Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github.

Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \quad \text{Christoffel symbol of the second kind}$$

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Curvature of space

1.1 p109 - Exercise 6

For an orthogonal coordinates system in a V_2 we have

$$ds^2 = a_{11}\left(dx^1\right)^2 + a_{22}\left(dx^2\right)^2$$

Show that

$$\frac{1}{a}R_{1212} = -\frac{1}{2}\frac{1}{\sqrt{a}}\left[\partial_1\left(\frac{1}{\sqrt{a}}\partial_1 a_{22}\right) + \partial_2\left(\frac{1}{\sqrt{a}}\partial_2 a_{11}\right)\right]$$

We have

$$(a_{mn}) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \quad (a_{mn}) = \frac{1}{a}\begin{pmatrix} a_{22} & 0 \\ 0 & a_{11} \end{pmatrix} \quad a = a_{11}a_{22} \tag{1}$$

We have also

$$R = -\frac{2}{a}R_{1212} \tag{2}$$

$$R = a^{mn}R_{mn} \quad \Rightarrow \quad R = a^{11}R_{11} + a^{22}R_{22} \tag{3}$$

Looking at the pattern generated by equations (2) and (3) suggests that using these equations could lead to the proposed equation. Let's have a try ...

$$\begin{cases} \Gamma^1_{11} = \frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11} & \Gamma^1_{22} = -\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22} \\ \\ \Gamma^2_{11} = -\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11} & \Gamma^2_{22} = \frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22} \\ \\ \Gamma^1_{12} = \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} & \Gamma^2_{12} = \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} \end{cases} \tag{4}$$

$$3.205. \quad \Rightarrow \quad R_{rm} = \frac{1}{2}\partial_{rm}\log a - \frac{1}{2}\Gamma^p_{rm}\partial_p \log a - \partial_n\Gamma^n_{rm} + \Gamma^p_{rn}\Gamma^n_{pm} \tag{5}$$

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\Gamma^1_{11}\partial_1\log a - \frac{1}{2}\Gamma^2_{11}\partial_2\log a \\ -\partial_1\Gamma^1_{11} - \partial_2\Gamma^2_{11} + \\ \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^2_{21} \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a - \frac{1}{2}\Gamma^1_{22}\partial_1\log a - \frac{1}{2}\Gamma^2_{22}\partial_2\log a \\ -\partial_1\Gamma^1_{22} - \partial_2\Gamma^2_{22} + \\ \Gamma^1_{21}\Gamma^1_{12} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{21}\Gamma^1_{22} + \Gamma^2_{22}\Gamma^2_{22} \end{cases} \tag{6}$$

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}\partial_1 \log a - \frac{1}{2}(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11})\partial_2 \log a \\ -\partial_1(\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}) - \partial_2(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11}) + \\ \frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11} + \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11}(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11}) + \\ (-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11})\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} + \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22}\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a - \frac{1}{2}(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22})\partial_1 \log a - \frac{1}{2}\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}\partial_2 \log a \\ -\partial_1(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22}) - \partial_2(\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}) + \\ \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11}\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} + (-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22})\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} + \\ \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22}(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22}) + \frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22} \\ \end{cases} \tag{7}$$

Simplifying the notational burden by replacing a_{11} by γ and a_{22} by η:

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\partial_1\gamma\partial_1 \log a + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\partial_2\gamma\partial_2 \log a \\ -\frac{1}{2}\partial_1(\frac{1}{\gamma}\partial_1\gamma) + \frac{1}{2}\partial_2(\frac{1}{\eta}\partial_2\gamma) \\ +\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_1\gamma\partial_1\gamma - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\gamma \\ -\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\gamma + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_1\eta\partial_1\eta \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a + \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\partial_1\eta\partial_1 \log a - \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\partial_2\eta\partial_2 \log a \\ +\frac{1}{2}\partial_1(\frac{1}{\gamma}\partial_1\eta) - \frac{1}{2}\partial_2(\frac{1}{\eta}\partial_2\eta) \\ +\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_2\gamma\partial_2\gamma - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\eta\partial_1\eta \\ -\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\eta\partial_1\eta + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_2\eta\partial_2\eta \\ \end{cases} \tag{8}$$

