

Tensor Calculus  
J.L. Synge and A.Schild (Dover Publication)  
Solutions to exercices

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## Remarks and warnings

### Some notation conventions

$$\partial_r a_{mn} \equiv \frac{\partial a_{mn}}{\partial x^r}$$

$$\Gamma_{mn}^r \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

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# Spaces and Tensors

## 1.1 p5-exercise

The parametric equations of a hypersurface in  $V_n$  are

$$\begin{aligned} x^1 &= a \cos(u^1) \\ x^2 &= a \sin(u^1) \cos(u^2) \\ x^3 &= a \sin(u^1) \sin(u^2) \cos(u^3) \\ &\vdots \\ x^{N-1} &= a \sin(u^1) \sin(u^2) \sin(u^3) \dots \sin(u^{N-2}) \cos(u^{N-1}) \\ x^N &= a \sin(u^1) \sin(u^2) \sin(u^3) \dots \sin(u^{N-2}) \sin(u^{N-1}) \end{aligned}$$

where  $a$  is a constant. Find the single equation of the hyperspace in the form 1.103.

We have:

$$\begin{aligned} (x^N)^2 + (x^{N-1})^2 &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i) (\cos^2(u^{N-1}) + \sin^2(u^{N-1})) \\ &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) \sin^2(u^{N-2}) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) (1 - \cos^2(u^{N-2})) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - a^2 \prod_{i=1}^{N-3} \sin^2(u^i) \cos^2(u^{N-2}) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - (x^{N-2})^2 \end{aligned}$$

giving

$$(x^N)^2 + (x^{N-1})^2 + (x^{N-2})^2 = a^2 \prod_{i=1}^{N-3} \sin^2(u^i)$$

In general, by recursion

$$\sum_{i=0}^k (x^{N-i})^2 = a^2 \prod_{i=1}^{N-k-1} \sin^2(u^i) \quad (k \leq N-2)$$

be  $k = N - 2$  ( $N - k - 1 = 1$ ) and in the left term put  $j = N - i$  ( $j$  goes from 2 to  $N$ ), we get

$$\begin{aligned}\sum_{j=2}^N (x^j)^2 &= a^2 \prod_{i=1}^1 \sin^2(u^i) \\ &= a^2 (1 - \cos^2(u^1)) \\ &= a^2 - (x^1)^2\end{aligned}$$

and thus the equation of the hyperspace is given by

$$\sum_{j=1}^N (x^j)^2 - a^2 = 0$$

Determine whether the points  $(\frac{1}{2}a, 0, 0, \dots, 0)$ ,  $(0, 0, \dots, 0, 2a)$  lie on the same or opposite sides of the hyperspace.

For  $(\frac{1}{2}a, 0, 0, \dots, 0)$  we have  $\sum_{j=1}^N (x^j)^2 - a^2 = -\frac{3a^2}{4} < 0$  and for  $(0, 0, \dots, 0, 2a)$  we have  $\sum_{j=1}^N (x^j)^2 - a^2 = \frac{3a^2}{4} > 0$ .

So the points lie on opposite sides of the hyperplane.



## 1.2 p6-exercise

Let  $U_2$  and  $W_2$  be subspaces of  $V_N$ . Show that if  $N = 3$  they will in general intersect in a curve; if  $N = 4$  they will in general intersect in a finite number of points; and if  $N > 4$  they will not in general intersect at all.

We have (see 1.102 page 5):  $x^r = f^r(u^1, u^2, \dots, u^M) \quad (r = 1, 2, \dots, N)$

Case  $N=3$ :

For  $U_2$  we have:

$$x^r = \phi^r(u^1, u^2) \quad (r = 1, 2, 3)$$

For  $W_2$  we have:

$$x^r = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

The intersect of the two hyperplanes is given by the  $N$  equations:

$$\phi^r(u^1, u^2) = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

So we have 3 equations in 4 unknown  $u^1, u^2, v^1, v^2$  and can choose (fix) one e.g.  $u^1$  and solve the set of equations for  $u^2, v^1, v^2$  giving

$$x^r = \theta^r(u^1) \quad (r = 1, 2, 3)$$

This is an equation of a curve in space (1 parameter equation)

Case  $N=4$ :

Using the same reasoning as with  $N=3$ , we get 4 equations for 4 unknown  $u^1, u^2, v^1, v^2$ .

Provided that the set of equation does not degenerate, these 4 equations will determine  $u^1, u^2, v^1, v^2$  without any degree of freedom. So we get points as solutions. This solution does not to be unique, e.g. if the  $\phi^r(u^1, u^2)$  are quadratic form, then the solutions

$$(u^1, u^2, v^1, v^2)$$

$$(-u^1, u^2, v^1, v^2)$$

$$(u^1, -u^2, v^1, v^2)$$

$$(-u^1, -u^2, v^1, v^2)$$

are possible.

Case  $N=5$ : There are more equations than variables. If the equations are not linear dependent, no solutions will be found.





### 1.3 p8-exercise

Show that  $(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = 3a_{rst}x^r x^s x^t$

$(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = a_{rst}x^r x^s x^t + a_{rts}x^r x^s x^t + a_{srt}x^r x^s x^t$  so by just renaming the dummy indices e.g. for the second term  $r \mapsto s$ ,  $s \mapsto t$  and  $t \mapsto r$  we get the desired result.



## 1.4 p8-exercise

If  $\phi = a_{rs}x^r x^s$ , show that

$$\frac{\partial \phi}{\partial x^r} = (a_{rs} + a_{sr})x^s \quad , \quad \frac{\partial^2 \phi}{\partial x^r \partial x^s} = a_{rs} + a_{sr}$$

Simplify these expressions in the case where  $a^{rs} = a^{sr}$

We have

$$\frac{\partial \phi}{\partial x^t} = \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs} \frac{\partial x^r}{\partial x^t} x^s + a_{rs} x^r \frac{\partial x^s}{\partial x^t} \quad (1)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs} \delta_t^r x^s + a_{rs} x^r \delta_t^s \quad (2)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts} x^s + a_{rt} x^r \quad (3)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts} x^s + a_{st} x^s \quad (\text{rename dummy variable in third term}) \quad (4)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + (a_{ts} + a_{st})x^s \quad (5)$$

Replace  $x^t$  by  $x^r$ , we get

$$\frac{\partial \phi}{\partial x^r} = \frac{\partial a_{rs}}{\partial x^r} x^r x^s + (a_{rs} + a_{sr})x^s \quad (6)$$

So the asked expression is only true if  $a_{rs}$  is not a function of the  $x^s$ . Assuming that  $a_{rs}$  is not a function of the  $x^s$ , take the partial derivative of (6) with respect to  $x^t$ , we get

$$\frac{\partial^2 \phi}{\partial x^r \partial x^t} = (a_{rs} + a_{sr}) \frac{\partial x^s}{\partial x^t} \quad (7)$$

$$= (a_{rs} + a_{sr}) \delta_t^s \quad (8)$$

$$= (a_{rt} + a_{tr}) \quad (9)$$

Replace  $x^t$  by  $x^s$ , and we get the proposed expression.



## 1.5 p8-clarification on expression 1.210

$$\frac{\partial^2 x^{,q}}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0$$

From 1.209:

$$\frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} + \frac{\partial x^r}{\partial x^{,n}} \frac{\partial^2 x^{,n}}{\partial x^p \partial x^s} = 0 \quad (1)$$

multiply (1) with  $\frac{\partial x^{,q}}{\partial x^r}$

$$\frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} + \frac{\partial x^r}{\partial x^{,n}} \frac{\partial^2 x^{,n}}{\partial x^p \partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0 \quad (2)$$

$$\Leftrightarrow \frac{\partial x^r}{\partial x^{,n}} \frac{\partial^2 x^{,n}}{\partial x^p \partial x^s} \frac{\partial x^{,q}}{\partial x^r} + \frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0 \quad (3)$$

$$\text{in the first term we get} \quad \frac{\partial x^{,q}}{\partial x^r} \frac{\partial x^r}{\partial x^{,n}} = \frac{\partial x^{,q}}{\partial x^{,n}} = \delta_n^q \quad (4)$$

(3) becomes

$$\frac{\partial^2 x^{,n}}{\partial x^p \partial x^s} \delta_n^q + \frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0 \quad (5)$$

$$\Leftrightarrow \frac{\partial^2 x^{,q}}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0 \quad (6)$$



## 1.6 p9-exercise

If  $A_s^r$  are the elements of a determinant A, and  $B_s^r$  the elements of a determinant B, show that the element of the product determinant is  $A_n^r B_s^n$ . Hence show that the product of the two jacobians

$$J = \left| \frac{\partial x^r}{\partial x^s} \right|, \quad J' = \left| \frac{\partial x'^r}{\partial x^s} \right|$$

is unity.

Remark: Some nitpick about the formulation:  $A_s^r$  are not the elements of a determinant A, but elements of the matrix A which gives  $\det\{A\}$  provided that A is square (which is not explicitly mentioned.). The same remark for B and  $A_n^r B_s^n$ .

Be  $A_k^i$  the elements of matrix A and  $B_j^k$  the elements of matrix B and  $C = A.B$  the resulting matrix of the multiplication of A and B, then

$$C_j^i = A_k^i B_j^k$$

are the elements of matrix C. Now, put  $A_k^i = \frac{\partial x^i}{\partial x'^k}$  and  $B_j^k = \frac{\partial x'^k}{\partial x^j}$  then,

$$C_j^i = A_k^i B_j^k \tag{1}$$

$$= \frac{\partial x^i}{\partial x'^k} \frac{\partial x'^k}{\partial x^j} \tag{2}$$

$$= \delta_k^i \tag{3}$$

So  $C = JJ'$  becomes the unity matrix.



