Tensor Calculus
J.L. Synge and A.Schild (Dover Publication)
Solutions to exercises

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github.

Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \quad \text{Christoffel symbol of the second kind}$$

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Special types of space

1.1 p112 - Exercise

Deduce from 4.110. that the Gaussian curvature of a V_2 positive-definite metric is given by

$$G = \frac{R_{1212}}{a_{11}a_{22} - a_{12}^2}$$

From p. 86 (exercise) we know that all the components of R_{mnrs} can be expressed as terms of R_{1212} (or vanish).
So by 4.110.,

$$K\left(a_{11}a_{22} - a_{12}a_{21}\right) = R_{1212} \tag{1}$$

and from page 96 3.415. we know that for V_2, $K = G$. Hence,

$$G = \frac{R_{1212}}{\left(a_{11}a_{22} - a_{12}a_{21}\right)} \tag{2}$$

♦

1.2 p113 - Exercise

Prove that, in a space V_N of constant curvature K,

$$\textbf{4.115.} \qquad R_{mn} = -(N-1)\,K a_{mn}, \quad R = -N\,(N-1)\,K$$

We have

$$R_{mn} = R^{s}_{.mns} = a^{sk} R_{kmns} \tag{1}$$

From 4.114.

$$\begin{aligned} R_{kmns} &= K\left(a_{kn}a_{ms} - a_{ks}a_{mn}\right) &(2)\\ &= K\left(\underbrace{\delta^{s}_{n}a_{ms}}_{amn} - N a_{mn}\right) &(3)\\ &= K\,(1-N)\,a_{mn} &(4) \end{aligned}$$

and

$$\begin{aligned} R &= R^{n}_{.n} &(5)\\ &= a^{kn}Rkn &(6)\\ &= -\underbrace{a^{kn}a_{kn}}_{N}(N-1)\,K &(7)\\ &= -N\,(N-1)\,K &(8) \end{aligned}$$

♦

1.3 p113 - Clarification

4.117. $$\frac{\delta^2 \eta^r}{\delta s^2} + \epsilon K \eta^r = 0$$

We have

$$\begin{aligned} R^r_{.smn} &= a^{rk} R_{ksmn} && (1) \\ (4.114) \Rightarrow \quad &= a^{rk} K \left(a_{km} a_{sn} - a_{kn} a_{sm}\right) && (2) \\ &= K \left(\delta^r_m a_{sn} - \delta^r_n a_{sm}\right) && (3) \end{aligned}$$

$$\begin{aligned} (3.311) \text{ and } (3) \quad & 0 = \frac{\delta^2 \eta^r}{\delta s^2} + K \left(\delta^r_m a_{sn} - \delta^r_n a_{sm}\right) p^s \eta^m p^n && (4) \\ \Leftrightarrow \quad & 0 = \frac{\delta^2 \eta^r}{\delta s^2} + K \left(\delta^r_m \eta^m \underbrace{a_{sn} p^s p^n}_{=\epsilon} - \delta^r_n \underbrace{a_{sm} p^s \eta^m}_{=0} p^n\right) && (5) \\ \Leftrightarrow \quad & 0 = \frac{\delta^2 \eta^r}{\delta s^2} + K \epsilon \underbrace{\delta^r_m \eta^m}_{=\eta^r} && (6) \\ \Leftrightarrow \quad & 0 = \frac{\delta^2 \eta^r}{\delta s^2} + K \epsilon \eta^r && (7) \end{aligned}$$

♦

1.4 p114 - Clarification

4.118. $$\frac{d^2\left(X_r\eta^r\right)}{ds^2}+\epsilon K\left(X_r\eta^r\right)=0$$

We know that $\frac{\delta X_r}{\delta s}=0$ (parallel transport)

$$\frac{\delta\left(X_r\eta^r\right)}{\delta s}=\eta^r\underbrace{\frac{\delta X_r}{\delta s}}_{=0}+X_r\frac{\delta\eta^r}{\delta s} \tag{1}$$

$$\Rightarrow \qquad \frac{\delta^2\left(X_r\eta^r\right)}{\delta s}=\underbrace{\frac{\delta X_r}{\delta s}}_{=0}\frac{\delta\eta^r}{\delta s}+X_r\frac{\delta^2\eta^r}{\delta s^2} \tag{2}$$

$$\Rightarrow \qquad X_r\frac{\delta^2\eta^r}{\delta s^2}=\frac{\delta^2\left(X_r\eta^r\right)}{\delta s^2} \tag{3}$$

$$\text{but} \qquad \frac{\delta^2\left(X_r\eta^r\right)}{\delta s^2}=\frac{d^2X_r\eta^r}{ds^2} \quad \text{as } X_r\eta^r \text{ is an invariant} \tag{4}$$

$$\Rightarrow X_r\frac{\delta^2\eta^r}{\delta s^2}=\frac{d^2X_r\eta^r}{ds^2} \tag{5}$$

$$\text{and so} \qquad \frac{\delta^2\left(\eta^r\right)}{\delta s^2}X_r+\epsilon K\left(X_r\eta^r\right)=\frac{d^2X_r\eta^r}{ds^2}+\epsilon K\left(X_r\eta^r\right)=0 \tag{6}$$

♦

1.5 p115 - Exercise

By taking an orthonormal set of N unit vectors propagated parallely along the geodesic, deduce from 4.120a that the magnitude η of the vector η^r is given by

$$\eta = C\left|\sin\left(s\sqrt{\epsilon K}\right)\right|$$

where C is a constant.

We have by 4.120a

$$X_r\eta^r = A\sin\left(s\sqrt{\epsilon K}\right) \tag{1}$$

We choose N different $X_r^{(k)}$ $(k = 1, 2, \ldots, N)$ which are orthonormal. Applying (1) N times with the different $X_r^{(k)}$ $(k = 1, 2, \ldots, N)$, we get

$$X_r^{(k)}\eta^r = A^{(k)}\sin\left(s\sqrt{\epsilon K}\right) \tag{2}$$

But as the $X_r^{(k)}$ are orthonormal and are used as a basis at the considered point of the geodesic we have

$$X_r^{(k)} = \delta_r^k \tag{3}$$

So, (2) becomes

$$\eta^k = A^{(k)}\sin\left(s\sqrt{\epsilon K}\right) \tag{4}$$

which are the components of the displacement vector in the orthonormal basis. By **2.301**. :

$$Y^2 = \epsilon a_{mn}Y^mY^n \tag{5}$$

$$\Rightarrow \qquad \eta^2 = \epsilon a_{mn}A^{(m)}A^{(n)}\sin^2\left(s\sqrt{\epsilon K}\right) \tag{6}$$

$$\Rightarrow \qquad \eta = C\left|\sin\left(s\sqrt{\epsilon K}\right)\right| \tag{7}$$

$$\text{with} \qquad C = \sqrt{\left|\epsilon a_{mn}A^{(m)}A^{(n)}\right|} \tag{8}$$

♦

1.6 p118 - Exercise

Examine the limit of the form **4.130** as R tends to infinity, and interpet the result.

We have

$$\textbf{4.130}. \qquad ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

But $\sin\epsilon \approx \epsilon$ for $\epsilon \ll 1$. So

$$\lim_{R\to\infty} ds^2 = dr^2 + R^2\left(\frac{r}{R}\right)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

$$= dr^2 + \left(rd\theta^2\right) + \left(r\sin\theta d\phi\right)^2 \tag{2}$$

This is the metric form for an Euclidean 3-space with spherical polar coordinates (see **2.532**. page 54).

♦

1.7 p119 - Exercise

Show that a transformation of a homogeneous coordinate system into another homogeneous system is necessarily linear. (Use the transformation equation **2.507** for Christoffel symbols, noting that all Christoffel symbols vanish when the coordinates are homogeneous).

