Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

Part I

Chapters I to IV

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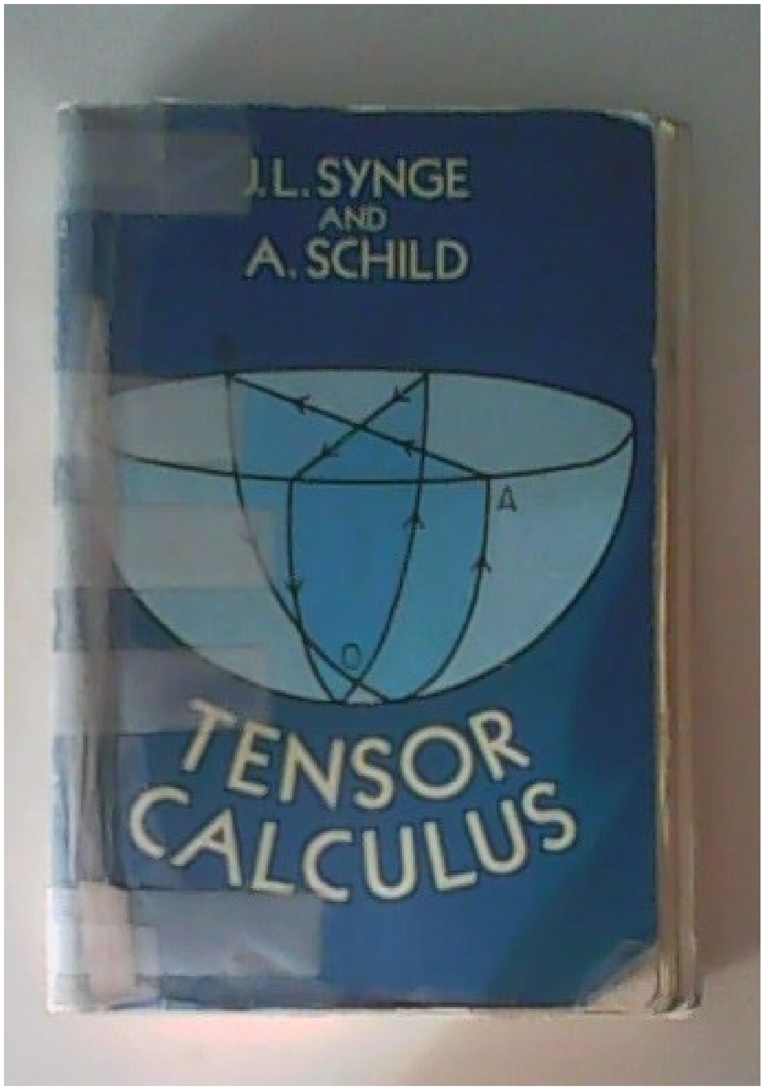

Figure 1: My copy, falling apart...

Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github. An overview of the material covered in the book can be found in the separate document "Synge overview.pdf".

References: I used following books

D. Beklémichev, *Cours de Géométrie analytique et d'Algèbre Linéaire*, 1988, Editions Mir-Moscou
L. Pontriaguine, *Equations différentielles ordinaires*, 1969, Editions Mir-Moscou
S.M. Selby (Ed.), *CRC Standard Mathematical Tables* 22^{nd} *edition*, 1974, CRC Press, Cleveland

Some notation conventions

†	means that the exercise has only been solved partially or contains i.m.o. a doubtful step
††	means that the exercise has not been solved as it should.
♦	end of an exercise or proof.
♢	end of Lemma or sub-task of an exercise.
	As a rule, I followed the notation used in the book, except some which where easier to type in Latex.
∂_r	$\equiv \frac{\partial}{\partial x^r}$
∂^2_{rs}	$\equiv \frac{\partial^2}{\partial x^r \partial x^s}$
$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix}$	Christoffel symbol of the second kind until chapter VII. In chapter VIII Γ^r_{mn} will represent a (general) linear connection

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Spaces and Tensors

1.1 p5-exercise

The parametric equations of a hypersurface in V_n are

$$
\begin{aligned}
x^1 &= a\cos\left(u^1\right)\\
x^2 &= a\sin\left(u^1\right)\cos\left(u^2\right)\\
x^3 &= a\sin\left(u^1\right)\sin\left(u^2\right)\cos\left(u^3\right)\\
&\vdots\\
x^{N-1} &= a\sin\left(u^1\right)\sin\left(u^2\right)\sin\left(u^3\right)\ldots\sin\left(u^{N-2}\right)\cos\left(u^{N-1}\right)\\
x^N &= a\sin\left(u^1\right)\sin\left(u^2\right)\sin\left(u^3\right)\ldots\sin\left(u^{N-2}\right)\sin\left(u^{N-1}\right)
\end{aligned}
$$

where a is a constant. Find the single equation of the hyperspace in the form 1.103.

We have:

$$
\begin{aligned}
(x^N)^2 + (x^{N-1})^2 &= a^2\prod_{i=1}^{N-2}\sin^2(u^i)(\cos^2(u^{N-1}) + \sin^2(u^{N-1}))\\
&= a^2\prod_{i=1}^{N-2}\sin^2(u^i)\\
&= a^2\prod_{i=1}^{N-3}\sin^2(u^i)\sin^2(u^{N-2})\\
&= a^2\prod_{i=1}^{N-3}\sin^2(u^i)(1-\cos^2(u^{N-2})\\
&= a^2\prod_{i=1}^{N-3}\sin^2(u^i) - a^2\prod_{i=1}^{N-3}\sin^2(u^i)\cos^2(u^{N-2})\\
&= a^2\prod_{i=1}^{N-3}\sin^2(u^i) - (x^{N-2})^2
\end{aligned}
$$

giving

$$(x^N)^2 + (x^{N-1})^2 + (x^{N-2})^2 = a^2\prod_{i=1}^{N-3}\sin^2(u^i)$$

In general, by recursion

$$\sum_{i=0}^{k}(x^{N-i})^2 = a^2\prod_{i=1}^{N-k-1}\sin^2(u^i) \quad (k \le N-2)$$

be k = N - 2 (N - k - 1 = 1) and in the left term put j = N - i (j goes from 2 to N), we get

$$\begin{aligned}\sum_{j=2}^{N}(x^j)^2 &= a^2\prod_{i=1}^{1}\sin^2(u^i)\\ &= a^2(1-\cos^2(u^1))\\ &= a^2-(x^1)^2\end{aligned}$$

and thus the equation of the hyperspace is given by

$$\sum_{j=1}^{N}(x^j)^2 - a^2 = 0$$

Determine whether the points $(\frac{1}{2}a, 0, 0, ...0)$, $(0, 0, ..., 0, 2a)$ lie on the same or opposite sides of the hyperspace.

For $(\frac{1}{2}a, 0, 0, ...0)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = -\frac{3a^2}{4} < 0$ and for $(0, 0, ..., 0, 2a)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = \frac{3a^2}{4} > 0$.
So the points lie on opposite sides of the hyperplane.

♦

1.2 p6-exercise

Let U_2 and W_2 be subspaces of V_N. Show that if N = 3 they will in general intersect in a curve; if N = 4 they will in general intersect in a finite number of points; and if $N > 4$ they will not in general intersect at all.

We have (see (**1.102**) page 5): $x^r = f^r(u^1, u^2, ..., u^M) \quad (r = 1, 2, ..., N)$

Case N=3:

For U_2 we have:

$$x^r = \phi^r(u^1, u^2) \quad (r = 1, 2, 3)$$

For W_2 we have:

$$x^r = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

The intersect of the two hyperplanes is given by the N equations:

$$\phi^r(u^1, u^2) = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

So we have 3 equations in 4 unknown u^1, u^2, v^1, v^2 and can choose (fix) one e.g. u^1 and solve the set of equations for u^2, v^1, v^2 giving

$$x^r = \theta^r(u^1) \quad (r = 1, 2, 3)$$

This is an equation of a curve in space (1 parameter equation)

Case N=4:
Using the same reasoning as with N=3, we get 4 equations for 4 unknown u^1, u^2, v^1, v^2.
Provided that the set of equation does not degenerate, these 4 equations will determine u^1, u^2, v^1, v^2 without any degree of freedom. So we get points as solutions. This solution does not to be unique, e.g. if the $\phi^r(u^1, u^2)$ are quadratic form, then the following solutions are possible.

$$(u^1, u^2, v^1, v^2)$$

$$(-u^1, u^2, v^1, v^2)$$

$$(u^1, -u^2, v^1, v^2)$$

$$(-u^1, -u^2, v^1, v^2)$$

Case N=5: There are more equations than variables. If the equations are not linear dependent, no solutions will be found.

♦

1.3 p8-exercise

Show that $(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = 3a_{rst}x^r x^s x^t$

$(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = a_{rst}x^r x^s x^t + a_{rts}x^r x^s x^t + a_{srt}x^r x^s x^t$ so by just renaming the dummy indices e.g. for the second term $r \mapsto s$, $s \mapsto t$ and $t \mapsto r$, we get the desired result.

♦

1.4 p8-exercise

If $\phi = a_{rs}x^r x^s$, show that

$$\frac{\partial \phi}{\partial x^r} = (a_{rs} + a_{sr})x^s \quad , \quad \frac{\partial^2 \phi}{\partial x^r \partial x^s} = a_{rs} + a_{sr}$$

Simplify these expressions in the case where $a^{rs} = a^{sr}$

We have

$$\begin{aligned}
\frac{\partial \phi}{\partial x^t} &= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs}\frac{\partial x^r}{\partial x^t} x^s + a_{rs}x^r \frac{\partial x^s}{\partial x^t} && (1)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs}\delta^r_t x^s + a_{rs}x^r \delta^s_t && (2)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts}x^s + a_{rt}x^r && (3)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts}x^s + a_{st}x^s \quad \text{(rename dummy variable in third term)} && (4)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + (a_{ts} + a_{st})x^s && (5)
\end{aligned}$$

Replace x^t by x^r, we get

$$\frac{\partial \phi}{\partial x^r} = \frac{\partial a_{rs}}{\partial x^r} x^r x^s + (a_{rs} + a_{sr})x^s \tag{6}$$

So the asked expression is only true if a_{rs} is not a function of the x^s. Assuming that a_{rs} is not a function of the x^s, take the partial derivative of (6) with respect to x^t, we get

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x^r \partial x^t} &= (a_{rs} + a_{sr})\frac{\partial x^s}{\partial x^t} && (7)\\
&= (a_{rs} + a_{sr})\delta^s_t && (8)\\
&= (a_{rt} + a_{tr}) && (9)
\end{aligned}$$

Replace x^t by x^s, and we get the proposed expression.

♦

1.5 p8-clarification on expression 1.210

$$\frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0$$

From (**1.209**):

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} + \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} = 0 \tag{1}$$

multiply (1) with $\frac{\partial x'^q}{\partial x^r}$

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} + \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{2}$$

$$\Leftrightarrow \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} \frac{\partial x'^q}{\partial x^r} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{3}$$

in the first term we get $$\frac{\partial x'^q}{\partial x^r} \frac{\partial x^r}{\partial x'^n} = \frac{\partial x'^q}{\partial x'^n} = \delta^q_n \tag{4}$$

(3) becomes

$$\frac{\partial^2 x'^n}{\partial x^p \partial x^s} \delta^q_n + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{5}$$

$$\Leftrightarrow \frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{6}$$

♦

1.6 p9-exercise

If A^r_s are the elements of a determinant A, and B^r_s the elements of a determinant B, show that the element of the product determinant is $A^r_n B^n_s$. Hence show that the product of the two jacobians

$$J = \left|\frac{\partial x^r}{\partial x'^s}\right|, \quad J' = \left|\frac{\partial x'^r}{\partial x^s}\right|$$

is unity.

Remark: Some nitpick about the formulation: A^r_s are not the elements of a determinant A, but elements of the matrix A which gives $\det\{A\}$ provided that A is square (which is not explicitly mentioned.). The same remark for B and $A^r_n B^n_s$.
Be A^i_k the elements of matrix A and B^k_j the elements of matrix B and C = A.B the resulting matrix of the multiplication of A and B, then

$$C^i_j = A^i_k B^k_j$$

are the elements of matrix C. Now, put $A^i_k = \frac{\partial x^i}{\partial x'^k}$ and $B^k_j = \frac{\partial x'^k}{\partial x^j}$ then,

$$\begin{aligned} C^i_j &= A^i_k B^k_j \\ &= \frac{\partial x^i}{\partial x'^k}\frac{\partial x'^k}{\partial x^j} \\ &= \delta^i_k \end{aligned}$$

So $C = JJ'$ becomes the unity matrix.

♦

1.7 p11-exercise

Show that a finite contravariant vector determines the ratios of the components of an infinitesimal displacement. (Consider the transformation of the equation $dx^r = \theta T^r$, where θ is an arbitrary factor which does not change under the transformation. Alternatively, show that the equations $T^r dx^s - T^s x^r = 0$ remain true when we transform the coordinates.)

Be T^q a contravariant vector.

$$T^{'q} = T^r \frac{\partial x^{'q}}{\partial x^r} \quad \text{(by definition)} \tag{1}$$

Be θ a small infinitesimal factor invariant for a coordinate transformation, define

$$dx^r = \theta T^r \tag{2}$$

$$\tag{3}$$

then

$$\frac{dx^r}{dx^s} = \frac{\theta T^r}{\theta T^s} \tag{4}$$

$$\Leftrightarrow T^s dx^r - T^r dx^s = 0 \tag{5}$$

Alternatively, multiply (5) with $\partial_{x^r} x^{'q}$, then

$$\frac{\partial x^{'q}}{\partial x^r} dx^r T^s - \frac{\partial x^{'q}}{\partial x^r} dx^s T^r = 0 \tag{6}$$

$$\Leftrightarrow \frac{\partial x^{'q}}{\partial x^r} dx^r T^s - dx^s T^{'q} = 0 \quad \text{(use (1) in the second term)} \tag{7}$$

$$\Leftrightarrow dx^{'q} T^s - dx^s T^{'q} = 0 \tag{8}$$

$$\tag{9}$$

Multiply (8) with $\partial_{x^s} x^{'p}$, then

$$dx^{'q} T^s \partial_{x^s} x^{'p} - dx^s T^{'q} \partial_{x^s} x^{'p} = 0 \tag{10}$$

$$\Leftrightarrow \quad T^{'p} dx^q - T^{'q} dx^{'p} = 0 \quad \text{(use (1) in the first term)} \tag{11}$$

and thus

$$\frac{dx^{'q}}{dx^{'p}} = \frac{T^{'q}}{T^{'p}}$$

♦

1.8 p12-exercise

Write down the equation of transformation, analogous to (**1.305**), of a contravariant tensor of the third order. Solve the equation so as to express the unprimed components in terms of the primed components.

Be

$$T'^{uvw} = T^{rst} \frac{\partial x'^u}{\partial x^r} \frac{\partial x'^v}{\partial x^s} \frac{\partial x'^w}{\partial x^t} \quad \text{(by definition)} \tag{1}$$

a contravariant vector.
Multiply (1) by $\frac{\partial x^n}{\partial x'^u}$

$$T'^{uvw} \frac{\partial x^n}{\partial x'^u} = T^{rst} \frac{\partial x'^u}{\partial x^r} \frac{\partial x^n}{\partial x'^u} \frac{\partial x'^v}{\partial x^s} \frac{\partial x'^w}{\partial x^t} \tag{2}$$

$$\Leftrightarrow T'^{uvw} \frac{\partial x^n}{\partial x'^u} = T^{rst} \delta^n_r \frac{\partial x'^v}{\partial x^s} \frac{\partial x'^w}{\partial x^t} \tag{3}$$

$$\Leftrightarrow T'^{uvw} \frac{\partial x^n}{\partial x'^u} = T^{nst} \frac{\partial x'^v}{\partial x^s} \frac{\partial x'^w}{\partial x^t} \tag{4}$$

Multiply (4) by $\frac{\partial x^m}{\partial x'^v}$

$$T'^{uvw} \frac{\partial x^n}{\partial x'^u} \frac{\partial x^m}{\partial x'^v} = T^{nst} \frac{\partial x'^v}{\partial x^s} \frac{\partial x^m}{\partial x'^v} \frac{\partial x'^w}{\partial x^t} \tag{5}$$

$$\Leftrightarrow \ T'^{uvw} \frac{\partial x^n}{\partial x'^u} \frac{\partial x^m}{\partial x'^v} = T^{nst} \delta^m_s \frac{\partial x'^w}{\partial x^t} \tag{6}$$

$$\Leftrightarrow \ T'^{uvw} \frac{\partial x^n}{\partial x'^u} \frac{\partial x^m}{\partial x'^v} = T^{nmt} \frac{\partial x'^w}{\partial x^t} \tag{7}$$

Multiply (7) by $\frac{\partial x^p}{\partial x'^w}$

$$T'^{uvw} \frac{\partial x^n}{\partial x'^u} \frac{\partial x^m}{\partial x'^v} \frac{\partial x^p}{\partial x'^w} = T^{nmt} \frac{\partial x'^w}{\partial x^t} \frac{\partial x^p}{\partial x'^w} \tag{8}$$

$$\Leftrightarrow \ T'^{uvw} \frac{\partial x^n}{\partial x'^u} \frac{\partial x^m}{\partial x'^v} \frac{\partial x^p}{\partial x'^w} = T^{nmt} \delta^p_t \tag{9}$$

$$\Leftrightarrow \ T'^{uvw} \frac{\partial x^n}{\partial x'^u} \frac{\partial x^m}{\partial x'^v} \frac{\partial x^p}{\partial x'^w} = T^{nmp} \tag{10}$$

Giving

$$T^{nmp} = T'^{uvw} \frac{\partial x^n}{\partial x'^u} \frac{\partial x^m}{\partial x'^v} \frac{\partial x^p}{\partial x'^w}$$

♦

1.9 p14-exercise

For a transformation from one set of rectangular Cartesian coordinates to another in Euclidean 3-space, show that the law of transformation of a contravariant vector is precisely the same as that of a covariant vector. Can this statements be extended to cover tensor of higher orders?

We have to prove, given that,

$$T^{'i} = T^j \frac{\partial x^{'i}}{\partial x^j} \quad T^{'}_i = T_j \frac{\partial x^j}{\partial x^{'i}}$$

that also

$$T^{'i} = T^j \frac{\partial x^j}{\partial x^{'i}} \quad T^{'}_i = T_j \frac{\partial x^{'i}}{\partial x^j} \tag{1}$$

$$\Leftrightarrow \frac{\partial x^j}{\partial x^{'i}} = \frac{\partial x^{'i}}{\partial x^j} \tag{2}$$

Be

$$\hat{e'}_i = g_i^k \hat{e_k} \quad \text{and } \hat{e}_i = h_i^k \hat{e'}_k \tag{3}$$

the transformation rules from one set of (rectangular Cartesian) basis vectors to another set of (rectangular Cartesian) basis vectors. Then,

$$\langle \hat{e'}_i, \hat{e'}_j \rangle = \langle g_i^k \hat{e}_k, g_j^k \hat{e}_k \rangle \text{ and } \langle \hat{e}_i, \hat{e}_j \rangle = \langle h_i^k \hat{e}^{'}_k, h_j^k \hat{e}^{'}_k \rangle \tag{4}$$

$$\Leftrightarrow \delta_p^j = g_p^k g_j^k \text{ and } \delta_p^j = h_p^k h_j^k \tag{5}$$

Be $\vec{v}$ a random vector in the Euclidean space,

$$\vec{v} = x^j \hat{e}_j = x^{'j} \hat{e}^{'}_j$$

then

$$\begin{array}{llccc}
(3) & \Rightarrow & x^j \hat{e}_j = x^j h_j^k \hat{e}^{'}_k & \text{and} & x^{'j} \hat{e}^{'}_{'j} = x^{'j} g_j^k \hat{e}_k \\
& \Rightarrow & x^{'j} = x^m h_j^m & \text{and} & x^m = x^{'j} g_m^j \\
& \Rightarrow & x^{'j} = x^{'i} g_i^m h_m^j & \text{and} & x^m = x^k h_k^j g_j m \\
& \Rightarrow & \delta_p^j = g_p^k h_k^j & \text{and} & \delta_p^j = g_k^j h_p^k \\
(5) & \Rightarrow & g_p^k g_j^k = g_p^k h_k^j & \text{and} & h_p^k h_j^k = g_k^j h_p^k \\
& \Rightarrow & g_j^k = h_k^j & \text{and} & h_j^k = g_k^j
\end{array} \tag{6}$$

Conclusion

$$g_j^k = h_k^j \tag{6}$$

From (6) we have further,

$$
\begin{array}{cccc}
 & x^j = {x'}^m g^j_m & \text{and} & {x'}^k = x^n h^k_n \\
\Rightarrow & \frac{\partial {x'}^k}{\partial x^j} = \frac{\partial x^n}{\partial x^j} h^k_n & \text{and} & \frac{\partial x^j}{\partial {x'}^k} = \frac{\partial {x'}^m}{\partial {x'}^k} g^j_m \\
\Leftrightarrow & \frac{\partial {x'}^k}{\partial x^j} = \delta^n_j h^k_n & \text{and} & \frac{\partial x^j}{\partial {x'}^k} = \delta^m_k g^j_m \\
\Leftrightarrow & \frac{\partial {x'}^k}{\partial x^j} = h^k_j & \text{and} & \frac{\partial x^j}{\partial {x'}^k} = g^j_k \\
(6): & \frac{\partial {x'}^k}{\partial x^j} & = & \frac{\partial x^j}{\partial {x'}^k}
\end{array}
\tag{7}
$$

So (7) matches (2), proving the assertion.

Can this statements be extended to cover tensor of higher orders? Consider

$$
{T'}^{i,j,\ldots,n} = T^{r,s,\ldots w} \frac{\partial {x'}^i}{\partial x^r} \frac{\partial {x'}^j}{\partial x^s} \cdots \frac{\partial {x'}^n}{\partial x^w} \text{ and } T^{r,s,\ldots w} = {T'}^{i,j,\ldots,n} \frac{\partial x^r}{\partial {x'}^i} \frac{\partial x^s}{\partial {x'}^j} \cdots \frac{\partial x^w}{\partial {x'}^n}
$$

Using the same reasoning as in (1) to (2) we need

$$
\frac{\partial {x'}^i}{\partial x^r} \frac{\partial {x'}^j}{\partial x^s} \cdots \frac{\partial {x'}^n}{\partial x^w} = \frac{\partial x^r}{\partial {x'}^i} \frac{\partial x^s}{\partial {x'}^j} \cdots \frac{\partial x^w}{\partial {x'}^n}
$$

As the conclusion (7) is independent of the order of the tensor, it is obvious that the above equality yields. Hence, the answer is YES.

♦

1.10 p16-exercise

In a space of 4 dimensions, the tensor A_{rst} is skew-symmetric in the last pair of suffixes. Show that only 24 of the 64 components may be chosen arbitrarily. If the further condition $A_{rst} + A_{str} + A_{trs} = 0$ is imposed, show that that only 20 components may be chosen arbitrarily.

We have, as A is skew-symmetric in the last pair of suffixes

$$A_{rst} = -A_{rts} \Rightarrow s = t\colon\ A_{rst} = 0$$

So, for each r (4 possible choices as N = 4) we have 4x4/2 - 4 = 6 degrees of freedom (we have the term 4x4/2 as the tensor is (skew-)symmetric, e.g. once we choose element a_{12}, then a_{21} is also known. The term -4 takes into account the diagonal element which are 0 and thus cannot be chosen.)
So, we have 4x6 = 24 degrees of freedom.
What about the supplementary constraint $A_{rst} + A_{str} + A_{trs} = 0 \quad (1)$:
Consider the two possible excluding cases:

i) $r = s \neq t$ **(** $\Leftrightarrow r = t \neq s$**)**
This case gives - without the additional constraint (1) - 4x(4x3/2-4) = 8 out of the 24 degrees of freedom. Does the constraint (1) reduces this degree of freedom?
We have,

$$A_{rst} + A_{str} + A_{trs} = 0 \tag{1}$$

$$\Rightarrow \quad \underbrace{A_{rrt} + A_{rtr}}_{= 0 \text{ (non-diagonal terms)}} \quad + \quad \underbrace{A_{trr}}_{= 0 \text{ (diagonal terms)}} \quad = 0 \tag{2}$$

So, no additional constraints are added by (1) to the restriction definend by i) and the DOF remains 8.

ii) $t \neq r \neq s \neq t$
This case means that we have to choose a set of 3 elements out of 4 elements without repetition. This a *variation* of 3 elements out of 4.

$$V_k^n = \frac{n!}{(n-k)!} \quad \text{giving} \quad V_3^4 = \frac{4!}{(4-3)!} = 24$$

The constraint (1) gives us 24 equations but as $A_{rst} = -A_{rts}$ only 12 equations have to be considered. So, with the constraint (1) the DOF becomes 24-12 = 12.
As **i)** and **ii)** are independent and excluding events we can add the DOF of both events and we get 8+12 = 20 DOF.

♦

1.11 p16-exercise

If A^{rs} is skew-symmetric and B_{rs} is symmetric, prove that $A^{rs}B_{rs} = 0$. Hence show that the quadratic form $a_{ij}x^i x^j$ is unchanged if a_{ij} is replaced by its symmetric part.

We can split the summation $A^{rs}B_{rs}$ in three subsummations:

$$\begin{aligned} A^{rs}B_{rs} =& A^{rs}B_{rs}|_{r=s} & (1)\\ &+A^{rs}B_{rs}|_{r>s} & (2)\\ &+A^{rs}B_{rs}|_{r<s} & (3)\end{aligned}$$

We have:
(1) $= 0$ as $A^{kk} = 0$ (skew-symmetric)
(2)+(3) $= A^{rs}B_{rs}|_{r>s} \ + A^{rs}B_{rs}|_{r<s}$
As $A^{rs} = -A^{sr}$ and $B^{rs} = B^{sr}$ we can write (2)+(3) as :

$$A^{rs}B_{rs}|_{r>s} \ + (-A^{sr})B_{sr}|_{r>s} = 0$$

So, $A^{rs}B_{rs} = 0$

Consider the quadratic form $\phi = a_{ij}x^i x^j$
Be $A_{ij} = (a_{ij})$ and $B_{ij} = (x^i x^j)$, then it is obvious that B_{ij} is symmetric and that $C_{ij} = -A_{ij}$ is the form where $-a_{ij}$ is replaced by its symmetric part (skew-symmetric). Hence $\phi = a_{ij}x^i x^j = a_{ij}b^{ij} = 0$ and so is $\phi = c_{ij}b^{ij} = 0$

♦

1.12 p18-exercise

What are the values (in a space of N dimensions) of the folllowing contractions formed from the Kronecker delta?

$$\delta^m_m, \delta^m_n \delta^n_m, \delta^m_n \delta^n_r \delta^r_m$$

$$\delta^m_m = N \tag{1}$$

$$\delta^m_n \delta^n_m = \delta^m_m = N \tag{2}$$

$$\delta^m_n \delta^n_r \delta^r_m = \delta^m_n \delta^n_m = \delta^m_m = N \tag{3}$$

♦

1.13 p19-exercise

If X^r, Y^r are arbitrary contravariant vectors and $a_{rs}X^rY^s$ is an invariant, then a_{rs} are the components of a covariant tensor of the second order.

We have to prove that

$$a'_{rs} = a_{ij}\frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^s} \text{ or } a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{1}$$

$a_{rs}X^rY^s$ is an invariant, means

$$a'_{rs}X'^rY'^s = a_{rs}X^rY^s \tag{2}$$

As X^r, Y^r are arbitrary contravariant vectors, we have

$$X'^r = X^i\frac{\partial x'^r}{\partial x^i} \text{ and } Y'^s = Y^j\frac{\partial x'^s}{\partial x^j} \tag{3}$$

(3) in (2) gives

$$\begin{aligned}
& a'_{rs}X^i\frac{\partial x'^r}{\partial x^i}Y^j\frac{\partial x'^s}{\partial x^j} = a_{rs}X^rY^s \\
\Leftrightarrow\quad & a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j}X^iY^j = a_{ij}X^iY^j \\
\Leftrightarrow\quad & (a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij})X^iY^j = 0
\end{aligned}$$

As X^r, Y^r are arbitrary contravariant vectors, we conclude that

$$a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij} = 0$$

giving

$$a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{4}$$

Which is the required transformation rule (1).

♦

1.14 p19-exercise

If X_{rs} is an arbitrary covariant tensor of the second order, and $A_r^{mn}X_{mn}$ is a covariant vector, then A_r^{mn} has the mixed tensor character indicated by the positions of its suffixes

We have to prove that

$$A_r^{'vw} = A_k^{mn}\frac{\partial x^k}{\partial x'^r}\frac{\partial x'^v}{\partial x^m}\frac{\partial x'^w}{\partial x^n} \tag{1}$$

We have

$$P_r = A_r^{mn}X_{mn} \tag{2}$$

is a covariant vector

$$\Rightarrow P_r^{'} = A_k^{mn}X_{mn}\frac{\partial x^k}{\partial x'^r} \tag{3}$$

but X_{mn} is a covariant tensor

$$\Rightarrow X_{mn} = X_{ps}^{'}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n} \tag{4}$$

So (4) in (3) gives

$$P_r^{'} = A_k^{mn}X_{ps}^{'}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n}\frac{\partial x^k}{\partial x'^r} \tag{5}$$

$$\Leftrightarrow P_r^{'} = \underbrace{A_k^{mn}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n}\frac{\partial x^k}{\partial x'^r}}_{(*)}X_{ps}^{'} \tag{6}$$

Putting (*) as $A_r^{'ps} = A_k^{mn}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n}\frac{\partial x^k}{\partial x'^r}$ we see that (6) has the form (2) and that $A_r^{'ps}$ obeys the rule of a mixed tensor (1).

♦

1.15 p21-exercise

If A_{rs} is a skew-symmetric covariant tensor, prove that B_{rst} defined as

$$B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$$

is a covariant tensor, and that it is skew-symmetric in all pairs of suffixes.

We have A_{rs} is a covariant tensor

$$A_{ij} = A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^i}\frac{\partial x^\beta}{\partial x^j} \tag{1}$$

$$\Rightarrow B_{rst} = \partial_r(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}) + \partial_s(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r}) + \partial_t(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}) \tag{2}$$

Note that

$$\partial_k(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}) = \partial_k(A_{\alpha\beta})\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \ A_{\alpha\beta}\partial_k(\frac{\partial x^\alpha}{\partial x^s})\frac{\partial x^\beta}{\partial x^t} + \ A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\partial_k(\frac{\partial x^\beta}{\partial x^t}) \tag{3}$$

so,

$$\begin{aligned}
B_{rst} &= \partial_r A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^s}\tfrac{\partial x^\beta}{\partial x^t} + \underbrace{A_{\alpha\beta}\partial_r\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}}_{*} + \underbrace{A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\partial_r\frac{\partial x^\beta}{\partial x^t}}_{**} \\
&+\partial_s A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^t}\tfrac{\partial x^\beta}{\partial x^r} + \underbrace{A_{\alpha\beta}\partial_s\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r}}_{***} + \underbrace{A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\partial_s\frac{\partial x^\beta}{\partial x^r}}_{*} \\
&+\partial_t A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^r}\tfrac{\partial x^\beta}{\partial x^s} + A_{\alpha\beta}\underbrace{\partial_t\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}}_{**} + \underbrace{A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\partial_t\frac{\partial x^\beta}{\partial x^s}}_{***}
\end{aligned} \tag{4}$$

In (4) consider the two terms with (*)

$$\begin{aligned}
T &= A_{\alpha\beta}\partial_r\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\partial_s\frac{\partial x^\beta}{\partial x^r} \\
&= A_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x^s\partial x^r}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial^2 x^\beta}{\partial x^r\partial x^s} \\
&= A_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x^s\partial x^r}\frac{\partial x^\beta}{\partial x^t} + A_{\beta\alpha}\frac{\partial x^\beta}{\partial x^t}\frac{\partial^2 x^\alpha}{\partial x^r\partial x^s} \text{ (by renaming dummy variables)}
\end{aligned}$$

As $A_{ij} = -A_{ji}$ (skew-symmetric tensor), we get $T = 0$. The same yields for the (**) and (***) terms. So, B_{rst} reduces to

$$\begin{aligned}
B_{rst} &= \partial_r A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \partial_s A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r} + \partial_t A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s} \\
\Leftrightarrow\ B_{rst} &= \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^r}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^s}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^t}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}
\end{aligned}$$

By adequate renaming of the dummy variable in the 3 terms:

$$\begin{bmatrix} 1^{st} term \\ 2^{nd} term \\ 3^{rd} term \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma \to \alpha & \alpha \to \beta & \beta \to \gamma \\ \beta \to \alpha & \gamma \to \beta & \alpha \to \gamma \\ \alpha \to \alpha & \beta \to \beta & \gamma \to \gamma \end{bmatrix}$$

we get

$$\begin{aligned} B_{rst} &= (\frac{\partial A_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial A_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma})\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}\frac{\partial x^\gamma}{\partial x^t} \\ \Leftrightarrow B_{rst} &= (\underbrace{\partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta}}_{(****)})\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}\frac{\partial x^\gamma}{\partial x^t} \end{aligned}$$

The expression (****) has exactly the required form $B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$ and is transformed according the rules of a covariant tensor.
Let's prove now that it is skew-symmetric in all pairs of suffixes. We have to consider the following permutations

$$\begin{bmatrix} rst \\ rts \\ srt \\ str \\ trs \\ tsr \end{bmatrix}$$

E.g. *srt*

$$\begin{aligned} B_{rts} &= \partial_r A_{ts} + \partial_t A_{sr} + \partial_s A_{rt} \\ &= -\partial_r A_{st} - \partial_t A_{rs} - \partial_s A_{tr} \\ &= -B_{rst} \end{aligned}$$

The same calculations can be done for the other permutations.

♦

1.16 p23-exercise 1.

In a V_4 there are two 2-spaces with equations

$$x^r = f^r(u^1, u^2),\; x^r = g^r(u^3, u^4)$$

Prove that if these 2-spaces have a curve of intersection, then the determinal equation

$$\left|\frac{\partial x^r}{\partial u^s}\right| = 0$$

is satisfied along the curve.

Having a curve means that one of the parameters u^i can be freely chosen while the other 3 are determined by the chosen parameter.
We have,

$$\left|\frac{\partial x^r}{\partial u^s}\right| = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} & \frac{\partial x^1}{\partial u^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^4}{\partial u^1} & \frac{\partial x^4}{\partial u^2} & \frac{\partial x^4}{\partial u^3} & \frac{\partial x^4}{\partial u^4} \end{vmatrix} \tag{1}$$

Suppose we choose u^4 as parameter. This means $u^i = \phi^i(u^4)$ for i=1,2,3 and thus we can write

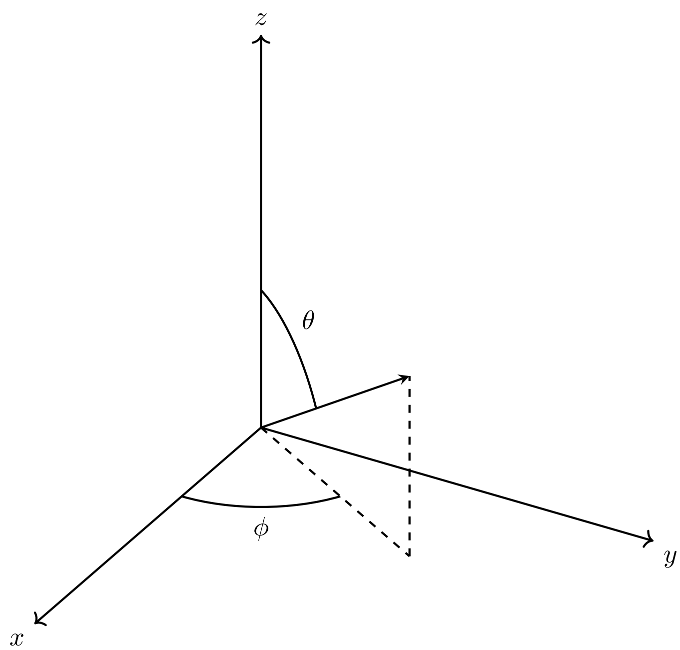
$$\begin{aligned} \frac{\partial x^i}{\partial u^4} &= \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} + \frac{\partial x^i}{\partial u^4} \quad \text{with j=1,2,3} \quad \text{i = 1,2,3,4} \\ &\Rightarrow \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} = 0 \end{aligned}$$

This means that in (1) the three first columns a not linearly independent and thus have $\left|\frac{\partial x^r}{\partial u^s}\right| = 0$.

♦

1.17 p23-exercise 2.

In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates x, y, z and spherical polar coordinates r,θ,ϕ.
Find the Jacobian of the transformation. Where is it zero or infinite?



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

Figure 1.1: Spherical coordinate system

Partial differentiating of (x,y,z) with respect to (r,θ,ϕ) gives the Jacobian

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \tag{1}$$

$$= r^2\sin\theta(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi) \tag{2}$$

$$= r^2\sin\theta \tag{3}$$

J=0: for $r = 0$ or $\theta = 0_{|r\epsilon(-\infty,+\infty)}$ and $J \to \pm\infty$ or $\mp\infty$ for $r \to \pm\infty|_{\theta\neq 0}$. But what about the case $r \to \pm\infty|_{\theta\to 0}$? This case is not determined as long as no path is chosen in the (r, θ) configuration space.

♦

1.18 p23-exercise 3.

If X, Y, Z are the components of a contravariant vector for rectangular Cartesian coordinates in Euclidean 3-space, find it's components for spherical polar coordinates.

Be x^α the components of a contravariant vector in spherical polar coordinates and x^i it's components in rectangular Cartesian coordinates. As we have

$$\begin{array}{r} x^\rho = \sqrt{x^j x^j} \\ x^\theta = \operatorname{atan} \frac{x^2}{x^1} \\ x^\phi = \operatorname{asin} \frac{x^3}{\sqrt{x^j x^j}} \end{array} \qquad \text{and} \qquad A^\alpha = A^i \frac{\partial x^\alpha}{\partial x^i} \tag{1}$$

$$\Rightarrow [A^\alpha] = \left[A^i \frac{\partial x^\alpha}{\partial x^i} \right] = \begin{bmatrix} \frac{x^1}{\sqrt{x^j x^j}} & \frac{x^2}{\sqrt{x^j x^j}} & \frac{x^3}{\sqrt{x^j x^j}} \\ -\frac{x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} & 0 \\ -\frac{x^3 x^1}{(x^j x^j)\sqrt{(x^1)^2+(x^2)^2}} & -\frac{x^3 x^2}{(x^j x^j)\sqrt{(x^1)^2+(x^2)^2}} & \frac{\sqrt{(x^1)^2+(x^2)^2}}{(x^j x^j)} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} \tag{2}$$

♦

1.19 p23-exercise 4.

In a space of three dimensions, how many different expressions are represented by the product $A^m_{np}B^{pq}_{rs}C^s_{tu}$? How many terms occur in each such expression, when written out explicitly?

As we have V_3 and considering that in $A^m_{np}B^{pq}_{rs}C^s_{tu}$ the six indices m, n, q, r, t, u are not dummy indices, we get 3^6 different expressions (first choose m: you have three choices, then n: also three choices giving 3x3 possibilities, etc for q, r, t, u).
For the second question, as in $A^m_{np}B^{pq}_{rs}$ there is only summation over index (p) we get three terms for this part. As the summation with $A^m_{np}B^{pq}_{rs}$ and C^s_{tu} occurs only on one index also (s) we get 3x3 terms in the expression.

♦

1.20 p23-exercise 5.

If A is an invariant in V_n, are the second derivatives $\frac{\partial^2 A}{\partial x^r \partial x^s}$ the components of a tensor?

As A is invariant (note: different alphabets in the indices indicates different coordinate systems):

$$A(x^\rho) = A(x^i) \tag{1}$$

$$\Rightarrow \frac{\partial A(x^\rho)}{\partial x^i} = \frac{\partial A(x^j)}{\partial x^i} \tag{2}$$

To simplify the notation, we put $A(x^\rho) = A^{'}$ and $A(x^j) = A^{'}$ then (2) can be written as

$$\frac{\partial A^{'}}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^i} = \frac{\partial A}{\partial x^i} \tag{3}$$

Conclusion: $\frac{\partial A}{\partial x^i}$ is a covariant tensor.

Consider now $\frac{\partial A}{\partial x^i} = \frac{\partial A^{'}}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^i}$. Then,

$$\frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A^{'}}{\partial x^\rho \partial x^j}\frac{\partial x^\rho}{\partial x^i} + \frac{\partial A^{'}}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \tag{4}$$

$$\Leftrightarrow \frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A^{'}}{\partial x^\rho \partial x^\gamma}\frac{\partial x^\gamma}{\partial x^j}\frac{\partial x^\rho}{\partial x^i} + \frac{\partial A^{'}}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \tag{5}$$

The first term on the right side, behaves as covariant tensor but the presence of the second term makes that generally, $\frac{\partial^2 A}{\partial x^i \partial x^j}$ has not a tensor character. This is only when $\frac{\partial A^{'}}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} = 0$, which means that x^ρ, x^i are a linear map of each other.

♦

1.21 p23-exercise 6.

Suppose that in V_2 the components of a contravariant tensor field T^{mn} in a coordinate system x^r are

$$T^{11} = 1 \quad T^{12} = 0$$

$$T^{21} = 1 \quad T^{22} = 0$$

Find the components $T^{'mn}$ in a coordinate system $x^{'r}$, where

$$x^{'1} = (x^1)^2 \quad x^{'2} = (x^2)^2$$

Write down the values of these components in particular at the point $x^1 = 1,\ x^2 0 =.$

As we have a contravariant tensor field :

$$T^{'mn} = T^{ij} \frac{\partial x^{'m}}{\partial x^i} \frac{\partial x^{'n}}{\partial x^j}$$

$$\begin{array}{lll} x^{'1} = (x^1)^2 \Rightarrow & \frac{\partial x^{'1}}{\partial x^1} = 2x^1 & \frac{\partial x^{'1}}{\partial x^2} = 0 \\ x^{'2} = (x^2)^2 \Rightarrow & \frac{\partial x^{'2}}{\partial x^1} = 0 & \frac{\partial x^{'2}}{\partial x^2} = 2x^2 \end{array}$$

$$\begin{aligned} &\Rightarrow T^{'11} = 4(x^1)^2 + 4(x^2)^2 \\ &\Rightarrow T^{'12} = T^{'21} = 0 \\ &\Rightarrow T^{'22} = 4(x^1)^2 + 4(x^2)^2 \end{aligned}$$

The components in at the point $x^1 = 1, x^2 = 0$ are

$$T^{'}(1,0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

♦

1.22 p24-exercise 7.

Given that if T_{mnrs} is a covariant tensor, and

$$T_{mnrs} + T_{mnsr} = 0$$

in a coordinate system x^p, establish directly that

$$T^{'}_{mnrs} + T^{'}_{mnsr} = 0$$

in any other coordinate system $x^{'q}$.

Note: in the following, different alphabets in the indices indicates different coordinate systems. As T_{mnrs} is a covariant tensor :

$$T_{\alpha\beta\gamma\delta} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} \tag{1}$$

$$\Rightarrow T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} + T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\delta}\frac{\partial x^s}{\partial x^\gamma} \tag{2}$$

Now, swap the dummy indices r and s in the second term on the right and as $T_{mnrs} = -T_{mnsr}$:

$$\begin{aligned} T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} &= T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} + T_{mnsr}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^s}{\partial x^\delta}\frac{\partial x^r}{\partial x^\gamma} && (3)\\ &= (T_{mnrs} + T_{mnsr})\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^s}{\partial x^\delta}\frac{\partial x^r}{\partial x^\gamma} && (4)\\ &= 0 && (5)\end{aligned}$$

♦

1.23 p24-exercise 8.

Prove that if A_r is a covariant vector, then $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ is a skew-symmetric covariant tensor of the second order (use the notation of 1.7).

Be $B_{rs} = \frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$.
i) B_{rs} is skew-symmetric: It is obvious that:

$$-B_{rs} = -\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r} = \frac{\partial A_s}{\partial x^r} - \frac{\partial A_r}{\partial x^s} \equiv B_{sr}$$

ii) B_{rs} is covariant:
Note: in the following, different alphabets in the indices indicates different coordinate systems.
Let

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta. \tag{1}$$

We know that $A_i = A_\gamma X^\gamma_i$ as A_i is covariant. Hence,

$$\partial_j A_i = \partial_j A_\gamma X^\gamma_i + A_\gamma \partial_j X^\gamma_i \tag{2}$$
$$= \partial_\alpha A_\gamma X^\alpha_j X^\gamma_i + A_\gamma \partial_j X^\gamma_i \tag{3}$$

Using (3), we compute the first term in (1)

$$\partial_s A_r X^r_\alpha X^s_\beta = \partial_\rho A_\gamma X^\rho_s X^\gamma_r X^r_\alpha X^s_\beta + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{4}$$
$$= \partial_\rho A_\gamma X^\rho_\beta X^\gamma_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{5}$$
$$= \partial_\rho A_\gamma \delta^\rho_\beta \delta^\gamma_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{6}$$
$$= \partial_\beta A_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{7}$$

In the same way, we get for the second term in (1)

$$\partial_r A_s X^s_\alpha X^r_\beta = \partial_\alpha A_\beta + A_\gamma \partial_r X^\gamma_s X^r_\alpha X^s_\beta \tag{8}$$

And thus,

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta = \partial_\beta A_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta - \partial_\alpha A_\beta - A_\gamma \partial_r X^\gamma_s X^r_\alpha X^s_\beta \tag{9}$$
$$\Rightarrow \partial_\beta A_\alpha - \partial_\alpha A_\beta = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta \tag{10}$$

So, i) and (10) proves that $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ is skew-symmetric tensor of the second order.

♦

1.24 p24-exercise 9.

Let $x^r, \overline{x}^r, y^r, \overline{y}^r$ be four systems of coordinates. Examine the tensor character of $\frac{\partial x^r}{\partial y^s}$ with respect to the following transformations:
i) A transformation $x^r = f^r(\overline{x}^1, \ldots, \overline{x}^N)$, with y^r unchanged;
ii) A transformation $y^r = g^r(\overline{y}^1, \ldots, \overline{y}^N)$, with x^r unchanged;

Note: in the following, different alphabets in the indices indicates different coordinate systems.

i) Let's compute the expression $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta}$. Obviously, the right side is an expression of a (possible) mixed tensor of the second order ($\frac{\partial x^r}{\partial y^s}$) under transformation from the (r) coordinate system to the (α) coordinate system. Then,

$$\begin{aligned}
A(\alpha, \beta) &= \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta} && (1)\\
&= \frac{\partial x^\alpha}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} && (2)\\
&= \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} && (3)
\end{aligned}$$

If we consider the $\overline{y}^r$ coordinate system as the y^ρ coordinate system and as $\overline{y}^r = y^r$ then $\frac{\partial y^\rho}{\partial y^s} = \delta^\rho_s$ and we get from (3)

$$\begin{aligned}
A(\alpha, \beta) &= \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} && (4)\\
&= \frac{\partial x^\alpha}{\partial y^\rho}\delta^\rho_s\frac{\partial x^s}{\partial x^\beta} && (5)\\
&= \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial x^\rho}{\partial x^\beta} && (6)\\
&= \frac{\partial x^\alpha}{\partial y^\rho}\delta^\rho_\beta && (7)\\
&= \frac{\partial x^\alpha}{\partial y^\beta} && (8)\\
\text{(1) and (8)} \Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} &= \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta} && (9)
\end{aligned}$$

So $A(r, s) = \frac{\partial x^r}{\partial y^s}$ is a mixed tensor of type A^r_s

ii) Let's compute the expression $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta}$. Obviously, the right side is an expression of a (possible) mixed tensor of the second order ($\frac{\partial x^r}{\partial y^s}$) under transformation from the (r) coordinate system to the (α) coordinate system. Then,

$$A(\alpha,\beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{10}$$

$$= \frac{\partial x^r}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{11}$$

$$= \frac{\partial x^r}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{12}$$

$$= \frac{\partial x^r}{\partial y^\rho}\delta^\rho_\beta\frac{\partial y^\alpha}{\partial y^r} \tag{13}$$

$$= \frac{\partial x^r}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{14}$$

$$= \frac{\partial x^r}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{15}$$

If we consider the $\overline{x}^r$ coordinate system as the x^σ coordinate system and as $\overline{x}^r = x^r$ then $\frac{\partial x^\sigma}{\partial x^r} = \delta^\sigma_r$ and we get from (15)

$$A(\alpha,\beta) = \frac{\partial x^r}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{16}$$

$$= \delta^r_\sigma\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{17}$$

$$= \frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^\sigma} \tag{18}$$

$$= \frac{\partial x^\sigma}{\partial y^\beta}\delta^\alpha_\sigma \tag{19}$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \tag{20}$$

$$\text{(10) and (19)} \;\Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{21}$$

So $A(r,s) = \frac{\partial x^r}{\partial y^s}$ is a mixed tensor of type A^r_s

♦

1.25 p24-exercise 10.

If x^r, y^r, z^r are three systems of coordinates, prove the follwoing rule for the multiplication of Jacobians.

$$\left|\frac{\partial x^m}{\partial y^n}\right|\left|\frac{\partial y^r}{\partial z^s}\right| = \left|\frac{\partial x^t}{\partial z^u}\right|$$

As we have

$$\frac{\partial x^t}{\partial z^u} = \frac{\partial x^t}{\partial y^k}\frac{\partial y^k}{\partial z^u} \tag{1}$$

$$\begin{bmatrix} \frac{\partial x^1}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial z^N} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial y^k}\frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial y^k}\frac{\partial y^k}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^k}\frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial y^k}\frac{\partial y^k}{\partial z^N} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^1} & \cdots & \frac{\partial x^N}{\partial y^k} \end{bmatrix}\begin{bmatrix} \frac{\partial y^1}{\partial z^1} & \cdots & \frac{\partial y^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial y^N}{\partial z^1} & \cdots & \frac{\partial y^N}{\partial z^N} \end{bmatrix} \tag{3}$$

$$\Rightarrow \left|\frac{\partial x^m}{\partial y^n}\right|\left|\frac{\partial y^r}{\partial z^s}\right| = \left|\frac{\partial x^t}{\partial z^u}\right| \tag{4}$$

♦

1.26 p24-exercise 11.

Prove that with respect to transformations

$$x'^r = C_{rs}x^s$$

where the coefficients are constants satisfying

$$C_{mr}C_{ms} = \delta^r_s$$

contravariant and covariant vectors have the same formula of transformation

$$A'^r = C_{rs}A^s,\ A_{,r} = C_{rs}A_s$$

i) $A'^r = C_{rs}A^s$

Be $A'^r = A^s\frac{\partial x'^r}{\partial x^s}$ and as $x'^r = C_{rs}x^s$ we have $\frac{\partial x'^r}{\partial x^s} = C_{rs}$. Hence,

$$A'^r = C_{rs}A^s$$

.

i) $A_{,r} = C_{rs}A_s$
Be $A_{,r} = A_s\frac{\partial x^s}{\partial x'^r}$ and as $x'^r = C_{rs}x^s$ we have

$$\frac{\partial x'^r}{\partial x'^t} = C_{rs}\frac{\partial x^s}{\partial x'^t} \tag{1}$$

$$\Rightarrow \delta^r_t = C_{rs}\frac{\partial x^s}{\partial x'^t} \tag{2}$$

Now, multiply (2) by C_{rq}. We get,

$$\delta^r_t C_{rq} = C_{rq}C_{rs}\frac{\partial x^s}{\partial x'^t} \tag{3}$$

$$C_{tq} = C_{rq}C_{rs}\frac{\partial x^s}{\partial x'^t} \tag{4}$$

$$\text{as } C_{mr}C_{ms} = \delta^r_s \Rightarrow \qquad C_{tq} = \delta^q_s\frac{\partial x^s}{\partial x'^t} \tag{5}$$

$$\Rightarrow C_{tq} = \frac{\partial x^q}{\partial x'^t} \text{ or } C_{rs} = \frac{\partial x^s}{\partial x'^r} \tag{6}$$

$$\text{as } A_{,r} = A_s\frac{\partial x^s}{\partial x'^r} \Rightarrow A_{,r} = C_{rs}\frac{\partial x^s}{\partial x'^r} \tag{7}$$

♦

1.27 p25-exercise 12.

Prove that

$$\frac{\partial ln\left|\frac{\partial x^m}{\partial y^n}\right|}{\partial x^r} = \frac{\partial^2 y^m}{\partial x^r \partial x^n}\frac{\partial x^n}{\partial y^m}$$

.

Lemma *Be A a square matrix $N \times N$; Be f a C^1 function $f : \mathbb{R}^{NxN} \to \mathbb{R}$. Define $A^{'}$ as $(A^{'}_{ij}) = \frac{df}{dA_{ij}}$. Then,*

$$(ln\,|A|)^{'} = (A^{-1})^T \;\; wih\; f = |A|$$

Proof:
By definition of the determinant, we have

$$|A| = A_{iK}C^i_K \qquad (development\ along\ column\ K,\ no\ summation\ on\ K!) \tag{1}$$

with $(C^i_K) = (-1)^{i+K}M^i_K$ being the cofactor of element A_{iK} and M^i_K the minor $(N-1)\times(N-1)$ matrix associated with the cofactor A_{K^i}. Be $C = (C_{ij})$ the $N \times N$ matrix formed with all possible cofactor elements C^i_j $(i, j = 1\ldots, N)$.
We have

$$A^{-1} = \frac{C^T}{|A|} \tag{2}$$

$$\Rightarrow (A^{-1})^T = \frac{C}{|A|} \tag{3}$$

$$differentiating\ (1)\ \Rightarrow \frac{\partial\,|A|}{\partial A_{mn}} = \frac{\partial A_{iK}}{\partial A_{mn}}C^i_K + A_{iK}\frac{\partial C^i_K}{\partial A_{mn}} \tag{4}$$

$$we\ have\ for\ i = m \quad \begin{array}{ll} \frac{\partial A_{iK}}{\partial A_{mn}} = 1 & K = n \\ \frac{\partial A_{iK}}{\partial A_{mn}} = 0 & K \neq n \end{array} \tag{5}$$

Also, $\forall K : \frac{\partial C^i_K}{\partial A_{in}} = 0$ as by definition of the cofactor matrix, A_{ij} is not contained in C_{ij}.
Hence, (4) *becomes*

$$\frac{\partial\,|A|}{\partial A_{ij}} = C^i_j \tag{6}$$

$$But, \qquad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{\frac{\partial|A|}{\partial A_{ij}}}{|A|} \tag{7}$$

$$(6)\ and\ (7)\ gives \qquad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{C^i_j}{|A|} \tag{8}$$

$$(3)\ and\ (8)\ gives \qquad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{(A^{-1}_{ij})^T\,|A|}{|A|} = (A^{-1}_{ij})^T \tag{9}$$

$$\Rightarrow \qquad (ln\,|A|)^{'} = (A^{-1})^T \tag{10}$$

⋄

Now the main proof:

Let,

$$A \equiv [a_{mn}] = \left[\frac{\partial y^m}{\partial x^n}\right] \tag{11}$$

$$\Rightarrow \qquad \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial ln\,|A|}{\partial a_{mn}}\frac{\partial a_{mn}}{\partial x^r} \tag{12}$$

$$\text{from (10) we get} \qquad \frac{\partial ln\,|A|}{\partial a_{mn}} = (A^{-1})^T_{mn} \tag{13}$$

$$\text{But A is a Jacobian, so} \qquad (A^{-1})_{mn} = \frac{\partial x^m}{\partial y^n} \tag{14}$$

$$\text{and thus} \qquad (A^{-1})^T_{mn} = \frac{\partial x^n}{\partial y^m} \tag{15}$$

$$\text{(13) can be written as} \qquad \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m}\frac{\partial a_{mn}}{\partial x^r} \tag{16}$$

$$\Rightarrow \qquad \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m}\frac{\partial^2 y^m}{\partial x^r \partial x^n} \tag{17}$$

♦

1.28 p25-exercise 13.

Consider the quantities $\frac{dx^r}{dt}$ for a particle moving in the plane. If x^r are the rectangular Cartesian coordinates, are these quantities the components of a contravariant or covariant vector with respect to rotation of the axes? Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?

Note: we suppose that by a rotation of the axes, the problem means a fixed rotation and not a rotation varying in time.

i) Be $v^r = \frac{dx^r}{dt}$ and consider v^α the same object but in another the coordinate system. A rotation of the axes implies the linear form

$$x^\alpha = R^\alpha_k x^k \quad \text{with } R^\alpha_k \neq R^\alpha_k(x^k) \tag{1}$$

$$\Rightarrow \quad \frac{\partial x^\alpha}{\partial x^r} = R^\alpha_k \delta^k_r \tag{2}$$

$$\Rightarrow \quad R^\alpha_r = \frac{\partial x^\alpha}{\partial x^r} \tag{3}$$

Consider $v^\alpha = \frac{dx^\alpha}{dt}$

$$v^\alpha = \frac{dx^\alpha}{dt} \tag{4}$$

$$(1) \quad \Rightarrow \quad v^\alpha = R^\alpha_k \frac{dx^k}{dt} \tag{5}$$

$$\Rightarrow \quad v^\alpha = R^\alpha_k v^k \tag{6}$$

$$(3) \quad \Rightarrow \quad v^\alpha = v^k \frac{\partial x^\alpha}{\partial x^r} \tag{7}$$

Conclusion: v^k is a contravariant vector.

ii) Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?
We know that

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^r} dx^r \tag{8}$$

$$\Rightarrow \quad \frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r} \frac{dx^r}{dt} \tag{9}$$

$$\Rightarrow \quad v^\alpha = v^r \frac{\partial x^\alpha}{\partial x^r} \tag{10}$$

So v^r is a contravariant vector in general. Note that this proof is more straightforward than the prove in i).

♦

1.29 p25-exercise 14.

Consider the question raised in No. 13 for the acceleration $\frac{d^2x^r}{dt^2}$.

From exercise 13. we know that

$$\begin{aligned} \frac{dx^\alpha}{dt} &= \frac{\partial x^\alpha}{\partial x^r}\frac{dx^r}{dt} && (1)\\ \Rightarrow \quad \frac{d^2x^\alpha}{dt^2} &= \frac{d^2x^r}{dt^2}\frac{\partial x^\alpha}{\partial x^r} + \frac{d\frac{\partial x^\alpha}{\partial x^r}}{dt}\frac{dx^r}{dt} && (2)\\ &= \frac{d^2x^r}{dt^2}\frac{\partial x^\alpha}{\partial x^r} + \frac{\partial^2 x^\alpha}{\partial x^r \partial x^m}\frac{dx^m}{dt}\frac{dx^r}{dt} && (3) \end{aligned}$$

The second term on the right does not vanish in general, hence $\frac{d^2x^r}{dt^2}$ has not a tensor character.

♦

1.30 p25-exercise 15.

It is well known that the equation of an ellipse may be written

$$ax^2 + 2hxy + by^2 = 1$$

What is the tensor character of a, h, b with respect to transformation to any Cartesian coordinates (rectangular or oblique) in the plane?