## 1.7 p11-exercise

Show that a finite contravariant vector determines the ratios of the components of an infinitesimal displacement. (Consider the transformation of the equation  $dx^r = \theta T^r$ , where  $\theta$  is an arbitrary factor which does not change under the transformation. Alternatively, show that the equations  $T^r dx^s - T^s dx^r = 0$  remain true when we transform the coordinates.)

Be  $T^q$  a contravariant vector.

$$T^{,q} = T^r \frac{\partial x^{,q}}{\partial x^r} \quad (\text{by definition}) \quad (1)$$

Be  $\theta$  a small infinitesimal factor invariant for a coordinate transformation, define

$$dx^r = \theta T^r \quad (2)$$

$$(3)$$

then

$$\frac{dx^r}{dx^s} = \frac{\theta T^r}{\theta T^s} \quad (4)$$

$$\Leftrightarrow T^s dx^r - T^r dx^s = 0 \quad (5)$$

Alternatively, multiply (5) with  $\partial_{x^r} x^{,q}$ , then

$$\frac{\partial x^{,q}}{\partial x^r} dx^r T^s - \frac{\partial x^{,q}}{\partial x^r} dx^s T^r = 0 \quad (6)$$

$$\Leftrightarrow \frac{\partial x^{,q}}{\partial x^r} dx^r T^s - dx^s T^{,q} = 0 \quad (\text{use (1) in the second term}) \quad (7)$$

$$\Leftrightarrow dx^{,q} T^s - dx^s T^{,q} = 0 \quad (8)$$

$$(9)$$

Multiply (8) with  $\partial_{x^s} x^{,p}$ , then

$$dx^{,q} T^s \partial_{x^s} x^{,p} - dx^s T^{,q} \partial_{x^s} x^{,p} = 0 \quad (10)$$

$$\Leftrightarrow T^{,p} dx^{,q} - T^{,q} dx^{,p} = 0 \quad (\text{use (1) in the first term}) \quad (11)$$

and thus

$$\frac{dx^{,q}}{dx^{,p}} = \frac{T^{,q}}{T^{,p}}$$



## 1.8 p12-exercise

Write down the equation of transformation, analogous to 1.305, of a contravariant tensor of the third order. Solve the equation so as to express the unprimed components in terms of the primed components.

Be

$$T^{,uvw} = T^{rst} \frac{\partial x^{,u}}{\partial x^r} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (\text{by definition}) \quad (1)$$

a contravariant vector.

Multiply (1) by  $\frac{\partial x^n}{\partial x^{,u}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{rst} \frac{\partial x^{,u}}{\partial x^r} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (2)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{rst} \delta_r^n \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (3)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{nst} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (4)$$

Multiply (4) by  $\frac{\partial x^m}{\partial x^{,v}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nst} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^{,w}}{\partial x^t} \quad (5)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nst} \delta_s^m \frac{\partial x^{,w}}{\partial x^t} \quad (6)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nmt} \frac{\partial x^{,w}}{\partial x^t} \quad (7)$$

Multiply (7) by  $\frac{\partial x^p}{\partial x^{,w}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmt} \frac{\partial x^{,w}}{\partial x^t} \frac{\partial x^p}{\partial x^{,w}} \quad (8)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmt} \delta_t^p \quad (9)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmp} \quad (10)$$

Giving

$$T^{nmp} = T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}}$$



## 1.9 p14-exercise

For a transformation from on set of rectangular Cartesian coordinates to another in Euclidean 3-space, show that the law of transformation of a contravariant vector is precisely the same as that of a covariant vector. Can this statements be extended to cover tensor of higher orders?

We have to prove that, given that,

$$T^{,i} = T^j \frac{\partial x^{,i}}{\partial x^j} \quad T_i = T_j \frac{\partial x^j}{\partial x^{,i}}$$

that also

$$T^{,i} = T^j \frac{\partial x^j}{\partial x^{,i}} \quad T_i = T_j \frac{\partial x^{,i}}{\partial x^j} \quad (1)$$

$$\Leftrightarrow \frac{\partial x^j}{\partial x^{,i}} = \frac{\partial x^{,i}}{\partial x^j} \quad (2)$$

Be

$$\hat{e}^{,i} = g_k^i \hat{e}^k \quad \text{and} \quad \hat{e}^i = h_k^i \hat{e}^{,k} \quad (3)$$

the transformation rules from one set of (rectangular Cartesian) basis vectors to another set of (rectangular Cartesian) basis vectors. Then,

$$\langle \hat{e}^{,i}, \hat{e}^{,j} \rangle = \langle g_k^i \hat{e}^k, g_k^j \hat{e}^k \rangle \quad \text{and} \quad \langle \hat{e}^i, \hat{e}^j \rangle = \langle h_k^i \hat{e}^{,k}, h_k^j \hat{e}^{,k} \rangle \quad (4)$$

$$\Leftrightarrow \delta_j^p = g_k^p g_k^j \quad \text{and} \quad \delta_j^p = h_k^p h_k^j \quad (5)$$

$$(6)$$

Be  $\vec{v}$  a random vector in the Euclidean space,

$$\vec{v} = x^j \hat{e}^j = x^{,j} \hat{e}^{,j} \quad (7)$$

then

$$(3) \Rightarrow x^j \hat{e}^j = x^j h_k^j \hat{e}^{,k} \quad \text{and} \quad x^{,j} \hat{e}^{,j} = x^{,j} g_k^j \hat{e}^k \quad (8)$$

$$\Rightarrow x^{,j} = x^m h_j^m \quad \text{and} \quad x^m = x^{,j} g_m^j \quad (9)$$

$$\Rightarrow x^{,j} = x^{,i} g_m^i h_j^m \quad \text{and} \quad x^m = x^k h_j^k g_m^j \quad (10)$$

$$\Rightarrow \delta_j^p = g_k^p h_j^k \quad \text{and} \quad \delta_j^p = g_j^k h_k^p \quad (11)$$

$$(5) \Rightarrow g_k^p g_k^j = g_k^p h_j^k \quad \text{and} \quad h_k^p h_k^j = g_j^k h_k^p \quad (12)$$

$$\Rightarrow g_k^j = h_j^k \quad \text{and} \quad h_k^j = g_j^k \quad (13)$$

From (9)

$$x^j = x^m g_j^m \text{ and } x^k = x^n h_k^n \quad (14)$$

$$\Rightarrow \frac{\partial x^k}{\partial x^j} = \frac{\partial x^n}{\partial x^j} h_k^n \text{ and } \frac{\partial x^j}{\partial x^k} = \frac{\partial x^m}{\partial x^k} g_j^m \quad (15)$$

$$\Leftrightarrow \frac{\partial x^k}{\partial x^j} = \delta_j^n h_k^n \text{ and } \frac{\partial x^j}{\partial x^k} = \delta_k^m g_j^m \quad (16)$$

$$\Leftrightarrow \frac{\partial x^k}{\partial x^j} = h_k^j \text{ and } \frac{\partial x^j}{\partial x^k} = g_j^k \quad (17)$$

$$(13) \Rightarrow \frac{\partial x^k}{\partial x^j} = \frac{\partial x^j}{\partial x^k} \quad (18)$$

So (13) matches (2), proving the assertion.

Can this statements be extended to cover tensor of higher orders? Consider

$$T^{i,j,\dots,n} = T^{r,s,\dots,w} \frac{\partial x^i}{\partial x^r} \frac{\partial x^j}{\partial x^s} \dots \frac{\partial x^n}{\partial x^w} \text{ and } T^{r,s,\dots,w} = T^{i,j,\dots,n} \frac{\partial x^r}{\partial x^i} \frac{\partial x^s}{\partial x^j} \dots \frac{\partial x^w}{\partial x^n}$$

Using the same reasoning as in (1) to (2) we need

$$\frac{\partial x^i}{\partial x^r} \frac{\partial x^j}{\partial x^s} \dots \frac{\partial x^n}{\partial x^w} = \frac{\partial x^r}{\partial x^i} \frac{\partial x^s}{\partial x^j} \dots \frac{\partial x^w}{\partial x^n}$$

As the conclusion (18) is independent of the order of the tensor, it is obvious that the above equality yields. Hence, the answer is YES.





## 1.10 p16-exercise

In a space of 4 dimensions, the tensor  $A_{rst}$  is skew-symmetric in the last pair of suffixes. Show that only 24 of the 64 components may be chosen arbitrarily. If the further condition  $A_{rst} + A_{str} + A_{trs} = 0$  is imposed, show that that only 20 components may be chosen arbitrarily.

We have, as  $A$  is skew-symmetric in the last pair of suffixes

$$A_{rst} = -A_{rts} \Rightarrow s = t: A_{rst} = 0$$

So, for each  $r$  (4 possible choices as  $N = 4$ ) we have  $4 \times 4 / 2 - 4 = 6$  degrees of freedom. [we have the term  $4 \times 4 / 2$  as the tensor is (skew-)symmetric, e.g. once we choose element  $a_{12}$ , then  $a_{21}$  is also known. The term  $-4$  takes into account the diagonal element which are 0 and thus cannot be chosen.] So, we have  $4 \times 6 = 24$  degrees of freedom.

What about the supplementary constraint  $A_{rst} + A_{str} + A_{trs} = 0$  :

Consider the two possible excluding cases:

$$\text{i) } r = s \neq t \quad (\Leftrightarrow r = t \neq s)$$

This case gives - without the additional constraint (1) -  $4 \times (4 \times 3 / 2 - 4) = 8$  degrees of freedom. Does the constraint (1) reduces this degree of freedom?

