

Tensor Calculus  
J.L. Synge and A.Schild (Dover Publication)  
Solutions to exercises  
Part II  
Chapters V to VIII

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## Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github. An overview of the material covered in the book can be found in the separate document "Synge overview.pdf".

## Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma_{mn}^r \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

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# Applications to Classical Mechanics

## 5.1 p153 - Exercise

If  $\mu^\alpha$  are the contravariant components of a unit vector in a surface  $S$ , show that  $\mu^\alpha f_\alpha$  is the physical component of acceleration in the direction tangent to  $S$  defined by  $\mu^\alpha$ .

As we are in an Euclidean space we can interpret  $a_{mn}\mu^\alpha f^\alpha$  as  $|\mu||f|\cos\theta$  with  $\theta$  the angle between the two vectors. As  $|\mu| = 1$  we have

$$a_{mn}\mu^\alpha f^\alpha = \mu^\alpha f_\alpha \tag{1}$$

$$= |f|\cos\theta \tag{2}$$

which is the projection of the vector  $f$  on the unit vector  $\mu$ .



## 5.2 p154 - Clarification to 5.226.

$$5.226. \quad \mathbf{v} \frac{d\mathbf{v}}{ds} = \mathbf{0}, \quad \bar{\kappa} \mathbf{v}^2 = \mathbf{0}$$

Assuming that the particle is not at rest  $v \neq 0$ , and therefore  $\bar{\kappa} = 0$ . ***Since this implies that the curve is a geodesic...***

The assertion in bold is a direct consequence

$$2.513. \quad \frac{\delta \frac{dx^r}{ds}}{\delta s} = 0$$

As in **5.233** we have  $\frac{\delta \lambda^\alpha}{\delta s} = \frac{\delta \frac{dx^\alpha}{ds}}{\delta s} = 0$ , the considered curve follows the geodesic curve.



### 5.3 p155 - Exercise

Show that in relativity the force 4-vector  $X^r$  lies along the first normal of the trajectory in space-time. Express the first curvature in terms of the proper mass  $m$  of the particle and the magnitude  $X$  of  $X^r$ .

Let us recall the first Frenet formula **2.705** without forgetting that the metric form is not positive-definite,

$$\frac{\delta \lambda^r}{\delta s} = \kappa \nu^r, \quad \epsilon_{(1)} \nu_n \nu^n = 1$$

As **5.299**

$$m \frac{\delta \lambda^r}{\delta s} = X^r$$

it is clear that  $X^r = m \kappa \nu^r$  and is collinear with the first normal.

$$X^r = m \kappa \nu^r \tag{1}$$

$$\times \quad a_{mr} X^m \quad \Rightarrow \quad \underbrace{a_{mr} X^m X^r}_{=(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2} = m \kappa \underbrace{a_{mr} \nu^m \nu^r}_{=\epsilon_{(1)}} \tag{2}$$

$$\Rightarrow \quad \kappa = \epsilon_{(1)} \frac{(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2}{m}$$





## 5.4 p156 - Clarification

Interpretation of

$$\mathbf{5.231.} \quad M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r$$

What do the  $M_{rs}$  represent?