By 2.507 we have the transformation rule

$$\Gamma'^{a}_{bc} = \Gamma^{r}_{mn} \partial_r z^a \partial_b z^m \partial_c z^n + \partial_r z^a \frac{\partial^2 z^r}{\partial z^b \partial z^c} \tag{1}$$

But, as both coordinate system are homogeneous, all Christoffel symbols vanish and so

$$\Gamma'^{a}_{bc} = \partial_r z^a \frac{\partial^2 z^r}{\partial z^b \partial z^c} = 0 \tag{2}$$

$$\Rightarrow \qquad \partial_r z^a \frac{\partial^2 z^r}{\partial z^b \partial z^c} = 0 \tag{3}$$

As the Jacobian can't vanish the possibility of having $\partial_r z^a = 0\ \forall a, r$ is excluded. Hence we must have $\frac{\partial^2 z^r}{\partial z^b \partial z^c} = 0$. And have a linear solution of the form

$$z^r = A_k z'^k + C \tag{4}$$

♦

1.8 p120 - Exercise

If $z_r, z_r^{'}$ are two systems of rectangular Cartesian coordinates in Euclidean 3-space, what is the geometrical interpretation of the constants in **4.204** and of the orthogonality conditions **4.209** ?

We have

$$z_m^{'} = A_{mn} z_n + A_m \tag{1}$$

As we assume that the Jacobian of the transformation does not vanish and thus the mapping is bijective, in an Euclidean 3-space A_m will perform a *translation* while A_{mn} can be interpreted as a combination rotation/reflection/stretching/contraction/shearing. I.e. the mapping is an *affine* transformation.
The condition **4.209** restricts the action of A_{mn} to a combination of rotation/reflection. Indeed, a rotation/reflection can be represented by $R = R_x(\gamma) \circ R_y(\beta) \circ R_z(\alpha)$ with

$$R_x = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} R_y = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & \pm 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} R_z = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \tag{2}$$

Note that for every axis, $R_k^T R_k = \mathbb{I}_3$.
Be $A_{mn} = R_x(\gamma) \circ R_y(\beta) \circ R_z(\alpha)$ We have by **4.209**, $A_{mq} A_{mq} = \delta_{pq}$ which can be expressed as

$$A^T A = \mathbb{I}_3 \tag{3}$$

$$\Rightarrow \qquad \mathbb{I}_3 = (R_x R_y R_z)^T R_x R_y R_z \tag{4}$$

$$= \underbrace{R_z^T \underbrace{R_y^T \underbrace{R_x^T R_x}_{=\mathbb{I}} R_y}_{=\mathbb{I}} R_z}_{=\mathbb{I}_3} \tag{5}$$

The identity yields, and interpret the coefficients of the orthogonal transformation as an Euclidean orthogonal transformation.

♦

1.9 p123 - Clarification

If $A_n A_n = 0$ it follows from **2.445** and **2.446** that the straight line is a geodesic null line.

We have

$$\begin{aligned}
&(2.446) && a_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 && \frac{dx^m}{du} = \frac{dz_m}{du} = A_m && (1)\\
&(4.215) && a_{mn} = \delta_{mn} && && (2)\\
&(1),(2) \Rightarrow && \delta_{mn}\frac{dz_m}{du}\frac{dz_n}{du} = 0 && && (3)\\
&\Rightarrow && A_n A_n = 0 && && (4)
\end{aligned}$$

♦

1.10 p123 - Clarification

It is easy to see ... viz.,

the straight line joining any two points in a plane lies entirely in the plane.

The plane is identified by

$$A_n z_n + B = 0 \tag{1}$$

and a line by

$$z_n = C_n u + D_n \tag{2}$$

Take two points at $u = 0$ and $u = p$ lying in the plane:

$$\begin{cases} A_n C_n p + A_n D_n + B = 0 \\ A_n D_n + B = 0 \end{cases} \tag{3}$$

$$\Rightarrow \qquad \begin{cases} A_n C_n p = 0 \\ A_n D_n + B = 0 \end{cases} \tag{4}$$

And as $p \neq 0 \;\Rightarrow A_n C_n = 0$. So for any arbitrary u of this line we have

$$\underbrace{A_n C_n}_{=0} u + \underbrace{A_n D_n + B}_{=0} = 0 \tag{5}$$

hence, all points of the line lie in the plane.

♦

1.11 p123 - Exercise

Show that a one-flat is a straight line.

A one-flat means $(N-1)$ equations

$$A_n^{(k)} z_n + B^{(k)} = 0 \quad k = 1, \ldots, N-1 \quad n = 1, \ldots, N \tag{1}$$

This is a set of $(N-1)$ linear equation in N unknown z_n. So we have one degree of freedom. E.g. put $z_N = u$ with u the free parameter. then,

$$A_\alpha^{(k)} z_\alpha + A_N^{(k)} u + B^{(k)} = 0 \quad \alpha = 1, \ldots, N-1 \tag{2}$$

If $det A_\alpha^{(k)} \neq 0$ we get a solution of the et of equation

$$Az = B \quad \text{with} \quad B \text{ a linear function in } u \tag{3}$$

$$\Rightarrow \qquad z_m = \left(A^{-1}B\right)_m \tag{4}$$

with $\left(A^{-1}B\right)_m$ of the form $C_m u + D_m$

♦

1.12 p126 - Exercise

Show that the null cone with vertex at the origin in space-time has the equation

$$y_1^2 + y_2^2 + y_3^2 - y_4^2 = 0$$

Prove that this null cone divides space-time into three regions such that
a. Any two points (events) both lying in one region can be joined by a continuous curve which does not cut the null cone.
b. All continuous curves joining two given points (events) which lie in different regions, cut the null cone.
Show further that the three regions may be further classified into past, present, and future as follows: If A and B are any two points in the past, then the straight segment AB lies entirely in the past. If A and B are any two points in the future, then the straight segment AB lies entirely in the future. If A is any point in the present, there exist at least one point B in the present that the straight segment AB cuts the null cone.

We first prove

$$y_1^2 + y_2^2 + y_3^2 - y_4^2 = 0$$

The null geodesic equations:

$$\begin{cases} \frac{\delta^2 x^r}{\delta u^2} = \frac{d^2 x_r}{du^2} = 0 \quad \text{as we use homogeneous coordinates} \\ \\ a_{mn} \frac{dx_m}{du} \frac{dx_n}{du} = 0 \end{cases} \tag{1}$$

$$\Rightarrow \quad \begin{cases} x_r = A_r u + B_r \quad \text{(put } B_r = 0 \text{ by aqequate choice of the origin)} \\ \\ A_1^2 + A_2^2 + A_1^2 + A_3^2 - A_4^2 = 0 \end{cases} \tag{2}$$

$$\Rightarrow \quad \frac{\left((x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2\right)}{u^2} = 0 \quad \text{for } u \neq 0 \tag{3}$$

$$\Rightarrow \quad (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2 = 0 \tag{4}$$

$\Diamond$

About the existence of three regions. First let's investigate the case in a V_3 space-time manifold in order to have a more intuitive grasp.
Consider a family of events (p_1, p_2, u) with $u \in (-\infty, \infty)$ (see the line $P^{'}P$" in figure 1.1.)