We have,

$$A_{rst} + A_{str} + A_{trs} = 0 \tag{1}$$

$$\Rightarrow \underbrace{A_{rrt} + A_{rtt}}_{= 0 \text{ (non-diagonal terms)}} + \underbrace{A_{trr}}_{= 0 \text{ (diagonal terms)}} = 0 \tag{2}$$

So, no additional constraints are added by (1) to the restriction i) and the DOF remains 8.

$$\text{ii) } t \neq r \neq s \neq t$$

This case means that we have to choose a set of 3 elements out of 4 elements without repetition. This a *variation* of 3 elements out of 4.

$$V_k^n = \frac{n!}{(n-k)!} \text{ giving } V_3^4 = \frac{4!}{(4-3)!} = 24$$

The constraint (1) gives us 24 equations but as  $A_{rst} = -A_{rts}$  only 12 equations have to be considered. So, with the additional constraints the DOF becomes  $24 - 12 = 12$ .

As i) and ii) are independent and excluding events we can add the DOF of both events and we get  $8 + 12 = 20$  DOF.



## 1.11 p16-exercise

If  $A^{rs}$  is skew-symmetric and  $B_{rs}$  is symmetric, prove that  $A^{rs}B_{rs} = 0$ . Hence show that the quadratic form  $a_{ij}x^i x^j$  is unchanged if  $a_{ij}$  is replaced by its symmetric part.

We can split the summation  $A^{rs}B_{rs}$  in three subsummations:

$$A^{rs}B_{rs} = A^{rs}B_{rs}|_{r=s} \tag{1}$$

$$+ A^{rs}B_{rs}|_{r>s} \tag{2}$$

$$+ A^{rs}B_{rs}|_{r<s} \tag{3}$$

We have:

$$(1) = 0 \text{ as } A^{kk} = 0 \text{ (skew-symmetric)}$$

$$(2)+(3) = A^{rs}B_{rs}|_{r>s} + A^{rs}B_{rs}|_{r<s}$$

As  $A^{rs} = -A^{sr}$  and  $B^{rs} = B^{sr}$  we can write (2)+(3) as :

$$A^{rs}B_{rs}|_{r>s} + (-A^{sr})B_{sr}|_{r>s} = 0$$

So,  $A^{rs}B_{rs} = 0$

Consider the quadratic form  $\phi = a_{ij}x^i x^j$

Be  $A_{ij} = (a_{ij})$  and  $B_{ij} = (x^i x^j)$ , then it is obvious that  $B_{ij}$  is symmetric and that  $C_{ij} = -A_{ij}$  is the form where  $-a_{ij}$  is replaced by its symmetric part (skew-symmetric). Hence  $\phi = a_{ij}x^i x^j = a_{ij}b^{ij} = 0$  and so is  $\phi = c_{ij}b^{ij} = 0$



## 1.12 p18-exercise

What are the values (in a space of  $N$  dimensions) of the following contractions formed from the Kronecker delta?

$$\delta_m^m, \delta_n^m \delta_m^n, \delta_n^m \delta_r^n \delta_m^r$$

We can split the summation  $A^{rs} B_{rs}$  in three subsummations:

$$\delta_m^m = N \tag{1}$$

$$\delta_n^m \delta_m^n = \delta_m^m = N \tag{2}$$

$$\delta_n^m \delta_r^n \delta_m^r = \delta_n^m \delta_m^n = \delta_m^m = N \tag{3}$$



### 1.13 p19-exercise

If  $X^r, Y^r$  are arbitrary contravariant vectors and  $a_{rs}X^rY^s$  is an invariant, then  $a_{rs}$  are the components of a covariant tensor of the second order.

We have to prove that

$$a'_{rs} = a_{ij} \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^s} \text{ or } a_{ij} = a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} \quad (1)$$

$a_{rs}X^rY^s$  is an invariant, means

$$a'_{rs}X'^rY'^s = a_{rs}X^rY^s \quad (2)$$

As  $X^r, Y^r$  are arbitrary contravariant vectors, we have

$$X'^r = X^i \frac{\partial x'^r}{\partial x^i} \text{ and } Y'^s = Y^j \frac{\partial x'^s}{\partial x^j} \quad (3)$$

(3) in (2) gives

$$a'_{rs}X^i \frac{\partial x'^r}{\partial x^i} Y^j \frac{\partial x'^s}{\partial x^j} = a_{rs}X^rY^s \quad (4)$$

$$\Leftrightarrow a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} X^i Y^j = a_{ij} X^i Y^j \quad (5)$$

$$\Leftrightarrow (a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} - a_{ij}) X^i Y^j = 0 \quad (6)$$

As  $X^r, Y^r$  are arbitrary contravariant vectors, we conclude that

$$a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} - a_{ij} = 0 \quad (7)$$

$$\Leftrightarrow a_{ij} = a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} \quad (8)$$

(8) = (1): OK



## 1.14 p19-exercise

If  $X_{rs}$  is an arbitrary covariant tensor of the second order, and  $A_r^{mn} X_{mn}$  is a covariant vector, then  $A_r^{mn}$  has the mixed tensor character indicated by the positions of its suffixes

We have to prove that

$$A_r^{vw} = A_k^{mn} \frac{\partial x^k}{\partial x^{,r}} \frac{\partial x^{,v}}{\partial x^m} \frac{\partial x^{,w}}{\partial x^n} \quad (1)$$

We have

$$P_r = A_r^{mn} X_{mn} \quad (2)$$

is a covariant vector

$$\Rightarrow P_r^{,} = A_k^{mn} X_{mn} \frac{\partial x^k}{\partial x^{,r}} \quad (3)$$

but  $X_{mn}$  is a covariant tensor

$$\Rightarrow X_{mn} = X_{ps} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \quad (4)$$

So (4) in (3) gives

$$P_r^{,} = A_k^{mn} X_{ps} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}} \quad (5)$$

$$\Leftrightarrow P_r^{,} = A_k^{mn} \underbrace{\frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}}_{(*)} X_{ps} \quad (6)$$

Putting (\*) as  $A_r^{ps} = A_k^{mn} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}$  we see that (6) has the form (2) and that  $A_r^{ps}$  obeys the rule of a mixed tensor (1).



## 1.15 p21-exercise

If  $A_{rs}$  is a skew-symmetric covariant tensor, prove that  $B_{rst}$  defined as

$$B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$$

is a covariant tensor, and that it is skew-symmetric in all pairs of suffixes.

We have  $A_{rs}$  is a covariant tensor

$$A_{ij} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^i} \frac{\partial x^\beta}{\partial x^j} \quad (1)$$

$$\Rightarrow B_{rst} = \partial_r (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}) + \partial_s (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r}) + \partial_t (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s}) \quad (2)$$

Note that

$$\partial_k (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}) = \partial_k (A_{\alpha\beta}) \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \partial_k (\frac{\partial x^\alpha}{\partial x^s}) \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \partial_k (\frac{\partial x^\beta}{\partial x^t}) \quad (3)$$

$$(4)$$

so,

$$\begin{aligned} B_{rst} = & \partial_r A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \underbrace{A_{\alpha\beta} \partial_r \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}}_{*} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \partial_r \frac{\partial x^\beta}{\partial x^t}}_{**} \\ & + \partial_s A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \underbrace{A_{\alpha\beta} \partial_s \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r}}_{***} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \partial_s \frac{\partial x^\beta}{\partial x^r}}_{*} \\ & + \partial_t A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} + \underbrace{A_{\alpha\beta} \partial_t \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s}}_{**} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \partial_t \frac{\partial x^\beta}{\partial x^s}}_{***} \end{aligned} \quad (5)$$

In (5) consider the two terms with (\*)

$$T = A_{\alpha\beta} \partial_r \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \partial_s \frac{\partial x^\beta}{\partial x^r} \quad (6)$$

$$= A_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x^s \partial x^r} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial^2 x^\beta}{\partial x^r \partial x^s} \quad (7)$$

$$= A_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x^s \partial x^r} \frac{\partial x^\beta}{\partial x^t} + A_{\beta\alpha} \frac{\partial x^\beta}{\partial x^t} \frac{\partial^2 x^\alpha}{\partial x^r \partial x^s} \quad (\text{by renaming dummy variables}) \quad (8)$$

As  $A_{ij} = -A_{ji}$  (skew-symmetric tensor), we get  $T = 0$ . The same yields for the (\*\*) and (\*\*\*) terms. So,  $B_{rst}$  reduces to

$$B_{rst} = \partial_r A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \partial_s A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \partial_t A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \quad (9)$$

$$\Leftrightarrow B_{rst} = \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^r} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^s} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^t} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \quad (10)$$

By adequate renaming of the dummy variable in the 3 terms:

$$\begin{bmatrix} 1^{st} term \\ 2^{nd} term \\ 3^{rd} term \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma \rightarrow \alpha & \alpha \rightarrow \beta & \beta \rightarrow \gamma \\ \beta \rightarrow \alpha & \gamma \rightarrow \beta & \alpha \rightarrow \gamma \\ \alpha \rightarrow \alpha & \beta \rightarrow \beta & \gamma \rightarrow \gamma \end{bmatrix}$$

we get

$$B_{rst} = \left( \frac{\partial A_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial A_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \right) \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \frac{\partial x^\gamma}{\partial x^t} \quad (11)$$

$$\Leftrightarrow B_{rst} = \underbrace{(\partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta})}_{(****)} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \frac{\partial x^\gamma}{\partial x^t} \quad (12)$$

The expression (\*\*\*\*) has exactly the required form  $B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$  and is transformed (12) according the rules of a covariant tensor.

Let's prove now that it is skew-symmetric in all pairs of suffixes. We have to consider the following permutations

$$\begin{bmatrix} rst \\ rts \\ srt \\ str \\ trs \\ tsr \end{bmatrix}$$

E.g.  $srt$

$$B_{rts} = \partial_r A_{ts} + \partial_t A_{sr} + \partial_s A_{rt} \quad (13)$$

$$= -\partial_r A_{st} - \partial_t A_{rs} - \partial_s A_{tr} \quad (14)$$

$$= -B_{rst} \quad (15)$$

The same calculations can be done for the other permutations.