Consider the transformation from a (w, z) coordinate system to a (x, y) coordinate system. For the considered type of transformation we have

$$\begin{aligned} x &= \alpha w + \beta z \\ y &= \gamma w + \delta z \end{aligned} \tag{1}$$

$$\text{consider} \qquad \begin{aligned} ax^2 + 2hxy + by^2 = 1 \\ pw^2 + 2qwz + rz^2 = 1 \end{aligned} \tag{2}$$

the two representations of the same ellipse in the respective coordinate systems. Plugging (1) in (2):

$$a\alpha^2 w^2 + a2\alpha\beta wz + \alpha\beta^2 z^2 + 2h\alpha\gamma w^2 + \beta\delta z^2 + 2h(\alpha\delta + \gamma\beta)wz + b\gamma^2 w^2 + 2b\gamma\delta wz + \delta^2 z^2 b = 1 \tag{3}$$

$$\tag{4}$$

Rearranging and equating the terms in w^2, wz, z^2 in (2) gives

$$p = a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \tag{5}$$

$$q = a\alpha\beta + h(\alpha\delta + \gamma\beta) + \gamma d \tag{6}$$

$$r = a\beta^2 + 2h\beta\delta + b\delta^2 \tag{7}$$

Consider the following objects

$$(A_{ij}) = \begin{bmatrix} a & h \\ h & b \end{bmatrix} \tag{8}$$

$$(A_{ij})^{'} = \begin{bmatrix} a^{'}_{11} & a^{'}_{12} \\ a^{'}_{21} & a^{'}_{21} \end{bmatrix} \tag{9}$$

$$\text{we calculate} \quad A^{'}_{ij} = A_{km} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} \tag{10}$$

with $(x^{'1}, x^{'2}) = (w, z)$ and $(x^1, x^2) = (x, y)$. We have,

$$\frac{\partial x^1}{\partial x'^1} = \alpha, \frac{\partial x^1}{\partial x'^2} = \beta, \frac{\partial x^2}{\partial x'^1} = \gamma, \frac{\partial x^2}{\partial x'^2} = \delta \tag{11}$$

$$(10) \text{ and } (11) \quad \Rightarrow \quad \begin{aligned} a^{'}_{11} &= a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \\ a^{'}_{22} &= a\beta^2 + 2h\delta\beta + b\delta^2 \\ a^{'}_{12} = a^{'}_{21} &= a\alpha\beta + h(\alpha\delta + \gamma\beta) + b\gamma\delta \end{aligned} \tag{12}$$

Combining (5), (6), (7) and (12) we get

$$p = a^{'}_{11}, r = a^{'}_{22}, q = a^{'}_{12} = a^{'}_{21}$$

and so (9) becomes

$$(A_{ij})^{'} = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$$

Considering (10) we see that $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ transforms to $\begin{bmatrix} p & q \\ q & r \end{bmatrix}$ according the rules of a covariant tensor of order two.

♦

1.31 p25-exercise 16.

Matter is distributed in a plane and A, B, H are the moments and product of inertia with respect to rectangular aces $0xy$ in a plane. Examine the tensor character of the set of quantities A, B, H under rotation of the axes. What notation would you suggest for moments and product of inertia in order to exhibit the tensor character? What simple invariant can be formed from A, B, H ?

Consider the transformation from a (x^1, x^2) coordinate system to a (y^1, y^2) coordinate system. For the considered type of transformation we have

$$\begin{aligned} y^1 &= \alpha x^1 + \beta x^2 \\ y^2 &= \gamma x^1 + \delta x^2 \end{aligned} \tag{1}$$

$$\text{Be} \quad \begin{aligned} A &= \textstyle\sum_\rho m_\rho (x^{2,\rho})^2 \\ B &= \textstyle\sum_\rho m_\rho (x^{1,\rho})^2 \\ H &= \textstyle\sum_\rho m_\rho x^{1,\rho} x^{2,\rho} \end{aligned} \quad \text{and} \quad \begin{aligned} A^{'} &= \textstyle\sum_\rho m_\rho (y^{2,\rho})^2 \\ B^{'} &= \textstyle\sum_\rho m_\rho (y^{1,\rho})^2 \\ H^{'} &= \textstyle\sum_\rho m_\rho y^{1,\rho} y^{2,\rho} \end{aligned} \tag{2}$$

the moments and product of inertia, ρ being the index of summation over all the points with mass m_ρ.
For the sake of notational simplicty we consider only one point of mass as the linearity of A, B, H related to ρ ensures the validity of the nexts steps for all points in the plane.
From (1) and (2) we have:

$$\frac{A^{'}}{m_\rho} = \gamma^2 (x^1)^2 + 2\gamma\delta x^1 x^2 + \delta^2 (x^2)^2 \tag{3}$$

$$\frac{B^{'}}{m_\rho} = \alpha^2 (x^1)^2 + 2\alpha\beta x^1 x^2 + \beta^2 (x^2)^2 \tag{4}$$

$$\frac{H^{'}}{m_\rho} = \alpha\gamma (x^1)^2 + (\gamma\beta + \alpha\delta) x^1 x^2 + \beta\delta (x^2)^2 \tag{5}$$

$$\text{Note that} \quad \begin{matrix} \frac{\partial y^1}{\partial x^1} = \alpha & \frac{\partial y^1}{\partial x^2} = \beta \\ \frac{\partial y^2}{\partial x^1} = \gamma & \frac{\partial y^2}{\partial x^2} = \delta \end{matrix} \tag{6}$$

$$\text{(6) in (4):} \quad \frac{B^{'}}{m_\rho} = (x^1)^2 \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + 2(x^1)(x^2)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} \tag{7}$$

$$= (x^1)^2 \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + (x^1)(x^2)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)(x^1)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} \tag{8}$$

Repeating the same calculations for $\frac{A^{'}}{m_\rho}$ and $\frac{H^{'}}{m_\rho}$ gives:

$$\begin{aligned} \frac{A^{'}}{m_\rho} &= (x^1)^2 \frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^1} + (x^1)(x^2)\frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^2} + (x^2)(x^1)\frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^1} + (x^2)^2 \frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^2} \\ \frac{B^{'}}{m_\rho} &= (x^1)^2 \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + (x^1)(x^2)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)(x^1)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} \\ \frac{H^{'}}{m_\rho} &= (x^1)^2 \frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^1} + (x^1)(x^2)\frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1} + (x^2)(x^1)\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^2} \end{aligned} \tag{9}$$

Now, define

$$(K_{ij}) = \begin{bmatrix} (x^1)^2 & (x^1)(x^2) \\ \\ (x^2)(x^1) & (x^2)^2 \end{bmatrix} \qquad (K_{ij})^{'} = \begin{bmatrix} (y^1)^2 & (y^1)(y^2) \\ \\ (y^2)(y^1) & (y^2)^2 \end{bmatrix} \tag{10}$$

Then (9) can be written as

$$\begin{aligned} \tfrac{A^{'}}{m_\rho} &= (y^1)^2 = K^{11}\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + K^{12}\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + K^{21}\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + K^{22}\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \\ \tfrac{B^{'}}{m_\rho} &= (y^2)^2 = K^{11}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^1} + K^{12}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + K^{21}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + K^{22}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^1}{\partial x^2} \\ \tfrac{H^{'}}{m_\rho} &= (y^1)(y^2) = K^{11}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + K^{12}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + K^{21}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + K^{22}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \end{aligned} \tag{11}$$

Hence,

$$\begin{aligned} K^{'11} &= K^{11}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^1} + K^{12}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + K^{21}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + K^{22}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^1}{\partial x^2} \\ K^{'22} &= K^{11}\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + K^{12}\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + K^{21}\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + K^{22}\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \\ K^{'12} = K^{'21} &= K^{11}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + K^{12}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + K^{21}\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + K^{22}\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \end{aligned} \tag{12}$$

So the object $(K_{ij}) \equiv K^{ij}$ transforms according (12) like a contravariant second order tensor.

Now, consider $|K^{ij}|$, obviously $|K^{ij}| = (x^1)^2(x^2)^2 - (x^1)(x^2)(x^2)(x^1) = 0$, but so is also $|K^{'ij}|$.

$$\Rightarrow \quad |K^{ij}| \text{ is an invariant under the considered transformation}$$

.

♦

1.32 p25-exercise 17.

Given a tensor S_{nmr} skew-symmetric tensor in the first two indices, find a tensor f_{mnr} skew-symmetric in the last two suffixes and satisfying the relation

$$-f_{mnr} + f_{nmr} = S_{mnr}$$

Answer: $f_{mnr} = \frac{1}{2}\left(-S_{rmn} + S_{nrm} - S_{mnr}\right)$

From exercise 13. we know that

$$-f_{mnr} + f_{nmr} = S_{mnr} \tag{1}$$

Swap the indices three times

$$\text{i)} \quad n \leftrightarrow r : (1) \Rightarrow -f_{mrn} + f_{rmn} = S_{mrn} \tag{2}$$

$$\Leftrightarrow \underbrace{f_{mnr}}_{*} + \underbrace{f_{rmn}}_{**} = -S_{rmn} \tag{3}$$

$$\text{ii)} \quad m \leftrightarrow r : (1) \Rightarrow -f_{rnm} + f_{nrm} = S_{rnm} \tag{4}$$

$$\Leftrightarrow \underbrace{f_{rmn}}_{**} + \underbrace{f_{nrm}}_{***} = -S_{nrm} \tag{5}$$

$$\text{iii)} \quad m \leftrightarrow n : (1) \Rightarrow -f_{nmr} + f_{mnr} = S_{nmr} \tag{6}$$

$$\Leftrightarrow \underbrace{f_{nrm}}_{***} + \underbrace{f_{mnr}}_{*} = -S_{mnr} \tag{7}$$

$$\text{(3) - (5) + (7):} \quad 2\underbrace{f_{mnr}}_{*} = -S_{rmn} + S_{nrm} - S_{mnr} \tag{8}$$

$$\Leftrightarrow f_{mnr} = \frac{-S_{rmn} + S_{nrm} - S_{mnr}}{2} \tag{9}$$

♦

Basic Operations in Riemannian Space

2.1 p27-exercise

Take polar coordinates r, θ in a plane. Draw the infinitesimal triangle with vertices (r, θ), $(r + dr, \theta)$, $(r, \theta + d\theta)$. Evaluate the square on the hypotenuse of this infinitesimal triangle, and so obtain the metric tensor for the plan for the coordinates(r, θ).

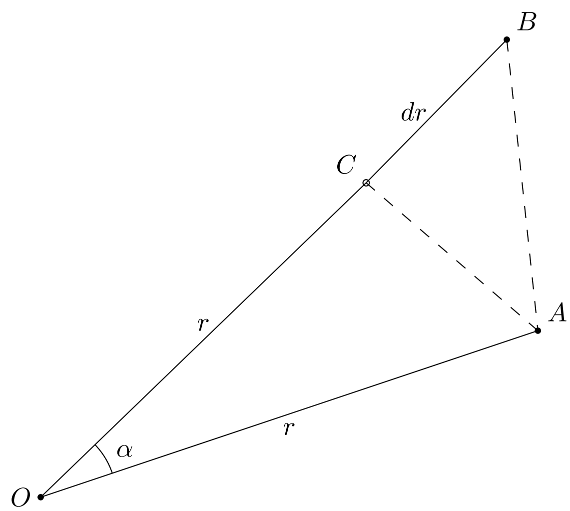


Figure 2.1: Metric tensor in polar coordinate system

$$\begin{aligned}
ds^2 &= |AB|^2 \\
&= dr^2 + |CA|^2 \\
|CA| &= r\sin(d\theta) \approx r d\theta \\
\Rightarrow ds^2 &= dr^2 + r^2 d\theta^2 \\
\Rightarrow (a_{mn}) &= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}
\end{aligned}$$

♦

2.2 p27-exercise

Show that if $x^1 = r, x^2 = \theta, x^3 = \phi$, in the usual notation of spherical polar coordinates, then

$$a_{11} = 1, a_{22} = r^2, a_{33} = r^2 \sin^2 \theta$$

and the other components vanish.

One can choose to start from $ds^2 = dx^2 + dy^2 + dz^2$ and then expanding the dx^i along (r, θ, ϕ) but this a rather tedious way. So we use a more geometrical way of deriving the metric
Consider an infinitesimal displacement of point E to J with $(dr, d\theta, d\phi)$.

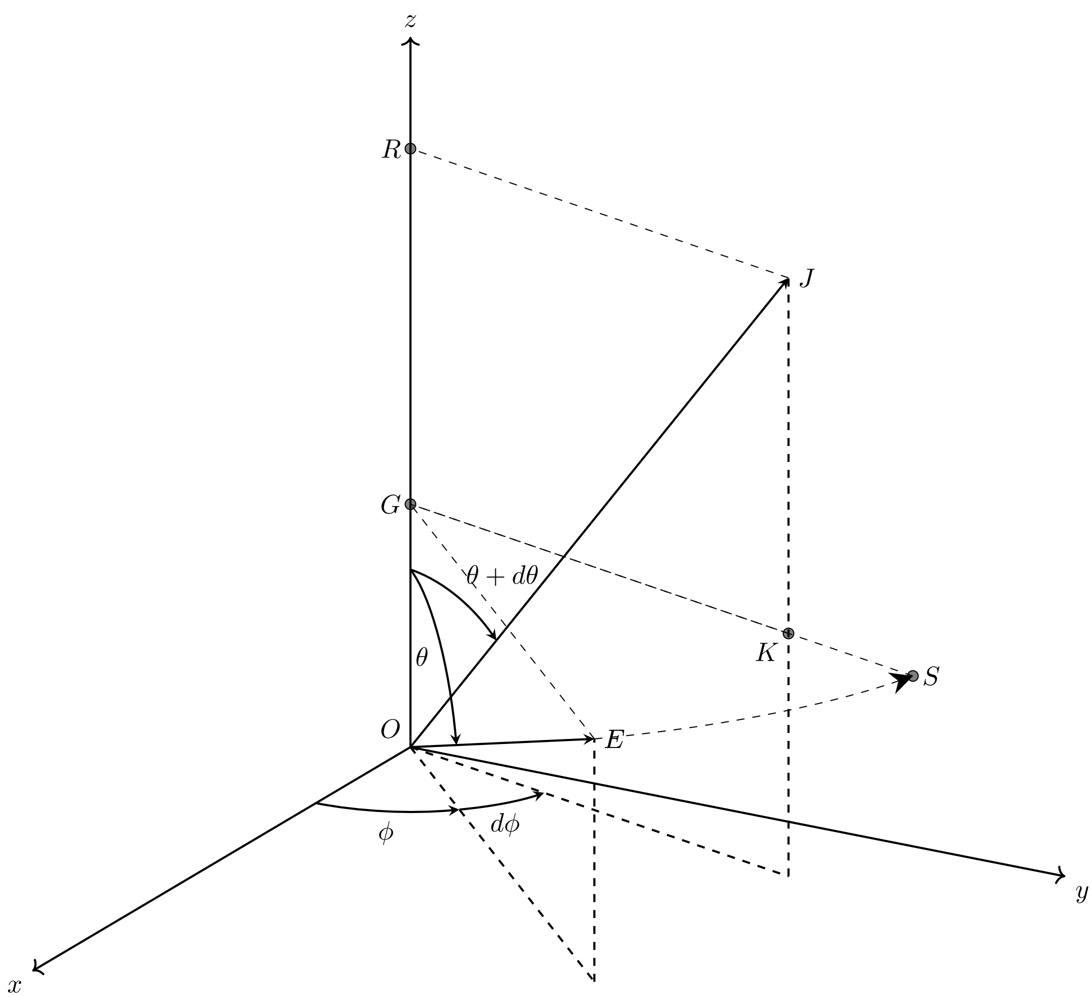


Figure 2.2: Metric tensor in spherical coordinate system

$$ds^2 = |EJ|^2 \tag{1}$$

As we use infinitesimal displacements we can assume that, omitting 2^{nd} order terms,

$$|ES| \perp |GK| \perp |JK| \perp |ES|$$

. Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \tag{2}$$

We have the following relationships

$$\left.\begin{gathered}
|GE| = |GS| = r\sin\theta \\
\\
|ES| = |GE|d\phi = r\sin\theta d\phi \\
\\
|GK| = |RJ| = (r+dr)\sin(\theta+d\theta) \\
= (r+dr)(\cos(\theta)\sin(d\theta)+\sin(\theta)\cos(d\theta)) \\
= (r+dr)(\cos(\theta)d\theta+\sin(\theta)) \\
= r\cos(\theta)d\theta + r\sin(\theta) + \sin(\theta)dr \\
\\
|OR| = (r+dr)\cos(\theta+d\theta) \\
= (r+dr)(\cos(\theta)\cos(d\theta)-\sin(\theta)\sin(d\theta)) \\
= (r+dr)(\cos(\theta)-\sin(\theta)d\theta) \\
= r\cos(\theta) - r\sin(\theta)d\theta + \cos(\theta)dr \\
\\
|OG| = r\cos(\theta) \\
\\
|JK| = |OR| - |OG| = \cos(\theta)dr - r\sin(\theta)d\theta \\
\\
|SK| = |GK| - |GS| = r\cos(\theta)d\theta + \sin(\theta)dr
\end{gathered}\right\} \tag{3}$$

$$\left.\begin{gathered}
|ES|^2 = r^2\sin^2(\theta)d\phi^2 \\
|SK|^2 = r^2\cos^2(\theta)d\theta^2 + \sin^2(\theta)dr^2 + 2r\cos(\theta)\sin(\theta)drd\theta \\
|JK|^2 = \cos^2(\theta)dr^2 + r^2\sin^2(\theta)d\theta^2 - 2r\cos(\theta)\sin(\theta)drd\theta
\end{gathered}\right\} \tag{4}$$

Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \tag{5}$$

$$= \begin{cases} r^2\sin^2(\theta)d\phi^2 \\ +r^2\cos^2(\theta)d\theta^2 + \sin^2(\theta)dr^2 + 2r\cos(\theta)\sin(\theta)drd\theta \\ +r^2\sin^2(\theta)d\theta^2 + \cos^2(\theta)dr^2 - 2r\cos(\theta)\sin(\theta)drd\theta \end{cases} \tag{6}$$

$$\Rightarrow ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 \tag{7}$$

$$\Rightarrow (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{8}$$

♦

2.3 p27-exercise

Starting from (**2.103**), show that

$$a_{mn} = \frac{\partial y^1}{\partial x^m}\frac{\partial y^1}{\partial x^n} + \frac{\partial y^2}{\partial x^m}\frac{\partial y^2}{\partial x^n} + \frac{\partial y^3}{\partial x^m}\frac{\partial y^3}{\partial x^n}$$

and calculate the quantities for a sphere, taking as curvilinear coordinates on he sphere

$$x^1 = y^1, x^2 = y^2$$

We have

$$\begin{aligned}
(\mathbf{2.103}) \quad & \Rightarrow y^1 = x^1, y^2 = x^2, y^3 = f^3(x^1, x^2) && (1)\\
\text{surface} = \text{sphere} \quad & \Rightarrow y^3 = \pm\sqrt{R^2 - (x^1)^2 - (x^2)^2} && (2)\\
ds^2 &= (dx^1)^2 + (dx^2)^2 + (dx^3)^2 && (3)\\
(1) \text{ and } (2) \quad & \Rightarrow \begin{cases} dy^1 = dx^1 \\ dy^2 = dx^2 \\ dy^3 = \pm\frac{1}{2}\frac{-2x^1dx^1 - 2x^2dx^2}{\sqrt{R^2-(x^1)^2-(x^2)^2}} \end{cases} && (4)\\
& \Rightarrow ds^2 = (dx^1)^2 + (dx^2)^2 + \frac{(x^1)^2(dx^1)^2 + (x^2)^2(dx^2)^2 + 2x^1x^2dx^1dx^2}{R^2 - (x^1)^2 - (x^2)^2} && (5)\\
& \Leftrightarrow ds^2 = \frac{(R^2 - (x^2)^2)(dx^1)^2 + (R^2 - (x^1)^2)(dx^2)^2 + 2x^1x^2dx^1dx^2}{R^2 - (x^1)^2 - (x^2)^2} && (6)\\
& \Rightarrow (a_{mn}) = \frac{1}{R^2 - (x^1)^2 - (x^2)^2}\begin{pmatrix} R^2 - (x^2)^2 & x^1x^2 \\ x^1x^2 & R^2 - (x^1)^2 \end{pmatrix} && (7)
\end{aligned}$$

♦

2.4 p30-clarification 2.202

$$a_{mr}\Delta^{ms} = a_{rm}\Delta^{sm} = \delta^s_r a$$

Case 1: $r = s$
We have, $a_{Rm}\Delta^{Rm}$ (no summation on R) is the definition of the determinant of A developed along the row R: OK.

Case 2: $r \neq s$
Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \tag{1}$$

and consider the matrix $A^{'}$

$$A^{'} = \begin{array}{cl} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{R1} & a_{R2} & \dots & a_{RN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{R1} & a_{R2} & \dots & a_{RN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} & \begin{array}{l} \qquad\vdots \\ \qquad\vdots \\ \leftarrow S^{th} \text{ row} \\ \qquad\vdots \\ \leftarrow R^{th} \text{ row} \\ \qquad\vdots \\ \qquad\vdots \end{array} \end{array} \tag{2}$$

This matrix corresponds to the way $a_{Rm}\Delta^{Sm}$ is computed. Indeed with the factor a_{Rm} is not associated it's own cofactor Δ^{Rm} but the cofactor of the m^{th} column in row S. Replacing the S^{th} row with the row R and calculating it's determinant is the same as calculating $a_{Rm}\Delta^{Sm}$
But, $|A^{'}| = 0$ as we have two identical rows. So, $a_{Rm}\Delta^{Sm} = 0$

Conclusion : The same reasoning can be applied when expanding the determinant along the columns instead of the rows we have indeed $\quad a_{mr}\Delta^{ms} = a_{rm}\Delta^{sm} = \delta^s_r a.$

♦

2.5 p31-exercise

Show that if $a_{mn} = 0$ for $m \neq n$, then

$$a^{11} = \frac{1}{a_{11}}, a^{22} = \frac{1}{a_{22}}, \ldots, a^{12} = 0, \ldots$$

We have to prove that:

$$a^{ij} = \begin{cases} \frac{1}{a_{ij}} & : i = j \\ 0 & : i \neq j \end{cases}$$

From 2.204:

$$a_{mR} a^{mS} = \delta^S_R \tag{1}$$

i) Be $R \neq S$

$$\begin{aligned} (1) \quad \Rightarrow \quad & a_{mR} a^{mS} = 0 \\ \text{but} \quad & a_{mR} = 0 \quad \forall m \neq R \\ \Rightarrow \quad & a_{RR} a^{RS} = 0 \end{aligned}$$

but $a_{RR} \neq 0$ (a_{RR} can't be 0 as the metric tensor would degenerate if $a_{mn} = 0 \quad \forall m \neq n$

$$\Rightarrow \quad a^{Rs} = 0$$

i) Be $R = S$

$$\begin{aligned} (1) \quad \Rightarrow \quad & a_{mR} a^{mR} = 1 \\ \text{but} \quad & a_{mR} = 0 \quad \forall m \neq R \\ \Rightarrow \quad & a_{RR} a^{RR} = 1 \\ \Rightarrow \quad & a^{RR} = \frac{1}{a_{RR}} \end{aligned}$$

♦

2.6 p31-exercise

Find the components of a^{mn} for spherical polar coordinates in Eulidean 3-space.

We have (see exercise page 27):

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

As $a_{mn} = 0 \quad \forall m \neq n$ we deduce (see previous exercise p. 31)

$$(a^{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta)} \end{pmatrix}$$

♦

2.7 p32-exercise

Find the mixed metric tensor $a_m^{\cdot n}$ obtained from a_{mn} by raising the second subscript

We have :

$$\begin{aligned}
a_i^{\cdot j} &= a_{in}a^{nj} \\
&= a_{in}a^{jn} \quad a^{jn} \text{ is symmetric} \\
&= \delta_i^j \quad \text{(see 2.205 pg. 30)} \\
&\Rightarrow a_i^{\cdot j} = \delta_i^j
\end{aligned}$$

♦

2.8 p32-clarification 2.214

$$\frac{\partial a}{\partial a_{mn}} = a a^{mn}$$

By definition, we have

$$a \equiv |a_{mn}| = a_{Mk}\Delta^{Mk} \quad \text{(develop determinant along row M)} \tag{1}$$

$$\Rightarrow \quad \frac{\partial a}{\partial a_{mn}} = \frac{\partial a_{Mk}}{\partial a_{mn}}\Delta^{Mk} + a_{Mk}\frac{\partial \Delta^{Mk}}{\partial a_{mn}} \tag{2}$$

$$\text{but} \quad \frac{\partial a_{Mk}}{\partial a_{mn}} = \begin{cases} 1 & \text{if} \quad k = N \\ 0 & \text{if} \quad k \neq N \end{cases} \tag{3}$$

$$\text{and} \quad \frac{\partial \Delta^{Mk}}{\partial a_{mn}} = 0 \quad \forall k \text{ as } \Delta^{Mk} \text{does not contain the row with } a_{mn} \text{ as element.} \tag{4}$$

$$\text{(3) and (4)} \quad \Rightarrow \quad \frac{\partial a}{\partial a_{mn}} = \Delta^{MN} \tag{5}$$

$$a^{mn} = \frac{\Delta^{mn}}{a} \quad \text{by definition (see 2.203 page 30)} \tag{6}$$

$$\Rightarrow \quad \frac{\partial a}{\partial a_{mn}} = a a^{mn} \tag{7}$$

♦

2.9 p32-exercise

Prove that $a_{mn}a^{mn} = N$.

From 2.204, we have

$$a_{mr}a^{ms} = \delta_r^s \tag{1}$$

$$\text{Consider} \quad a_{mR}a^{mR} = 1 \tag{2}$$

$$\text{We can repeat (2) for R = 1,2,...,N} \quad \Rightarrow a_{mr}a^{mr} = N \tag{3}$$

♦

2.10 p33-exercise

Show that in Euclidean 3-space with rectangular Cartesian coordinates, the definition 2.301 coincides with the usual definition of the magnitude of a vector.

The length of an arbitrary vector in Euclidean 3-space with rectangular Cartesian coordinates, is

$$ds^2 = (dy^1)^2 + (dy^2)^2(dy^2)^2$$

From 2.301, it is obvious that the metrixc tensor can be expressed as,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

♦

2.11 p34-exercise

A curve in Euclidean 3-space has the equations

$$x^1 = a\cos(u), x^2 = a\sin(u), x^3 = bu$$

where x^1, x^2, x^3 are rectangular Cartesian coordinates,u is a parameter, and a, b are positive constants. Find the length of this curve between the point $u = 0$ and $u = 2\pi$.

The metric tensor has the following form,

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$\text{and (2.306)} \qquad s = \int_0^{2\pi} [\epsilon a_{mn} p^m p^n]^{\frac{1}{2}} du \tag{2}$$

with

$$p^1 = \frac{dx^1}{du} = -a\sin(u), \quad p^2 = \frac{dx^2}{du} = a\cos(u), \quad p^3 = \frac{dx^3}{du} = b$$

Hence (2) becomes

$$\begin{aligned} s &= \int_0^{2\pi} \epsilon[a^2\sin^2(u) + a^2\cos^2(u) + b^2]^{\frac{1}{2}} du \\ &= \int_0^{2\pi} \epsilon[a^2 + b^2]^{\frac{1}{2}} du \\ &= [a^2 + b^2]^{\frac{1}{2}} \ u\big|_0^{2\pi} \\ &= 2\pi[a^2 + b^2]^{\frac{1}{2}} \end{aligned}$$

♦

2.12 p36-clarification 2.314

Going from 2.313 to 2.314 yields because both X^m and Y^m are unit vectors and by definition of the magnitude (see 2.301) both $a_{mn}X^mX^n$ and $a_{mn}Y^mY^n$ are 1 (also due to the fact that only a positive definite metric tensor is considered, $\epsilon = 1$).

♦

2.13 p37-exercise

Show that the small angle between unit vectors X^r and $X^r + dX^r$ (these increments being infenitesimal) is given by

$$\theta^2 = a_{mn}X^m X^n$$

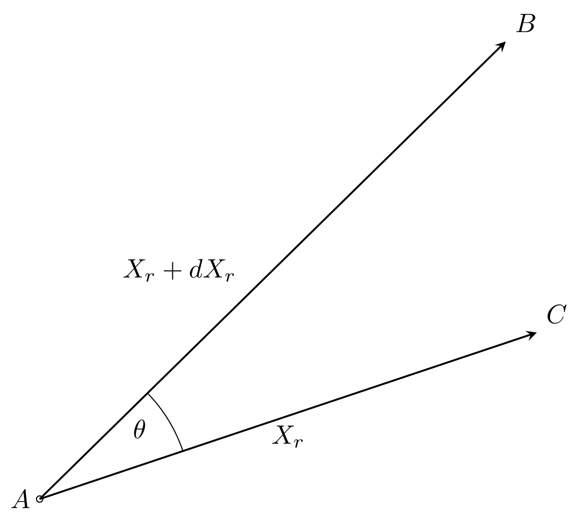


Figure 2.3: Small angle expression

By definition (**2.302** page 33)

$$|BC|^2 = \epsilon a_{mn} dX^m dX^n$$

We can drop $\epsilon = 1$ as the considered space is positive definite.
As θ is infinitesimal, we can state

$$\begin{aligned} & |BC| \approx |AC|\theta \\ \text{and} \quad & |AC| = X^r = 1 \quad (\text{as } X^r \text{ is a unit vector}) \\ \Rightarrow \quad & \theta^2 = a_{mn} dX^m dX^n \end{aligned}$$

♦

2.14 p39-clarification 2.409

We clarify the integration by parts in the derivation of the general geodesic equation.

We have

$$\int d(A.B) = \int AdB + \int BdA \tag{1}$$

$$\Rightarrow \int AdB = \int d(A.B) - \int BdA \tag{2}$$

Now, substitute 2.407 in 2.406, we get

$$\frac{dL}{dv} = \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial p^r}{\partial v}du \tag{3}$$

$$= \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial \frac{\partial x^r}{\partial v}}{\partial u}du \tag{4}$$

$$= \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}d(\frac{\partial x^r}{\partial v}) \tag{5}$$

To integrate by parts the second term in (5) we put in (2)

$$A = \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} \quad \text{and } B = \frac{\partial x^r}{\partial v}$$

$$\int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}d(\frac{\partial x^r}{\partial v}) = \int AdB \tag{6}$$

$$= \int d(A.B) - \int BdA \tag{7}$$

$$= \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial x^r}{\partial v}\Big|_{u_0}^{u_1} - \int \frac{\partial x^r}{\partial v}d(\frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}) \tag{8}$$

$$= \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial x^r}{\partial v}\Big|_{u_0}^{u_1} - \int \frac{\partial x^r}{\partial v}\frac{\partial \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r})}{\partial u}du \tag{9}$$

Replacing (9) in (5) gives the formula (**2.409**).

♦

2.15 p41-exercise

Prove the following identities:

$$[mn, r] = [nm, r], \quad [rm, n] + [rn, m] = \partial_r a_{mn}$$

$$\begin{aligned}
[mn, r] &= \frac{1}{2}(\partial_n a_{mr} + \partial_m a_{nr} - \partial_r a_{mn}) \\
&= \frac{1}{2}(\partial_m a_{nr} + \partial_n a_{mr} - \partial_r a_{nm}) \\
&= [nm, r]
\end{aligned}$$

and

$$\begin{aligned}
[rm, n] + [rn, m] &= \frac{1}{2}(\partial_r a_{mn} + \partial_m a_{rn} - \partial_n a_{rm} + \partial_n a_{rm} + \partial_r a_{mn} - \partial_m a_{rn}) \\
&= \frac{1}{2}(\partial_r a_{mn} + \partial_r a_{mn}) \\
&= \partial_r a_{mn}
\end{aligned}$$

♦

2.16 p42-exercise

Prove that

$$[mn, r] = a_{rs}\Gamma^s_{mn}$$

$$\begin{aligned} a_{rs}\Gamma^s_{mn} &= a_{rs}a^{sk}[mn, k] \\ &= \delta^k_r [mn, k] \\ &= [mn, r] \end{aligned}$$

♦

2.17 p42-clarification on 2.430

> ... This may proved without difficulty by starting with 2.427, in which λ is a known function of u , and defining s by the relation
>
> $$s = \int_{u_0}^{u} (exp \int_{v_0}^{v} \lambda(w)dw)dv$$
>
> u_0, v_0 being constants....

$$\text{see 2.428 :} \qquad \lambda(u) = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{1}$$

We suppose $u(s)$ continuous by parts with continuous inverse.

$$\Rightarrow \qquad \frac{ds}{du} = \frac{1}{\frac{du}{ds}} \tag{2}$$

$$\Rightarrow \qquad \frac{d^2s}{du^2} = \frac{d(\frac{1}{\frac{du}{ds}})}{ds}\frac{ds}{du} \tag{3}$$

$$= -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2}\frac{ds}{du} \tag{4}$$

$$\Rightarrow \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{5}$$

By definition (2.428)

$$\lambda(u) = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{6}$$

$$\text{hence by (5) and (6):} \qquad \lambda(u) = \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} \tag{7}$$

$$\text{in (8) put} \qquad y = \frac{ds}{du} \tag{8}$$

$$\text{and so} \qquad \lambda(w) = \frac{y'}{y} \tag{9}$$

$$\Rightarrow \int \frac{y'}{y}dw = \int \lambda(w)dw \tag{10}$$

$$\Leftrightarrow \qquad \int d(lny) = \int \lambda(w)dw \tag{11}$$

$$\Rightarrow ln(y)|_{v_0}^{v} = \int_{v_0}^{v} \lambda(w)dw \tag{12}$$

$$\Rightarrow y \qquad = exp(\int_{v_0}^{v} \lambda(w)dw) + C \tag{13}$$

Taking into account (8), we get:

$$\frac{ds}{dv} = exp(\int_{v_0}^{v} \lambda(w)dw) + C \tag{14}$$

$$\Rightarrow \qquad s|_{u_0}^{u} = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv + Cu + B^{'} \tag{15}$$

$$\Leftrightarrow \qquad s = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv + Cu + B \tag{16}$$

We show that we have to put $C = 0$ and can drop the constant B. Remember by (7)

$$\lambda(u) = \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} \tag{17}$$

$$\text{by (17)} \qquad \frac{ds}{du} = exp(\int_{u_0}^{u} \lambda(w)dw) + C \tag{18}$$

$$\text{and} \qquad \frac{d^2s}{du^2} = \lambda(u)exp(\int_{u_0}^{u} \lambda(w)dw) \tag{19}$$

$$\text{hence by (17), (18) and (19):} \qquad \lambda(u) = \frac{\lambda(u)exp(\int_{u_0}^{u} \lambda(w)dw)}{exp(\int_{u_0}^{u} \lambda(w)dw) + C} \tag{20}$$

So, whatever the constant B, the relation (17) is correct on the condition that C=0. So, indeed, we can choose the independent variable s as

$$s = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv$$

♦

2.18 p42-clarification on 2.430

After 2.430 it is stated:
*"No matter what values these constants have, (**2.424**) is satisfied, and by adjusting the constant v_0, we can ensure that $a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \pm 1$ along C, so that s is actually the arc length."*

We first prove that (**2.424**) is satisfied, no matter what values the constants take. We have

$$(2.430) \qquad s = \int_{u_0}^{u} (exp \int_{v_0}^{v} \lambda(w)dw)dv \tag{1}$$

$$\text{and } (2.427) \qquad \frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = \lambda\frac{dx^r}{du} \tag{2}$$

In (2) we can write the first term as

$$\frac{d^2x^r}{du^2} = \frac{d(\frac{dx^r}{du})}{ds}\frac{ds}{du} \tag{3}$$

$$\text{with} \quad \frac{d(\frac{dx^r}{du})}{ds} = \frac{d(\frac{dx^r}{ds}\frac{ds}{du})}{ds} = \frac{d^2x^r}{ds^2}\frac{ds}{du} + \frac{dx^r}{ds}\frac{d(\frac{ds}{du})}{ds} \tag{4}$$

Assuming the curve smooth, we have

$$\frac{d(\frac{ds}{du})}{ds} = \frac{d(\frac{1}{\frac{du}{ds}})}{ds} = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} = \lambda \tag{5}$$

Putting (4) and (5) in (3) we get

$$\frac{d^2x^r}{du^2} = \frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \lambda\frac{dx^r}{ds}\frac{ds}{du} \tag{6}$$

Plugging (6) in 2.427 gives:

$$\frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \lambda\frac{dx^r}{ds}\frac{ds}{du} + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \lambda\frac{dx^r}{ds}\frac{ds}{du} \tag{7}$$

$$\Leftrightarrow \frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = 0 \tag{8}$$

We can assume that $\frac{ds}{du}$ does not become 0 or $\pm\infty$ along the curve by choosing an adequate constant v_0. Indeed, from (2.430) we get

$$\frac{ds}{du} = exp(\int_{u_0}^{u} \lambda(w)dw) \tag{9}$$

$$= \frac{\phi(u)}{\phi(u_0)} \tag{10}$$

with $\phi(u_0) = e^{\theta(u_0)}$, $\theta(u)$ being the indefinite integral $\int \lambda(w)dw$.
So, it is sufficient to choose v_0 so that $\theta(u_0)$ does not become $\pm\infty$ to ensure that $\frac{ds}{du} \neq 0$ or $\neq \pm\infty$

along the curve and so we have from (8)

$$\frac{d^2x^r}{ds^2} + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = 0 \tag{11}$$

which is the definition (2.424) of a geodesic.

The same reasoning about $\frac{ds}{du} \neq 0$ can be made to prove that $a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \pm 1$ along C. Indeed, by definition (2.305):

$$ds = \left[\epsilon a_{mn}\frac{dx^m}{du}\frac{dx^n}{du}\right]^{\frac{1}{2}} \tag{12}$$

$$\text{equating with (9)} \quad \left[\epsilon a_{mn}\frac{dx^m}{du}\frac{dx^n}{du}\right]^{\frac{1}{2}} = exp(\int_{u_0}^{u}\lambda(w)dw) \tag{13}$$

$$\Rightarrow \quad \epsilon a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \left[exp(\int_{u_0}^{u}\lambda(w)dw)\right]^2 \tag{14}$$

$$\text{but} \quad \frac{ds}{du} = exp(\int_{u_0}^{u}\lambda(w)dw) \tag{15}$$

$$\text{and so, (9) becomes} \quad \epsilon a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \left(\frac{ds}{du}\right)^2 \tag{16}$$

$$\Rightarrow \quad a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \epsilon \tag{17}$$

♦

2.19 p43-clarification

$$\boxed{\lambda = \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + 2\Gamma^N_{\mu N}\frac{dx^\mu}{dx^N} + \Gamma^N_{NN}}$$

We start with (**2.427**) with $r = N$

$$\frac{d^2x^N}{dx^{N\,2}} + \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} = \lambda\frac{dx^N}{dx^N} \tag{1}$$

$$\Rightarrow \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} = \lambda \quad (\text{as} \quad \frac{d^2x^N}{dx^{N\,2}} = 0 \quad \frac{dx^N}{dx^N} = 1) \tag{2}$$

But (2) is only valid with the dummy indices μ and ν spanning the whole dimension $(1, 2, \ldots N)$, but by choice $\mu, \nu \quad \epsilon \quad (1, 2, \ldots, N-1)$. We have thus to add in the left term of (2) the cases

$$\begin{cases} \Gamma^N_{N\nu} & \nu = (1, 2, \ldots, N-1) \\ \\ \Gamma^N_{\mu N} & \mu = (1, 2, \ldots, N-1) \\ \\ \Gamma^N_{NN} \end{cases}$$

$$\text{(2) becomes} \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{N\nu}\frac{dx^N}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{\mu N}\frac{dx^\mu}{dx^N}\frac{dx^N}{dx^N} + \Gamma^N_{NN}\frac{dx^N}{dx^N}\frac{dx^N}{dx^N} = \lambda$$

$$\Rightarrow \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{N\nu}\frac{dx^\nu}{dx^N} + \Gamma^N_{\mu N}\frac{dx^\mu}{dx^N} + \Gamma^N_{NN} = \lambda$$

As $\Gamma^N_{\mu N}$ is symmetric on the lower indices and μ, ν being dummy indices:

$$\lambda = \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + 2\Gamma^N_{N\nu}\frac{dx^\nu}{dx^N} + \Gamma^N_{NN}$$

The other $N-1$ equation for $r = 1, \ldots, N-1$ can be deduced following the same reasoning.

♦

2.20 p45-clarification

$f_r \equiv a_{rm}\frac{d^2x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du}$ is covariant and $f^r \equiv \frac{d^2x^m}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du}$ is contravariant.

$$f_r \equiv a_{rm}\frac{d^2x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \tag{1}$$

$$\text{multiply with} \quad a^{sr} \quad \Rightarrow \quad f_r a^{sr} = a^{sr}a_{rm}\frac{d^2x^m}{du^2} + a^{sr}[mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \tag{2}$$

$$\Rightarrow \quad f^s = \delta^s_m\frac{d^2x^m}{du^2} + \underbrace{a^{sr}[mn,r]}_{\Gamma^s_{mn}}\frac{dx^m}{du}\frac{dx^n}{du} \tag{3}$$

$$\Rightarrow \quad f^s = \frac{d^2x^s}{du^2} + \Gamma^s_{mn}\frac{dx^m}{du}\frac{dx^n}{du} \tag{4}$$

By lifting the index of f_r we get a contravariant vector confirming that (4) is contravariant.

♦

2.21 p45-clarification

(2.443) and (2.444)

$$p^r \frac{\partial w}{\partial p^r} - w = C^t \quad \Rightarrow \quad w = C^t$$

By definition

$$w = a_{mn} p^m p^n \tag{1}$$

$$\Rightarrow \quad \frac{\partial w}{\partial p^r} = a_{mn}\left(\frac{\partial p^m}{\partial p^r} p^n + p^m \frac{\partial p^n}{\partial p^r}\right) \tag{2}$$

$$= a_{mn}(\delta^m_r p^n + p^m \delta^n_r) \tag{3}$$

$$= a_{rn} p^n + a_{mr} p^m \tag{4}$$

$$= 2 a_{mr} p^m \quad \text{(as } a_{mn} \quad \text{is symmetric)} \tag{5}$$

$$(4) \quad \Rightarrow \quad p^r \frac{\partial w}{\partial p^r} = 2 a_{mr} p^r p^m \tag{6}$$

$$= 2w \tag{7}$$

$$(2.443) \quad \Rightarrow \quad p^r \frac{\partial w}{\partial p^r} - w = 2w - w = w = C^t \tag{8}$$

$$\Rightarrow \quad w \equiv a_{mn} p^m p^n = C^t \tag{9}$$

♦

2.22 p47-exercise

The class of all parameters u, for which the equations of a geodesic null line assume the simple form (**2.445**), are obtained from any one such parameter by linear transformation

$$\overline{u} = au + b$$

a and b being constants.

The simple form (**2.445**) is :

$$\frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \tag{1}$$

$$\tag{2}$$

The general form of a geodesic is (**2.447**)

$$\frac{d^2x^r}{d\overline{u}^2} + \Gamma^r_{mn}\frac{dx^m}{d\overline{u}}\frac{dx^n}{d\overline{u}} = \lambda\frac{dx^r}{d\overline{u}} \tag{3}$$

$$\tag{4}$$

So (**2.447**) can only of the form (**2.445**) if $\lambda = 0$

$$\lambda = -\frac{\frac{d^2\overline{u}}{du^2}}{\left(\frac{d\overline{u}}{du}\right)^2} = 0 \tag{5}$$

$$\tag{6}$$

We can state that $\frac{d\overline{u}}{du} \neq 0$ as $\overline{u}$ can't be a constant (being a parameter of a curve). So,

$$\frac{d^2\overline{u}}{du^2} = 0 \tag{7}$$

$$\Rightarrow \quad \frac{d\overline{u}}{du} = a \tag{8}$$

$$\Rightarrow \quad \overline{u} = au + b \tag{9}$$

♦

2.23 p47-exercise

Consider a 3-space with coordinates x, y, z and a metric form $\Phi = (dx)^2 + (dy)^2 - (dz)^2$. prove that the geodesic null lines may be represented by the equations

$$x = au + a^{'} \quad y = bu + b^{'} \quad z = cu + c^{'}$$

where u is a parameter and $a, a^{'}, b, b^{'}, c, c^{'}$ are constants which are arbitrary except for the relation $a^2 + b^2 - c^2 = 0$.

Given is

$$\Phi = (dx)^2 + (dy)^2 - (dz)^2 \tag{1}$$

From the previous exercise we have already proven that x, y, z are of the form

$$x^i = q_i u + q_i^{'} \tag{2}$$

To be a null geodesic null line we need to have (2.448)

$$a_{mn} \frac{dx^m}{du} \frac{dx^n}{du} = 0 \tag{3}$$

$$\text{from (1) we deduce} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4}$$

$$(3) \quad \Rightarrow \quad (dx)^2 + (dy)^2 - (dz)^2 = 0 \tag{5}$$

$$(2) \quad \Rightarrow \quad (q_1)^2 + (q_2)^2 - (q_3)^2 = 0 \tag{6}$$

♦

2.24 p48-exercise

Prove that the Christoffel symbols of the first kind transform according the equation

$$[mn,r]' = [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^s}{\partial x'^r} + a_{pq}\frac{\partial x^p}{\partial x'^r}\frac{\partial^2 x^q}{\partial x'^m \partial x'^n}$$

From 2.438 page 45, we have

$$f_r \equiv a_{rm}\frac{d^2 x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \quad \text{is covariant} \tag{1}$$

$$\Rightarrow \quad f_r' = f_s \frac{\partial x^s}{\partial x'^r} \tag{2}$$

$$\text{with} \quad f_r' = a_{rm}'\frac{d^2 x'^m}{du^2} + [mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} \tag{3}$$

Combining (1), (2) and (3) gives

$$a_{rm}'\frac{d^2 x'^m}{du^2} + [mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} = (a_{sm}\frac{d^2 x^m}{du^2} + [mn,s]\frac{dx^m}{du}\frac{dx^n}{du})\frac{\partial x^s}{\partial x'^r} \tag{4}$$

We rewrite (4) as

$$[mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} = -\underbrace{a_{rm}'\frac{d^2 x'^m}{du^2}}_{(*)} + \underbrace{a_{sm}\frac{d^2 x^m}{du^2}\frac{\partial x^s}{\partial x'^r}}_{(**)} + \underbrace{[mn,s]\frac{dx^m}{du}\frac{dx^n}{du}\frac{\partial x^s}{\partial x'^r}}_{(***)} \tag{5}$$

$$(***) \Leftrightarrow [mn,s]\frac{\partial x^m}{\partial x'^p}\frac{dx'^p}{du}\frac{\partial x^n}{\partial x'^q}\frac{dx'^q}{du}\frac{\partial x^s}{\partial x'^r} \tag{6}$$

In (6) renaming the dummy indices m, n, p, q gives

$$(***) \Leftrightarrow [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{dx'^m}{du}\frac{\partial x^q}{\partial x'^n}\frac{dx'^n}{du}\frac{\partial x^s}{\partial x'^r} \tag{7}$$

$$\Leftrightarrow [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^s}{\partial x'^r}\left(\frac{dx'^m}{du}\frac{dx'^n}{du}\right) \tag{8}$$

Also,

$$(**) \Leftrightarrow a_{sm}\frac{d^2 x^m}{du^2}\frac{\partial x^s}{\partial x'^r} \tag{9}$$

$$\text{As we have also} \quad \frac{d^2 x^m}{du^2} = \frac{d(\frac{\partial x^m}{\partial x'^p}\frac{dx'^p}{du})}{du} \tag{10}$$

$$= \frac{\partial x^m}{\partial x'^p}\frac{d^2 x'^p}{du^2} + \frac{dx'^p}{du}\frac{\partial^2 x^m}{\partial x'^p \partial x'^q}\frac{dx'^q}{du} \tag{11}$$

(11) and (9) gives by changing the dummy indices ($m \to t$, $p \to m$, $q \to n$)

$$(**) = \underbrace{a_{st} \frac{\partial x^t}{\partial x'^p} \frac{d^2 x'^p}{du^2} \frac{\partial x^s}{\partial x'^r}}_{(****)} + a_{pq} \frac{\partial^2 x^q}{\partial x'^m \partial x'^n} \frac{\partial x^p}{\partial x'^r} \left(\frac{dx'^m}{du} \frac{dx'^m}{du} \right) \tag{12}$$

$$\text{with } (****) \quad = a_{st} \left(\frac{\partial x^t}{\partial x'^m} \frac{\partial x^s}{\partial x'^r} \right) \frac{d^2 x'^m}{du^2} \tag{13}$$

But a_{st} is a covariant tensor, so

$$a'_{rm} = a_{st} \frac{\partial x^t}{\partial x'^m} \frac{\partial x^s}{\partial x'^r} \tag{14}$$

$$\text{(13) becomes} \quad (****) = a'_{rm} \frac{d^2 x'^m}{du^2} \tag{15}$$

$$\text{and from (5) we have} \quad (*) = -a'_{rm} \frac{d^2 x'^m}{du^2} \tag{16}$$

$$\tag{17}$$

and both terms cancel each other in equation (5). So adding $(*), (**)$ and $(***)$ in (5) , we get

$$[mn, r]' \frac{dx'^m}{du} \frac{dx'^n}{du} = a_{pq} \frac{\partial^2 x^q}{\partial x'^m \partial x'^n} \frac{\partial x^p}{\partial x'^r} \left(\frac{dx'^m}{du} \frac{dx'^m}{du} \right) + [pq, s] \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n} \frac{\partial x^s}{\partial x'^r} \left(\frac{dx'^m}{du} \frac{dx'^n}{du} \right) \tag{18}$$

$$\Rightarrow [mn, r]' = [pq, s] \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n} \frac{\partial x^s}{\partial x'^r} + a_{pq} \frac{\partial x^p}{\partial x'^r} \frac{\partial^2 x^q}{\partial x'^m \partial x'^n} \tag{19}$$

♦

2.25 p50-clarification 2.515

$$\frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du}S^r + T_r\frac{dS^r}{du} = \left(\frac{dT_r}{du} - \Gamma^m_{rn}T_m\frac{dx^n}{du}\right)S^r$$

with

$$\frac{\delta T_r}{\delta u} = \frac{dT_r}{du} - \Gamma^m_{rn}\frac{dx^n}{du}T_m$$

a covariant vector.

It is given that S^r is a tensor propagated parallelly along the curve. Then by (**2.5212**) we have

$$\frac{dS^r}{du} + \Gamma^r_{mn}S^m\frac{dx^n}{du} = 0 \qquad (1)$$

$$\frac{dS^r}{du} = -\Gamma^r_{mn}S^m\frac{dx^n}{du} \qquad (2)$$

$$\text{and} \quad \frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du}S^r + T_r\frac{dS^r}{du} \qquad (3)$$

$$= \frac{dT_r}{du}S^r - \Gamma^r_{mn}S^m\frac{dx^n}{du}T_r \qquad (4)$$

$$(5)$$

Swap dummy indices r and m in the second term:

$$\frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du}S^r - \Gamma^m_{rn}S^r\frac{dx^n}{du}T_m \qquad (6)$$

$$= \left(\frac{dT_r}{du} - \Gamma^m_{rn}\frac{dx^n}{du}T_m\right)S^r \qquad (7)$$

$$(8)$$

As $T_r S^r$ is an invariant and thus is also $\frac{d(T_r S^r)}{du}$ and as S^r can be chosen arbitrarily (as long it is a contravariant tensor), implies that

$$\frac{dT_r}{du} - \Gamma^m_{rn}\frac{dx^n}{du}T_m$$

is covariant and thus also

$$\frac{\delta T_r}{\delta u} = \frac{dT_r}{du} - \Gamma^m_{rn}\frac{dx^n}{du}T_m$$

♦

2.26 p50-clarification 2.516

$$\frac{\delta T_{rs}}{\delta u} \equiv \frac{dT_{rs}}{du} - \Gamma^m_{rn} T_{ms} \frac{dx^n}{du} - \Gamma^m_{sn} T_{rm} \frac{dx^n}{du}$$

is a covariant vector.

We build an invariant $T_{rs}S^rU^s$ with S^r and U^s arbitrary contravariant tensors. Then we know that $\frac{d(T_{rs}S^rU^s)}{du}$ is also an invariant. We have

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = \frac{dT_{rs}}{du} S^rU^s + T_{rs}\frac{dS^r}{du}U^s + T_{rs}S^r\frac{dU^s}{du} \tag{1}$$

with with S^r and U^s propagated parallelly along the curve. Then,

$$\frac{dS^r}{du} = -\Gamma^r_{mn} S^m \frac{dx^n}{du} \tag{2}$$

$$\frac{dU^s}{du} = -\Gamma^s_{mn} Y^m \frac{dx^n}{du} \tag{3}$$

(2), (3) in (1) gives

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = \frac{dT_{rs}}{du} S^rU^s - T_{rs}U^s\Gamma^r_{mn}S^m\frac{dx^n}{du} - T_{rs}S^r\Gamma^s_{mn}U^m\frac{dx^n}{du} \tag{4}$$

Changing the dummy indices in the second and third term gives:

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = (\frac{dT_{rs}}{du} - \Gamma^m_{rn}T_{ms}\frac{dx^n}{du} - \Gamma^m_{sn}T_{rm}\frac{dx^n}{du})S^rU^s \tag{5}$$

As the left term is an invariant and S^r and U^s are arbitrary contravariant tensors, means that the expression in the brackets in the right part of the equation, is a covariant tensor.

♦

2.27 p51-exercise

Find the absolute derivative of T^r_{st}.

Define the invariant $I = D^r_{st} R^r S_s T_t$

$$I = D^r_{st} R^r S_s T_t \tag{1}$$
$$\Rightarrow A = \frac{dI}{du} = \frac{d\left(D^r_{st}\right)}{du} R^r S_s T_t + \ D^r_{st} S_s T_t \frac{d\left(R^r\right)}{du} + \ D^r_{st} R_r T_t \frac{d\left(S^s\right)}{du} + \ D^r_{st} R_r S_s \frac{d\left(T^t\right)}{du} \tag{2}$$

Reminder, performing a parallel propagation of a covariant and contravariant vector gives as equations

$$\frac{dV^v}{du} = -\Gamma^v_{mn} V^m \frac{dx^n}{du} \tag{3}$$
$$\frac{dW_w}{du} = +\Gamma^m_{wn} W^m \frac{dx^n}{du} \tag{4}$$

So (2) becomes:

$$A = \begin{cases} \frac{d(D^r_{st})}{du} R^r S_s T_t \\ \\ -D^r_{st} S_s T_t \Gamma^r_{mn} R^m \frac{dx^n}{du} \\ \\ +D^r_{st} R_r T_t \Gamma^m_{sn} S^m \frac{dx^n}{du} \\ \\ +D^r_{st} R_r S_s \Gamma^m_{tn} T^m \frac{dx^n}{du} \end{cases} \tag{5}$$

In (5) apply the following renaming of dummy variables

$$\begin{cases} 2^{nd} line: & r \to m, m \to r \\ 3^{rd} line: & s \to m, m \to s \\ 4^{th} line: & t \to m, m \to t \end{cases}$$

and regrouping terms with $R^r S_s T_t$, (5) becomes then

$$A = \left[\frac{d\left(D^r_{st}\right)}{du} + (\ D^r_{mt} \Gamma^s_{mn} + \ D^r_{sm} \Gamma^t_{mn} - \ D^m_{st} \Gamma^m_{rn}) \frac{dx^n}{du} \right] R^r S_s T_t$$

But A is an invariant, so the expression in the square parenthesis is a tensor of the form T^r_{st} and we define the absolute derivative of T^r_{st} as:

$$\frac{\delta T^r_{st}}{\delta u} = \frac{d\left(T^r_{st}\right)}{du} + \Gamma^s_{mn} T^r_{mt} \frac{dx^n}{du} + \Gamma^t_{mn} T^r_{sm} \frac{dx^n}{du} - \Gamma^m_{rn} T^m_{st} \frac{dx^n}{du}$$

♦

2.28 p53-exercise

Prove that

$$\delta^r_{s|t} = 0, \quad a^{rs}_{|t} = 0$$

i) $\delta^t_{s|t} = 0$

$$\text{(2.524) gives:} \quad \delta^r_{s|t} = \underbrace{\frac{\partial \delta^r_s}{\partial x^t}}_{=0} + \Gamma^r_{mt}\delta^m_s - \Gamma^m_{st}\delta^r_m \tag{1}$$

$$= \Gamma^r_{st} - \Gamma^r_{st} = 0 \tag{2}$$

ii) $a^{rs}_{|t} = 0$ We know that

$$\delta^r_{s|t} = a_{sk}a^{kr}|_t \tag{3}$$

$$\Rightarrow \delta^r_{s|t} = \frac{\partial a_{sk}}{\partial x^t}a^{kr} + a_{sk}\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a_{sk}a^{km} - \Gamma^m_{st}a_{mk}a^{kr} \tag{4}$$

Rearrange (4) and add $\Gamma^k_{mt}a_{ks}a^{mr}$ and subtract $\Gamma^m_{kt}a_{ms}a^{kr}$ (as $\Gamma^k_{mt}a_{ks}a^{mr} - \Gamma^m_{kt}a_{ms}a^{kr} = 0$)

$$\delta^r_{s|t} = (\frac{\partial a_{sk}}{\partial x^t} - \Gamma^m_{st}a_{mk} - \Gamma^m_{kt}a_{ms})a^{kr} + (\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a^{km} + \Gamma^k_{mt}a^{mr})a_{sk} \tag{5}$$

$$\text{but} \quad a_{sk|t} = (\frac{\partial a_{sk}}{\partial x^t} - \Gamma^m_{st}a_{mk} - \Gamma^m_{kt}a_{ms}) \tag{6}$$

$$\text{and as (2.526)} \quad a_{sk|t} = 0 \tag{7}$$

$$\text{(5) becomes} \quad \delta^r_{s|t} = \underbrace{(\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a^{km} + \Gamma^k_{mt}a^{mr})}_{a^{kr}_{|t}}a_{sk} \tag{8}$$

$$= a^{kr}_{|t}a_{sk} \tag{9}$$

$$= 0 \quad \text{as} \quad \delta^r_{s|t} = 0 \quad \text{(see first part of this exercise)} \tag{10}$$

As all a_{ks} can't be zero and as we didn't choose any special Riemannian space, we can conclude from $a^{kr}_{|t}a_{sk} = 0$ that

$$a^{rs}_{|t} = 0$$

♦

2.29 p54-exercise

Prove that

$$\frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s}$$

$$\frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{d\lambda^n}{ds} \tag{1}$$

By definition of the absolute derivative, we have:

$$\frac{\delta\lambda^n}{\delta s} = \frac{d\lambda^n}{ds} + \Gamma^n_{pk}\lambda^p\frac{dx^k}{ds} \tag{2}$$

$$\text{(2) in (1)}\quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m(\frac{\delta\lambda^n}{\delta s} - \Gamma^n_{pk}\lambda^p\frac{dx^k}{ds}) \tag{3}$$

$$= \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2a_{mn}\lambda^m\Gamma^n_{pk}\lambda^p\frac{dx^k}{ds} \tag{4}$$

$$\text{we have}\quad \Gamma^n_{pk} = a^{ns}[pk,s] \tag{5}$$

$$\text{so,}\quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2\underbrace{a_{mn}a^{ns}}_{=\delta^s_m}[pk,s]\lambda^m\lambda^p\frac{dx^k}{ds} \tag{6}$$

$$= \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2[pk,m]\lambda^m\lambda^p\frac{dx^k}{ds} \tag{7}$$

$$\text{but}\quad 2[pk,m]\lambda^m\lambda^p = [pk,m]\lambda^m\lambda^p + [pk,m]\lambda^m\lambda^p \tag{8}$$

$$= [pk,m]\lambda^m\lambda^p + [mk,p]\lambda^m\lambda^p \tag{9}$$

$$\text{we have also}\quad \begin{cases} [pk,m] = \frac{1}{2}(\partial_k a_{pm} + \partial_p a_{km} - \partial_m a_{pk}) \\ [mk,p] = \frac{1}{2}(\partial_k a_{pm} + \partial_m a_{pk} - \partial_p a_{mk}) \end{cases} \tag{10}$$

$$\Rightarrow 2[pk,m] = \partial_k a_{pm} \tag{11}$$

$$\text{so}\quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n - \underbrace{\partial_k a_{pm}\frac{dx^k}{ds}}_{=\frac{da_{mn}}{ds}}\lambda^m\lambda^p + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} \tag{12}$$

$$\Rightarrow\quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} \tag{13}$$

♦

2.30 p54-exercise

Prove that

$$(T^r S_s)_{|n} = T^r_{|n} S_s + T^r S_{|n}$$

$$\begin{aligned}
(T^r S_s)_{|n} &= \partial_n (T^r S_s) + \Gamma^r_{nm} T^m S_s - \Gamma^m_{sn} T^r S_m && (1)\\
&= \partial_n (T^r) S_s + T^r \partial_n (S_s) + S_s \Gamma^r_{nm} T^m - T^r \Gamma^m_{sn} S_m && (2)\\
&= T^r \underbrace{(\partial_n (S_s) - \Gamma^m_{sn} S_m)}_{S_{s|n}} + S_s \underbrace{(\partial_n (T^r) + \Gamma^r_{nm} T^m)}_{T^r_{|n}} && (3)\\
&= T^r S_{s|n} + S_s T^r_{|n} && (4)
\end{aligned}$$

♦

2.31 p57-exercise

Compute the Christoffel symbols in 2.540 directly from the definitions 2.421 and 2.422. Check that all Christoffels symbols not shown explicitly in 2.540 vanish.

