# Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

Part II

Chapters V to VIII

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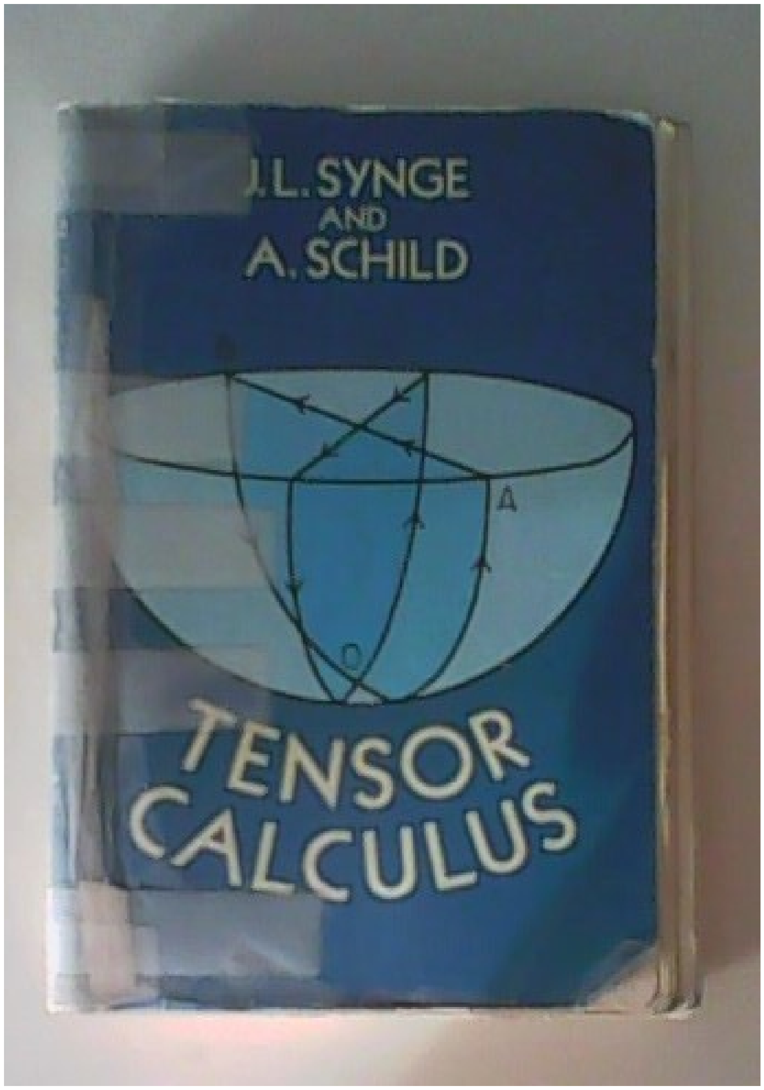



Figure 1: My copy, falling apart...

## Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github. An overview of the material covered in the book can be found in the separate document "Synge overview.pdf".

## Some notation conventions

| | |
|---|---|
| † | means that the exercise has only been solved partially or contains i.m.o. a doubtful step |
| †† | means that the exercise has not been solved as it should. |
| ♦ | end of an exercise or proof. |
| ◊ | end of Lemma or sub-task of an exercise. |

**As a rule, I followed the notation used in the book, except some which where easier to type in Latex.**

$\partial_r \quad \equiv \frac{\partial}{\partial x^r}$

$\partial^2_{rs} \quad \equiv \frac{\partial^2}{\partial x^r \partial x^s}$

$\Gamma^r_{mn} \quad \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix}$ Christoffel symbol of the second kind

# Contents

# List of Figures

# Applications to Classical Mechanics

## 5.1 p153 - Exercise

If $\mu^\alpha$ are the contra-variant components of a unit vector in a surface $S$, show that $\mu^\alpha f_\alpha$ is the physical component of acceleration in the direction tangent to $S$ defined by $\mu^\alpha$.

As we are in an Euclidean space we can interpret $a_{mn}\mu^\alpha f^\alpha$ as $|\mu|\,|f|\cos\theta$ with $\theta$ the angle between the two vectors. As $|\mu| = 1$ we have

$$a_{mn}\mu^\alpha f^\alpha = \mu^\alpha f_\alpha \quad (1)$$

$$= |f|\cos\theta \quad (2)$$

which is the projection of the vector $f$ on the unit vector $\mu$.

♦

## 5.2 p154 - Clarification to 5.226.

5.226. $$\mathbf{v}\frac{\mathbf{dv}}{\mathbf{ds}} = \mathbf{0}, \quad \overline{\kappa}\mathbf{v^2} = \mathbf{0}$$

Assuming that the particle is not at rest $v \neq 0$, and therefore $\overline{\kappa} = 0$. ***Since this implies that the curve is a geodesic***...

The assertion in bold is a direct consequence

2.513. $$\frac{\delta \frac{dx^r}{ds}}{\delta s} = 0$$

As in **5.233** we have $\frac{\delta\lambda^\alpha}{\delta s} = \frac{\delta \frac{dx^\alpha}{ds}}{\delta s} = 0$, the considered curve follows the geodesic curve.

♦

## 5.3 p155 - Exercise

Show that in relativity the force 4-vector $X^r$ lies along the first normal of the trajectory in space-time. Express the first curvature in terms of the proper mass $m$ of the particle and the magnitude $X$ of $X^r$.

Let us recall the first Frenet formula **2.705** without forgetting that the metric form is not positive-definite,

$$\frac{\delta\lambda^r}{\delta s} = \kappa\nu^r, \quad \epsilon_{(1)}\nu_n\nu^n = 1$$

As **5.299**

$$m\frac{\delta\lambda^r}{\delta s} = X^r$$

it is clear that $X^r = m\kappa\nu^r$ and is collinear with the first normal.

$$X^r = m\kappa\nu^r \tag{1}$$

$$\times \quad a_{mr}X^m \quad \Rightarrow \quad \underbrace{a_{mr}X^mX^r}_{=(X^1)^2+(X^2)^2+(X^3)^2-(X^4)^2} = m\kappa\underbrace{a_{mr}\nu^m\nu^r}_{=\epsilon_{(1)}} \tag{2}$$

$$\Rightarrow \qquad \kappa = \epsilon_{(1)}\frac{\left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2 - \left(X^4\right)^2}{m}$$

♦

## 5.4 p156 - Clarification to 5.231

Interpretation of

$$\textbf{5.231}. \qquad M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r$$

What do the $M_{rs}$ represent?