## 1.16 p23-exercise 1.

In a  $V_4$  there are two 2-spaces with equations

$$x^r = f^r(u^1, u^2), \quad x^r = g^r(u^3, u^4)$$

Prove that if these 2-spaces have a curve of intersection, then the determinantal equation

$$\left| \frac{\partial x^r}{\partial u^s} \right| = 0$$

is satisfied along the curve.

Having a curve means that one of the parameters  $u^i$  can be freely chosen while the other 3 are determined by the chosen parameter.

We have,

$$\left| \frac{\partial x^r}{\partial u^s} \right| = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} & \frac{\partial x^1}{\partial u^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^4}{\partial u^1} & \frac{\partial x^4}{\partial u^2} & \frac{\partial x^4}{\partial u^3} & \frac{\partial x^4}{\partial u^4} \end{vmatrix} \quad (1)$$

Suppose we choose  $u^4$  as parameter. This means  $u^i = \phi^i(u^4)$  for  $i=1,2,3$  and thus we can write

$$\frac{\partial x^i}{\partial u^4} = \frac{\partial x^i}{\partial u^j} \frac{d\phi^j}{du^4} + \frac{\partial x^i}{\partial u^4} \quad \text{with } j=1,2,3 \quad i = 1,2,3,4 \quad (2)$$

$$\Rightarrow \frac{\partial x^i}{\partial u^j} \frac{d\phi^j}{du^4} = 0 \quad (3)$$

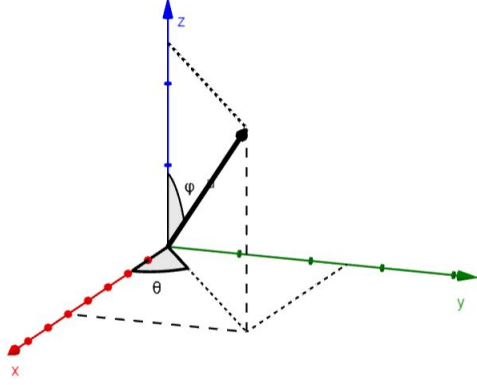
This means that in (1) the three first columns are not linearly independent and thus have  $\left| \frac{\partial x^r}{\partial u^s} \right| = 0$





## 1.17 p23-exercise 2.

In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates  $x, y, z$  and spherical polar coordinates  $r, \theta, \phi$ . Find the Jacobian of the transformation. Where is it zero or infinite?



We use the latitude  $\psi$  instead of the co-latitude  $\phi$ .

$$\begin{cases} x = r \cos(\psi) \cos(\theta) \\ y = r \cos(\psi) \sin(\theta) \\ z = r \sin(\psi) \end{cases}$$

Partial differentiating of  $(x, y, z)$  with respect to  $(r, \psi, \theta)$  gives the Jacobian

$$J = \begin{vmatrix} \cos(\psi) \cos(\theta) & -r \sin(\psi) \cos(\theta) & -r \cos(\psi) \sin(\theta) \\ \cos(\psi) \sin(\theta) & -r \sin(\psi) \sin(\theta) & r \cos(\psi) \cos(\theta) \\ \sin(\psi) & r \cos(\psi) & 0 \end{vmatrix} \quad (1)$$

$$J = \cos(\psi) \cos(\theta) (-r^2) \cos^2(\psi) \cos(\theta) \quad (2)$$

$$+ r \sin(\psi) \cos(\theta) (-r \cos(\psi) \cos(\theta) \sin(\psi)) \quad (3)$$

$$- r \cos(\psi) \sin(\theta) (r \cos^2(\psi) \sin(\theta) + r \sin^2(\psi) \sin(\theta)) \quad (4)$$

$$= -r^2 \cos^3(\psi) \cos^2(\theta) - r^2 \sin^2(\psi) \cos^2(\theta) \cos(\psi) - r^2 \cos(\psi) \sin^2(\theta) \quad (5)$$

Noting that the 2<sup>nd</sup> term in (5) can be written as  $-r^2 \cos^2(\theta) \cos(\psi) + r^2 \cos^2(\theta) \cos^3(\psi)$ , we get

$$J = -r^2 (\cos^3(\psi) \cos^2(\theta) + \cos^2(\theta) \cos(\psi) - \cos^3(\psi) \cos^2(\theta) + \cos(\psi) \sin^2(\theta)) \quad (6)$$

$$= -r^2 \cos(\psi) \quad (7)$$

$J=0$ : for  $r = 0$  or  $\psi = \frac{\pi}{2} |_{r \in (-\infty, +\infty)}$  and  $J \rightarrow \pm\infty$  or  $\mp\infty$  for  $r \rightarrow \pm\infty |_{\psi \neq 0}$ . But what about the case  $r \rightarrow \pm\infty |_{\psi \rightarrow 0}$ ? This case is not determined as long as no path is chosen in the  $(r, \psi)$  configuration space.



### 1.18 p23-exercise 3.

If  $X, Y, Z$  are the components of a contravariant vector for rectangular Cartesian coordinates in Euclidean 3-space, find its components for spherical polar coordinates.

Be  $x^\alpha$  the components of a contravariant vector in spherical polar coordinates and  $x^i$  its components in rectangular Cartesian coordinates. As we have

$$\begin{aligned} x^\rho &= \sqrt{x^j x^j} \\ x^\theta &= \text{atan} \frac{x^2}{x^1} \\ x^\phi &= \text{asin} \frac{x^3}{\sqrt{x^j x^j}} \end{aligned} \quad \text{and} \quad A^\alpha = A^i \frac{\partial x^\alpha}{\partial x^i} \quad (1)$$

$$\Rightarrow [A^\alpha] = \left[ A^i \frac{\partial x^\alpha}{\partial x^i} \right] = \begin{bmatrix} \frac{x^1}{\sqrt{x^j x^j}} & \frac{x^2}{\sqrt{x^j x^j}} & \frac{x^3}{\sqrt{x^j x^j}} \\ -\frac{x^2}{(x^1)^2 + (x^2)^2} & \frac{x^1}{(x^1)^2 + (x^2)^2} & 0 \\ -\frac{x^3 x^1}{(x^j x^j) \sqrt{(x^1)^2 + (x^2)^2}} & -\frac{x^3 x^2}{(x^j x^j) \sqrt{(x^1)^2 + (x^2)^2}} & \frac{\sqrt{(x^1)^2 + (x^2)^2}}{(x^j x^j)} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} \quad (2)$$

◆

### 1.19 p23-exercise 4.

In a space of three dimensions, how many different expressions are represented by the product  $A_{np}^m B_{rs}^{pq} C_{tu}^s$ ? How many terms occur in each such expression, when written out explicitly?

As we have  $V_3$  and considering that in  $A_{np}^m B_{rs}^{pq} C_{tu}^s$  the six indices  $m, n, q, r, t, u$  are not dummy indices, we get  $3^6$  different expressions (first choose  $m$ : you have three choices, then  $n$ : also three choices giving  $3 \times 3$  possibilities, etc for  $q, r, t, u$ ).

For the second question, as in  $A_{np}^m B_{rs}^{pq}$  there is only summation on over index ( $p$ ) we get three terms for this part. As the summation with  $A_{np}^m B_{rs}^{pq}$  and  $C_{tu}^s$  occurs only on one index also ( $s$ ) we get  $3 \times 3$  terms in the expression.



## 1.20 p23-exercise 5.

If  $A$  is an invariant in  $V_n$ , are the second derivatives  $\frac{\partial^2 A}{\partial x^r \partial x^s}$  the components of a tensor?

As  $A$  is invariant (note: different alphabets in the indices indicates different coordinate systems):

$$A(x^\rho) = A(x^i) \quad (1)$$

$$\Rightarrow \frac{\partial A(x^\rho)}{\partial x^i} = \frac{\partial A(x^j)}{\partial x^i} \quad (2)$$

To simplify the notation, we put  $A(x^\rho) = A'$  and  $A(x^j) = A'$  then (2) can be written as

$$\frac{\partial A'}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^i} = \frac{\partial A}{\partial x^i} \quad (3)$$

Conclusion:  $\frac{\partial A}{\partial x^i}$  is a covariant tensor.

Consider now  $\frac{\partial A}{\partial x^i} = \frac{\partial A'}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^i}$ . Then,

$$\frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A'}{\partial x^\rho \partial x^j} \frac{\partial x^\rho}{\partial x^i} + \frac{\partial A'}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \quad (4)$$

$$\Leftrightarrow \frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A'}{\partial x^\rho \partial x^\gamma} \frac{\partial x^\gamma}{\partial x^j} \frac{\partial x^\rho}{\partial x^i} + \frac{\partial A'}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \quad (5)$$

The first term on the right side, behaves as covariant tensor but the presence of the second term makes that generally,  $\frac{\partial^2 A}{\partial x^i \partial x^j}$  has not a tensor character. This is only when  $\frac{\partial A'}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} = 0$ , which means that  $x^\rho, x^i$  are a linear map of each other.



## 1.21 p23-exercise 6.

Suppose that in  $V_2$  the components of a contravariant tensor field  $T^{mn}$  in a coordinate system  $x^r$  are

$$T^{11} = 1 \quad T^{12} = 0$$

$$T^{21} = 1 \quad T^{22} = 0$$

Find the components  $T^{mn}$  in a coordinate system  $x'^r$ , where

$$x'^1 = (x^1)^2 \quad x'^2 = (x^2)^2$$

Write down the values of these components in particular at the point  $x^1 = 1, x^2 = 0$ .