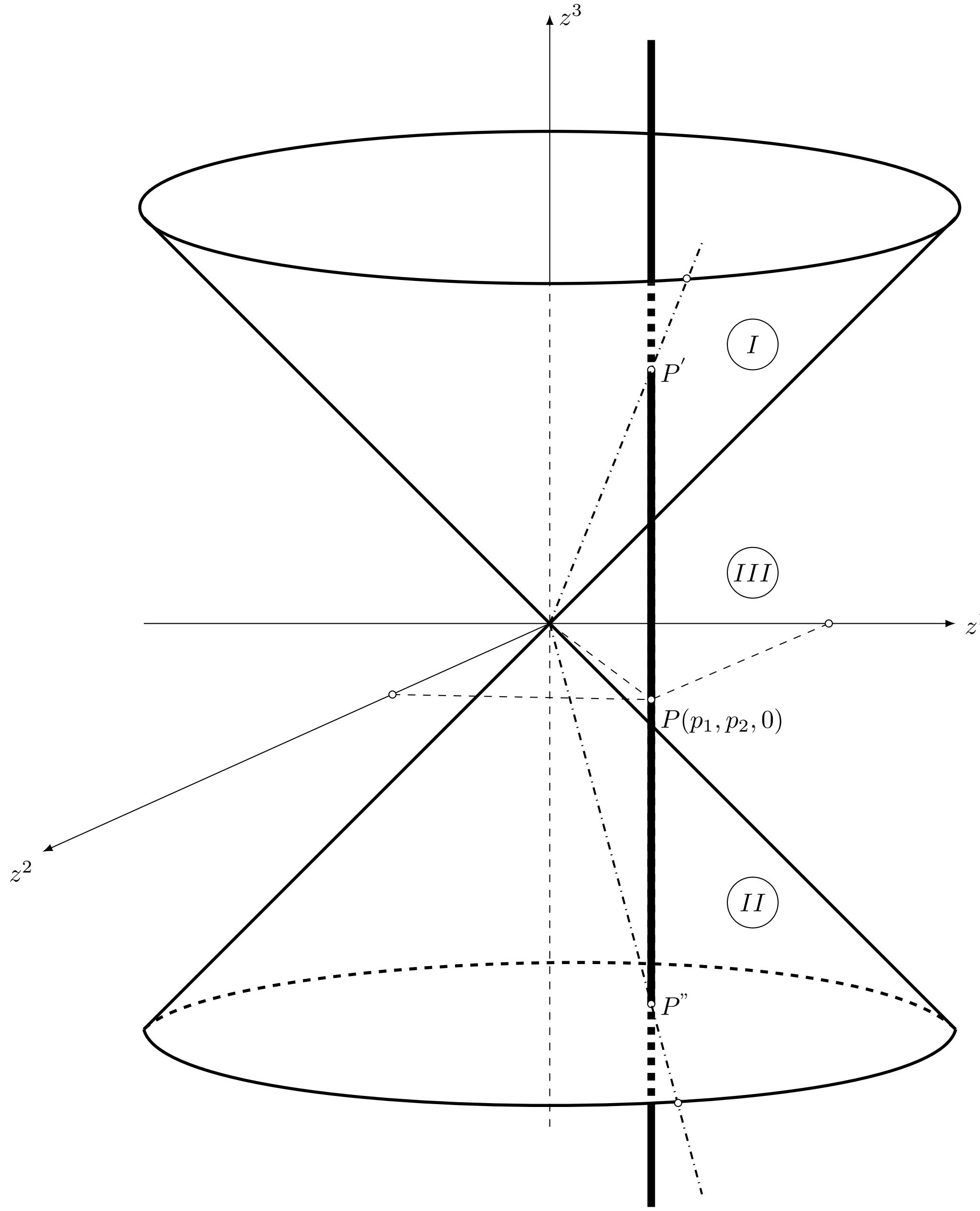


Figure 1.1: Regions delimited by the light cone in a V_3 space-time manifold

Be $R^2 = y_1^2 + y_2^2$. The light cone has the the equation $R^2 - y_3^2 = 0$. Only the events at $u_0 = \pm R(p_1, p_2)$ will lie on the light-cone.
We can distinguish three regions:
Region I where $u > u_0 = R$: the events (p_1, p_2, u) will lie on the line above point $P^{'}$.
Region II where $u < -u_0 = -R$: the events (p_1, p_2, u) will lie on the line below point P".
Region III where $-R = -u_0 < u < u_0 = R$: the events (p_1, p_2, u) will lie on the segment $P^{'}P$".

Let's generalize this now for a V_4 space-time manifold.
Put $R^2 = (y_1)^2 + (y_2)^2 + (y_3)^2$ and consider $\phi(y_1, y_2, y_3, y_4) = (y_1)^2 + (y_2)^2 + (y_3)^2 - (y_4)^2$ so $\phi(y_1, y_2, y_3, y_4) = R^2 - (y_4)^2$.
For $\phi = 0$ we lie on the light-cone.
For $\phi > 0 \quad \Rightarrow R^2 > y_4^2$ and so $-R < y_4 < R$ defines one region (region III).
For $\phi < 0 \quad \Rightarrow R^2 < y_4^2$ and so $y_4 > R$ and $y_4 < -R$ define two regions (region I and II).

$\Diamond$

We now show statement $a.$ of the exercise.
Consider 2 events $P_0,\ P_1$ with coordinates $(y_1^{(0)}, y_2^{(0)}, y_3^{(0)}, y_4^{(0)})$ and $(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)})$ and a curve defined by

$$y_i = \pm\sqrt{\left(\left(y_i^{(1)}\right)^2 - \left(y_i^{(0)}\right)^2\right) u + \left(y_i^{(0)}\right)^2} \qquad u \in [0,1] \tag{5}$$

where the $\pm$ is chosen so that $y_i(0) = y_i^{(0)}$ and $y_i(1) = y_i^{(1)}$ and that the sign only changes when $y_i(u) = 0$ and $sign\left(y_i^{(0)}\right) \neq sign\left(y_i^{(1)}\right)$. Such curve will be continuous. Put

$$R^2 = (y_1)^2 + (y_2)^2 + (y_3)^2 \tag{6}$$
$$R_0{}^2 = \left(y_1^0\right)^2 + \left(y_2^0\right)^2 + \left(y_3^0\right)^2 \tag{7}$$
$$R_1{}^2 = \left(y_1^1\right)^2 + \left(y_2^1\right)^2 + \left(y_3^1\right)^2 \tag{8}$$

For the points on the curve defined by (5), R^2 can then be written as

$$R^2 = \left(R_1^2 - R_0^2\right) u + R_0^2 \tag{9}$$

Be $\phi(u) = R^2 - y_4^2$.
Case a1: P_0, P_1 both lie in region I or both in region II. Then,

$$\phi(u) < 0 \quad \forall\ u \in [0,1] \tag{10}$$
$$\Rightarrow \quad R_0^2 < (y_4^0)^2 \quad \wedge \quad R_1^2 < (y_4^1)^2 \tag{11}$$
$$\text{with } (y_4^0 > 0\ \wedge\ y_4^1 > 0) \text{ in Region I} \quad \vee \quad (y_4^0 < 0 \quad \wedge \quad y_4^1 < 0) \text{ in Region II} \tag{12}$$

Then,

$$\nexists\ u \in [0,1] : \phi(u) = 0$$

Indeed,

$$\phi(u) = \left(R_1^2 - R_0^2\right) u + R_0^2 + \left(\left(y_4^{(0)}\right)^2 - \left(y_4^{(0)}\right)^2\right) u - \left(y_4^{(0)}\right)^2 \tag{13}$$

$$\phi(u) = 0 \qquad \Rightarrow u = -\frac{R_0^2 - \left(y_4^{(0)}\right)^2}{R_1^2 - R_0^2 - \left(y_4^{(1)}\right)^2 + \left(y_4^{(0)}\right)^2} \tag{14}$$

Let's simplify notationally the last equation. Put $R_0^2 - \left(y_4^{(0)}\right)^2 = -\tau$ and $R_1^2 - \left(y_4^{(1)}\right)^2 = -\sigma$ with both $\tau, \sigma > 0$. (14) can be written as

$$u = \frac{\tau}{\tau - \sigma} \tag{15}$$

$$= \frac{1}{1 - \frac{\sigma}{\tau}} \tag{16}$$

$$\Rightarrow \qquad |u| > 1 \qquad \text{as } \frac{\sigma}{\tau} > 0 \tag{17}$$

Note, that in the case $\tau = \sigma$ we have $\phi(u) = \tau =$ constant and can't reach 0. So, there exist no $u \in [0, 1]$ for which $\phi(u) = 0$ and the curve does not intersect the null cone.

Case a2: P_0, P_1 both lie in region III. Then,

$$\phi(u) > 0 \quad \forall\, u \in [0, 1] \tag{18}$$

$$\Rightarrow \quad R_0^2 > (y_4^0)^2 \quad \wedge \quad R_1^2 > (y_4^1)^2 \tag{19}$$

Then,

$$\nexists\, u \in [0, 1] : \phi(u) = 0$$

Indeed, Let's simplify notationally the equation (14) by now by puting $R_0^2 - \left(y_4^{(0)}\right)^2 = \tau$ and $R_1^2 - \left(y_4^{(1)}\right)^2 = \sigma$ with both $\tau, \sigma > 0$. (14) can be written again as

$$u = \frac{1}{1 - \frac{\sigma}{\tau}} \tag{20}$$

and follow the same reasoning as in case 1. So, there exist no $u \in [0, 1]$ for which $\phi(u) = 0$ and the curve does not intersect the null cone.

$\Diamond$

We now show statement b. of the exercise.

Case b1: P_0 lies in region I, P_1 lies in region II.
Those two regions are separated by the 3-flat (plane) $y_4 = 0$. So it's suffice that $R^2 = 0$ for $\phi(u)$ being zero. Hence $y_i = 0$, $i = 1, 2, 3$, and the cruve will cut the cone at it's apex.