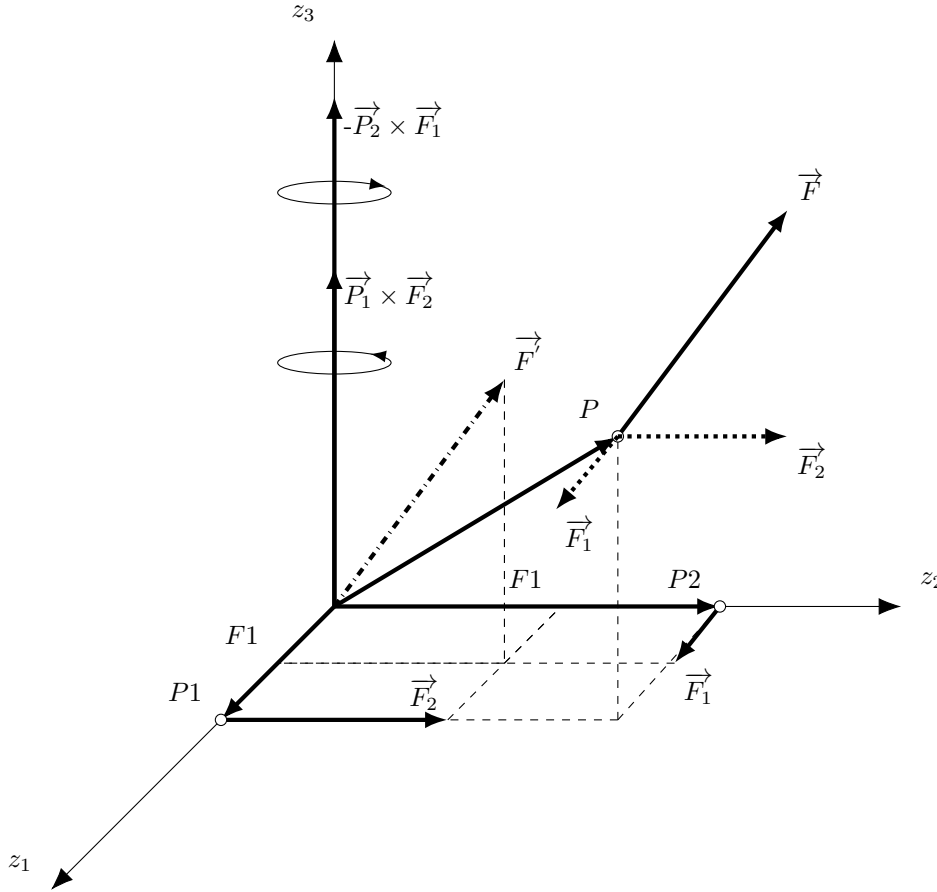


Figure 5.1: Interpretation of the tensor moment  $M_{12}$

Let's consider a mass point  $P$  on which a force  $\vec{F}$  is acting. The force has components  $(F_x, F_y, F_z)$  in the space  $V'_3$  (which is by the way not the space  $V_3$  of the considered mass point).

Let's investigate the element  $M_{12}$  of the *tensor moment*.

$P_1 F_2 \vec{e}_3$  is the vector product  $\vec{P}_1 \times \vec{F}_2$  and is as such the torque of the component  $F_2$  of  $\vec{F}$  acting on the mass point situated at  $P_1$ . The origin being fixed,  $\vec{F}_2$  tries to move  $P_1$ , clockwise along the  $z_3$  axis. The same is true for the component  $\vec{F}_1$  acting on the mass point situated at  $P_2$ , and is represented here by the vector  $-\vec{P}_2 \times \vec{F}_1$  ( $\vec{F}_1$  tries to move  $P_2$ , counter clockwise along the  $z_3$  axis). Hence,  $P_1 F_2 - P_2 F_1$  is the net force trying to move the point  $P$  along the  $z_3$  axis (i.e. in the plane  $\parallel$  with the  $z_3 = 0$  plane).



## 5.5 p156 - Clarification

$$\mathbf{5.234.} \quad \frac{dh_r}{dt} = M_r$$

$$h_r = m\epsilon_{rmn}z_mv_n \tag{1}$$

$$\Rightarrow \quad \frac{dh_r}{dt} = m\epsilon_{rmn} \frac{dz_m}{dt} v_n + m\epsilon_{rmn} z_m \frac{dv_n}{dt} \tag{2}$$

$$= m \underbrace{\epsilon_{rmn} v_m v_n}_{=0} + \underbrace{\epsilon_{rmn} z_m F_n}_{=M_r} \tag{3}$$

$$= M_r \tag{4}$$



## 5.6 p158-159 - Clarification

$$\mathbf{5.313.} \quad \omega_{rs} = -\omega_{sr}$$

From 5.310 and the vector character of  $v_r$  and  $z_r$  (for transformations which do not change the origin), **it follows that  $\omega_{rs}$  is a Cartesian tensor of second order.**

Be

$$v_r = -\omega_{rn} z_n \quad (1)$$

Considering orthogonal transformation in a flat space  $z'_m = A_{mr} z_r + B_m$  with  $B_m = 0$  as we consider only transformations which do not change the origin. Differentiation with the parameter  $t$  gives

$$v'_m = A_{mr} v_r \quad (2)$$

$$= -\omega_{rn} A_{mr} z_n \quad (3)$$

$$(4)$$

But  $z'_q = A_{qr} z_r \Rightarrow A_{qn} z'_q = A_{qn} A_{qr} z_r \Rightarrow A_{qn} z'_q = z_n$  Hence

$$v'_m = -\omega_{rn} A_{mr} z_n \quad (5)$$

$$= -\underbrace{\omega_{rn} A_{mr} A_{qn}}_{\stackrel{\text{def}}{=} \omega'_{mq}} z'_q \quad (6)$$

$$v'_m = -\omega'_{mq} z'_q \quad (7)$$



## 5.7 p159 - Exercise

Show that if a rigid body rotates about the point  $z_r = b_r$  as fixed point, the velocity of a general point of the body is given by

$$v_r = -\omega_{rm} (z_m - b_m)$$

By 5.302.:

$$\left(z_m^{(1)} - z_m^{(2)}\right) \left(dz_m^{(1)} - dz_m^{(2)}\right) = 0 \quad (1)$$