As we have a contravariant tensor field :

$$T'^{mn} = T^{ij} \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^n}{\partial x^j} \quad (1)$$

$$\begin{aligned} x'^1 = (x^1)^2 &\Rightarrow \frac{\partial x'^1}{\partial x^1} = 2x^1 & \frac{\partial x'^1}{\partial x^2} &= 0 \\ x'^2 = (x^2)^2 &\Rightarrow \frac{\partial x'^2}{\partial x^1} &= 0 & \frac{\partial x'^2}{\partial x^2} = 2x^2 \end{aligned} \quad (2)$$

$$(3)$$

$$\Rightarrow T'^{11} = 4(x^1)^2 + 4(x^2)^2 \quad (4)$$

$$\Rightarrow T'^{12} = T'^{21} = 0 \quad (5)$$

$$\Rightarrow T'^{22} = 4(x^1)^2 + 4(x^2)^2 \quad (6)$$

The components in at the point  $x^1 = 1, x^2 = 0$  are

$$T'(1, 0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$



## 1.22 p24-exercise 7.

Given that if  $T_{mnr s}$  is a covariant tensor, and

$$T_{mnr s} + T_{mnsr} = 0$$

in a coordinate system  $x^p$ , establish directly that

$$T_{mnr s} + T_{mnsr} = 0$$

in any other coordinate system  $x, q$ .

Note: in the following, different alphabets in the indices indicates different coordinate systems.  
As we  $T_{mnr s}$  is a covariant tensor :

$$T_{\alpha\beta\gamma\delta} = T_{mnr s} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^r}{\partial x^\gamma} \frac{\partial x^s}{\partial x^\delta} \quad (1)$$

$$\Rightarrow T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnr s} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^r}{\partial x^\gamma} \frac{\partial x^s}{\partial x^\delta} + T_{mnr s} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^r}{\partial x^\delta} \frac{\partial x^s}{\partial x^\gamma} \quad (2)$$

Now, swap the dummy indices r and s in the second term on the right and as  $T_{mnr s} = -T_{mnsr}$ :

$$T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnr s} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^r}{\partial x^\gamma} \frac{\partial x^s}{\partial x^\delta} + T_{mnsr} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^s}{\partial x^\delta} \frac{\partial x^r}{\partial x^\gamma} \quad (3)$$

$$= (T_{mnr s} + T_{mnsr}) \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^s}{\partial x^\delta} \frac{\partial x^r}{\partial x^\gamma} \quad (4)$$

$$= 0 \quad (5)$$



## 1.23 p24-exercise 8.

Prove that if  $A_r$  is a covariant vector, then  $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$  is a skew-symmetric covariant tensor of the second order (use the notation of 1.7).

Be  $B_{rs} = \frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ .

i)  $B_{rs}$  is skew-symmetric: It is obvious that:

$$-B_{rs} = -\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r} = \frac{\partial A_s}{\partial x^r} - \frac{\partial A_r}{\partial x^s} \equiv B_{sr}$$

ii)  $B_{rs}$  is covariant:

*Note: in the following, different alphabets in the indices indicates different coordinate systems.*

Let

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X_\alpha^r X_\beta^s. \quad (1)$$

We know that  $A_i = A_\gamma X_i^\gamma$  as  $A_i$  is covariant. Hence,

$$\partial_j A_i = \partial_j A_\gamma X_i^\gamma + A_\gamma \partial_j X_i^\gamma \quad (2)$$

$$= \partial_\alpha A_\gamma X_j^\alpha X_i^\gamma + A_\gamma \partial_j X_i^\gamma \quad (3)$$

Using (3), we compute the first term in (1)

$$\partial_s A_r X_\alpha^r X_\beta^s = \partial_\rho A_\gamma X_s^\rho X_r^\gamma X_\alpha^r X_\beta^s + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s \quad (4)$$

$$= \partial_\rho A_\gamma X_\beta^\rho X_\alpha^\gamma + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s \quad (5)$$

$$= \partial_\rho A_\gamma \delta_\beta^\rho \delta_\alpha^\gamma + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s \quad (6)$$

$$= \partial_\beta A_\alpha + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s \quad (7)$$

In the same way, we get for the second term in (1)

$$\partial_r A_s X_\alpha^s X_\beta^r = \partial_\alpha A_\beta + A_\gamma \partial_r X_s^\gamma X_\alpha^r X_\beta^s \quad (8)$$

And thus,

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X_\alpha^r X_\beta^s = \partial_\beta A_\alpha + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s - \partial_\alpha A_\beta - A_\gamma \partial_r X_s^\gamma X_\alpha^r X_\beta^s \quad (9)$$

$$\Rightarrow \partial_\beta A_\alpha - \partial_\alpha A_\beta = (\partial_s A_r - \partial_r A_s) X_\alpha^r X_\beta^s \quad (10)$$

So, i) and (10) proves that  $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$  is skew-symmetric tensor of the second order.



## 1.24 p24-exercise 9.

Let  $x^r, \bar{x}^r, y^r, \bar{y}^r$  be four systems of coordinates. Examine the tensor character of  $\frac{\partial x^r}{\partial y^s}$  with respect to the following transformations:

- i) A transformation  $x^r = f^r(\bar{x}^1, \dots, \bar{x}^N)$ , with  $y^r$  unchanged;
- ii) A transformation  $y^r = g^r(\bar{y}^1, \dots, \bar{y}^N)$ , with  $x^r$  unchanged;

*Note: in the following, different alphabets in the indices indicates different coordinate systems.*

i) Let's compute the expression  $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta}$ . Obviously, the right side is an expression of a (possible) mixed tensor of the second order ( $\frac{\partial x^r}{\partial y^s}$ ) under transformation from the (r) coordinate system to the ( $\alpha$ ) coordinate system. Then,

$$A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta} \quad (1)$$

$$= \frac{\partial x^\alpha}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \quad (2)$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \quad (3)$$

If we consider the  $\bar{y}^r$  coordinate system as the  $y^\rho$  coordinate system and as  $\bar{y}^r = y^r$  then  $\frac{\partial y^\rho}{\partial y^s} = \delta_s^\rho$  and we get from (3)

$$A(\alpha, \beta) = \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \quad (4)$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \delta_s^\rho \frac{\partial x^s}{\partial x^\beta} \quad (5)$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial x^\rho}{\partial x^\beta} \quad (6)$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \delta_\beta^\rho \quad (7)$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \quad (8)$$

$$(1) \text{ and } (8) \Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta} \quad (9)$$

So  $A(r, s) = \frac{\partial x^r}{\partial y^s}$  is a mixed tensor of type  $A_s^r$

ii) Let's compute the expression  $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta}$ . Obviously, the right side is an expression of a (possible) mixed tensor of the second order ( $\frac{\partial x^r}{\partial y^s}$ ) under transformation from the (r) coordinate



system to the  $(\alpha)$  coordinate system. Then,

$$A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta} \quad (10)$$

$$= \frac{\partial x^r}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta} \quad (11)$$

$$= \frac{\partial x^r}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (12)$$

$$= \frac{\partial x^r}{\partial y^\rho} \delta_\beta^\rho \frac{\partial y^\alpha}{\partial y^r} \quad (13)$$

$$= \frac{\partial x^r}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (14)$$

$$= \frac{\partial x^r}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (15)$$

If we consider the  $\bar{x}^r$  coordinate system as the  $x^\sigma$  coordinate system and as  $\bar{x}^r = x^r$  then  $\frac{\partial x^\sigma}{\partial x^r} = \delta_r^\sigma$  and we get from (15)

$$A(\alpha, \beta) = \frac{\partial x^r}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (16)$$

$$= \delta_\sigma^r \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (17)$$

$$= \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^\sigma} \quad (18)$$

$$= \frac{\partial x^\sigma}{\partial y^\beta} \delta_\sigma^\alpha \quad (19)$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \quad (20)$$

$$(10) \text{ and } (19) \Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta} \quad (21)$$

So  $A(r, s) = \frac{\partial x^r}{\partial y^s}$  is a mixed tensor of type  $A_s^r$



## 1.25 p24-exercise 10.

If  $x^r, y^r, z^r$  are three systems of coordinates, prove the following rule for the multiplication of Jacobians.

$$\left| \frac{\partial x^m}{\partial y^n} \right| \left| \frac{\partial y^r}{\partial z^s} \right| = \left| \frac{\partial x^t}{\partial z^u} \right|$$

As we have

$$\frac{\partial x^t}{\partial z^u} = \frac{\partial x^t}{\partial y^k} \frac{\partial y^k}{\partial z^u} \quad (1)$$

$$\begin{bmatrix} \frac{\partial x^1}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial z^N} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial y^k} \frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial y^k} \frac{\partial y^k}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^k} \frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial y^k} \frac{\partial y^k}{\partial z^N} \end{bmatrix} \quad (2)$$

$$= \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^1} & \cdots & \frac{\partial x^N}{\partial y^N} \end{bmatrix} \begin{bmatrix} \frac{\partial y^1}{\partial z^1} & \cdots & \frac{\partial y^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial y^N}{\partial z^1} & \cdots & \frac{\partial y^N}{\partial z^N} \end{bmatrix} \quad (3)$$

$$\Rightarrow \left| \frac{\partial x^m}{\partial y^n} \right| \left| \frac{\partial y^r}{\partial z^s} \right| = \left| \frac{\partial x^t}{\partial z^u} \right| \quad (4)$$