Case b2: P_0 lies in region I or II, P_1 lies in region III.
We have

$$R_0^2 > (y_4^0)^2 \quad R_1^2 < (y_4^1)^2 \tag{21}$$

Put $R_0^2 - (y_4^0)^2 = \tau$ and $R_1^2 - (y_4^1)^2 = -\sigma$ with $\tau, \sigma > 0$. We get

$$\phi(u) = 0 \tag{22}$$

$$\Rightarrow \qquad u = -\frac{\tau}{-\sigma - \tau} \tag{23}$$

$$= \frac{1}{1 + \frac{\sigma}{\tau}} \tag{24}$$

So, there is a solution $u \in [0, 1]$ for which $\phi(u) = 0$ and the curve intersects the null cone.

$\Diamond$

We now investigate the straight segment questions.
Case 1: A and B both lie in the the present (region I) or in the past (region II)
Consider 2 events $P_0,\ P_1$ with coordinates $(y_1^{(0)}, y_2^{(0)}, y_3^{(0)}, y_4^{(0)})$ and $(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)})$ and a segment defined by

$$y_i = \left(y_i^{(1)} - y_i^{(0)}\right) u + y_i^{(0)} \qquad\qquad u \in [0, 1] \tag{25}$$

Be $\phi(u) = y_1^2 + y_2^2 + y_3^2 - y_4^2$.

Then,

$$\phi(u) = \begin{cases} \left[\left(y_1^{(1)} - y_1^{(0)}\right) u + y_1^{(0)}\right]^2 \\ + \left[\left(y_2^{(1)} - y_2^{(0)}\right) u + y_2^{(0)}\right]^2 \\ + \left[\left(y_3^{(1)} - y_3^{(0)}\right) u + y_3^{(0)}\right]^2 \\ - \left[\left(y_4^{(1)} - y_4^{(0)}\right) u + y_4^{(0)}\right]^2 \end{cases} \tag{26}$$

$$= \begin{cases} \left(y_1^{(1)}\right)^2 u^2 + \left(y_1^{(0)}\right)^2 u^2 - 2y_1^{(1)}y_1^{(0)}u^2 + 2y_1^{(1)}y_1^{(0)}u - 2\left(y_1^{(0)}\right)^2 u + \left(y_1^{(0)}\right)^2 \\ + \left(y_2^{(1)}\right)^2 u^2 + \left(y_2^{(0)}\right)^2 u^2 - 2y_2^{(1)}y_2^{(0)}u^2 + 2y_2^{(1)}y_2^{(0)}u - 2\left(y_2^{(0)}\right)^2 u + \left(y_2^{(0)}\right)^2 \\ + \left(y_3^{(1)}\right)^2 u^2 + \left(y_3^{(0)}\right)^2 u^2 - 2y_3^{(1)}y_3^{(0)}u^2 + 2y_3^{(1)}y_3^{(0)}u - 2\left(y_3^{(0)}\right)^2 u + \left(y_3^{(0)}\right)^2 \\ - \left(y_4^{(1)}\right)^2 u^2 - \left(y_4^{(0)}\right)^2 u^2 + 2y_4^{(1)}y_4^{(0)}u^2 - 2y_4^{(1)}y_4^{(0)}u + 2\left(y_4^{(0)}\right)^2 u - \left(y_4^{(0)}\right)^2 \end{cases} \tag{27}$$

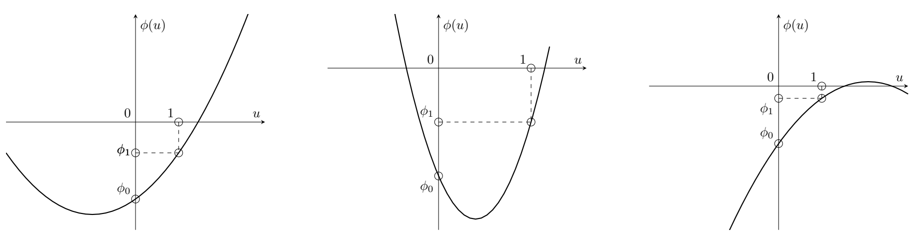
Put $\phi_0 = \left(y_1^{(0)}\right)^2 + \left(y_2^{(0)}\right)^2 + \left(y_3^{(0)}\right)^2 - \left(y_4^{(0)}\right)^2$ and $\phi_1 = \left(y_1^{(1)}\right)^2 + \left(y_2^{(1)}\right)^2 + \left(y_3^{(1)}\right)^2 - \left(y_4^{(1)}\right)^2$. Then,

$$\phi(u) = \begin{cases} \phi_1 u^2 + \phi_0 u^2 - 2\left(y_1^{(1)}y_1^{(0)} + y_2^{(1)}y_2^{(0)} + y_3^{(1)}y_3^{(0)} - y_4^{(1)}y_4^{(0)}\right) u^2 \\ -2\phi_0 u + 2\left(y_1^{(1)}y_1^{(0)} + y_2^{(1)}y_2^{(0)} + y_3^{(1)}y_3^{(0)} - y_4^{(1)}y_4^{(0)}\right) u \\ +\phi_0 \end{cases} \tag{28}$$

Let's put $\kappa = y_1^{(1)}y_1^{(0)} + y_2^{(1)}y_2^{(0)} + y_3^{(1)}y_3^{(0)} - y_4^{(1)}y_4^{(0)}$, we get the expression

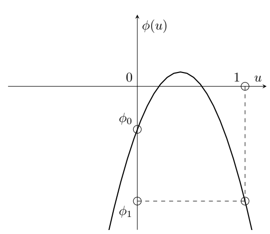
$$\phi(u) = (\phi_1 + \phi_0 - 2\kappa)\, u^2 + 2\,(\kappa - \phi_0)\, u + \phi_0 \tag{29}$$

The function $\phi(u)$ is a parabola.



(a) (b) (c)

Figure 1.2: Non problematic null cone parametric functions.



(a)

Figure 1.3: Problematic null cone parametric functions.

From the parabola's in figure 1.2. it is clear that the function can't reach 0 in $u \in (0,1)$ and hence that the segment will not intersect the null cone.
One problematic could occur as represented in figure 1.3. For such a parabola, we have :

$$\begin{cases} \phi(u) = au^2 + bu + c \\ a = (\phi_1 + \phi_0 - 2\kappa) < 0 \\ b = 2\,(\kappa - \phi_0) \\ c = \phi_0 \\ a + b = \phi_1 - \phi_0 \\ \phi_0 < 0 \\ \phi_1 < 0 \end{cases} \tag{30}$$

From the 3 known inequalities

$$\begin{cases} a < 0 \\ \phi_0 < 0 \\ \phi_1 < 0 \end{cases} \tag{31}$$

we get

$$\begin{cases} a < 0 \quad \Rightarrow \quad \kappa > \frac{\phi_1 + \phi_0}{2} \\ b = 2\,(\kappa - \phi_0) \quad \Rightarrow \quad b > \phi_1 - \phi_0 \end{cases} \tag{32}$$

Note that there is nothing special in the choice of $\phi_1,\ \phi_0$ as the segment is not oriented and as $\phi_1, \phi_0 < 0$ we can put arbitrarily $b \geq 0$ with $\phi_1 \geq \phi_0$.
Let's examine if there is a value $u_{max} \in (0,1)$ such that $\phi(u_{max}) \geq 0$ and let us investigate whether the relation between the coefficients a, b does not lead to a contradiction for such value of $u_{max} \in (0,1)$.