At the fixed point we have  $z_m^{(2)} = b_m$  and  $dz_m^{(2)} = 0$ , hence

$$\left(z_m^{(1)} - b_m\right) \left(dz_m^{(1)}\right) = 0 \quad (2)$$

$$\Rightarrow z_m^{(1)} dz_m^{(1)} = b_m dz_m^{(1)} \quad (3)$$

As this is true for any point of the rigid mass, expanding (1) and using (3) we get when dividing by  $dt$

$$\left(z_m^{(2)} - b_m\right) v_m^{(1)} + \left(z_m^{(1)} - b_m\right) v_m^{(2)} = 0 \quad (4)$$

Taking twice the partial derivative  $\frac{\partial^2}{\partial z_p^{(1)} \partial z_q^{(1)}}$  we get

$$\left(z_m^{(2)} - b_m\right) \frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \quad (5)$$

As this is true for any arbitrary point in the rigid body we get

$$\frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \quad (6)$$

$$\Rightarrow v_m = K_{mr} z_r + B_m \quad (7)$$

At the fixed point we have

$$K_{mr} b_r + B_m = 0 \quad (8)$$

Plugging this in (7)

$$v_m = K_{mr} (z_r - b_m) \quad (9)$$

Putting  $K_{mr} = -\omega_{mr}$  gives us indeed the asked expression.



## 5.8 p161 - Clarification

$$\mathbf{5.325.} \quad \Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p$$

and hence, since  $\Omega_{np}$  is arbitrary,

$$\mathbf{5.326.} \quad \sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

To be complete the following step should be inserted

$$\Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p \quad (1)$$

As  $\Omega_{np}$  is skew-symmetric:

$$- \Omega_{np} \sum (m f_p z_n) = - \Omega_{np} \sum F_p z_n \quad (2)$$

$$(1)+(2) \quad \Omega_{np} \sum m (f_n z_p - f_p z_n) = \Omega_{np} \sum (F_n z_p - F_p z_n) \quad (3)$$

and hence, since  $\Omega_{np}$  is arbitrary,

$$\mathbf{5.326.} \quad \sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$



## 5.9 p161 - Clarification

$$\begin{aligned} \mathbf{5.329.} \quad h_{np} &= \sum m (\omega_{nq} z_q z_p - \omega_{pq} z_q z_n) \\ &= J_{npqr} \omega_{rq} \end{aligned}$$

where

$$\mathbf{5.330.} \quad J_{npqr} = \sum m (\delta_{nr} z_q z_p - \delta_{pr} z_n z_q)$$

$$h_{np} = \sum m (\omega_{nq} z_q z_p - \omega_{pq} z_q z_n) \tag{1}$$

$$= \sum m (\omega_{rq} \delta_{rn} z_q z_p - \omega_{rq} \delta_{rp} z_q z_n) \tag{2}$$

$$= \omega_{rq} \sum m (\delta_{rn} z_q z_p - \delta_{rp} z_q z_n) \tag{3}$$

$$= J_{npqr} \omega_{rq} \tag{4}$$



## 5.10 p166 - Exercise

Deduce immediately from **5.420.** that the Coriolis force is perpendicular to the velocity.

$$G'_s = 2m\omega'_{sm}(S', S)v'_m(S') \quad (1)$$

$$\times v'_s(S') \quad : \quad G'_s v'_s(S') = m \left( \omega'_{sm}(S', S)v'_m(S')v'_s(S') + \omega'_{ms}(S', S)v'_m(S')v'_s(S') \right) \quad (2)$$

$$= 0 \quad \text{as } \omega'_{ms} \text{ is skew-symmetric} \quad (3)$$



## 5.11 p166 - Exercise

Show that if  $N = 3$  and  $\dot{\omega}'_r(S', S) = 0$ , then the centrifugal force may be written

$$\mathbf{5.422.} \quad C'_s = m\omega'_n(S', S)\omega'_n(S', S)z'_s - m\omega'_n(S', S)z'_n\omega'_s(S', S)$$

Deduce that  $C'_s$  is coplanar with the vectors  $\omega'_s(S', S)$  and  $z'_n$  and perpendicular to the former.