Easy but very tedious, not reproduced yet, later perhaps

♦

2.32 p57-exercise

Show that for the spherical polar metric 2.532, we have $ln\sqrt{a} = 2ln(x^1) + ln(sin(x^2))$ and

$$\textbf{2.544} \quad \Gamma^n_{1n} = \frac{2}{x^1}, \quad \Gamma^n_{2n} = \cot\big(x^2\big), \quad \Gamma^n_{3n} = 0$$

The spherical polar metric 2.532 is,

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1\sin\big(x^2\big))^2 \end{pmatrix} \tag{1}$$

$$\Rightarrow |a_{mn}| = \big[(x^1)^2\sin\big(x^2\big)\big]^2 \tag{2}$$

$$\Rightarrow ln(\sqrt{|a_{mn}|}) = 2ln(x^1) + ln(\sin\big(x^2\big)) \tag{3}$$

$$\Rightarrow \quad \begin{cases} \Gamma^n_{1n} = \partial_1(ln(\sqrt{a})) = \frac{2}{x^1} \\ \\ \Gamma^n_{2n} = \partial_2(ln(\sqrt{a})) = \frac{\cos\big(x^2\big)}{\sin(x^2)} = \cot\big(x^2\big) \\ \\ \Gamma^n_{3n} = 0 \end{cases} \tag{4}$$

♦

2.33 p58-exercise

Show that for the spherical polar metric

$$\textbf{2.546} \quad T^n_{\ |n} = \frac{1}{r^2}\partial_r(r^2\, T^1) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\, T^2) + \partial_\phi T^3$$

Obtain a similar expression for the "Laplacian" ΔV of an invariant V defined as

$$\textbf{2.547} \quad \Delta V = (a^{mn}\partial_m V)_{|n}$$

We have

$$\textbf{(2.545)} \quad T^n_{\ |n} = \frac{1}{\sqrt{a}}\partial_n(\sqrt{a}\, T^n) \tag{1}$$

$$\text{and from the previous exercise p.58:} \quad \sqrt{a} = (x^1)^2\sin\big(x^2\big) \tag{2}$$

$$\Rightarrow \quad T^n_{\ |n} = \frac{1}{\sqrt{a}}\left(\sin\big(x^2\big)\partial_1[(x^1)^2T^1] + (x^1)^2\partial_2[\sin\big(x^2\big)T^2] + (x^1)^2\sin\big(x^2\big)\partial_3T^3\right) \tag{3}$$

$$= \frac{1}{x^1}\partial_1[(x^1)^2T^1] + \frac{1}{\sin(x^2)}\partial_2[\sin\big(x^2\big)T^2] + \partial_3T^3 \tag{4}$$

Replace in (4) $\quad x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$

$$T^n_{\ |n} = \frac{1}{r^2}\partial_r[r^2\, T^r] + \frac{1}{\sin\theta}\partial_\theta[\sin\theta\, T^\theta] + \partial_\phi T^\phi \tag{5}$$

Let's calculate the Laplacian.

$$\Delta V = (a^{mn}\partial_m V)_{|n} \tag{6}$$

$$\text{be} \qquad G^n = a^{mn}\partial_m V \tag{7}$$

$$\text{then (see exercise p.32)} \qquad \begin{cases} G^1 = \partial_r V \\ G^2 = \frac{1}{r^2}\partial_\theta V \\ G^1 = \frac{1}{r^2\sin^2\theta}\partial_\phi V \end{cases} \tag{8}$$

and by the previous result of this exercise

$$\Delta V = G^n_{|n} = \frac{1}{r^2}\partial_r[r^2\, G^1] + \frac{1}{\sin\theta}\partial_\theta[\sin\theta\, G^2] + \partial_\phi G^3 \tag{9}$$

$$= \frac{1}{r^2}\partial_r[r^2\partial_r V] + \frac{1}{r^2\sin\theta}\partial_\theta[\sin\theta\ \partial_\theta V] + \frac{1}{r^2\sin^2\theta}\partial^2_\phi V \tag{10}$$

♦

2.34 p60 - clarification for 2.609

$$A^r_{.mns} = -\partial_s \Gamma^r_{mn} + 2\Gamma^r_{sp}\Gamma^p_{mn}$$

We have (2.608):

$$\frac{d^2x^r}{ds^2} = -\Gamma^r_{mn} p^m p^n \tag{1}$$

$$\Rightarrow \quad \frac{d^3x^r}{ds^3} = -\frac{d\Gamma^r_{mn}}{ds} p^m p^n - \Gamma^r_{mn}\left(p^m \frac{dp^n}{ds} + p^n \frac{dp^m}{ds}\right) \tag{2}$$

$$= -\partial_k \Gamma^r_{mn} \underbrace{\frac{dx^k}{du}}_{=p^k} p^m p^n - \Gamma^r_{mn}\left(p^m \frac{dp^n}{ds} + p^n \frac{dp^m}{ds}\right) \tag{3}$$

$$\text{as} \quad \frac{dp^g}{ds} = \frac{d^2x^g}{ds^2} = -\Gamma^g_{ik} p^i p^k \quad \Rightarrow \quad = -\partial_k \Gamma^r_{mn} \underbrace{\frac{dx^k}{ds}}_{=p^k} p^m p^n + \Gamma^r_{mn} p^m \Gamma^n_{ik} p^i p^k + \Gamma^r_{mn} p^n \Gamma^m_{ki} p^k p^i \tag{4}$$

In the second and third terms, rename the dummy indices : $m \leftrightarrow k, n \leftrightarrow i$ and $m \leftrightarrow i, n \leftrightarrow k$.
So (4) becomes

$$\frac{d^3x^r}{ds^3} = -\partial_k \Gamma^r_{mn} p^m p^n p^k + \Gamma^r_{ki} p^k \Gamma^i_{nm} p^n p^m + \Gamma^r_{ik} p^k \Gamma^i_{nm} p^n p^m \tag{5}$$

$$= \left(-\partial_k \Gamma^r_{mn} + \Gamma^r_{ik}\Gamma^i_{mn} + \Gamma^r_{ik}\Gamma^i_{nm}\right) p^m p^n p^k \tag{6}$$

$$= \left(-\partial_k \Gamma^r_{mn} + 2\Gamma^r_{ik}\Gamma^i_{mn}\right) p^m p^n p^k \tag{7}$$

♦

2.35 p62-exercise

Prove that if a pair of vectors are unit orthogonal vectors at a point on a curve, and if they are both propagated parallelly along the curve, then they remain unit orthogonal vectors along the curve.

Given is, a pair of vectors U^m and V^m which are unit orthogonal vectors at a point on a curve. So,

$$\begin{array}{rl} \text{U is a unit vector (2.302)} & a_{mn}U^mU^n = \epsilon \\ \text{V is a unit vector (2.302)} & a_{mn}V^mV^n = \epsilon \\ \text{U, V are orthogonal (2.317)} & a_{mn}U^mV^n = 0 \end{array} \tag{1}$$

at one point on the curve.
We have to prove that the above properties are valid along the curve (i.e. $\forall$ points on the curve) provided that the vectors are propagated // along the curve, which means

$$\textbf{(2.512)} \qquad \frac{\delta U^r}{\delta u} = \frac{dU^r}{du} + \Gamma^r_{mn}U^m\frac{dx^n}{du} = 0 \tag{2}$$

for both vectors U, V. Thus,

$$\frac{dU^r}{du} = -\Gamma^r_{mn}U^m\frac{dx^n}{du} \tag{3}$$

i) Consider the magnitude M at a random point on the curve

$$M = a_{mn}U^mU^n \tag{4}$$

$$\Rightarrow \qquad \frac{dM}{ds} = \frac{da_{mn}}{ds}U^mU^n + 2a_{mn}U^m\frac{dU^n}{ds} \tag{5}$$

Obviously M and $\frac{dM}{ds}$ are invariants. Also, we can choose at any point on the curve a Riemannian coordinate system (RCS) for which the Christoffel symbols vanish at that point. Hence, $\frac{dU^r}{ds} = 0$ at that point and the second term in the right part of (5) vanish. (5) becomes then,

$$\frac{dM}{ds} = \frac{\partial a'_{mn}}{\partial x'^k}\frac{dx'^k}{ds}U'^mU'^n \tag{6}$$

We also know **(2.425. page 41)** that $[km,n]' + [kn,m]' = \frac{\partial a'_{mn}}{\partial x'^k}$. But in the chosen coordinate system, $[km,n] = 0$ at the origin of this coordinate system. So by (6) we get $\frac{dM}{ds} = 0$.
So the magnitude is constant along the curve and as we know that a certain point $M = 1$:

U,V are unit vectors along the curve

ii) Consider now the angle between the vectors U, V. Be $A = \cos\ \theta$. By definition

$$A = a_{mn} U^m V^n \tag{7}$$

$$\Rightarrow \qquad \frac{dA}{ds} = \frac{da_{mn}}{du} U^m V^n + a_{mn} (V^m \frac{dU^n}{du} + U^m \frac{dV^n}{du}) \tag{8}$$

We follow the same reasoning as in i) and so

$$\frac{dA}{ds} = 0$$

So, the angle is constant and we know it is $\frac{\pi}{2}$ at a certain point. So,

U,V are orthogonal along the curve

♦

2.36 p62-exercise

Given that λ^r is a unit vector field, prove that

$$\lambda^r{}_{|s}\lambda_r = 0 \quad \text{and} \quad \lambda^r\lambda_{r|s} = 0$$

Is the relation $\lambda^r{}_{|s}\lambda_s = 0$ true for a general unit vector field?

To simplify the calculation, we choose a random element in the unit vector field and use at that point a Riemannian coordinate system (RCS). So, we have

$$\lambda^r{}_{|s} = \partial_s\lambda^r \quad \text{and} \quad \lambda_{r|s} = \partial_s\lambda_r \tag{1}$$

$$\text{as we have a unit vector fields:} \qquad a_{mn}\lambda^m\lambda^n = 1 \tag{2}$$

$$\Leftrightarrow \lambda_n\lambda^n = 1 \qquad \text{(by lowering the index m)} \tag{3}$$

$$\Rightarrow \qquad \lambda^n\partial_s\lambda_n + \lambda_n\partial_s\lambda^n = 0 \tag{4}$$

We prove that

$$\lambda^n\partial_s\lambda_n = \lambda_n\partial_s\lambda^n \quad \forall \text{ vector fields}$$

We have the trivial identity

$$\partial_s(\lambda^r\lambda_r) = \partial_s(\lambda^r\lambda_r) \tag{5}$$

$$\Leftrightarrow \qquad \partial_s(a^{rm}\lambda_m\lambda_r) = \partial_s(a_{rm}\lambda^m\lambda^r) \tag{6}$$

Lemma *: $\partial_s a^{rm} = 0$ in a Riemannian coordinate system (i.e. at the origin)*

We have

$$a^{rm}a_{ms} = \delta^r_s \tag{7}$$

$$\Rightarrow \qquad a_{ms}\partial_k a^{rm} + a^{rm}\partial_k a_{ms} = 0 \tag{8}$$

$$\textit{we know (2.618)} \qquad \partial_r a_{mn} = 0 \qquad \textit{at the origin of a RCS} \tag{9}$$

$$\textit{so (8) becomes} \qquad a_{ms}\partial_k a^{rm} = 0 \tag{10}$$

$$\textit{multiply (10) by} \quad a^{ns} \quad \Rightarrow \qquad \underbrace{a^{ns}a_{ms}}_{=\delta^n_m}\partial_k a^{rm} = 0 \tag{11}$$

$$\Rightarrow \qquad \partial_k a^{rn} = 0 \tag{12}$$

◇

Now, expanding (6) and using **2.618** and the lemma:

$$a^{rm}\lambda_m\partial_s\lambda_r + a^{rm}\lambda_r\partial_s\lambda_m = a_{rm}\lambda^m\partial_s\lambda^r + a_{rm}\lambda^r\partial_s\lambda^m \tag{13}$$

$$\text{renaming dummy indices:} \qquad a^{rm}\lambda_r\partial_s\lambda_m = a_{rm}\lambda^r\partial_s\lambda^m \tag{14}$$

$$\Rightarrow \qquad \lambda^m\partial_s\lambda_m = \lambda_m\partial_s\lambda^m \tag{15}$$

Considering (5) and (15) we conclude:

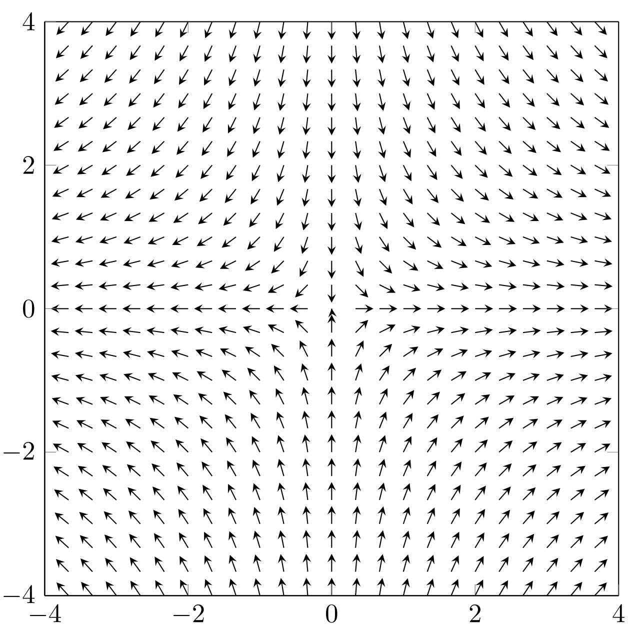
$$\lambda^n\partial_s\lambda_n = \lambda_n\partial_s\lambda^n = 0 \tag{16}$$

and as $\partial_s\lambda_n = \lambda_{n|s}$ and $\partial_s\lambda^n = \lambda^n{}_{|s}$ at the origin of the considered coordinate system, we have:

$$\lambda^m\lambda_{n|s} = \lambda_m\lambda^n{}_{|s} = 0 \tag{17}$$

◇

Is the relation $\lambda^r_{|s}\lambda_s = 0$ true for a general unit vector field?
The answer is NO. Let's consider the following unit vector field in a Cartesian Coordinate system:



$$V : \mathbb{R}^2_* \to \mathbb{R}^2 | V(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

Figure 2.4: Vector field for which $\lambda^r_{|s}\lambda_s = 0$ does not hold

Put $r = \sqrt{x^2+y^2}$, we get (as we have a Cartesian Coordinate system, the Christoffel symbols vanish and the covariant components of the vectors are equal to their contravariant part):

$$\begin{cases} V^1 = V_1 = +\frac{x}{r} \\ V^2 = V_2 = -\frac{y}{r} \end{cases} \tag{18}$$

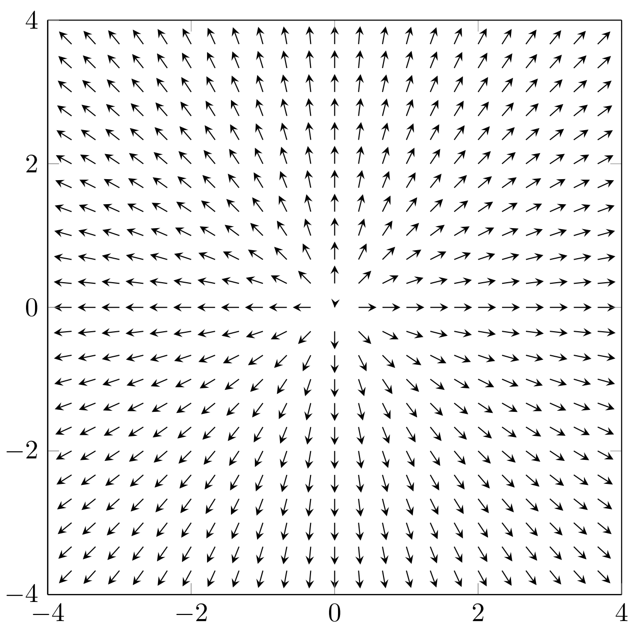
$$\begin{cases} V^1_{|1} = V_{1|1} = \frac{y^2}{r^3} \quad V^1_{|2} = V_{1|2} = -\frac{xy}{r^3} \\ A\ V^2_{|1} = V_{2|1} = \frac{xy}{r^3} \quad V^2_{|2} = V_{2|2} = -\frac{x^2}{r^3} \end{cases} \tag{19}$$

$$\Rightarrow \quad \begin{cases} V^1_{|s}V_s = V^1_{|1}V_1 + V^1_{|2}V_2 = \frac{y^2}{r^3}\frac{x}{r} + (-\frac{xy}{r^3})(-\frac{y}{r}) = \frac{xy^2}{r^4} \neq 0 \\ V^2_{|s}V_s = V^2_{|1}V_1 + V^2_{|2}V_2 = \frac{xy}{r^3}\frac{x}{r} + (-\frac{y}{r})(-\frac{y}{r}) = \frac{x^2y}{r^4} \neq 0 \end{cases} \tag{20}$$

Just as a check, we calculate $V^r_{|s}V_r$ which should be zero:

$$\Rightarrow \quad \begin{cases} V^s_{|1}V_s = V^1_{|1}V_1 + V^2_{|1}V_2 = (+\frac{y^2}{r^3})\frac{x}{r} + (+\frac{xy}{r^3})(-\frac{y}{r}) = 0 \\ V^s_{|2}V_s = V^1_{|2}V_1 + V^2_{|2}V_2 = (-\frac{xy}{r^3})\frac{x}{r} + (-\frac{y}{r})(-\frac{x^2}{r^3}) = 0 \end{cases} \tag{21}$$

Now, let's consider another unit vector field in a Cartesian Coordinate system:



$$V : \mathbb{R}^2_* \to \mathbb{R}^2 | V(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

Figure 2.5: Vector field for which $\lambda^r_{|s}\lambda_s = 0$ hold

$$\begin{cases} V^1 = V_1 = +\frac{x}{r} \\ V^2 = V_2 = +\frac{y}{r} \end{cases} \tag{22}$$

$$\begin{cases} V^1_{\ |1} = V_{1|1} = \frac{y^2}{r^3} & V^1_{\ |2} = V_{1|2} = -\frac{xy}{r^3} \\ V^2_{\ |1} = V_{2|1} = -\frac{xy}{r^3} & V^2_{\ |2} = V_{2|2} = +\frac{x^2}{r^3} \end{cases} \tag{23}$$

$$\Rightarrow \quad \begin{cases} V^1_{\ |s}V_s = V^1_{\ |1}V_1 + V^1_{\ |2}V_2 = (+\frac{y^2}{r^3})\frac{x}{r} + (-\frac{xy}{r^3})(+\frac{y}{r}) = 0 \\ V^2_{\ |s}V_s = V^2_{\ |1}V_1 + V^2_{\ |2}V_2 = (-\frac{xy}{r^3})\frac{x}{r} + (+\frac{y}{r})(+\frac{y}{r}) = 0 \end{cases} \tag{24}$$

Just as a check, we calculate $V^r_{\ |s}V_r$ which should be zero:

$$\Rightarrow \quad \begin{cases} V^s_{\ |1}V_s = V^1_{\ |1}V_1 + V^2_{\ |1}V_2 = (+\frac{y^2}{r^3})(+\frac{x}{r}) + (-\frac{xy}{r^3})(+\frac{y}{r}) = 0 \\ V^s_{\ |2}V_s = V^1_{\ |2}V_1 + V^2_{\ |2}V_2 = (-\frac{xy}{r^3})(+\frac{x}{r}) + (+\frac{y}{r}) + \frac{x^2}{r^3}) = 0 \end{cases} \tag{25}$$

So, in the second example the relationship $\lambda^r_{\ |s}\lambda_s = 0$ holds.
Question (to investigate further and later) : does the fact that in the first case $\boldsymbol{\nabla} \times \overline{V} \neq 0$ and in the second case $\boldsymbol{\nabla} \times \overline{V} = 0$, means that there is some relation with this expression?

♦

2.37 p64-clarification 2.625

2.625 $\frac{dx^r}{dx^N} = \frac{X^r}{X^N}$

Be C a surface defined by the function $F(x^1, \ldots, x^{N-1}) = C$ and $c_\perp$ the curve intersecting the surface C perpendicularly at a point p.
Along the curve at that point p we have

$$\text{as} \quad \begin{cases} \frac{dx^r}{ds} & \text{is the tangent vector along} \quad c_\perp \\ X^r = a^{rn}\frac{\partial F}{\partial x^n} & \text{is orthogonal on the surface (2.623)} \quad C \end{cases}$$
$$\Rightarrow \quad \frac{dx^r}{ds} = kX^r$$

So, $\frac{dx^r}{ds}$ is proportional to X^r (as the curve intersects the surface orthogonally). This means that also all the components (coordinates) of both quantities are proportional. And so,

$$\frac{\frac{dx^r}{ds}}{\frac{dx^N}{ds}} = \frac{kX^r}{kX^N}$$
$$\Rightarrow \quad \frac{dx^r}{dx^N} = \frac{X^r}{X^N}$$

♦

2.38 p65-exercise

Deduce from **2.629** that

$$a^{N\rho} = 0 \qquad a^{NN} = \frac{1}{a_{NN}}$$

We have (see 2.629):

$$a_{N\rho} = 0 \tag{1}$$

$$\text{and also} \qquad a_{Nm}a^{ms} = \delta^s_N \tag{2}$$

In (2) split the m index in the subspace and the remaining coordinate N

$$a_{N\rho}a^{\rho s} + a_{NN}a^{Ns} = \delta^s_N$$

$$\text{as} \quad a_{N\rho} = 0 \Rightarrow \qquad a_{NN}a^{Ns} = \delta^s_N$$

Case 1: $s \neq N$

$$a_{NN}a^{Ns} = 0 \quad \Leftrightarrow \quad a_{NN}a^{N\rho} = 0 \qquad (\text{as} \quad s \neq N)$$

$$\text{as we suppose} \quad a_{NN} \neq 0 \quad \Rightarrow \quad a^{N\rho} = 0$$

Case 2: $s = N$

$$a_{NN}a^{NN} = 1$$

$$\Rightarrow \quad a^{NN} = \frac{1}{a_{NN}}$$

♦

2.39 p69-clarification on 2.645

In 2.645 we have

$$T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\alpha\beta} T_N$$

and

$$T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N$$

Indeed,

$$\begin{aligned} T^N &= a^{mN} T_m \\ &= a^{\alpha N} T_\alpha + a^{NN} T_N \\ \text{but (2.631)} \qquad a^{\alpha N} &= 0 \\ \Rightarrow \qquad T^N &= a^{NN} T_N \\ \text{as} \quad a^{NN} = \frac{1}{a_{NN}} \quad \Rightarrow \qquad T_{\alpha|\beta} &= T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N \end{aligned}$$

♦

2.40 p69-exercise

Show that

$$\textbf{2.648} \qquad T^{\alpha}_{\ |\beta} = T^{\alpha}_{\ ||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N$$

$$\textbf{2.649} \qquad T^{N}_{\ |\alpha} = \partial_\alpha T^N - \frac{1}{2a_{NN}}\partial_N a_{\alpha\mu}T^\mu + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$$

$$\textbf{2.650} \qquad T^{\alpha}_{\ |N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$$

i) $T^{\alpha}_{\ |\beta} = T^{\alpha}_{\ ||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N$

$$\textbf{(2.520)} \qquad T^{\alpha}_{\ |\beta} = \partial_\beta T^\alpha + \Gamma^\alpha_{m\beta}T^m \qquad (m = 1, \ldots, N) \tag{1}$$

$$\Leftrightarrow \qquad T^{\alpha}_{\ |\beta} = \underbrace{\partial_\beta T^\alpha + \Gamma^\alpha_{\mu\beta}T^\mu}_{T^{\alpha}_{\ ||\beta}} + \Gamma^\alpha_{N\beta}T^N \tag{2}$$

$$\textbf{(2.639)} \qquad \Gamma^\alpha_{N\beta} = \frac{1}{2}a^{\alpha\mu}\partial_N a^{\mu\beta} \tag{3}$$

$$\text{(2) and (3):} \qquad T^{\alpha}_{\ |\beta} = T^{\alpha}_{\ ||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N \tag{4}$$

$\diamond$

Remark: We also use **(2.639)** for the two other identities.

ii) $T^{N}_{\ |\alpha} = \partial_\alpha T^N - \frac{1}{2a_{NN}}\partial_N a_{\alpha\mu}T^\mu + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$

$$\textbf{(2.520)} \quad \Rightarrow \quad T^{N}_{\ |\alpha} = \partial_\alpha T^N + \Gamma^N_{m\alpha}T^m \qquad (m = 1, \ldots, N)$$

$$\Leftrightarrow \quad T^{N}_{\ |\alpha} = \partial_\beta T^\alpha + \underbrace{\Gamma^N_{\sigma\alpha}}_{-\frac{1}{2a_{NN}}\partial_N a_{\sigma\alpha}} T^\sigma + \underbrace{\Gamma^N_{N\alpha}}_{\frac{1}{2a_{NN}}\partial_\alpha a_{NN}} T^N$$

$$\Rightarrow \quad T^{N}_{\ |\alpha} = \partial_\beta T^\alpha - \frac{1}{2a_{NN}}\partial_N a_{\sigma\alpha}T^\sigma + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$$

iii) $T^{\alpha}_{\ |N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$

$$\textbf{(2.520)} \quad \Rightarrow \quad T^{\alpha}_{\ |N} = \partial_N T^\alpha + \Gamma^\alpha_{mN}T^m \qquad (m = 1, \ldots, N)$$

$$\Leftrightarrow \quad T^{\alpha}_{\ |N} = \partial_N T^\alpha + \underbrace{\Gamma^\alpha_{\sigma N}}_{\frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}} T^\sigma + \underbrace{\Gamma^\alpha_{NN}}_{-\frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}} T^N$$

$$\Rightarrow \quad T^{\alpha}_{\ |N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$$

$\blacklozenge$

2.41 p71-exercise

Write down equation 2.643 tot 2.650 for the special cae of a geodesic normal coordinate system.

$$\textbf{(2.643)} \qquad T_{\alpha||\beta} = \partial_\beta T_\alpha - \Gamma^\gamma_{\alpha\beta} T_\gamma \qquad \text{(does not change)} \tag{1}$$

$$\textbf{(2.644)} \qquad T_{\alpha|\beta} = T_{\alpha||\beta} - \underbrace{\Gamma^N_{\alpha\beta}}_{=\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}} T_N \tag{2}$$

$$= T_{\alpha||\beta} + \frac{1}{2}\epsilon \partial_N a_{\alpha\beta} T_N \tag{3}$$

$$\textbf{(2.645)} \qquad T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N \qquad \text{(does not change)} \tag{4}$$

$$\textbf{(2.646)} \qquad T_{N|\alpha} = \partial_\alpha T_N - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N \tag{5}$$

$$= \partial_\alpha T_N - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu \tag{6}$$

$$\textbf{(2.647)} \qquad T_{\alpha|N} = \partial_N T_\alpha - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N \tag{7}$$

$$= \partial_N T_\alpha - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu \tag{8}$$

$$\textbf{(2.648)} \qquad T^\alpha_{\ |\beta} = T^\alpha_{\ ||\beta} + \frac{1}{2} a^{\alpha\mu}\partial_N a_{\mu\beta} T_N \qquad \text{(does not change)} \tag{9}$$

$$\textbf{(2.649)} \qquad T^N_{\ |\alpha} = \partial_\alpha T^N - \frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\frac{1}{a_{NN}}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N \tag{10}$$

$$= \partial_\alpha T^N - \frac{1}{2}\epsilon \partial_N a_{\mu\alpha} T^\mu \tag{11}$$

$$\textbf{(2.650)} \qquad T^\alpha_{\ |N} = \partial_N T^\alpha + \frac{1}{2} a^{\alpha\mu}\partial_N a_{\mu\sigma} T^\sigma - \frac{1}{2} a^{\alpha\mu}\underbrace{\partial_\mu a_{NN}}_{=0} T^N \tag{12}$$

$$= \partial_N T^\alpha + \frac{1}{2} a^{\alpha\mu}\partial_N a_{\mu\sigma} T^\sigma \tag{13}$$

To investigate: note (3) and (4) which suggest that $\epsilon T_N = T^N$. Prove formally?

♦

2.42 p73-Clarification 2.706

... Let us now define a unit vector $\lambda^r_{(2)}$ and a positive invariant $\kappa_{(2)}$ by the equation

$$\begin{cases} \frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \epsilon\epsilon_{(1)}\kappa_{(1)}\lambda^r \\ \epsilon_{(2)}\lambda^n_{(2)}\lambda_{(2)n} = 1 \end{cases} \tag{1}$$

We can state that $\kappa_{(2)}$ is an invariant but one has to check whether the expression (1) implies that $\kappa_{(2)}$ is indeed invariant.
What we know is that $\lambda^r, \frac{\delta\lambda^r}{\delta s}, \frac{\delta\lambda^r_{(1)}}{\delta s}, \lambda^r_{(2)}$ are contravariant vectors. Also $\kappa_{(1)}$ is an invariant as $\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ and the magnitude of $\frac{\delta\lambda^r}{\delta s}$ does not depend on the coordinate system. So,

$$\begin{aligned}
(1) \times \lambda_{(2)r} \quad \Rightarrow \quad & \frac{\delta\lambda^r_{(1)}}{\delta s}\lambda_{(2)r} = \kappa_{(2)}\lambda^r_{(2)}\lambda_{(2)r} - \epsilon\epsilon_{(1)}\kappa_{(1)}\lambda^r\lambda_{(2)r} \\
\Rightarrow \quad & \kappa_{(2)}\underbrace{\lambda^r_{(2)}\lambda_{(2)r}}_{\text{invariant}} = \underbrace{\frac{\delta\lambda^r_{(1)}}{\delta s}\lambda_{(2)r}}_{\text{invariant}} + \underbrace{\epsilon\epsilon_{(1)}}_{\text{invariant}}\underbrace{\kappa_{(1)}}_{\text{invariant}}\underbrace{\lambda^r\lambda_{(2)r}}_{\text{invariant}} \\
\Rightarrow \quad & \kappa_{(2)} = \text{invariant}
\end{aligned}$$

♦

2.43 p74-Clarification 2.710

$$\textbf{2.710} \qquad \begin{cases} \frac{\delta\lambda^r_{(M-1)}}{\delta s} = \kappa_{(M)}\lambda^r_{(M)} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\lambda^r_{(M-2)} \\ \epsilon_{(M-1)}\lambda^n_{(M-1)}\lambda_{(M-1)n} = 1 \qquad \text{(M=1,2,...,N)} \end{cases} \tag{1}$$

... It is easily proved by mathematical induction that the whole sequence of vectors defined by 2.710 are perpendicular to the tangent and to one another ...

We already know from 2.703 to 2.709 that $\lambda^r, \lambda^r_{(1)}, \lambda^r_{(2)}, \lambda^r_{(3)}$, satisfying equations (1), are all mutually perpendicular. Let us assume that the orthogonality for the set $\{\lambda^r_{(k)} : k = 0, 1, 2, 3, ..., M-1\}$ has been verified. We prove by induction that then, $\lambda^r_{(M)}$ will be orthogonal to all elements of the set.

i) Consider the set $\{\lambda^r_{(k)} : k = 0, 1, 2, 3, ..., M-3\}$ where we already know that $\lambda^r_{(k)}$ are mutually perpendicular and also $\lambda^r_{(k)} \perp \lambda^r_{(M-1)}$, $\lambda^r_{(k)} \perp \lambda^r_{(M-2)}$ and $\lambda^r_{(M-1)} \perp \lambda^r_{(M-1)} \quad \forall k$.

$$(1) \times \lambda_{(k)r} \quad \Rightarrow \qquad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(k)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(k)r} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\lambda^r_{(M-2)}\lambda_{(k)r}}_{=0} \tag{2}$$

$$\text{We have} \qquad \lambda_{(k)r}\lambda^r_{(M-1)} = 0 \tag{3}$$

$$\Rightarrow \qquad \frac{\delta\lambda_{(k)r}\lambda^r_{(M-1)}}{\delta s} = \lambda^r_{(M-1)}\frac{\delta\lambda_{(k)r}}{\delta s} + \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = 0 \tag{4}$$

$$\Rightarrow \qquad \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = -\lambda^r_{(M-1)}\frac{\delta\lambda_{(k)r}}{\delta s} \tag{5}$$

$$\text{We have} \qquad \frac{\delta\lambda_{(k)r}}{\delta s} = \kappa_{(k+1)}\lambda_{(k+1)r} - \epsilon_{(k)}\epsilon_{(k-1)}\kappa_{(k)}\lambda_{(k-1)r} \tag{6}$$

$$(5) \text{ and } (6) \Rightarrow \qquad \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = -\kappa_{(k+1)}\underbrace{\lambda_{(k+1)r}\lambda^r_{(M-1)}}_{=0} - \epsilon_{(k)}\epsilon_{(k-1)}\kappa_{(k)}\underbrace{\lambda_{(k-1)r}\lambda^r_{(M-1)}}_{=0} \tag{7}$$

$$\text{From } (2) \Rightarrow \qquad \kappa_{(M)}\lambda^r_{(M)}\lambda_{(k)r} = 0 \tag{8}$$

$$\Rightarrow \qquad \lambda^r_{(M)} \perp \lambda_{(k)r} \qquad \forall k = 0, 1, 2, 3, ..., M-3 \tag{9}$$

ii) Consider the case $k = M - 1$

$$(1) \times \lambda_{(M-1)r} \quad \frac{\delta \lambda^r_{(M-1)}}{\delta s} \lambda_{(M-1)r} = \kappa_{(M)} \lambda^r_{(M)} \lambda_{(M-1)r} - \epsilon_{(M-2)} \epsilon_{(M-1)} \kappa_{(M-1)} \underbrace{\lambda^r_{(M-2)} \lambda_{(M-1)r}}_{=0} \tag{10}$$

$$\text{from (2.530):} \quad \frac{\delta \lambda^r_{(M-1)}}{\delta s} \lambda_{(M-1)r} = \frac{1}{2} \underbrace{\frac{\delta \lambda_{(M-1)r} \lambda^r_{(M-1)}}{\delta s}}_{=0 \text{ as } \lambda_{(M-1)r} \lambda^r_{(M-1)} = \epsilon_{(M-1)}} \tag{11}$$

$$\Rightarrow \quad \kappa_{(M)} \lambda^r_{(M)} \lambda_{(M-1)r} = 0 \tag{12}$$

$$\Rightarrow \quad \lambda_{(M-1)r} \perp \lambda_{(M)r} \tag{13}$$

iii) Consider the case $k = M - 2$

$$(1) \times \lambda_{(M-2)r} \quad \frac{\delta \lambda^r_{(M-1)}}{\delta s} \lambda_{(M-2)r} = \kappa_{(M)} \lambda^r_{(M)} \lambda_{(M-2)r} - \epsilon_{(M-2)} \epsilon_{(M-1)} \kappa_{(M-1)} \underbrace{\lambda^r_{(M-2)} \lambda_{(M-2)r}}_{=\epsilon_{(M-2)}} \tag{14}$$

$$\Rightarrow \quad \frac{\delta \lambda^r_{(M-1)}}{\delta s} \lambda_{(M-2)r} = \kappa_{(M)} \lambda^r_{(M)} \lambda_{(M-2)r} - \epsilon_{(M-1)} \kappa_{(M-1)} \underbrace{\epsilon_{(M-2)} \epsilon_{(M-2)}}_{=1} \tag{15}$$

$$\Rightarrow \quad \frac{\delta \lambda^r_{(M-1)}}{\delta s} \lambda_{(M-2)r} = \kappa_{(M)} \lambda^r_{(M)} \lambda_{(M-2)r} - \epsilon_{(M-1)} \kappa_{(M-1)} \tag{16}$$

$$\text{We have} \quad \lambda^r_{(M-1)} \lambda_{(M-2)r} = 0 \tag{17}$$

$$\Rightarrow \quad \lambda_{(M-2)r} \frac{\delta \lambda^r_{(M-1)}}{\delta s} = -\lambda^r_{(M-1)} \frac{\delta \lambda_{(M-2)r}}{\delta s} \tag{18}$$

$$\text{We have also} \quad \frac{\delta \lambda_{(M-2)r}}{\delta s} = \kappa_{(M-1)} \lambda_{(M-1)r} - \epsilon_{(M-3)} \epsilon_{(M-2)} \kappa_{(M-2)} \lambda_{(M-3)r} \tag{19}$$

$$(19) \times \lambda^r_{(M-1)} \text{ and (18):} \quad \lambda_{(M-2)r} \frac{\delta \lambda^r_{(M-1)}}{\delta s} = -\kappa_{(M-1)} \underbrace{\lambda^r_{(M-1)} \lambda_{(M-1)r}}_{=\epsilon_{(M-1)}} - \epsilon_{(M-3)} \epsilon_{(M-2)} \kappa_{(M-2)} \underbrace{\lambda^r_{(M-3)} \lambda_{(M-1)r}}_{=0} \tag{20}$$

$$\Rightarrow \quad \lambda_{(M-2)r} \frac{\delta \lambda^r_{(M-1)}}{\delta s} = -\kappa_{(M-1)} \epsilon_{(M-1)} \tag{21}$$

$$\text{(16) and (21):} \quad -\kappa_{(M-1)} \epsilon_{(M-1)} = \kappa_{(M)} \lambda^r_{(M)} \lambda_{(M-2)r} - \epsilon_{(M-1)} \kappa_{(M-1)} \tag{22}$$

$$\Rightarrow \quad \kappa_{(M)} \lambda^r_{(M)} \lambda_{(M-2)r} = 0 \tag{23}$$

$$\Rightarrow \quad \lambda_{(M-2)r} \perp \lambda_{(M)r} \tag{24}$$

With, i), ii), iii) all possible cases are covered which makes the proof complete.

♦

2.44 p75-Clarification 2.714

2.714 $(\kappa_{(1)})^2 = \epsilon_{(1)} a_{mn} \frac{\delta\lambda^m}{\delta s} \frac{\delta\lambda^n}{\delta s}, \qquad \epsilon_{(1)} = \pm 1$

$$\frac{\delta\lambda^n}{\delta s} = \kappa_{(1)} \lambda^n_{(1)} \tag{1}$$

$$(1) \times (1) \quad \Rightarrow \quad \frac{\delta\lambda^m}{\delta s} \frac{\delta\lambda^n}{\delta s} = (\kappa_{(1)})^2 \lambda^m_{(1)} \lambda^n_{(1)} \tag{2}$$

$$(2) \times a_{mn} \Rightarrow \quad a_{mn} \frac{\delta\lambda^m}{\delta s} \frac{\delta\lambda^n}{\delta s} = a_{mn} (\kappa_{(1)})^2 \lambda^m_{(1)} \lambda^n_{(1)} \tag{3}$$

$$= (\kappa_{(1)})^2 \underbrace{\lambda_{(1)m} \lambda^n_{(1)}}_{=\epsilon_{(1)}} \tag{4}$$

$$= (\kappa_{(1)})^2 \tag{5}$$

♦

2.45 p75-exercise

For positive definite metric forms, write out explicitly the Frenet formulae for the case N=2, 3 and 4.

The general Frenet formulae are

$$\begin{cases} \frac{\delta\lambda^r_{(M-1)}}{\delta s} = \kappa_{(M)}\lambda^r_{(M)} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\lambda^r_{(M-2)} \\ \epsilon_{(M-1)}\lambda^n_{(M-1)}\lambda_{(M-1)n} = 1 \qquad \text{(M=1,2,...,N)} \end{cases} \tag{1}$$

As Φ is positive definite, we have $\epsilon_{(k)} = 1 \quad \forall k$

N=2	N=3	N=4
$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(3)}}{\delta s} = \kappa_{(4)}\lambda^r_{(4)} - \kappa_{(3)}\lambda^r_{(2)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$ $\lambda^n_{(3)}\lambda_{(3)n} = 1$

Taking into account that $\kappa_{(N)} = 0$ for a space V_N, we get,

N=2	N=3	N=4
$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = -\kappa_{(1)}\lambda^r$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = -\kappa_{(2)}\lambda^r_{(1)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(3)}}{\delta s} = -\kappa_{(3)}\lambda^r_{(2)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$ $\lambda^n_{(3)}\lambda_{(3)n} = 1$

♦

2.46 p76-exercise

In an Euclidean space V_N, the fundamental form is given as $\Phi = dx^n dx^n$. Show that a curve which has $\kappa_{(2)} = 0$ and $\kappa_{(1)} =$ constant satisfies equations of the form

$$x^r = A^r \cos \kappa_{(1)} s + B^r \sin \kappa_{(1)} s + C^r$$

where A^r, B^r, C^r are constants satisfying

$$A^r A^R = B^r B^r = \frac{1}{\kappa_{(1)}^2}, \quad A^r B^r = 0$$

so that A^r and B^r are vectors of equal magnitude and perpendicular to one another. (This curve is a circle in the N-space)

$$\text{What we know} \qquad \Phi = dx^n dx^n \tag{1}$$

$$\Rightarrow \qquad (a_{mn}) = (\delta_n^m) \tag{2}$$

$$\text{and given} \qquad \kappa_{(1)} = \text{constant} \quad \kappa_{(2)} = 0 \quad \epsilon_{(1)} = \epsilon_{(2)}, \cdots = 1 \tag{3}$$

$$\text{we have (2.705)} \qquad \frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \qquad \text{with} \quad \lambda^r = \frac{dx^r}{ds} \tag{4}$$

$$\text{but} \quad (a_{mn}) = (\delta_n^m) \quad \Rightarrow \qquad \frac{\delta \lambda^r}{\delta s} = \frac{d\lambda^r}{ds} \tag{5}$$

$$\text{(4) and (5)} \quad \Rightarrow \qquad \frac{d\lambda^r}{ds} = \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \tag{6}$$

$$\text{also} \qquad \frac{\delta \lambda_{(1)}^r}{\delta s} = \underbrace{\kappa_{(1)}}_{=0} \frac{\delta \lambda_{(1)}^r}{\delta s} - \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \tag{7}$$

Hence we get the following set of equations

$$\begin{cases} \frac{dx^r}{ds} = \lambda^r \\ \frac{d\lambda^r}{ds} = \kappa_{(1)}\lambda^r_{(1)} \\ \frac{d\lambda^r_{(1)}}{ds} = -\kappa_{(1)}\lambda^r_{(1)} \\ \kappa_{(1)} = \kappa \quad (=\text{constant}) \\ \kappa_{(2)} = 0 \\ \lambda^n\lambda_n = 1 \\ \lambda^n_{(1)}\lambda_{(1)n} = 1 \end{cases} \tag{8}$$

$$(8) \quad \Rightarrow \quad \frac{d^2\lambda^r_{(1)}}{ds^2} + \kappa^2\lambda^r_{(1)} = 0 \tag{9}$$

Solving the ODE (9). Put $e^{rs} = \lambda^k_{(1)}$

$$(9): \qquad r^2 + \kappa^2 = 0 \tag{10}$$

$$\Rightarrow \qquad r = \pm\imath\kappa \tag{11}$$

$$\text{Hence, a general solution of (9) is of the form:} \qquad \lambda^r_{(1)} = p^r e^{\imath\kappa s} + q^r e^{-\imath\kappa s} \tag{12}$$

$$\text{put} \quad p^r + q^r = A'^r \quad \text{and} \quad p^r - q^r = B'^r \tag{13}$$

$$\Leftrightarrow \qquad p^r = \frac{A'^r + B'^r}{2} \text{and} \quad p^r = \frac{A'^r - B'^r}{2} \tag{14}$$

$$\text{(12) can then be written as} \quad \lambda^r_{(1)} = A'^r \frac{e^{\imath\kappa s} + e^{-\imath\kappa s}}{2} + B'^r \frac{e^{\imath\kappa s} - e^{-\imath\kappa s}}{2} \tag{15}$$

$$\text{or} \quad \lambda^r_{(1)} = A'^r \cos\kappa s + B'^r \sin\kappa s \tag{16}$$

$$\text{We have (8)} \qquad \lambda^r = -\,\kappa_{(1)} \frac{d\lambda^r_{(1)}}{ds} \tag{17}$$

$$\frac{d(16)}{ds} \quad \text{and (17)} \Rightarrow \qquad \lambda^r = A'^r \sin\kappa s - B'^r \cos\kappa s \tag{18}$$

$$\text{as} \quad \lambda^r = \frac{dx^r}{ds} \quad \text{with (18)} \quad \Rightarrow \qquad x^r = -\frac{A'^r}{\kappa}\cos\kappa s - \frac{B'^r}{\kappa}\sin\kappa s + C^r \tag{19}$$

Replace $-\frac{A'^r}{\kappa}$ with A^r and $-\frac{B'^r}{\kappa}$ with B^r, we get then the following set of equations,

$$\begin{cases} x^r = A^r \cos\kappa s + B'^r \sin\kappa s + C^r \\ \lambda^r = -\kappa A^r \sin\kappa s + \kappa B^r \cos\kappa s \\ \lambda^r_{(1)} = -\kappa A^r \cos\kappa s - \kappa B^r \sin\kappa s \end{cases} \tag{20}$$

$$\text{with the following constraints} \qquad \lambda^n \lambda_n = 1 \qquad \lambda^n_{(1)}\lambda_{(1)n} = 1 \tag{21}$$

$$\lambda^n \lambda_n = 1 \quad \Rightarrow \kappa^2 A^r A^r \sin^2\kappa s + \kappa^2 B^r B^r \cos^2\kappa s - 2\kappa^2 A^r B^r \sin\kappa s \cos\kappa s = 1 \tag{22}$$

$$\text{or} \qquad A^r A^r \sin^2\kappa s + B^r B^r \cos^2\kappa s - 2A^r B^r \sin\kappa s \cos\kappa s = \frac{1}{\kappa^2} \tag{23}$$

$$\lambda^n_{(1)}\lambda_{(1)n} = 1 \quad \Rightarrow \kappa^2 A^r A^r \cos^2\kappa s + \kappa^2 B^r B^r \sin^2\kappa s + 2\kappa^2 A^r B^r \sin\kappa s \cos\kappa s = 1 \tag{24}$$

$$\text{or} \qquad A^r A^r \cos^2\kappa s + B^r B^r \sin^2\kappa s + 2A^r B^r \sin\kappa s \cos\kappa s = \frac{1}{\kappa^2} \tag{25}$$

$$\text{Choose} \quad \kappa s = \frac{\pi}{2} \quad \text{and} \quad \kappa s = 0 \tag{26}$$

$$\Rightarrow \qquad A^r A^r = \frac{1}{\kappa^2} \quad \text{and} \quad B^r B^r = \frac{1}{\kappa^2} \tag{27}$$

$$\text{Morover considering (26)-(24) and (28)} \quad \Rightarrow \qquad 2A^r B^r \sin\kappa s \cos\kappa s = 0 \quad \forall \kappa s \tag{28}$$

$$\Rightarrow \qquad A^r B^r = 0 \tag{29}$$

Note: when deriving expressions (23) and (26) we use the fact that $(a_{mn}) = (a^{mn}) = (\delta^m_n)$

♦

2.47 p78-exercise 1

For cylindrical coordinates in Euclidean 3-space, write down the metric form by inspection of a diagram showing a general infinitesimal displacement, and calculate all the Christoffel symbols of both kinds.

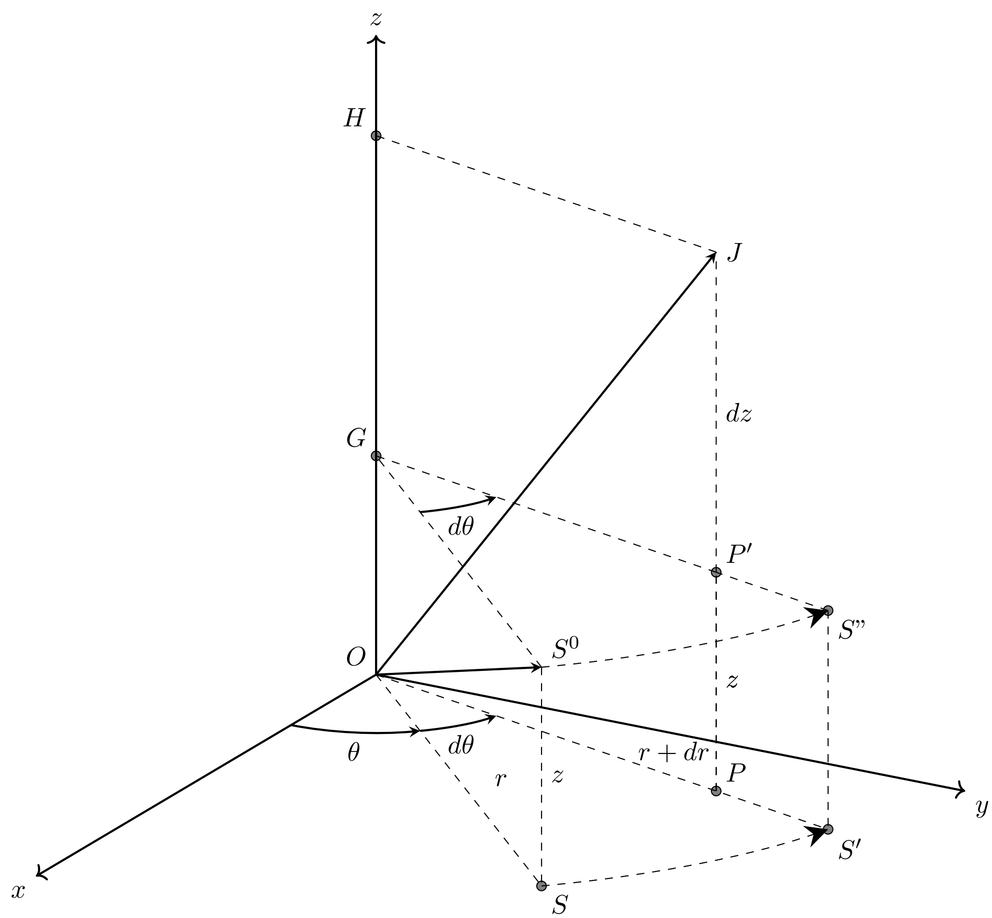


Figure 2.6: Cylindrical coorinates.

From the figure we may (assuming an infinitesimal displacement), we may approximate $\left|\overrightarrow{SS'}\right|$

with the arclength $rd\theta$ and assume $\left|\overrightarrow{SS^{'}},\right| \perp \left|\overrightarrow{GS^{'}}\right|$) Hence, the infinitesimal displacement from S

$$ds^2 = \left|\overrightarrow{SS^{'}}\right|^2 + \left|\overrightarrow{S^{'}\cdot P^{'}}\right|^2 + \left|\overrightarrow{P^{'}J}\right|^2 \tag{1}$$

$$= dr^2 + ((r+dr)d\theta)^2 + dz^2 \tag{2}$$

$$= dr^2 + r^2 d\theta^2 + dz^2 \tag{3}$$

$$\text{Hence} \qquad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (a^{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

Note that all $a_{mn} = 0 \quad \forall m \neq n$. So,

$$\begin{cases} [r\ r, r] = [\theta\ \theta, \theta] = [z\ z, z] = 0 \\ [r\ \theta, r] = [r\ r, \theta] = [r\ r, z] = 0 \\ [r\ z, \theta] = [z\ \theta, r] = [z\ \theta, z] = 0 \end{cases} \tag{5}$$

$$\text{But:} \qquad [\theta\ \theta, r] = -r \quad \text{and} \quad [r\ \theta, \theta] = r \tag{6}$$

$$\text{Hence} \qquad \begin{cases} \Gamma^m_{nk} = 0 \quad \forall \quad (nk) \neq (r,\theta), (\theta,\theta) \\ \\ \Gamma^\theta_{r\theta} = \frac{1}{r} \quad \text{and} \quad \Gamma^r_{\theta\theta} = -r \end{cases} \tag{7}$$

♦

2.48 p78-exercise 2

If a_{rs} and b_{rs} are covariant tensors, show that the roots of the determinant equation

$$|Xa_{rs} - b_{rs}| = 0$$

are invariants.

$$\begin{aligned}
\text{Be} \qquad & c_{rs} = Xa_{rs} - b_{rs} && (1)\\
\text{Given} \qquad & a_{rs} = a'_{mk}\frac{\partial x'^m}{\partial x^r}\frac{\partial x'^k}{\partial x^s} && (2)\\
\text{and} \qquad & b_{rs} = b'_{mk}\frac{\partial x'^m}{\partial x^r}\frac{\partial x'^k}{\partial x^s} && (3)\\
(1) \quad \Rightarrow \qquad & c_{rs} = \underbrace{(Xa'_{mk} - b'_{mk})}_{=c'_{km}}\frac{\partial x'^m}{\partial x^r}\frac{\partial x'^k}{\partial x^s} && (4)\\
& \quad = c'_{km}\frac{\partial x'^m}{\partial x^r}\frac{\partial x'^k}{\partial x^s} && (5)\\
\text{Be} \qquad & J = \left|\frac{\partial x'^m}{\partial x^r}\right| = \left|\frac{\partial x'^k}{\partial x^s}\right| && (6)\\
\text{In (5) put} \qquad & d_{kr} = c'_{km}\frac{\partial x'^m}{\partial x^r} && (7)\\
\Rightarrow \qquad & c_{rs} = d'_{kr}\frac{\partial x'^k}{\partial x^s} && (8)\\
\text{or in matrix form} \qquad & C = D^T J \quad \text{with} \quad D = C' J && (9)\\
\Rightarrow \qquad & |C| = \left|(C' J)^T J\right| && (10)\\
\Leftrightarrow \qquad & |C| = \left|C'\right| |J| \, |J| && (11)
\end{aligned}$$

As $|J| \neq 0$ (J is the Jacobian of the transformation, and thus can't be zero), then

$$|C| = 0 \Rightarrow \left|C'\right| = 0$$

.

So, the root of $|C| = 0$ is also a root of $\left|C'\right| = 0$ and is as a consequence, invariant.

♦

2.49 p78-exercise 3

Is the form $dx^2 + 3dxdy + 4dy^2 + dz^2$ positive definite?

$$\Phi = dx^2 + 3dxdy + 4dy^2 + dz^2 \tag{1}$$

Put (1) in the form $$\Phi = X^2 + 3XY + 4Y^2 + Z^2 \tag{2}$$

Z has only a positive contribution: so put $Z = 0 \quad \Rightarrow \quad \Phi = X^2 + 3XY + 4Y^2$ (3)

(3) can only be zero or negative if $XY < 0$:put $Y = -aX \quad (a > 0)$ (4)

$$\Rightarrow \Phi = X^2 - 3aX^2 + 4a^2X^2 \tag{5}$$

The roots of (5) are $$a_{1,2} = \frac{3 \pm \sqrt{9-16}}{8} \tag{6}$$

So, by (6) we can't get a $a \in \mathbb{R}_*$, so that (1) can be 0 or negative. Hence,

$$\text{The form} \quad \Phi \quad \text{is positive definite}$$

♦

2.50 p78-exercise 4

If X^r, Y^r are unit vectors inclined at an angle θ, prove that

$$\sin^2\theta = (a_{rm}a_{sn} - a_{rs}a_{mn})X^r Y^s X^m Y^n$$

$X^r Y^s$ are unit vectors. So,

$$\begin{aligned}
a_{rm}X^r X^m = 1 \quad \text{and} \quad a_{sn}Y^s Y^n = 1 \\
\text{We have} \qquad \sin^2\theta &= 1 - \cos^2\theta \\
\text{and (2.312)} \qquad \cos\theta &= a_{mn}X^m Y^n \\
\Rightarrow \qquad \sin^2\theta &= a_{rm}X^r X^m a_{sn}Y^s Y^n - a_{mn}X^m Y^n a_{rs}X^r Y^s \\
&= (a_{rm}a_{sn} - a_{mn}a_{rs})X^r Y^s X^m Y^n
\end{aligned}$$

♦

2.51 p78-exercise 5

Show that, if θ is the angle between the normals to the surfaces $x^1 = C^{st}, x^2 = C^{st}$, then

$$\cos\theta = \frac{a^{12}}{\sqrt{a^{11}a^{22}}}$$

$$\text{be} \qquad \phi_1^{'}(x^1, x^2, \ldots, x^N) = C^{st} \quad \phi_2^{'}(x^1, x^2, \ldots, x^N) = C^{st} \tag{1}$$

the two equations representing S_1, S_2 (see page 63). We can rewrite (1) as:

$$x^1 = \phi_1^{'}(x^1, x^2, \ldots, x^N) = C^{st} \tag{2}$$
$$x^2 = \phi_2^{'}(x^1, x^2, \ldots, x^N) = C^{st} \tag{3}$$

$$\text{From (2.622) we know that} \quad X^m = a^{mn}\partial_n\phi_1,\ Y^m = a^{mn}\partial_n\phi_2 \text{ are } \perp \text{ vectors to the surfaces } \phi_1, \phi_2 \tag{4}$$

$$\text{We know also} \qquad |X^m|^2 = a^{mk}X^mX^k \tag{5}$$
$$= a_{mk}a^{mn}\partial_n\phi_1 a^{kp}\partial_p\phi_1 \tag{6}$$
$$= \delta_k^k a^{kp}\partial_n\phi_1\partial_p\phi_1 \tag{7}$$
$$= a^{np}\partial_n\phi_1\partial_p\phi_1 \tag{8}$$
$$\text{as } \phi_1 = x^1 = C^{st} \ \Rightarrow \ = a^{np}\delta_n^1\delta_p^1 \tag{9}$$
$$= \epsilon a^{11} \tag{10}$$
$$\text{Analog, we have} \qquad |Y^m|^2 = \epsilon a^{22} \tag{11}$$
$$\text{By definition:} \qquad \cos\theta = \frac{a_{mn}X^mY^n}{|X^r|\,|Y^s|} \tag{12}$$
$$\text{and} \qquad a_{mn}X^mY^n = a_{mn}a^{mk}\partial_k\phi_1\ a^{np}\partial_p\phi_2 \tag{13}$$
$$= \delta_n^k a^{np}\partial_k\phi_1\ \partial_p\phi_2 \tag{14}$$
$$= a^{kp}\partial_k\phi_1\ \partial_p\phi_2 \tag{15}$$
$$\text{as } \phi_1 = x^1 = C^{st},\ \phi_2 = x^2 = C^{st} \ \Rightarrow \ = a^{kp}\delta_k^1\delta_p^2 \tag{16}$$
$$= a^{12} \tag{17}$$
$$\text{So (12) becomes with (10), (11) and (17)} \qquad \cos\theta = \frac{a_{12}}{\sqrt{\epsilon a^{11}\epsilon a^{22}}} \tag{18}$$
$$= \frac{a_{12}}{\sqrt{a^{11}a^{22}}} \tag{19}$$

♦

2.52 p78-exercise 6

Let $x^1,\ x^2,\ x^3$ be rectangular Cartesian coordinates in Euclidean 3-space, and let $x^1,\ x^2$ be taken as coordinates on a surface $x^3 = f(x^1,\ x^2)$. Show that the Christoffel symbols of the second kind for the surface are

$$\Gamma^r_{mn} = \frac{f_r f_{mn}}{1 + f_n f_p}$$

the suffixes taking the values $1,\ 2$ and the subscripts indicating partial derivatives.