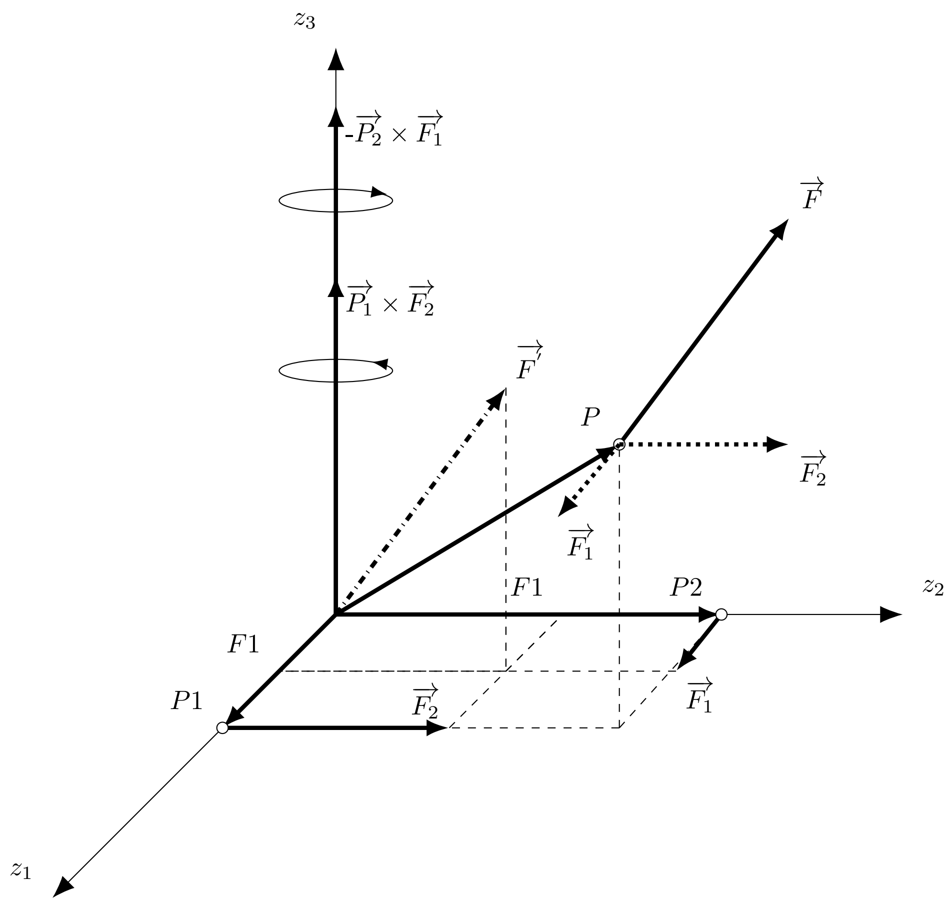



Figure 5.1: Interpretation of the tensor moment $M_{12}$

Let's consider a mass point $P$ on which a force $\overrightarrow{F}$ is acting. The force has components $(F_x, F_y, F_z)$ in the space $V_3'$ (which is by the way not the space $V_3$ of the considered mass point).
Let's investigate the element $M_{12}$ of the *tensor moment*.
$P_1 F_2 \overrightarrow{e_3}$ is the vector product $\overrightarrow{P_1} \times \overrightarrow{F_2}$ and is as such the torque of the component $F_2$ of $\overrightarrow{F}$ acting on the mass point situated at $P_1$. The origin being fixed, $\overrightarrow{F_2}$ tries to move $P_1$, clockwise along the $z_3$ axis. The same is true for the component $\overrightarrow{F_1}$ acting on the mass point situated at $P_2$, and is represented here by the vector $-\overrightarrow{P_2} \times \overrightarrow{F_1}$ ($\overrightarrow{F_1}$ tries to move $P_2$, counter clockwise along the $z_3$ axis). Hence, $P_1F_2 - P_2F_1$ is the net force trying to move the point $P$ along the $z_3$ axis (i.e. in the plane $\parallel$ with the $z_3 = 0$ plane).

♦

## 5.5 p156 - Clarification to 5.234

**5.234.** $$\frac{dh_r}{dt} = M_r$$

$$h_r = m\epsilon_{rmn} z_m v_n \tag{1}$$

$$\Rightarrow \quad \frac{dh_r}{dt} = m\epsilon_{rmn}\frac{dz_m}{dt}v_n + m\epsilon_{rmn} z_m \frac{dv_n}{dt} \tag{2}$$

$$= m\underbrace{\epsilon_{rmn} v_m v_n}_{=0} + \underbrace{\epsilon_{rmn} z_m F_n}_{=M_r} \tag{3}$$

$$= M_r \tag{4}$$

♦

## 5.6 p158-159 - Clarification to 5.313

**5.313.** $\omega_{rs} = -\omega_{sr}$

From 5.310 and the vector character of $v_r$ and $z_r$ (for transformations which do not change the origin), **it follows that $\omega_{rs}$ is a Cartesian tensor of second order**.

Be

$$v_r = -\omega_{rn} z_n \tag{1}$$

Considering orthogonal transformation in a flat space $z_m^{'} = A_{mr} z_r + B_m$ with $B_m = 0$ as we consider only transformations which do not change the origin. Differentiation with the parameter $t$ gives

$$\begin{aligned} v_m^{'} &= A_{mr} v_r &(2)\\ &= -\omega_{rn} A_{mr} z_n &(3)\\ & &(4) \end{aligned}$$

But $z_q^{'} = A_{qr} z_r \quad \Rightarrow \quad A_{qn} z_q^{'} = A_{qn} A_{qr} z_r \quad \Rightarrow \quad A_{qn} z_q^{'} = z_n$ Hence

$$\begin{aligned} v_m^{'} &= -\omega_{rn} A_{mr} z_n &(5)\\ &= -\underbrace{\omega_{rn} A_{mr} A_{qn}}_{\stackrel{\text{def}}{=} \omega_{mq}^{'}} z_q^{'} &(6)\\ v_m^{'} &= -\omega_{mq}^{'} z_q^{'} &(7) \end{aligned}$$

♦

## 5.7 p159 - Exercise

Show that if a rigid body rotates about the point $z_r = b_r$ as fixed point, the velocity of a general point of the body is given by

$$v_r = -\omega_{rm}\left(z_m - b_m\right)$$

By **5.302**.:

$$\left(z_m^{(1)} - z_m^{(2)}\right)\left(dz_m^{(1)} - dz_m^{(2)}\right) = 0 \tag{1}$$

At the fixed point we have $z_m^{(2)} = b_m$ and $dz_m^{(2)} = 0$, hence

$$\left(z_m^{(1)} - b_m\right)\left(dz_m^{(1)}\right) = 0 \tag{2}$$

$$\Rightarrow \qquad z_m^{(1)} dz_m^{(1)} = b_m dz_m^{(1)} \tag{3}$$

As this is true for any point of the rigid mass, expanding (1) and using (3) we get when dividing by $dt$

$$\left(z_m^{(2)} - b_m\right) v_m^{(1)} + \left(z_m^{(1)} - b_m\right) v_m^{(2)} = 0 \tag{4}$$

Taking twice the partial derivative $\frac{\partial^2}{\partial z_p^{(1)} \partial z_q^{(1)}}$ we get

$$\left(z_m^{(2)} - b_m\right) \frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \tag{5}$$

As this is true for any arbitrary point in the rigid body we get

$$\frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \tag{6}$$

$$\Rightarrow \qquad v_m = K_{mr} z_r + B_m \tag{7}$$

At the fixed point we have

$$K_{mr} b_r + B_m = 0 \tag{8}$$

Plugging this in (7)

$$v_m = K_{mr}\left(z_r - b_m\right) \tag{9}$$

Putting $K_{mr} = -\omega_{mr}$ gives us indeed the asked expression.