By **5.420.** with  $\dot{\omega}'_r(S', S) = 0$  and using **5.316.** ( $\omega'_{rs} = \epsilon_{rsn}\omega'_n$ )

$$C'_s = m\omega'_{sm}(S', S)\omega'_{nm}(S', S)z'_n \quad (1)$$

$$= m\epsilon_{smk}\omega'_k(S', S)\epsilon_{nmp}\omega'_p(S', S)z'_n \quad (2)$$

$$= m\epsilon_{msk}\epsilon_{mnp}\omega'_k(S', S)\omega'_p(S', S)z'_n \quad (3)$$

$$= m(\delta_{sn}\delta_{kp} - \delta_{sp}\delta_{kn})\omega'_k(S', S)\omega'_p(S', S)z'_n \quad (4)$$

$$= m\delta_{sn}\delta_{kp}\omega'_k(S', S)\omega'_p(S', S)z'_n - m\delta_{sp}\delta_{kn}\omega'_k(S', S)\omega'_p(S', S)z'_n \quad (5)$$

$$= m\omega'_p(S', S)\omega'_p(S', S)z'_s - m\omega'_n(S', S)\omega'_s(S', S)z'_n \quad (6)$$

To deduce that  $C'_s$  is coplanar with the vectors  $\omega'_s(S', S)$  and  $z'_n$  we calculate the mixed triple product

$$P = \epsilon_{spr}C'_s\omega'_p(S', S)z'_r \quad (7)$$

$$= m \underbrace{\epsilon_{spr}\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_p(S', S)z'_r}_{=0} - \underbrace{m\epsilon_{spr}\omega'_n(S', S)\omega'_s(S', S)z'_n\omega'_p(S', S)z'_r}_{=0} \quad (8)$$

$$= 0 \quad (9)$$

Both terms vanish: the first by the presence of the terms  $\epsilon_{spr}z'_s z'_r$  which cancel each other and for the second by the terms  $\epsilon_{spr}\omega'_s(S', S)\omega'_p(S', S)$ . As  $P = 0$ , the three vectors are coplanar.

We now calculate the inner product  $C'_s\omega'_s(S', S)$

$$P = m\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_s(S', S) - \underbrace{m\omega'_n(S', S)\omega'_s(S', S)z'_n\omega'_s(S', S)}_{\Leftrightarrow m\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_s(S', S)} \quad (10)$$

$$= 0 \quad (11)$$





## 5.12 p181 and p182 - Clarification Figures 13., 14. and 15.

There are several ways to get a homeomorphism of the configuration space of a rigid body with fixed point.