We have (rectangular Cartesian coordinates in Euclidean 3-space)

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1}$$

with
$$x^3 = f(x^1,\ x^2) \tag{2}$$

and thus
$$dx^3 = \partial_1 f\ dx^1 + \partial_2 f\ dx^2 \tag{3}$$

$\Rightarrow$
$$ds^2 = (1 + (\partial_1 f)^2)(dx^1)^2 + (1 + (\partial_2 f)^2)(dx^2)^2 + 2\partial_1 f\ \partial_2 f\ dx^1 dx^2 \tag{4}$$

put
$$\begin{cases} f_1 = \partial_1 f \\ f_2 = \partial_2 f \\ f_{11} = \partial_{11} f \\ f_{22} = \partial_{22} f \\ f_{12} = f_{21} = \partial_{12} f \end{cases} \tag{5}$$

from (4)
$$(a_{mn}) = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix} \tag{6}$$

$\Rightarrow$
$$|a_{mn}| = (1 + f_1^2)(1 + f_2^2) - (f_1 f_2)^2 \tag{7}$$

$$= 1 + f_1^2 + f_2^2 \tag{8}$$

also
$$(a^{mn}) = \frac{1}{1 + f_1^2 + f_2^2} \begin{pmatrix} 1 + f_2^2 & -f_1 f_2 \\ -f_1 f_2 & 1 + f_1^2 \end{pmatrix} \tag{9}$$

Calculating the Christoffels symbols: $[mn,\ k] = \frac{1}{2}(\partial_m a_{nk} + \partial_n a_{mk} - \partial_k a_{mn})$ (10)

$$\Rightarrow \quad \begin{cases} [11,\ 1] = f_1 f_{11} \\ [11,\ 2] = f_2 f_{11} \\ [12,\ 1] = f_1 f_{12} \\ [12,\ 2] = f_2 f_{21} \\ [22,\ 2] = f_2 f_{22} \end{cases} \tag{11}$$

From (11) we can see that the general form is: $[mn, s] = f_{mn} f_s$ (12)

Calculating the Christoffels symbols: $\Gamma^r_{mn} = a_{rs}[mn, s]$ (13)

(13) with (12): $\Gamma^r_{mn} = a_{rs} f_{mn} f_s = f_{mn}(a^{rs} f_s)$ (14)

put $\Delta = \dfrac{1}{1 + f_1^2 + f_2^2} = \dfrac{1}{1 + f_p f_p}$ (15)

$$\Rightarrow \quad \begin{cases} \Gamma^1_{mn} & = (a_{11} f_1 + a_{12} f_2) f_{mn} \\ & = \Delta(f_1 + f_2^2 f_1 - f_1 f_2^2) f_{mn} \\ & = \Delta f_1 f_{mn} \\ & \\ \Gamma^2_{mn} & = (a_{21} f_1 + a_{22} f_2) f_{mn} \\ & = \Delta(-f_1^2 f_2 + f_2 + f_2 f_1^2) f_{mn} \\ & = \Delta f_2 f_{mn} \end{cases} \tag{16}$$

$$\Rightarrow \quad \Gamma^r_{mn} = \frac{f_r f_{mn}}{1 + f_p f_p}$$

♦

2.53 p78-exercise 7

Write down the differential equations of the geodesics on a sphere, using colatitude θ and the azimuth ϕ as coordinates. Integrate the differential equations and obtain a finite equation

$$A\sin\theta\cos\phi + B\sin\theta\sin\phi + C\cos\theta = 0$$

where A, B, C are arbitrary constants.

We will use two different approaches to determine the relation and finally use a geometrical reasoning allowing us to avoid solving the ODE's resulting from the above mentioned approaches.

We will first find the solution, starting from the variational principle, defining a geodesic. In spherical coordinates we have (see exercise page 27) $ds^2 = dr^2 + r^2\sin^2(\theta)d\phi^2 + r^2 d\theta^2$.
As $r = R = C^{st}$ this reduces to

$$ds^2 = R^2\sin^2(\theta)d\phi^2 + R^2 d\theta^2 \tag{1}$$

So the length of a curve on the sphere from a point P_1 to another point P_2 , the curve being determined by $\theta = \theta(u) \quad \phi = \phi(u)$ is:

$$L = R\int_{P_1}^{P_2} = \sqrt{\sin^2(\theta)d\phi^2 + d\theta^2}du \tag{2}$$

$$\text{Be} \quad \theta = u \quad \phi = \phi(\theta) \tag{3}$$

$$\Rightarrow \quad L = R\int_{\theta_1}^{\theta_2} = \sqrt{1 + \sin^2(\theta)(\frac{d\phi}{d\theta})^2}d\theta \tag{4}$$

Applying the variational principle on L and using the Euler-Langrange equations:

$$\frac{d\frac{\partial\mathcal{L}}{\partial\dot{\phi}}}{d\theta} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \text{with} \quad \mathcal{L} = \sqrt{1+\sin^2(\theta)(\dot{\phi})^2} \tag{5}$$

$$\text{as} \quad \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{6}$$

$$\text{(5) becomes:} \quad \frac{d\frac{\partial\mathcal{L}}{\partial\dot{\phi}}}{d\theta} = 0 \tag{7}$$

$$\Leftrightarrow \quad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = C \qquad (= \text{constant}) \tag{8}$$

$$\text{with} \quad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\sqrt{1+\sin^2\theta\ \dot{\phi}^2}}{\partial\dot{\phi}} = \frac{\sin^2\theta\ \dot{\phi}}{\sqrt{1+\sin^2\theta\ \dot{\phi}^2}} \tag{9}$$

$$\text{(9) and (10):} \quad C^2 = \frac{\sin^2\theta\ \dot{\phi}}{\sqrt{1+\sin^2\theta\ \dot{\phi}^2}} \tag{10}$$

$$\text{or} \quad \dot{\phi} = \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} \tag{11}$$

Solving the ODE (11). Put $u = \cot\theta \quad \Rightarrow \quad du = -\csc^2 d\theta = -\frac{1}{\sin^2\theta}d\theta$. So,

$$\phi = -C\int \frac{\sin\theta}{\sqrt{\sin^2\theta - C^2}}du \tag{12}$$

$$= -C\int \frac{du}{\sqrt{1-\frac{C^2}{\sin^2\theta}}} \tag{13}$$

$$= -C\int \frac{du}{\sqrt{1-C^2-C^2\cot^2\theta}} \tag{14}$$

$$= -C\int \frac{du}{\sqrt{1-C^2-C^2u^2}} \tag{15}$$

$$\text{be} \quad a = \frac{\sqrt{1-C^2}}{C} \tag{16}$$

$$\text{(15) becomes} \quad \phi = -\int \frac{1}{\sqrt{a^2-u^2}}du \tag{17}$$

$$\text{put} \quad u = av \tag{18}$$

$$\text{(17) becomes} \quad \phi = -\int \frac{1}{\sqrt{1-v^2}}dv \tag{19}$$

$$= -\arccos v + C^{st} \tag{20}$$

$$= -\arccos\frac{u}{a} + \phi_0 \tag{21}$$

$$\text{or:} \quad \frac{u}{a} = \cos(\phi - \phi_0) \ \text{(by choosing an adequate } \phi_0\text{)} \tag{22}$$

$$\text{so :} \quad \cot\theta = a\cos(\phi-\phi_0) \tag{23}$$

$$\text{expanding } \cos(\phi-\phi_0) \text{ gives:} \quad \frac{\cos\theta}{\sin\theta} = A\cos\phi + B\sin\phi \tag{24}$$

$$\text{or:} \quad A\cos\phi\sin\theta + B\sin\phi\sin\theta - \cos\theta = 0 \tag{25}$$

◇

Finding the geodesics from the tensorial formula's.
Note: For ease of notation we put $R = 1$ without losing any general solutions.

$$\text{from (1) we get:} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{26}$$

$$\text{and} \quad (a^{mn}) = \begin{pmatrix} \frac{1}{\sin^2\theta} & 0 \\ 0 & 1 \end{pmatrix} \tag{27}$$

$$\text{hence:} \quad \begin{cases} \Gamma^1_{11} = 0 & \Gamma^2_{11} = 0 \\ \Gamma^1_{12} = 0 & \Gamma^2_{12} = \cot\theta \\ \Gamma^1_{22} = -\cos\theta\sin\theta & \Gamma^2_{22} = 0 \end{cases} \tag{28}$$

Finding the geodesics from the tensorial formula's, implies solving 2^{nd} order ODE's. In order to find the simpliest form to solve , we write down three possible forms of the geodesic equations:

$$\text{arc-length s as independent variable} \quad \begin{cases} (a) \quad \frac{d^2\phi}{ds^2} + 2\cot\theta\frac{d\phi}{ds}\frac{d\theta}{ds} = 0 \\ \\ (b) \quad \frac{d^2\theta}{ds^2} - \sin\theta\cos\theta(\frac{d\phi}{ds})^2 = 0 \\ \\ (c) \quad (\frac{d\theta}{ds})^2 + \sin^2\theta(\frac{d\phi}{ds})^2 = 1 \end{cases} \tag{29}$$

$$\theta \text{ as independent variable} \quad \begin{cases} \lambda = -\sin\theta\cos\theta(\frac{d\phi}{d\theta})^2 \\ \\ \frac{d^2\phi}{d\theta^2} = \lambda\frac{d\phi}{d\theta} - 2\cot\theta\frac{d\phi}{d\theta} \end{cases} \tag{30}$$

$$\Rightarrow \quad \frac{d^2\phi}{d\theta^2} = -\sin\theta\cos\theta(\frac{d\phi}{d\theta})^3 - 2\cot\theta\frac{d\phi}{d\theta} \tag{31}$$

$$\phi \text{ as independent variable} \quad \begin{cases} \lambda = 2\cot\theta\frac{d\theta}{d\phi} \\ \\ \frac{d^2\theta}{d\phi^2} = \lambda\frac{d\theta}{d\phi} + \sin\theta\cos\theta(\frac{d\theta}{d\phi})^2 \end{cases} \tag{32}$$

$$\Rightarrow \quad \frac{d^2\theta}{d\phi^2} = 2\cot\theta(\frac{d\theta}{d\phi})^2 + \sin\theta\cos\theta(\frac{d\theta}{d\phi})^2 \tag{33}$$

Inspection shows hat the expression (31) and (33) are quite complicated while using (29b) and (29c) we can get an expression of the form,

$$\ddot{\theta} - \sin\theta\cos\theta\left(\frac{1-\dot{\theta}^2}{\sin^2\theta}\right) = 0 \tag{34}$$

$$\text{or:}\quad \ddot{\theta} - \cot\theta\left(1-\dot{\theta}^2\right) = 0 \tag{35}$$

$$\text{Put}\quad u(\theta) = \dot{\theta} \quad\Rightarrow\quad \ddot{\theta} = \dot{u}u \tag{36}$$

$$\text{(36) can be written as:}\quad \dot{u}u - \cot\theta\left(1-u^2\right) = 0 \tag{37}$$

$$\text{or}\quad \frac{\dot{u}u}{(1-u^2)} = \cot\theta \tag{38}$$

$$\text{or}\quad \frac{u}{(1-u^2)}du = \frac{\cos\theta}{\sin\theta}d\theta \tag{39}$$

$$\text{or}\quad -\frac{1}{2}\frac{1}{(1-u^2)}d(1-u^2) = \frac{1}{\sin\theta}d(\sin\theta) \tag{40}$$

$$\text{hence}\quad -d(\log\left(\sqrt{1-u^2}\right)) = d(\log(\sin\theta)) \tag{41}$$

$$\Rightarrow\quad d(\log\sqrt{1-u^2} + \log(\sin\theta)) = 0 \tag{42}$$

$$\Leftrightarrow\quad d(\log\left(\sqrt{1-u^2}\sin\theta\right)) = 0 \tag{43}$$

$$\Rightarrow\quad (1-\dot{\theta}^2)\sin^2\theta = C^2 \tag{44}$$

$$\Rightarrow\quad \dot{\theta}^2 = 1 - \frac{C^2}{\sin^2\theta} \tag{45}$$

$$\text{we have (29c)}\quad \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta = 1 \tag{46}$$

$$\text{and so (45):}\quad 1 - \frac{C^2}{\sin^2\theta} + \dot{\phi}^2\sin^2\theta = 1 \tag{47}$$

$$\Rightarrow\quad \dot{\phi}^2 = \frac{C^2}{\sin^4\theta} \tag{48}$$

$$\Rightarrow\quad \dot{\phi} = \frac{C}{\sin^2\theta} \tag{49}$$

$$\text{we have}\quad \dot{\phi} = \frac{d\phi}{d\theta}\dot{\theta} \tag{50}$$

$$\text{so}\quad d\phi = \frac{C}{\sin^2\theta\sqrt{1-\frac{C^2}{\sin^2\theta}}}d\theta \tag{51}$$

$$\text{so}\quad d\phi = \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}}d\theta \tag{52}$$

Note that expression (52) is exactly the expression (11) we found by applying directly the variational principle to find the general expression. So applying steps (12) to (25) gives us the same expression.

◇

Instead of solving the ODE's (45) and (52) we can use geometrical considerations to get the asked expression.
Due to the invariance of a sphere regarding rotation of the axes, we can choose a reference axis system XYZ (from which θ, ϕ are measured) so that at $s = 0$ of the geodesic, corresponds the point $r = 1,\ \theta = 0$. As from (45), $(1 - \dot{\theta}^2)\sin^2\theta\Big|_{s=0} = C^2$ follows that $C = 0$ (because $\theta|_{s=0} = 0$). So we get $(1 - \dot{\theta}^2)\sin^2\theta = 0\ \forall\ \theta : \Rightarrow\ \dot{\theta} = 1$ and $\theta = s$. Then from (37) $\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta = 1$ follows that $\dot{\phi} = 0$ and thus $\phi = C^{st}$. Again, considering symmetry we can choose the axis system so that $\phi = 0$. The set of equations $\theta = s,\ \phi = 0$ represents a circle on the sphere generated by the intersection of the sphere with the XZ plane. Again, considering symmetry, we can conclude that any circle on the sphere generated by the intersection a plane going through the origin of the axis system, is also a geodesic curve. So, be $\widehat{n} = (A, B, C)$ the normal vector defining a plane going through the origin and $\widehat{p} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ a point on the sphere, then the intersection of the plane and the sphere is given by

$$\langle\widehat{n}|\widehat{p}\rangle = A\cos\phi\sin\theta + B\sin\phi\sin\theta + C\cos\theta = 0$$

♦

2.54 p79-exercise 8

Find in integrated form the geodesic null lines in a V_3 for which the metric form is

$$(dx^1)^2 - R^2[(dx^2)^2 + (dx^3)^2]$$

R being a function of x^1 only.

We have,

$$\Phi = (dx^1)^2 - R^2[(dx^2)^2 + (dx^3)^2] \tag{1}$$

$$\text{Hence,} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -R^2 & 0 \\ 0 & 0 & -R^2 \end{pmatrix} \tag{2}$$

$$\text{and,} \quad (a^{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{R^2} & 0 \\ 0 & 0 & -\frac{1}{R^2} \end{pmatrix} \tag{3}$$

$$\text{Hence,} \quad \begin{cases} [22,1] = R\,\partial_1 R & [12,2] = -R\,\partial_1 R \\ [33,1] = R\,\partial_1 R & [13,3] = -R\,\partial_1 R \end{cases} \tag{4}$$

$$\text{and,} \quad \begin{cases} \Gamma^1_{22} = R\,\partial_1 R & \Gamma^2_{12} = \frac{1}{R}\partial_1 R \\ \Gamma^1_{33} = R\,\partial_1 R & \Gamma^3_{13} = \frac{1}{R}\partial_1 R \end{cases} \tag{5}$$

$$\text{with all other } [mn,s] \text{ and } \Gamma^s_{mn} \text{ being zero.} \tag{6}$$

The equations of null geodesics give:

$$\begin{cases} \frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \\ a_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \end{cases} \tag{7}$$

$\boldsymbol{x^r = x^1}$ gives:

$$\frac{d^2x^1}{du^2} + R\partial_1 R(\frac{dx^2}{du})^2 + R\partial_1 R(\frac{dx^3}{du})^2 = 0 \tag{8}$$

Put $u = x^1$

$$\text{from (8):} \quad R\,\partial_1 R\left[(\frac{dx^2}{du})^2 + (\frac{dx^3}{du})^2\right] = 0 \tag{9}$$

If, $R = R(x^1) \neq C^{st}$, the form (9) can only be zero if $(\frac{dx^2}{du})^2 + (\frac{dx^3}{du})^2 = 0$ and thus $\frac{dx^2}{du} = \frac{dx^3}{du} = 0$ and hence $x^2,\ x^3$ are constant.

Conclusion: the null geodesics are the bundle of rays parallel with the x^1 axis with vector equation $\widehat{p} = (s, A, B), s \in (-\infty, +\infty)$ and A, B arbitrary constants.

♦

2.55 p79-exercise 9

Show that, for normal coordinate system, the Christoffel symbols

$$[\rho N,\ \sigma],\ [\rho\sigma,\ N],\ [\rho N,\ N],\ [NN,\ N]$$

$$\Gamma^{\rho}_{N\sigma},\ \Gamma^{N}_{\rho\sigma},\ \Gamma^{\rho}_{NN},\ \Gamma^{N}_{N\rho},\ \Gamma^{N}_{NN}$$

have tensor character with respect to the transformation of the coordinates $x^1,\ldots,x^{N-1}$

We know that $a_{\rho\sigma} = a^{'}_{mn}\partial_\rho x^{'m}\partial_\sigma x^{'n}$ with $\rho,\sigma = 1,\ldots,N-1$ and $m,n = 1,\ldots,N$. We have also $x^N = x^{'N}$.

$[\rho N,\ \sigma]$

$$[\rho N,\ \sigma] = \frac{1}{2}\partial_N a_{\rho\sigma} \quad \text{see (2.639)} \tag{1}$$

$$= \frac{1}{2}\partial_N(a^{'}_{mn}\partial_\rho x^{'m}\partial_\sigma x^{'n}) \tag{2}$$

$$= \begin{cases} \frac{1}{2}(\partial_N a^{'}_{mn}\partial_\rho x^{'m}\partial_\sigma x^{'n} \\ +a^{'}_{mn}\partial_\sigma x^{'n}\partial_{N\rho}x^{'m} \\ +a^{'}_{mn}\partial_\rho x^{'m}\partial_{N\sigma}x^{'n}) \end{cases} \tag{3}$$

$$\text{We have} \quad \partial_N x^{'m} = \delta^m_N \Rightarrow\ \partial_{N\rho}x^{'m} = \partial_{N\sigma}x^{'n} = 0 \tag{4}$$

$$\Rightarrow \quad [\rho N,\ \sigma] = \frac{1}{2}\partial_N a^{'}_{mn}\partial_\rho x^{'m}\partial_\sigma x^{'n} \tag{5}$$

$$= \begin{cases} \frac{1}{2}(\partial_N a^{'}_{\alpha\beta}\partial_\rho x^{'\alpha}\partial_\sigma x^{'\beta} \\ +\partial_N a^{'}_{NN}\underbrace{\partial_\rho x^{'N}}_{=0}\underbrace{\partial_\sigma x^{'N}}_{=0} \\ +\partial_N a^{'}_{\alpha N}\partial_\rho x^{'\alpha}\underbrace{\partial_\sigma x^{'N}}_{=0} \\ +\partial_N a^{'}_{\beta N}\partial_\rho x^{'\alpha}\underbrace{\partial_\sigma x^{'N}}_{=0}) \end{cases} \tag{6}$$

$$\Rightarrow \quad [\rho N,\ \sigma] = \underbrace{\frac{1}{2}\partial_N a^{'}_{\alpha\beta}}_{=[\alpha N,\ \beta]'}\partial_\rho x^{'\alpha}\partial_\sigma x^{'\beta} \tag{7}$$

$$= [\alpha N,\ \beta]^{'}\partial_\rho x^{'\alpha}\partial_\sigma x^{'\beta} \tag{8}$$

This confirms the tensor character of $[\rho N,\ \sigma]$

⋄

$[\rho\sigma,\ N]$ this follows immediately from the previous and considering $[\rho\sigma,\ N] = -[\rho N,\ \sigma]$ see(2.639)

⋄

$[\boldsymbol{\rho N},\ \boldsymbol{N}]$

We prove the case for $[\boldsymbol{NN},\ \boldsymbol{\rho}]$ as $[\rho N,\ N] = -[NN,\ \rho]$

$$[NN,\ \rho] = \frac{1}{2}\partial_\rho a_{NN} \quad \text{see (2.639)} \tag{9}$$

$$= \frac{1}{2}\partial_\rho (a'_{mn}\partial_N x'^m \partial_N x'^n) \tag{10}$$

$$= \begin{cases} \frac{1}{2}(\partial_\rho a'_{mn}\partial_N x'^m \partial_N x'^n \\ +a'_{mn}\partial_\rho x'^n \underbrace{\partial_{N\rho} x'^m}_{=0} \\ +a'_{mn}\partial_\rho x'^m \underbrace{\partial_{N\rho} x'^n}_{=0}) \end{cases} \tag{11}$$

$$= \frac{1}{2}\partial_\rho a'_{mn}\partial_N x'^m \partial_N x'^n \tag{12}$$

$$= \begin{cases} \frac{1}{2}(\partial_\rho a'_{\alpha\beta} \underbrace{\partial_N x'^\alpha}_{=0} \underbrace{\partial_N x'^\beta}_{=0} \\ +\partial_\rho a'_{NN} \underbrace{\partial_N x'^N}_{=1} \underbrace{\partial_N x'^N}_{=1} \\ +\partial_\rho a'_{\alpha N} \underbrace{\partial_N x'^\alpha}_{=0} \underbrace{\partial_N x'^N}_{=1} \\ +\partial_\rho a'_{\beta N} \underbrace{\partial_N x'^\alpha}_{=0} \underbrace{\partial_N x'^N}_{=1}) \end{cases} \tag{13}$$

$$\Rightarrow \quad [NN,\ \rho] = \frac{1}{2}\partial_\rho a'_{NN} \tag{14}$$

$$\Leftrightarrow \quad [NN,\ \rho] = \underbrace{\frac{1}{2}\partial_\alpha a'_{NN}}_{=[NN,\ \alpha]'} \partial_\rho x'^\alpha \tag{15}$$

$$\Rightarrow \quad [NN,\ \rho] = [NN,\ \alpha]' \partial_\rho x'^\alpha \tag{16}$$

This confirms the tensor character of $[NN,\ \rho]$ and consequently of $[\rho N,\ N]$

$\diamond$

$[NN,\ N]$

$$[NN,\ N] = \frac{1}{2}\partial_N a_{NN} \quad \text{see (2.639)} \tag{17}$$

$$= \frac{1}{2}\partial_N (a^{'}_{mn}\partial_N x^{'m}\partial_N x^{'n}) \tag{18}$$

$$= \begin{cases} \frac{1}{2}(\partial_N a^{'}_{mn}\underbrace{\partial_N x^{'m}}_{\delta^m_N}\underbrace{\partial_N x^{'n}}_{\delta^n_N} \\ \\ +a^{'}_{mn}\partial_N x^{'n}\underbrace{\partial_{NN} x^{'m}}_{=0} \\ \\ +a^{'}_{mn}\partial_N x^{'m}\underbrace{\partial_{NN} x^{'n}}_{=0}) \end{cases} \tag{19}$$

$$= \frac{1}{2}\partial_N a^{'}_{mn}\delta^m_N\delta^n_N \tag{20}$$

$$= \underbrace{\frac{1}{2}\partial_N a^{'}_{NN}}_{=[NN,\ N]'} \tag{21}$$

$$\Rightarrow \quad [NN,\ N] = [NN,\ N]^{'} \tag{22}$$

This confirms the tensor character of $[NN,\ N]$ as an invariant under transformation of the coordinates $x^1, \ldots, x^{N-1}$

◇

$$\Gamma^\rho_{N\sigma},\ \Gamma^N_{\rho\sigma},\ \Gamma^\rho_{NN},\ \Gamma^N_{N\rho},\ \Gamma^N_{NN}$$

For the Christoffel symbols of the second kind we use:

$$\Gamma^r_{st} = a^{rk}[st,\ k] \tag{23}$$

$$a^{rk} = a^{'mn}\frac{\partial x^r}{\partial x^{'m}}\frac{\partial x^k}{\partial x^{'n}} \tag{24}$$

$$\text{(2.631) page 65:} \qquad a^{N\rho} = 0 \tag{25}$$

$\boldsymbol{\Gamma^{\rho}_{N\sigma}}$

$$\Gamma^{\rho}_{N\sigma} = a^{\rho k}[N\sigma,\ k] \tag{26}$$
$$= a^{\rho\tau}[N\sigma,\ \tau] + \underbrace{a^{\rho N}}_{=0}[N\sigma,\ N] \tag{27}$$
$$(24):\quad = a^{'mn}\frac{\partial x^{\rho}}{\partial x'^{m}}\frac{\partial x^{\tau}}{\partial x'^{n}}[N\sigma,\ \tau] \tag{28}$$
$$\text{also (see previous results):}\quad [N\sigma,\ \tau] = [N\mu,\ \nu]^{'}\partial_{\sigma}x^{'\mu}\partial_{\tau}x^{'\nu} \tag{29}$$

And so,

$$\Gamma^{\rho}_{N\sigma} = a^{'mn}\frac{\partial x^{\rho}}{\partial x'^{m}}\frac{\partial x^{\tau}}{\partial x'^{n}}[N\mu,\ \nu]^{'}\partial_{\sigma}x^{'\mu}\partial_{\tau}x^{'\nu} \tag{30}$$
$$= a^{'mn}\frac{\partial x^{\rho}}{\partial x'^{m}}\underbrace{\frac{\partial x^{'\nu}}{\partial x'^{n}}}_{=\delta^{\nu}_{n}}[N\mu,\ \nu]^{'}\partial_{\sigma}x^{'\mu} \tag{31}$$
$$= a^{'\theta\nu}[N\mu,\ \nu]^{'}\frac{\partial x^{\rho}}{\partial x'^{\theta}}\frac{\partial x^{'\mu}}{\partial x^{\sigma}} + \underbrace{a^{'N\nu}}_{=0}[N\mu,\ \nu]^{'}\frac{\partial x^{\rho}}{\partial x'^{N}}\frac{\partial x^{'\mu}}{\partial x^{\sigma}} \tag{32}$$
$$= \underbrace{a^{'\theta\nu}[N\mu,\ \nu]^{'}}_{=\Gamma'^{\theta}_{N\mu}}\frac{\partial x^{\rho}}{\partial x'^{\theta}}\frac{\partial x^{'\mu}}{\partial x^{\sigma}} \tag{33}$$
$$\Rightarrow\quad \Gamma^{\rho}_{N\sigma} = \Gamma'^{\theta}_{N\mu}\frac{\partial x^{\rho}}{\partial x'^{\theta}}\frac{\partial x^{'\mu}}{\partial x^{\sigma}} \tag{34}$$

So, $\Gamma^{\rho}_{N\sigma}$ is a 2^{nd} order mixed tensor (contravariant in ρ, covariant in σ.)

⋄

$\boldsymbol{\Gamma^{N}_{\rho\sigma}}$

$$\Gamma^{N}_{\rho\sigma} = a^{Nk}[\rho\sigma,\ k] \tag{35}$$
$$= \underbrace{a^{N\tau}}_{=0}[\rho\sigma,\ \tau] + a^{NN}[\rho\sigma,\ N] \tag{36}$$
$$(8):\quad = a^{'NN}[\alpha\beta,\ N]^{'}\partial_{\rho}x^{'\alpha}\partial_{\sigma}x^{'\beta} \tag{37}$$
$$\text{considering :}\quad a^{'N\tau} = 0 \quad \text{we can write this as:} \tag{38}$$
$$\Gamma^{N}_{\rho\sigma} = a^{'N\tau}[\alpha\beta,\ \tau]^{'}\partial_{\rho}x^{'\alpha}\partial_{\sigma}x^{'\beta} + a^{'NN}[\alpha\beta,\ N]^{'}\partial_{\rho}x^{'\alpha}\partial_{\sigma}x^{'\beta} \tag{39}$$
$$= \underbrace{a^{'Nk}[\alpha\beta,\ k]^{'}}_{=\Gamma'^{N}_{\alpha\beta}}\partial_{\rho}x^{'\alpha}\partial_{\sigma}x^{'\beta} \tag{40}$$
$$\Rightarrow\quad \Gamma^{N}_{\rho\sigma} = \Gamma'^{N}_{\alpha\beta}\partial_{\rho}x^{'\alpha}\partial_{\sigma}x^{'\beta} \tag{41}$$

So, $\Gamma^{N}_{\rho\sigma}$ is a 2^{nd} order mixed tensor (covariant in both indices)

⋄

$\boldsymbol{\Gamma^{\rho}_{NN}}$

$$\begin{aligned}
\Gamma^{\rho}_{NN} &= a^{\rho k}[NN,\ k] && (42)\\
&= a^{\rho\tau}[NN,\ \tau] + \underbrace{a^{\rho N}}_{=0}[NN,\ N] && (43)\\
(16): \quad &= a^{'mn}\frac{\partial x^{\rho}}{\partial x'^{m}}\frac{\partial x^{\tau}}{\partial x'^{n}}[NN,\ \alpha]^{'}\partial_{\tau}x^{'\alpha} && (44)\\
&= a^{'mn}\frac{\partial x^{\rho}}{\partial x'^{m}}\underbrace{\frac{\partial x^{\alpha}}{\partial x'^{n}}}_{=\delta^{\alpha}_{n}}[NN,\ \alpha]^{'} && (45)\\
&= a^{'\tau\alpha}[NN,\ \alpha]^{'}\frac{\partial x^{\rho}}{\partial x'^{\tau}} + \underbrace{a^{'N\alpha}}_{=0}[NN,\ \alpha]^{'}\frac{\partial x^{\rho}}{\partial x'^{N}} && (46)\\
&= \underbrace{a^{'\tau\alpha}[NN,\ \alpha]^{'}}_{=\Gamma'^{\tau}_{NN}}\frac{\partial x^{\rho}}{\partial x'^{\tau}} && (47)\\
\Rightarrow \quad \Gamma^{\rho}_{NN} &= \Gamma'^{\tau}_{NN}\frac{\partial x^{\rho}}{\partial x'^{\tau}} && (48)
\end{aligned}$$

So, Γ^{ρ}_{NN} is a 1^{st} order contravariant tensor.

⋄

$\boldsymbol{\Gamma^{N}_{N\rho}}$

$$\begin{aligned}
\Gamma^{N}_{N\rho} &= a^{Nk}[N\rho,\ k] && (49)\\
&= a^{NN}[N\rho,\ N] + \underbrace{a^{N\tau}}_{=0}[N\rho,\ \tau] && (50)\\
(16): \quad &= a^{'NN}[N\alpha,\ N]^{'}\partial_{\rho}x^{'\alpha} && (51)\\
\text{considering}: \quad a^{'N\tau} &= 0 \quad \text{we can write this as:} && (52)\\
&= a^{'N\tau}[N\alpha,\ \tau]^{'}\partial_{\rho}x^{'\alpha} + a^{'NN}[N\alpha,\ N]^{'}\partial_{\rho}x^{'\alpha} && (53)\\
&= \underbrace{a^{'Nk}[N\alpha,\ k]^{'}}_{=\Gamma'^{N}_{N\alpha}}\partial_{\rho}x^{'\alpha} && (54)\\
\Rightarrow \quad \Gamma^{N}_{N\rho} &= \Gamma'^{N}_{N\alpha}\partial_{\rho}x^{'\alpha} && (55)
\end{aligned}$$

So, $\Gamma^{N}_{N\rho}$ is a 1^{st} order covariant tensor.

⋄

$\boldsymbol{\Gamma^{N}_{NN}}$

$$\Gamma^{N}_{NN} = a^{Nk}[NN,\ k] \tag{56}$$

$$= a^{NN}[NN,\ N] + \underbrace{a^{N\tau}}_{=0}[NN,\ \tau] \tag{57}$$

$$= a^{'NN}[NN,\ N] \tag{58}$$

$$(22): \quad = a^{'NN}[NN,\ N]^{'} \tag{59}$$

$$\Rightarrow \quad \Gamma^{N}_{NN} = \Gamma^{'N}_{NN} \tag{60}$$

So, Γ^{N}_{NN} is an invariant.

♦

2.56 p79-exercise 10

If θ, ϕ are colatitude and azimuth on a sphere, and we take

$$x^1 = \theta\cos\phi,\ x^2 = \theta\sin\phi$$

Calculate the Christoffels symbols for the coordinate system x^1, x^2 and show that they vanish at the point $\theta = 0$

We have, page 48 (2.507) : $$\Gamma'^r_{mn} = \Gamma^s_{pq}\frac{\partial x'^r}{\partial x^s}\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n} + \frac{\partial x'^r}{\partial x^s}\frac{\partial^2 x^s}{\partial x'^m \partial x'^n} \tag{1}$$

The calculations are really basic but lengthy and only train your skills in basic calculus. So I will only calculate Γ'^1_{11}.
We know that in a spherical coordinate system all Γ'^s_{pq} vanish except for $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^\phi_{\phi\theta} = \cot\theta$.

$$\boldsymbol{\Gamma'^1_{11} = \Gamma^\theta_{\phi\phi}\partial_\theta x^1\partial_1\phi\partial_1\phi + 2\Gamma^\phi_{\phi\theta}\partial_\phi x^1\partial_1\theta\partial_1\phi + \partial_\theta x^1\partial^2_{11}\theta + \partial_\phi x^1\partial^2_{11}\phi} \tag{2}$$

$$\text{we have}\ \begin{cases} x^1 = \theta\cos\phi \\ x^2 = \theta\sin\phi \end{cases} \Rightarrow \begin{cases} \theta = \sqrt{(x^1)^2 + (x^2)^2} \\ \phi = \arctan\frac{x^2}{x^1} \end{cases} \tag{3}$$

$$\text{so}\ \begin{cases} \partial_1\theta = \frac{x^1}{\theta} = \cos\phi & \partial_1\phi = -\frac{x^2}{\theta^2} = -\frac{\sin\phi}{\theta} \\ \partial_\theta x^1 = \cos\phi & \partial_\phi x^1 = -\theta\sin\phi \\ \partial^2_{11}\theta = \frac{\theta^2-(x^1)^2}{\theta^3} = \frac{\sin^2\phi}{\theta} & \partial^2_{11}\phi = 2\frac{x^1x^2}{\theta^4} = 2\frac{\cos\phi\sin\phi}{\theta^2} \end{cases} \tag{4}$$

$$\Gamma'^1_{11} = \Gamma^\theta_{\phi\phi}\cos\phi\frac{\sin^2\phi}{\theta^2} + 2\Gamma^\phi_{\phi\theta}(-\theta\sin\phi)\cos\phi(-\frac{\sin\phi}{\theta}) + \cos\phi\frac{\sin^2\phi}{\theta} + (-\theta\sin\phi)(2\frac{\cos\phi\sin\phi}{\theta^2}) \tag{5}$$

$$= -\sin\theta\cos\theta\frac{\cos\phi\sin^2\phi}{\theta^2} + 2\cot\theta\cos\phi\sin^2\phi + \frac{\cos\phi\sin^2\phi}{\theta} - 2\frac{\cos\phi\sin^2\phi}{\theta} \tag{6}$$

$$= (2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2})\cos\phi\sin^2\phi \tag{7}$$

◇

Does $\mathbf{\Gamma'^1_{11}}$ vanish for $\theta \to 0$?
The problematic term in (7) is $2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2}$ for which $\lim_{\theta\to 0} = \pm\infty \mp \infty \mp \infty$ is undefined.

$$\text{Consider} \quad L_+ = \lim_{\theta\to 0_+} 2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} \tag{8}$$

$$\text{We have} \quad \sin\theta, \theta \geq 0 \quad \sin\theta \leq \theta \quad \tfrac{1}{\theta} \leq \tfrac{1}{\sin\theta} \quad 0 \geq \cos\theta \leq 1 \tag{9}$$

$$\text{so} \quad L_+ = \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\sin\theta} - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} \tag{10}$$

$$\geq \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\theta} - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} \tag{11}$$

$$\geq \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\theta} - \frac{1}{\theta} - \frac{\cancel{\theta}\cos\theta}{\theta^{\cancel{2}}} \tag{12}$$

$$\geq \lim_{\theta\to 0_+} \frac{\cos\theta - 1}{\theta} \tag{13}$$

$$\text{(l'Hospitale rule)} \Rightarrow \quad L_+ \geq 0 \tag{14}$$

$$\text{also} \quad L_+ = \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\sin\theta} - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} \tag{15}$$

$$\leq \lim_{\theta\to 0_+} 2\frac{\cos\theta}{\sin\theta} - \frac{\cos\theta}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2} \tag{16}$$

$$\leq \lim_{\theta\to 0_+} \cos\theta(\frac{2}{\sin\theta} - \frac{1}{\theta} - \frac{\sin\theta}{\theta^2}) \tag{17}$$

$$\frac{\sin\theta}{\theta} \leq 1 \quad \Rightarrow \quad \leq \lim_{\theta\to 0_+} \cos\theta(\frac{2}{\sin\theta} - \frac{1}{\theta} - \frac{1}{\theta}) \tag{18}$$

$$\frac{1}{\sin\theta} \geq \frac{1}{\theta} \quad \Rightarrow \quad \leq \lim_{\theta\to 0_+} \cos\theta(\frac{2}{\theta} - \frac{1}{\theta} - \frac{1}{\theta}) \tag{19}$$

$$\leq 0 \tag{20}$$

$$\text{(14) and ((20)} \quad \Rightarrow L_+ = 0 \tag{21}$$

Considering that

$$L_- = \lim_{\theta\to 0_-} 2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2}$$

is equivalent to

$$L_+^\alpha = \lim_{\alpha\to 0_+} -(2\cot\alpha - \frac{1}{\alpha} - \frac{\sin\alpha\cos\alpha}{\alpha^2})$$

(substitute $\theta = -\alpha$) we conclude that

$$L_- = -L_+ = 0$$

And so

$$\Gamma'^1_{11}\Big|_{\theta=0} = (2\cot\theta - \frac{1}{\theta} - \frac{\sin\theta\cos\theta}{\theta^2})\cos\phi\sin^2\phi\Big|_{\theta=0} = 0$$

♦

2.57 p79-exercise 11

If vectors T^r and S_r undergo parallel propagation along a curve, show that $T^n S_n$ is constant along the curve.

Parallel propagation of T^r and S_r along a curve means

$$
\begin{aligned}
\frac{\delta T^r}{\delta u} &\equiv \frac{dT^r}{du} + \Gamma^r_{mn} T^m \frac{dx^n}{du} = 0 \\
\frac{\delta S_r}{\delta u} &\equiv \frac{dS_r}{du} - \Gamma^m_{rn} S_m \frac{dx^n}{du} = 0 \\
\text{Hence} \quad \frac{dT^r S_r}{du} &= S_r \left(-\Gamma^r_{mn} T^m \frac{dx^n}{du} \right) + T^r \left(\Gamma^m_{rn} S_m \frac{dx^n}{du} \right) \\
&= \frac{dx^n}{du} \left(\Gamma^m_{rn} S_m T^r - \Gamma^m_{rn} T^r S_m \right) \\
&= 0 \\
\Rightarrow \qquad T^r S_r &= C^{st}
\end{aligned}
$$

♦

2.58 p79-exercise 12

Deduce from 2.201 that the determinant $a = |a_{mn}|$ transforms according to

$$a^{'} = aJ^2 \quad , \quad J = \left| \frac{\partial x^r}{\partial x^{'s}} \right|$$

$$a^{'}_{rs} = a_{mk} \frac{\partial x^m}{\partial x^{'r}} \frac{\partial x^k}{\partial x^{'s}} \tag{1}$$

$$\text{Be} \qquad (J_{mr}) = \left(\frac{\partial x^m}{\partial x^{'r}} \right) \tag{2}$$

$$\text{In (1) put} \qquad c_{kr} = a_{mk} \frac{\partial x^m}{\partial x^{'r}} = a_{km} \frac{\partial x^m}{\partial x^{'r}} \tag{3}$$

$$\Rightarrow \qquad a^{'}_{rs} = c_{kr} \frac{\partial x^k}{\partial x^{'s}} \tag{4}$$

$$\text{or in matrix form} \qquad A^{'} = C^T J \quad \text{with} \quad C = AJ \tag{5}$$

$$\Rightarrow \qquad \left| A^{'} \right| = \left| (AJ)^T J \right| \tag{6}$$

$$\Leftrightarrow \qquad \left| A^{'} \right| = \left| J^T A^T J \right| \tag{7}$$

$$\Leftrightarrow \qquad \left| A^{'} \right| = |A| \, |J| \, |J| \tag{8}$$

$$\Leftrightarrow \qquad \left| A^{'} \right| = |A| \, |J|^2 \tag{9}$$

♦

2.59 p79-exercise 13

Using local Cartesians and applying the result of the previous exercise (Ex. 12), prove that, if the metric form is positive-definite, then the determinant $a = |a_{mn}|$ is always positive.

Using local Cartesian coordinates we have:

$$\Phi = \epsilon_i (dy^i)^2 \quad \text{with} \quad \Phi > 0 \tag{1}$$

$$\text{so} \quad |a_{mn}| = \begin{vmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & \epsilon_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \epsilon_N \end{vmatrix} = \prod_{i=1}^{N} \epsilon_i > 0 \tag{2}$$

$$\tag{3}$$

Going from (a_{mn}) to any arbitrary coordinate system, we have (see exercise 12 page 79)

$$\left|a'_{mn}\right| = \underbrace{|a_{mn}|}_{>0} \underbrace{J^2}_{>0} \tag{4}$$

$$\Rightarrow \quad \left|a'_{mn}\right| > 0 \tag{5}$$

♦

2.60 p79-exercise 14

In a plane, let $x^1,\ x^2$ be the distances of a general point from the point with rectangular coordinates $(1,0)$, $(-1,0)$, respectively. (These are bipolar coordinates.) Find the line element for these coordinates, and find the conjugate tensor a^{mn}.

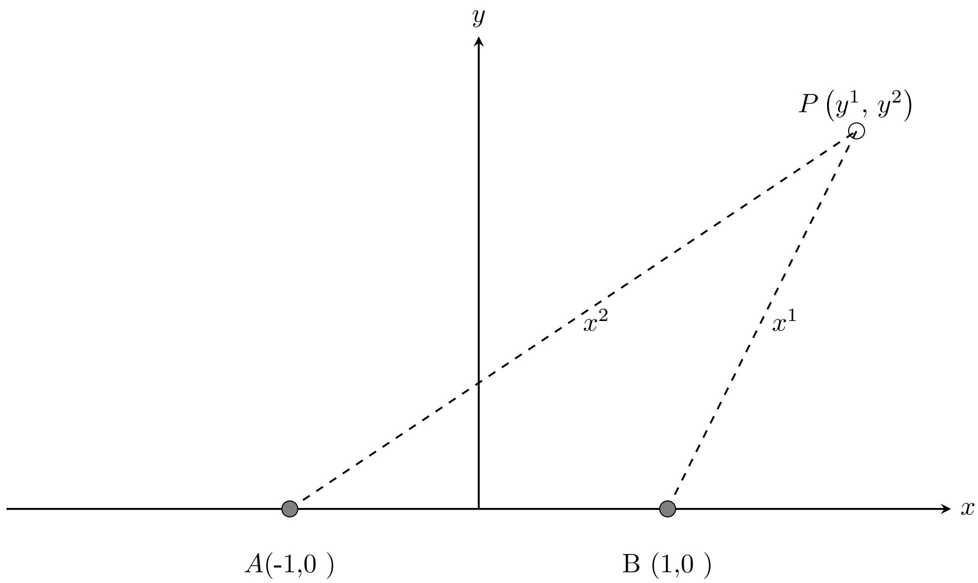


Figure 2.7: Bipolar coordinates

$$(x^1)^2 = \left(y^1-1\right)^2 + \left(y^2\right)^2 \tag{1}$$
$$(x^2)^2 = \left(y^1+1\right)^2 + \left(y^2\right)^2 \tag{2}$$
$$\partial_1(1) \quad \Rightarrow \quad x^1 = \left(y^1-1\right)\partial_1 y^1 + \left(y^2\right)\partial_1 y^2 \tag{3}$$
$$\partial_2(2) \quad \Rightarrow \quad x^2 = \left(y^1+1\right)\partial_2 y^1 + \left(y^2\right)\partial_2 y^2 \tag{4}$$
$$\partial_2(1) \quad \Rightarrow \quad 0 = \left(y^1-1\right)\partial_2 y^1 + \left(y^2\right)\partial_2 y^2 \tag{5}$$
$$\partial_1(2) \quad \Rightarrow \quad 0 = \left(y^1+1\right)\partial_1 y^1 + \left(y^2\right)\partial_1 y^2 \tag{6}$$
$$\Rightarrow \begin{cases} \text{(3)-(6):} \quad \partial_1 y^1 = -\frac{x^1}{2} \\ \\ \text{(4)-(5):} \quad \partial_2 y^1 = \frac{x^2}{2} \\ \\ \text{(6):} \qquad \partial_1 y^2 = \frac{y^1+1}{2y^2}x^1 \\ \\ \text{(5):} \qquad \partial_2 y^2 = -\frac{y^1-1}{2y^2}x^2 \end{cases} \tag{7}$$
$$\text{Hence,} \quad J = \left(\frac{\partial y^m}{\partial x^n}\right) = \begin{pmatrix} -\frac{x^1}{2} & \frac{x^2}{2} \\ \frac{y^1+1}{2y^2}x^1 & -\frac{y^1-1}{2y^2}x^2 \end{pmatrix} \tag{8}$$

Be A the metric tensor in Cartesian coordinate system. Then, going to an arbitrary coordinate system gives a metric tensor according to

$$A^{'} = J^T A J = J^T J \quad \text{as} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

$$\Rightarrow \quad A^{'} = \begin{pmatrix} -\frac{x^1}{2} & \frac{y^1+1}{2y^2}x^1 \\ \frac{x^2}{2} & -\frac{y^1-1}{2y^2}x^2 \end{pmatrix} \begin{pmatrix} -\frac{x^1}{2} & \frac{x^2}{2} \\ \frac{y^1+1}{2y^2}x^1 & -\frac{y^1-1}{2y^2}x^2 \end{pmatrix} \tag{10}$$

$$\Rightarrow \quad A^{'} = \frac{x^1 x^2}{4(y^2)^2} \begin{pmatrix} x^1 x^2 & -\left[(y^1)^2 + (y^2)^2 - 1)\right] \\ -\left[(y^1)^2 + (y^2)^2 - 1)\right] & x^1 x^2 \end{pmatrix} \tag{11}$$

We use plane geometry to express expression (11) as a function of only (x^1, x^2). For the triangle APB in the figure below, we have

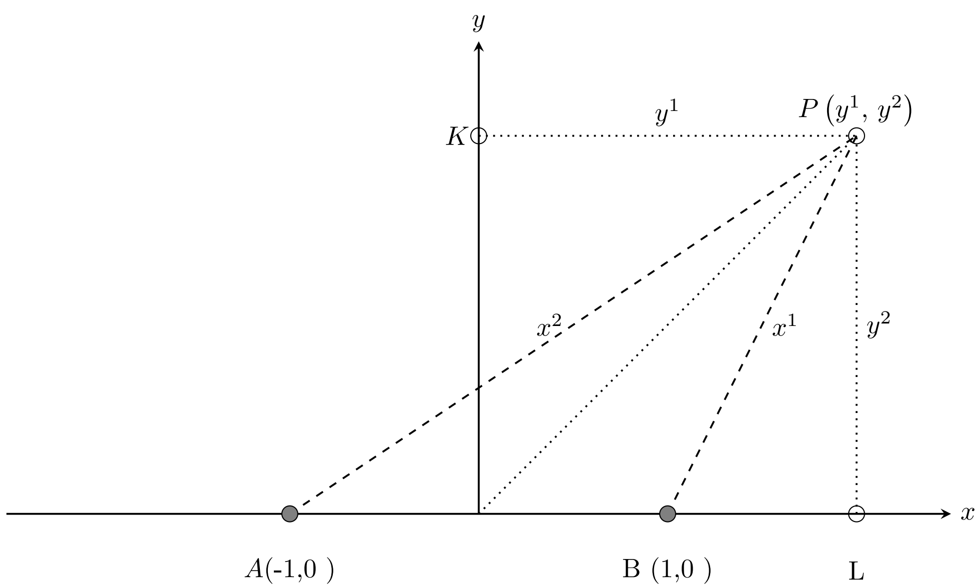


Figure 2.8: Bipolar coordinates versus Cartesian coordinates

$$|OP|^2 = \frac{(x^1)^2 + (x^2)^2}{2} - \underbrace{\frac{|AB|^2}{4}}_{=1}$$

$$\Rightarrow \quad (y^1)^2 + (y^2)^2 = \frac{(x^1)^2 + (x^2)^2}{2} - 1$$

The area K of the triangle can be expressed in two ways

$$\begin{aligned} K &= \frac{1}{2}y^2\,|AB| = y^2 \\ \text{and}\quad K &= \sqrt{s(s-x^1)(s-x^2)(s-2)} \\ \text{with}\quad s &= \frac{1}{2}(x^1+x^2+2) \\ \text{hence}\quad (y^2)^2 &= \frac{1}{16}(x^1+x^2+2)(x^1+2)(x^2+2)(x^1+x^2) \end{aligned}$$

So we get from (11):

$$A' = \frac{4x^1x^2}{(x^1+x^2+2)(x^1+2)(x^2+2)(x^1+x^2)}\begin{pmatrix} x^1x^2 & 2-\frac{(x^1)^2+(x^2)^2}{2} \\ 2-\frac{(x^1)^2+(x^2)^2}{2} & x^1x^2 \end{pmatrix}$$

For the conjugate metric tensor we start from expression (11) and invert the metric tensor

$$\left|A'\right| = \left[\frac{x^1x^2}{4(y^2)^2}\right]^2\left[(x^1x^2)^2 - \left((y^1)^2+(y^2)^2-1\right)^2\right] \tag{12}$$

$$\text{hence}\quad \left(a'^{mn}\right) = \frac{1}{|A'|}\frac{x^1x^2}{4(y^2)^2}\begin{pmatrix} x^1x^2 & \left[(y^1)^2+(y^2)^2-1\right] \\ \left[(y^1)^2+(y^2)^2-1\right] & x^1x^2 \end{pmatrix} \tag{13}$$

$$= \frac{4(y^2)^2\begin{pmatrix} x^1x^2 & \left[(y^1)^2+(y^2)^2-1\right] \\ \left[(y^1)^2+(y^2)^2-1\right] & x^1x^2 \end{pmatrix}}{x^1x^2\left[(x^1x^2)^2-\left((y^1)^2+(y^2)^2-1\right)^2\right]} \tag{14}$$

$$\text{with}\quad x^1x^2 = \sqrt{\left[(y^1-1)^2+(y^2)^2\right]\left[(y^1+1)^2+(y^2)^2\right]} \tag{15}$$

♦

2.61 p79-exercise 15

Given $\Phi = a_{mn}dx^m dx^n$, with $a_{11} = a_{22} = 0$ but $a_{12} \neq 0$, show that Φ may be written in the form

$$\Phi = \epsilon\Psi_1^2 - \epsilon\Psi_2^2 + \Phi_2$$

where Φ_2 is a homogeneous quadratic form in $dx^3, dx^4, \ldots, dx^N$, $\epsilon = \pm 1$, and where

$$\Psi_1 = \frac{1}{\sqrt{(2\epsilon a_{12})}}\left[a_{12}(dx^1 + dx^2) + (a_{13} + a_{23})dx^3 + \cdots + (a_{1N} + a_{2N})dx^N\right]$$

$$\Psi_2 = \frac{1}{\sqrt{(2\epsilon a_{12})}}\left[a_{12}(-dx^1 + dx^2) + (a_{13} - a_{23})dx^3 + \cdots + (a_{1N} - a_{2N})dx^N\right]$$

Using local Cartesian coordinates: Be

$$\Phi = a_{mn}dx^m dx^n \tag{1}$$

and consider the following sequences of terms:

$$\Psi_1 = b_{11}dx^1 + b_{12}dx^2 + b_{13}dx^3 + \cdots + b_{1N}dx^N \tag{2}$$

$$\Psi_2 = b_{21}dx^1 + b_{22}dx^2 + b_{23}dx^3 + \cdots + b_{2N}dx^N \tag{3}$$

The expressions (2) and (3) contain all the terms of Φ with $(dx^1)^2, (dx^2)^2$ and dx^1dx^2. So, Φ can be expressed as

$$\Phi = \Psi_1^2 - \Psi_2^2 + \Phi_2 \tag{4}$$

With Φ_2 being a homogeneous form containing only terms in $dx^i dx^j \quad i,j > 2$.
We can express $\Psi_1^2 - \Psi_2^2$ as

$$\Psi_1^2 - \Psi_2^2 = \begin{cases} (b_{11}^2 - b_{21}^2)(dx^1)^2 + (b_{12}^2 - b_{22}^2)(dx^2)^2 + (b_{13}^2 - b_{23}^2)(dx^3)^2 + \cdots + (b_{1N}^2 - b_{2N}^2)(dx^N)^2 + \\ 2(b_{11}b_{12} - b_{21}b_{22})dx^1dx^2 + 2(b_{11}b_{13} - b_{21}b_{23})dx^1dx^3 + \cdots + 2(b_{11}b_{1N} - b_{21}b_{2N})dx^1dx^N + \\ 2(b_{12}b_{13} - b_{22}b_{23})dx^2dx^3 + 2(b_{12}b_{14} - b_{22}b_{24})dx^2dx^4 + \cdots + 2(b_{12}b_{1N} - b_{21}b_{2N})dx^2dx^N + \\ + \ldots 2(b_{1j}b_{1k} - b_{2j}b_{2k})dx^kdx^j + \ldots \quad j > 2,\ k \neq j \end{cases} \tag{5}$$

Equating the terms in (1) and (5) and taking into account (=given) $a_{11} = a_{22} = 0$ and $a_{12} \neq 0$

$$\begin{cases} b_{11}^2 - b_{21}^2 = 0 \\ b_{12}^2 - b_{22}^2 = 0 \\ 2\left(b_{11}b_{12} - b_{21}b_{22}\right) = a_{12} + a_{21} \end{cases}$$

$$\Rightarrow \quad \begin{cases} b_{11}^2 = b_{21}^2 \\ b_{12}^2 = b_{22}^2 \\ \left(b_{11}b_{12} - b_{21}b_{22}\right) = a_{12} \end{cases}$$

$$\Rightarrow \quad b_{11} = \pm b_{21} \quad b_{22} = \pm b_{12}$$

$$\Rightarrow \quad \begin{cases} \pm b_{11}b_{22} \mp b_{11}b_{22} = a_{12} \\ \text{or} \\ \pm b_{12}b_{21} \mp b_{12}b_{21} = a_{12} \end{cases}$$

As $a_{12} \neq 0$, the signs in the above expressions must be the same. Hence,

$$\begin{cases} 2b_{11}b_{22} = \pm a_{12} \\ 2b_{12}b_{21} = \pm a_{12} \end{cases}$$

(6) can be satisfied with an infinite combination of $b_{11}, b_{12}, b_{21}, b_{22}$. Choose,

$$b_{11} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{22} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{12} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{21} = -\sqrt{\frac{\epsilon a_{12}}{2}}$$
$$\text{with} \quad \epsilon = \pm 1 \quad \text{so that} \quad \epsilon a_{12} \geq 0$$

Put $\xi = \sqrt{\frac{\epsilon a_{12}}{2}}$. Then, (2) and (3) can be expressed as

$$\Psi_1 = \xi dx^1 + \xi dx^2 + b_{13}dx^3 + \cdots + b_{1N}dx^1 dx^N$$
$$\Psi_2 = -\xi dx^1 + \xi dx^2 + b_{23}dx^3 + \cdots + b_{2N}dx^1 dx^N$$

What are the $b_{.j} \quad (j > 2)$?
From (5), e.g. for $j = 3$, identifying the terms in dx^1dx^3 and dx^2dx^3 we see that

$$\begin{cases} \xi b_{13} + \xi b_{23} = a_{13} \\ \xi b_{13} - \xi b_{23} = a_{23} \end{cases}$$

$$\Rightarrow \quad \begin{cases} b_{13} = \frac{a_{13}+a_{23}}{2\xi} \\ b_{23} = \frac{a_{13}-a_{23}}{2\xi} \end{cases}$$

$$\text{or, in general} \quad \begin{cases} b_{1j} = \frac{a_{1j}+a_{2j}}{2\xi} \\ \\ b_{2j} = \frac{a_{1j}-a_{2j}}{2\xi} \end{cases} \qquad (j > 2)$$

Bring $\frac{1}{\xi}$ out in Ψ_1, Ψ_2. This gives

$$\Psi_1 = \frac{1}{\xi}\left(\xi^2(dx^1+dx^2)+\frac{a_{13}+a_{23}}{2}dx^3+\cdots+\frac{a_{1N}+a_{2N}}{2}dx^N\right)$$
$$\Psi_2 = \frac{1}{\xi}\left(\xi^2(-dx^1+dx^2)+\frac{a_{13}-a_{23}}{2}dx^3+\cdots+\frac{a_{1N}-a_{2N}}{2}dx^N\right)$$
$$\Rightarrow \quad \begin{cases} \Psi_1 = \frac{\sqrt{2}}{\sqrt{\epsilon a_{12}}}\left(\frac{\epsilon a_{12}}{2}(dx^1+dx^2)+\frac{a_{13}+a_{23}}{2}dx^3+\cdots+\frac{a_{1N}+a_{2N}}{2}dx^N\right) \\ \\ \Psi_2 = \frac{\sqrt{2}}{\epsilon\sqrt{\epsilon a_{12}}}\left(\frac{\epsilon a_{12}}{2}(-dx^1+dx^2)+\frac{a_{13}-a_{23}}{2}dx^3+\cdots+\frac{a_{1N}-a_{2N}}{2}dx^N\right) \end{cases}$$
$$\Leftrightarrow \quad \begin{cases} \Psi_1 = \frac{1}{\sqrt{2\epsilon a_{12}}}\left(\epsilon a_{12}(dx^1+dx^2)+(a_{13}+a_{23})dx^3+\cdots+(a_{1N}+a_{2N})dx^N\right) \\ \\ \Psi_2 = \frac{1}{\sqrt{2\epsilon a_{12}}}\left(\epsilon a_{12}(-dx^1+dx^2)+(a_{13}-a_{23})dx^3+\cdots+(a_{1N}-a_{2N})dx^N\right) \end{cases}$$

What about the factor $\epsilon=\pm1$ in the terms $\epsilon a_{12}(dx^1+dx^2)$ and $\epsilon a_{12}(-dx^1+dx^2)$? Consider the alternate form

$$\begin{cases} \Psi_1^{'} = \frac{1}{\sqrt{2\epsilon a_{12}}}\left(a_{12}(dx^1+dx^2)+(a_{13}+a_{23})dx^3+\cdots+(a_{1N}+a_{2N})dx^N\right) \\ \\ \Psi_2^{'} = \frac{1}{\sqrt{2\epsilon a_{12}}}\left(a_{12}(-dx^1+dx^2)+(a_{13}-a_{23})dx^3+\cdots+(a_{1N}-a_{2N})dx^N\right) \end{cases}$$

Let's have a look at the terms p_{ij} in $dx^i dx^j$ in $\Psi_1'^2 - \Psi_2'^2$

$$
\begin{aligned}
p_{11}(dx^1)^2 + 2p_{12}dx^1dx^2 + p_{22}(dx^2)^2 &= \frac{1}{2\epsilon a_{12}}\left((a_{12})^2(dx^1+dx^2)^2 - (a_{12})^2(-dx^1+dx^2)^2\right) && (6)\\
&= \frac{\epsilon a_{12}}{2}\left((dx^1+dx^2)^2 - (-dx^1+dx^2)^2\right) && (7)\\
&= \frac{\epsilon a_{12}}{2}\left((dx^1)^2 + 2dx^1dx^2 + (dx^2)^2 - (dx^1)^2 + 2dx^1dx^2 - (dx^2)^2\right) && (8)\\
&= 2\epsilon a_{12}dx^1dx^2 && (9)\\
2p_{13}dx^1dx^3 &= \frac{1}{2\epsilon a_{12}}(2a_{12}(a_{13}+a_{23})dx^1dx^3 + 2a_{12}(a_{13}-a_{23})dx^1dx^3) && (10)\\
&= \frac{1}{2\epsilon a_{12}}(4a_{12}a_{13})dx^1dx^3 && (11)\\
&= 2\epsilon a_{13}dx^1dx^3 && (12)\\
2p_{23}dx^2dx^3 &= \frac{1}{2\epsilon a_{12}}(2a_{12}(a_{13}+a_{23})dx^2dx^3 - 2a_{12}(a_{13}-a_{23})dx^2dx^3) && (13)\\
&= \frac{1}{2\epsilon a_{12}}(4a_{12}a_{23})dx^2dx^3 && (14)\\
&= 2\epsilon a_{23}dx^2dx^3 && (15)\\
2p_{34}dx^3dx^4 &= \frac{1}{2\epsilon a_{12}}\left(\begin{array}{c} 2(a_{13}+a_{23})(a_{14}+a_{24}) \\ -2(a_{13}-a_{23})(a_{14}-a_{24}) \end{array}\right)dx^3dx^4 && (16)\\
&= \frac{2\epsilon}{a_{12}}\left(a_{23}a_{14} + a_{13}a_{24}\right)dx^3dx^4 && (17)
\end{aligned}
$$

We rewrite now the metric form Φ as

$$\Phi = \epsilon\Psi_1^{'2} - \epsilon\Psi_2^{'2} + \Phi_2$$

From (9), (12) and (15) we see that the terms in dx^1, dx^2 in $\Phi = \epsilon\Psi_1^{'2} - \epsilon\Psi_2^{'2} + \Phi_2$ correspond to the expected metric form and that the other terms in $dx^j, dx^k,\ j,k \neq 1,2$ can be corrected in the remaining term Φ_2.