♦

## 5.8 p161 - Clarification to 5.325 and 5.326

**5.325.** $$\Omega_{np}\sum(mf_n z_p) = \Omega_{np}\sum F_n z_p$$

and hence, since $\Omega_{np}$ is arbitrary,

**5.326.** $$\sum m\,(f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

To be complete the following step should be inserted

$$\Omega_{np}\sum(mf_n z_p) = \Omega_{np}\sum F_n z_p \tag{1}$$

As $\Omega_{np}$ is skew-symmetric: $$-\,\Omega_{np}\sum(mf_p z_n) = -\Omega_{np}\sum F_p z_n \tag{2}$$

(1)+(2) $$\Omega_{np}\sum m\,(f_n z_p - f_p z_n) = \Omega_{np}\sum (F_n z_p - F_p z_n) \tag{3}$$

and hence, since $\Omega_{np}$ is arbitrary,

**5.326.** $$\sum m\,(f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

♦

## 5.9 p161 - Clarification to 5.329 and 5.330

**5.329**.
$$\begin{aligned} h_{np} &= \sum m \left(\omega_{nq} z_q z_p - \omega_{pq} z_q z_n\right) \\ &= J_{npqr} \omega_{rq} \end{aligned}$$

where

**5.330**.
$$J_{npqr} = \sum m \left(\delta_{nr} z_q z_p - \delta_{pr} z_n z_q\right)$$

$$\begin{aligned} h_{np} &= \sum m \left(\omega_{nq} z_q z_p - \omega_{pq} z_q z_n\right) && (1) \\ &= \sum m \left(\omega_{rq} \delta_{rn} z_q z_p - \omega_{rq} \delta_{rp} z_q z_n\right) && (2) \\ &= \omega_{rq} \sum m \left(\delta_{rn} z_q z_p - \delta_{rp} z_q z_n\right) && (3) \\ &= J_{npqr} \omega_{rq} && (4) \end{aligned}$$

♦

## 5.10 p166 - Exercise

Deduce immediately from **5.420**. that the Coriolis force is perpendicular to the velocity.

$$G_s^{'} = 2m\omega_{sm}^{'}(S^{'}, S)v_m^{'}(S^{'}) \tag{1}$$

$$\times v_s^{'}(S^{'}) \quad : \qquad G_s^{'} v_s^{'}(S^{'}) = m\left(\omega_{sm}^{'}(S^{'}, S)v_m^{'}(S^{'})v_s^{'}(S^{'}) + \omega_{ms}^{'}(S^{'}, S)v_m^{'}(S^{'})v_s^{'}(S^{'})\right) \tag{2}$$

$$= 0 \quad \text{as } \omega_{ms}^{'} \text{ is skew-symmetric} \tag{3}$$

♦

## 5.11 p166 - Exercise

Show that if $N = 3$ and $\dot{\omega}_r^{'}(S^{'}, S) = 0$, then the centrifugal force may be written

$$\textbf{5.422}. \qquad C_s^{'} = m\omega_n^{'}(S^{'}, S)\omega_n^{'}(S^{'}, S)z_s^{'} - m\omega_n^{'}(S^{'}, S)z_n^{'}\omega_s^{'}(S^{'}, S)$$

Deduce that $C_s^{'}$ is coplanar with the vectors $\omega_s^{'}(S^{'}, S)$ and $z_n^{'}$ and perpendicular to the former.

By **5.420**. with $\dot{\omega}_r^{'}(S^{'}, S) = 0$ and using **5.316**. $\quad (\omega_{rs}^{'} = \epsilon_{rsn}\omega_n^{'})$

$$\begin{aligned}
C_s^{'} &= m\omega_{sm}^{'}(S^{'}, S)\omega_{nm}^{'}(S^{'}, S)z_n^{'} && (1)\\
&= m\epsilon_{smk}\omega_k^{'}(S^{'}, S)\epsilon_{nmp}\omega_p^{'}(S^{'}, S)z_n^{'} && (2)\\
&= m\epsilon_{msk}\epsilon_{mnp}\omega_k^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_n^{'} && (3)\\
&= m\left(\delta_{sn}\delta_{kp} - \delta_{sp}\delta_{kn}\right)\omega_k^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_n^{'} && (4)\\
&= m\delta_{sn}\delta_{kp}\omega_k^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_n^{'} - m\delta_{sp}\delta_{kn}\omega_k^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_n^{'} && (5)\\
&= m\omega_p^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_s^{'} - m\omega_n^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)z_n^{'} && (6)
\end{aligned}$$

To deduce that $C_s^{'}$ is coplanar with the vectors $\omega_s^{'}(S^{'}, S)$ and $z_n^{'}$ we calculate the mixed triple product

$$\begin{aligned}
P &= \epsilon_{spr}C_s^{'}\omega_p^{'}(S^{'}, S)z_r^{'} && (7)\\
&= m\underbrace{\epsilon_{spr}\omega_n^{'}(S^{'}, S)\omega_n^{'}(S^{'}, S)z_s^{'}\omega_p^{'}(S^{'}, S)z_r^{'}}_{=0} - \underbrace{m\epsilon_{spr}\omega_n^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)z_n^{'}\omega_p^{'}(S^{'}, S)z_r^{'}}_{=0} && (8)\\
&= 0 && (9)
\end{aligned}$$

Both terms vanish: the first by the presence of the terms $\epsilon_{spr}z_s^{'}z_r^{'}$ which cancel each other and for the second by the terms $\epsilon_{spr}\omega_s^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)$. As $P = 0$, the three vectors are coplanar.
We now calculate the inner product $C_s^{'}\omega_s^{'}(S^{'}, S)$

$$\begin{aligned}
P &= m\omega_n^{'}(S^{'}, S)\omega_n^{'}(S^{'}, S)z_s^{'}\omega_s^{'}(S^{'}, S) - \underbrace{m\omega_n^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)z_n^{'}\omega_s^{'}(S^{'}, S)}_{\Leftrightarrow\ m\omega_n^{'}(S^{'}, S)\omega_n^{'}(S^{'}, S)z_s^{'}\omega_s^{'}(S^{'}, S)} && (10)\\
&= 0 && (11)
\end{aligned}$$

♦

## 5.12 p168 - Exercise

Taking $N = 3$, show that **5.424** may be reduced to the usual Euler equations:

$$I_{11}\frac{d\omega_1^{'}(S^{'},S)}{dt} - \left(I_{22} - I_{33}^{'}\right)\omega_2(S^{'},S)\omega_3^{'}(S^{'},S) = M_1^{'}$$

and two similar equations.