$$\frac{d\phi(u)}{du} = 0 \tag{33}$$

$$\Rightarrow \qquad u_{max} = -\frac{b}{2a} \quad \text{with } 0 < b < -2a \tag{34}$$

$$\phi(u_{max}) = -\frac{b^2}{4a} + \phi_0 \tag{35}$$

Is it possible to have $\phi(u_{max}) \geq 0$ with $u_{max} \in (0,1)$? One straightforward condition to have this possible is that the discriminant of the quadratic equation os $b^2 - 4ac >= 0$. Hence,

$$D = [2(\kappa - \phi_0)]^2 - 4(\phi_1 + \phi_0 - 2\kappa)\phi_0 \tag{36}$$

$$= 4\kappa^2 + 4\phi_0^2 - 8\kappa\phi_0 - 4\phi_1\phi_0 - 4\phi_0^2 + 8\kappa\phi_0 \tag{37}$$

$$= 4\left(\kappa^2 - \phi_1\phi_0\right) \tag{38}$$

$$= \begin{cases} 4\left[\left(y_1^{(0)}y_4^{(1)} - y_4^{(0)}y_1^{(1)}\right)^2 + \left(y_4^{(0)}y_2^{(1)} - y_2^{(0)}y_4^{(1)}\right)^2 + \left(y_4^{(0)}y_3^{(1)} - y_3^{(0)}y_4^{(1)}\right)^2\right] \\ -4\left[\left(y_2^{(0)}y_1^{(1)} - y_1^{(0)}y_2^{(1)}\right)^2 + \left(y_3^{(0)}y_1^{(1)} - y_1^{(0)}y_3^{(1)}\right)^2 + \left(y_3^{(0)}y_2^{(1)} - y_2^{(0)}y_3^{(1)}\right)^2\right] \end{cases} \tag{39}$$

Obviously, this path of proving leads to nothing as D can be positive even for a segment lying entirely in zone I or II. To see that, take two events, stationary in the space at the point $P\left(0, 0, y_3^{(0)}\right)$. The negative term in (39) disappears and obviously $D > 0$.

' WHAT IS THE ANALYTICAL WAY TO PROVE THIS? IS $a > 0$ A CONDITION?

♦

1.13 p133 - Exercise

In a space of two dimensions prove the relation

4.318. $$\in_{mp}\in_{mq}=\delta_{pq}$$

Suppose $p=q$, then in the summation the term is 0 if $m=p=q$ and the remaining term is 1×1 or -1×-1 giving indeed $\delta_{pq}=1$.
If $p\neq q$ we get either $m=p$ or $m=q$ in each term of the summation and hence all terms vanish.

♦

1.14 p135 - Exercise

Write out the six independent non-zero components of P_{mn} as given by **4.324**.

We have

$$\textbf{4.324}. \qquad P_{mn} = \in_{mnrs} X^r Y^s \tag{1}$$

$$\text{with} \qquad m = n \quad \Rightarrow \quad P_{mn} = 0 \tag{2}$$

So, the six independent components are in the set $\{mn\} = \{12, 13, 14, 23, 24, 34\}$ as $P_{nm} = -P_{mn}$.

$$\begin{cases} P_{12} = \in_{1234} X^3Y^4 + \in_{1243} X^4Y^3 \\ P_{13} = \in_{1324} X^2Y^4 + \in_{1342} X^4Y^2 \\ P_{14} = \in_{1423} X^2Y^3 + \in_{1432} X^3Y^2 \\ P_{23} = \in_{2314} X^1Y^4 + \in_{2341} X^4Y^1 \\ P_{24} = \in_{2413} X^1Y^3 + \in_{2431} X^3Y^1 \\ P_{34} = \in_{3412} X^1Y^2 + \in_{3421} X^2Y^1 \end{cases} \tag{3}$$

$$\begin{cases} P_{12} = X^3Y^4 - X^4Y^3 \\ P_{13} = -X^2Y^4 + X^4Y^2 \\ P_{14} = X^2Y^3 - X^3Y^2 \\ P_{23} = X^1Y^4 - X^4Y^1 \\ P_{24} = -X^1Y^3 + X^3Y^1 \\ P_{34} = X^1Y^2 - X^2Y^1 \end{cases} \tag{4}$$

♦

1.15 p135 - Exercise

Translate the well-known vector relations

$$A \times (B \times C) = B(A.C) - C(A.B)$$

$$\nabla \times (\nabla \times V) = \nabla(\nabla.V) - \nabla^2 V$$

into Cartesian tesnor form, and prove the by use of 4.329.

We have

$$\textbf{4.329}. \qquad \in_{mrs}\in_{mpq} = \delta_{rp}\delta_{sq} - \delta_{rq}\delta_{sp} \tag{1}$$

The first identity

$$A \times (B \times C) = B(A.C) - C(A.B) \tag{2}$$

$$\Leftrightarrow \qquad \in_{npm}\in_{mrs} A_p B_r C_s = A_p \left(B_n C_p - C_n B_p\right) \tag{3}$$

Indeed,

$$(B \times C)_m = \in_{mrs} B_r C_s \tag{4}$$

$$\Rightarrow \qquad (A \times (B \times C))_n = \in_{npm} A_p \in_{mrs} B_r C_s \tag{5}$$

$$= - \in_{mpn}\in_{mrs} A_p \in_{mrs} B_r C_s \tag{6}$$

$$= -\delta_{pr}\delta_{ns} A_p B_r C_s + \delta_{ps}\delta_{nr} A_p B_r C_s \tag{7}$$

$$= A_p B_n C_p - A_p B_p C_n \tag{8}$$

$$\Leftrightarrow \qquad B(A.C) - C(A.B) \tag{9}$$

The second identity

$$\nabla \times (\nabla \times V) = \nabla(\nabla.V) - \nabla^2 V \tag{10}$$

$$\Leftrightarrow \qquad \in_{nrm}\in_{mpq} V_{q,pr} = V_{p,pn} - V_{n,pp} \tag{11}$$

Indeed,

$$(\nabla \times V)_m = \in_{mpq} V_{q,p} \tag{12}$$

$$\Rightarrow \qquad (\nabla \times (\nabla \times V))_n = \in_{nrm} \left(\in_{mpq} V_{q,p}\right)_{,r} \tag{13}$$

$$= \in_{nrm}\in_{mpq} V_{q,pr} \tag{14}$$

$$= \delta_{rq}\delta_{np} V_{q,pr} - \delta_{pr}\delta_{nq} V_{q,pr} \tag{15}$$

$$= V_{p,pn} - V_{n,pp} \tag{16}$$

We have also

$$(\nabla V) = V_{p,p} \tag{17}$$

$$\Rightarrow \qquad (\nabla(\nabla.V))_n = (V_{p,p})_n \tag{18}$$

$$= V_{p,pn} \tag{19}$$

and

$$\nabla^2 V_n \equiv V_{n,pp} \tag{20}$$

$$\Rightarrow \qquad (\nabla(\nabla.V))_n - \nabla^2 V_n = V_{p,pn} - V_{n,pp} \tag{21}$$

which corresponds to (15). So the tensor expression in Cartesian tensor form can be written as

$$\in_{nrm} \in_{mpq} V_{q,pr} = V_{p,pn} - V_{n,pp}$$

♦

1.16 p139 - Exercise 1.

Show that, in a 3-space of constant curvature $-\frac{1}{R^2}$ and positive definite metric form, the line element in polar coordinate is

$$ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

We have by 4.120c

$$X_r\eta^r = A\sinh\left(s\sqrt{-\epsilon K}\right) \tag{1}$$

We choose N different $X_r^{(k)}$ $(k = 1, 2, \ldots, N)$ which are orthonormal. Applying (1) N times with the different $X_r^{(k)}$ $(k = 1, 2, \ldots, N)$, we get

$$X_r^{(k)}\eta^r = A^{(k)}\sinh\left(s\sqrt{-\epsilon K}\right) \tag{2}$$

But as the $X_r^{(k)}$ are orthonormal and are used as a basis at the considered point of the geodesic we have

$$X_r^{(k)} = \delta_r^k \tag{3}$$

So, (2) becomes

$$\eta^k = A^{(k)}\sinh\left(s\sqrt{-\epsilon K}\right) \tag{4}$$

which are the components of the displacement vector in the orthonormal basis. By **2.301**. :

$$Y^2 = \epsilon a_{mn} Y^m Y^n \tag{5}$$

$$\Rightarrow \qquad \eta^2 = \epsilon a_{mn} A^{(m)} A^{(n)} \sinh^2\left(s\sqrt{-\epsilon K}\right) \tag{6}$$

$$\Rightarrow \qquad \eta = C\left|\sinh\left(s\sqrt{-\epsilon K}\right)\right| \tag{7}$$

$$\text{with} \qquad C = \sqrt{\left|\epsilon a_{mn} A^{(m)} A^{(n)}\right|} \tag{8}$$

As $\epsilon = 1$ (positive-definite metric) and $K = -\frac{1}{R^2}$ we have

$$\eta = C\left|\sinh\left(\frac{s}{R}\right)\right| \tag{9}$$

From this and using the very same reasoning from 4.126. to 4.130. (pages 117-119) we get

$$ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

♦

1.17 p139 - Exercise 2.