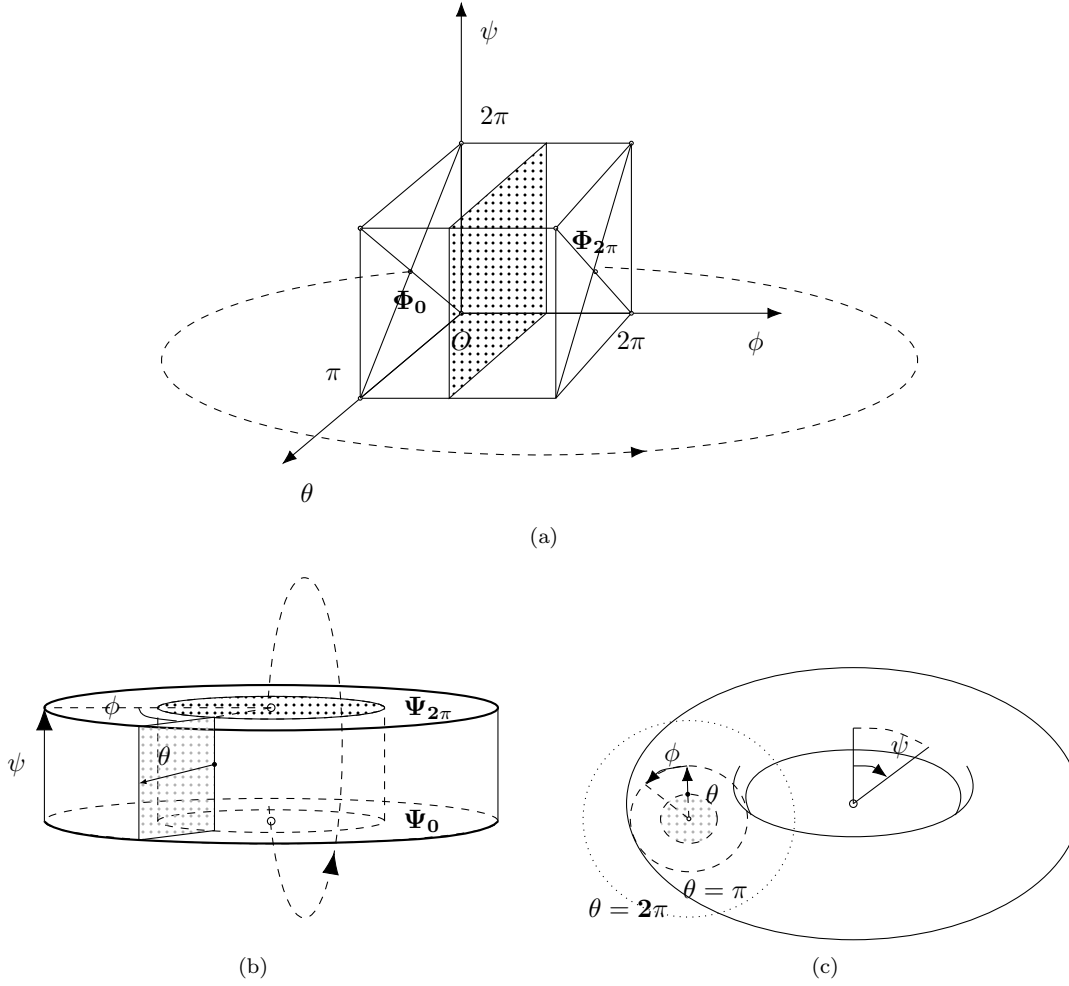


Figure 5.2: Homeomorphism of the configuration space of a rigid body with fixed point.

Consider figure 5.2(a). We can stretch like an accordion the cuboid along the  $\phi$  axis and bent it so that the planes  $\phi = 0$  and  $\phi = 2\pi$  join. We get (b), a torus with square sections. The dimension  $\phi$  is dealt with as a point  $P(\theta, \phi, \psi)$  in the configuration space returns to the same point when varying  $\phi$  to  $\phi + 2k\pi$ .

We can apply the same procedure of stretching and bending for the  $\psi$  dimension so that the planes  $\Psi = 0$  and  $\Psi = 2\pi$  join. We get (c), a torus-like object.

The only dimension left is  $\theta$  which our multi-dimensional crippled mind can't find a way to reshape this pseudo-torus so that when varying  $\theta$  we can come back to the same point as started.



### 5.13 p186 - Exercise 1

If a vector at the point with coordinates  $(1, 1, 1)$  in Euclidean 3-space has components  $(3, -1, 2)$ , find the contravariant, covariant and physical components in spherical polar coordinates.

The tensor  $T_n$  to consider is  $(3, -1, 2) - (1, 1, 1) = (2, -2, 1)$ .

The Jacobian matrix for the transformation  $z^n \rightarrow x^k$ , evaluated at the point  $(1, 1, 1)$  is

$$J_{(1,1,1)} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (2)$$

$$\Rightarrow \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'n} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \quad (4)$$

We have the metric tensor evaluated at  $(1, 1, 1)$

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (5)$$

$$\Rightarrow \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \quad (6)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \quad (7)$$

And the physical components

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \quad (8)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \quad (9)$$

Another way to find the physical components is to project orthogonally the tensor on the unit vectors of a local Cartesian coordinate system, oriented along the unit vectors  $\bar{e}_r, \bar{e}_\theta, \bar{e}_\phi$  corresponding to the vector  $P(1, 1, 1)$  with modulus  $|P| = \sqrt{3}$ . We have for the tensor  $T_n(2, -2, 1)$  with modulus  $|T_n| = 3$  as component along  $\bar{e}_r$ :

$$|T_n| \cos \alpha = |T_n| \frac{\langle T_n, P \rangle}{|T_n| |P|} \quad (10)$$

$$= |T_n| \frac{2 - 2 + 1}{|T_n| |P|} \quad (11)$$

$$= \frac{1}{\sqrt{3}} \quad (12)$$

For the component along  $\bar{e}_\theta$  we first have to determine the vector  $\bar{e}_\theta$ . As first equation we have the

orthogonality condition with  $\bar{e}_r$  and putting  $\bar{e}_\theta = (a, b, c)$ , get  $\langle \bar{e}_r, \bar{e}_\theta \rangle = a + b + c = 0$ . As  $\bar{e}_\theta$  lies in the plane  $(1, 1, 0) - (0, 0, 0) - (0, 0, 1)$  we can put  $a = b$  and get  $\bar{e}_\theta = \frac{1}{\sqrt{6}}(1, 1, -2)$  and get for the tensor  $T_n(2, -2, 1)$  as component along  $\bar{e}_\theta$ :