We first begin with an approach which leads to nothing. I probably made a reasoning error. I give here the whole calculation as this was interesting and also to, later, find my mistake. After this buggy solution, I will give a second version, which works.

**5.424**:

$$M_{ab}^{'} = J_{abrq}^{'}\frac{d\omega_{rq}^{'}(S^{'},S)}{dt} + J_{cdrq}^{'}\left(\delta_{ac}\delta_{du}\delta_{bv} + \delta_{bd}\delta_{cu}\delta_{av}\right)\omega_{rq}^{'}(S^{'},S)\omega_{uv}^{'}(S^{'},S) = \tag{1}$$

$$\times\epsilon_{sab}: \quad 2M_s^{'} = \epsilon_{sab}J_{abrq}^{'}\frac{d\omega_{rq}^{'}(S^{'},S)}{dt} + \epsilon_{sab}J_{cdrq}^{'}\left(\delta_{ac}\delta_{du}\delta_{bv} + \delta_{bd}\delta_{cu}\delta_{av}\right)\omega_{rq}^{'}(S^{'},S)\omega_{uv}^{'}(S^{'},S) \tag{2}$$

Using $\omega_{rq}^{'}(S^{'},S) = \epsilon_{rqt}\omega_t^{'}(S^{'},S)$ and $I_{st} = \frac{1}{2}J_{abrq}^{'}\epsilon_{abs}\epsilon_{rqt}$

$$2M_s^{'} = 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} + \epsilon_{sab}\epsilon_{rqi}\epsilon_{uvj}J_{cdrq}^{'}\left(\delta_{ac}\delta_{du}\delta_{bv} + \delta_{bd}\delta_{cu}\delta_{av}\right)\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \tag{3}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ + \left(\epsilon_{sab}\epsilon_{rqi}\epsilon_{uvj}J_{cdrq}^{'}\delta_{ac}\delta_{du}\delta_{bv} + \epsilon_{sab}\epsilon_{rqi}\epsilon_{uvj}J_{cdrq}^{'}\delta_{bd}\delta_{cu}\delta_{av}\right)\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{4}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ + \left(\epsilon_{scb}\epsilon_{rqi}\epsilon_{dbj}J_{cdrq}^{'} + \epsilon_{sad}\epsilon_{rqi}\epsilon_{caj}J_{cdrq}^{'}\right)\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{5}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ + \left(\epsilon_{bcs}\epsilon_{bdj}\right)\epsilon_{rqi}J_{cdrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\ + \left(\epsilon_{asd}\epsilon_{acj}\right)\epsilon_{rqi}J_{cdrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{6}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ + \left(\delta_{cd}\delta_{sj} - \delta_{cj}\delta_{sd}\right)\epsilon_{rqi}J_{cdrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\ + \left(\delta_{sc}\delta_{dj} - \delta_{sj}\delta_{dc}\right)\epsilon_{rqi}J_{cdrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{7}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ \\ +\epsilon_{rqi}J_{ccrq}^{'}\omega_i^{'}(S^{'},S)\omega_s^{'}(S^{'},S) \\ \\ -\epsilon_{rqi}J_{jsrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\ \\ +\epsilon_{rqi}J_{sjrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\ \\ -\epsilon_{rqi}J_{ccrq}^{'}\omega_i^{'}(S^{'},S)\omega_s^{'}(S^{'},S) \end{cases} \tag{8}$$

giving

$$2M_s^{'} = \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ \\ +\epsilon_{rqi}J_{sjrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\ \\ -\epsilon_{rqi}J_{jsrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{9}$$

For $s = 1$:

| | $+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$ | $-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$ |
|---|---|---|
| $\epsilon_{123}$ | $+J_{1112}\omega_3\omega_1 + J_{1212}\omega_3\omega_2 + J_{1312}\omega_3\omega_3$ | $-J_{1112}\omega_3\omega_1 - J_{2112}\omega_3\omega_2 - J_{3112}\omega_3\omega_3$ |
| $\epsilon_{132}$ | $-J_{1113}\omega_2\omega_1 - J_{1213}\omega_2\omega_2 - J_{1313}\omega_2\omega_3$ | $+J_{1113}\omega_2\omega_1 + J_{2113}\omega_2\omega_2 + J_{3113}\omega_2\omega_3$ |
| $\epsilon_{213}$ | $-J_{1121}\omega_3\omega_1 - J_{1221}\omega_3\omega_2 - J_{1321}\omega_3\omega_3$ | $+J_{1121}\omega_3\omega_1 + J_{2121}\omega_3\omega_2 + J_{3121}\omega_3\omega_3$ |
| $\epsilon_{231}$ | $+J_{1123}\omega_1\omega_1 + J_{1223}\omega_1\omega_2 + J_{1323}\omega_1\omega_3$ | $-J_{1123}\omega_1\omega_1 - J_{2123}\omega_1\omega_2 - J_{3123}\omega_1\omega_3$ |
| $\epsilon_{321}$ | $-J_{1132}\omega_1\omega_1 - J_{1232}\omega_1\omega_2 - J_{1332}\omega_1\omega_3$ | $+J_{1132}\omega_1\omega_1 + J_{2132}\omega_1\omega_2 + J_{3132}\omega_1\omega_3$ |
| $\epsilon_{312}$ | $+J_{1131}\omega_2\omega_1 + J_{1231}\omega_2\omega_2 + J_{1331}\omega_2\omega_3$ | $-J_{1131}\omega_2\omega_1 - J_{2131}\omega_2\omega_2 - J_{3131}\omega_2\omega_3$ |

Taking into account that $J_{abcd} = 0$ for $a \neq c \wedge b \neq d$

| | $+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$ | $-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$ |
|---|---|---|
| $\epsilon_{123}$ | $+\cancel{J_{1112}\omega_3\omega_1} + J_{1212}\omega_3\omega_2 + J_{1312}\omega_3\omega_3$ | $-\cancel{J_{1112}\omega_3\omega_1}$ |
| $\epsilon_{132}$ | $-\cancel{J_{1113}\omega_2\omega_1} - J_{1213}\omega_2\omega_2 - J_{1313}\omega_2\omega_3$ | $+\cancel{J_{1113}\omega_2\omega_1}$ |
| $\epsilon_{213}$ | $-\cancel{J_{1121}\omega_3\omega_1}$ | $+\cancel{J_{1121}\omega_3\omega_1} + J_{2121}\omega_3\omega_2 + J_{3121}\omega_3\omega_3$ |
| $\epsilon_{231}$ | $+J_{1323}\omega_1\omega_3$ | $-J_{2123}\omega_1\omega_2$ |
| $\epsilon_{321}$ | $-J_{1232}\omega_1\omega_2$ | $+J_{3132}\omega_1\omega_3$ |
| $\epsilon_{312}$ | $+\cancel{J_{1131}\omega_2\omega_1}$ | $-\cancel{J_{1131}\omega_2\omega_1} - J_{2131}\omega_2\omega_2 - J_{3131}\omega_2\omega_3$ |