Show that the volume of an antipodal 3-space of positive-definite metric form and positive constant curvature $\frac{1}{R^2}$ is $2\pi^2 R^3$. (Use the equation 4.130. to find the area of a sphere $r =$ constant in polar coordinates. Multiply by dr and integrate for $0 \leq r \leq \pi R$ to get the volume). What is the volume if the space is polar?

We have 4.130

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

Having a positive-definite metric form, the space can be locally considered as Euclidean and an elementary area of a surface with constant $r(\rightarrow dr = 0)$ can be calculated as $dS = ds_{d\theta=0} ds_{d\phi=0}$ and get by (1)

$$dS = R^2 \sin^2\left(\frac{r}{R}\right)\sin\theta d\phi d\theta \tag{2}$$

$$\Rightarrow \quad \frac{S}{8} = R^2 \sin^2\left(\frac{r}{R}\right)\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin\theta d\phi d\theta \tag{3}$$

$$= R^2 \sin^2\left(\frac{r}{R}\right)\frac{\pi}{2}\left(-\cos\theta\right)\Big|_0^{\frac{\pi}{2}} \tag{4}$$

$$\Rightarrow \quad S = 4\pi R^2 \sin^2\left(\frac{r}{R}\right) \tag{5}$$

We see that the area is a cyclic function of r having zeros' at $r = k\frac{\pi}{R}, k = 1, 2, \ldots$

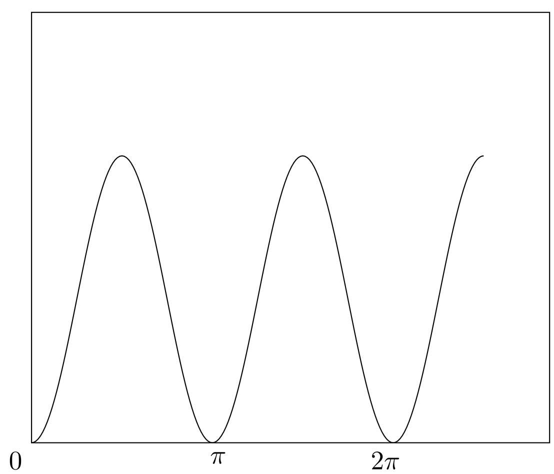


Figure 1.4: Area of an antipodal 3-space of positive-definite metric form

So, there are good reasons to restrict r to $[0, \pi R]$ as otherwise all space of that type would have infinite volume, whatever it's curvature. So, we get as volume

$$V = 4\pi R^2 \int_0^{\pi R} \sin^2\left(\frac{r}{R}\right) dr \tag{6}$$

$$= 4\pi R^3 \int_0^{\pi R} \sin^2\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) \tag{7}$$

$$= 4\pi R^3 \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^{\pi} \tag{8}$$

$$= 2\pi^2 R^3 \tag{9}$$

Fo a polar space, the volume would be half of that of an antipodal one (with same curvature of course) as in (3) we would consider only 4 quadrants instead of 8.

♦

1.18 p139 - Exercise 3.

By direct calculation of the tensor R_{rsmn} verify that **4.130**. is the metric form of a space of constant curvature.

We have 4.130

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

$$\Rightarrow \qquad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2\sin^2\left(\frac{r}{R}\right) & 0 \\ 0 & 0 & R^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \end{pmatrix} \tag{2}$$

Now for R_{rsmn} we refer to exercise 7 page 109 of chapter 3, where the curvature tensor was calculated for a general case of the form

$$ds^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2$$

where h_1, h_2, h_3 are functions of the three coordinates. We have then for our case

$$\begin{cases} h_1 = 1 \\ h_2 = R\sin\left(\frac{r}{R}\right) \\ h_3 = R\sin\left(\frac{r}{R}\right)\sin\theta \end{cases} \tag{3}$$

In the exercise we got for the non-vanishing curvature tensors

$$R_{1212} = -h_2\partial_{11}^2(h_2) - h_1\partial_{22}^2(h_1) + \frac{h_2}{h_1}\partial_1 h_1\partial_1 h_2 + \frac{h_1}{h_2}\partial_2 h_1\partial_2 h_2 - \frac{h_1 h_2}{h_3^2}\partial_3 h_1\partial_3 h_2 \tag{4}$$

$$R_{2323} = -h_3\partial_{22}^2(h_3) - h_2\partial_{33}^2(h_2) + \frac{h_3}{h_2}\partial_2 h_2\partial_2 h_3 + \frac{h_2}{h_3}\partial_3 h_2\partial_3 h_3 - \frac{h_2 h_3}{h_1^2}\partial_1 h_2\partial_1 h_3 \tag{5}$$

$$R_{1313} = -h_3\partial_{11}^2(h_3) - h_1\partial_{33}^2(h_1) + \frac{h_3}{h_1}\partial_1 h_1\partial_1 h_3 + \frac{h_1}{h_3}\partial_3 h_1\partial_3 h_3 - \frac{h_1 h_3}{h_2^2}\partial_2 h_1\partial_2 h_3 \tag{6}$$

$$R_{1213} = -h_1\partial_{32}^2(h_1) + \frac{h_1}{h_3}\partial_2 h_3\partial_3 h_1 + \frac{h_1}{h_2}\partial_2 h_1\partial_3 h_2 \tag{7}$$

$$R_{1223} = h_2\partial_{31}^2(h_2) - \frac{h_2}{h_1}\partial_1 h_2\partial_3 h_1 - \frac{h_2}{h_3}\partial_3 h_2\partial_1 h_3 \tag{8}$$

$$R_{1323} = -h_3\partial_{21}^2(h_3) + \frac{h_3}{h_1}\partial_1 h_3\partial_3 h_1 + \frac{h_3}{h_2}\partial_2 h_3\partial_1 h_2 \tag{9}$$

Clearly $\partial_k^2(h_1) = 0$ and $\partial_{mn}^2(h_1) = 0$ and considering $h_2 = h_2(r),\ h_3 = h_3(r,\theta)$ we can simplify

$$R_{1212} = -h_2\partial_{11}^2(h_2) \tag{10}$$

$$R_{2323} = -h_3\partial_{22}^2(h_3) - \frac{h_2h_3}{h_1^2}\partial_1h_2\partial_1h_3 \tag{11}$$

$$R_{1313} = -h_3\partial_{11}^2(h_3) \tag{12}$$

$$R_{1213} = 0 \tag{13}$$

$$R_{1223} = 0 \tag{14}$$

$$R_{1323} = -h_3\partial_{21}^2(h_3) + \frac{h_3}{h_2}\partial_2h_3\partial_1h_2 \tag{15}$$

with

$$\begin{cases} \partial_1h_2 = \cos\left(\frac{r}{R}\right) \\ \partial_1h_3 = \cos\left(\frac{r}{R}\right)\sin\theta \\ \partial_2h_3 = R\sin\left(\frac{r}{R}\right)\cos\theta \\ \partial_{11}^2h_2 = -\frac{1}{R}\sin\left(\frac{r}{R}\right) \\ \partial_{11}^2h_3 = -\frac{1}{R}\sin\left(\frac{r}{R}\right)\sin\theta \\ \partial_{21}^2h_3 = \cos\left(\frac{r}{R}\right)\cos\theta \\ \partial_{22}^2h_3 = -R\sin\left(\frac{r}{R}\right)\sin\theta \end{cases} \tag{16}$$

giving

$$R_{1212} = \sin^2\left(\frac{r}{R}\right) \tag{17}$$

$$R_{2323} = R^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta - R^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta\cos^2\left(\frac{r}{R}\right) \tag{18}$$

$$= R^2\sin^4\left(\frac{r}{R}\right)\sin^2\theta \tag{19}$$

$$R_{1313} = \sin^2\left(\frac{r}{R}\right)\sin^2\theta \tag{20}$$

$$R_{1213} = 0 \tag{21}$$

$$R_{1223} = 0 \tag{22}$$

$$R_{1323} = -R\sin\left(\frac{r}{R}\right)\sin\theta\cos\left(\frac{r}{R}\right)\cos\theta + \sin\theta R\sin\left(\frac{r}{R}\right)\cos\theta\cos\left(\frac{r}{R}\right) \tag{23}$$

$$= 0 \tag{24}$$

and considering the symmetries

$$R_{1212} = -R_{1221} = -R_{2112} = \sin^2\left(\frac{r}{R}\right) \tag{25}$$
$$R_{2323} = -R_{2332} = -R_{3223} = R^2\sin^4\left(\frac{r}{R}\right)\sin^2\theta \tag{26}$$
$$R_{1313} = -R_{1331} = -R_{3113} = \sin^2\left(\frac{r}{R}\right)\sin^2\theta \tag{27}$$
$$\tag{28}$$