## 1.26 p24-exercise 11.

Prove that with respect to transformations

$$x^{,r} = C_{rs}x^s$$

where the coefficients are constants satisfying

$$C_{mr}C_{ms} = \delta_s^r$$

contravariant and covariant vectors have the same formula of transformation

$$A^{,r} = C_{rs}A^s, A_{,r} = C_{rs}A_s$$

i)  $A^{,r} = C_{rs}A^s$

Be  $A^{,r} = A^s \frac{\partial x^{,r}}{\partial x^s}$  and as  $x^{,r} = C_{rs}x^s$  we have  $\frac{\partial x^{,r}}{\partial x^s} = C_{rs}$ . Hence,

$$A^{,r} = C_{rs}A^s$$

.

i)  $A_{,r} = C_{rs}A_s$

Be  $A_{,r} = A_s \frac{\partial x^s}{\partial x^{,r}}$  and as  $x^{,r} = C_{rs}x^s$  we have

$$\frac{\partial x^{,r}}{\partial x^{,t}} = C_{rs} \frac{\partial x^s}{\partial x^{,t}} \quad (1)$$

$$\Rightarrow \delta_t^r = C_{rs} \frac{\partial x^s}{\partial x^{,t}} \quad (2)$$

Now, multiply (2) by  $C_{rq}$ . We get,

$$\delta_t^r C_{rq} = C_{rq} C_{rs} \frac{\partial x^s}{\partial x^{,t}} \quad (3)$$

$$C_{tq} = C_{rq} C_{rs} \frac{\partial x^s}{\partial x^{,t}} \quad (4)$$

$$\text{as } C_{mr}C_{ms} = \delta_s^r \Rightarrow C_{tq} = \delta_s^q \frac{\partial x^s}{\partial x^{,t}} \quad (5)$$

$$\Rightarrow C_{tq} = \frac{\partial x^q}{\partial x^{,t}} \text{ or } C_{rs} = \frac{\partial x^s}{\partial x^{,r}} \quad (6)$$

$$\text{as } A_{,r} = A_s \frac{\partial x^s}{\partial x^{,r}} \Rightarrow A_{,r} = C_{rs} \frac{\partial x^s}{\partial x^{,r}} \quad (7)$$



## 1.27 p25-exercise 12.

Prove that

$$\frac{\partial \ln \left| \frac{\partial x^m}{\partial y^n} \right|}{\partial x^r} = \frac{\partial^2 y^m}{\partial x^r \partial x^n} \frac{\partial x^n}{\partial y^m}$$

Lemma:

Be  $A$  a square matrix  $N \times N$ ; Be  $f$  a  $C^1$  function  $f : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$ .

Define  $A'$  as  $(A'_{ij}) = \frac{df}{dA_{ij}}$ . Then,

$$(\ln |A|)' = (A^{-1})^T \text{ with } f = |A|$$

Proof:

By definition of the determinant, we have

$$|A| = A_{iK} C_K^i \quad (\text{no summation on } K!) \quad (1)$$

with  $(C_K^i) = (-1)^{i+K} M_K^i$  being the cofactor of element  $A_{iK}$  and  $M_K^i$  the minor  $(N-1) \times (N-1)$  matrix associated with the cofactor  $A_{iK}$ . Be  $C = (C_{ij})$  the  $N \times N$  matrix formed with all possible cofactor elements  $C_j^i$  ( $i, j = 1 \dots, N$ ).

We have

$$A^{-1} = \frac{C^T}{|A|} \quad (2)$$

$$\Rightarrow (A^{-1})^T = \frac{C}{|A|} \quad (3)$$

$$\text{differentiating (1)} \Rightarrow \frac{\partial |A|}{\partial A_{mn}} = \frac{\partial A_{iK}}{\partial A_{mn}} C_K^i + A_{iK} \frac{\partial C_K^i}{\partial A_{mn}} \quad (4)$$

$$\text{we have for } i = m \quad \begin{aligned} \frac{\partial A_{iK}}{\partial A_{mn}} &= 1 & K &= n \\ \frac{\partial A_{iK}}{\partial A_{mn}} &= 0 & K &\neq n \end{aligned} \quad (5)$$

Also,  $\forall K : \frac{\partial C_K^i}{\partial A_{in}} = 0$  as by definition of the cofactor matrix,  $A_{ij}$  is not contained in  $C_{ij}$ .

Hence, (4) becomes

$$\frac{\partial |A|}{\partial A_{ij}} = C_j^i \quad (6)$$

$$\text{But, } \frac{\partial \ln |A|}{\partial A_{ij}} = \frac{\frac{\partial |A|}{\partial A_{ij}}}{|A|} \quad (7)$$

$$(6) \text{ and } (7) \text{ gives } \frac{\partial \ln |A|}{\partial A_{ij}} = \frac{C_j^i}{|A|} \quad (8)$$

$$(3) \text{ and } (8) \text{ gives } \frac{\partial \ln |A|}{\partial A_{ij}} = \frac{(A_{ij}^{-1})^T |A|}{|A|} = (A_{ij}^{-1})^T \quad (9)$$

$$\Rightarrow (\ln |A|)' = (A^{-1})^T \quad (10)$$

Now the main proof:

Let,

$$A \equiv [a_{mn}] = \left[ \frac{\partial y^m}{\partial x^n} \right] \quad (11)$$

$$\Rightarrow \frac{\partial \ln |A|}{\partial x^r} = \frac{\partial \ln |A|}{\partial a_{mn}} \frac{\partial a_{mn}}{\partial x^r} \quad (12)$$

$$\text{from (10) we get } \frac{\partial \ln |A|}{\partial a_{mn}} = (A^{-1})_{mn}^T \quad (13)$$

$$\text{But } A \text{ is a Jacobian, so } (A^{-1})_{mn} = \frac{\partial x^m}{\partial y^n} \quad (14)$$

$$\text{and thus } (A^{-1})_{mn}^T = \frac{\partial x^n}{\partial y^m} \quad (15)$$

$$(13) \text{ can be written as } \frac{\partial \ln |A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m} \frac{\partial a_{mn}}{\partial x^r} \quad (16)$$

$$\Rightarrow \frac{\partial \ln |A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m} \frac{\partial^2 y^m}{\partial x^r \partial x^n} \quad (17)$$

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## 1.28 p25-exercise 13.

Consider the quantities  $\frac{dx^r}{dt}$  for a particle moving in the plane. If  $x^r$  are the rectangular Cartesian coordinates, are these quantities the components of a contravariant or covariant vector with respect to rotation of the axes? Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?

Note: we suppose that by a rotation of the axes, the problem means a fixed rotation and not a rotation varying in time.

i) Be  $v^r = \frac{dx^r}{dt}$  and consider  $v^\alpha$  the same object but in another the coordinate system. A rotation of the axes implies the linear form

$$x^\alpha = R^\alpha_k x^k \quad \text{with } R^\alpha_k \neq R^\alpha_k(x^k) \quad (1)$$

$$\Rightarrow \frac{\partial x^\alpha}{\partial x^r} = R^\alpha_k \delta_r^k \quad (2)$$

$$\Rightarrow R^\alpha_r = \frac{\partial x^\alpha}{\partial x^r} \quad (3)$$

Consider  $v^\alpha = \frac{dx^\alpha}{dt}$

$$v^\alpha = \frac{dx^\alpha}{dt} \quad (4)$$

$$(1) \Rightarrow v^\alpha = R^\alpha_k \frac{dx^k}{dt} \quad (5)$$

$$\Rightarrow v^\alpha = R^\alpha_k v^k \quad (6)$$

$$(3) \Rightarrow v^\alpha = v^k \frac{\partial x^\alpha}{\partial x^r} \quad (7)$$

Conclusion:  $v^k$  is a contravariant vector.

ii) Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?

We know that

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^r} dx^r \quad (8)$$

$$\Rightarrow \frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r} \frac{dx^r}{dt} \quad (9)$$

$$\Rightarrow v^\alpha = v^r \frac{\partial x^\alpha}{\partial x^r} \quad (10)$$

So  $v^r$  is a contravariant vector in general. Note that this proof is more straightforward than the prove in i).



## 1.29 p25-exercise 14.

Consider the question raised in No. 13 for the acceleration  $\frac{d^2 x^r}{dt^2}$ .

From exercise 13. we know that

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r} \frac{dx^r}{dt} \quad (1)$$

$$\Rightarrow \frac{d^2 x^\alpha}{dt^2} = \frac{d^2 x^r}{dt^2} \frac{\partial x^\alpha}{\partial x^r} + \frac{d \frac{\partial x^\alpha}{\partial x^r}}{dt} \frac{dx^r}{dt} \quad (2)$$

$$= \frac{d^2 x^r}{dt^2} \frac{\partial x^\alpha}{\partial x^r} + \frac{\partial^2 x^\alpha}{\partial x^r \partial x^m} \frac{dx^m}{dt} \frac{dx^r}{dt} \quad (3)$$

The second term on the right does not vanish in general, hence  $\frac{d^2 x^r}{dt^2}$  has not a tensor character.



### 1.30 p25-exercise 15.

It is well known that the equation of an ellipse may be written

$$ax^2 + 2hxy + by^2 = 1$$

What is the tensor character of  $a, h, b$  with respect to transformation to any Cartesian coordinates (rectangular or oblique) in the plane?