Opposite sign terms vanish, giving

| | $+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$ | $-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$ |
|---|---|---|
| $\epsilon_{123}$ | $+J_{1212}\omega_3\omega_2 + J_{1312}\omega_3\omega_3$ | |
| $\epsilon_{132}$ | $-J_{1213}\omega_2\omega_2 - J_{1313}\omega_2\omega_3$ | |
| $\epsilon_{213}$ | | $+J_{2121}\omega_3\omega_2 + J_{3121}\omega_3\omega_3$ |
| $\epsilon_{231}$ | $+J_{1323}\omega_1\omega_3$ | $-J_{2123}\omega_1\omega_2$ |
| $\epsilon_{321}$ | $-J_{1232}\omega_1\omega_2$ | $+J_{3132}\omega_1\omega_3$ |
| $\epsilon_{312}$ | | $-J_{2131}\omega_2\omega_2 - J_{3131}\omega_2\omega_3$ |

Considering $J_{abcd} = -J_{badc}$

| | $+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$ | $-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$ |
|---|---|---|
| $\epsilon_{123}$ | $+\cancel{J_{1212}}\omega_3\omega_2 + \cancel{J_{1312}}\omega_3\omega_3$ | |
| $\epsilon_{132}$ | $-\cancel{J_{1213}}\omega_2\omega_2 - \cancel{J_{1313}}\omega_2\omega_3$ | |
| $\epsilon_{213}$ | | $+\cancel{J_{2121}}\omega_3\omega_2 + \cancel{J_{3121}}\omega_3\omega_3$ |
| $\epsilon_{231}$ | $+\cancel{J_{1323}}\omega_1\omega_3$ | $-\cancel{J_{2123}}\omega_1\omega_2$ |
| $\epsilon_{321}$ | $-\cancel{J_{1232}}\omega_1\omega_2$ | $+\cancel{J_{3132}}\omega_1\omega_3$ |
| $\epsilon_{312}$ | | $-\cancel{J_{2131}}\omega_2\omega_2 - \cancel{J_{3131}}\omega_2\omega_3$ |

We get

$$m_s^{'} = I_{st}\frac{d\omega_t^{'}(S^{'}, S)}{dt}$$

?????

◊

Let's try another approach. Start with **5.332**.: $\frac{d}{dt}\left(I_{st}\omega_t\right) = M_s$

$$\frac{d}{dt}\left(I_{st}(S^{'},S)\omega_t(S^{'},S)\right) = M_s(S^{'},S) \tag{10}$$

Cf. **5.408**.

$$\begin{aligned}
& & \omega^{'}_u(S^{'},S) &= A_{uq}\omega_q(S^{'},S) & (11)\\
\times A_{ut} \quad &\rightarrow & A_{ut}\omega^{'}_u(S^{'},S) &= A_{ut}A_{uq}\omega_q(S^{'},S) & (12)\\
& & &= \omega_t(S^{'},S) & (13)\\
& & \omega_t(S^{'},S) &= A_{ut}\omega^{'}_u(S^{'},S) & (14)\\
\textbf{(10)} \quad &\Rightarrow & M_s(S^{'},S) &= \frac{d}{dt}\left(I_{st}(S^{'},S)A_{ut}\omega^{'}_u(S^{'},S)\right) & (15)\\
\times A_{ps} \quad &\Rightarrow & M^{'}_p(S^{'},S) &= A_{ps}\frac{d}{dt}\left(I_{st}(S^{'},S)A_{ut}\omega^{'}_u(S^{'},S)\right) & (16)\\
& & I_{st}(S^{'},S) &= A_{as}A_{bt}I^{'}_{ab}(S^{'},S) & (17)\\
\textbf{(16)} \quad &\Rightarrow & M^{'}_p(S^{'},S) &= A_{ps}\frac{d}{dt}\left(A_{as}A_{bt}I^{'}_{ab}(S^{'},S)A_{ut}\omega^{'}_u(S^{'},S)\right) & (18)\\
& & &= A_{ps}\frac{d}{dt}\left(A_{as}I^{'}_{ak}(S^{'},S)\omega^{'}_k(S^{'},S)\right) & (19)
\end{aligned}$$

As we transformed $I_{st}(S^{'},S)$ to a coordinate system fixed to the body we have that the elements of $I^{'}_{ab}(S^{'},S)$ are constants.
Hence,

$$\begin{aligned}
M^{'}_p(S^{'},S) &= I^{'}_{ak}S^{'},S)A_{ps}\frac{d}{dt}\left(A_{as}\omega^{'}_k(S^{'},S)\right) & (20)\\
&= I^{'}_{ak}(S^{'},S)A_{ps}\left(\dot{A}_{as}\omega^{'}_k(S^{'},S) + A_{as}\dot{\omega}^{'}_k(S^{'},S)\right) & (21)\\
&= I^{'}_{ak}(S^{'},S)A_{ps}A_{as}\dot{\omega}^{'}_k(S^{'},S) + I^{'}_{ak}(S^{'},S)A_{ps}\dot{A}_{as}\omega^{'}_k(S^{'},S) & (22)\\
&= I^{'}_{pk}(S^{'},S)\dot{\omega}^{'}_k(S^{'},S) + I^{'}_{ak}(S^{'},S)A_{ps}\dot{A}_{as}\omega^{'}_k(S^{'},S) & (23)
\end{aligned}$$

$$\begin{aligned}
\textbf{5.408.} \quad &\Rightarrow & A_{ps}\dot{A}_{as} &= \omega^{'}_{ap}(S^{'},S) & (24)\\
\textbf{(23)} \quad &\Rightarrow & M^{'}_p(S^{'},S) &= I^{'}_{pk}(S^{'},S)\dot{\omega}^{'}_k(S^{'},S) + I^{'}_{ak}(S^{'},S)\omega^{'}_{ap}(S^{'},S)\omega^{'}_k(S^{'},S) & (25)
\end{aligned}$$

Let's now calculate the last expression for $p = 1$

$$M^{'}_1(S^{'},S) = I^{'}_{1k}(S^{'},S)\dot{\omega}^{'}_k(S^{'},S) + I^{'}_{ak}(S^{'},S)\omega^{'}_{a1}(S^{'},S)\omega^{'}_k(S^{'},S) \tag{26}$$

As we want an arbitrary, fixed to the body of course, coordinate system, it is possible to chose one so that the $I^{'}_{kj}(S^{'},S) = 0$ for $k \neq j$ i.e. $I^{'}_{kj}(S^{'},S)$ is diagonal. This is possible because $I^{'}_{kj}(S^{'},S)$ is symmetric (the finite-dimensional spectral theorem says that any symmetric matrix whose entries are real can be diagonalized by an orthogonal matrix).
We get, noticing that $\omega^{'}_{ab}(S^{'},S)$ is skew-symmetric and hence $\omega^{'}_{11}(S^{'},S) = 0$ :

$$M^{'}_1(S^{'},S) = I^{'}_{11}(S^{'},S)\dot{\omega}^{'}_1(S^{'},S) + I^{'}_{22}(S^{'},S)\omega^{'}_{21}(S^{'},S)\omega^{'}_2(S^{'},S) + I^{'}_{33}(S^{'},S)\omega^{'}_{31}(S^{'},S)\omega^{'}_3(S^{'},S) \tag{27}$$