Noting that $\partial_{ii}\log a = \partial_i\left(\frac{1}{a_{11}}\partial_i a_{11}\right) + \partial_i\left(\frac{1}{a_{22}}\partial_i a_{22}\right)$ and $\partial_i \log a = \frac{1}{a_{11}}\partial_i a_{11} + \frac{1}{a_{22}}\partial_i a_{22} \quad (i = 1, 2)$, we get:

$$
\left|
\begin{array}{c|c}
2R_{11} = & 2R_{22} = \\
\underbrace{\partial_1\left(\frac{1}{\gamma}\partial_1\gamma\right)}_{*} + \partial_1\left(\tfrac{1}{\eta}\partial_1\eta\right) & \partial_2\left(\tfrac{1}{\gamma}\partial_2\gamma\right) + \underbrace{\partial_2\left(\frac{1}{\eta}\partial_2\eta\right)}_{*} \\
-\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_1\gamma)^2}_{-} - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_1\gamma\partial_1\eta & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}\partial_1\gamma\partial_1\eta + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_1\eta)^2 \\
 & -\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta - \underbrace{\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_2\eta)^2}_{-} \\
+\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_2\gamma)^2 + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta & \\
-\underbrace{\partial_1(\frac{1}{\gamma}\partial_1\gamma)}_{*} + \partial_2(\tfrac{1}{\eta}\partial_2\gamma) & +\partial_1(\tfrac{1}{\gamma}\partial_1\eta) - \underbrace{\partial_2(\frac{1}{\eta}\partial_2\eta)}_{*} \\
+\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_1\gamma)^2}_{-} - \underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2}_{+} & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}(\partial_2\gamma)^2 - \underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2}_{+} \\
-\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2}_{+} + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}(\partial_1\eta)^2 & -\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2}_{+} + \underbrace{\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_2\eta)^2}_{-}
\end{array}
\right| \tag{9}
$$

$$
\Rightarrow
\left|
\begin{array}{c|c}
2R_{11} = & 2R_{22} = \\
\partial_1\left(\tfrac{1}{\eta}\partial_1\eta\right) + \partial_2(\tfrac{1}{\eta}\partial_2\gamma) & \partial_1(\tfrac{1}{\gamma}\partial_1\eta) + \partial_2\left(\tfrac{1}{\gamma}\partial_2\gamma\right) \\
+\tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}(\partial_1\eta)^2 - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_2\gamma)^2 & -\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}(\partial_1\eta)^2 + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}(\partial_2\gamma)^2 \\
-\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_1\gamma\partial_1\eta + \tfrac{1}{2}\tfrac{1}{\eta}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta & +\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\gamma}\partial_1\gamma\partial_1\eta - \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{\eta}\partial_2\gamma\partial_2\eta
\end{array}
\right| \tag{10}
$$

Be $R = \frac{1}{\gamma}R_{11} + \frac{1}{\eta}R_{22}$, all first order derivatives vanish and we get,

$$
\frac{1}{\gamma}R_{11} + \frac{1}{\eta}R_{22} = \frac{1}{2}\left[\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)\right] + \frac{1}{2}\left[\frac{1}{\gamma}\partial_2(\frac{1}{\gamma}\partial_2\gamma) + \frac{1}{\eta}\partial_2(\frac{1}{\eta}\partial_2\gamma)\right] \tag{11}
$$

We further simplify this expression. Looking at the symmetry we only explicit the calculations for the first terms in ∂_1.

$$\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta)+\frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta) = \frac{1}{\eta}\partial_1(\frac{1}{\sqrt{\gamma}}\frac{1}{\sqrt{\gamma}}\frac{\sqrt{\eta}}{\sqrt{\eta}}\partial_1\eta)+\frac{1}{\gamma}\partial_1(\frac{1}{\sqrt{\eta}}\frac{1}{\sqrt{\eta}}\frac{\sqrt{\gamma}}{\sqrt{\gamma}}\partial_1\eta) \tag{12}$$

$$= \frac{1}{\eta}\partial_1\left[\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}\frac{1}{\sqrt{a}}\partial_1\eta\right]+\frac{1}{\gamma}\partial_1\left[\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{a}}\partial_1\eta\right] \tag{13}$$

$$= \begin{cases} \underbrace{\frac{1}{\eta}\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}}_{=\frac{1}{\sqrt{a}}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1\eta\right]+\underbrace{\frac{1}{\gamma}\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}}_{=\frac{1}{\sqrt{a}}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1\eta\right] \\ +\frac{1}{\sqrt{a}}\partial_1\eta\underbrace{\left[\frac{1}{\eta}\partial_1\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}+\frac{1}{\gamma}\partial_1\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}\right]}_{=0} \end{cases} \tag{14}$$

$$= 2\frac{1}{\sqrt{a}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1 a_{22}\right] \tag{15}$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta)+\frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)\right] = \frac{1}{\sqrt{a}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1 a_{22}\right] \tag{16}$$

Using (16) and the same calculations for the terms in ∂_2 and using (2) and (3) we get

$$\frac{1}{a}R_{1212} = -\frac{1}{2}\frac{1}{\sqrt{a}}\left[\partial_1\left(\frac{1}{\sqrt{a}}\partial_1 a_{22}\right)+\partial_2\left(\frac{1}{\sqrt{a}}\partial_2 a_{11}\right)\right]$$