♦

2.62 p80-exercise 16

Find the null geodesics of a 4-space with line element

$$ds^2 = \epsilon\gamma\left(dx^2 + dy^2 + dz^2 - dt^2\right)$$

where γ is an arbitrary function of $x,\ y,\ z,\ t$.

We have:

$$(a_{mn}) = \gamma\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \quad (a^{mn}) = \frac{1}{\gamma}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{1}$$

$$\tag{2}$$

Be $(x^1, x^2, x^3, x4) \equiv (x, y, z, t)$.
The general conditions for a null geodesic are

$$\begin{cases} \frac{d^2x^r}{dv^2} + \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = 0 \\ \\ a_{mn}dx^m dx^n = 0 \end{cases} \tag{3}$$

When calculating $[mn, r] = \frac{1}{2}\left(\partial_n a_{mr} + \partial_m a_{nr} - \partial_r a_{mn}\right)$ we note that (a_{mn}) is a diagonal matrix, ans so is also (a^{mn}). Hence Γ^r_{mn} will contain only one term:

$$\Gamma^r_{mn} = a^{RR}[mn, R] \tag{4}$$

$$\Rightarrow \begin{cases} i) & m, n \neq R \ \wedge \ m \neq n & : & [mn, R] = 0 \\ \\ ii) & m \neq n = R \ \vee \ n \neq m = R & : & [mn, R] = \frac{1}{2}\partial_m a_{RR} \\ \\ iii) & m = n \neq R & : & [mn, R] = -\frac{1}{2}\partial_R a_{mn} \\ \\ iv) & m = n = R & : & [mn, R] = \frac{1}{2}\partial_R a_{RR} \end{cases} \tag{5}$$

and for the Γ^r_{mn}:

$$\Rightarrow \begin{cases} i) & m,n \neq R \ \wedge \ m \neq n & : & \Gamma^R_{mn} = 0 \\ \\ ii) & m \neq n = R \ \vee \ n \neq m = R & : & \Gamma^R_{nR} = \frac{1}{2\gamma}\partial_n\gamma \\ \\ iii) & m = n \neq R = \ 1,2,3 & : & \Gamma^R_{kk} = -\frac{1}{2\gamma}\partial_R\gamma \\ & m = n \neq R = 4 & : & \Gamma^4_{kk} = \frac{1}{2\gamma}\partial_R\gamma \\ \\ iv) & m = n = R & : & \Gamma^R_{RR} = \frac{1}{2\gamma}\partial_R\gamma \end{cases} \tag{6}$$

Let's compute

$$A_r \equiv \frac{d^2x^r}{dv^2} + \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} \tag{7}$$

and take $v = x^4$, so $x^1, \ x^2, \ x^3 = f(x^4)$. (In the following $d_kx^j \equiv \frac{dx^j}{d(x^k)}$, $d_k^2x^j \equiv \frac{d^2x^j}{d(x^k)^2}$)

$$\begin{cases} A_1 = & d_4^2x^1 + \Gamma^1_{mn}d_4x^md_4x^n \\ A_2 = & d_4^2x^2 + \Gamma^2_{mn}d_4x^md_4x^n \\ A_3 = & d_4^2x^3 + \Gamma^3_{mn}d_4x^md_4x^n \\ A_4 = & \Gamma^4_{mn}d_4x^md_4x^n \end{cases} \tag{8}$$

and get from (8) and (6):

$$A_1 - d_4^2x^1 = \begin{cases} \frac{1}{\gamma}\partial_2\gamma \ d_4x^1d_4x^2 + \frac{1}{\gamma}\partial_3\gamma \ d_4x^1d_4x^3 + \frac{1}{\gamma}\partial_4\gamma \ d_4x^1d_4x^4 \\ \\ + \quad \underbrace{\frac{1}{2\gamma}\partial_1\gamma(d_4x^1)^2}_{=\begin{cases} \frac{1}{\gamma}\partial_1\gamma(d_4x^1)(d_4x^1) \\ -\frac{1}{2\gamma}\partial_1\gamma(d_4x^1)^2 \end{cases}} \quad -\frac{1}{2\gamma}\partial_1\gamma(d_4x^2)^2 - \frac{1}{2\gamma}\partial_1\gamma(d_4x^3)^2 + \frac{1}{2\gamma}\partial_1\gamma(d_4x^4)^2 \end{cases} \tag{9}$$

$$= \begin{cases} \frac{1}{\gamma}\partial_1\gamma d_4x^1d_4x^1 + \frac{1}{\gamma}\partial_2\gamma \ d_4x^1d_4x^2 + \frac{1}{\gamma}\partial_3\gamma \ d_4x^1d_4x^3 + \frac{1}{\gamma}\partial_4\gamma \ d_4x^1d_4x^4 \\ \\ -\frac{1}{2\gamma}\partial_1\gamma(d_4x^1)^2 - \frac{1}{2\gamma}\partial_1\gamma(d_4x^2)^2 - \frac{1}{2\gamma}\partial_1\gamma(d_4x^3)^2 + \frac{1}{2\gamma}\partial_1\gamma(d_4x^4)^2 \end{cases} \tag{10}$$

$$\tag{11}$$

$$= \begin{cases} \frac{1}{\gamma}d_4x^1\left(\partial_1\gamma d_4x^1 + \partial_2\gamma \ d_4x^2 + \partial_3\gamma \ d_4x^3 + \partial_4\gamma \ d_4x^4\right) \\ \\ -\frac{1}{2\gamma}\partial_1\gamma\left(\underbrace{(d_4x^1)^2 + (d_4x^2)^2 + (d_4x^3)^2 - (d_4x^4)^2}_{=a_{mn}dx^mdx^n=0}\right) \end{cases} \tag{12}$$

$$\Rightarrow \quad A_1 = d_4^2x^1 + \frac{1}{\gamma}d_4x^1 < \boldsymbol{\nabla}\gamma|\partial_4\overline{x} > \tag{13}$$

with $\boldsymbol{\nabla}\gamma = (\partial_x\gamma, \ \partial_y\gamma, \ \partial_z\gamma, \ \partial_t\gamma)$ and $\partial_4\overline{x} = (\partial_tx, \ \partial_ty, \ \partial_tz, \ \partial_tt)$.

So for the geodesic with get for the first coordinate

$$d_4^2 x^1 + \frac{1}{\gamma} d_4 x^1 < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} >= 0$$

Doing analogous calculations give the following set of equations

$$d_4^2 x^k + \frac{1}{\gamma} d_4 x^k < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} >= 0 \tag{14}$$

Note that for $k = 4$ we have as $d_4 x^4 = 1$ and $d_4^2 x^4 = 0$:

$$\frac{1}{\gamma} < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} >= 0 \tag{15}$$

$$\Rightarrow \quad < \boldsymbol{\nabla}\gamma | \partial_4 \overline{x} >= 0 \tag{16}$$

Note that this means that $\boldsymbol{\nabla}\gamma$ and $\partial_4 \overline{x}$ are orthogonal 4-vectors.
So the set of equations in (14) reduce to the following set of 2^{nd} order differential equations:

$$\begin{cases} x^{'}{'} = 0 \\ y^{'}{'} = 0 \\ z^{'}{'} = 0 \\ t = t \end{cases} \Rightarrow \begin{cases} x = x_1 t + x_0 \\ y = y_1 t + y_0 \\ z = z_1 t + z_0 \\ t = t \end{cases} \tag{17}$$

This is the equations of 4-space cone, which was expected as the metric is localy a Minkowksi-like metric.

♦

2.63 p80-exercise 17

In a space V_N the metric tensor is a_{mn}. Show that the null geodesics are unchanged if the metric tensor is changed to $b_{mn} = \gamma a_{mn}$, γ being a function of the coordinates.

Null geodesics are determined by the following set of equations

$$\begin{cases} \frac{d^2x^r}{dv^2} + \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = 0 \\ \\ a_{mn}dx^m dx^n = 0 \end{cases} \tag{1}$$

$$\text{We have} \quad \Gamma^r_{mn} = a^{rs}[mn, s] \tag{2}$$

$$= \frac{1}{\gamma}a^{rs}\gamma[mn, s] \tag{3}$$

$$\text{We have also} \quad b^{rs} = \frac{1}{\gamma}a^{rs} \tag{4}$$

$$\text{Indeed} \quad a^{rs}a_{rs} = \delta^r_t \tag{5}$$

$$\Rightarrow \quad \frac{1}{\gamma}a^{rs}\gamma a_{rs} = \delta^r_t \tag{6}$$

$$\Rightarrow \quad \frac{1}{\gamma}a^{rs}b_{rs} = \delta^r_t \tag{7}$$

$$\Rightarrow \quad \frac{1}{\gamma}a^{rs} = b_{rs} \tag{8}$$

$$\text{So (3) becomes} \quad \Gamma^r_{mn} = b^{rs}\gamma[mn, s] \tag{9}$$

Be $[mn, s]'$ the Christoffel symbol associated with the metric $b_{mn} = \gamma a_{mn}$. Then with

$$[mn, s]' = \frac{1}{2}\left(\partial_n\gamma a_{ms} + \partial_m\gamma a_{ns} - \partial_s\gamma a_{mn}\right)$$

we get

$$[mn, s]' = \gamma[mn, s] + \frac{1}{2}\left(a_{ms}\partial_n\gamma + a_{ns}\partial_m\gamma - a_{mn}\partial_s\gamma\right) \tag{10}$$

Substitute $\gamma[mn, s]$ from (10) in (9) we get

$$\Gamma^r_{mn} = \underbrace{b^{rs}\gamma[mn, s]'}_{=\Gamma'^r_{mn}} - \underbrace{\frac{1}{2} b^{rs} \left(a_{ms}\partial_n\gamma + a_{ns}\partial_m\gamma - a_{mn}\partial_s\gamma\right)}_{:=Q} \tag{11}$$

$$\text{with} \quad Q = \frac{1}{2} b^{rs} \left(a_{ms}\partial_n\gamma + a_{ns}\partial_m\gamma - a_{mn}\partial_s\gamma\right) \tag{12}$$

$$= \frac{1}{2}\gamma \left(\underbrace{a^{rs}a_{ms}}_{=\delta^r_m} \partial_n\gamma + \underbrace{a^{rs}a_{ns}}_{=\delta^r_n} \partial_m\gamma - a^{rs}a_{mn}\partial_s\gamma \right) \tag{13}$$

$$Q \times \frac{dx^m}{dv}\frac{dx^n}{dv} = \frac{1}{2}\gamma \left(\underbrace{\delta^r_m \frac{dx^m}{dv}}_{=\frac{dx^r}{dv}} \underbrace{\partial_n\gamma \frac{dx^n}{dv}}_{=\frac{d\gamma}{dv}} + \underbrace{\delta^r_n \frac{dx^n}{dv}}_{=\frac{dx^r}{dv}} \underbrace{\partial_m\gamma \frac{dx^m}{dv}}_{=\frac{d\gamma}{dv}} - a^{rs}\partial_s\gamma \underbrace{a_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv}}_{=0 \text{ by}(1)} \right) \tag{14}$$

$$= \frac{1}{2}\gamma \left(\frac{dx^r}{dv}\frac{d\gamma}{dv} + \frac{dx^r}{dv}\frac{d\gamma}{dv} \right) \tag{15}$$

$$= \left(\gamma \frac{d\gamma}{dv}\right) \frac{dx^r}{dv} \tag{16}$$

So from (11) and (16), (1) can be expressed as

$$\frac{d^2x^r}{dv^2} + \Gamma'^r_{mn} \frac{dx^m}{dv}\frac{dx^n}{dv} = \left(\gamma \frac{d\gamma}{dv}\right) \frac{dx^r}{dv} \tag{17}$$

By (2.449) page 46 we see that the vector $\frac{d^2x^r}{dv^2} + \Gamma'^r_{mn} \frac{dx^m}{dv}\frac{dx^n}{dv}$ is collinear to $\frac{dx^r}{dv}$. Also $b_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = 0$. And hence (17) determines the geodesic null lines. So both expressions (1) and (17) are equivalent to determine the same geodesic null lines in the space V_N equipped with the metric a_{mn} or b_{mn}

♦

2.64 p80-exercise 18

Are the relations

$$T_{|rs} = T_{|sr}$$

$$T_{r|sk} = T_{r|ks}$$

true (a) in curvilinear coordinates in Euclidean space, (b) in a general Riemannian space?

$$\boldsymbol{T_{|rs} \stackrel{?}{=} T_{|sr}}$$

We have $T|_r = \partial_r T$ (see (2.528) page 53)

$$\begin{array}{ccc} \boldsymbol{T_{|rs}} & & \boldsymbol{T_{|sr}} \\ & \leftrightarrow & \\ T|_{rs} = \partial^2_{rs} T - \Gamma^m_{rs} T|_m & & T|_{sr} = \partial^2_{sr} T - \Gamma^m_{sr} T|_m \end{array} \tag{1}$$

As $\partial^2_{rs} = \partial^2_{sr}$ and $\Gamma^m_{rs} = \Gamma^m_{sr}$ we can conclude that $\boldsymbol{T_{|rs} = T_{|sr}}$ in both cases (a) and (b).

$\diamond$

$$\boldsymbol{T_{r|sk} \stackrel{?}{=} T_{r|ks}}$$

$$\begin{array}{lcl} \boldsymbol{T_{r|sk}} & & \boldsymbol{T_{r|ks}} \\ A_{rs} \coloneqq T_{r|s} = \partial_s T_r - \Gamma^m_{rs} T_m & & B_{rk} \coloneqq T_{r|k} = \partial_k T_r - \Gamma^m_{rk} T_m \\ A_{rs|k} = \partial_k A_{rs} - \Gamma^m_{rk} A_{ms} - \Gamma^m_{sk} A_{rm} & & B_{rk|s} = \partial_s B_{rk} - \Gamma^m_{rs} B_{mk} - \Gamma^m_{ks} B_{rm} \\ A_{ms} = \partial_s T_m - \Gamma^n_{ms} T_n & & B_{mk} = \partial_k T_m - \Gamma^n_{mk} T_n \\ A_{rm} = \partial_m T_r - \Gamma^n_{rm} T_n & \leftrightarrow & B_{rm} = \partial_m T_r - \Gamma^n_{rm} T_n \\ A_{rs|k} = \begin{cases} \underbrace{\partial^2_{sk} T_r}_{*} - T_m \partial_k \Gamma^m_{rs} - \underbrace{\Gamma^m_{rs} \partial_k T_m}_{**} \\ - \underbrace{\Gamma^m_{rk} \partial_s T_m}_{***} + \Gamma^m_{rk} \Gamma^n_{ms} T_n \\ - \underbrace{\Gamma^m_{sk} \partial_m T_r}_{****} + \underbrace{\Gamma^m_{sk} \Gamma^n_{rm} T_n}_{*****} \end{cases} & & B_{rk|s} = \begin{cases} \underbrace{\partial^2_{ks} T_r}_{*} - T_m \partial_s \Gamma^m_{rk} - \underbrace{\Gamma^m_{rk} \partial_s T_m}_{***} \\ - \underbrace{\Gamma^m_{rs} \partial_k T_m}_{**} + \Gamma^m_{rs} \Gamma^n_{mk} T_n \\ - \underbrace{\Gamma^m_{ks} \partial_m T_r}_{****} + \underbrace{\Gamma^m_{ks} \Gamma^n_{rm} T_n}_{*****} \end{cases} \end{array} \tag{2}$$

$$\Rightarrow \quad A_{rs|k} - B_{rk|s} = T_m \left(\underbrace{\partial_s \Gamma^m_{rk} - \partial_k \Gamma^m_{rs} + \Gamma^n_{rk}\Gamma^m_{ns} - \Gamma^n_{rs}\Gamma^m_{nk}}_{:=R^s.rmn} \right) \tag{3}$$

So, $T_{r|sk} = T_{r|ks}$ only if $T_m R^s_{.rmn} = 0$ and as T_m is an arbitrary tensor, $R^s_{.rmn}$ must vanish for $T_{r|sk} = T_{r|ks}$.

Note:

Although Γ^n_{rk} is not a tensor, the quantity $R^s_{.rmn}$ is. Indeed, as both $A_{rs|k} - B_{rk|s}$ and T_m have the tensor character, this implies that $R^s_{.rmn}$ is a tensor. Now, for an Euclidean space equipped with Cartesian coordinates all Γ^n_{rk} are constant and vanish. So, $R^s_{.rmn} = 0$. Let's consider a change of coordinate system from Cartesian to curvilinear coordinate system. Then, by the tensor character of $R'^a_{.bcd}$ we have,

$$R'^a_{.bcd} = R^s_{.rmn} \partial_s x'^a \partial_{('b)} x^r \partial_{('c)} x^m \partial_{('d)} x^n \tag{4}$$

but $R^s_{.rmn} = 0$ in the Cartesian coordinate system and so is $R'^a_{.bcd}$

Conclusion:

In a general Riemannian space $T_{r|sk} \neq T_{r|ks}$ but $T_{r|sk} = T_{r|ks}$ in a curvilinear Euclidean space.

♦

2.65 p80-exercise 19

Consider a V_N with indefinite metric form. For all points P lying on the cone of geodesic null lines drawn from O, the definition 2.611 for Riemannian coordinates apparently breaks down. Revise the definition of Riemannian coordinates so as to include such points.

For geodesic null lines we have (2.445 page 46)

$$\begin{cases} \frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \\ a_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \end{cases} \tag{1}$$

or (2.448 page 46)

$$\begin{cases} \frac{d^2x^r}{dv^2} + \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = \lambda(v)\frac{dx^r}{dv} \\ a_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} = 0 \end{cases} \tag{2}$$

where by suitable choice of the parameter v , $\lambda(v)$ can be made any preassigned function of v.

$$(2) \quad \Rightarrow \quad \frac{d^2x^r}{dv^2} = \lambda\frac{dx^r}{dv} - \Gamma^r_{mn}\frac{dx^m}{dv}\frac{dx^n}{dv} \tag{3}$$

$$\Rightarrow \quad \frac{d^3x^r}{dv^3} = \frac{d\lambda}{dv}\frac{dx^r}{dv} + \lambda\frac{d^2x^r}{dv^2} + A^r_{.mns}\frac{dx^m}{dv}\frac{dx^n}{dv}\frac{dx^s}{dv} \tag{4}$$

$$\text{with} \quad \Rightarrow \quad A^r_{.mns} = -\partial_s\Gamma^r_{mn} + 2\Gamma^r_{sp}\Gamma^p_{mn} \tag{5}$$

Expanding x^r in a Taylor series around a point $O(a^r)$ we get (for ease of notation we put $p^r := \frac{dx^r}{dv}$)

$$x^r = a^r + vp^r + \frac{1}{2}v^2\lambda p^r - \frac{1}{2}v^2\Gamma^r_{mn}p^mp^n + \frac{1}{6}v^3\frac{d\lambda}{dv}p^r + \frac{1}{6}v^3\lambda\frac{dp^r}{dv} + \frac{1}{6}v^3A^r_{.mns}p^mp^np^s + \dots \tag{6}$$

$$= a^r + \left(v + \frac{1}{2}v^2\lambda + \frac{1}{6}v^3\frac{d\lambda}{dv}\right)p^r - \frac{1}{2}v^2\Gamma^r_{mn}p^mp^n + \frac{1}{6}v^3\lambda\frac{dp^r}{dv} + \frac{1}{6}v^3A^r_{.mns}p^mp^np^s + \dots \tag{7}$$

Put $x'^r := v\xi(v)p^r$ with $\xi(v) = \left(1 + \frac{1}{2}v\lambda + \frac{1}{6}v^2\frac{d\lambda}{dv}\right)$. Hence (7) becomes

$$x^r = a^r + x'^r + \frac{\lambda v^3}{6}\left(\frac{p'^r}{v\xi} - \frac{\xi + v\xi'}{v^2\xi^2}x'^r\right) - \frac{\Gamma^r_{mn}}{2\xi^2}x'^mx'^n + \frac{A^r_{.mns}}{6\xi^3}x'^mx'^nx'^s + \dots \tag{8}$$

$$= a^r + \tau(v)x'^r + \frac{\lambda v^2}{6\xi}p'^r - \frac{\Gamma^r_{mn}}{2\xi^2}x'^mx'^n + \frac{A^r_{.mns}}{6\xi^3}x'^mx'^nx'^s + \dots \tag{9}$$

$$\text{with} \qquad \tau(v) := 1 - \frac{\lambda v}{6\xi^2}\left(\xi + v\xi'\right) \tag{10}$$

Is the Jacobian non-zero?

$$\frac{\partial x^r}{\partial x'^q} = \tau\delta^r_q + \frac{\lambda v^2}{6\xi}\frac{d\delta^r_q}{dv} - \frac{\Gamma^r_{mq}}{\xi^2}x'^m + \frac{A^r_{.mnq}}{2\xi^3}x'^mx'^n + \dots \tag{11}$$

In the infinitesimal neighbourhood of O we have

$$\begin{cases} v \to 0 \\ \xi \to 1 \\ \xi^{'} \to \frac{\lambda}{2} \\ x^{'m} \to 0 \\ \tau \to 1 \end{cases} \tag{12}$$

$$\tag{13}$$

so that the Jacobian determinant becomes

$$\left| \frac{\partial x^r}{\partial x^{'q}} \right| = \left| \tau \delta_q^r \right| = \tau^N = 1 \tag{14}$$

Conclusion:
So in order to define a Riemannian coordinates system which is still valid on the null geodesics, it is sufficient to define the Riemannian coordinates around O as

$$x^{'r} \coloneqq v\xi p^r$$

with

$$\xi(v) = \left(1 + \frac{\lambda}{2}v + \frac{1}{6}\frac{d\lambda}{dv}v^2\right)$$

and λ, by suitable choice of v, being any pre-defined function of v.

♦

Curvature of space

3.1 p82 - Exercise

Explain why the surfaces of an ordinary cylinder and an ordinary cone are to be regarded as "flat" in the sense of our definition.

The reason is because those surfaces can be "unwrapped" like the figure below shows.

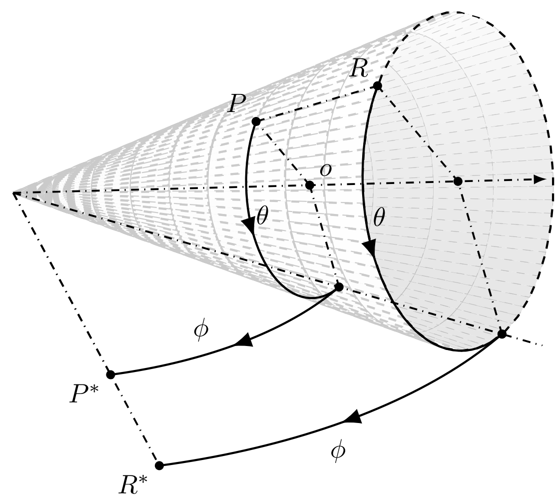


Figure 3.1: Unwrapping of a cone

For the cone, we can for each point P on the cone, lying on a distance h from the apex and making an angle θ, associate on a plane, tangent to the cone, a point P^* lying at the same distance h from the apex, taken as origin for the coordinate system, and making an angle $\phi = \theta \sin \alpha$ with α the angle of the cone. This pair of coordinates are polar coordinates with $r \in (-\infty, +\infty)$ and $\theta \in [0, 2k\pi)$. The same reasoning can be applied to a cylinder which is a cone with the apex at ∞. In that case the coordinate system becomes a Cartesian coordinate system.
As a continuous mapping exist from polar to orthogonal Cartesian coordinates both coordinate system can be written under the required form (3.101) and so can be called "flat".

♦

3.2 p83 - Exercise

What are the values of $R^{s}_{.rmn}$ in an Euclidean plane, the coordinates being rectangular Cartesians? Deduce the values of the components of this tensor for polar coordinates from its tensor character, or else by direct calculation.

$$R^{s}_{rmn} = 0$$

also in polar coordinates (see exercise 18 of Chapter II).

♦

3.3 p86 - Exercise

Show that in a V_2 all the components of the covariant curvature tensor either vanish or are expressible in terms of R_{1212}.

We have (3.115) and (3.116)

$$\begin{cases} R_{rsmn} = -R_{srmn} \\ R_{rsmn} = -R_{rsnm} \\ R_{rsmn} = R_{mnrs} \\ R_{rsmn} + R_{rmns} + R_{rnsm} = 0 \end{cases} \tag{1}$$

It is clear from the two first identities that in the tuples (rs) and (mn) both indices have to be different when the tensor is not 0. So we only have to consider R_{1212}, R_{1221}, R_{2112} and R_{2121}.
The two first identities gives us:

$$R_{1221} = -R_{1212} \tag{2}$$

$$R_{2112} = -R_{1212} \tag{3}$$

$$R_{2121} = -R_{2112} = R_{1212} \tag{4}$$

The third identity doesn't give us any additional information. The fourth identity gives us only trivial statements:

$$R_{1212} + \underbrace{R_{1122}}_{=0} + \underbrace{R_{1221}}_{=-R_{1212}} = 0 \tag{5}$$

$$\underbrace{R_{1221}}_{=-R_{1212}} + R_{1212} + \underbrace{R_{1122}}_{=0} = 0 \tag{6}$$

$$\underbrace{R_{2112}}_{=-R_{1212}} + \underbrace{R_{2121}}_{=R_{1212}} + \underbrace{R_{2211}}_{=0} = 0 \tag{7}$$

$$\underbrace{R_{2121}}_{=R_{1212}} + \underbrace{R_{2211}}_{=0} + \underbrace{R_{2112}}_{=-R_{1212}} = 0 \tag{8}$$

Conclusion:
We get the identities (2), (3) and (4) in function of R_{1212} and all vanish if $R_{1212} = 0$

♦

3.4 p86-87 - clarification

The number of independent components of the covariant curvature tensor in a space of N dimensions is

$$\frac{1}{12}N^2\left(N^2-1\right)$$

We have (3.115) and (3.116)

$$\begin{cases} R_{rsmn}=-R_{srmn} \\ R_{rsmn}=-R_{rsnm} \\ R_{rsmn}=R_{mnrs} \\ R_{rsmn}+R_{rmns}+R_{rnsm}=0 \end{cases} \tag{1}$$

It is clear from the two first identities that in the tuple (rs) and (mn) both indices have to be different when the component is not 0. So we only have to consider the component with the pair of tuples (r,s) and (m,n) with $r\neq s$ and $m\neq n$. For the tuple (r,s) we have N possibilities to draw an index for r but for s only $N-1$ indices remain as $r\neq s$. So for the tuple (r,s) we get $N(N-1)$ possibilities. But note by the first identity $R_{rsmn}=-R_{srmn}$ that we only have to consider the half of this quantity as once we have chosen a tuple (r,s) we also know the component for the tuple (s,r). So the total number of possibilities we have for (r,s) is $M=\frac{1}{2}N(N-1)$. The same yields for the tuple (mn). So, we get in total M^2 possibilities according to the two first identities.
The third identity $R_{rsmn}=R_{mnrs}$ puts an extra constraint on the number of possibilities as we have to subtract from M^2 the number of possibilities covered by this third identity. Note that, once we have chosen a tuple (rs) we have to exclude the tuple $(m,n)=(r,s)$ as the identity $R_{rsrs}=R_{rsrs}$ becomes trivial.. So for the first tuple we have M possibilities, but once chosen, only $M-1$ remain for the second tuple. So we get $M(M-1)$ possibilities. But, again we only have to take half of these possibilities as the identities $R_{rsmn}=R_{mnrs}$ and $R_{mnrs}=R_{rsmn}$ are equivalent.
So the total number of possibilities reduces to

$$M^2-\frac{1}{2}M(M-1) \quad \text{with} \quad M=\frac{1}{2}N(N-1)$$

What about the fourth identity

$$R_{rsmn}+R_{rmns}+R_{rnsm}=0$$

First we note that this identity implies that all indices are different as it becomes trivial in the other cases. This is a consequence of the first 3 identities. Indeed, we know already that

$$\begin{cases} r\neq s \\ m\neq n \\ (r,s)\neq(m,n) \end{cases} \tag{2}$$

Let's consider the following cases

$$\begin{cases} r=m & \to m\neq s \ m\neq n \ r\neq n & \to & R_{rsrn}+\underbrace{R_{rrns}}_{=0}+\underbrace{R_{rnsr}}_{=-R_{rnrs}=-R_{rsrn}}=0 \\ r=n & \to n\neq s \ m\neq n \ r\neq s & \to & R_{rsmr}+\underbrace{R_{rmrs}}_{=-R_{mrrs}=-R_{rsmr}}+\underbrace{R_{rrsm}}_{=0}=0 \\ s=m & \to m\neq r \ n\neq s \ r\neq s & \to & R_{rssn}+\underbrace{R_{rsns}}_{=-R_{rssn}}+\underbrace{R_{rnss}}_{=0}=0 \\ s=n & \to r\neq s \ m\neq s \ m\neq n & \to & R_{rsms}+\underbrace{R_{rmss}}_{=0}+\underbrace{R_{rssm}}_{=-R_{rsms}}=0 \end{cases}$$

So indeed, once two indices are equal, the fourth identity becomes trivial and does not put extra constraints to the number of possibilities. For the tuple (r,s,m,n) we have N possibilities to draw an index for r, for s only $N-1$, for m only $N-2$ and for n only $N-3$ indices remain as $r\neq s\neq m\neq n$. The maximum number of constraint generated by the fourth identity is thus

$$N(N-1)(N-2)(N-3)$$

But here again double counts occur. Indeed the fourth identity is true for the 6 tuples

$$(rsmn),(rmsn),(rmns),(rsnm),(rnsm),(rnms)$$

as first entry in the identity. The same reasoning is valid for the tuples $(n...)$, $(s...)$ $(m...)$.
So in total we get $6\times 4=24$ equivalent identities and the number of constraints generated by the fourth identity reduces to

$$\frac{1}{24}N(N-1)(N-2)(N-3)$$

Note that this number of constraints vanish for $N\leq 3$.
Putting it all together the number of independent components of R_{rsmn} becomes

$$\begin{aligned} \mho &= M^2-\frac{1}{2}M(M-1)-\frac{1}{24}N(N-1)(N-2)(N-3) \\ &= \frac{1}{2}M(M+1)-\frac{1}{24}N(N-1)(N-2)(N-3) \\ &= \frac{1}{8}N(N-1)\left(N(N-1)+2\right)-\frac{1}{24}N(N-1)(N-2)(N-3) \\ &= \frac{N}{24}\left(3N^2-6N^2+9N-6-N^3+3N^2+3N^2-9N-2N+6\right) \\ &= \frac{1}{12}N^2\left(N^2-1\right) \end{aligned}$$

♦

3.5 p87 - Exercise

Using the fact that the absolute derivative of the fundamental tensor vanishes, prove that 3.107 may be written

$$\frac{\delta^2 T_r}{\delta u \delta v} - \frac{\delta^2 T_r}{\delta v \delta u} = R_{rpmn} T^p \partial_u x^m \partial_v x^n$$

By 2.519 and 2.619 we have

$$\begin{aligned}
\frac{\delta T_r}{\delta u} &= \frac{\delta(a_{rk}T^k)}{\delta u} = \underbrace{\frac{\delta(a_{rk})}{\delta u}}_{=0} T^k + a_{rk}\frac{\delta(T^k)}{\delta u} \\
&= a_{rk} T^k_{|n} \partial_u x^n \\
\Rightarrow \quad \frac{\delta^2 T_r}{\delta u \delta v} &= \frac{\delta(a_{rk}T^k_{|n}\partial_u x^n)}{\delta v} \\
&= \underbrace{\frac{\delta(a_{rk})}{\delta v}}_{=0} T^k_{|n}\partial_u x^n + a_{rk}\frac{\delta(T^k_{|n})}{\delta v}\partial_u x^n + a_{rk}T^k_{|n}\delta\frac{(\partial_u x^n)}{\delta v} \\
&= a_{rk} \underbrace{\frac{\delta(T^k_{|n})}{\delta v}}_{=T^k_{|nm}\partial_v x^m} \partial_u x^n + a_{rk}T^k_{|n} \underbrace{\delta\frac{(\partial_u x^n)}{\delta v}}_{=(\partial_u x^n)_{|m}\partial_v x^m} \\
&= a_{rk}T^k_{|nm}\partial_v x^m \partial_u x^n + a_{rk}T^k_{|n} \underbrace{(\partial_u x^n)_{|m}}_{=\partial_m(\partial_u x^n)+\Gamma^n_{pm}\partial_u x^p} \partial_v x^m \\
&= a_{rk}T^k_{|nm}\partial_v x^m \partial_u x^n + a_{rk}T^k_{|n}\left(\underbrace{\partial_m(\partial_u x^n)}_{=\partial_u(\delta^n_m)=0} + \Gamma^n_{pm}\partial_u x^p\right)\partial_v x^m \\
&= a_{rk}T^k_{|nm}\partial_v x^m \partial_u x^n + a_{rk}T^k_{|n}\Gamma^n_{pm}\partial_u x^p \partial_v x^m
\end{aligned}$$

Hence we have

$$\begin{aligned}
\frac{\delta^2 T_r}{\delta v \delta u} &= a_{rk}T^k_{|nm}\partial_u x^m \partial_v x^n + a_{rk}T^k_{|n}\Gamma^n_{pm}\partial_v x^p \partial_u x^m \\
&= a_{rk}T^k_{|mn}\partial_u x^n \partial_v x^m + a_{rk}T^k_{|n}\Gamma^n_{mp}\partial_v x^m \partial_u x^p \\
\Rightarrow \quad \frac{\delta^2 T_r}{\delta u \delta v} - \frac{\delta^2 T_r}{\delta v \delta u} &= a_{rk}T^k_{|nm}\partial_v x^m \partial_u x^n - a_{rk}T^k_{|mn}\partial_u x^n \partial_v x^m \\
&= \left(a_{rk}T^k_{|nm} - a_{rk}T^k_{|mn}\right)\partial_u x^n \partial_v x^m \\
&= \left(\underbrace{T_{r|nm} - T_{r|mn}}_{=-R_{rpmn}T^p}\right)\partial_u x^n \partial_v x^m \\
&= -\underbrace{R_{rpmn}}_{=-R_{rpnm}} T^p \partial_u x^n \partial_v x^m = R_{rpmn}T^p\partial_u x^m \partial_v x^n
\end{aligned}$$

♦

3.6 p91 - Exercise

Would the study of geodesic deviation enable us to distinguish between a plane and a right circular cylinder?

For geodesic lines we have

$$\frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0$$

with the fundamental tensor for a cylinder (see exercise page 27)

$$(a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

As no element of this tensor is a function of the coordinates, it is clear that all Christoffel symbols vanish and the geodesic curve are solutions of the simple system of 2^{nd} order differential equations

$$\frac{d^2x^r}{du^2} = 0$$

Hence

$$x^r = \kappa^r u + \mu^r \tag{1}$$
$$\text{or} \quad x^r = \kappa^r u \tag{2}$$

by choosing the origin of the coordinates system with the initial condition position of the point. Choosing polar coordinates ϕ, z as coordinates, the distance one walks when following a geodesic is given by

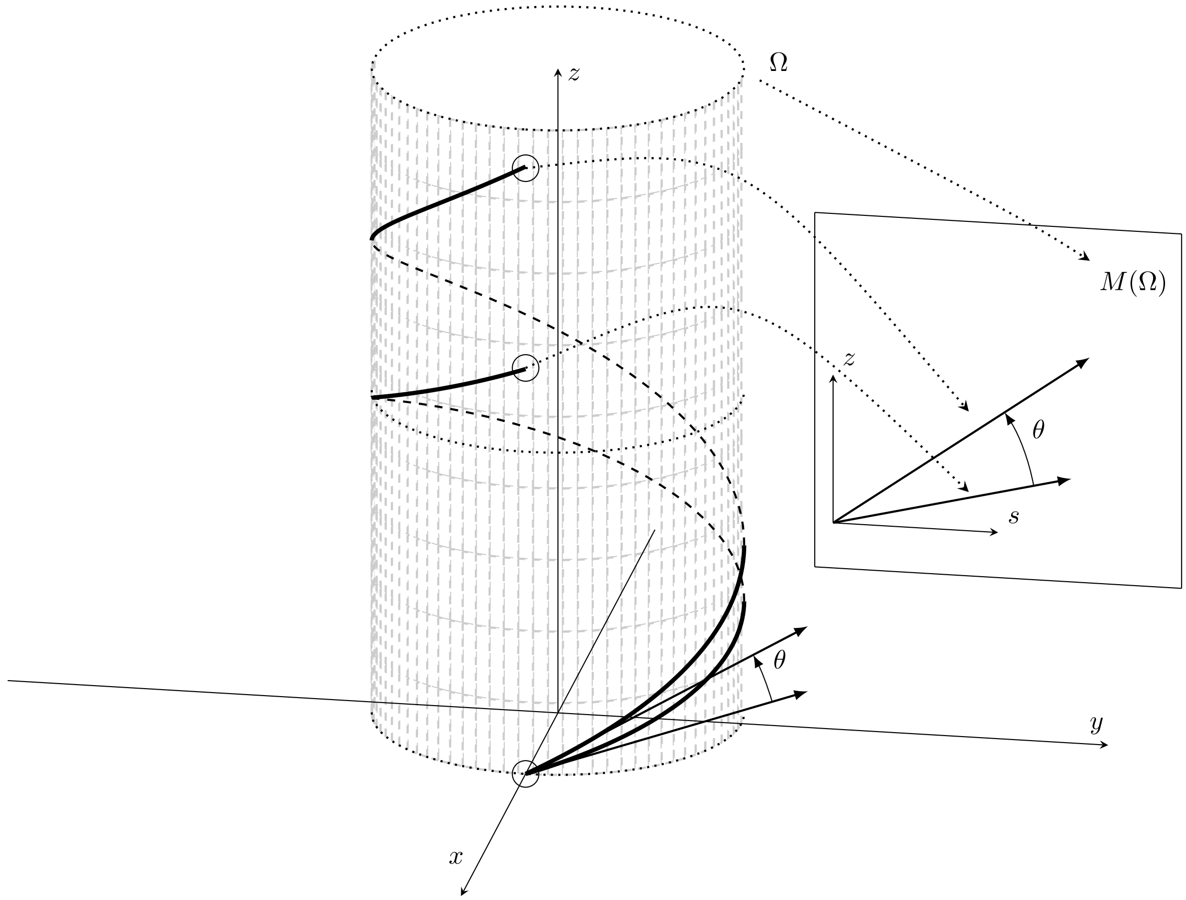
$$s = \int_{u_0}^{u_1} \sqrt{(r^2(d\phi)^2 + (dz)^2)}$$

and if we take $u = \phi$ as independent a parameter, by (2) we get

$$s - s_0 = \int_{\psi_0}^{\psi_1} \sqrt{(r^2 + \kappa^2)} d\psi \equiv g(\psi - \psi_0)$$
$$\text{or} \quad s = \int_{\psi_0}^{\psi_1} \sqrt{(r^2 + \kappa^2)} d\psi \equiv g\psi$$

by introducing transformed coordinates $s^{'} = s - s_0$ and $\psi^{'} = \psi - \psi_0$.
and thus we get

$$z = ms^{'} \tag{3}$$



;

Figure 3.2: Geodesics on a cylinder

The above figure illustrates what an observer living in the manifold Ω sees when walking along geodesics on the cylinder. He only can measure the distance $s^{'}$ and the displacement along z and by (3) can only draw a chart like the one seen on the right of the cylinder. A "flatlander" living in the mapped manifold $M(\Omega)$ would see the same chart when walking along geodesics in his plane.

Conclusion: No, studying the geodesic deviation on a right circular cylinder does not enable us to say on which surface we are.

♦

3.7 p93 - Exercise

For rectangular cartesians in Euclidean 3-space, show that the general solution of 3.311 is $\eta^r = A^r s + B^r$, where A, B^r are constants. Verify this by elementary geometry.

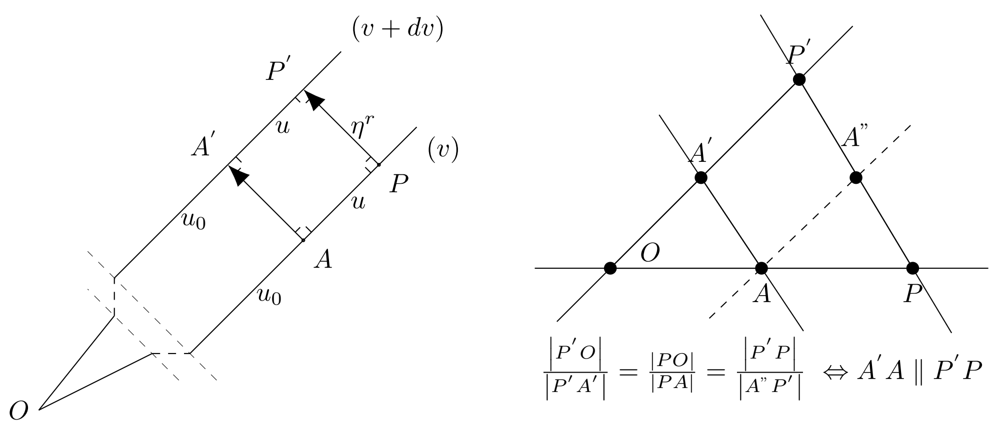
We have equation 3.111

$$\frac{\delta^2 \eta^r}{\delta s^2} + R^r_{.smn} p^s \eta^m p^n = 0 \tag{1}$$

From exercise on page 83 we know that $R^r_{.smn} = 0$ in an Euclidean space. Also, in such spaces, the Christoffels symbols vanish and equation (1) reduces to $\frac{d^2\eta}{ds^2} = 0$. And so,

$$\eta^r = A^r s + B^r$$

This is also easily deducted from a geometrical point of view. In an Euclidean space, the geodesics are straight lines. For an infinitesimal change in the geodesic family parameter v, we can assume that a vector, going perpendicular from 1 point from one geodesic with parameter v to another infinitesimal close geodesic with parameter $v + dv$, will also be perpendicular on this geodesic. This situation is depicted in fig. 3.3(a). We conclude that $\overrightarrow{AA'} \parallel \overrightarrow{PP'}$. This can also be deducted from Thales theorem (see fig. 3.3 (b)).



(a) (b)

Figure 3.3: Geometrical deduction of the geodesical deviation equation in an Euclidean space.

Hence, we than can say that,

$$\frac{\left|AA'\right|}{u_0} = \frac{\left|PP'\right|}{u}$$

or,

$$\eta^r = A^r u + B^r$$

as the reference point A can be chosen arbitrarily on the line AP.

♦

3.8 p96 - Clarification

> ... But under parallel propagation along a geodesic, a vector makes a constant angle with the geodesic; following the vector round the small quadrilateral, it is easy to see that the angle through which the vector has turned on completion of the circuit is E, the excess of the angle-sum of four right angles...

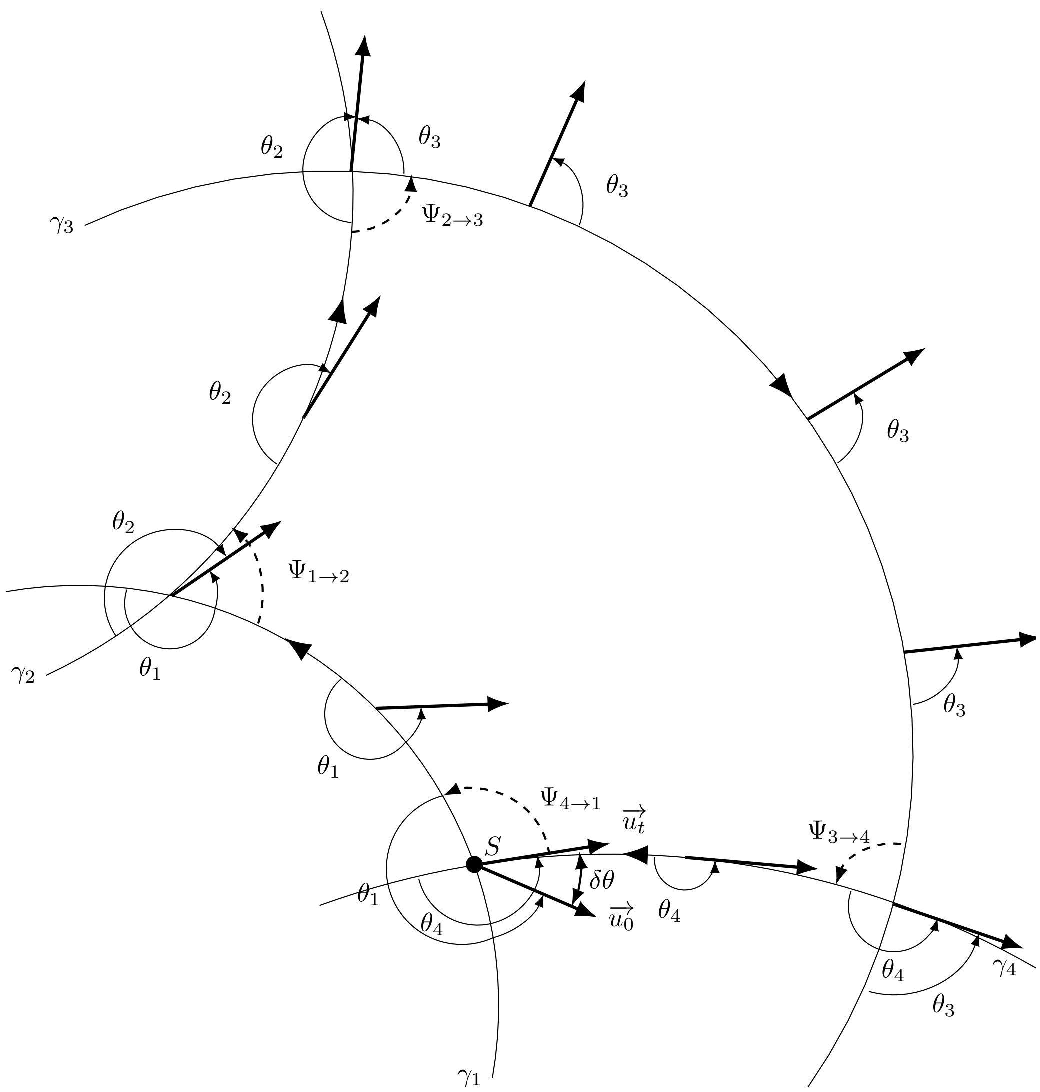


Figure 3.4: Parallel transportation along a closed path

Consider 4 geodesics $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ close to each other so that they form a small quadrilateral. At each intersection they form an angle $\Psi_{i\to i+1}$. A vector $\overrightarrow{u_0}$ is transported parallely along the path starting at the intersection S of γ_1, γ_4 and ends as vector $\overrightarrow{u_t}$ at the same point S. In general $\overrightarrow{u_0} \neq \overrightarrow{u_t}$ and will differ by a small angle $\delta\theta$. Let's investigate the relationship between $\delta\theta$ and the $\Psi_{i\to i+1}$.

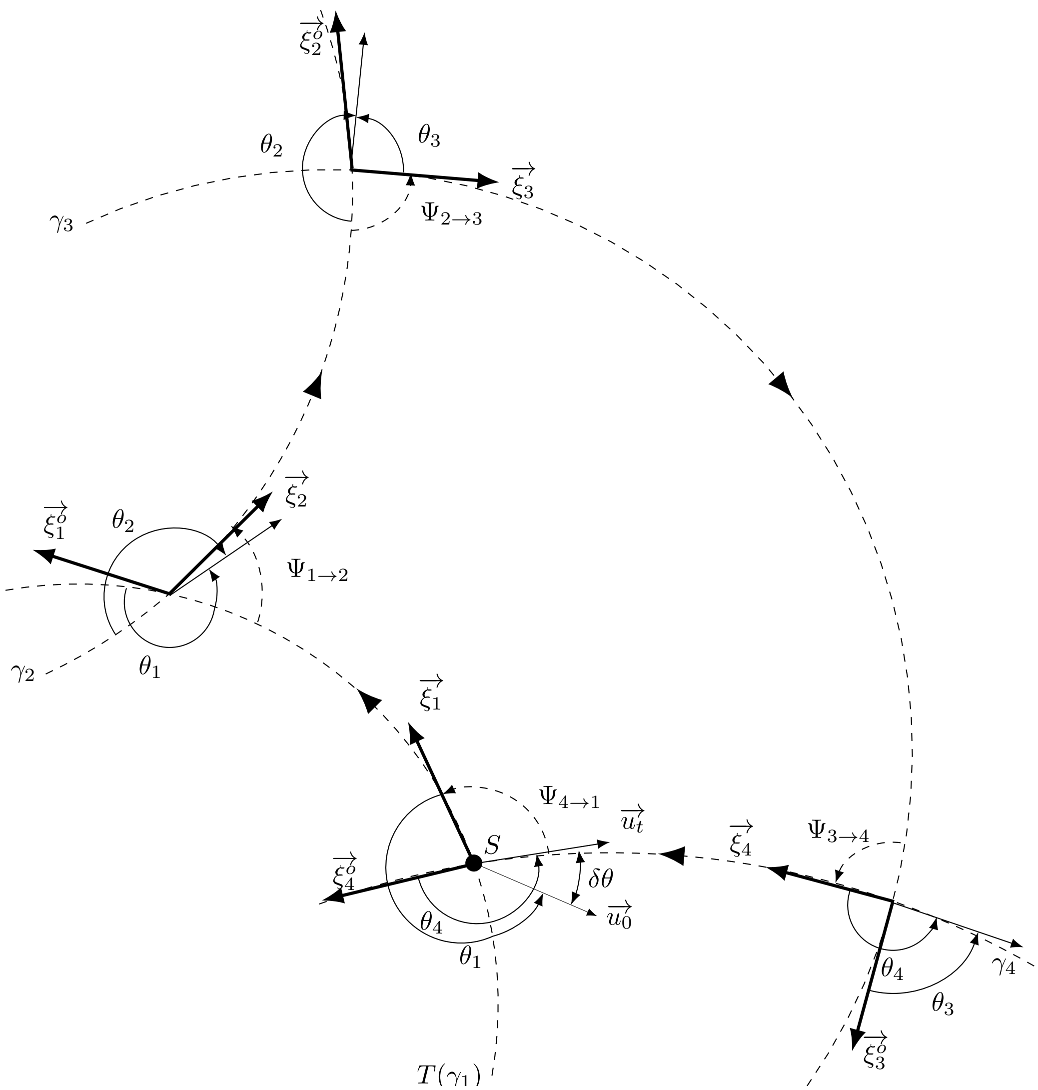


Figure 3.5: Relationship between parallel transportation along a closed path and the excess of the angle-sum over four right angles of a quadilateral.

Let ξ_i and ξ_i^o $(i = 1, 2, 3, 4)$ be respectively, tangent unit vectors to the geodesics at the beginning and at the end of the intersections of the geodesics. Be $\widehat{\xi}_i$ and $\widehat{\xi}_i^o$ $(i = 1, 2, 3, 4)$ the angles of these vectors relative to an arbitrary reference vector and be $\widehat{\tau}_i$ and $\widehat{\tau}_i^o$ $(i = 0, 2, 3, 4)$ the angles of the transported vector (relative to this arbitrary reference vector) at the intersections of the geodesics. Then,

$$\begin{cases} \widehat{\tau}_0 = \widehat{\xi}_1 + \theta_1 & \\ \widehat{\tau}_1 = \widehat{\xi}_1^o + \theta_1 & \widehat{\tau}_1 = \widehat{\xi}_2 + \theta_2 \\ \widehat{\tau}_2 = \widehat{\xi}_2^o + \theta_1 & \widehat{\tau}_2 = \widehat{\xi}_3 + \theta_3 \\ \widehat{\tau}_3 = \widehat{\xi}_3^o + \theta_3 & \widehat{\tau}_3 = \widehat{\xi}_4 + \theta_4 \\ \widehat{\tau}_4 = \widehat{\xi}_4^o + \theta_4 & \end{cases} \tag{1}$$

We have also

$$\begin{cases} \widehat{\xi}_2 - \widehat{\xi}_1^o = \Psi_{1\to 2} \\ \widehat{\xi}_3 - \widehat{\xi}_2^o = \Psi_{2\to 3} \\ \widehat{\xi}_4 - \widehat{\xi}_3^o = \Psi_{3\to 4} \\ \widehat{\xi}_1 - \widehat{\xi}_4^o = \Psi_{4\to 1} \\ \widehat{\tau}_4 - \widehat{\tau}_0 = \delta\theta \end{cases} \tag{2}$$

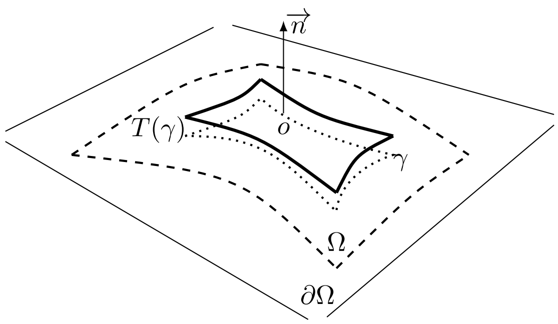
Combining (1) and (2)

$$\begin{cases} \Psi_{1\to 2} = \theta_1 - \theta_2 \\ \Psi_{2\to 3} = \theta_2 - \theta_3 \\ \Psi_{3\to 4} = \theta_3 - \theta_4 \\ \Psi_{4\to 1} = \theta_4 - \theta_1 - \delta\theta \end{cases} \tag{3}$$

and so

$$\delta\theta = -(\Psi_{1\to 2} + \Psi_{2\to 3} + \Psi_{3\to 4} + \Psi_{4\to 1}) \tag{4}$$

Note that these relationships are valid on the (curved) V_2 manifold. In order to go further we map the quadrilateral, on the manifold, on it's tangent plane (see fig. 3.6 (a) hereunder) - supposing that the quadrilateral is infinitesimally small and that we can find a conformal map from γ to $T(\gamma)$.



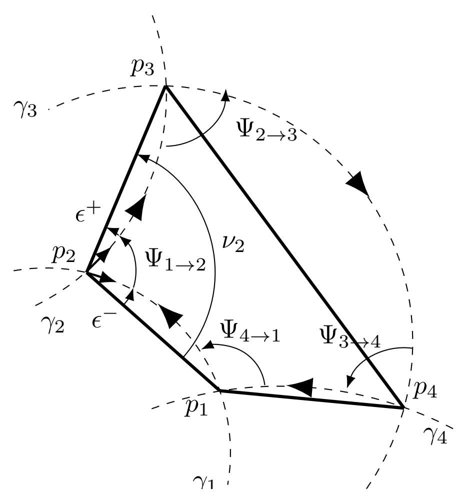


Figure 3.6: Relationship between parallel transportation along a closed path and the excess of the angle-sum over four right angles of a quadrilateral.

Let's look at the point p_2 on $\partial\Omega$. We have $\nu_2 = \epsilon^- + \Psi_{1\to 2} + \epsilon^+ = \Psi_{1\to 2} + \epsilon$. In general,

$$\sum_{i=1}^{4} \nu_i = 2\pi \tag{5}$$

$$\underbrace{\sum_{i=1}^{4} \Psi_{i\to i+1}}_{=-\delta\theta} + \sum_{i=1}^{4} \epsilon_i = \sum_{i=1}^{4} \frac{\pi}{2} \tag{6}$$

$$\Rightarrow \quad -\delta\theta = \sum_{i=1}^{4} (\frac{\pi}{2} - \epsilon_i) \tag{7}$$

Calling $\frac{\pi}{2} - \epsilon_i$ the excess, we get the assertion made.

♦

3.9 p98 - Clarification

> ... it is easy to see that the expansion takes the form
>
> **3.425.** $$\eta = \theta\left(s - \frac{1}{6}\epsilon K s^3 + \dots\right)$$

Expanding η in a power series gives

$$\eta = \underbrace{\eta|_0}_{=0} + \underbrace{\left.\frac{d\eta}{ds}\right|_0}_{=\theta} s - \frac{1}{2}\underbrace{\left.\frac{d^2\eta}{ds^2}\right|_0}_{=0} s^2 + \frac{1}{6}\left.\frac{d^3\eta}{ds^3}\right|_0 s^3 + \dots \quad (1)$$

$$\frac{d^2(1)}{ds^2} \quad \Rightarrow \quad \frac{d^2\eta}{ds^2} = \left.\frac{d^3\eta}{ds^3}\right|_0 s + \dots \quad (2)$$

$$\text{for } \lim_{s\to 0} \quad \text{we have } \eta \approx \theta s \quad \text{so (2)} \quad \Rightarrow \quad \frac{d^2\eta}{ds^2} = \left.\frac{d^3\eta}{ds^3}\right|_0 \frac{\eta}{\theta} + \dots \quad (3)$$

$$\lim_{s\to 0} \quad \Rightarrow \quad \left.\frac{d^3\eta}{ds^3}\right|_0 = \theta \underbrace{\lim_{s\to 0}\frac{1}{\eta}\frac{d^2\eta}{ds^2}}_{=-\epsilon K} \quad (4)$$

$$\Rightarrow \quad \eta = \theta\left(s - \frac{1}{6}\epsilon K s^3 + \dots\right) \quad (5)$$

♦

3.10 p102 - Clarification

Still using the same notation and Fig. 7, we have at B the three vectors $(T^r)_1,\ (T^r)_2,\ Y_r$ ... it follows that

$$\textbf{3.516.} \qquad (\Delta T^r)_A (Y_r)_{A'2} = -(\Delta T^r)_B (Y_r)_B$$

First we note that for an invariant "propagated parallely" along a curve we have

$$\frac{\delta(T^r Y_r)}{\delta u} = \frac{\delta T^r}{\delta u} Y_r + T^r \frac{\delta Y_r}{\delta u} = 0$$

In fig. 3.7 we use the following convention:

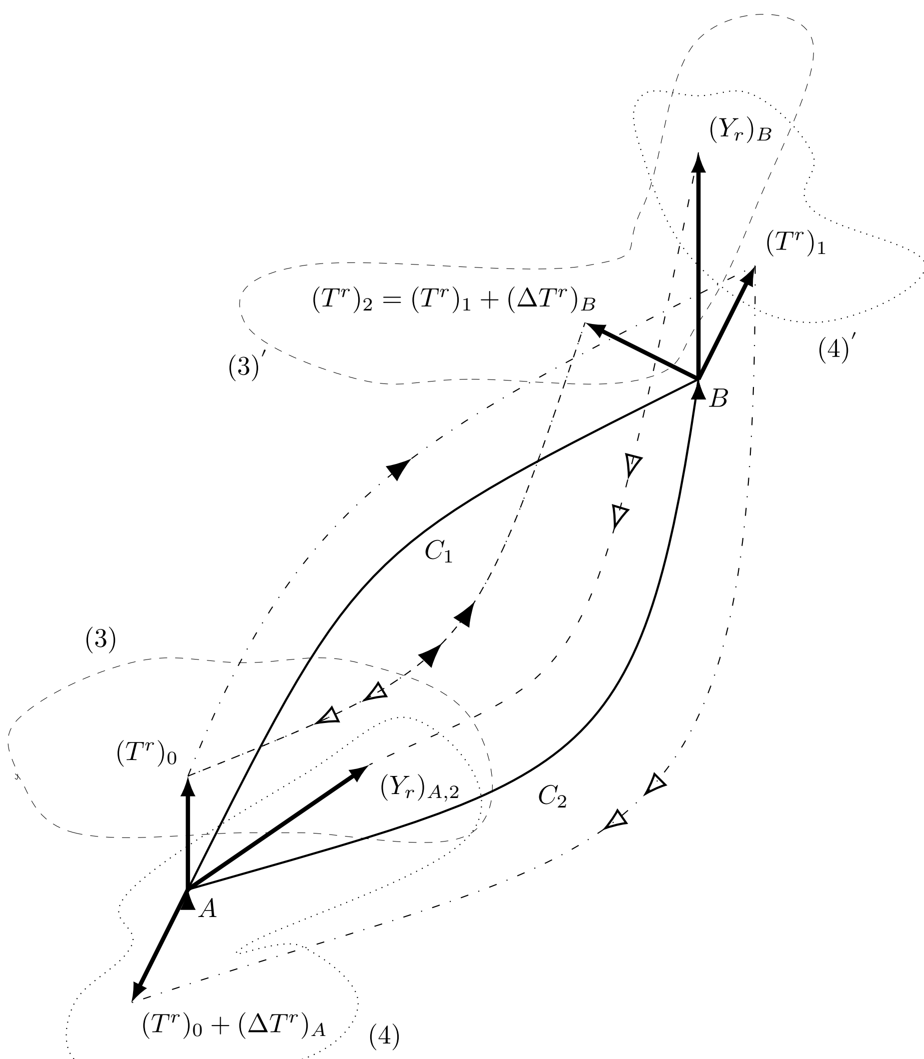


Figure 3.7: Parallel transportation along a closed path

- a one black arrowed line means a forward propagation from A to B along C_1

- a double black arrowed line means a forward propagation from A to B along C_2
- a double open arrowed line means a backward propagation from B to A along C_2

In order to find the angular displacement of the vector $(T^r)_0$ when propagated parallely from A to B and back to A along different paths, we follow the dash dotted line in Fig. 1.7. Following that path, we end with the vector $(T^r)_0 + (\Delta T^r)_A)$ in A and have the vector $(T^r)_1$ as intermediate forward propagation from A to B along C_1, that vector being transported backwards to A along C_2.