Using **5.317**: $\omega^{'}_{21}(S^{'},S) = -\omega^{'}_{3}(S^{'},S)$ and $\omega^{'}_{31}(S^{'},S) = \omega^{'}_{2}(S^{'},S)$ we get the asked expression

$$M^{'}_{1}(S^{'},S) = I^{'}_{11}(S^{'},S)\dot{\omega}^{'}_{1}(S^{'},S) - \left(I^{'}_{22}(S^{'},S) - I^{'}_{33}(S^{'},S)\right)\omega^{'}_{2}(S^{'},S)\omega^{'}_{3}(S^{'},S) \tag{28}$$

♦

## 5.13 p169 - Exercise

Assign convenient generalized coordinates for the three systems $(a)$, $(b)$, and $(c)$ mentioned at the beginning of this section, and calculate the kinematical metric form in each case

$(a)$ **a particle on a surface** $(N = 2)$

No need here for fancy general coordinates: the $V_2$ coordinate system in the plane is the metric form of choice. Indeed $|v|^2 = a_{mn}v_m v_n$ and for a $V_2$

$$ds^2 = \left(a_{11}\left(v^1\right)^2 + 2a_{12}v^1v^2 + a_{22}\left(v^2\right)^2\right)dt^2$$

and if the space is Euclidean and the plane smooth, we can choose an orthogonal system where $a_{12}$ will vanish.

$(b)$ **a rigid body which can turn about a fixed point, as in the preceding section** $(N = 3)$

For a rigid body we can choose a coordinate system $S^{'}$ fixed to the body to describe the geometry of the rigid body. The kinetic energy referenced to a 'non-moving' (abuse of language) coordinate system $S$ is

$$T = \frac{1}{2}\sum\rho v_n^{'}(S)v_n^{'}(S) \qquad \text{(summation over all masses in the rigid body)} \tag{1}$$

We know by **5.409**: $v_n^{'}(S) = v_n^{'}(S^{'}) + \omega_{mn}^{'}(S^{'}, S)z_m^{'}$. As the $v_n^{'}(S^{'})$ are fixed, we have $v_n^{'}(S^{'}) = 0$ giving

$$T = \frac{1}{2}\sum\rho z_m^{'}z_k^{'}\omega_{mn}^{'}(S^{'}, S)\omega_{kn}^{'}(S^{'}, S) \tag{2}$$

Note in (2) that we bring $\omega_{mn}^{'}(S^{'}, S)$ out of the summation as this expression is the same for all masses in the body.

$$\begin{aligned}
\omega_{mn}(S^{'}, S) &= \epsilon_{mnt}\omega_t^{'}(S^{'}, S) && (3)\\
\Rightarrow \quad T &= \frac{1}{2}\sum\rho\epsilon_{mnt}\epsilon_{kns}z_m^{'}z_k^{'}\omega_t^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S) && (4)\\
&= \frac{1}{2}\sum\rho\left(\delta_{mk}\delta_{ts} - \delta_{ms}\delta_{kt}\right)z_m^{'}z_k^{'}\omega_t^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S) && (5)\\
&= \frac{1}{2}\sum\rho\left(z_m^{'}z_m^{'}\omega_t^{'}(S^{'}, S)\omega_t^{'}(S^{'}, S) - z_s^{'}z_t^{'}\omega_t^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)\right) && (6)\\
&= \frac{1}{2}\sum\rho\left(\delta_{st}z_m^{'}z_m^{'}\omega_s^{'}(S^{'}, S)\omega_t^{'}(S^{'}, S) - z_s^{'}z_t^{'}\omega_t^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)\right) && (7)\\
&= \frac{1}{2}\sum\rho\left(\delta_{st}z_m^{'}z_m^{'} - z_s^{'}z_t^{'}\right)\omega_s^{'}(S^{'}, S)\omega_t^{'}(S^{'}, S) && (8)
\end{aligned}$$

By **5.335**. we have $I_{st} = \delta_{st}\sum\rho z_m z_m - \sum\rho z_s z_t$ and so (8) can be written as

$$T = \frac{1}{2}I_{st}\omega_s^{'}(S^{'}, S)\omega_t^{'}(S^{'}, S) \tag{9}$$

So we can choose the three angles $\Omega'_s(S', S)$ with $(\omega'_s(S', S) = \frac{d\Omega'_s(S',S)}{dt})$ as generalized coordinates and define

$$ds^2 = I_{st} d\Omega'_s(S', S) d\Omega'_t(S', S)$$

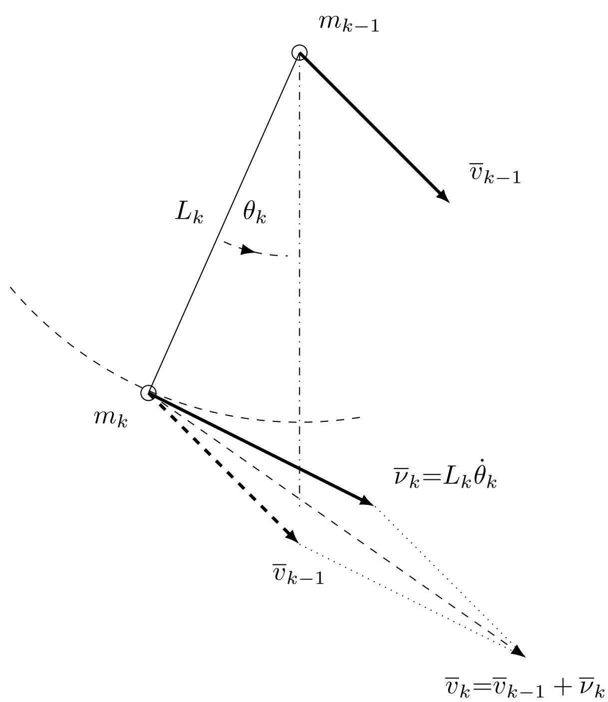
with

$$a_{mn} = I_{mn}$$

having constants as elements. Some check on consistency of the metric tensor defined by (14):
**Positive definite ?** : Yes, as $T$ is positive by construction.
**Symmetric ?** : Yes, as $a_{mn} = I_{km}$ and $I_{km}$ is symmetric.

$(c)$ **a chain of six rods smoothly hinged together, with one end fixed and all moving on a smooth plane** $(N = 6)$

To simplify the notation we will assume that the mass $m_k$ of each rod (with length $L_k$) is concentrated at it's endpoint .
First we note that the velocity of a rod is composed of two vectors, one (labelled as $\overline{\nu}_k$) generated by its own rotation relative to the previous rod and the other (labelled as $\overline{v}_{k-1}$) generated by the velocity of the endpoint of the rod to which it is attached (see.fig. 5.2).