By **4.114**.

$$R_{rsmn} = K\left(a_{rm}a_{sn} - a_{rn}a_{sm}\right) \tag{29}$$

$$\Rightarrow \quad \begin{cases} R_{1212} = K\left(a_{11}a_{22} - a_{12}a_{21}\right) \\ R_{2323} = K\left(a_{22}a_{33} - a_{23}a_{32}\right) \\ R_{1313} = K\left(a_{11}a_{33} - a_{13}a_{31}\right) \end{cases} \tag{30}$$

$$\Rightarrow \quad \begin{cases} R_{1212} = KR^2\sin^2\left(\frac{r}{R}\right) \\ R_{2323} = KR^2\sin^2\left(\frac{r}{R}\right)R^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \\ R_{1313} = KR^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \end{cases} \tag{31}$$

$$\Rightarrow \quad \begin{cases} R_{1212} = KR^2\sin^2\left(\frac{r}{R}\right) \\ R_{2323} = KR^4\sin^4\left(\frac{r}{R}\right)\sin^2\theta \\ R_{1313} = KR^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \end{cases} \tag{32}$$

replacing (25), (26) and (27) in (32) we get

$$\begin{cases} \sin^2\left(\frac{r}{R}\right) = KR^2\sin^2\left(\frac{r}{R}\right) \\ R^2\sin^4\left(\frac{r}{R}\right)\sin^2\theta = KR^4\sin^4\left(\frac{r}{R}\right)\sin^2\theta \\ \sin^2\left(\frac{r}{R}\right)\sin^2\theta = KR^2\sin^2\left(\frac{r}{R}\right)\sin^2\theta \end{cases} \tag{33}$$

giving indeed for the three curvature tensors $K = \frac{1}{R^2}$
With this, the question of the exercise is answered but we go a little bit further and investigate for this practical case the equations of **4.115**. and calculate $R = a^{mn}R_{mn}$. With R, the curvature invariant. To avoid confusion with the curvature R itself we use $\mathfrak{R}$ for the curvature invariant.
As the metric tensor is diagonal:

$$\mathfrak{R} = a^{11}R_{11} + a^{22}R_{22} + a^{33}R_{33} \tag{34}$$
$$\tag{35}$$

and

$$R_{mn} = a^{sn} R_{srmn} \tag{36}$$

$$\Rightarrow \begin{cases} R_{11} = a^{11} R_{1111} + a^{22} R_{2112} + a^{33} R_{3113} \\ R_{22} = a^{11} R_{1221} + a^{22} R_{2222} + a^{33} R_{3223} \\ R_{33} = a^{11} R_{1331} + a^{22} R_{2332} + a^{33} R_{3333} \end{cases} \tag{37}$$

$$\Rightarrow \begin{cases} R_{11} = -a^{22} \sin^2\left(\frac{r}{R}\right) - a^{33} \sin^2\left(\frac{r}{R}\right) \sin^2\theta \\ R_{22} = -a^{11} \sin^2\left(\frac{r}{R}\right) - a^{33} R^2 \sin^4\left(\frac{r}{R}\right) \sin^2\theta \\ R_{33} = -a^{11} \sin^2\left(\frac{r}{R}\right) \sin^2\theta - a^{22} R^2 \sin^4\left(\frac{r}{R}\right) \sin^2\theta \end{cases} \tag{38}$$

$$\Rightarrow \begin{cases} R_{11} = -\frac{\sin^2\left(\frac{r}{R}\right)}{R^2 \sin^2\left(\frac{r}{R}\right)} - \frac{\sin^2\left(\frac{r}{R}\right) \sin^2\theta}{R^2 \sin^2\left(\frac{r}{R}\right) \sin^2\theta} \\ R_{22} = -\sin^2\left(\frac{r}{R}\right) - \frac{R^2 \sin^4\left(\frac{r}{R}\right) \sin^2\theta}{R^2 \sin^2\left(\frac{r}{R}\right) \sin^2\theta} \\ R_{33} = -\sin^2\left(\frac{r}{R}\right) \sin^2\theta - \frac{R^2 \sin^4\left(\frac{r}{R}\right) \sin^2\theta}{R^2 \sin^2\left(\frac{r}{R}\right)} \end{cases} \tag{39}$$

giving

$$\begin{cases} R_{11} = -\frac{2}{R^2} \\ R_{22} = -2 \sin^2\left(\frac{r}{R}\right) \\ R_{33} = -2 \sin^2\left(\frac{r}{R}\right) \sin^2\theta \end{cases} \tag{40}$$

hence

$$\mathfrak{R} = a^{11} R_{11} + a^{22} R_{22} + a^{33} R_{33} \tag{41}$$

$$\Rightarrow \quad R = -\frac{2}{R^2} - 2\frac{\sin^2\left(\frac{r}{R}\right)}{R^2 \sin^2\left(\frac{r}{R}\right)} - 2\frac{\sin^2\left(\frac{r}{R}\right) \sin^2\theta}{R^2 \sin^2\left(\frac{r}{R}\right) \sin^2\theta} \tag{42}$$

$$\mathfrak{R} = -\frac{6}{R^2} \tag{43}$$

The equations in (40) and (43) are indeed in accordance with **4.115** .

♦

1.19 p139 - Exercise 4

Show that if V_N has positive-definite metric form and constant positive curvature K, then coordinates y^r exist so that

$$ds^2 = \frac{dy^m dy^m}{\left(1 + \frac{1}{4} y^n y^n\right)^2}$$

(Starting with a coordinate system x^r which is locally Cartesian at O, take at any point P the coordinates

$$y^r = p^r \frac{2}{\sqrt{K}} \tan\left(\frac{1}{2} r \sqrt{K}\right)$$

where p^r ar the the components of the unit tangent vector $\left(\frac{dx^r}{ds}\right)$ at O to the geodesic OP and r is the geodesic distance OP.)

Let us first understand what happens. Fig.1.5 for a V_3 will help us understand. Let P and $P + dP$ be two points separated by an infinitesimal distance. Consider the two geodesics initiated from the origin and joining these two points. Be X and $X^{'}$ the two tangents unit vectors to these geodesics. Those vectors have components $p^r = \frac{dx^r}{ds}$ taken along their respective geodesics. By the considered transformation the points P and $P + dP$ are mapped on the points $\tau(P)$ and $\tau(P + dP)$ with $|O\tau(P)|$ and $|O\tau(P + dP)|$ collinear with the two tangents unit vectors X and $X^{'}$.
Observe also the segment PN which corresponds to the geodesic displacement η for the geodesic distance $r = OP$. As the metric form is positive-definite, we can consider that the infinitesimal triangle $\left|P\widehat{NP + d}P\right|$ lies in an infinitesimally Euclidean space and we can express $ds^2 = \eta^2 + dr^2$ as $|NP + dP| = dr$. Observe now, the triangle $\left|\tau(P)\tau(\widehat{N)\tau(}P + dP)\right|$. There also we have $|\tau(P + dP)\tau(P)|^2 = |\tau(P)\tau(N)|^2 + |\tau(P + dP)\tau(N)|^2$. Can we find a relationship between these two triangle?