Note that for a vector the forward and backward propagation along the same curve is a null operation:

$$(T^r)_0 \underset{\underset{C_2}{A \to B}}{\to} (T^r)_2 \underset{\underset{C_2}{B \to A}}{\to} (T^r)_0$$

Also, we have in Fig.1.7.

$$(T^r)_0 \underset{\underset{C_1}{A \to B}}{\to} (T^r)_1 \underset{\underset{C_2}{B \to A}}{\to} (T^r)_0 + (\Delta T^r)_A$$

$$(Y_r)_B \underset{\underset{C_2}{B \to A}}{\to} (Y_r)_{A,2}$$

At A we form the following invariants

$$\begin{cases} ((T^r)_0 + (\Delta T^r)_A)\ (Y_r)_{A,2} \\ (T^r)_0\ (Y_r)_{A,2} \end{cases} \tag{1}$$

and at B

$$\begin{cases} (T^r)_1\ (Y_r)_B \\ (T^r)_2\ (Y_r)_B = ((T^r)_1 + (\Delta T^r)_B)\ (Y_r)_B \end{cases} \tag{2}$$

Due to the null effect of parallel propagation on invariants, we get

$$(T^r)_1\ (Y_r)_B = ((T^r)_0 + (\Delta T^r)_A)\ (Y_r)_{A,2} \tag{3}$$

$$((T^r)_1 + (\Delta T^r)_B)\ (Y_r)_B = (T^r)_0\ (Y_r)_{A,2} \tag{4}$$

$$(3)\text{-}(4) \quad \Rightarrow \quad -(\Delta T^r)_B)\ (Y_r)_B = (\Delta T^r)_A)\ (Y_r)_{A,2} \tag{5}$$

♦

3.11 p105 - Clarification

3.521.
$$\frac{dI}{dv} = \int_{u_1}^{u_2} \partial_v \left(T_n \partial_u x^n\right) du$$
$$= \int_{u_1}^{u_2} \frac{\delta T_n}{\delta v} \partial_u x^n du + \int_{u_1}^{u_2} T_n \frac{\delta \left(\partial_u x^n\right)}{\delta v} du.$$

Now $\frac{\delta T_n}{\delta v} = 0$, since T_r is propagated along *all* curves in V_n.

To better understand this last statement recall that from 3.515, we have

$$\left(\Delta T^r\right)_B \left(Y_r\right)_B = \int\int Y_r R^r_{.pmn} T^p \partial_u x^m \partial_v x^n du dv$$
$$= 0 \quad \text{as} \qquad R^r_{.pmn} = 0$$

As $\left(Y_r\right)_B$ is arbitrary we have $\left(\Delta T^r\right)_B = 0$. Consider fig. 3.8 below.

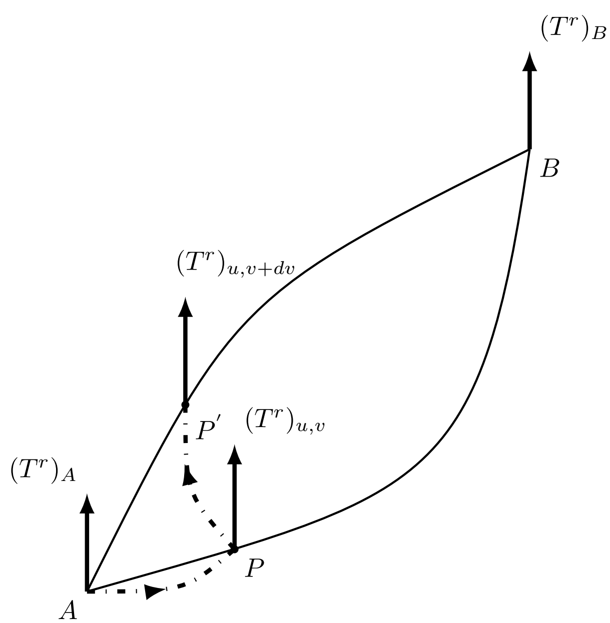


Figure 3.8: Parallel transportation along a path in a space with zero curvature tensor

Consider the path $A \rightarrow P \rightarrow P'$, P being situated at the parametric coordinates (u, v) and P' at $(u, v + dv)$. For this path we have also $\left(\Delta T^r\right)_{P,P'} = 0$. So $(T^r)_{u,v} = (T^r)_A$ and $(T^r)_{u,v+dv} = (T^r)_{u,v}$ and thus $\frac{\delta T_r}{\delta v} = 0$.

♦

3.12 p108 - Exercise 1

Taking polar coordinates on a sphere of radius a, calculate the curvature tensor, the Ricci tensor, and the curvature invariant.

We have

$$\Phi = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2 \tag{1}$$

We only have to calculate R_{1212} (see exercise page 86).

$$\begin{aligned} R_{\theta\phi\theta\phi} &= \partial_\theta \underbrace{[\phi\phi,\theta]}_{=-a^2\sin\theta\cos\theta} -\partial_\phi \underbrace{[\phi\theta,\theta]}_{=0} + \underbrace{\Gamma^\theta_{\phi\theta}}_{=0}[\theta\phi,\theta] + \underbrace{\Gamma^\phi_{\phi\theta}[\theta\phi,\phi]}_{=a^2\cos^2\theta} - \underbrace{\Gamma^\theta_{\phi\phi}}_{=0}[\theta\theta,\theta] - \underbrace{\Gamma^\phi_{\phi\phi}}_{=0}[\theta\theta,\phi] && (2)\\ &= a^2\sin^2\theta && (3)\end{aligned}$$

$$3.208.: \quad \frac{R_{11}}{a_{11}} = \frac{R_{22}}{a_{22}} = -\frac{R_{\theta\phi\theta\phi}}{det(a_{mn})} \quad \Rightarrow \quad \begin{cases} R_{11} = -1 \\ R_{12} = 0 \\ R_{22} = -\sin^2\theta \end{cases} \tag{4}$$

$$3.210.: \quad R = -\frac{2}{det(a_{mn})} R_{\theta\phi\theta\phi} \quad \Rightarrow \quad R = -\frac{2}{a^2} \tag{5}$$

♦

3.13 p108 - Exercise 2

Take as manifold V_2 the surface of an ordinary right circular cone, and consider one of the circular sections. A vector in V_2 is propagated parallely round this circle. Show that its direction is changed on completion of the circuit. Can you reconcile this result with the fact that V_2 is flat?

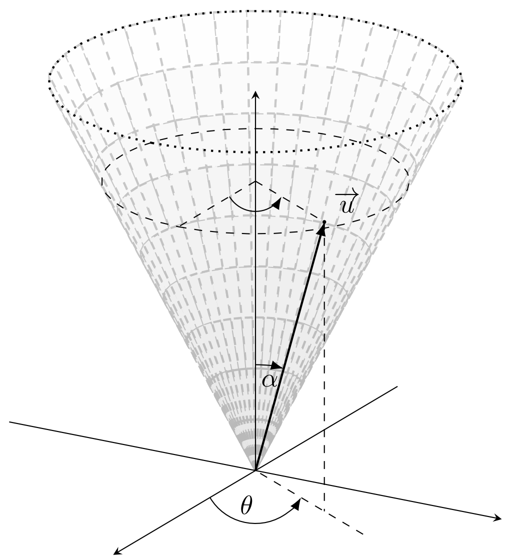


Figure 3.9: Intrinsic coordinates on a cone

We take as coordinate system (u, θ), embedded in the manifold, u being the distance of the generator of the considered point to the apex of the cone and θ the angle with a arbitrary vector laying in a plane perpendicular to the axis of the cone.
It is not hard to see that the fundamental form for this manifold is

$$\Phi = du^2 + \underbrace{k}_{= \sin^2 \alpha} u^2 d\theta^2$$

We have

$$(a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & ku^2 \end{pmatrix} \quad (a^{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{ku^2} \end{pmatrix}$$

$$\begin{pmatrix} [mn, u] \\ [mn, \theta] \end{pmatrix} = \begin{pmatrix} 0 & 0 & -ku \\ ku & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^u_{mn} \\ \Gamma^\theta_{mn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -ku \\ 0 & \frac{1}{u} & 0 \end{pmatrix}$$

Let's calculate the curvature tensor. From the exercise on page 86 we know that for a V_2 all components of the curvature tensor can be expressed in terms of R_{1212}. We have

$$R_{u\theta u\theta}=\begin{cases}\frac{1}{2}\left(\partial_{\theta u}a_{u\theta}+\partial_{u\theta}a_{\theta u}-\partial_{\theta\theta}a_{uu}-\partial_{uu}a_{\theta\theta}\right)\\ +a^{pq}\left([u\theta,p][\theta u,q]-[uu,p][\theta\theta,q]\right)\end{cases}=\begin{cases}-k\\ +a^{uu}\left(\underbrace{[u\theta,u][\theta u,u]-[uu,u][\theta\theta,u]}_{=0}\right)\\ +\underbrace{a^{u\theta}}_{=0}\left([u\theta,u][\theta u,\theta]-[uu,u][\theta\theta,\theta]\right)\\ +\underbrace{a^{\theta u}}_{=0}\left([u\theta,\theta][\theta u,u]-[uu,\theta][\theta\theta,u]\right)\\ +\underbrace{a^{\theta\theta}}_{=\frac{1}{ku^2}}\left(\underbrace{[u\theta,\theta][\theta u,\theta]}_{=k^2u^2}-\underbrace{[uu,\theta][\theta\theta,\theta]}_{=0}\right)\end{cases}$$

So indeed all components of the curvature tensor vanish and hence V_2 is flat.

Let's now calculate the parallel transportation of a vector T^r along a circle somewhere on the cone. Taking θ as the parameter of the curve, the equation of the curve is $(u=u_0,\ \theta)\quad \theta\in\ [0,2\pi)$. We have for parallel transportation along that curve $\frac{\delta T^r}{\delta\theta}=0$ and get

$$\begin{cases}\frac{dT^u}{d\theta}+\Gamma^u_{\theta\theta}T^\theta\frac{d\theta}{d\theta}=0\\ \frac{dT^\theta}{d\theta}+\Gamma^\theta_{u\theta}T^u\frac{d\theta}{d\theta}+\Gamma^\theta_{\theta u}T^\theta\underbrace{\frac{du}{d\theta}}_{=0}=0\quad\left(\frac{du}{d\theta}=0\quad\text{as}\quad u=C^{st}\right)\end{cases}\tag{1}$$

$$\Rightarrow\begin{cases}\dot{T^u}-ku_0T^\theta=0\\ \dot{T^\theta}+\frac{1}{u_0}T^u=0\end{cases}\tag{2}$$

$$\Rightarrow\begin{cases}\frac{\ddot{T^u}}{T^u}=-k\\ \dot{T^\theta}=-\frac{T^u}{u_0}\end{cases}\tag{3}$$

From (3a) we deduce that a solution can be of the form
$T^u=p^{'}\left(e^{(a\theta+b^{'})}+e^{-(a\theta+b^{'})}\right)$. Substituting in (3) we see that $a^2=-k\rightarrow a=\pm i\sqrt{k}$. Replacing $b^{'}$

by ib the solution for the system of differential equations (3) becomes

$$T^u = p^{'}\left(e^{i(\sqrt{k}\theta+b)} + e^{-i(\sqrt{k}\theta+b)}\right) \tag{4}$$

$$= p\cos\left(\sqrt{k}\theta + b\right) \tag{5}$$

$$\Leftrightarrow \quad = C_1 \sin\sqrt{k}\theta + C_2\cos\sqrt{k}\theta \tag{6}$$

$$\text{(6) in (3) gives:} \quad \begin{cases} T^u = C_1\sin\sqrt{k}\theta + C_2\cos\sqrt{k}\theta \\ \\ T^\theta = \frac{1}{\sqrt{k}u_0}\left(C_1\cos\sqrt{k}\theta - C_2\sin\sqrt{k}\theta\right) \end{cases} \tag{7}$$

$$\text{with} \quad \begin{cases} C_1 = \sqrt{k}u_0 \; T^\theta\big|_{\theta=0} \\ \\ C_2 = T^u\big|_{\theta=0} \end{cases} \tag{8}$$

$$\Rightarrow \quad \begin{cases} T^u = \sqrt{k}u_0T_0^\theta\sin\sqrt{k}\theta + T_0^u\cos\sqrt{k}\theta \\ \\ T^\theta = T_0^\theta\cos\sqrt{k}\theta - \frac{T_0^u}{\sqrt{k}u_0}\sin\sqrt{k}\theta \end{cases} \tag{9}$$

Let's now compute the angle ϕ between the starting vector T_0^r and the vector T^r parallely transported over an angle θ on the circle. We have (see **(2.301.)** and **(2.312.)**):

$$\begin{cases} \left|T_0^r\right|^2 = \left(T_0^u\right)^2 + ku_0^2\left(T_0^\theta\right)^2 \\ \\ \left|T^r\right|^2 = \left(T_0^u\right)^2 + ku_0^2\left(T_0^\theta\right)^2 \\ \\ \cos\phi = \frac{T_0^uT^u + ku_0^2T_0^\theta T^\theta}{\left(T_0^u\right)^2 + ku_0^2\left(T_0^\theta\right)^2} \end{cases}$$

The last equation in (19) becomes

$$\cos\phi = \cos\sqrt{k}\theta$$
$$\Rightarrow \quad \phi = \sqrt{k}\theta + 2m\pi \qquad (m = 0, \pm1, \pm2, \dots)$$

So for $\theta = 2\pi$ the angle between the starting vector and the transported vector is not $2m\pi$.

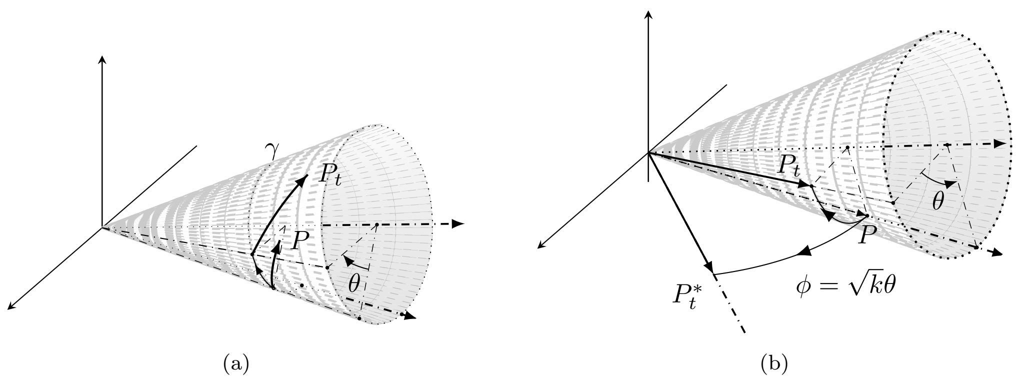


Figure 3.10: Relationship between parallel transportation along a circle of a cone and unwrapping a cone .

Fig. 3.10 illustrates the analogy between
(a): the $\parallel$ transportation of a vector P along a circular curve γ on the cone over an angle θ giving a vector P_t making an angle $\phi = \sqrt{k}\theta$ with the starting vector, and
(b): the result of "unwrapping" a cone. Be two vectors $\overrightarrow{OP}$ and $\overrightarrow{OP_t}$, P and P_t being two points placed at a distance $r\theta$ along a circle with radius r. Placing the vector $\overrightarrow{OP}$ in the XY-plane and unwrapping the cone over an angle θ will map the vector $\overrightarrow{OP_t}$ in the XY-plane to a vector $\overrightarrow{OP_t^*}$ making an angle $\phi = \sqrt{k}\theta$ with the vector $\overrightarrow{OP}$.
The transported vector will only coincide with the initial vector for $\sqrt{k} = \sin\alpha = \frac{1}{n} \quad (n = 1, 2, \dots)$.
Fig. 3.11 illustrates this in the $X - Y$-plane, for n=3. Only after a transportation over three periods, will the transported vector coincide with the initial vector. The dotted area represents the unwrapped cone.

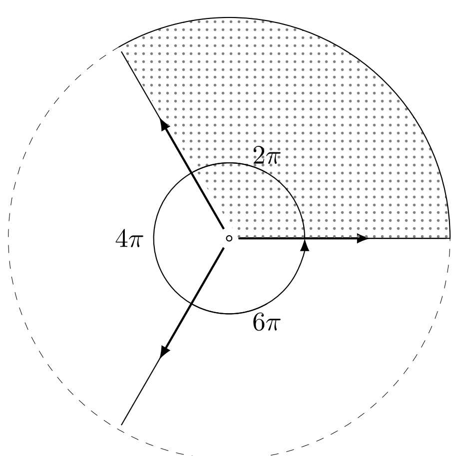


Figure 3.11: Parallel transportation along a circle of a cone with $\sin\alpha = \frac{1}{3}$.

♦

3.14 p109 - Exercise 3 and 4

Exercise 3.

Consider the equations

$$\left(R_{mn}-\theta a_{mn}\right)X^n=0$$

where R_{mn} is the Ricci tensor in a V_N $(N>2)$, θ an invariant, and X^n a vector. Show that, if these equations are to be consistent, θ must have one of a certain set of N values, and that the vectors X^n corresponding to different values of θ are perpendicular to one another. (The directions of these vectors are called the *Ricci principal directions*).

$$\left(R_{mn}-\theta a_{mn}\right)X^n=0 \tag{1}$$

$$(1)\times\left(a^{mp}\right)\quad\Rightarrow\quad a^{mp}R_{mn}X^n-\theta\underbrace{a^{mp}a_{mn}}_{=\delta_n^p}X^n=0 \tag{2}$$

$$\Rightarrow\quad a^{mp}R_{mn}X^n-\theta X^p=0 \tag{3}$$

Define $T_n^p=a^{mp}R_{mn}$, then (3) can be written in matrix form with $\mathbf{T}\equiv\left(T_n^p\right)$, $\mathbf{X}\equiv\left(X^p\right)$ and $\mathbf{I}\equiv\left(\delta_j^i\right)$

$$\left(\mathbf{T}-\theta\mathbf{I}\right)\mathbf{X}=0 \tag{4}$$

This is an eigenvector equation with $\mathbf{T}$ being Hermitian i.e. $\mathbf{T}^\dagger=\mathbf{T}$. Indeed, obviously the complex conjugat $\overline{\mathbf{T}}=\mathbf{T}$ and

$$\begin{aligned}\mathbf{T}^T&=\left(\mathbf{AR}\right)^T\\&=\mathbf{R}^T\mathbf{A}^T\\\Leftrightarrow\left(T_i^j\right)&=\left(R_{kj}\right)^T\left(a^{ik}\right)^T\\&=\left(R_{jk}\right)\left(a^{ki}\right)\end{aligned}$$

as both $R_{jk},\ a^{ki}$ are symmetric we have

$$\begin{aligned}\left(T_i^j\right)&=\left(R_{kj}\right)\left(a^{ik}\right)\\&=\left(T_j^i\right)\\\Rightarrow\qquad\mathbf{T}^\dagger&=\mathbf{T}\end{aligned}$$

This means that the N roots of $det\left(\mathbf{T}-\theta\mathbf{I_n}\right)=0$, which is a necessary condition to have equation (4) consistent, are real. Hence θ will take N values, being the eigenvalues of the transformation matrix $\mathbf{T}$. If all eigenvalues have multiplicity one, then the N eigenvectors in (4) corresponding to the N eigenvalues will be orthogonal to each other. But, can eigenvalues with algebraic multiplicity $m>1$

occur? The answer is yes. Let's rewrite $P(\theta) = det\left(\mathbf{T} - \theta \mathbf{I_n}\right)$ as $\theta^N + q_i \theta^{n-i} \quad (i = N-1, N-2, \ldots, 1)$ with q_i functions of the Ricci tensor components. The condition for eigenvalues with algebraic multiplicity $m > 1$ to occur is that the determinant of the Sylvester matrix of the two following polynomial should be zero.

$$\begin{cases} P(\theta) = \theta^N + q_i \theta^{n-i} \quad (i = N-1, N-2, \ldots, 1) \\ \frac{dP(\theta)}{d\theta} = N\theta^{N-1} + (n-i) q_i \theta^{n-i-1} \quad (i = N-1, N-2, \ldots, 1) \end{cases} \tag{5}$$

The associated Sylvester matrix with these two polynomials will be of the form

$$S\left(P(\theta), \frac{dP(\theta)}{d\theta}\right) = \begin{pmatrix} 1 & q_{N-1} & \ldots & q_1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & q_{N-1} & \ldots & q_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & q_{N-1} & \ldots & q_2 & q_1 \\ N & (N-1)q_{N-1} & \ldots & q_1 & 0 & 0 & \ldots & 0 \\ 0 & N & (N-1)q_{N-1} & \ldots & q_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & N & (N-1)q_{N-1} & \ldots & 2q_2 & q_1 \end{pmatrix}$$

If the determinant of this matrix is not zero, then there will be no algebraic multiplicity. In the other case, one has to check whether in the eigenspace, related to the eigenvalues with algebraic multiplicity $m > 1$, m linear independent eigenvectors can be found.

Exercise 4.

What becomes of the Ricci principal directions (see above) if $N = 2$?

From **3.208**. we have

$$\frac{R_{11}}{a_{11}} = \frac{R_{12}}{a_{12}} = \frac{R_{22}}{a_{22}} = -\frac{R_{1212}}{a}$$

$$\Rightarrow \begin{cases} T_1^1 = a^{11} R_{11} + a^{12} R_{21} \\ T_2^1 = a^{11} R_{12} + a^{12} R_{22} \\ T_2^2 = a^{21} R_{12} + a^{22} R_{22} \end{cases}$$

$$\text{put } K = -\frac{R_{1212}}{a} \quad \Rightarrow \begin{cases} T_1^1 = K a^{11} a_{11} + K a^{12} a_{12} \\ T_2^1 = K a^{11} a_{12} + K a^{12} a_{22} \\ T_2^2 = K a^{21} a_{12} + K a^{22} a_{22} \end{cases} \Rightarrow \begin{cases} T_1^1 = K \delta_1^1 = K \\ T_2^1 = K \delta_2^1 = 0 \\ T_2^2 = K \delta_2^2 = K \end{cases}$$

hence the characteristic equation $det\left(\mathbf{T} - \theta \mathbf{I_n}\right) = 0$ becomes $(K - \theta)^2 = 0$. So only one value of θ exists as $\theta = K$. Equation (4) becomes $\mathbf{0X} = 0$. So we can chose any pair of linear independent vectors as eigenvectors and can make them perpendicular to one another.

♦

3.15 p109 - Exercise 5

Suppose that two spaces V_N, $V_N^{'}$ have metric tensors a_{mn}, $a_{mn}^{'}$ such that $a_{mn}^{'} = k\ a_{mn}$, where k is a constant. Write down the relations between the curvature tensors, the Ricci tensors, and the curvature invariants of the two spaces.

We have

$$ds^2 = a_{mn}dx^m dx^n$$
$$ds^{'2} = a_{mn}^{'}dx^{'m}dx^{'n}$$

But let's be careful: there is no reason to assume that $dx^m = dx^{'m}$. Let's embed the two spaces in a space V_{N+1}. If an observer in that space sees two displacements ds^2 and $ds^{'2}$ which for him have the same magnitude, we have

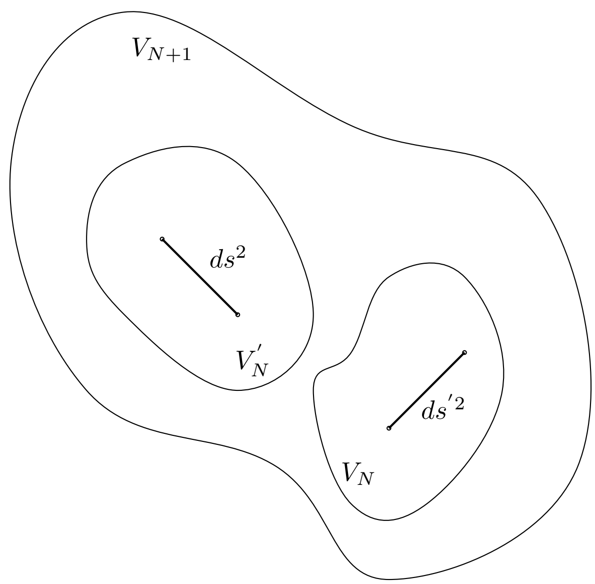


Figure 3.12: Embedded V_N spaces

$$ds^{'2} = ds^2$$
$$\Rightarrow \qquad a_{mn}^{'}dx^{'m}dx^{'n} = a_{mn}dx^m dx^n$$
$$\Rightarrow \qquad ka_{mn}dx^{'m}dx^{'n} = a_{mn}dx^m dx^n$$
$$\Rightarrow \qquad dx^{'m} = \frac{1}{\sqrt{k}}dx^m$$

We have also

$$
\begin{aligned}
& & a'_{mk}a'^{kn} &= \delta^n_m \\
&\Rightarrow & ka_{mp}a'^{pn} &= \delta^n_m \\
&\Rightarrow & a'^{pn} &= \frac{1}{k}a^{pn}
\end{aligned}
$$

And get the following relations

$$
\begin{cases}
[mn,r]' = \sqrt{k}^3[mn,r] \\
\\
\Gamma'^r_{.mn} = \sqrt{k}\Gamma^r_{.mn}
\end{cases}
$$

$$
\begin{aligned}
& \Rightarrow & R'^s_{.rmn} &= \frac{\partial \Gamma'^s_{.rn}}{\partial x'^m} + \dots \\
& \Rightarrow & R'^s_{.rmn} &= \frac{\sqrt{k}\partial \Gamma^s_{.rn}}{\partial\left(\frac{x^m}{\sqrt{k}}\right)} + \dots \\
& \Rightarrow & R'^s_{.rmn} &= kR^s_{.rmn} \\
\times\, a'^{ks} & \Rightarrow & a'^{ks}R'^s_{.rmn} &= ka'^{ks}R^s_{.rmn} \\
& \Rightarrow & R'_{krmn} &= k\frac{1}{k}a^{ks}R^s_{.rmn} \\
& \Rightarrow & R'_{krmn} &= R_{krmn} \\
R_{rm} = R^n_{.rmn} & \Rightarrow & R'_{mn} &= kR_{rmn} \\
R = a^{mn}R_{mn} & \Rightarrow & R' &= a'^{mn}R'_{mn} \\
& \Rightarrow & R' &= \frac{1}{k}a^{mn}kR_{mn} \\
& \Rightarrow & R' &= R
\end{aligned}
$$

Summary

$$
\begin{aligned}
R'^s_{.rmn} &= kR^s_{.rmn} \\
R'_{krmn} &= R_{krmn} \\
R'_{mn} &= kR_{rmn} \\
R' &= R
\end{aligned}
$$

♦

3.16 p109 - Exercise 6

For an orthogonal coordinates system in a V_2 we have

$$ds^2 = a_{11}\left(dx^1\right)^2 + a_{22}\left(dx^2\right)^2$$

Show that
$$\frac{1}{a}R_{1212} = -\frac{1}{2}\frac{1}{\sqrt{a}}\left[\partial_1\left(\frac{1}{\sqrt{a}}\partial_1 a_{22}\right) + \partial_2\left(\frac{1}{\sqrt{a}}\partial_2 a_{11}\right)\right]$$

We have

$$(a_{mn}) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \quad (a^{mn}) = \frac{1}{a}\begin{pmatrix} a_{22} & 0 \\ 0 & a_{11} \end{pmatrix} \quad a = a_{11}a_{22} \tag{1}$$

We have also

$$R = -\frac{2}{a}R_{1212} \tag{2}$$
$$R = a^{mn}R_{mn} \quad \Rightarrow \quad R = a^{11}R_{11} + a^{22}R_{22} \tag{3}$$

Looking at the pattern generated by equations (2) and (3) suggests that using these equations could lead to the proposed equation. Let's have a try ...

$$\begin{cases} \Gamma^1_{11} = \frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11} & \Gamma^1_{22} = -\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22} \\ \\ \Gamma^2_{11} = -\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11} & \Gamma^2_{22} = \frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22} \\ \\ \Gamma^1_{12} = \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} & \Gamma^2_{12} = \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} \end{cases} \tag{4}$$

$$3.205. \quad \Rightarrow \quad R_{rm} = \frac{1}{2}\partial_{rm}\log a - \frac{1}{2}\Gamma^p_{rm}\partial_p \log a - \partial_n\Gamma^n_{rm} + \Gamma^p_{rn}\Gamma^n_{pm} \tag{5}$$

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\Gamma^1_{11}\partial_1\log a - \frac{1}{2}\Gamma^2_{11}\partial_2\log a \\ -\partial_1\Gamma^1_{11} - \partial_2\Gamma^2_{11} + \\ \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^2_{21} \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a - \frac{1}{2}\Gamma^1_{22}\partial_1\log a - \frac{1}{2}\Gamma^2_{22}\partial_2\log a \\ -\partial_1\Gamma^1_{22} - \partial_2\Gamma^2_{22} + \\ \Gamma^1_{21}\Gamma^1_{12} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{21}\Gamma^1_{22} + \Gamma^2_{22}\Gamma^2_{22} \\ \end{cases} \tag{6}$$

$$
\Rightarrow \begin{cases}
R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}\partial_1 \log a - \frac{1}{2}(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11})\partial_2 \log a \\
-\partial_1(\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}) - \partial_2(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11}) + \\
\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11} + \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11}(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11}) + \\
(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11})\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} + \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22}\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} \\
\\
R_{22} = \frac{1}{2}\partial_{22}\log a - \frac{1}{2}(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22})\partial_1 \log a - \frac{1}{2}\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}\partial_2 \log a \\
-\partial_1(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22}) - \partial_2(\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}) + \\
\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11}\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} + (-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22})\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} + \\
\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22}(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22}) + \frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}
\end{cases} \tag{7}
$$

Simplifying the notational burden by replacing a_{11} by γ and a_{22} by η:

$$
\Rightarrow \begin{cases}
R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\partial_1\gamma\partial_1\log a + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\partial_2\gamma\partial_2\log a \\
-\frac{1}{2}\partial_1(\frac{1}{\gamma}\partial_1\gamma) + \frac{1}{2}\partial_2(\frac{1}{\eta}\partial_2\gamma) \\
+\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_1\gamma\partial_1\gamma - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\gamma \\
-\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\gamma + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_1\eta\partial_1\eta \\
\\
R_{22} = \frac{1}{2}\partial_{22}\log a + \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\partial_1\eta\partial_1\log a - \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\partial_2\eta\partial_2\log a \\
+\frac{1}{2}\partial_1(\frac{1}{\gamma}\partial_1\eta) - \frac{1}{2}\partial_2(\frac{1}{\eta}\partial_2\eta) \\
+\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_2\gamma\partial_2\gamma - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\eta\partial_1\eta \\
-\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\eta\partial_1\eta + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_2\eta\partial_2\eta
\end{cases} \tag{8}
$$

Noting that $\partial_{ii}\log a = \partial_i\left(\frac{1}{a_{11}}\partial_i a_{11}\right) + \partial_i\left(\frac{1}{a_{22}}\partial_i a_{22}\right)$ and $\partial_i \log a = \frac{1}{a_{11}}\partial_i a_{11} + \frac{1}{a_{22}}\partial_i a_{22} \quad (i = 1, 2)$, we get:

$$
\begin{array}{|c|c|}
2R_{11} = & 2R_{22} = \\
\underbrace{\partial_1\left(\frac{1}{\gamma}\partial_1\gamma\right)}_{*} + \partial_1\left(\tfrac{1}{\eta}\partial_1\eta\right) & \partial_2\left(\tfrac{1}{\gamma}\partial_2\gamma\right) + \underbrace{\partial_2\left(\frac{1}{\eta}\partial_2\eta\right)}_{*} \\
-\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_1\gamma)^2}_{-} - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_1\gamma\partial_1\eta & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}\partial_1\gamma\partial_1\eta + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_1\eta)^2 \\
 & -\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta - \underbrace{\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_2\eta)^2}_{-} \\
+\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_2\gamma)^2 + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta & \\
-\underbrace{\partial_1(\frac{1}{\gamma}\partial_1\gamma)}_{*} + \partial_2(\tfrac{1}{\eta}\partial_2\gamma) & +\partial_1(\tfrac{1}{\gamma}\partial_1\eta) - \underbrace{\partial_2(\frac{1}{\eta}\partial_2\eta)}_{*} \\
+\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_1\gamma)^2}_{-} - \underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2}_{+} & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}(\partial_2\gamma)^2 - \underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2}_{+} \\
-\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2}_{+} + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}(\partial_1\eta)^2 & -\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2}_{+} + \underbrace{\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_2\eta)^2}_{-}
\end{array} \tag{9}
$$

$$
\Rightarrow \quad
\begin{array}{|c|c|}
2R_{11} = & 2R_{22} = \\
\partial_1\left(\tfrac{1}{\eta}\partial_1\eta\right) + \partial_2(\tfrac{1}{\eta}\partial_2\gamma) & \partial_1(\tfrac{1}{\gamma}\partial_1\eta) + \partial_2\left(\tfrac{1}{\gamma}\partial_2\gamma\right) \\
+\tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}(\partial_1\eta)^2 - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_2\gamma)^2 & -\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_1\eta)^2 + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}(\partial_2\gamma)^2 \\
-\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_1\gamma\partial_1\eta + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}\partial_1\gamma\partial_1\eta - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta
\end{array} \tag{10}
$$

Be $R = \frac{1}{\gamma}R_{11} + \frac{1}{\eta}R_{22}$, all first order derivatives vanish and we get,

$$
\frac{1}{\gamma}R_{11} + \frac{1}{\eta}R_{22} = \frac{1}{2}\left[\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)\right] + \frac{1}{2}\left[\frac{1}{\gamma}\partial_2(\frac{1}{\gamma}\partial_2\gamma) + \frac{1}{\eta}\partial_2(\frac{1}{\eta}\partial_2\gamma)\right] \tag{11}
$$

We further simplify this expression. Considering the symmetry of (11) we only explicit the calculations for the first terms in ∂_1.

$$\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta)+\frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)=\frac{1}{\eta}\partial_1(\frac{1}{\sqrt{\gamma}}\frac{1}{\sqrt{\gamma}}\frac{\sqrt{\eta}}{\sqrt{\eta}}\partial_1\eta)+\frac{1}{\gamma}\partial_1(\frac{1}{\sqrt{\eta}}\frac{1}{\sqrt{\eta}}\frac{\sqrt{\gamma}}{\sqrt{\gamma}}\partial_1\eta) \tag{12}$$

$$=\frac{1}{\eta}\partial_1\left[\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}\frac{1}{\sqrt{a}}\partial_1\eta\right]+\frac{1}{\gamma}\partial_1\left[\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{a}}\partial_1\eta\right] \tag{13}$$

$$=\begin{cases}\underbrace{\frac{1}{\eta}\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}}_{=\frac{1}{\sqrt{a}}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1\eta\right]+\underbrace{\frac{1}{\gamma}\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}}_{=\frac{1}{\sqrt{a}}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1\eta\right]\\[2ex] +\frac{1}{\sqrt{a}}\partial_1\eta\underbrace{\left[\frac{1}{\eta}\partial_1\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}+\frac{1}{\gamma}\partial_1\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}\right]}_{=0}\end{cases} \tag{14}$$

$$=2\frac{1}{\sqrt{a}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1 a_{22}\right] \tag{15}$$

$$\Rightarrow\quad \frac{1}{2}\left[\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta)+\frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)\right]=\frac{1}{\sqrt{a}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1 a_{22}\right] \tag{16}$$

Using (16) and the same calculations for the terms in ∂_2 and using (2) and (3) we get

$$\frac{1}{a}R_{1212}=-\frac{1}{2}\frac{1}{\sqrt{a}}\left[\partial_1\left(\frac{1}{\sqrt{a}}\partial_1 a_{22}\right)+\partial_2\left(\frac{1}{\sqrt{a}}\partial_2 a_{11}\right)\right]$$

♦

3.17 p109 - Exercise 7

Suppose that in a V_3 the metric is :

$$ds^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2$$

where h_1, h_2, h_3 are functions of the three coordinates. Calculate the curvature tensor in terms of the $h^{'}s$ and their derivatives. Check your result by nothing that the curvature tensor will vanish if h_1 is a function of x^1 only, h_2 a function of x^2 only, and h_3 a function of x^3 only.

From **3.115**. and **3.115**. we get for the non vanishing components of the covariant curvature tensor(6 independent components to calculate):

$$R_{1212} = \begin{cases} -R_{1221} \\ -R_{2112} \\ R_{2121} \end{cases} \quad R_{2323} = \begin{cases} -R_{2332} \\ -R_{3223} \\ R_{3232} \end{cases} \quad R_{1313} = \begin{cases} -R_{1331} \\ -R_{3113} \\ R_{3131} \end{cases}$$

$$R_{1213} = \begin{cases} -R_{1231} \\ R_{1312} \\ -R_{1321} \\ -R_{2113} \\ R_{2131} \\ -R_{3112} \\ R_{3121} \end{cases} \quad R_{1223} = \begin{cases} -R_{1232} \\ -R_{2123} \\ R_{2132} \\ R_{2312} \\ -R_{2321} \\ R_{3212} \\ -R_{3221} \end{cases} \quad R_{1323} = \begin{cases} -R_{1332} \\ R_{2313} \\ -R_{2331} \\ -R_{3123} \\ R_{3132} \\ -R_{3213} \\ R_{3231} \end{cases}$$

The metric tensors:

$$(a_{mn}) = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \quad (a^{mn}) = \begin{pmatrix} \frac{1}{h_1^2} & 0 & 0 \\ 0 & \frac{1}{h_2^2} & 0 \\ 0 & 0 & \frac{1}{h_3^2} \end{pmatrix}$$

The Christoffel symbols:

$$\begin{array}{lll} [11,1] = h_1\partial_1 h_1 & [11,2] = -h_1\partial_2 h_1 & [11,3] = -h_1\partial_3 h_1 \\ [12,1] = h_1\partial_2 h_1 & [12,2] = h_2\partial_1 h_2 & [12,3] = 0 \\ [22,1] = -h_2\partial_1 h_2 & [22,2] = h_2\partial_2 h_2 & [22,3] = -h_2\partial_3 h_2 \\ [23,1] = 0 & [23,2] = h_2\partial_3 h_2 & [23,3] = h_3\partial_2 h_3 \\ [33,1] = -h_3\partial_1 h_3 & [33,2] = -h_3\partial_2 h_3 & [33,3] = -h_3\partial_3 h_3 \\ [31,1] = h_1\partial_3 h_1 & [31,2] = 0 & [31,3] = h_3\partial_1 h_3 \end{array}$$

$$\begin{array}{lll}
\Gamma^1_{11} = \frac{1}{h_1}\partial_1 h_1 & \Gamma^2_{11} = -\frac{h_1}{h_2^2}\partial_2 h_1 & \Gamma^3_{11} = -\frac{h_1}{h_3^2}\partial_3 h_1 \\
\Gamma^1_{12} = \frac{1}{h_1}\partial_2 h_1 & \Gamma^2_{12} = \frac{1}{h_2}\partial_1 h_2 & \Gamma^3_{12} = 0 \\
\Gamma^1_{22} = -\frac{h_2}{h_1^2}\partial_1 h_2 & \Gamma^2_{22} = \frac{1}{h_2}\partial_2 h_2 & \Gamma^3_{22} = -\frac{h_2}{h_3^2}\partial_3 h_2 \\
\Gamma^1_{23} = 0 & \Gamma^2_{23} = \frac{1}{h_2}\partial_3 h_2 & \Gamma^3_{23} = \frac{1}{h_3}\partial_2 h_3 \\
\Gamma^1_{33} = -\frac{h_3}{h_1^2}\partial_1 h_3 & \Gamma^2_{33} = -\frac{h_3}{h_2^2}\partial_2 h_3 & \Gamma^3_{33} = \frac{1}{h_3}\partial_3 h_3 \\
\Gamma^1_{31} = \frac{1}{h_1}\partial_3 h_1 & \Gamma^2_{31} = 0 & \Gamma^3_{31} = \frac{1}{h_3}\partial_1 h_3
\end{array}$$

We use 3.113.

$$R_{rsmn} = \partial_m[sn,r] - \partial_n[sm,r] + \Gamma^p_{sm}[rn,p] - \Gamma^p_{sn}[rm,p]$$

Note that we only have to perform the full calculation for two curvature tensors e.g. R_{1212} and R_{1213} as the others can be retrieved by using adequate indices renaming and use of the identities 3.115.

$$\begin{aligned}
R_{1212} &= -h_2\partial^2_{11}(h_2) - h_1\partial^2_{22}(h_1) + \frac{h_2}{h_1}\partial_1 h_1\partial_1 h_2 + \frac{h_1}{h_2}\partial_2 h_1\partial_2 h_2 - \frac{h_1 h_2}{h_3^2}\partial_3 h_1\partial_3 h_2 \\
R_{2323} &= -h_3\partial^2_{22}(h_3) - h_2\partial^2_{33}(h_2) + \frac{h_3}{h_2}\partial_2 h_2\partial_2 h_3 + \frac{h_2}{h_3}\partial_3 h_2\partial_3 h_3 - \frac{h_2 h_3}{h_1^2}\partial_1 h_2\partial_1 h_3 \\
R_{1313} &= -h_3\partial^2_{11}(h_3) - h_1\partial^2_{33}(h_1) + \frac{h_3}{h_1}\partial_1 h_1\partial_1 h_3 + \frac{h_1}{h_3}\partial_3 h_1\partial_3 h_3 - \frac{h_1 h_3}{h_2^2}\partial_2 h_1\partial_2 h_3
\end{aligned}$$

$$\begin{aligned}
R_{1213} &= -h_1\partial^2_{32}(h_1) + \frac{h_1}{h_3}\partial_2 h_3\partial_3 h_1 + \frac{h_1}{h_2}\partial_2 h_1\partial_3 h_2 \\
R_{1223} &= h_2\partial^2_{31}(h_2) - \frac{h_2}{h_1}\partial_1 h_2\partial_3 h_1 - \frac{h_2}{h_3}\partial_3 h_2\partial_1 h_3 \\
R_{1323} &= -h_3\partial^2_{21}(h_3) + \frac{h_3}{h_1}\partial_1 h_3\partial_3 h_1 + \frac{h_3}{h_2}\partial_2 h_3\partial_1 h_2
\end{aligned}$$

And, indeed, all curvature tensors vanish when the h_i are only a function of the indices' dimension.

♦

3.18 p109 - Exercise 8

In relativity we encounter the metric form

$$\Phi = e^{\alpha} + e^{x^1}\left[\left(dx^2\right)^2 + \sin^2 x^2 \left(dx^3\right)^2\right] - e^{\gamma}\left(dx^4\right)^2$$

where α and γ are functions of x^1 and x^4 only.
Show that the complete set of non-zero components of the Einstein tensor (see equation (3.214)) for the form given above are as follows

$$\begin{aligned}
G^1_{.1} &= e^{-\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_1\right) + e^{-x^1} \\
G^2_{.2} &= e^{\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_1\gamma_1\right) \\
&\quad + e^{\gamma}\left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4\right) \\
G^3_{.3} &= G^2_{.2} \\
G^4_{.4} &= e^{-\alpha}\left(-\frac{3}{4} - \frac{1}{2}\alpha_1\right) + e^{-x^1} \\
e^{\alpha}G^4_{.1} &= -e^{\gamma}G^4_{.1} = -\frac{1}{2}\alpha_4
\end{aligned}$$

The subscript on α and γ indicate partial derivatives with respect to x^1 and x^4.

We have

$$(a_{mn}) = \begin{pmatrix} e^{\alpha} & 0 & 0 & 0 \\ 0 & e^{x^1} & 0 & 0 \\ 0 & 0 & e^{x^1}\sin^2 x^2 & 0 \\ 0 & 0 & 0 & -e^{\gamma} \end{pmatrix} \quad (a^{mn}) = \begin{pmatrix} e^{-\alpha} & 0 & 0 & 0 \\ 0 & e^{-x^1} & 0 & 0 \\ 0 & 0 & \frac{e^{-x^1}}{\sin^2 x^2} & 0 \\ 0 & 0 & 0 & -e^{-\gamma} \end{pmatrix} \tag{1}$$

And will use the following definitions:

$$G^n_{.t} = R^n_{.t} - \frac{1}{2}\delta^n_t R \tag{2}$$

$$R^n_{.t} = a^{nk}R_{kt} \tag{3}$$

$$R_{kt} = a^{sn}R_{sktn} \tag{4}$$

$$R = a^{kt}R_{kt} \tag{5}$$

Considering that the non-diagonal components of a_{mn} vanish and as $R_{sktn} = 0$ when $s = k$ or $t = n$, we can write :

$$\begin{pmatrix} R_{11} \\ R_{22} \\ R_{33} \\ R_{44} \end{pmatrix} = \begin{pmatrix} 0 & R_{2112} & R_{3113} & R_{4114} \\ R_{1221} & 0 & R_{3223} & R_{4224} \\ R_{1331} & R_{2332} & 0 & R_{4334} \\ R_{1441} & R_{2442} & R_{3443} & 0 \end{pmatrix} \begin{pmatrix} a^{11} \\ a^{22} \\ a^{33} \\ a^{44} \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} R_{12} \\ R_{13} \\ R_{14} \\ R_{23} \\ R_{24} \\ R_{34} \end{pmatrix} = \begin{pmatrix} 0 & 0 & R_{3123} & R_{4124} \\ 0 & R_{2132} & 0 & R_{4134} \\ 0 & R_{2142} & R_{3143} & 0 \\ R_{1231} & 0 & 0 & R_{4234} \\ R_{1241} & 0 & R_{3243} & 0 \\ R_{1341} & R_{2342} & 0 & 0 \end{pmatrix} \begin{pmatrix} a^{11} \\ a^{22} \\ a^{33} \\ a^{44} \end{pmatrix} \tag{7}$$

$$\tag{8}$$

The Christoffel symbols of the first kind are:

$$\begin{array}{llll}
[11,1] = \frac{1}{2}\alpha_1 e^{\alpha} & [11,2] = 0 & [11,3] = 0 & [11,4] = -\frac{1}{2}\alpha_4 e^{\alpha} \\
[12,1] = 0 & [12,2] = \frac{1}{2}e^{x^1} & [12,3] = 0 & [12,4] = 0 \\
[13,1] = 0 & [13,2] = 0 & [13,3] = \frac{1}{2}e^{x^1}\sin^2 x^2 & [13,4] = 0 \\
[14,1] = \frac{1}{2}\alpha_4 e^{\alpha} & [14,2] = 0 & [14,3] = 0 & [14,4] = -\frac{1}{2}\gamma_1 e^{\gamma} \\
[22,1] = -\frac{1}{2}e^{x^1} & [22,2] = 0 & [22,3] = 0 & [22,4] = 0 \\
[23,1] = 0 & [23,2] = 0 & [23,3] = \frac{1}{2}e^{x^1}\sin 2x^2 & [23,4] = 0 \\
[24,1] = 0 & [24,2] = 0 & [24,3] = 0 & [24,4] = 0 \\
[33,1] = -\frac{1}{2}e^{x^1} sin^2 x^2 & [33,2] = -\frac{1}{2}e^{x^1}\sin 2x^2 & [33,3] = 0 & [33,4] = 0 \\
[34,1] = 0 & [34,2] = 0 & [34,3] = 0 & [34,4] = 0 \\
[44,1] = \frac{1}{2}\gamma_1 e^{\gamma} & [44,2] = 0 & [44,3] = 0 & [44,4] = -\frac{1}{2}\gamma_4 e^{\gamma}
\end{array} \tag{9}$$

We use 3.114. and considering that $a_{mn} = a^{mn} = 0$ for $m \neq n$:

$$R_{rsmn} = \begin{cases} \frac{1}{2}\left(\partial^2_{sm} a_{rn} + \partial^2_{rn} a_{sm}\right) \\ +\frac{1}{e^{\alpha}}\left([rn,1][sm,1] - [rm,1][sn,1]\right) \\ +\frac{1}{e^{x^1}}\left([rn,2][sm,2] - [rm,2][sn,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([rn,3][sm,3] - [rm,3][sn,3]\right) \\ -\frac{1}{e^{\gamma}}\left([rn,4][sm,4] - [rm,4][sn,4]\right) \end{cases}$$

Giving:

$$R_{2112} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{22} + \partial^2_{22} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][11,1] - [21,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][11,2] - [21,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][11,3] - [21,3][12,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][11,4] - [21,4][12,4]\right) \end{cases} \tag{10}$$

(11)

$$R_{3113} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{33} + \partial^2_{33} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][11,1] - [31,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][11,2] - [31,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][11,3] - [31,3][13,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][11,4] - [31,4][13,4]\right) \end{cases} \tag{12}$$

(13)

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{44} + \partial^2_{44} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][11,1] - [41,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][11,2] - [41,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][11,3] - [41,3][14,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][11,4] - [41,4][14,4]\right) \end{cases} \tag{14}$$

(15)

$$R_{3223} = \begin{cases} \frac{1}{2}\left(\partial_{22}^2 a_{33} + \partial_{33}^2 a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][22,1] - [32,1][23,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][22,2] - [32,2][23,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][22,3] - [32,3][23,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][22,4] - [32,4][23,4]\right) \end{cases} \tag{16}$$

(17)

$$R_{4224} = \begin{cases} \frac{1}{2}\left(\partial_{22}^2 a_{44} + \partial_{44}^2 a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][22,1] - [42,1][24,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][22,2] - [42,2][24,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][22,3] - [42,3][24,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][22,4] - [42,4][24,4]\right) \end{cases} \tag{18}$$

(19)

$$R_{4334} = \begin{cases} \frac{1}{2}\left(\partial_{33}^2 a_{44} + \partial_{44}^2 a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][33,1] - [43,1][34,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][33,2] - [43,2][34,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][33,3] - [43,3][34,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][33,4] - [43,4][34,4]\right) \end{cases} \tag{20}$$

(21)

$$R_{3123} = \begin{cases} \frac{1}{2}\left(\partial^2_{12} a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][12,1]-[32,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][12,2]-[32,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][12,3]-[32,3][13,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][12,4]-[32,4][13,4]\right) \end{cases} \tag{22}$$

(23)

$$R_{4124} = \begin{cases} \frac{1}{2}\left(\partial^2_{12} a_{44}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][12,1]-[42,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][12,2]-[42,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][12,3]-[42,3][14,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][12,4]-[42,4][14,4]\right) \end{cases} \tag{24}$$

(25)

$$R_{2132} = \begin{cases} \frac{1}{2}\left(\partial^2_{13} a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][13,1]-[23,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][13,2]-[23,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][13,3]-[23,3][12,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][13,4]-[23,4][12,4]\right) \end{cases} \tag{26}$$

(27)

$$R_{4134} = \begin{cases} \frac{1}{2}\left(\partial^2_{13} a_{44}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][13,1] - [43,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][13,2] - [43,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][13,3] - [43,3][14,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][13,4] - [43,4][14,4]\right) \end{cases} \tag{28}$$

(29)

$$R_{2142} = \begin{cases} \frac{1}{2}\left(\partial^2_{14} a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][14,1] - [24,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][14,2] - [24,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][14,3] - [24,3][12,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][14,4] - [24,4][12,4]\right) \end{cases} \tag{30}$$

(31)

$$R_{3143} = \begin{cases} \frac{1}{2}\left(\partial^2_{14} a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][14,1] - [34,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][14,2] - [34,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][14,3] - [34,3][13,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][14,4] - [34,4][13,4]\right) \end{cases} \tag{32}$$

(33)

$$R_{1231} = \begin{cases} \frac{1}{2}\left(\partial^2_{23} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][23,1]-[13,1][21,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][23,2]-[13,2][21,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][23,3]-[13,3][21,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][23,4]-[13,4][21,4]\right) \end{cases} \tag{34}$$

(35)

$$R_{4234} = \begin{cases} \frac{1}{2}\left(\partial^2_{23} a_{44}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][23,1]-[43,1][24,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][23,2]-[43,2][24,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][23,3]-[43,3][24,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][23,4]-[43,4][24,4]\right) \end{cases} \tag{36}$$

(37)

$$R_{1241} = \begin{cases} \frac{1}{2}\left(\partial^2_{24} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][24,1]-[14,1][21,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][24,2]-[14,2][21,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][24,3]-[14,3][21,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][24,4]-[14,4][21,4]\right) \end{cases} \tag{38}$$

(39)

$$R_{3243} = \begin{cases} \frac{1}{2}\left(\partial^2_{24} a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][24,1] - [34,1][23,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][24,2] - [34,2][23,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][24,3] - [34,3][23,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][24,4] - [34,4][23,4]\right) \end{cases} \tag{40}$$

(41)

$$R_{1341} = \begin{cases} \frac{1}{2}\left(\partial^2_{34} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][34,1] - [14,1][31,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][34,2] - [14,2][31,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][34,3] - [14,3][31,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][34,4] - [14,4][31,4]\right) \end{cases} \tag{42}$$

(43)

$$R_{2342} = \begin{cases} \frac{1}{2}\left(\partial^2_{34} a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][34,1] - [24,1][32,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][34,2] - [24,2][32,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][34,3] - [24,3][32,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][34,4] - [24,4][32,4]\right) \end{cases} \tag{44}$$

(45)

Considering that $[mn, q] = 0$ for $m \neq n \neq q \neq m$ and replacing the remaining Christoffels symbols:

$$R_{2112} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{22} + \partial^2_{22} a_{11}\right) \\ -\frac{1}{4}\alpha_1 e^{x^1} \end{cases} \tag{46}$$

(47)

$$R_{3113} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{33} + \partial_{33}^2 a_{11}\right) \\ \\ -\frac{1}{4}\left(1+\alpha_1\right) e^{x^1} sin^2 x^2 \end{cases} \tag{48}$$

(49)

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{44} + \partial_{44}^2 a_{11}\right) \\ \\ +\frac{1}{4}\left(\alpha_1\gamma_1 e^{\gamma} - \alpha_4^2 e^{\alpha}\right) + \frac{1}{4}\left(\alpha_4\gamma_4 e^{\alpha} - \gamma_1^2 e^{\gamma}\right) \end{cases} \tag{50}$$

(51)

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{44} + \partial_{44}^2 a_{11}\right) \\ \\ +\frac{1}{4}\left(\alpha_1\gamma_1 e^{\gamma} - \alpha_4^2 e^{\alpha}\right) - \frac{1}{4}\left(\alpha_4\gamma_4 e^{\alpha} - \gamma_1^2 e^{\gamma}\right) \end{cases} \tag{52}$$

(53)

$$R_{3223} = \begin{cases} \frac{1}{2}\left(\partial_{22}^2 a_{33} + \partial_{33}^2 a_{22}\right) \\ \\ +\frac{1}{4}\frac{e^{x^1}}{e^{2\alpha}} sin^2 x^2 - e^{x^1}\cos^2 x^2 \end{cases} \tag{54}$$

(55)

$$R_{4224} = \begin{cases} \frac{1}{2}\left(\partial_{22}^2 a_{44} + \partial_{44}^2 a_{22}\right) \\ \\ -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} \end{cases} \tag{56}$$

(57)

$$R_{4334} = \begin{cases} \frac{1}{2}\left(\partial_{33}^2 a_{44} + \partial_{44}^2 a_{33}\right) \\ \\ -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} \sin^2 x^2 \end{cases} \tag{58}$$

(59)

$$R_{3123} = \frac{1}{2}\left(\partial^2_{12} a_{33}\right) = 0 \tag{60}$$
$$R_{4124} = \frac{1}{2}\left(\partial^2_{12} a_{44}\right) = 0 \tag{61}$$
$$R_{2132} = \frac{1}{2}\left(\partial^2_{13} a_{22}\right) = 0 \tag{62}$$
$$R_{4134} = \frac{1}{2}\left(\partial^2_{13} a_{44}\right) = 0 \tag{63}$$
$$R_{2142} = e^{-\alpha}\left([22,1][14,1]\right) \tag{64}$$
$$R_{3143} = e^{-\alpha}\left([33,1][14,1]\right) \tag{65}$$
$$R_{1231} = \frac{1}{2}\left(\partial^2_{23} a_{11}\right) = 0 \tag{66}$$
$$R_{4234} = \frac{1}{2}\left(\partial^2_{23} a_{44}\right) = 0 \tag{67}$$
$$R_{1241} = \frac{1}{2}\left(\partial^2_{24} a_{11}\right) = 0 \tag{68}$$
$$R_{3243} = \frac{1}{2}\left(\partial^2_{24} a_{33}\right) = 0 \tag{69}$$
$$R_{1341} = \frac{1}{2}\left(\partial^2_{34} a_{11}\right) = 0 \tag{70}$$
$$R_{2342} = \frac{1}{2}\left(\partial^2_{34} a_{22}\right) = 0 \tag{71}$$

Giving:

$$R_{2112} = \frac{1}{4}\left(1-\alpha_1\right) e^{x^1} \tag{72}$$
$$R_{3113} = \frac{1}{4}\left(1-\alpha_1\right) e^{x^1} \sin^2 x^2 \tag{73}$$
$$R_{4114} = \begin{cases} \frac{1}{2} e^{\alpha}\left(\alpha_{44} + \frac{1}{2}\alpha_4^2 - \frac{1}{2}\alpha_4\gamma_4\right) \\ -\frac{1}{2} e^{\gamma}\left(\gamma_{11} + \frac{1}{2}\gamma_1^2 - \frac{1}{2}\alpha_1\gamma_1\right) \end{cases} \tag{74}$$
$$R_{3223} = \left(\frac{1}{4}\frac{e^{x^1}}{e^{\alpha}} - 1\right) e^{x^1} \sin^2 x^2 \tag{75}$$
$$R_{4224} = -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} \tag{76}$$
$$R_{4334} = -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} sin^2 x^2 \tag{77}$$
$$R_{2142} = -\frac{1}{4}\alpha_4 e^{x^1} \tag{78}$$
$$R_{3143} = -\frac{1}{4}\alpha_4 e^{x^1} \sin^2 x^2 \tag{79}$$

As all other curvature components vanish, we get

$$\begin{pmatrix} R_{11} \\ \\ R_{22} \\ \\ R_{33} \\ \\ R_{44} \end{pmatrix} = P \begin{pmatrix} e^{-\alpha} \\ \\ e^{-x^1} \\ \\ \frac{e^{-x^1}}{\sin^2 x^2} \\ \\ -e^{-\gamma} \end{pmatrix} \tag{80}$$

With

$$P = \begin{pmatrix} 0 & \frac{1}{4}\left(1-\alpha_1\right)e^{x^1} & \frac{1}{4}\left(1-\alpha_1\right)e^{x^1}\sin^2 x^2 & \begin{matrix} \frac{1}{2}e^{\alpha}\left(\alpha_{44}+\frac{1}{2}\alpha_4^2-\frac{1}{2}\alpha_4\gamma_4\right) \\ -\frac{1}{2}e^{\gamma}\left(\gamma_{11}+\frac{1}{2}\gamma_1^2-\frac{1}{2}\alpha_1\gamma_1\right) \end{matrix} \\ \\ . & 0 & \left(\frac{1}{4}\frac{e^{x^1}}{e^{\alpha}}-1\right)e^{x^1}\sin^2 x^2 & -\frac{1}{4}\gamma_1\frac{e^{\gamma}e^{x^1}}{e^{\alpha}} \\ \\ . & . & 0 & -\frac{1}{4}\gamma_1\frac{e^{\gamma}e^{x^1}}{e^{\alpha}}sin^2 x^2 \\ \\ . & . & . & 0 \end{pmatrix} \tag{81}$$

and

$$\begin{pmatrix} R_{12} \\ R_{13} \\ R_{14} \\ R_{23} \\ R_{24} \\ R_{34} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4}\alpha_4 e^{x^1} & -\frac{1}{4}\alpha_4 e^{x^1}\sin^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-\alpha} \\ \\ e^{-x^1} \\ \\ \frac{e^{-x^1}}{\sin^2 x^2} \\ \\ -e^{-\gamma} \end{pmatrix} \tag{82}$$

Finally we can compute the Einstein tensor:

$$R = a^{11}R_{11} + a^{22}R_{22} + a^{33}R_{33} + a^{44}R_{44} \tag{83}$$

$$= \begin{cases} e^{-\alpha}\left(\gamma_{11}+\frac{1}{2}\gamma_1^2-\frac{1}{2}\alpha_1\gamma_1\right) \\ \\ e^{-\gamma}\left(\alpha_{44}+\frac{1}{2}\alpha_4^2-\frac{1}{2}\alpha_4\gamma_4\right) \\ \\ +e^{-\alpha}\left(\frac{3}{2}+\gamma_1-\alpha_1\right) \\ \\ -2e^{-x^1} \end{cases} \tag{84}$$

$$G^{1}_{.1} = e^{-\alpha} R_{11} - \frac{1}{2} R \tag{85}$$
$$= e^{-\alpha} \left(-\frac{1}{4} - \frac{1}{2}\gamma_1 \right) + e^{-x^1} \tag{86}$$

$$G^{2}_{.2} = e^{-x^1} R_{22} - \frac{1}{2} R \tag{87}$$
$$= \begin{cases} e^{-\alpha} \left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1\gamma_1 + \frac{1}{4}\alpha_1 \right) \\ \\ +e^{-\gamma} \left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4 \right) \end{cases} \tag{88}$$

$$G^{3}_{.3} = \frac{e^{-x^1}}{\sin^2 x^2} R_{33} - \frac{1}{2} R \tag{89}$$
$$= \begin{cases} e^{-\alpha} \left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1\gamma_1 + \frac{1}{4}\alpha_1 \right) \\ \\ +e^{-\gamma} \left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4 \right) \end{cases} \tag{90}$$

$$G^{4}_{.4} = -e^{-\gamma} R_{44} - \frac{1}{2} R \tag{91}$$
$$= e^{-\alpha} \left(-\frac{3}{4} + \frac{1}{2}\alpha_1 \right) + e^{-x^1} \tag{92}$$

$$G^{1}_{.4} = e^{-\alpha} R_{14} \tag{93}$$
$$= -\frac{1}{2} e^{-\alpha} \alpha_4 \tag{94}$$

$$G^{4}_{.1} = -e^{-\gamma} R_{14} \tag{95}$$
$$= \frac{1}{2} e^{-\gamma} \alpha_4 \tag{96}$$

♦

3.19 p110 - Exercise 9

If we change the metric tensor from a_{mn} to $a_{mn} + b_{mn}$ where b_{mn} is small, calculate the principal parts of the increment in the components of the curvature tensor.