(a)

Figure 5.2: Composition of absolute and relative velocities of a chain of rods

If we take Cartesian coordinates it is easy to see that rod (1) will have components

$$\left(L_1\dot{\theta}_1\cos\theta, L_1\dot{\theta}_1\sin\theta_1\right)$$

rod (2)

$$\left(L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2, L_1\dot{\theta}_1\sin\theta_1 + L_2\dot{\theta}_2\sin\theta_2\right)$$

$$\vdots$$

rod (k)

$$\left(\sum_{i=1}^{k} L_i\dot{\theta}_i\cos\theta_i, \sum_{i=1}^{k} L_i\dot{\theta}_i\sin\theta_i\right)$$

and so

$$\left(v^{(k)}\right)^2 = \left(\sum_{i=1}^{k} L_i\dot{\theta}_i\cos\theta_i\right)^2 + \left(\sum_{i=1}^{k} L_i\dot{\theta}_i\sin\theta_i\right)^2 \tag{10}$$

$$= \sum_{i=1}^{k}\left(L_i\dot{\theta}_i\right)^2 + 2\sum_{i=1}^{k}\sum_{j=1}^{k-i}\left(L_iL_{i+j}\dot{\theta}_i\dot{\theta}_{i+j}\cos\left(\theta_i - \theta_{i+j}\right)\right) \tag{11}$$

So the kinetic energy of one rod and the total kinetic energy of the system are

$$T^{(k)} = \frac{1}{2}m_k\left[\sum_{i=1}^{k}\left(L_i\dot{\theta}_i\right)^2 + 2\sum_{i=1}^{k}\sum_{j=1}^{k-i}\left(L_iL_{i+j}\dot{\theta}_i\dot{\theta}_{i+j}\cos\left(\theta_i - \theta_{i+j}\right)\right)\right] \tag{12}$$

$$T = \sum_{k=1}^{N} T^{(k)} \tag{13}$$

For $N = 6$ we get

| rod | $T^{(k)}$ |
|---|---|
| 1 | $\frac{1}{2}m_1\left[\left(L_1\dot{\theta}_1\right)^2\right]$ |
| 2 | $\frac{1}{2}m_2\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right)\right]$ |
| 3 | $\frac{1}{2}m_3\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + 2L_1L_3\dot{\theta}_1\dot{\theta}_3\cos\left(\theta_1 - \theta_3\right) + \ldots\right]$ |
| 4 | $\frac{1}{2}m_4\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + 2L_1L_3\dot{\theta}_1\dot{\theta}_3\cos\left(\theta_1 - \theta_3\right) + \ldots\right]$ |
| 5 | $\frac{1}{2}m_5\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + \left(L_5\dot{\theta}_5\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + \ldots\right]$ |
| 6 | $\frac{1}{2}m_6\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + \left(L_5\dot{\theta}_5\right)^2 + \left(L_6\dot{\theta}_6\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + \ldots\right]$ |

Giving for $T$

$$2T = \begin{cases} (m_1+m_2+m_3+m_4+m_5+m_6)\left(L_1\dot{\theta}_1\right)^2 \\ +(m_2+m_3+m_4+m_5+m_6)\left(L_2\dot{\theta}_2\right)^2 \\ +(m_3+m_4+m_5+m_6)\left(L_3\dot{\theta}_3\right)^2 \\ +(m_4+m_5+m_6)\left(L_4\dot{\theta}_4\right)^2 \\ +(m_5+m_6)\left(L_5\dot{\theta}_5\right)^2 \\ +(m_6)\left(L_6\dot{\theta}_6\right)^2 \\ +2\left(m_2+m_3+m_4+m_5+m_6\right)L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right) \\ +2\left(m_3+m_4+m_5+m_6\right)L_1L_3\dot{\theta}_1\dot{\theta}_3\cos\left(\theta_1-\theta_3\right) \\ +2\left(m_3+m_4+m_5+m_6\right)L_2L_3\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) \\ +2\left(m_4+m_5+m_6\right)L_1L_4\dot{\theta}_1\dot{\theta}_4\cos\left(\theta_1-\theta_4\right) \\ +2\left(m_4+m_5+m_6\right)L_2L_4\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right) \\ +2\left(m_4+m_5+m_6\right)L_3L_4\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right) \\ +2\left(m_5+m_6\right)L_1L_5\dot{\theta}_1\dot{\theta}_5\cos\left(\theta_1-\theta_5\right) \\ +2\left(m_5+m_6\right)L_2L_5\dot{\theta}_2\dot{\theta}_5\cos\left(\theta_2-\theta_5\right) \\ +2\left(m_5+m_6\right)L_3L_5\dot{\theta}_3\dot{\theta}_5\cos\left(\theta_3-\theta_5\right) \\ +2\left(m_5+m_6\right)L_4L_5\dot{\theta}_4\dot{\theta}_5\cos\left(\theta_4-\theta_5\right) \\ +2\left(m_6\right)L_1L_6\dot{\theta}_1\dot{\theta}_6\cos\left(\theta_1-\theta_6\right) \\ +2\left(m_6\right)L_2L_6\dot{\theta}_2\dot{\theta}_6\cos\left(\theta_2-\theta_6\right) \\ +2\left(m_6\right)L_3L_6\dot{\theta}_3\dot{\theta}_6\cos\left(\theta_3-\theta_6\right) \\ +2\left(m_6\right)L_4L_6\dot{\theta}_4\dot{\theta}_6\cos\left(\theta_4-\theta_6\right) \\ +2\left(m_6\right)L_5L_6\dot{\theta}_5\dot{\theta}_6\cos\left(\theta_5-\theta_6\right) \end{cases} \tag{14}$$

We define as general coordinates the angles $\theta^i$ and express $ds^2$ as

$$ds^2 = 2Tdt^2$$

and see that $ds^2$ is of the required form

$$ds^2 = a_{mn}d\theta^m d\theta^n$$

The metric tensor $a_{mn}$ contains elements depending on the $\theta_k$ chosen as general coordinates of the system and is a good candidate as metric tensor. Some check on consistency of the metric tensor defined by (14):

**Positive definite ?** : Yes, as $T$ is positive by definition

**Symmetric ?** : Yes, as the non-diagonal term $a_{ij}$ contains $\cos(\theta_i - \theta_j) = \cos(\theta_j - \theta_i)$

**Number of elements** : the metric tensor $a_{mn}$ for $N = 6$ should contain 6 diagonal elements and $\frac{6\times 6-6}{2} = 15$ independent non-diagonal elements. Checking (8), one can find that the numbers yield.

♦

## 5.14 p174 - Exercise

Establish the general result

$$v\frac{dv}{ds} = X_r\lambda^r, \quad \kappa v^2 = X_r\nu^r$$

Deduce that, if no forces at on the system, the trajectory is a geodesic in configuration space and the magnitude of the velocity is constant.