Consider the transformation from a  $(w, z)$  coordinate system to a  $(x, y)$  coordinate system. For the considered type of transformation we have

$$\begin{aligned} x &= \alpha w + \beta z \\ y &= \gamma w + \delta z \end{aligned} \quad (1)$$

$$\text{consider} \quad \begin{aligned} ax^2 + 2hxy + by^2 &= 1 \\ pw^2 + 2qwz + rz^2 &= 1 \end{aligned} \quad (2)$$

the two representations of the same ellipse in the respective coordinate systems. Plugging (1) in (2):

$$a\alpha^2 w^2 + 2a\alpha\beta wz + \alpha\beta^2 z^2 + 2h\alpha\gamma w^2 + \beta\delta z^2 + 2h(\alpha\delta + \gamma\beta)wz + b\gamma^2 w^2 + 2b\gamma\delta wz + \delta^2 z^2 = 1 \quad (3)$$

$$(4)$$

Rearranging and equating the terms in  $w^2, wz, z^2$  in (2) gives

$$p = a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \quad (5)$$

$$q = a\alpha\beta + h(\alpha\delta + \gamma\beta) + \gamma\delta \quad (6)$$

$$r = a\beta^2 + 2h\beta\delta + b\delta^2 \quad (7)$$

Consider the following objects

$$(A_{ij}) = \begin{bmatrix} a & h \\ h & b \end{bmatrix} \quad (8)$$

$$(A_{ij})' = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix} \quad (9)$$

$$\text{we calculate} \quad A'_{ij} = A_{km} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} \quad (10)$$

with  $(x^1, x^2) = (w, z)$  and  $(x'^1, x'^2) = (x, y)$ . We have,

$$\frac{\partial x^1}{\partial x'^1} = \alpha, \frac{\partial x^1}{\partial x'^2} = \beta, \frac{\partial x^2}{\partial x'^1} = \gamma, \frac{\partial x^2}{\partial x'^2} = \delta \quad (11)$$

$$\begin{aligned} (10) \text{ and } (11) \quad \Rightarrow \quad \begin{aligned} a'_{11} &= a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \\ a'_{22} &= a\beta^2 + 2h\beta\delta + b\delta^2 \\ a'_{12} &= a'_{21} = a\alpha\beta + h(\alpha\delta + \gamma\beta) + b\gamma\delta \end{aligned} \end{aligned} \quad (12)$$



Combining (5), (6), (7) and (12) we get

$$p = a'_{11}, r = a'_{22}, q = a'_{12} = a'_{21}$$

and so (9) becomes

$$(A_{ij})' = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$$

Considering (10) we see that  $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$  transforms to  $\begin{bmatrix} p & q \\ q & r \end{bmatrix}$  according to the rules of a covariant tensor of order two.



### 1.31 p25-exercise 16.

Matter is distributed in a plane and  $A, B, H$  are the moments and product of inertia with respect to rectangular axes  $Oxy$  in a plane. Examine the tensor character of the set of quantities  $A, B, H$  under rotation of the axes. What notation would you suggest for moments and product of inertia in order to exhibit the tensor character? What simple invariant can be formed from  $A, B, H$ ?

Consider the transformation from a  $(x^1, x^2)$  coordinate system to a  $(y^1, y^2)$  coordinate system. For the considered type of transformation we have

$$\begin{aligned} y^1 &= \alpha x^1 + \beta x^2 \\ y^2 &= \gamma x^1 + \delta x^2 \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Be } A &= \sum_{\rho} m_{\rho} (x^{2,\rho})^2 & A' &= \sum_{\rho} m_{\rho} (y^{2,\rho})^2 \\ B &= \sum_{\rho} m_{\rho} (x^{1,\rho})^2 & B' &= \sum_{\rho} m_{\rho} (y^{1,\rho})^2 \\ H &= \sum_{\rho} m_{\rho} x^{1,\rho} x^{2,\rho} & H' &= \sum_{\rho} m_{\rho} y^{1,\rho} y^{2,\rho} \end{aligned} \quad (2)$$

the moments and product of inertia,  $\rho$  being the index of summation over all the points with mass  $m_{\rho}$ .

For the sake of notational simplicity we consider only one point of mass as the linearity of  $A, B, H$  related to  $\rho$  ensures the validity of the next steps for all points in the plane.

From (1) and (2) we have:

$$\frac{A'}{m_{\rho}} = \gamma^2 (x^1)^2 + 2\gamma\delta x^1 x^2 + \delta^2 (x^2)^2 \quad (3)$$

$$\frac{B'}{m_{\rho}} = \alpha^2 (x^1)^2 + 2\alpha\beta x^1 x^2 + \beta^2 (x^2)^2 \quad (4)$$

$$\frac{H'}{m_{\rho}} = \alpha\gamma (x^1)^2 + (\gamma\beta + \alpha\delta) x^1 x^2 + \beta\delta (x^2)^2 \quad (5)$$

$$\begin{aligned} \text{Note that } \frac{\partial y^1}{\partial x^1} &= \alpha & \frac{\partial y^1}{\partial x^2} &= \beta \\ \frac{\partial y^2}{\partial x^1} &= \gamma & \frac{\partial y^2}{\partial x^2} &= \delta \end{aligned} \quad (6)$$

$$(6) \text{ in } (4): \frac{B'}{m_{\rho}} = (x^1)^2 \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + 2(x^1)(x^2) \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \quad (7)$$

$$= (x^1)^2 \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + (x^1)(x^2) \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + (x^2)(x^1) \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \quad (8)$$

Repeating the same calculations for  $\frac{A'}{m_{\rho}}$  and  $\frac{H'}{m_{\rho}}$  gives:

$$\begin{aligned} \frac{A'}{m_{\rho}} &= (x^1)^2 \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^1} + (x^1)(x^2) \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^2} + (x^2)(x^1) \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^1} + (x^2)^2 \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^2} \\ \frac{B'}{m_{\rho}} &= (x^1)^2 \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + (x^1)(x^2) \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + (x^2)(x^1) \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \\ \frac{H'}{m_{\rho}} &= (x^1)^2 \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^1} + (x^1)(x^2) \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^1} + (x^2)(x^1) \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^2} \end{aligned} \quad (9)$$

Now, define

$$(K_{ij}) = \begin{bmatrix} (x^1)^2 & (x^1)(x^2) \\ (x^2)(x^1) & (x^2)^2 \end{bmatrix} \quad (K_{ij})' = \begin{bmatrix} (y^1)^2 & (y^1)(y^2) \\ (y^2)(y^1) & (y^2)^2 \end{bmatrix} \quad (10)$$

Then (9) can be written as

$$\begin{aligned} \frac{A'}{m_\rho} &= (y^1)^2 = K^{11} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^1} + K^{12} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^2} + K^{21} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^1} + K^{22} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^2} \\ \frac{B'}{m_\rho} &= (y^2)^2 = K^{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + K^{12} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + K^{21} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^1} + K^{22} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \\ \frac{H'}{m_\rho} &= (y^1)(y^2) = K^{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^1} + K^{12} \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^1} + K^{21} \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^2} + K^{22} \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^2} \end{aligned} \quad (11)$$

Hence,

$$\begin{aligned} K^{,11} &= K^{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + K^{12} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + K^{21} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^1} + K^{22} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \\ K^{,22} &= K^{11} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^1} + K^{12} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^2} + K^{21} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^1} + K^{22} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^2} \\ K^{,12} &= K^{,21} = K^{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^1} + K^{12} \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^1} + K^{21} \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^2} + K^{22} \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^2} \end{aligned} \quad (12)$$

So the object  $(K_{ij}) \equiv K^{ij}$  transforms according (12) like a contravariant second order tensor.

Now, consider  $|K^{ij}|$ , obviously  $|K^{ij}| = (x^1)^2(x^2)^2 - (x^1)(x^2)(x^2)(x^1) = 0$ , but so is also  $|K^{,ij}|$ .

$\Rightarrow |K^{ij}|$  is an invariant under the considered transformation



### 1.32 p25-exercise 17.

$S_{nmr}$  is a skew-symmetric tensor in the first two indices.  $-f_{mnr} + f_{nmr} = S_{mnr}$ .

From exercise 13. we know that

$$-f_{mnr} + f_{nmr} = S_{mnr} \quad (1)$$

Swap the indices three times

$$\text{i) } n \leftrightarrow r : (1) \Rightarrow -f_{mrn} + f_{rmn} = S_{mrn} \quad (2)$$

$$\Leftrightarrow \underbrace{f_{mnr}}_* + \underbrace{f_{rmn}}_{**} = -S_{rmn} \quad (3)$$

$$\text{ii) } m \leftrightarrow r : (1) \Rightarrow -f_{rnm} + f_{nrn} = S_{rnm} \quad (4)$$

$$\Leftrightarrow \underbrace{f_{rmn}}_{**} + \underbrace{f_{nrn}}_{***} = -S_{nrn} \quad (5)$$

$$\text{iii) } m \leftrightarrow n : (1) \Rightarrow -f_{nmr} + f_{mnr} = S_{nmr} \quad (6)$$

$$\Leftrightarrow \underbrace{f_{nrm}}_{***} + \underbrace{f_{mnr}}_* = -S_{mnr} \quad (7)$$

$$(3) - (5) + (7): \quad 2 \underbrace{f_{mnr}}_* = -S_{rmn} + S_{nrn} - S_{mnr} \quad (8)$$

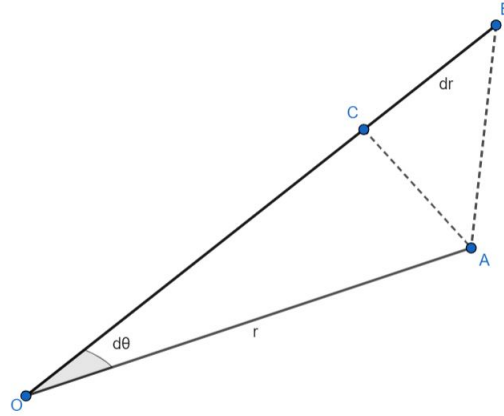
$$\Leftrightarrow f_{mnr} = \frac{-S_{rmn} + S_{nrn} - S_{mnr}}{2} \quad (9)$$



# Basic Operations in Riemannian Space

## 2.1 p27-exercise

Take polar coordinates  $r, \theta$  in a plane. Draw the infinitesimal triangle with vertices  $(r, \theta)$ ,  $(r + dr, \theta)$ ,  $(r, \theta + d\theta)$ . Evaluate the square on the hypotenuse of this infinitesimal triangle, and so obtain the metric tensor for the plan for the coordinates  $(r, \theta)$ .



$$ds^2 = |AB|^2 \quad (1)$$

$$= dr^2 + |CA|^2 \quad (2)$$

$$|CA| = r \sin(d\theta) \approx r d\theta \quad (3)$$

$$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 \quad (4)$$

$$\Rightarrow (a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad (5)$$

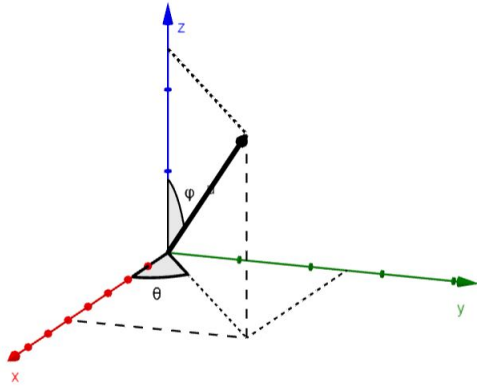


## 2.2 p27-exercise

Show that if  $x^1 = r, x^2 = \theta, x^3 = \phi$ , in the usual notation of spherical polar coordinates, then

$$a_{11} = 1, a_{22} = r^2, a_{33} = r^2 \sin^2 \theta$$

and the other components vanish.