$$|T_n| \cos \beta = |T_n| \frac{\langle T_n, \bar{e}_\theta \rangle}{|T_n|} \quad (13)$$

$$= |T_n| \frac{2 - 2 - 2}{|T_n| \sqrt{6}} \quad (14)$$

$$= -\frac{\sqrt{2}}{\sqrt{3}} \quad (15)$$

For the component along  $\bar{e}_\phi$  we first have to determine the vector  $\bar{e}_\phi$ . As first equation we have the orthogonality condition with the pair  $\bar{e}_r, \bar{e}_\theta$  and get  $\bar{e}_\phi = \bar{e}_r \times \bar{e}_\theta = \frac{1}{\sqrt{3}\sqrt{6}}(-3, 3, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ . For the tensor  $T_n(2, -2, 1)$  as component along  $\bar{e}_\phi$ :

$$|T_n| \cos \gamma = |T_n| \frac{\langle T_n, \bar{e}_\phi \rangle}{|T_n|} \quad (16)$$

$$= |T_n| \frac{-2 - 2}{|T_n| \sqrt{2}} \quad (17)$$

$$= -\frac{4}{\sqrt{2}} \quad (18)$$

$$= -2\sqrt{2} \quad (19)$$

giving

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} 1 \\ \frac{\sqrt{3}}{\sqrt{2}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \quad (20)$$

as in (9).

◆

## 5.14 p186 - Exercise 2

In cylindrical coordinates  $(r, \phi, z)$  in Euclidean 3-space, a vector field is such that the vector at each point points along the parametric line of  $\phi$ , in the sense of  $\phi$  increasing, and its magnitude is  $kr$ , where  $k$  is a constant. Find the contravariant, covariant and physical components of this vector field.

We can work backwards, with the physical components as starting point. Indeed, at a point  $P(r, \phi, z)$  the tensor of this vector field will have  $(0, kr, 0)$  as physical components in the cylindrical coordinates  $(r, \phi, z)$  system.

We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

Giving

$$\begin{cases} X_1 = h_1 X_1^{phys.} = 0 \\ X_2 = h_2 X_2^{phys.} = kr^2 \\ X_3 = h_3 X_3^{phys.} = 0 \end{cases} \quad (2)$$

and

$$\begin{cases} X^1 = \frac{X_1^{phys.}}{h_1} = 0 \\ X^2 = \frac{X_2^{phys.}}{h_2} = k \\ X^3 = \frac{X_3^{phys.}}{h_3} = 0 \end{cases} \quad (3)$$



### 5.15 p186 - Exercise 3

Find the physical components of velocity and acceleration along the parametric lines of cylindrical coordinates in terms of the and their derivatives with respect to time.

We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

and the contravariant velocities

$$\begin{cases} v^1 = \frac{dr}{dt} \\ v^2 = \frac{d\phi}{dt} \\ v^3 = \frac{dz}{dt} \end{cases} \quad (2)$$

giving by  $v_K^{phys.} = h_K v^K$

$$\begin{cases} v_r = \frac{dr}{dt} \\ v_\phi = r \frac{d\phi}{dt} \\ v_z = \frac{dz}{dt} \end{cases} \quad (3)$$

For the acceleration using  $f^r = \frac{\delta v^r}{\delta t}$  and the Christoffel symbols being

$$\begin{cases} \Gamma_{nk}^m = 0 \quad \forall \quad (nk) \neq (r, \theta), (\theta, \theta) \\ \Gamma_{r\theta}^\theta = \frac{1}{r} \quad \text{and} \quad \Gamma_{\theta\theta}^r = -r \end{cases} \quad (4)$$

we have

$$\left\{ \begin{array}{l} f^1 = \frac{dv^1}{dt} - \underbrace{r v^2 \frac{dx^2}{dt}}_{=(v^2)^2} \\ f^2 = \frac{dv^2}{dt} + \underbrace{\frac{1}{r} v^1 \frac{dx^2}{dt} + \frac{1}{r} v^2 \frac{dx^2 1}{dt}}_{=\frac{2}{r} v^1 v^2} \\ f^3 = \frac{dv^3}{dt} \end{array} \right. \quad (5)$$

giving by  $f_K^{phys.} = h_K f^K$

$$\left\{ \begin{array}{l} f_r = \frac{dv^1}{dt} - r (v^2)^2 \\ f_{phi} = r \frac{dv^2}{dt} + r \frac{2}{r} v^1 v^2 \\ f_z = \frac{dv^3}{dt} \end{array} \right. \quad (6)$$

$$\Rightarrow \left\{ \begin{array}{l} f_r = \frac{d^2 r}{dt^2} - r \left( \frac{d\phi}{dt} \right)^2 \\ f_{phi} = r \frac{d^2 \phi}{dt^2} + 2 \frac{dr}{dt} \frac{d\phi}{dt} \\ f_z = \frac{d^2 z}{dt^2} \end{array} \right. \quad (7)$$

◆

## 5.16 p186 - Exercise 4

A particle moves on a sphere under the action of gravity. Find the contravariant and covariant components of the force, using colatitude and azimuth, and write down the equation of motion.