Let's start with one form of the curvature tensor

$$R_{rsmn} = \begin{cases} \frac{1}{2}\left(\partial^2_{sm}a_{rn} + \partial^2_{rn}a_{sm} - \partial^2_{sn}a_{rm} - \partial^2_{rm}a_{sn}\right) \\ +a^{pq}\left([rn,p][sm,q] - [rm,p][sn,q]\right) \end{cases} \tag{1}$$

Be R'_{rsmn} and a'_{rn} the components of the tensors after changing the metric form to $a_{mn} + b_{mn}$. Then

$$R'_{rsmn} = \begin{cases} \frac{1}{2}\left(\partial^2_{sm}a'_{rn} + \partial^2_{rn}a'_{sm} - \partial^2_{sn}a'_{rm} - \partial^2_{rm}a'_{sn}\right) \\ +a'^{pq}\left([rn,p]'[sm,q]' - [rm,p]'[sn,q]'\right) \end{cases} \tag{2}$$

At the point where we want to calculate the increment of the curvature tensor, we can choose Riemannian coordinates related to the metric a'_{mn}. Then, the Christoffel symbols vanish at that point as origin and (2) becomes

$$R'_{rsmn} = \frac{1}{2}\left(\partial^2_{sm}a'_{rn} + \partial^2_{rn}a'_{sm} - \partial^2_{sn}a'_{rm} - \partial^2_{rm}a'_{sn}\right) \tag{3}$$

$$= \begin{cases} \frac{1}{2}\left(\partial^2_{sm}a_{rn} + \partial^2_{rn}a_{sm} - \partial^2_{sn}a_{rm} - \partial^2_{rm}a_{sn}\right) \\ \\ +\frac{1}{2}\left(\partial^2_{sm}b_{rn} + \partial^2_{rn}b_{sm} - \partial^2_{sn}b_{rm} - \partial^2_{rm}b_{sn}\right) \end{cases} \tag{4}$$

$$= \begin{cases} R_{rsmn} - a^{pq}\left([rn,p][sm,q] - [rm,p][sn,q]\right) \\ \\ +\frac{1}{2}\left(\partial^2_{sm}b_{rn} + \partial^2_{rn}b_{sm} - \partial^2_{sn}b_{rm} - \partial^2_{rm}b_{sn}\right) \end{cases} \tag{5}$$

$$\Rightarrow \quad \Delta R_{rsmn} = \begin{cases} -a^{pq}\left([rn,p][sm,q] - [rm,p][sn,q]\right) \\ \\ +\frac{1}{2}\left(\partial^2_{sm}b_{rn} + \partial^2_{rn}b_{sm} - \partial^2_{sn}b_{rm} - \partial^2_{rm}b_{sn}\right) \end{cases} \tag{6}$$

As $[rn,p]' = 0$ we have $[rn,p]" = -[rn,p]$ (the suffix " referring to b_{mn}). So we have for $[rn,s]"$ and $[ms,r]"$

$$\partial_r b_{sn} + \partial_n b_{rs} - \partial_s b_{rn} = -\partial_r a_{sn} - \partial_n a_{rs} + \partial_s a_{rn} \tag{7}$$

$$\partial_m b_{rs} + \partial_s b_{rm} - \partial_r b_{sm} = -\partial_m a_{rs} - \partial_s a_{rm} + \partial_r a_{sm} \tag{8}$$

$$\partial_m(7) \quad \Rightarrow \quad \partial^2_{rm}b_{sn} + \partial^2_{mn}b_{rs} - \partial^2_{sm}b_{rn} = -\partial^2_{rm}a_{sn} - \partial^2_{mn}a_{rs} + \partial^2_{sm}a_{rn} \tag{9}$$

$$\partial_n(8) \quad \Rightarrow \quad \partial^2_{mn}b_{rs} + \partial^2_{sn}b_{rm} - \partial^2_{rn}b_{sm} = -\partial^2_{mn}a_{rs} - \partial^2_{sn}a_{rm} + \partial^2_{rn}a_{sm} \tag{10}$$

Combining (9) and (10) in (6), we get

$$\Delta R_{rsmn} = \begin{cases} -a^{pq}\left([rn,p][sm,q]-[rm,p][sn,q]\right) \\ +\frac{1}{2}\left(\partial^2_{rm}a_{sn}+\partial^2_{mn}a_{rs}-\partial^2_{sm}a_{rn}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}a_{rs}+\partial^2_{sn}a_{rm}-\partial^2_{rn}a_{sm}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}b_{rs}+\partial^2_{mn}b_{rs}\right) \end{cases} \tag{11}$$

$$= \begin{cases} a^{pq}\left([rm,p][sn,q]-[rn,p][sm,q]\right) \\ +\frac{1}{2}\left(\partial^2_{rm}a_{sn}+\partial^2_{sn}a_{rm}-\partial^2_{sm}a_{rn}-\partial^2_{rn}a_{sm}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}a_{rs}+\partial^2_{mn}a_{rs}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}b_{rs}+\partial^2_{mn}b_{rs}\right) \end{cases} \tag{12}$$

$$= \begin{cases} R_{rsnm} \\ +\frac{1}{2}\left(\partial^2_{mn}a_{rs}+\partial^2_{mn}a_{rs}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}b_{rs}+\partial^2_{mn}b_{rs}\right) \end{cases} \tag{13}$$

$$= -R_{rsmn} + \partial^2_{mn}a_{rs} + \partial^2_{mn}b_{rs} \tag{14}$$

Can I simplify further ??

♦

3.20 p110 - Exercise 10

If we use normal coordinates in a Riemannian V_n, the metric form is as in equation 2.630. For this coordinate system, express the curvature tensor, the Ricci tensor, and the curvature invariant in terms of the corresponding quantities for the $(N-1)-$ space $x^N = C^{st}$ and certain additional terms. Check these additional terms by nothing that they must have tensor character with respect to transformations of the coordinates $x^1, x^2, \ldots, x^{N-1}$.

$$R_{rsmn} = \partial_m[sn,r] - \partial_n[sm,r] + \Gamma^p_{sm}[rn,p] - \Gamma^p_{sn}[rm,p]$$

We indicate by $R^{'}_{\ldots}$ the quantity generated by the previous definition, but restricted to the subspace V_{N-1}. Calculating $R_{\alpha\beta\gamma\delta}$ restricted to the subspace V_{N-1} gives:

$$R_{\alpha\beta\gamma\delta} = \underbrace{\partial_\gamma[\beta\delta,\alpha] - \partial_\delta[\beta\gamma,\alpha] + \Gamma^\nu_{\beta\gamma}[\alpha\delta,\nu] - \Gamma^\nu_{\beta\delta}[\alpha\gamma,\nu]}_{=R^{'}_{\alpha\beta\gamma\delta}} + \Gamma^N_{\beta\gamma}[\alpha\delta,N] - \Gamma^N_{\beta\delta}[\alpha\gamma,N] \quad (1)$$

$$= R^{'}_{\alpha\beta\gamma\delta} + \underbrace{\Gamma^N_{\beta\gamma}}_{-\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\beta\gamma}} \underbrace{[\alpha\delta,N]}_{-\frac{1}{2}\partial_N a_{\alpha\delta}} - \underbrace{\Gamma^N_{\beta\delta}}_{-\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\beta\delta}} \underbrace{[\alpha\gamma,N]}_{-\frac{1}{2}\partial_N a_{\alpha\gamma}} \quad \text{(see page 66/67)} \quad (2)$$

$$\Rightarrow \quad R_{\alpha\beta\gamma\delta} = R^{'}_{\alpha\beta\gamma\delta} + \frac{1}{4}\frac{1}{a_{NN}}\left(\partial_N a_{\beta\gamma}\partial_N a_{\alpha\delta} - \partial_N a_{\beta\delta}\partial_N a_{\alpha\gamma}\right) \quad (3)$$

We calculate $R_{\alpha\beta}$:

$$R_{\alpha\beta} = a^{sn} R_{s\alpha\beta n} \quad (4)$$

$$= \underbrace{a^{\nu\mu} R_{\nu\alpha\beta\mu}}_{R^{'}_{\alpha\beta}} + \underbrace{a^{N\mu}}_{=0} R_{N\alpha\beta\mu} + \underbrace{a^{\nu N}}_{=0} R_{\nu\alpha\beta N} + \underbrace{a^{NN}}_{=\frac{1}{a_{NN}}} R_{N\alpha\beta N} \quad (5)$$

$$(5) \Rightarrow \quad R_{\alpha\beta} = R^{'}_{\alpha\beta} + \frac{1}{a_{NN}} R_{N\alpha\beta N} \quad (6)$$

$$R_{N\alpha\beta N} = \begin{cases} \partial_\beta[\alpha N,N] - \partial_N[\alpha\beta,N] \\ +\Gamma^\nu_{\alpha\beta}[NN,\nu] - \Gamma^\nu_{\alpha N}[N\beta,\nu] \\ +\Gamma^N_{\alpha\beta}[NN,N] - \Gamma^N_{\alpha N}[N\beta,N] \end{cases} \quad (7)$$

$$= \begin{cases} R^{'}_{N\alpha\beta N} \\ + \underbrace{\Gamma^N_{\alpha\beta}}_{-\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}} \underbrace{[NN,N]}_{\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{NN}} - \underbrace{\Gamma^N_{\alpha N}}_{\frac{1}{2}\frac{1}{a_{NN}}\partial_\alpha a_{NN}} \underbrace{[N\beta,N]}_{\frac{1}{2}\partial_\beta a_{NN}} \end{cases} \quad (8)$$

$$= R^{'}_{N\alpha\beta N} - \frac{1}{4}\frac{1}{a_{NN}}\left(\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}\partial_N a_{NN} + \partial_\alpha a_{NN}\partial_\beta a_{NN}\right) \quad (9)$$

$$(6) \text{ and } (9) \Rightarrow \qquad R_{\alpha\beta} = \begin{cases} R'_{\alpha\beta} + \frac{1}{a_{NN}} R'_{N\alpha\beta N} \\ -\frac{1}{4}\frac{1}{a_{NN}^2}\left(\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}\partial_N a_{NN} + \partial_\alpha a_{NN}\partial_\beta a_{NN}\right) \end{cases} \tag{10}$$

And the invariant curvature:

$$R = a^{mn} R_{mn} \tag{11}$$

$$= \underbrace{a^{\mu\nu} R_{\mu\nu}}_{=R'} + \underbrace{a^{N\nu}}_{=0} R_{N\nu} + \underbrace{a^{\mu N}}_{=0} R_{\mu N} + a^{NN} R_{NN} \tag{12}$$

$$= R' + a^{NN} R_{NN} \tag{13}$$

$$R_{NN} = a_{sn} R_{sNNn} \tag{14}$$

$$= \underbrace{a_{\mu\nu} R_{\mu NN\nu}}_{=R'_{NN}} + \underbrace{a_{N\nu} R_{NNN\nu}}_{=0} + \underbrace{a_{\mu N} R_{\mu NNN}}_{=0} + \underbrace{a_{NN} R_{NNNN}}_{=0} \tag{15}$$

$$(13) \text{ and } (15) \Rightarrow \qquad R = R' + \frac{1}{a_{NN}} R'_{NN} \tag{16}$$

We now check the tensor character of the residuals in (3), (10) and (16) Let's define the following coordinate transformations $a = a(\alpha, \beta, \gamma, \dots),\ b = b(\alpha, \beta, \gamma, \dots), \dots$ and $N = N$. Hence,

$$a_{\alpha\beta} = a_{ab}\partial_\alpha a \partial_\beta b \tag{17}$$

Now,

$$i) \qquad \mathbf{R}_{\alpha\beta\gamma\delta} = \mathbf{R}'_{\alpha\beta\gamma\delta} + \frac{\mathbf{1}}{\mathbf{4}}\frac{\mathbf{1}}{\mathbf{a_{NN}}} \underbrace{\left(\partial_N a_{\beta\gamma}\partial_N a_{\alpha\delta} - \partial_N a_{\beta\delta}\partial_N a_{\alpha\gamma}\right)}_{\mathbf{T}_{\alpha\beta\gamma\delta}=\mathbf{tensor}?}$$

$$T_{\alpha\beta\gamma\delta} = \begin{cases} \partial_N\left(a_{ab}\partial_\beta a \partial_\gamma b\right)\partial_N\left(a_{cd}\partial_\alpha c \partial_\delta d\right) \\ -\partial_N\left(a_{ab}\partial_\beta a \partial_\delta b\right)\partial_N\left(a_{cd}\partial_\alpha c \partial_\gamma d\right) \end{cases} \tag{18}$$

$$= \begin{cases} \partial_N a_{ab}\partial_N a_{cd}\partial_\beta a \partial_\gamma b \partial_\alpha c \partial_\delta d \\ -\partial_N a_{ab}\partial_N a_{cd}\partial_\beta a \partial_\delta b \partial_\alpha c \partial_\gamma d \\ +\text{other terms} \end{cases} \tag{19}$$

$$= \begin{cases} \left(\underbrace{\partial_N a_{cb}\partial_N a_{ad} - \partial_N a_{bd}\partial_N a_{ac}}_{\equiv T_{abcd}}\right)\partial_\alpha a \partial_\beta b \partial_\gamma g \partial_\delta d \\ +\text{other terms} \end{cases} \tag{20}$$

So we see that T_{abcd} transforms to $T_{\alpha\beta\gamma\delta}$ as a tensor provided that the other terms in (20) vanish. This is the case as the other terms will all have elements containing factors like $\partial^2_{N\nu x}$ which are zero as the x are no explicit function of N.

$$ii) \qquad \mathbf{R}_{\alpha\beta} = \begin{cases} R'_{\alpha\beta} + \frac{1}{a_{NN}} R'_{N\alpha\beta N} \\ \\ -\frac{1}{4}\frac{1}{a_{NN}^2}\left(\underbrace{\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}\partial_N a_{NN} + \partial_\alpha a_{NN}\partial_\beta a_{NN}}_{\equiv T_{\alpha\beta}=tensor?}\right) \end{cases}$$

$$T_{\alpha\beta} = \frac{1}{a_{NN}}\partial_N\left(a_{ab}\partial_\alpha a\partial_\beta b\right)\partial_N a_{NN} + \partial_a a_{NN}\partial_b a_{NN}\partial_\alpha a\partial_\beta b \tag{21}$$

$$= \frac{1}{a_{NN}}\partial_N a_{ab}\partial_\alpha a\partial_\beta b\partial_N a_{NN} + \partial_a a_{NN}\partial_b a_{NN}\partial_\alpha a\partial_\beta b + \text{ other terms} \tag{22}$$

$$= \left(\underbrace{\frac{1}{a_{NN}}\partial_N a_{ab}\partial_N a_{NN} + \partial_a a_{NN}\partial_b a_{NN}}_{\equiv T_{ab}}\right)\partial_\alpha a\partial_\beta b + \text{ other terms} \tag{23}$$

Again the other terms vanish - see previous curvature tensor - and so T_{ab} transforms to $T_{\alpha\beta}$ as a tensor.

$$iii) \qquad \mathbf{R} = \mathbf{R}' + \frac{\mathbf{1}}{\mathbf{a_{NN}}}\mathbf{R}'_{\mathbf{NN}}$$

The last term is an invariant and by definition at tensor.

♦

3.21 p110 - Exercise 11

Prove that

$$T^{mn}{}_{|mn} = T^{mn}{}_{|nm}$$

where T^{mn} is not necessarily symmetric.

$$T^{mn}{}_{|s} = \partial_s T^{mn} + \Gamma^m_{ks} T^{kn} + \Gamma^n_{ks} T^{mk} \tag{1}$$
$$T^{mn}{}_{|st} = (\partial_s T^{mn})_{|t} + (\Gamma^m_{ks})_{|t} T^{kn} + (\Gamma^n_{ks})_{|t} T^{mk} + \Gamma^m_{ks} (T^{kn})_{|t} + \Gamma^n_{ks} (T^{mk})_{|t}) \tag{2}$$

We choose Riemannian coordinates and take as origin the point where we want to check the asked identity. At that point the Christoffel symbols vanish and (2) becomes

$$T^{mn}{}_{|st} = (\partial_s T^{mn})_{|t} + (\Gamma^m_{ks})_{|t} T^{kn} + (\Gamma^n_{ks})_{|t} T^{mk} \tag{3}$$

$$= \begin{cases} \partial^2_{st} T^{mn} + \quad \text{terms in} \quad \Gamma\text{'s } (=0) \\ +T^{kn} (\Gamma^m_{ks})_{|t} \\ +T^{mk} (\Gamma^n_{ks})_{|t} \end{cases} \tag{4}$$

$$= \begin{cases} \partial^2_{st} T^{mn} \\ +T^{kn} \left[\underbrace{a^{mp}{}_{|t}}_{=0} [ks, p] + \frac{1}{2} a^{mp} \left((\partial_k a_{sp})_{|t} + (\partial_s a_{kp})_{|t} - (\partial_p a_{ks})_{|t} \right) \right] \\ +T^{mk} \left[\underbrace{a^{np}{}_{|t}}_{=0} [ks, p] + \frac{1}{2} a^{np} \left((\partial_k a_{sp})_{|t} + (\partial_s a_{kp})_{|t} - (\partial_p a_{ks})_{|t} \right) \right] \end{cases} \tag{5}$$

$$= \begin{cases} \partial^2_{st} T^{mn} \\ +\frac{1}{2} a^{mp} \left(\partial^2_{kt} a_{sp} + \partial^2_{st} a_{kp} - \partial^2_{pt} a_{ks} \right) T^{kn} \\ +\frac{1}{2} a^{np} \left(\partial^2_{kt} a_{sp} + \partial^2_{st} a_{kp} - \partial^2_{pt} a_{ks} \right) T^{mk} \end{cases} \tag{6}$$

In the same way we get

$$T^{mn}{}_{|ts} = \begin{cases} \partial^2_{st} T^{mn} \\ +\frac{1}{2} a^{mp} \left(\partial^2_{ks} a_{tp} + \partial^2_{st} a_{kp} - \partial^2_{ps} a_{kt} \right) T^{kn} \\ +\frac{1}{2} a^{np} \left(\partial^2_{ks} a_{tp} + \partial^2_{st} a_{kp} - \partial^2_{ps} a_{kt} \right) T^{mk} \end{cases} \tag{7}$$

Hence

$$2\left(T^{mn}{}_{|st} - T^{mn}{}_{|ts}\right) = \begin{cases} \quad a^{mp}\left(\partial^2_{kt}a_{sp} + \partial^2_{st}a_{kp} - \partial^2_{pt}a_{ks} - \partial^2_{ks}a_{tp} - \partial^2_{st}a_{kp} + \partial^2_{ps}a_{kt}\right)T^{kn} \\ \\ +a^{np}\left(\partial^2_{kt}a_{sp} + \partial^2_{st}a_{kp} - \partial^2_{pt}a_{ks} - \partial^2_{ks}a_{tp} - \partial^2_{st}a_{kp} + \partial^2_{ps}a_{kt}\right)T^{mk} \end{cases} \tag{8}$$

$$= \begin{cases} \quad a^{mp}\left(\partial^2_{kt}a_{sp} - \partial^2_{pt}a_{ks} - \partial^2_{ks}a_{tp} + \partial^2_{ps}a_{kt}\right)T^{kn} \\ \\ +a^{np}\left(\partial^2_{kt}a_{sp} - \partial^2_{pt}a_{ks} - \partial^2_{ks}a_{tp} + \partial^2_{ps}a_{kt}\right)T^{mk} \end{cases} \tag{9}$$

Putting $s = m$ and $t = n$:

$$2\left(T^{mn}{}_{|mn} - T^{mn}{}_{|nm}\right) = \begin{cases} \quad a^{mp}\left(\partial^2_{kn}a_{mp} - \partial^2_{pn}a_{km} - \partial^2_{km}a_{np} + \partial^2_{pm}a_{kn}\right)T^{kn} \\ \\ +a^{np}\left(\partial^2_{kn}a_{mp} - \partial^2_{pn}a_{km} - \partial^2_{km}a_{np} + \partial^2_{pm}a_{kn}\right)T^{mk} \end{cases} \tag{10}$$

In the last term of (10) we can swap the indices m and n giving

$$2\left(T^{mn}{}_{|mn} - T^{mn}{}_{|nm}\right) = \begin{cases} \quad a^{mp}\left(\partial^2_{kn}a_{mp} - \partial^2_{pn}a_{km} - \partial^2_{km}a_{np} + \partial^2_{pm}a_{kn}\right)T^{kn} \\ \\ +a^{mp}\left(\partial^2_{km}a_{np} - \partial^2_{pm}a_{kn} - \partial^2_{kn}a_{mp} + \partial^2_{pn}a_{km}\right)T^{nk} \end{cases} \tag{11}$$

In the second term of (11) we may again swap the indices n and k giving

$$2\left(T^{mn}{}_{|mn} - T^{mn}{}_{|nm}\right) = \begin{cases} \quad a^{mp}\left(\cancel{\partial^2_{kn}a_{mp}} - \partial^2_{pn}a_{km} - \partial^2_{km}a_{np} + \cancel{\partial^2_{pm}a_{kn}}\right)T^{kn} \\ \\ +a^{mp}\left(\partial^2_{nm}a_{kp} - \cancel{\partial^2_{pm}a_{kn}} - \cancel{\partial^2_{kn}a_{mp}} + \partial^2_{pk}a_{nm}\right)T^{kn} \end{cases} \tag{12}$$

$$\tag{13}$$

$$= \left(\underbrace{a^{mp}\partial^2_{nm}a_{kp}}_{\text{swap m and p}} + \underbrace{a^{mp}\partial^2_{pk}a_{nm}}_{\text{swap m and p}} - a^{mp}\partial^2_{pn}a_{km} - a^{mp}\partial^2_{km}a_{np}\right)T^{kn} \tag{14}$$

$$= \left(a^{mp}\partial^2_{np}a_{km} + a^{mp}\partial^2_{mk}a_{np} - a^{mp}\partial^2_{pn}a_{km} - a^{mp}\partial^2_{km}a_{np}\right)T^{kn} \tag{15}$$

$$= 0 \tag{16}$$

♦

3.22 p110 - Exercise 12 ††

Prove that the quantities

$$G^{mn} + \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]$$

can be expressed in terms of the metric tensor and its first derivatives.

Let's define

$$K^{mn} \equiv \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right] \tag{1}$$

$$T^{mn} \equiv G^{mn} + K^{mn} \tag{2}$$

The strategy to proof this, is to separate in both terms of the expression, the parts that can be expressed in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and the parts of higher order differentiation. We begin with the second term.
As the covariant derivatives of the metric tensor vanish, we get

$$K^{mn} = \frac{1}{2a}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|rs} \tag{3}$$

$$= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|r}}_{=0}\right]_{|s} \tag{4}$$

$$= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|s} \tag{5}$$

$$= \frac{1}{2a}\left[\underbrace{a_{|rs}}_{=\partial^2_{rs}a}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a_{|r}\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|s}}_{=0}\right] \tag{6}$$

Considering,

$$\partial^2_{rs}\ln a = \partial_s\left(\frac{1}{a}\partial_r a\right) \tag{7}$$

$$= \frac{1}{a}\partial^2_{rs}a - \frac{1}{a^2}\partial_r a\partial_s a \tag{8}$$

$$\Rightarrow \qquad \frac{1}{a}\partial^2_{rs}a = \partial^2_{rs}\ln a + \frac{1}{a^2}\partial_r a\partial_s a \tag{9}$$

$$\Rightarrow \quad K^{mn} = \frac{1}{2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial^2_{rs}\ln a + \mathcal{E}^{mn} \tag{10}$$

with $\mathcal{E}^{mn}$ being a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.

Note that $\mathcal{E}^{mn}$ is a symmetrical object in m, n. Indeed,

$$\mathcal{E}^{mn} = \frac{1}{2a^2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r a \partial_s a \tag{11}$$

$$\Rightarrow \qquad \mathcal{E}^{nm} = \frac{1}{2a^2}\left(a^{nm}a^{rs} - a^{nr}a^{ms}\right)\partial_r a \partial_s a \tag{12}$$

$$= \frac{1}{2a^2}\left(a^{mn}a^{sr} - a^{ns}a^{mr}\right)\partial_s a \partial_r a \tag{13}$$

$$= \mathcal{E}^{mn} \tag{14}$$

But by (2.541.) : $\partial_i \ln a = 2\Gamma^t_{it}$ hence,

$$\partial_{rs} \ln a = 2\partial_s \Gamma^t_{rt} = 2\partial_r \Gamma^t_{st} \tag{15}$$

$$\Rightarrow \quad K^{mn} = \left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_s \Gamma^t_{rt} + \mathcal{E}^{mn} \tag{16}$$

$$\tag{17}$$

Considering (10) and swapping dummy indices we get the following expressions for K^{mn}

$$K^{mn} = \left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_s \Gamma^t_{rt} + \mathcal{E}^{mn} \tag{18}$$

$$K^{mn} = \left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r \Gamma^t_{st} + \mathcal{E}^{mn} \tag{19}$$

$$K^{mn} = \left(a^{mn}a^{rs} - a^{ms}a^{nr}\right)\partial_s \Gamma^t_{rt} + \mathcal{E}^{mn} \tag{20}$$

$$K^{mn} = \left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r \Gamma^t_{st} + \mathcal{E}^{mn} \tag{21}$$

$$\tag{22}$$

We now rewrite G^{mn}.
We have:

$$G^{mn} = a^{nk} G^{m}_{\cdot\ k} \tag{23}$$

$$G^{m}_{\cdot\ k} = R^{m}_{\cdot\ k} - \frac{1}{2}\delta^m_k R \tag{24}$$

$$R^{m}_{\cdot\ k} = a^{mp} R_{pk} \tag{25}$$

$$R = a^{pk} R_{pk} \tag{26}$$

And by 3.203.

$$R_{pk} = \partial_k \Gamma^t_{pt} - \partial_t \Gamma^t_{pk} + \mathcal{F}^{pk} \tag{27}$$

With $\mathcal{F}^{pk}$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only. Note that $\mathcal{F}^{pk}$ is a symmetrical object in p, k as

R_{pk} is a symmetrical tensor in p, k. Hence by (15) to (19):

$$G^{mn} = a^{nr} R^{m}_{\cdot\, r} - \frac{1}{2} a^{nr} \delta^{m}_{k} R \tag{28}$$

$$= a^{nr} a^{ms} R_{pk} - \frac{1}{2} a^{nm} a^{rs} R_{rs} \tag{29}$$

$$= \left(a^{nr} a^{ms} - \frac{1}{2} a^{nm} a^{rs} \right) R_{rs} \tag{30}$$

$$= \left(a^{nr} a^{ms} - \frac{1}{2} a^{nm} a^{rs} \right) \left(\partial_r \Gamma^{t}_{st} - \partial_t \Gamma^{t}_{rs} + \mathcal{F}^{rs} \right) \tag{31}$$

$$= \left(a^{nr} a^{ms} - \frac{1}{2} a^{nm} a^{rs} \right) \left(\partial_r \Gamma^{t}_{st} - \partial_t \Gamma^{t}_{rs} \right) + \mathcal{H}^{mn} \tag{32}$$

$$= a^{nr} a^{ms} \partial_r \Gamma^{t}_{st} - \frac{1}{2} a^{nm} a^{rs} \partial_r \Gamma^{t}_{st} - a^{nr} a^{ms} \partial_t \Gamma^{t}_{rs} + \frac{1}{2} a^{nm} a^{rs} \partial_t \Gamma^{t}_{rs} + \mathcal{H}^{mn} \tag{33}$$

with $\mathcal{H}^{mn}$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.
Note that $\mathcal{H}^{mn}$ is a symmetrical object in m, n . Indeed,

$$\mathcal{H}^{mn} = \left(a^{nr} a^{ms} - \frac{1}{2} a^{nm} a^{rs} \right) \mathcal{F}^{rs} \tag{34}$$

$$\Rightarrow \qquad \mathcal{H}^{nm} = \left(a^{mr} a^{ns} - \frac{1}{2} a^{mn} a^{rs} \right) \mathcal{F}^{rs} \tag{35}$$

$$= \left(a^{ms} a^{nr} - \frac{1}{2} a^{mn} a^{rs} \right) \mathcal{F}^{rs} \tag{36}$$

$$= \mathcal{H}^{mn} \tag{37}$$

Putting (10) and (2) together with $\mathcal{L}^{mn} = \mathcal{E}^{mn} + \mathcal{H}^{mn}$ we get,

$$T^{mn} = Q^{mn} + \mathcal{L}^{mn} \tag{38}$$

with $\mathcal{L}^{mn}$ a symmetrical object in m, n and depending only on terms in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and

$$Q^{mn} \equiv \begin{cases} \cancel{a^{nr} a^{ms} \partial_r \Gamma^{t}_{st}} - \underbrace{\frac{1}{2} a^{nm} a^{rs} \partial_r \Gamma^{t}_{st}}_{*} \\[2em] -a^{nr} a^{ms} \partial_t \Gamma^{t}_{rs} + \tfrac{1}{2} a^{nm} a^{rs} \partial_t \Gamma^{t}_{rs} \\[2em] + \underbrace{a^{mn} a^{rs} \partial_s \Gamma^{t}_{rt}}_{*} - \cancel{a^{mr} a^{ns} \partial_s \Gamma^{t}_{rt}} \end{cases} \tag{39}$$

$$= \frac{1}{2} a^{mn} a^{rs} \left(\partial_s \Gamma^{t}_{rt} + \partial_t \Gamma^{t}_{rs} \right) - a^{nr} a^{ms} \partial_t \Gamma^{t}_{rs} \tag{40}$$

Q^{mn} a symmetrical object in m, n but still containing seconder order derivatives.
† And here, I'm stuck.†

♦

Special types of space

4.1 p112 - Exercise

Deduce from 4.110. that the Gaussian curvature of a V_2 positive-definite metric is given by

$$G = \frac{R_{1212}}{a_{11}a_{22} - a_{12}^2}$$

From p. 86 (exercise) we know that all the components of R_{mnrs} can be expressed as terms of R_{1212} (or vanish).
So by 4.110.,

$$K\left(a_{11}a_{22} - a_{12}a_{21}\right) = R_{1212} \tag{1}$$

and from page 96 (**3.415**) we know that for V_2, $K = G$. Hence,

$$G = \frac{R_{1212}}{\left(a_{11}a_{22} - a_{12}a_{21}\right)} \tag{2}$$

♦

4.2 p113 - Exercise

Prove that, in a space V_N of constant curvature K,

$$(\mathbf{4.115}) \qquad R_{mn} = -(N-1)\,K a_{mn}, \quad R = -N(N-1)\,K$$

We have

$$R_{mn} = R^{s}_{.mns} = a^{sk} R_{kmns}$$

From (**4.114**)

$$\begin{aligned} & R_{kmns} = K\left(a_{kn}a_{ms} - a_{ks}a_{mn}\right) \\ \Rightarrow \qquad & R_{kmns} = K\left(\underbrace{\delta^{s}_{n}a_{ms}}_{amn} - N a_{mn}\right) = K\left(1-N\right)a_{mn} \end{aligned}$$

and

$$\begin{aligned} R &= R^{n}_{.n} \\ &= a^{kn}R_{kn} \\ &= -\underbrace{a^{kn}a_{kn}}_{N}\left(N-1\right)K \\ &= -N\left(N-1\right)K \end{aligned}$$

♦

4.3 p113 - Clarification

4.117. $$\frac{\delta^2 \eta^r}{\delta s^2} + \epsilon K \eta^r = 0$$

We have

$$\begin{aligned}
& R^r_{.smn} = a^{rk} R_{ksmn} && (1)\\
(4.114) \Rightarrow \quad & \qquad = a^{rk} K \left(a_{km} a_{sn} - a_{kn} a_{sm}\right) && (2)\\
& \qquad = K \left(\delta^r_m a_{sn} - \delta^r_n a_{sm}\right) && (3)
\end{aligned}$$

$$\begin{aligned}
(3.311) \text{ and } (3) \qquad & 0 = \frac{\delta^2 \eta^r}{\delta s^2} + K \left(\delta^r_m a_{sn} - \delta^r_n a_{sm}\right) p^s \eta^m p^n \\
\Leftrightarrow \qquad & 0 = \frac{\delta^2 \eta^r}{\delta s^2} + K \left(\delta^r_m \eta^m \underbrace{a_{sn} p^s p^n}_{=\epsilon} - \delta^r_n \underbrace{a_{sm} p^s \eta^m}_{=0} p^n \right) \\
\Leftrightarrow \qquad & 0 = \frac{\delta^2 \eta^r}{\delta s^2} + K \epsilon \underbrace{\delta^r_m \eta^m}_{=\eta^r} \\
\Leftrightarrow \qquad & 0 = \frac{\delta^2 \eta^r}{\delta s^2} + K \epsilon \eta^r
\end{aligned}$$

♦

4.4 p114 - Clarification

4.118. $$\frac{d^2\left(X_r\eta^r\right)}{ds^2}+\epsilon K\left(X_r\eta^r\right)=0$$

We know that $\frac{\delta X_r}{\delta s}=0$ (parallel transport)

$$\begin{aligned}
& & \frac{\delta\left(X_r\eta^r\right)}{\delta s} &= \eta^r\underbrace{\frac{\delta X_r}{\delta s}}_{=0}+X_r\frac{\delta\eta^r}{\delta s} \\
&\Rightarrow & \frac{\delta^2\left(X_r\eta^r\right)}{\delta s} &= \underbrace{\frac{\delta X_r}{\delta s}}_{=0}\frac{\delta\eta^r}{\delta s}+X_r\frac{\delta^2\eta^r}{\delta s^2} \\
&\Rightarrow & X_r\frac{\delta^2\eta^r}{\delta s^2} &= \frac{\delta^2\left(X_r\eta^r\right)}{\delta s^2} \\
&\text{but} & \frac{\delta^2\left(X_r\eta^r\right)}{\delta s^2} &= \frac{d^2X_r\eta^r}{ds^2} \qquad \text{as } X_r\eta^r \text{ is an invariant} \\
&\Rightarrow & X_r\frac{\delta^2\eta^r}{\delta s^2} &= \frac{d^2X_r\eta^r}{ds^2} \\
&\text{and so} & \frac{\delta^2\left(\eta^r\right)}{\delta s^2}X_r+\epsilon K\left(X_r\eta^r\right) &= \frac{d^2X_r\eta^r}{ds^2}+\epsilon K\left(X_r\eta^r\right)=0
\end{aligned}$$

♦

4.5 p115 - Exercise

By taking an orthonormal set of N unit vectors propagated parallely along the geodesic, deduce from 4.120a that the magnitude η of the vector η^r is given by

$$\eta = C \left|\sin\left(s\sqrt{\epsilon K}\right)\right|$$

where C is a constant.

We have by 4.120a

$$X_r \eta^r = A \sin\left(s\sqrt{\epsilon K}\right) \tag{1}$$

We choose N different $X_r^{(k)}$ $(k = 1, 2, \ldots, N)$ which are orthonormal. Applying (1) N times with the different $X_r^{(k)}$ $(k = 1, 2, \ldots, N)$, we get

$$X_r^{(k)} \eta^r = A^{(k)} \sin\left(s\sqrt{\epsilon K}\right) \tag{2}$$

But as the $X_r^{(k)}$ are orthonormal and are used as a basis at the considered point of the geodesic we have

$$X_r^{(k)} = \delta_r^k$$

So, (2) becomes

$$\eta^k = A^{(k)} \sin\left(s\sqrt{\epsilon K}\right)$$

which are the components of the displacement vector in the orthonormal basis. By **2.301**. :

$$\begin{aligned}
& & Y^2 &= \epsilon a_{mn} Y^m Y^n \\
&\Rightarrow & \eta^2 &= \epsilon a_{mn} A^{(m)} A^{(n)} \sin^2\left(s\sqrt{\epsilon K}\right) \\
&\Rightarrow & \eta &= C \left|\sin\left(s\sqrt{\epsilon K}\right)\right| \\
&\text{with} & C &= \sqrt{\left|\epsilon a_{mn} A^{(m)} A^{(n)}\right|}
\end{aligned}$$

♦

4.6 p118 - Exercise

Examine the limit of the form **4.130** as R tends to infinity, and interpet the result.

We have

4.130. $$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

But $\sin\epsilon \approx \epsilon$ for $\epsilon \ll 1$. So

$$\lim_{R\to\infty} ds^2 = dr^2 + R^2\left(\frac{r}{R}\right)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

$$= dr^2 + \left(rd\theta^2\right) + (r\sin\theta d\phi)^2 \tag{2}$$

This is the metric form for an Euclidean 3-space with spherical polar coordinates (see **2.532**. page 54).

♦

4.7 p119 - Exercise

Show that a transformation of a homogeneous coordinate system into another homogeneous system is necessarily linear. (Use the transformation equation **2.507** for Christoffel symbols, noting that all Christoffel symbols vanish when the coordinates are homogeneous).

By 2.507 we have the transformation rule

$$\Gamma'^{a}_{bc} = \Gamma^{r}_{mn} \partial_r z^a \partial_b z^m \partial_c z^n + \partial_r z^a \frac{\partial^2 z^r}{\partial z^b \partial z^c} \tag{1}$$

But, as both coordinate system are homogeneous, all Christoffel symbols vanish and so

$$\Gamma'^{a}_{bc} = \partial_r z^a \frac{\partial^2 z^r}{\partial z^b \partial z^c} = 0 \tag{2}$$

$$\Rightarrow \qquad \partial_r z^a \frac{\partial^2 z^r}{\partial z^b \partial z^c} = 0 \tag{3}$$

As the Jacobian can't vanish the possibility of having $\partial_r z^a = 0 \ \forall a, r$ is excluded. Hence we must have $\frac{\partial^2 z^r}{\partial z^b \partial z^c} = 0$. And have a linear solution of the form

$$z^r = A_k z'^k + C \tag{4}$$

♦

4.8 p120 - Exercise

If $z_r, z_r^{'}$ are two systems of rectangular Cartesian coordinates in Euclidean 3-space, what is the geometrical interpretation of the constants in **4.204** and of the orthogonality conditions **4.209** ?

We have

$$z_m^{'} = A_{mn} z_n + A_m \tag{1}$$

As we assume that the Jacobian of the transformation does not vanish and thus the mapping is bijective, in an Euclidean 3-space A_m will perform a *translation* while A_{mn} can be interpreted as a combination rotation/reflection/stretching/contraction/shearing. I.e. the mapping is an *affine* transformation.
The condition **4.209** restricts the action of A_{mn} to a combination of rotation/reflection. Indeed, a rotation/reflection can be represented by $R = R_x(\gamma) \circ R_y(\beta) \circ R_z(\alpha)$ with

$$R_x = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} R_y = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & \pm 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} R_z = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \tag{2}$$

Note that for every axis, $R_k^T R_k = \mathbb{I}_3$.
Be $A_{mn} = R_x(\gamma) \circ R_y(\beta) \circ R_z(\alpha)$ We have by **4.209**, $A_{mq} A_{mq} = \delta_{pq}$ which can be expressed as

$$A^T A = \mathbb{I}_3 \tag{3}$$

$$\Rightarrow \qquad \mathbb{I}_3 = (R_x R_y R_z)^T R_x R_y R_z \tag{4}$$

$$= \underbrace{R_z^T \underbrace{R_y^T \underbrace{R_x^T R_x}_{=\mathbb{I}} R_y}_{=\mathbb{I}} R_z}_{=\mathbb{I}_3} \tag{5}$$

The identity yields, and interpret the coefficients of the orthogonal transformation as an Euclidean orthogonal transformation.

♦

4.9 p123 - Clarification

If $A_n A_n = 0$ it follows from **2.445** and **2.446** that the straight line is a geodesic null line.

We have

$$\begin{aligned}
&(2.446) && a_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \qquad \frac{dx^m}{du} = \frac{dz_m}{du} = A_m && (1)\\
&(4.215) && a_{mn} = \delta_{mn} && (2)\\
&(1),(2) \quad \Rightarrow && \delta_{mn}\frac{dz_m}{du}\frac{dz_n}{du} = 0 && (3)\\
&\Rightarrow && A_n A_n = 0 && (4)
\end{aligned}$$

♦

4.10 p123 - Clarification

It is easy to see ... viz.,

the straight line joining any two points in a plane lies entirely in the plane.

The plane is identified by

$$A_n z_n + B = 0$$

and a line by

$$z_n = C_n u + D_n$$

Take two points at $u = 0$ and $u = p$ lying in the plane:

$$\begin{cases} A_n C_n p + A_n D_n + B = 0 \\ A_n D_n + B = 0 \end{cases}$$

$$\Rightarrow \qquad \begin{cases} A_n C_n p = 0 \\ A_n D_n + B = 0 \end{cases}$$

And as $p \neq 0 \Rightarrow A_n C_n = 0$. So for any arbitrary u of this line we have

$$\underbrace{A_n C_n}_{=0} u + \underbrace{A_n D_n + B}_{=0} = 0$$

hence, all points of the line lie in the plane.

♦

4.11 p123 - Exercise

Show that a one-flat is a straight line.

A one-flat means $(N-1)$ equations

$$A_n^{(k)} z_n + B^{(k)} = 0 \quad k = 1, \ldots, N-1 \quad n = 1, \ldots, N \tag{1}$$

This is a set of $(N-1)$ linear equation in N unknown z_n. So we have one degree of freedom. E.g. put $z_N = u$ with u the free parameter. then,

$$A_\alpha^{(k)} z_\alpha + A_N^{(k)} u + B^{(k)} = 0 \quad \alpha = 1, \ldots, N-1 \tag{2}$$

If $det A_\alpha^{(k)} \neq 0$ we get a solution of the set of equation

$$Az = B \quad \text{with} \quad B \text{ a linear function in } u \tag{3}$$

$$\Rightarrow \qquad z_m = \left(A^{-1}B\right)_m \tag{4}$$

with $\left(A^{-1}B\right)_m$ of the form $C_m u + D_m$

♦

4.12 p126 - Exercise†

Show that the null cone with vertex at the origin in space-time has the equation

$$y_1^2 + y_2^2 + y_3^2 - y_4^2 = 0$$

Prove that this null cone divides space-time into three regions such that
a. Any two points (events) both lying in one region can be joined by a continuous curve which does not cut the null cone.
b. All continuous curves joining two given points (events) which lie in different regions, cut the null cone.
Show further that the three regions may be further classified into past, present, and future as follows: If A and B are any two points in the past, then the straight segment AB lies entirely in the past. If A and B are any two points in the future, then the straight segment AB lies entirely in the future. If A is any point in the present, there exist at least one point B in the present that the straight segment AB cuts the null cone.

We first prove

$$y_1^2 + y_2^2 + y_3^2 - y_4^2 = 0$$

The null geodesic equations:

$$\begin{cases} \frac{\delta^2 x^r}{\delta u^2} = \frac{d^2 x_r}{du^2} = 0 \quad \text{as we use homogeneous coordinates} \\ \\ a_{mn}\frac{dx_m}{du}\frac{dx_n}{du} = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x_r = A_r u + B_r \quad (\text{ put } B_r = 0 \text{ by adequate choice of the origin }) \\ \\ A_1^2 + A_2^2 + A_1^2 + A_3^2 - A_4^2 = 0 \end{cases}$$

$$\Rightarrow \quad \frac{\left((x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2\right)}{u^2} = 0 \quad \text{for } u \neq 0$$

$$\Rightarrow \quad (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2 = 0$$

◊

About the existence of three regions. First let's investigate the case in a V_3 space-time manifold in order to have a more intuitive grasp.
Consider a family of events (p_1, p_2, u) with $u \in (-\infty, \infty)$ (see the line $P^{'}P$" in figure 1.1.)

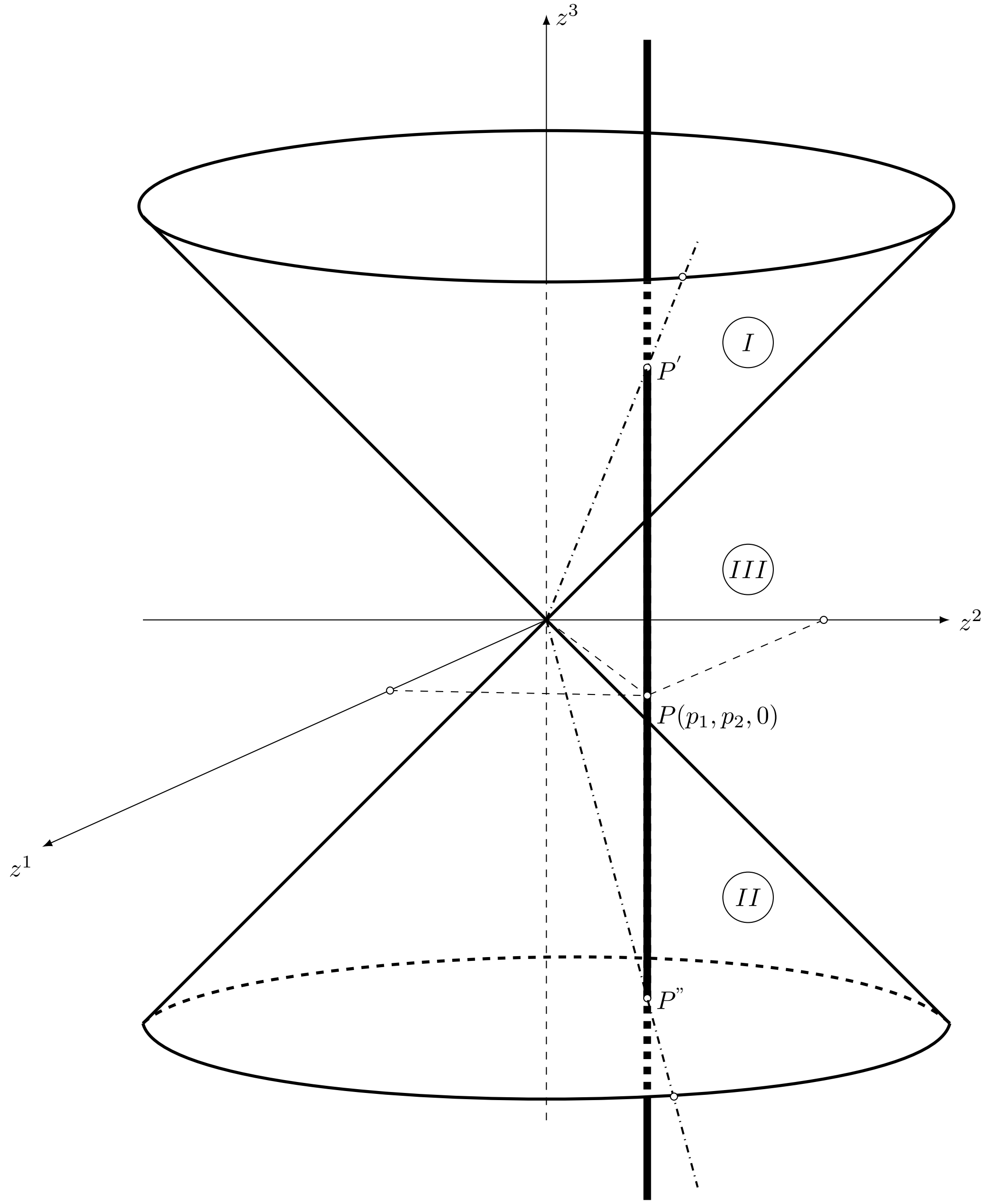


Figure 4.1: Regions delimited by the light cone in a V_3 space-time manifold

Be $R^2 = y_1^2 + y_2^2$. The light cone has the the equation $R^2 - y_3^2 = 0$. Only the events at $u_0 = \pm R(p_1, p_2)$ will lie on the light-cone.
We can distinguish three regions:
Region I where $u > u_0 = R$: the events (p_1, p_2, u) will lie on the line above point P'.
Region II where $u < -u_0 = -R$: the events (p_1, p_2, u) will lie on the line below point $P"$.
Region III where $-R = -u_0 < u < u_0 = R$: the events (p_1, p_2, u) will lie on the segment $P'P"$.

Let's generalize this now for a V_4 space-time manifold.
Put $R^2 = (y_1)^2 + (y_2)^2 + (y_3)^2$ and consider $\phi(y_1, y_2, y_3, y_4) = (y_1)^2 + (y_2)^2 + (y_3)^2 - (y_4)^2$ so

$$\phi(y_1, y_2, y_3, y_4) = R^2 - (y_4)^2 \tag{1}$$

For $\phi = 0$ we lie on the light-cone.
For $\phi > 0 \quad \Rightarrow R^2 > y_4^2$ and so $-R < y_4 < R$ defines one region (region III).
For $\phi < 0 \quad \Rightarrow R^2 < y_4^2$ and so $y_4 > R$ or $y_4 < -R$ define two regions (region I and II).

$\lozenge$

We now show statement a. of the exercise.
Consider 2 events $P_0,\ P_1$ with coordinates $(y_1^{(0)}, y_2^{(0)}, y_3^{(0)}, y_4^{(0)})$ and $(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)})$ and a curve defined by

$$y_i = \pm\sqrt{\left(\left(y_i^{(1)}\right)^2 - \left(y_i^{(0)}\right)^2\right) u + \left(y_i^{(0)}\right)^2} \qquad u \in [0,1] \tag{2}$$

where the $\pm$ is chosen so that $y_i(0) = y_i^{(0)}$ and $y_i(1) = y_i^{(1)}$ and that the sign only changes when $y_i(u) = 0$ and $sign\left(y_i^{(0)}\right) \neq sign\left(y_i^{(1)}\right)$. Such curve will be continuous.
Be $\phi_0 \equiv \phi(0)$, $\phi_1 \equiv \phi(1)$,

Then (2) in (1) gives

$$\phi(u) = (\phi_1 + \phi_0)\, u + \phi_0 \tag{3}$$

Case a1: P_0, P_1 both lie in region I or both in region II.
We have the following conditions:

$$\phi_0 < 0 \wedge \phi_1 < 0$$

then,

$$\nexists\ u \in [0,1] : \phi(u) = 0$$

Indeed, as $\phi_0 < 0 \wedge \phi_1 < 0$, we have in expression (3), $\phi_1 + \phi_0 < 0$ and $\phi_0 < 0$ meaning that for $\phi(u)$ to become 0, u must be negative, which is not in the domain of the defined curve.

So, there exist no $u \in [0,1]$ for which $\phi(u) = 0$ and the curve does not intersect the null cone.

Case a2: P_0, P_1 both lie in region III. Then,

$$\phi_0 > 0 \wedge \phi_1 > 0$$

Then,

$$\nexists\ u \in [0,1] : \phi(u) = 0$$

as we can follow the same reasoning as in case 1 but with $\phi_1 + \phi_0 > 0$ and $\phi_0 > 0$. So, there exist no $u \in [0, 1]$ for which $\phi(u) = 0$ and the curve does not intersect the null cone.

$\Diamond$

We now show statement b. of the exercise.

Case b1: P_0 lies in region I, P_1 lies in region II.
Those two regions are separated by the 3-flat (plane) $y_4 = 0$. So it's suffice that $R^2 = 0$ for $\phi(u)$ being zero. Hence $y_i = 0$, $i = 1, 2, 3$, and the curve will cut the cone at it's apex.

Case b2: P_0 lies in region I or II, P_1 lies in region III.
Put

$$\begin{aligned} R^2 &= (y_1)^2 + (y_2)^2 + (y_3)^2 \\ {R_0}^2 &= \left(y_1^0\right)^2 + \left(y_2^0\right)^2 + \left(y_3^0\right)^2 \\ {R_1}^2 &= \left(y_1^1\right)^2 + \left(y_2^1\right)^2 + \left(y_3^1\right)^2 \end{aligned}$$

For the points on the curve, R^2 can then be written as

$$R^2 = \left(R_1^2 - R_0^2\right) u + R_0^2$$

Then

$$\phi(u) = \left(R_1^2 - R_0^2\right) u + R_0^2 - \left[\left(y_4^{(1)}\right)^2 - \left(y_4^{(0)}\right)^2\right] u - \left(y_4^{(0)}\right)^2 \quad \text{(see (1))} \tag{4}$$

We have the following conditions:

$$\begin{cases} \phi_0 < 0 \quad \wedge \quad \phi_1 > 0 \\ \\ R_0^2 > (y_4^0)^2 \quad \wedge \quad R_1^2 < (y_4^1)^2 \end{cases}$$

Then,

$$\exists\ u \in [0, 1] : \phi(u) = 0$$

From (4) we have ,

$$\phi(u) = 0 \quad \Rightarrow \qquad = -\frac{R_0^2 - \left(y_4^{(0)}\right)^2}{R_1^2 - R_0^2 - \left(y_4^{(1)}\right)^2 + \left(y_4^{(0)}\right)^2} \tag{5}$$

Let's simplify notationally the last equation. As $\phi_0 < 0 \wedge \phi_1 > 0$, put $R_0^2 - \left(y_4^{(0)}\right)^2 = -\tau$ and $R_1^2 - \left(y_4^{(1)}\right)^2 = \sigma$ with both $\tau, \sigma > 0$. (5) can be written as

$$\begin{aligned} u &= \frac{\tau}{\sigma + \tau} \\ &= \frac{1}{1 + \frac{\sigma}{\tau}} \\ \Rightarrow \qquad & \exists\, u \in [0,1] : \phi(u) = 0 \qquad \text{as } \frac{\sigma}{\tau} > 0 \end{aligned}$$

So, there is a solution $u \in [0,1]$ for which $\phi(u) = 0$ and the curve intersects the null cone.

$\Diamond$

We now investigate the straight segment questions.

Case 1: A and B both lie in the present (region I) or in the past (region II)

Case 2: If A is any point in the present, there exist at least one point B in the present that the straight segment AB cuts the null cone

†

♦

4.13 p133 - Exercise

In a space of two dimensions prove the relation

$$\textbf{4.318.} \qquad \epsilon_{mp}\epsilon_{mq} = \delta_{pq}$$

Suppose $p = q$, then in the summation the term is 0 if $m = p = q$ and the remaining term is 1×1 or -1×-1 giving indeed $\delta_{pq} = 1$.
If $p \neq q$ we get either $m = p$ or $m = q$ in each term of the summation and hence all terms vanish.

♦

4.14 p135 - Clarification

> **4.324.** $P_{mn} = \epsilon_{mnrs} X_r Y_s$
>
> In 4.323 a skew-symmetric tensor is formed.