We use the latitude  $\psi$  instead of the co-latitude  $\phi$ .

$$\begin{cases} x = r \cos(\psi) \cos(\theta) \\ y = r \cos(\psi) \sin(\theta) \\ z = r \sin(\psi) \end{cases}$$

$$ds^2 = dx^2 + dy^2 + dz^2 \quad \text{with} \quad (1)$$

$$\left. \begin{aligned} dx &= dr \cos(\psi) \cos(\theta) - r \sin(\psi) d\psi \cos(\theta) - r \cos(\psi) \sin(\theta) d\theta \\ dy &= dr \cos(\psi) \sin(\theta) - r \sin(\psi) d\psi \sin(\theta) + r \cos(\psi) \cos(\theta) d\theta \\ dz &= dr \sin(\psi) + r \cos(\psi) d\psi \end{aligned} \right\} \quad (2)$$

$$\begin{aligned}
& \left. \begin{aligned}
dx^2 &= \cos^2(\psi) \cos^2(\theta) dr^2 \\
&- r^2 \sin^2(\psi) \cos^2(\theta) d\psi^2 \\
&- r^2 \cos^2(\psi) \sin^2(\theta) d\theta^2 \\
&- \cos(\psi) \cos(\theta) r \sin(\psi) \cos(\theta) dr d\psi \\
&- \cos(\psi) \cos(\theta) r \cos(\psi) \sin(\theta) dr d\theta \\
&+ r \sin(\psi) \cos(\theta) r \cos(\psi) \sin(\theta) d\psi d\theta
\end{aligned} \right\} \\
& \left. \begin{aligned}
dy^2 &= \cos^2(\psi) \sin^2(\theta) dr^2 \\
&+ r^2 \sin^2(\psi) \sin^2(\theta) d\psi^2 \\
&+ r^2 \cos^2(\psi) \cos^2(\theta) d\theta^2 \\
&- \cos(\psi) \sin(\theta) r \sin(\psi) \sin(\theta) dr d\psi \\
&- \cos(\psi) \sin(\theta) r \cos(\psi) \cos(\theta) dr d\theta \\
&- r \sin(\psi) \sin(\theta) r \cos(\psi) \cos(\theta) d\psi d\theta
\end{aligned} \right\} \quad (3) \\
dz^2 &= \sin^2(\psi) dr^2 + r^2 \cos^2(\psi) d\psi^2 + r \sin(\psi) \cos(\psi) dr d\psi
\end{aligned}$$

Rearrange terms:

$$\begin{aligned}
& \left. \begin{aligned}
dx^2 &= \cos^2(\psi) \cos^2(\theta) dr^2 \\
&+ r^2 \sin^2(\psi) \cos^2(\theta) d\psi^2 \\
&+ r^2 \cos^2(\psi) \sin^2(\theta) d\theta^2 \\
&- r \cos(\psi) \sin(\psi) \cos^2(\theta) dr d\psi \\
&- r \cos^2(\psi) \cos(\theta) \sin(\theta) dr d\theta \\
&+ r^2 \sin(\psi) \cos(\theta) \cos(\psi) \sin(\theta) d\psi d\theta
\end{aligned} \right\} \\
& \left. \begin{aligned}
dy^2 &= \cos^2(\psi) \sin^2(\theta) dr^2 \\
&+ r^2 \sin^2(\psi) \sin^2(\theta) d\psi^2 \\
&+ r^2 \cos^2(\psi) \cos^2(\theta) d\theta^2 \\
&- r \cos(\psi) \sin(\psi) \sin^2(\theta) dr d\psi \\
&- r \cos^2(\psi) \sin(\theta) \cos(\theta) dr d\theta \\
&- r^2 \sin(\psi) \sin(\theta) \cos(\psi) \cos(\theta) d\psi d\theta
\end{aligned} \right\} \quad (4) \\
dz^2 &= \sin^2(\psi) dr^2 + r^2 \cos^2(\psi) d\psi^2 + r \sin(\psi) \cos(\psi) dr d\psi
\end{aligned}$$

Grouping similar infinitesimal components and using basic trigonometric identities gives:

$$ds^2 = dr^2 + r^2 d\psi^2 + r^2 \cos^2(\psi) d\theta^2 \quad (5)$$

$$\text{replace } \psi \text{ with } \frac{\pi}{2} - \phi \Rightarrow ds^2 = dr^2 + r^2 d\phi^2 + r^2 \sin^2(\phi) d\theta^2 \quad (6)$$



As we use infinitesimal displacements we can assume that

. Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \quad (8)$$

We have the following relationships

$$\begin{aligned}
 |ES| &= r \sin(\phi) d\theta \\
 |GE| &= |GS| = r \sin(\phi) \\
 |GK| &= |RJ| = (r + dr) \sin(\phi + d\phi) \\
 &= (r + dr)(\cos(\phi) \sin(d\phi) + \sin(\phi) \cos(d\phi)) \\
 &= (r + dr)(\cos(\phi) d\phi + \sin(\phi)) \\
 &= r \cos(\phi) d\phi + r \sin(\phi) + \sin(\phi) dr \\
 |OR| &= (r + dr) \cos(\phi + d\phi) \\
 &= (r + dr)(\cos(\phi) \cos(d\phi) - \sin(\phi) \sin(d\phi)) \\
 &= (r + dr)(\cos(\phi) - \sin(\phi) d\phi) \\
 &= r \cos(\phi) - r \sin(\phi) d\phi + \cos(\phi) dr \\
 |OG| &= r \cos(\phi) \\
 |JK| &= |OR| - |OG| = \cos(\phi) dr - r \sin(\phi) d\phi \\
 |SK| &= |GK| - |GS| = r \cos(\phi) d\phi + \sin(\phi) dr
 \end{aligned} \tag{9}$$

$$\begin{aligned}
 |ES|^2 &= r^2 \sin^2(\phi) d\theta^2 \\
 |SK|^2 &= r^2 \cos^2(\phi) d\phi^2 + \sin^2(\phi) dr^2 + 2r \cos(\phi) \sin(\phi) dr d\phi \\
 |JK|^2 &= \cos^2(\phi) dr^2 + r^2 \sin^2(\phi) d\phi^2 - 2r \cos(\phi) \sin(\phi) dr d\phi
 \end{aligned} \tag{10}$$

Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \tag{11}$$

$$= \begin{cases} r^2 \sin^2(\phi) d\theta^2 \\ +r^2 \cos^2(\phi) d\phi^2 + \sin^2(\phi) dr^2 + 2r \cos(\phi) \sin(\phi) dr d\phi \\ +r^2 \sin^2(\phi) d\phi^2 + \cos^2(\phi) dr^2 - 2r \cos(\phi) \sin(\phi) dr d\phi \end{cases} \tag{12}$$

$$\Rightarrow ds^2 = dr^2 + r^2 d\phi^2 + r^2 \sin^2(\phi) d\theta^2 \tag{13}$$



2.3 p25-clarification 1.34

Be  $E = mc^4$  and given an Euclidean space, prove that ....blalbla ...blalbla ....blalbla ....blalbla

bma bla blaa

$$\sum_{i=100}^{\infty} a_i x^i \tag{1}$$

$$\sum_{i=100}^{\infty} a_i x^i \tag{2}$$



2.4 p25-exercise

Be  $E = mc^2$  and given an Euclidean space, prove that ....blalbla ...blalbla ....blalbla ....blalbla

bma bla blaa

$$\sum_{i=0}^{\infty} a_i x^i \tag{1}$$

$$\sum_{i=100}^{\infty} a_i x^i \tag{2}$$

$$\sum_{i=100}^{\infty} a_i x^i \tag{3}$$



2.5 p25-clarification 1.34

Be  $E = mc^4$  and given an Euclidean space, prove that ....blalbla ...blalbla ....blalbla ....blalbla

bma bla blaa

$$\sum_{i=100}^{\infty} a_i x^i \tag{1}$$

$$\sum_{i=100}^{\infty} a_i x^i \tag{2}$$



2.6 p25-clarification 1.34

Be  $E = mc^4$  and given an Euclidean space, prove that ....blalbla ...blalbla ....blalbla ....blalbla

bma bla blaa

$$\sum_{i=100}^{\infty} a_i x^i \tag{1}$$

$$\sum_{i=100}^{\infty} a_i x^i \tag{2}$$