We determine first the physical components of the force.

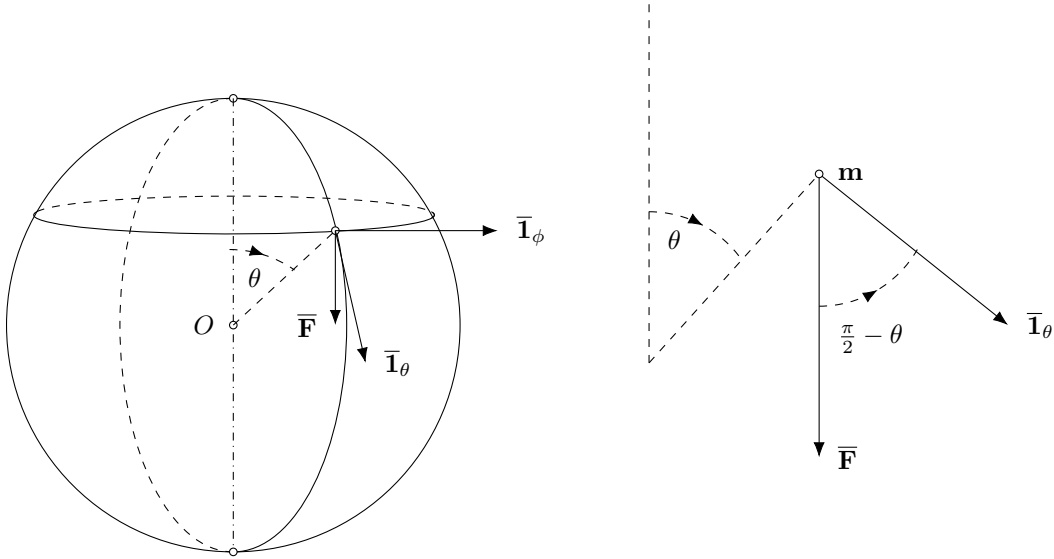


Figure 5.3: Physical components of the gravitational force tensor acting on a mass  $\mathbf{m}$  on a sphere

We note first that the unit vector  $\bar{\mathbf{l}}_\phi$  is perpendicular to the plane formed by the vectors  $\bar{\mathbf{l}}_\theta, \bar{\mathbf{F}}$  and so the force has no components projected on this vector. The vector  $\bar{\mathbf{F}}$  is parallel with the axis of reference of the sphere with radius  $R$  and so the physical components become

$$\Rightarrow \begin{cases} F_\phi^{phys} = 0 \\ F_\theta^{phys} = mg \sin \theta \end{cases} \quad (1)$$

$$\Rightarrow \begin{cases} F_\phi = 0 & F_\phi = 0 \\ F^\theta = \frac{1}{R} mg \sin \theta & F_\theta = R mg \sin \theta \end{cases} \quad (2)$$

We use equation 5.212.

$$\begin{cases} \frac{d}{dt} \frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ T = \frac{1}{2} m a_{pq} \dot{x}^p \dot{x}^q, \quad \dot{x}^s = \frac{dx^s}{dt} \end{cases} \quad (3)$$



with for our case

$$T = \frac{1}{2}mR^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) \quad (4)$$

and get the set of equation of motion (the second column gives the dimensional analysis as a check for consistency)

$$\left\{ \begin{array}{l} \frac{\ddot{\phi}}{\dot{\phi}} = -2 \cot \theta \dot{\theta} \quad : \quad \frac{[T]^{-2}}{[T]^{-1}} \cong [T]^{-1} \\ \ddot{\theta} - \left( \dot{\phi} \right)^2 \sin \theta \cos \theta = \frac{g}{R} \sin \theta \quad : \quad [T]^{-2} + ([T]^{-1})^2 \cong \frac{[L][T]^{-2}}{[L]} \end{array} \right. \quad (5)$$

Let's check the special case when  $\dot{\phi} = 0$ .