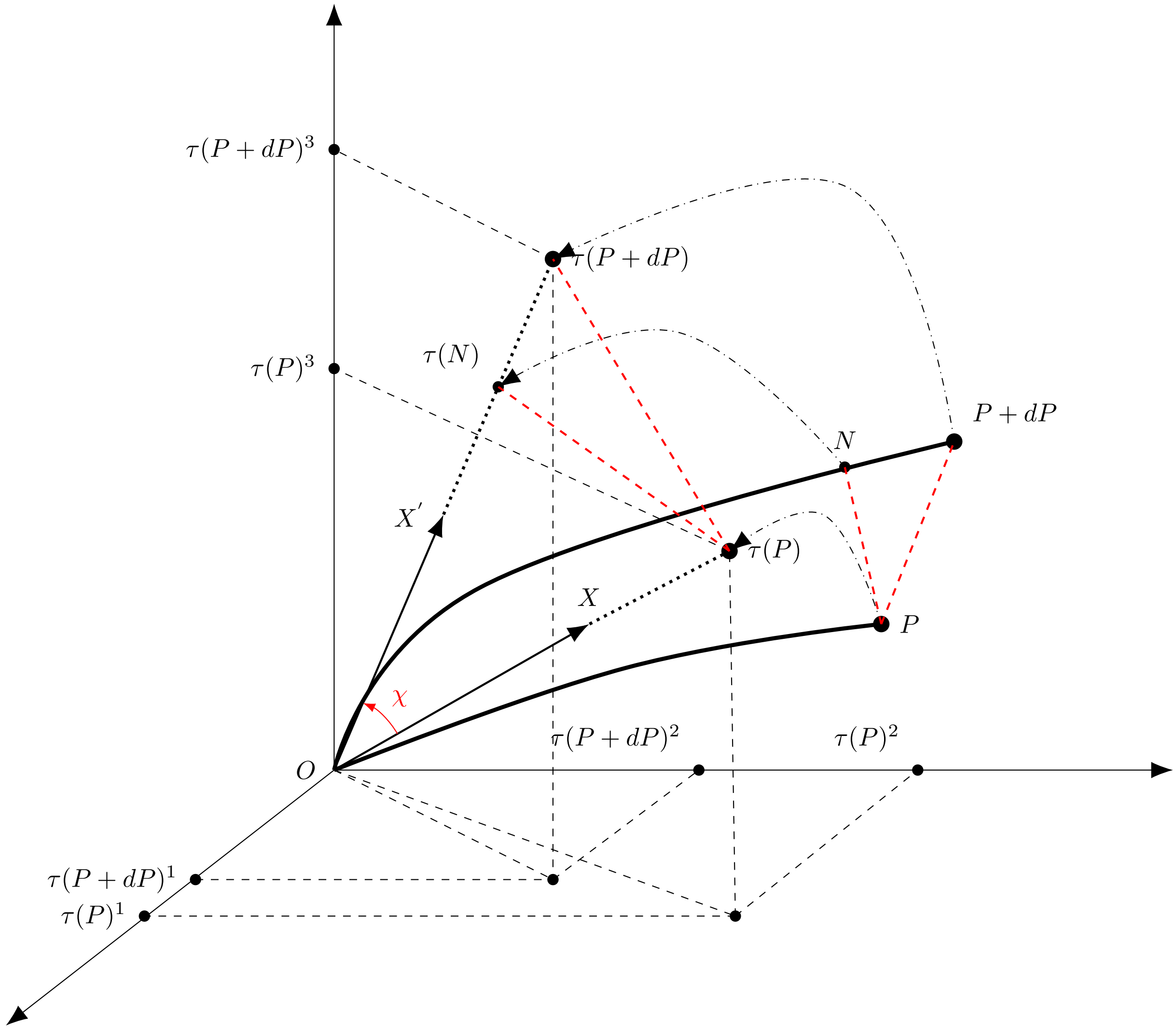


Figure 1.5: Coordinate system in constant curvature space

Let's define $\alpha(r) = \frac{2}{\sqrt{K}} \tan\left(\frac{1}{2} r\sqrt{K}\right)$ so that $y^k = \alpha(r)p^k$ We have

$$\begin{cases} \left|OX'\right| = |OX| = 1 \\ \\ |O\tau(N)| = |O\tau(P)| = \alpha(r) \\ \\ |O\tau(P+dP)| = \alpha(r+dr) \\ \end{cases} \tag{1}$$

Expanding the last equation in a Taylor series we get as first order term

$$|\tau(N)\tau(P+dP)| = \frac{1}{\cos^2(\frac{1}{2}r\sqrt{K})} dr \tag{2}$$

Also,

$$|\tau(N)\tau(P)| = 2\alpha(r)\sin\frac{\chi}{2} \approx \alpha(r)\chi \tag{3}$$

From **4.124** we have $\chi = \left(\frac{d\eta}{dr}\right)_{r=0} = C\sqrt{K}$. But note also that from **4.122** we have for the geodesic displacement $\eta = C\left|\sin r\sqrt{K}\right|$.

$$C = \frac{\eta}{\left|\sin r\sqrt{K}\right|} \tag{4}$$

$$\Rightarrow \qquad \chi = \frac{\eta}{\left|\sin r\sqrt{K}\right|}\sqrt{K} \tag{5}$$

$$\Rightarrow \qquad |\tau(N)\tau(P)| = \eta\frac{\sqrt{K}}{\left|\sin r\sqrt{K}\right|}\alpha(r) \tag{6}$$

Let's put $|\tau(N)\tau(P)| = \hat{\eta}$.

$$\hat{\eta} = \eta\frac{\sqrt{K}}{\left|\sin r\sqrt{K}\right|}\alpha(r) \tag{7}$$

At the point P we have $|PN| = \eta$ and so

$$ds^2 = \eta^2 + dr^2 \tag{8}$$

Let's put $|\tau(P+dP)\tau(P)|^2 = d\hat{s}^2$.

$$d\hat{s}^2 = \hat{\eta}^2 + |\tau(P+dP)\tau(N)|^2 \tag{9}$$

$$\text{(2) and (7)} \Rightarrow \qquad = \eta^2\frac{K}{\sin^2(r\sqrt{K})}\frac{4}{K}\frac{\sin^2(\frac{1}{2}r\sqrt{K})}{\cos^2(\frac{1}{2}r\sqrt{K})} + \frac{1}{\cos^4(\frac{1}{2}r\sqrt{K})}dr^2 \tag{10}$$

$$\sin r\sqrt{K} = 2\sin\left(\frac{1}{2}r\sqrt{K}\right)\cos\left(\frac{1}{2}r\sqrt{K}\right) \tag{11}$$

$$\Rightarrow \qquad d\hat{s}^2 = \eta^2\frac{1}{\cos^4(\frac{1}{2}r\sqrt{K})} + \frac{1}{\cos^4(\frac{1}{2}r\sqrt{K})}dr^2 \tag{12}$$

$$\Rightarrow \qquad \eta^2 + dr^2 = d\hat{s}^2\cos^4(\frac{1}{2}r\sqrt{K}) \tag{13}$$

$$\Rightarrow \qquad ds^2 = d\hat{s}^2\cos^4(\frac{1}{2}r\sqrt{K}) \tag{14}$$

It is easy to see that $d\hat{s}^2 = dy^k dy^k$ and also

$$\cos^4(\frac{1}{2}r\sqrt{K}) = \left(\cos^2(\frac{1}{2}r\sqrt{K})\right)^2 \tag{15}$$

$$\Leftrightarrow \qquad = \left(\frac{\cos^2(\frac{1}{2}r\sqrt{K})}{\cos^2(\frac{1}{2}r\sqrt{K}) + \sin^2(\frac{1}{2}r\sqrt{K})}\right)^2 \tag{16}$$

$$\Leftrightarrow \qquad = \left(\frac{1}{1 + \tan^2(\frac{1}{2}r\sqrt{K})}\right)^2 \tag{17}$$

We note that $\frac{2}{\sqrt{K}} \tan\left(\frac{1}{2}r\sqrt{K}\right)$ is the size of the vector $|O\tau(P)|$ and can express this as (as we use local Cartesian coordinates at the origin) $|O\tau(P)|^2 = y^k y^k$ and thus $\tan^2(\frac{1}{2}r\sqrt{K}) = \frac{K}{4}y^k y^k$. Combining this with (14) and (17) gives:

$$ds^2 = d\hat{s}^2 \left(\frac{1}{1 + \frac{K}{4}y^k y^k}\right)^2 \tag{18}$$

which gives as final expression

$$ds^2 = \frac{dy^k dy^k}{\left(1 + \frac{K}{4}y^k y^k\right)^2} \tag{19}$$

♦