Let's check whether P_{mn} is indeed a tensor and if it is an oriented one. Be an orthogonal transformation (proper or not)

$$X_r^{'} = A_{rm} X_m + A_r \tag{1}$$

and let's check how the expression $P_{mn}^{'} = P_{rs} \partial_m X_r \partial_n X_s$ behaves.

$$\begin{aligned} P_{mn}^{'} &= P_{rs} \partial_m z_r \partial_n z_s && (2) \\ (\mathbf{4.303}.) \ \Rightarrow \quad &= \epsilon_{rspq} X_p Y_q A_{mr} A_{ns} && (3) \end{aligned}$$

From (4.302.) we have

$$X_p = A_{kp} X_k^{'} - A_{kp} A_k \tag{4}$$

Replacing this in (3)

$$\begin{aligned} P_{mn}^{'} &= \epsilon_{rspq} \left(A_{kp} X_k^{'} - A_{kp} A_k \right) \left(A_{tq} Y_t^{'} - A_{tq} A_t \right) A_{mr} A_{ns} && (5) \\ &= \begin{cases} \epsilon_{rspq} A_{mr} A_{ns} A_{kp} A_{tq} X_k^{'} Y_t^{'} \\ \\ -\epsilon_{rspq} A_{mr} A_{ns} A_{kp} A_{tq} X_k^{'} A_t \\ \\ -\epsilon_{rspq} A_{mr} A_{ns} A_{kp} A_{tq} Y_t^{'} A_k \\ \\ +\epsilon_{rspq} A_{mr} A_{ns} A_{kp} A_{tq} A_k A_t \end{cases} && (6) \\ &= \epsilon_{rspq} A_{mr} A_{ns} A_{kp} A_{tq} \left(X_k^{'} - A_k \right) \left(Y_t^{'} - A_t \right) && (7) \end{aligned}$$

A analogous reasoning as in (4.316.) gives us $\epsilon_{mnkt} \left| A_{mr} \right| = \epsilon_{rspq} A_{mr} A_{ns} A_{kp} A_{tq}$ and so

$$P_{mn}^{'} = \epsilon_{mnkt} \left| A_{mr} \right| \left(X_k^{'} - A_k \right) \left(Y_t^{'} - A_t \right) \tag{8}$$

Apparently, even with $|A_{mr}| = 1$, P_{mn} does not behave like a tensor due to the $\left(X_k^{'} - A_k \right) \left(Y_t^{'} - A_t \right)$ components. But of course, this is consequence of the sloppy use of the transformation equation: equation (1) is the transformation rule for a point in the V_4 space but the object P_{mn} takes two vectors as input. If we consider a vector as an object defined by an ordered pair i.e. $X \equiv \left(z_{(X)}^{(1)}, z_{(X)}^{(0)} \right)$ then P_{mn} should be defined as $P_{mn} = \epsilon_{mnrs} \left(z_{(X)r}^{(1)} - z_{(X)r}^{(0)} \right) \left(z_{(Y)s}^{(1)} - z_{(Y)s}^{(0)} \right)$. This means that when using the transformation rule (1) we will get

$$X^{'} \equiv \left(z^{'(1)}_{(X)}, z^{'(0)}_{(X)}\right)$$

giving as components

$$\begin{aligned} X^{'}_{r} &= z^{'(1)}_{(X)r} - z^{'(0)}_{(X)r} && (9)\\ &= A_{rm} z^{(1)}_{(X)m} + A_r - A_{rm} z^{(0)}_{(X)m} - A_r && (10)\\ &= A_{rm}\left(z^{(1)}_{(X)m} - z^{(0)}_{(X)m}\right) && (11) \end{aligned}$$

Replacing all this we get as a more correct representation of p_{mn} and $P^{'}_{mn}$:

$$\begin{cases} P_{mn} = \epsilon_{mnrs}\left(z^{(1)}_{(X)r} - z^{(0)}_{(X)r}\right)\left(z^{(1)}_{(Y)s} - z^{(0)}_{(Y)s}\right) \\ \\ P^{'}_{mn} = \epsilon_{mnkt}\left|A_{mr}\right|\left(z^{'(1)}_{(X)r} - z^{'(0)}_{(X)r}\right)\left(z^{'(1)}_{(Y)s} - z^{'(0)}_{(Y)s}\right) \\ \end{cases} \qquad (12)$$

We see that indeed P_{mn} is an oriented Cartesian tensor.

♦

4.15 p135 - Exercise

Write out the six independent non-zero components of P_{mn} as given by **4.324**.

We have

$$\textbf{4.324}. \qquad P_{mn} = \in_{mnrs} X^r Y^s \tag{1}$$

$$\text{with} \qquad m = n \quad \Rightarrow \quad P_{mn} = 0 \tag{2}$$

So, the six independent components are in the set $\{mn\} = \{12, 13, 14, 23, 24, 34\}$ as $P_{nm} = -P_{mn}$.

$$\begin{cases} P_{12} = \in_{1234} X^3Y^4 + \in_{1243} X^4Y^3 \\ P_{13} = \in_{1324} X^2Y^4 + \in_{1342} X^4Y^2 \\ P_{14} = \in_{1423} X^2Y^3 + \in_{1432} X^3Y^2 \\ P_{23} = \in_{2314} X^1Y^4 + \in_{2341} X^4Y^1 \\ P_{24} = \in_{2413} X^1Y^3 + \in_{2431} X^3Y^1 \\ P_{34} = \in_{3412} X^1Y^2 + \in_{3421} X^2Y^1 \end{cases} \tag{3}$$

$$\begin{cases} P_{12} = X^3Y^4 - X^4Y^3 \\ P_{13} = -X^2Y^4 + X^4Y^2 \\ P_{14} = X^2Y^3 - X^3Y^2 \\ P_{23} = X^1Y^4 - X^4Y^1 \\ P_{24} = -X^1Y^3 + X^3Y^1 \\ P_{34} = X^1Y^2 - X^2Y^1 \end{cases} \tag{4}$$

♦

4.16 p136 - Exercise

Translate the well-known vector relations

$$A \times (B \times C) = B(A.C) - C(A.B)$$

$$\nabla \times (\nabla \times V) = \nabla(\nabla.V) - \nabla^2 V$$

into Cartesian tesnor form, and prove the by use of 4.329.

We have

$$\textbf{4.329}. \qquad \in_{mrs}\in_{mpq} = \delta_{rp}\delta_{sq} - \delta_{rq}\delta_{sp} \tag{1}$$

The first identity

$$A \times (B \times C) = B(A.C) - C(A.B) \tag{2}$$

$$\Leftrightarrow \qquad \in_{npm}\in_{mrs} A_p B_r C_s = A_p \left(B_n C_p - C_n B_p\right) \tag{3}$$

Indeed,

$$(B \times C)_m = \in_{mrs} B_r C_s \tag{4}$$

$$\Rightarrow \qquad \left(A \times (B \times C)\right)_n = \in_{npm} A_p \in_{mrs} B_r C_s \tag{5}$$

$$= - \in_{mpn}\in_{mrs} A_p \in_{mrs} B_r C_s \tag{6}$$

$$= -\delta_{pr}\delta_{ns} A_p B_r C_s + \delta_{ps}\delta_{nr} A_p B_r C_s \tag{7}$$

$$= A_p B_n C_p - A_p B_p C_n \tag{8}$$

$$\Leftrightarrow \qquad B(A.C) - C(A.B) \tag{9}$$

The second identity

$$\nabla \times (\nabla \times V) = \nabla(\nabla.V) - \nabla^2 V \tag{10}$$

$$\Leftrightarrow \qquad \in_{nrm}\in_{mpq} V_{q,pr} = V_{p,pn} - V_{n,pp} \tag{11}$$

Indeed,

$$(\nabla \times V)_m = \in_{mpq} V_{q,p} \tag{12}$$

$$\Rightarrow \qquad \left(\nabla \times (\nabla \times V)\right)_n = \in_{nrm} \left(\in_{mpq} V_{q,p}\right)_{,r} \tag{13}$$

$$= \in_{nrm}\in_{mpq} V_{q,pr} \tag{14}$$

$$= \delta_{rq}\delta_{np} V_{q,pr} - \delta_{pr}\delta_{nq} V_{q,pr} \tag{15}$$

$$= V_{p,pn} - V_{n,pp} \tag{16}$$

We have also

$$
\begin{aligned}
& & (\nabla V) &= V_{p,p} && (17)\\
&\Rightarrow & (\nabla(\nabla.V))_n &= (V_{p,p})_n && (18)\\
& & &= V_{p,pn} && (19)
\end{aligned}
$$

and

$$
\begin{aligned}
& & \nabla^2 V_n &\equiv V_{n,pp} && (20)\\
&\Rightarrow & (\nabla(\nabla.V))_n - \nabla^2 V_n &= V_{p,pn} - V_{n,pp} && (21)
\end{aligned}
$$

which corresponds to (15). So the tensor expression in Cartesian tensor form can be written as

$$\in_{nrm}\in_{mpq} V_{q,pr} = V_{p,pn} - V_{n,pp}$$

♦

4.17 p139 - Exercise 1.

Show that, in a 3-space of constant curvature $-\frac{1}{R^2}$ and positive definite metric form, the line element in polar coordinate is

$$ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

We have by 4.120c

$$X_r\eta^r = A\sinh\left(s\sqrt{-\epsilon K}\right) \tag{1}$$

We choose N different $X_r^{(k)}$ $(k = 1, 2, \ldots, N)$ which are orthonormal. Applying (1) N times with the different $X_r^{(k)}$ $(k = 1, 2, \ldots, N)$, we get

$$X_r^{(k)}\eta^r = A^{(k)}\sinh\left(s\sqrt{-\epsilon K}\right) \tag{2}$$

But as the $X_r^{(k)}$ are orthonormal and are used as a basis at the considered point of the geodesic we have

$$X_r^{(k)} = \delta_r^k \tag{3}$$

So, (2) becomes

$$\eta^k = A^{(k)}\sinh\left(s\sqrt{-\epsilon K}\right) \tag{4}$$

which are the components of the displacement vector in the orthonormal basis. By **2.301**. :

$$Y^2 = \epsilon a_{mn}Y^mY^n \tag{5}$$

$$\Rightarrow \qquad \eta^2 = \epsilon a_{mn}A^{(m)}A^{(n)}\sinh^2\left(s\sqrt{-\epsilon K}\right) \tag{6}$$

$$\Rightarrow \qquad \eta = C\left|\sinh\left(s\sqrt{-\epsilon K}\right)\right| \tag{7}$$

$$\text{with} \qquad C = \sqrt{\left|\epsilon a_{mn}A^{(m)}A^{(n)}\right|} \tag{8}$$

As $\epsilon = 1$ (positive-definite metric) and $K = -\frac{1}{R^2}$ we have

$$\eta = C\left|\sinh\left(\frac{s}{R}\right)\right| \tag{9}$$

From this and using the very same reasoning from 4.126. to 4.130. (pages 117-119) we get

$$ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

♦

4.18 p139 - Exercise 2.

Show that the volume of an antipodal 3-space of positive-definite metric form and positive constant curvature $\frac{1}{R^2}$ is $2\pi^2 R^3$. (Use the equation 4.130. to find the area of a sphere $r =$ constant in polar coordinates. Multiply by dr and integrate for $0 \leq r \leq \pi R$ to get the volume). What is the volume if the space is polar?

We have 4.130

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

Having a positive-definite metric form, the space can be locally considered as Euclidean and an elementary area of a surface with constant $r (\rightarrow dr = 0)$ can be calculated as $dS = ds_{d\theta=0} ds_{d\phi=0}$ and get by (1)

$$\begin{aligned}
& & dS &= R^2 \sin^2\left(\frac{r}{R}\right)\sin\theta d\phi d\theta & (2)\\
&\Rightarrow & \frac{S}{8} &= R^2 \sin^2\left(\frac{r}{R}\right)\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin\theta d\phi d\theta & (3)\\
& & &= R^2 \sin^2\left(\frac{r}{R}\right)\frac{\pi}{2}\left(-\cos\theta\right)\Big|_0^{\frac{\pi}{2}} & (4)\\
&\Rightarrow & S &= 4\pi R^2 \sin^2\left(\frac{r}{R}\right) & (5)
\end{aligned}$$

We see that the area is a cyclic function of r having zeros' at $r = k\frac{\pi}{R}, k = 1, 2, \ldots$

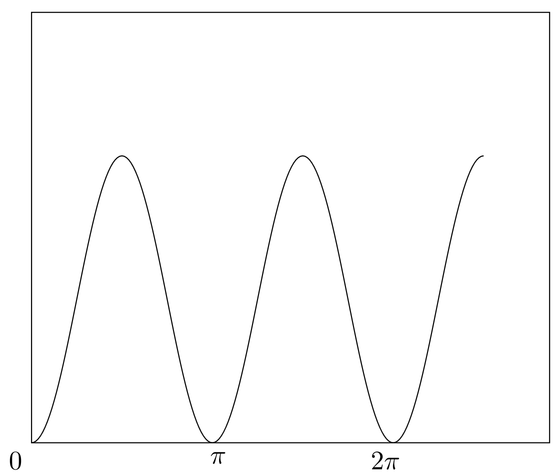


Figure 4.2: Area of an antipodal 3-space of positive-definite metric form

So, there are good reasons to restrict r to $[0, \pi R]$ as otherwise all space of that type would have infinite volume, whatever it's curvature. So, we get as volume

$$\begin{aligned} V &= 4\pi R^2 \int_0^{\pi R} \sin^2\left(\frac{r}{R}\right) dr && (6) \\ &= 4\pi R^3 \int_0^{\pi R} \sin^2\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) && (7) \\ &= 4\pi R^3 \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^{\pi} && (8) \\ &= 2\pi^2 R^3 && (9) \end{aligned}$$

For a polar space, the volume would be half of that of an antipodal one (with same curvature of course) as in (3) we would consider only 4 quadrants instead of 8.

♦

4.19 p139 - Exercise 3.

By direct calculation of the tensor R_{rsmn} verify that **4.130**. is the metric form of a space of constant curvature.

We have 4.130

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

$$\Rightarrow \qquad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2\sin^2\left(\frac{r}{R}\right) & 0 \\ 0 & 0 & R^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \end{pmatrix} \tag{2}$$

Now for R_{rsmn} we refer to exercise 7 page 109 of chapter 3, where the curvature tensor was calculated for a general case of the form

$$ds^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2$$

where h_1, h_2, h_3 are functions of the three coordinates. We have then for our case

$$\begin{cases} h_1 = 1 \\ h_2 = R\sin\left(\frac{r}{R}\right) \\ h_3 = R\sin\left(\frac{r}{R}\right)\sin\theta \end{cases} \tag{3}$$

In the exercise we got for the non-vanishing curvature tensors

$$R_{1212} = -h_2\partial^2_{11}(h_2) - h_1\partial^2_{22}(h_1) + \frac{h_2}{h_1}\partial_1 h_1\partial_1 h_2 + \frac{h_1}{h_2}\partial_2 h_1\partial_2 h_2 - \frac{h_1h_2}{h_3^2}\partial_3 h_1\partial_3 h_2 \tag{4}$$

$$R_{2323} = -h_3\partial^2_{22}(h_3) - h_2\partial^2_{33}(h_2) + \frac{h_3}{h_2}\partial_2 h_2\partial_2 h_3 + \frac{h_2}{h_3}\partial_3 h_2\partial_3 h_3 - \frac{h_2h_3}{h_1^2}\partial_1 h_2\partial_1 h_3 \tag{5}$$

$$R_{1313} = -h_3\partial^2_{11}(h_3) - h_1\partial^2_{33}(h_1) + \frac{h_3}{h_1}\partial_1 h_1\partial_1 h_3 + \frac{h_1}{h_3}\partial_3 h_1\partial_3 h_3 - \frac{h_1h_3}{h_2^2}\partial_2 h_1\partial_2 h_3 \tag{6}$$

$$R_{1213} = -h_1\partial^2_{32}(h_1) + \frac{h_1}{h_3}\partial_2 h_3\partial_3 h_1 + \frac{h_1}{h_2}\partial_2 h_1\partial_3 h_2 \tag{7}$$

$$R_{1223} = h_2\partial^2_{31}(h_2) - \frac{h_2}{h_1}\partial_1 h_2\partial_3 h_1 - \frac{h_2}{h_3}\partial_3 h_2\partial_1 h_3 \tag{8}$$

$$R_{1323} = -h_3\partial^2_{21}(h_3) + \frac{h_3}{h_1}\partial_1 h_3\partial_3 h_1 + \frac{h_3}{h_2}\partial_2 h_3\partial_1 h_2 \tag{9}$$

Clearly $\partial_k^2(h_1) = 0$ and $\partial_{mn}^2(h_1) = 0$ and considering $h_2 = h_2(r),\ h_3 = h_3(r,\theta)$ we can simplify

$$R_{1212} = -h_2\partial_{11}^2(h_2) \tag{10}$$
$$R_{2323} = -h_3\partial_{22}^2(h_3) - \frac{h_2h_3}{h_1^2}\partial_1 h_2\partial_1 h_3 \tag{11}$$
$$R_{1313} = -h_3\partial_{11}^2(h_3) \tag{12}$$

$$R_{1213} = 0 \tag{13}$$
$$R_{1223} = 0 \tag{14}$$
$$R_{1323} = -h_3\partial_{21}^2(h_3) + \frac{h_3}{h_2}\partial_2 h_3\partial_1 h_2 \tag{15}$$

with

$$\begin{cases} \partial_1 h_2 = \cos\left(\frac{r}{R}\right) \\ \partial_1 h_3 = \cos\left(\frac{r}{R}\right)\sin\theta \\ \partial_2 h_3 = R\sin\left(\frac{r}{R}\right)\cos\theta \\ \partial_{11}^2 h_2 = -\frac{1}{R}\sin\left(\frac{r}{R}\right) \\ \partial_{11}^2 h_3 = -\frac{1}{R}\sin\left(\frac{r}{R}\right)\sin\theta \\ \partial_{21}^2 h_3 = \cos\left(\frac{r}{R}\right)\cos\theta \\ \partial_{22}^2 h_3 = -R\sin\left(\frac{r}{R}\right)\sin\theta \end{cases} \tag{16}$$

giving

$$R_{1212} = \sin^2\left(\frac{r}{R}\right) \tag{17}$$
$$R_{2323} = R^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta - R^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta\cos^2\left(\frac{r}{R}\right) \tag{18}$$
$$= R^2\sin^4\left(\frac{r}{R}\right)\sin^2\theta \tag{19}$$
$$R_{1313} = \sin^2\left(\frac{r}{R}\right)\sin^2\theta \tag{20}$$
$$R_{1213} = 0 \tag{21}$$
$$R_{1223} = 0 \tag{22}$$
$$R_{1323} = -R\sin\left(\frac{r}{R}\right)\sin\theta\cos\left(\frac{r}{R}\right)\cos\theta + \sin\theta R\sin\left(\frac{r}{R}\right)\cos\theta\cos\left(\frac{r}{R}\right) \tag{23}$$
$$= 0 \tag{24}$$

and considering the symmetries

$$R_{1212} = -R_{1221} = -R_{2112} = \sin^2\left(\frac{r}{R}\right) \tag{25}$$

$$R_{2323} = -R_{2332} = -R_{3223} = R^2\sin^4\left(\frac{r}{R}\right)\sin^2\theta \tag{26}$$

$$R_{1313} = -R_{1331} = -R_{3113} = \sin^2\left(\frac{r}{R}\right)\sin^2\theta \tag{27}$$

$$\tag{28}$$

By **4.114**.

$$R_{rsmn} = K\left(a_{rm}a_{sn} - a_{rn}a_{sm}\right) \tag{29}$$

$$\Rightarrow \quad \begin{cases} R_{1212} = K\left(a_{11}a_{22} - a_{12}a_{21}\right) \\ R_{2323} = K\left(a_{22}a_{33} - a_{23}a_{32}\right) \\ R_{1313} = K\left(a_{11}a_{33} - a_{13}a_{31}\right) \end{cases} \tag{30}$$

$$\Rightarrow \quad \begin{cases} R_{1212} = KR^2\sin^2\left(\frac{r}{R}\right) \\ R_{2323} = KR^2\sin^2\left(\frac{r}{R}\right)R^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \\ R_{1313} = KR^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \end{cases} \tag{31}$$

$$\Rightarrow \quad \begin{cases} R_{1212} = KR^2\sin^2\left(\frac{r}{R}\right) \\ R_{2323} = KR^4\sin^4\left(\frac{r}{R}\right)\sin^2\theta \\ R_{1313} = KR^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \end{cases} \tag{32}$$

replacing (25), (26) and (27) in (32) we get

$$\begin{cases} \sin^2\left(\frac{r}{R}\right) = KR^2\sin^2\left(\frac{r}{R}\right) \\ R^2\sin^4\left(\frac{r}{R}\right)\sin^2\theta = KR^4\sin^4\left(\frac{r}{R}\right)\sin^2\theta \\ \sin^2\left(\frac{r}{R}\right)\sin^2\theta = KR^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \end{cases} \tag{33}$$

giving indeed for the three curvature tensors $K = \frac{1}{R^2}$
With this, the question of the exercise is answered but we go a little bit further and investigate for this practical case the equations of **4.115**. and calculate $R = a^{mn}R_{mn}$. With R, the curvature invariant. To avoid confusion with the curvature R itself we use $\mathfrak{R}$ for the curvature invariant.
As the metric tensor is diagonal:

$$\mathfrak{R} = a^{11}R_{11} + a^{22}R_{22} + a^{33}R_{33} \tag{34}$$

$$\tag{35}$$

and

$$R_{mn} = a^{sn} R_{srmn} \tag{36}$$

$$\Rightarrow \begin{cases} R_{11} = a^{11} R_{1111} + a^{22} R_{2112} + a^{33} R_{3113} \\ R_{22} = a^{11} R_{1221} + a^{22} R_{2222} + a^{33} R_{3223} \\ R_{33} = a^{11} R_{1331} + a^{22} R_{2332} + a^{33} R_{3333} \end{cases} \tag{37}$$

$$\Rightarrow \begin{cases} R_{11} = -a^{22} \sin^2\left(\frac{r}{R}\right) - a^{33} \sin^2\left(\frac{r}{R}\right) \sin^2\theta \\ R_{22} = -a^{11} \sin^2\left(\frac{r}{R}\right) - a^{33} R^2 \sin^4\left(\frac{r}{R}\right) \sin^2\theta \\ R_{33} = -a^{11} \sin^2\left(\frac{r}{R}\right) \sin^2\theta - a^{22} R^2 \sin^4\left(\frac{r}{R}\right) \sin^2\theta \end{cases} \tag{38}$$

$$\Rightarrow \begin{cases} R_{11} = -\frac{\sin^2\left(\frac{r}{R}\right)}{R^2 \sin^2\left(\frac{r}{R}\right)} - \frac{\sin^2\left(\frac{r}{R}\right) \sin^2\theta}{R^2 \sin^2\left(\frac{r}{R}\right) \sin^2\theta} \\ R_{22} = -\sin^2\left(\frac{r}{R}\right) - \frac{R^2 \sin^4\left(\frac{r}{R}\right) \sin^2\theta}{R^2 \sin^2\left(\frac{r}{R}\right) \sin^2\theta} \\ R_{33} = -\sin^2\left(\frac{r}{R}\right) \sin^2\theta - \frac{R^2 \sin^4\left(\frac{r}{R}\right) \sin^2\theta}{R^2 \sin^2\left(\frac{r}{R}\right)} \end{cases} \tag{39}$$

giving

$$\begin{cases} R_{11} = -\frac{2}{R^2} \\ R_{22} = -2 \sin^2\left(\frac{r}{R}\right) \\ R_{33} = -2 \sin^2\left(\frac{r}{R}\right) \sin^2\theta \end{cases} \tag{40}$$

hence

$$\mathfrak{R} = a^{11} R_{11} + a^{22} R_{22} + a^{33} R_{33} \tag{41}$$

$$\Rightarrow \quad \mathfrak{R} = -\frac{2}{R^2} - 2 \frac{\sin^2\left(\frac{r}{R}\right)}{R^2 \sin^2\left(\frac{r}{R}\right)} - 2 \frac{\sin^2\left(\frac{r}{R}\right) \sin^2\theta}{R^2 \sin^2\left(\frac{r}{R}\right) \sin^2\theta} \tag{42}$$

$$= -\frac{6}{R^2} \tag{43}$$

The equations in (40) and (43) are indeed in accordance with **4.115** .

♦

4.20 p139 - Exercise 4

Show that if V_N has positive-definite metric form and constant positive curvature K, then coordinates y^r exist so that

$$ds^2 = \frac{dy^m dy^m}{\left(1 + \frac{1}{4} y^n y^n\right)^2}$$

(Starting with a coordinate system x^r which is locally Cartesian at O, take at any point P the coordinates

$$y^r = p^r \frac{2}{\sqrt{K}} \tan\left(\frac{1}{2} r \sqrt{K}\right)$$

where p^r ar the the components of the unit tangent vector $\left(\frac{dx^r}{ds}\right)$ at O to the geodesic OP and r is the geodesic distance OP.)

Let us first understand what happens. Fig.1.5 for a V_3 will help us understand. Let P and $P + dP$ be two points separated by an infinitesimal distance. Consider the two geodesics initiated from the origin and joining these two points. Be X and $X^{'}$ the two tangents unit vectors to these geodesics. Those vectors have components $p^r = \frac{dx^r}{ds}$ taken along their respective geodesics. By the considered mapping, $y^r = p^r \frac{2}{\sqrt{K}} \tan\left(\frac{1}{2} r \sqrt{K}\right)$, the points P and $P + dP$ are mapped on the points $\tau(P)$ and $\tau(P + dP)$ with $|O\tau(P)|$ and $|O\tau(P + dP)|$ collinear with the two tangents unit vectors X and $X^{'}$.

Observe also the segment PN which corresponds to the geodesic displacement η for the geodesic distance $r = OP$. As the metric form is positive-definite, we can consider that the infinitesimal triangle $\left|P\widehat{NP + d}P\right|$ lies in an infinitesimally Euclidean space and we can express $ds^2 = \eta^2 + dr^2$ as $|NP + dP| = dr$. Observe now, the triangle $\left|\tau(P)\tau(\widehat{N)\tau(}P + dP)\right|$. There also we have $|\tau(P + dP)\tau(P)|^2 = |\tau(P)\tau(N)|^2 + |\tau(P + dP)\tau(N)|^2$. Can we find a relationship between these two triangle?

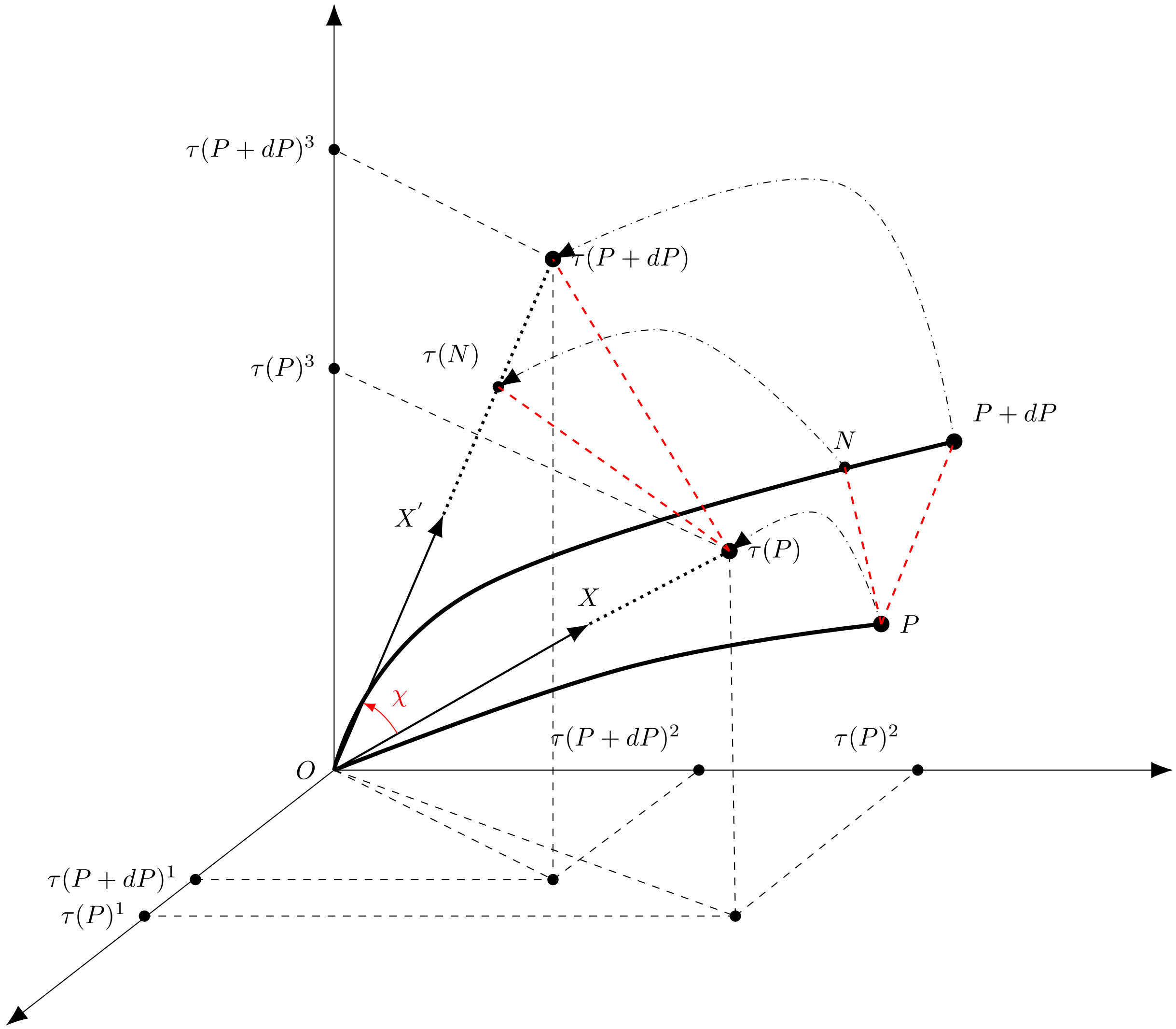


Figure 4.3: Coordinate system in constant curvature space

Let's define $\alpha(r) = \frac{2}{\sqrt{K}} \tan\left(\frac{1}{2} r\sqrt{K}\right)$ so that $y^k = \alpha(r)p^k$ We have

$$\begin{cases} \left|OX'\right| = |OX| = 1 \\ \\ |O\tau(N)| = |O\tau(P)| = \alpha(r) \\ \\ |O\tau(P+dP)| = \alpha(r+dr) \\ \end{cases} \tag{1}$$

Expanding the last equation in a Taylor series we get as first order term

$$|\tau(N)\tau(P+dP)| = \frac{1}{\cos^2(\frac{1}{2}r\sqrt{K})}dr \tag{2}$$

Also,

$$|\tau(N)\tau(P)| = 2\alpha(r)\sin\frac{\chi}{2} \approx \alpha(r)\chi \tag{3}$$

From **4.124** we have $\chi = \left(\frac{d\eta}{dr}\right)_{r=0} = C\sqrt{K}$. But note also that from **4.122** we have for the geodesic displacement $\eta = C\left|\sin r\sqrt{K}\right|$.

$$C = \frac{\eta}{\left|\sin r\sqrt{K}\right|} \tag{4}$$

$$\Rightarrow \qquad \chi = \frac{\eta}{\left|\sin r\sqrt{K}\right|}\sqrt{K} \tag{5}$$

$$\Rightarrow \qquad |\tau(N)\tau(P)| = \eta\frac{\sqrt{K}}{\left|\sin r\sqrt{K}\right|}\alpha(r) \tag{6}$$

Let's put $|\tau(N)\tau(P)| = \hat{\eta}$.

$$\hat{\eta} = \eta\frac{\sqrt{K}}{\left|\sin r\sqrt{K}\right|}\alpha(r) \tag{7}$$

At the point P we have $|PN| = \eta$ and so

$$ds^2 = \eta^2 + dr^2 \tag{8}$$

Let's put $|\tau(P+dP)\tau(P)|^2 = d\hat{s}^2$.

$$d\hat{s}^2 = \hat{\eta}^2 + |\tau(P+dP)\tau(N)|^2 \tag{9}$$

$$(2) \text{ and } (7) \quad \Rightarrow \qquad = \eta^2\frac{K}{\sin^2(r\sqrt{K})}\frac{4}{K}\frac{\sin^2(\frac{1}{2}r\sqrt{K})}{\cos^2(\frac{1}{2}r\sqrt{K})} + \frac{1}{\cos^4(\frac{1}{2}r\sqrt{K})}dr^2 \tag{10}$$

$$\sin r\sqrt{K} = 2\sin\left(\frac{1}{2}r\sqrt{K}\right)\cos\left(\frac{1}{2}r\sqrt{K}\right) \tag{11}$$

$$\Rightarrow \qquad d\hat{s}^2 = \eta^2\frac{1}{\cos^4(\frac{1}{2}r\sqrt{K})} + \frac{1}{\cos^4(\frac{1}{2}r\sqrt{K})}dr^2 \tag{12}$$

$$\Rightarrow \qquad \eta^2 + dr^2 = d\hat{s}^2\cos^4(\frac{1}{2}r\sqrt{K}) \tag{13}$$

$$\Rightarrow \qquad ds^2 = d\hat{s}^2\cos^4(\frac{1}{2}r\sqrt{K}) \tag{14}$$

It is easy to see that $d\hat{s}^2 = dy^k dy^k$ and also

$$\cos^4(\frac{1}{2}r\sqrt{K}) = \left(\cos^2(\frac{1}{2}r\sqrt{K})\right)^2 \tag{15}$$

$$= \left(\frac{\cos^2(\frac{1}{2}r\sqrt{K})}{\cos^2(\frac{1}{2}r\sqrt{K}) + \sin^2(\frac{1}{2}r\sqrt{K})}\right)^2 \tag{16}$$

$$= \left(\frac{1}{1 + \tan^2(\frac{1}{2}r\sqrt{K})}\right)^2 \tag{17}$$

We note that $\frac{2}{\sqrt{K}}\tan\left(\frac{1}{2}r\sqrt{K}\right)$ is the size of the vector $|O\tau(P)|$ and can express this as (as we use local Cartesian coordinates at the origin) $|O\tau(P)|^2 = y^k y^k$ and thus $\tan^2(\frac{1}{2}r\sqrt{K}) = \frac{K}{4}y^k y^k$. Combining this with (14) and (17) gives:

$$ds^2 = d\hat{s}^2 \left(\frac{1}{1 + \frac{K}{4}y^k y^k}\right)^2 \tag{18}$$

which gives as final expression

$$ds^2 = \frac{dy^k dy^k}{\left(1 + \frac{K}{4}y^k y^k\right)^2} \tag{19}$$

♦

4.21 p140 - Exercise 5

Show that in a flat V_n the straight line joining any two points of a P-flat ($P > N$) lies entirely in the P-flat.

For a P-flat we have by an appropriate re indexing of the variables z_k

$$A_{mp} z_p + B_p = 0 \qquad m = 1, \ldots, P \tag{1}$$

A straight line has as equation $z_p = C_p u + D_p$. As we have two points in the P-flat we have two u_1, u_2 for which yields

$$\begin{cases} A_{mp}\left(C_p u_1 + D_p\right) + B_p = 0 \\ \\ A_{mp}\left(C_p u_2 + D_p\right) + B_p = 0 \end{cases} \qquad m = 1, \ldots, P \tag{2}$$

Subtracting the corresponding equations in m for the two sets u_1, u_2 gives

$$A_{mp} C_p \left(u_1 - u_0\right) = 0 \tag{3}$$

$$\Rightarrow \qquad A_{mp} C_p = 0 \qquad \text{for } m = 1, \ldots, P \tag{4}$$

$$(4) \text{ in } (2) \ \Rightarrow \qquad A_{mp} D_p + B_p = 0 \qquad \text{for } m = 1, \ldots, P \tag{5}$$

So for an arbitrary u we get from (1),(4) and (5)

$$A_{mp}\left(C_p u + D_p\right) + B_p = \underbrace{A_{mp} C_p}_{=0} u + \underbrace{A_{mp} D_p + B_p}_{=0} \qquad \text{for } m = 1, \ldots, P \tag{6}$$

$$\Rightarrow \qquad A_{mp}\left(\underbrace{C_p u + D_p}_{z_p}\right) + B_p = 0 \qquad \text{for } m = 1, \ldots, P \tag{7}$$

So the points z_p lying on the line satisfy the conditions for the P-flat and lie therefore in the P-flat.

♦

4.22 p140 - Exercise 6

Show that in four dimensions the transformation

$$\begin{aligned}
z_1^{'} &= z_1 \cosh\phi + iz_4 \sinh\phi \\
z_2^{'} &= z_2 \\
z_3^{'} &= z_3 \\
z_4^{'} &= -iz_1 \sinh\phi + z_4 \cosh\phi
\end{aligned}$$

is orthogonal, ϕ being any constant. Putting $z_1 = x$, $z_2 = y$, $z_3 = z$, $z_4 = ict$, $\phi = \frac{v}{c}$, obtain the transformation connecting $(x^{'}, y^{'}, z^{'}, t^{'})$ and (x, y, z, t). This is the *Lorentz transformation* of the special theory of relativity.

We can represent the transformation with the matrix

$$(A_{mn}) = \begin{pmatrix} \cosh\phi & 0 & 0 & i\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \tag{1}$$

We use **4.210**. i.e. $A_{pm}A_{qm} = \delta_{pq}$ as a condition for the orthogonality of a transformation. This can be written in matrix form

$$(A_{mn})\,(A_{mn})^T = \mathbf{I} \tag{2}$$

and get

$$\begin{aligned}
(A_{mn})\,(A_{mn})^T &= \begin{pmatrix} \cosh\phi & 0 & 0 & i\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} \cosh\phi & 0 & 0 & -i\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} && (3) \\
&= \begin{pmatrix} \cosh^2\phi - \sinh^2\phi & 0 & 0 & -i\cosh\phi\sinh\phi + i\cosh\phi\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\cosh\phi\sinh\phi - i\cosh\phi\sinh\phi & 0 & 0 & -\sinh^2\phi + \cosh^2\phi \end{pmatrix} && (4) \\
&= \mathbf{I} && (5)
\end{aligned}$$

We now calculate the Lorentz transformation.

First note that

$$\begin{cases} \cosh y = \frac{e^y+e^{-y}}{2} \\ \sinh y = \frac{e^y-e^{-y}}{2} \\ \tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \end{cases} \tag{6}$$

$$\Rightarrow \qquad \begin{cases} \cosh\left(\tanh^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \\ \sinh\left(\tanh^{-1} x\right) = \frac{x}{\sqrt{1-x^2}} \end{cases} \tag{7}$$

Replacing x with $\tanh\phi = \frac{v}{c}$ gives

$$\begin{cases} \cosh\phi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\ \sinh\phi = \frac{v}{c\sqrt{1-\frac{v^2}{c^2}}} \end{cases} \tag{8}$$

and the transformation becomes

$$\begin{aligned} x' &= \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}$$

♦

4.23 p140 - Exercise 7

> Prove that in a flat space a plane, defined by **4.22**, is itself a flat space f $N-1$ dimensions.

For a $N-1$-flat we have **4.22**

$$A_r^{'} z_r + B^{'} = 0 \tag{1}$$

Suppose that $A_N \neq 0$ then we can express Z_N, by dividing the equation (1) by A_N as

$$z_N = A_\gamma z_\gamma + B \qquad \text{with } \gamma = 1, 2, \ldots, N-1 \tag{2}$$

Then $\phi = z_n z_n$ becomes

$$\phi = dz_\gamma dz_\gamma + dz_N dz_N \tag{3}$$
$$= dz_\gamma dz_\gamma + (A_\gamma dz_\gamma)(A_\tau dz_\tau) \tag{4}$$
$$= dz_\gamma dz_\gamma + A_{\gamma\tau} dz_\gamma dz_\tau \tag{5}$$

So ϕ can be expressed as $\phi = a_{\gamma\tau} dz_\gamma dz_\tau$.
But as $a_{\gamma\tau}$ are constants, the Christoffel symbols vanish and so does the curvature tensor $R^\alpha_{\beta\gamma\delta}$ in the $N-1$ space. Hence, the V_{N-1} space delimited by equation (1) is flat.
Question: can we find the right orthogonal transformation so that the metric form in (5) can be made homogeneous?
The metric form in (5) can be represented as

$$(a_{mn}) = \begin{pmatrix} 1 - A_1^2 & \frac{1}{2} A_1 A_2 & \ldots & \frac{1}{2} A_1 A_{N-1} \\ \frac{1}{2} A_1 A_2 & 1 - A_2^2 & \ldots & \frac{1}{2} A_2 A_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} A_1 A_{N-1} & \frac{1}{2} A_2 A_{N-1} & \ldots & 1 - A_{N-1}^2 \end{pmatrix} \tag{6}$$

♦

4.24 p140 - Exercise 8

Show that in a flat space of positive-definite metric form, a sphere of zero radius consists of a single point, but that if the metric form is indefinite, a sphere of zero radius extends to infinity.

A sphere is determined by $z_k z_k = \pm R^2$ with $+$ for a positive definite metric and $\pm$ if the metric is indefinite.
For a positive definite metric a zero radius sphere has the equation $z_n z_n = 0$. It is obvious that as $z_n = \sqrt{\epsilon} y_k = y_k$, each term in the summation is non-negative , and so only $z_k = 0\ \forall n$ holds this equation.
In an indefinite metric form space, at least two ϵ_n differ so the zero sphere can be written as $y_p y_p = y_n y_n$, the indices p, n regrouped in a way the left side has positive ϵ and the right side negative ϵ. So the y_k can span the whole real line.
Note that the case where all ϵ are negative means that the metric form is positive definite. Indeed, from the definition **2.105**., page 29 we have $ds^2 = \epsilon\phi = \epsilon a_{mn} dx^m dx^s, ds > 0$. So the epsilons are in fact an artefact to get ds^2 positive in any case and if all ϵ_n are -1 we can multiply them straight away with the ϵ of **2.105**. ensuring that $ds^2 > 0$.

♦

4.25 p140 - Exercise 9

Prove that in two dimensions

$$\epsilon_{mn}\epsilon_{pq} = \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}$$

Suppose first that $m = n$ or $p = q$: the left side will vanish but also the right side as we will have an expression $\delta_{Mp}\delta_{Mq} - \delta_{Mq}\delta_{Mp} = 0$ (we use capital indices to emphasise that no summation occurs with repeated indices).
Suppose now that $m \neq n$ and $p \neq q$.
If mn and pq are no permutation, the left side will be 1 but in the right side the negative term will vanish as $m = p$ and $n = q$ so $m \neq q$ while the left term will be 1. The same yields with mn and pq are both permutations as the same reasoning is valid for the right side and the left side is equal to $(-1)(-1) = 1$.
If only one of mn or pq is a permutation e.g. $m \neq p$ then the positive term in the right side will vanish while the negative will be -1 and the left term will be $(-1)(1) = -1$.

♦

4.26 p140 - Exercise 10

If, in a space of four dimensions, F_{mn} is a skew-symmetric Cartesian tensor, and

$$\hat{F}_{mn} = \frac{1}{2}\epsilon_{rsmn}F_{rs}$$

prove that the differential equations

$$F_{mn,r} + F_{nr,m} + F_{rm,n} = 0$$

may be written

$$\hat{F}_{mn,n} = 0$$

Let's us express F_{mn} as the result of expression (**4.324**) i.e

$$F_{mn} = \epsilon_{mnks}X_kY_s$$

or simplified

$$F_{mn} = \epsilon_{mnks}Z_{ks}$$

This expression gives indeed skew-symmetric tensors.

*NOTE: at first glance this way of representation is a restriction as the skew-symmetric Cartesian tensor F_{mn} should moreover be an oriented Cartesian tensor. This can be circumvent by the result of clarification (**4.14**) where we found that the tensor character of the quantities F_{mn} was only influenced by the determinant of an orthogonal transformation. So if in the case we are dealing with non-proper orthogonal transformation, we replace the given identity by $|A_{mn}|\,F_{mn,r} + |A_{mn}|\,F_{nr,m} + |A_{mn}|\,F_{rm,n} = 0$ and define $\hat{F}_{mn} = \frac{1}{2}\epsilon_{rsmn}\,|A_{mn}|\,F_{rs}$ and the following reasoning will still be valid.*

We have:

$$\begin{cases} F_{mn} = \epsilon_{mnks}Z_{ks} \\ \\ F_{nr} = \epsilon_{nrks}Z_{ks} \\ \\ F_{rm} = \epsilon_{rmks}Z_{ks} \\ \end{cases} \tag{1}$$

And so,

$$F_{mn,r} + F_{nr,m} + F_{rm,n} = \begin{cases} \epsilon_{mnks} Z_{ks,r} \\ +\epsilon_{nrks} Z_{ks,m} \\ +\epsilon_{rmks} Z_{ks,n} \end{cases} \tag{2}$$

Multiplying (2) with ϵ_{mnrt}

$$(F_{mn,r} + F_{nr,m} + F_{rm,n})\,\epsilon_{mnrt} = \begin{cases} \epsilon_{mnks}\epsilon_{mnrt} Z_{ks,r} \\ +\epsilon_{nrks}\epsilon_{mnrt} Z_{ks,m} \\ +\epsilon_{rmks}\epsilon_{mnrt} Z_{ks,n} \end{cases} \tag{3}$$

$$= 3\epsilon_{rmks}\epsilon_{rmnt} Z_{ks,n} \tag{4}$$

$$= 3\epsilon_{rmnt} \left(\underbrace{\epsilon_{rmks} Z_{ks}}_{=F_{rm}} \right)_{,n} \tag{5}$$

$$= 3 \left(\underbrace{\epsilon_{rmnt} F_{rm}}_{=2\hat{F}_{nt}} \right)_{,n} \tag{6}$$

$$= -6\hat{F}_{tn,n} \tag{7}$$

As $(F_{mn,r} + F_{nr,m} + F_{rm,n})\,\epsilon_{mnrt} = 0$ we have indeed

$$\hat{F}_{tn,n} = 0$$

♦

4.27 p140 - Exercise 11

Write out explicitly and simplify the expressions

$$F_{mn}F_{mn},\ \epsilon_{mnrs}F_{mn}F_{rs}$$

where F_{mn} is a skew-symmetric oriented Cartesian tensor.
What is the tensor character of these expressions?

REMARK: although not explicitly stated we assume that we are in a V_4-space.
Let's us express F_{mn} as the result of expression **4.324**. i.e

$$F_{mn} = \epsilon_{mnks}X_kY_s$$

or simplified

$$F_{mn} = \epsilon_{mnks}Z_{ks}$$

First we note that by a same reasoning for **4.329**. we have

$$\epsilon_{mnrs}\epsilon_{mnpq} = 2\left(\delta_{rp}\delta_{sq} - \delta_{rq}\delta_{sp}\right)$$

The factor 2 arising from the fact that we are dealing in V_4 with a sum over the ordered pair (mn).
We have for the first expression $F_{mn}F_{mn}$:

$$\begin{aligned}
\frac{1}{2}F_{mn}F_{mn} &= \frac{1}{2}\epsilon_{mnrs}\epsilon_{mnpq}Z_{rs}Z_{pq}\\
&= \delta_{rp}\delta_{sq}Z_{rs}Z_{pq} - \delta_{rq}\delta_{sp}Z_{rs}Z_{pq}\\
&= \delta_{sq}Z_{rs}Z_{rq} - \delta_{sp}Z_{rs}Z_{pr}\\
&= Z_{rs}Z_{rs} - Z_{rp}Z_{pr}\\
\Rightarrow \qquad F_{mn}F_{mn} &= 2\left(X_rX_rY_sY_s - \left(X_rY_r\right)^2\right)
\end{aligned}$$

$F_{mn}F_{mn}$ **is an oriented Cartesian invariant.**

We have for the second expression $\epsilon_{mnrs}F_{mn}F_{rs}$:

$$\begin{aligned}
\epsilon_{mnrs}F_{mn}F_{rs} &= \underbrace{\epsilon_{mnrs}\epsilon_{mnpq}}_{2(\delta_{rp}\delta_{sq}-\delta_{rq}\delta_{sp})}\ \epsilon_{rsuv}Z_{pq}Z_{uv}\\
&= 2\left(\delta_{rp}\delta_{sq}\epsilon_{rsuv} - \delta_{rq}\delta_{sp}\epsilon_{rsuv}\right)Z_{pq}Z_{uv}\\
&= 2\left(\epsilon_{pquv} - \epsilon_{qpuv}\right)Z_{pq}Z_{uv}\\
&= 4\epsilon_{pquv}Z_{pq}Z_{uv}\\
\Rightarrow \qquad \epsilon_{mnrs}F_{mn}F_{rs} &= 4\epsilon_{pquv}X_pY_qX_uY_v
\end{aligned}$$

$\epsilon_{mnrs}F_{mn}F_{rs}$ **is an oriented Cartesian invariant.**

♦

4.28 p141 - Exercise 12

Show that in a flat space with positive-definite metric form all spheres have positive constant curvature. Show that if the metric is indefinite then some spheres have positive constant curvature and some have negative constant curvature. Discuss the Riemannian curvature of the null-cone.

A sphere is determined by $z_k z_k = C$ (see **4.224**.).
For a **positive definite** metric it is obvious that as $z_k = \sqrt{\epsilon_k} y_k = y_k$, each term in the summation is non-negative , and so only $C > 0$ holds for this equation. From chapter **4.4** is follows that a sphere has constant curvature $\frac{1}{C} > 0$
In an **indefinite metric** form space, at least two ϵ_k differ so the zero sphere can be written as $y_p y_p = C + y_n y_n$, the indices p, n regrouped in a way the left side has positive $\epsilon' s$ and the right side negative $\epsilon' s$. So C can be either positive or negative while still representing a sphere in V_n.
For the **null-cone**, we have $C = 0$, so the Riemannian curvature becomes infinite as

$$K = \lim_{C \to 0} \frac{1}{C} = \infty$$

.

♦

4.29 p141 - Exercise 13

Show that in any space of three dimensions the permutation symbols transform according to

$$\epsilon'_{mnr} = \epsilon_{stu} J' \partial_m x^s \partial_n x^t \partial_r x^u, \qquad J' = \left| \frac{\partial x'^p}{\partial x^q} \right|$$

or

$$\epsilon'_{mnr} = \epsilon_{stu} J \partial_s x'^m \partial'^n_t \partial_u x'^r, \qquad J = \left| \frac{\partial x^p}{\partial x'^q} \right|$$

Using the result of Exercises II, 12, deduce that in a Riemannian 3-space the quantities η_{mnr} and η^{mnr} defined by

$$\eta_{mnr} = \epsilon_{mnr}\sqrt{a}, \quad \eta^{mnr} = \frac{\epsilon_{mnr}}{\sqrt{a}}, \quad a = |a_{pq}|$$

are components of covariant and contravariant oriented tensors .

First remember that $J' = \frac{1}{J}$.
The reasoning is completely analogous as to the reasoning from **4.312** till **4.317** except that the $\frac{\partial z^m}{\partial z'^s}$ are held and not replaced by the A_{mn}.
4.316 becomes

$$\epsilon'_{mnr} J = \epsilon_{stu} \partial_m x^s \partial_n x^t \partial_r x^u, \qquad J = \left| \frac{\partial x^p}{\partial x'^q} \right|$$

$$J = \frac{1}{J'} \quad \Rightarrow \quad \epsilon'_{mnr} = \epsilon_{stu} J' \partial_m x^s \partial_n x^t \partial_r x^u, \qquad J' = \left| \frac{\partial x'^p}{\partial x^q} \right|$$

Following Exercises II, 12 we have $a' = aJ^2$. So,

$$\begin{aligned}
\eta'_{mnr} &= \eta_{uvw} \partial_m x^u \partial_n x^v \partial_r x^w \\
&= \sqrt{a}\ \underbrace{\epsilon_{uvw} \partial_m x^u \partial_n x^v \partial_r x^w}_{=\frac{1}{J'}\epsilon'_{mnr}} \\
&= \sqrt{a}\ J \epsilon'_{mnr} \\
\sqrt{a'} = \sqrt{aJ^2} \quad \Rightarrow \quad &= \sqrt{a'}\ \epsilon'_{mnr}
\end{aligned}$$

The same reasoning applies to the contravariant counterpart.

♦

4.30 p141 - Exercise 14

Translate into Cartesian tensor form and thus verify the following well known vector relations.

$$\nabla.\left(\phi V\right)=\phi\nabla.V+V.\nabla\phi$$

$$\vdots$$

$$\nabla.\left(\phi\mathbf{V}\right)=\phi\nabla.\mathbf{V}+\mathbf{V}.\nabla\phi$$

$$\begin{aligned}\nabla.\left(\phi V\right)&\equiv\partial_k\phi V_k\\&=\phi\partial_k V_k+V_k\partial_k\phi\\&\equiv\phi\nabla.V+V.\nabla\phi\end{aligned}$$

$$\lozenge$$

$$\nabla\times\left(\phi\mathbf{V}\right)=\phi\nabla\times\mathbf{V}+\mathbf{V}\times\nabla\phi$$

$$\begin{aligned}\left(\nabla\times\left(\phi V\right)\right)_m&\equiv\epsilon_{mnr}\partial_n\phi V_r\\&=\phi\epsilon_{mnr}\partial_n V_r+\epsilon_{mnr}V_r\partial_n\phi\\&\equiv\phi\nabla\times V+\nabla\phi\times V\\&=\phi\nabla\times V-V\times\nabla\phi\end{aligned}$$

$$\lozenge$$

$$\nabla.\left(\mathbf{U}\times\mathbf{V}\right)=\mathbf{V}.\left(\nabla\times\mathbf{U}\right)-\mathbf{U}.\left(\nabla\times\mathbf{V}\right)$$

$$\begin{aligned}\nabla.\left(U\times V\right)&\equiv\partial_m\left(\epsilon_{mnr}U_nV_r\right)\\&=U_n\epsilon_{mnr}\partial_m V_r+V_r\epsilon_{mnr}\partial_m U_n\\&=-U_n\underbrace{\epsilon_{nmr}\partial_m V_r}_{\equiv\left(\nabla\times V\right)_n}+V_r\underbrace{\epsilon_{rmn}\partial_m U_n}_{\equiv\left(\nabla\times U\right)_r}\\&\equiv V.\left(\nabla\times U\right)-U.\left(\nabla\times V\right)\end{aligned}$$

$$\lozenge$$

$$\nabla\times\left(\mathbf{U}\times\mathbf{V}\right)=\mathbf{V}.\nabla\mathbf{U}-\mathbf{U}.\nabla\mathbf{V}+\mathbf{U}\nabla.\mathbf{V}-\mathbf{V}\nabla.\mathbf{U}$$

$$
\begin{aligned}
(\nabla \times (U \times V))_k &\equiv \epsilon_{kpm}\partial_p \epsilon_{mnr} U_n V_r \\
&= \underbrace{\epsilon_{kpm}}_{=\epsilon_{mkp}} \epsilon_{mnr} V_r \partial_p U_n + \underbrace{\epsilon_{kpm}}_{=\epsilon_{mkp}} \epsilon_{mnr} U_n \partial_p V_r \\
&= \delta_{kn}\delta_{pr} V_r \partial_p U_n - \delta_{kr}\delta_{pn} V_r \partial_p U_n + \delta_{kn}\delta_{pr} U_n \partial_p V_r - \delta_{kr}\delta_{pn} U_n \partial_p V_r \\
&= \underbrace{V_p \partial_p U_k}_{\equiv (V.\nabla U)_k} - \underbrace{V_k \partial_n U_n}_{\equiv (V\nabla.U)_k} + \underbrace{U_k \partial_r V_r}_{\equiv (U\nabla.V)_k} - \underbrace{U_p \partial_p V_k}_{\equiv (U.\nabla V)_k} \\
&\equiv V.\nabla U - U.\nabla V + U\nabla.V - V\nabla.U
\end{aligned}
$$

$$\Diamond$$

$$\nabla (\mathbf{U}.\mathbf{V}) = \mathbf{U}.\nabla\mathbf{V} + \mathbf{V}.\nabla\mathbf{U} + \mathbf{U} \times (\nabla \times \mathbf{V}) + \mathbf{V} \times (\nabla \times \mathbf{U})$$

$$
\begin{aligned}
(\nabla (U.V))_p &\equiv \partial_p U_k V_k \\
&= V_k \partial_p U_k + U_k \partial_p V_k \\
(U \times (\nabla \times V))_p &\equiv \epsilon_{pkm} U_k \epsilon_{mnr} \partial_n V_r \\
&= \epsilon_{mpk}\epsilon_{mnr} U_k \partial_n V_r \\
&= \delta_{pn}\delta_{kr} U_k \partial_n V_r - \delta_{pr}\delta_{kn} U_k \partial_n V_r \\
&= U_r \partial_p V_r - U_n \partial_n V_p \\
\Rightarrow \qquad (U \times (\nabla \times V))_p &\equiv U_r \partial_p V_r - \underbrace{U_n \partial_n V_p}_{\equiv (U.\nabla V)_p}
\end{aligned}
$$

Plugging (23) in (18) twice (with interchanging U and V) gives

$$\nabla (U.V) = U \times (\nabla \times V) + U.\nabla V + V \times (\nabla \times U) + V.\nabla U$$

$$\Diamond$$

$$\nabla \times (\nabla \phi) = \mathbf{0}$$

$$(\nabla \times (\nabla \phi))_k \equiv \epsilon_{kmn} \partial_m \partial_n \phi$$

We just have to note that if $m = n$ the terms in the sum are zero and that when $m \neq n$ we will have two terms which add as $\epsilon_{kMN}\partial_M \partial_N \phi + \epsilon_{kNM}\partial_N \partial_M \phi$ which obviously is zero. And so $\nabla \times (\nabla \phi) = 0$.

$$\Diamond$$

$$\nabla . (\nabla \times \mathbf{V}) = \mathbf{0}$$

$$\begin{aligned}\nabla.(\nabla\times V) &\equiv \partial_p\epsilon_{pmn}\partial_m V_n\\ &= \epsilon_{npm}\partial_p\partial_m V_n\end{aligned}$$

We apply the same reasoning as in the previous identity. And so $\nabla.(\nabla\times V)=0$.

$\Diamond$

$$\nabla\times(\nabla\times\mathbf{V}) = \nabla\nabla.\mathbf{V} - \nabla^{\mathbf{2}}\mathbf{V}$$

$$\begin{aligned}(\nabla\times(\nabla\times V))_m &\equiv \epsilon_{mkr}\partial_k\epsilon_{ruv}\partial_u V_v\\ &= \epsilon_{rmk}\epsilon_{ruv}\partial_k\partial_u V_v\\ &= \delta_{mu}\delta_{kv}\partial_k\partial_u V_v - \delta_{mv}\delta_{ku}\partial_k\partial_u V_v\\ &= \underbrace{\partial_m\partial_v V_v}_{(\nabla\nabla.V)_m} - \underbrace{\partial_k\partial_k V_m}_{(\nabla^2 V)_m}\end{aligned}$$

$\Diamond$

$$\nabla.\mathbf{r} = \mathbf{3}$$

where r is the vector with components equal to the Cartesian coordinates z_1, z_2, z_3

$$\begin{aligned}\nabla.r &\equiv \partial_k z_k\\ &= \delta_{kk}\\ &= 3\end{aligned}$$

$\Diamond$

$$\nabla\times\mathbf{r} = \mathbf{0}$$

where r is the vector with components equal to the Cartesian coordinates z_1, z_2, z_3

$$\begin{aligned}(\nabla\times r)_m &\equiv \epsilon_{mns}\partial_n z_s\\ &= \epsilon_{mns}\delta_{ns}\end{aligned}$$

This sum is zero as when $n=s$, $\delta_{ns}=1$ but $\epsilon_{mNN}=0$ and when $n\neq s$, $\delta_{ns}=0$.

$\Diamond$

$$\mathbf{V}.\nabla\mathbf{r} = \mathbf{0}$$

where r is the vector with components equal to the Cartesian coordinates z_1, z_2, z_3

$$\begin{aligned}(V.\nabla r)_m &\equiv V_n \partial_n r_m \\ &= V_n \delta_{nm} \\ &= V_m\end{aligned}$$

◊

♦

4.31 p141 - Exercise 15

Prove that

$$\epsilon_{amn}\epsilon_{ars} + \epsilon_{ams}\epsilon_{anr} = \epsilon_{amr}\epsilon_{ans}$$

$$\begin{aligned}
\epsilon_{amn}\epsilon_{ars} + \epsilon_{ams}\epsilon_{anr} &= (\delta_{mr}\delta_{ns} - \delta_{ms}\delta_{nr}) + (\delta_{mn}\delta_{sr} - \delta_{mr}\delta_{sn}) \\
&= \delta_{mn}\delta_{sr} - \delta_{ms}\delta_{nr} \\
&= \epsilon_{amr}\epsilon_{anr}
\end{aligned}$$

♦