The first equation can be rewritten and gives of course  $\phi = C$  while the second equation becomes

$$\ddot{\theta} = \frac{g}{R} \sin \theta$$

which is similar to the equation of the simple gravity pendulum.



## 5.17 p186 - Exercise 5

Consider the motion of a particle on a smooth torus under no forces except normal reaction. The geometrical line element may be written

$$ds^2 = (a - b \cos \theta)^2 d\phi^2 + b^2 d\theta^2$$

where  $\phi$  is an azimuthal angle and  $\theta$  an angular displacement from the equatorial plane. Show that the path of a particle satisfies the following two differential equations in which  $h$  is a constant

$$(a) \quad (a - b \cos \theta)^2 \frac{d\phi}{ds} = h$$

$$(b) \quad b^2 \left( \frac{d\theta}{d\phi} \right)^2 = \frac{(a - b \cos \theta)^4}{h^2} - (a - b \cos \theta)^2$$

We use equation **5.212**.

$$\begin{cases} \frac{d}{dt} \frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ T = \frac{1}{2} m a_{pq} \dot{x}^p \dot{x}^q, \quad \dot{x}^s = \frac{dx^s}{dt} \end{cases} \quad (1)$$

with for our case

$$T = \frac{1}{2} m \left( b^2 \dot{\theta}^2 + (a - b \cos \theta)^2 \dot{\phi}^2 \right) \quad (2)$$

giving

$$\begin{cases} \frac{\partial T}{\partial \dot{\phi}} = m (a - b \cos \theta)^2 \dot{\phi} & \frac{\partial T}{\partial \phi} = 0 \\ \frac{\partial T}{\partial \dot{\theta}} = m b^2 \dot{\theta} & \frac{\partial T}{\partial \theta} = m b (a - b \cos \theta) \dot{\phi}^2 \sin \theta \end{cases} \quad (3)$$

$$\Rightarrow \begin{cases} (a - b \cos \theta)^2 \ddot{\phi} + 2b (a - b \cos \theta) \dot{\theta} \dot{\phi} \sin \theta = 0 \\ b^2 \ddot{\theta} - b (a - b \cos \theta) \dot{\phi}^2 \sin \theta = 0 \end{cases} \quad (4)$$

$$\Rightarrow \begin{cases} (a - b \cos \theta) \ddot{\phi} = -2b \dot{\theta} \dot{\phi} \sin \theta \\ b^2 \ddot{\theta} - b (a - b \cos \theta) \dot{\phi}^2 \sin \theta = 0 \end{cases} \quad (5)$$

In the first equation, put  $y \equiv \dot{\phi}$  giving for the first equation:

$$\frac{dy}{y} = -2b \frac{\sin \theta d\theta}{(a - b \cos \theta)} \quad (6)$$

$$\Leftrightarrow \frac{dy}{y} = -2 \frac{d(a - b \cos \theta)}{(a - b \cos \theta)} \quad (7)$$

$$\Rightarrow \log y = -2 \log(a - b \cos \theta) + \log C \quad (8)$$

$$\Rightarrow \dot{\phi} = C (a - b \cos \theta)^{-2} \quad (9)$$

Note that  $\dot{\phi}$  is a time derivative. But as we are on a geodesic, **5.226.** stands and so  $v$  is constant as  $\frac{dv}{ds} = 0$ . Using  $v = \frac{ds}{dt}$ , (9) can be written as

$$(a - b \cos \theta)^2 \frac{d\phi}{dt} = C \quad (10)$$

$$\Leftrightarrow (a - b \cos \theta)^2 \frac{d\phi}{ds} \underbrace{\frac{ds}{dt}}_{=v} = C \quad (11)$$

$$\Leftrightarrow (a - b \cos \theta)^2 \frac{d\phi}{ds} = h \quad \text{with } h = \frac{C}{v} \quad (12)$$

We don't use the second equation in (5) but the line element equation instead

$$ds^2 = (a - b \cos \theta)^2 d\phi^2 + b^2 d\theta^2 \quad (13)$$

$$\Rightarrow \left( \frac{ds}{d\phi} \right)^2 = (a - b \cos \theta)^2 + b^2 \left( \frac{d\theta}{d\phi} \right)^2 \quad (14)$$

$$\Rightarrow b^2 \left( \frac{d\theta}{d\phi} \right)^2 = \left( \frac{d\phi}{ds} \right)^{-2} - (a - b \cos \theta)^2 \quad (15)$$

$$(12) \quad : \quad b^2 \left( \frac{d\theta}{d\phi} \right)^2 = \frac{(a - b \cos \theta)^4}{h^2} - (a - b \cos \theta)^2 \quad (16)$$

