(1)}\right)_{,n}}{\lambda_{(1)}}\right) Y^m = 0 \\ \\ \left(\Gamma'^r_{mn} + \delta^r_m \frac{\left(\lambda_{(2)}\right)_{,n}}{\lambda_{(2)}}\right) Y^m = 0 \end{cases} \tag{5}$$

$$\Rightarrow \qquad \Gamma'^r_{mn} = \Gamma^r_{mn} + \delta^r_m \left[\frac{\left(\lambda_{(1)}\right)_{,n}}{\lambda_{(1)}} - \frac{\left(\lambda_{(2)}\right)_{,n}}{\lambda_{(2)}}\right] \tag{6}$$

Put $\psi_n = \frac{1}{2}\left(\frac{\left(\lambda_{(1)}\right)_{,n}}{\lambda_{(1)}} - \frac{\left(\lambda_{(2)}\right)_{,n}}{\lambda_{(2)}}\right)$ and we get

$$\Gamma'^{r}_{mn} = \Gamma^{r}_{mn} + 2\delta^{r}_{m}\psi_n$$

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If Γ^r_{mn} is symmetric then,

$$\Gamma'^{r}_{mn} - \Gamma'^{r}_{nm} = \delta^{r}_{m}\left[\frac{\left(\lambda_{(1)}\right)_{,n}}{\lambda_{(1)}} - \frac{\left(\lambda_{(2)}\right)_{,n}}{\lambda_{(2)}}\right] - \delta^{r}_{n}\left[\frac{\left(\lambda_{(1)}\right)_{,m}}{\lambda_{(1)}} - \frac{\left(\lambda_{(2)}\right)_{,m}}{\lambda_{(2)}}\right]$$

which is of the form $\Gamma^r_{mn} - \Gamma^r_{nm} = \delta^r_m A_n - \delta^r_n A_m$ as required by Exercise 4.

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8.6 p310 - Exercise 15

In a space with symmetric connection, show that

$$T^r{}_{|mn} - T^r{}_{|nm} = -T^s R^r_{.smn}$$

We have

$$\begin{cases} T^r{}_{|mn} = \frac{\partial}{\partial x^n} T^r{}_{|m} + \Gamma^r_{qn} T^q{}_{|m} - \Gamma^q_{mn} T^r{}_{|q} \\ \\ T^r{}_{|nm} = \frac{\partial}{\partial x^m} T^r{}_{|n} + \Gamma^r_{qm} T^q{}_{|n} - \Gamma^q_{nm} T^r{}_{|q} \end{cases}$$

giving with $T^q{}_{|m} = T^q_{,m} + \Gamma^q_{km} T^k$ and $T^q{}_{|n} = T^q_{,n} + \Gamma^q_{kn} T^k$

$$T^r{}_{|mn} - T^r{}_{|nm} = \begin{cases} T^s \left(\frac{\partial}{\partial x^n} \Gamma^r_{ms} - \frac{\partial}{\partial x^m} \Gamma^r_{ns} \right) \\ \\ \cancel{+\Gamma^r_{mq} T^q_{,n} - \Gamma^r_{nq} T^q_{,m}} \\ \\ \cancel{+\Gamma^r_{qn} T^q_{,m} + \Gamma^r_{pn} \Gamma^p_{sm} T^s} \\ \\ +\Gamma^r_{pn} \Gamma^p_{sm} T^s - \Gamma^r_{pm} \Gamma^p_{sn} T^s \end{cases}$$

$$= -T^s R^r_{.smn} \qquad (\mathbf{8.214})$$

♦