In configuration space $f_r = X_r$. Hence by $\mathbf{5,515}$

$$X^r = v\frac{dv}{ds}\lambda^r + \kappa v^2\nu^r \tag{1}$$

$$\Rightarrow \qquad X^r\lambda_r = X_r\lambda^r = v\frac{dv}{ds} \quad \text{as } \lambda^r \perp \nu^r \tag{2}$$

$$\text{and} \qquad X^r\nu_r = X_r\nu^r = \kappa v^2 \quad \text{as } \lambda^r \perp \nu^r \tag{3}$$

$$\tag{4}$$

The trajectory is a geodesic if $\kappa = 0$ which is the case as $X_r = 0$ and

$$v\frac{dv}{ds} = 0 \Rightarrow \frac{dv}{ds} = 0 \Rightarrow v = C^t$$

♦

## 5.15 p174 - Clarification

> It is easy to see that the lines of force are the orthogonal trajectories of the equipotential surface $V = C^t$

Consider a curve given by $x^r = x^r(u)$.
Along that line we have $V = V(x^r(u))$. Take $u = s$ as parameter and let's impose that $V(s) = C^t$.
We have $\frac{dV}{ds} = \frac{\partial V}{\partial x^r}\frac{dx^r}{ds} = \frac{\partial V}{\partial x^r}\lambda^r = 0$ with $\lambda^r = \frac{dx^r}{ds}$ the tangent vector along that curve.
But $X_r = \frac{\partial V}{\partial x^r}$.
So, $X_r\lambda^r = 0$ and as $X_r$ is collinear with $dx^r$ (the infinitesimal line element of the line of force) we have $dx_n\lambda^n = a_{mn}dx^m\lambda^n = 0$ proving the perpendicularity of both curves.

♦

## 5.16 p176 - Exercise

For a spherical pendulum, show that the lines of force are geodesics on the sphere on which the particle is constrained to move. What does the theorem stated above tell us in this case?

For the spherical pendulum we have the following situation

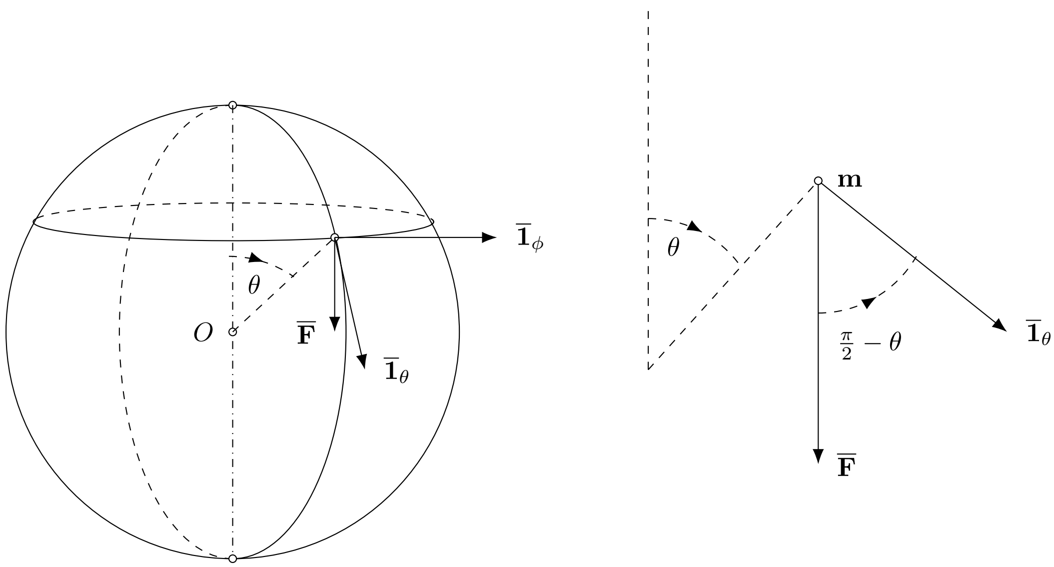



Figure 5.3: Physical components of the gravitational force tensor acting on a mass **m** on a sphere

From the figure it is clear that the only component of the gravitational force acting on the mass is restricted along the $\overline{1}_\theta$ vector which, with varying $\theta$ lays along a great circle of the sphere which is a geodesic. Hence the lines of force are great circle on the sphere.

For the theorem stated this means that as a mass is launched along a great circle, it will stay on that great circle.

♦

## 5.17 p176 - Exercise

A system starts from rest at a configuration $O$. Prove that the trajectory at $O$ is tangent to the line of force through $O$, and that the first curvature of the trajectory is one-third of the first curvature of the line of force.

From **5.533** we have

$$v\frac{dv}{ds} = X_r\lambda^r, \quad \kappa v^2 = X_r\nu^r \tag{1}$$

From the second expression we have as $v = 0$ at $O$ that $X_r\nu^r = 0$, meaning that $X_r$ is perpendicular to $\nu^r$. Also by **5.516**

$$f^r = \frac{dv}{dt}\lambda^r + \kappa v^2\nu^r \tag{2}$$

we know that the acceleration lies in the elementary two-space containing the tangent and the first normal to the trajectory implying by the previous result that $X_r$ and $\lambda^r$ are collinear. Note that from (1) we can not conclude (because $v = 0$) from the first expression that $X_r\lambda^r = 0$ . Indeed, $v\frac{dv}{ds}$ is a derived expression form of $\frac{dv}{dt}$. As $\frac{dv}{dt}$ is not necessarily 0 (otherwise the system would for ever stay on the configuration at $O$ meaning that $ds = 0$, making the expression $v\frac{dv}{ds}$ meaningless.)

Let's consider (2) with $f^r = X^r$:

$$\frac{dv}{dt}\lambda^r + \kappa v^2\nu^r = X^r \tag{3}$$

We know that at $O$, $X^r$ is tangent to the trajectory and so $X^r = X\lambda^r$. At the same point we can also define $X^r = X\lambda^{'r}$, with $\lambda^{'r}$ the tangent vector to the line of force. Multiplying (3) with $\lambda^r$ we see that $\frac{dv}{dt} = X$. So we get for (3)

$$X\lambda^r + \kappa v^2\nu^r = X\lambda^{'r} \tag{4}$$

$$\frac{\delta(4)}{\delta s} \quad \Rightarrow \quad \frac{dX}{ds}\lambda^r + X\underbrace{\frac{\delta\lambda^r}{\delta s}}_{\kappa\nu^r} + \frac{d\kappa}{ds}\underbrace{v^2}_{=0}\nu_r + 2\kappa\underbrace{v\frac{dv}{ds}}_{=\frac{dv}{dt}=X}\nu^r + \kappa\underbrace{v^2}_{=0}\frac{\delta\nu^r}{\delta s} = \frac{dX}{ds}\lambda^{'r} + X\underbrace{\frac{\delta\lambda^{'r}}{\delta s}}_{=\kappa^{'}\nu^{'r}} \Rightarrow 3\kappa \quad = \tag{5}$$

(we evaluate the expression at point $O$ and define $\kappa^{'}\nu^{'r}$ as the first curvature tensor of the line of force evaluated at 0)

$$\frac{dX}{ds}\lambda^r + X\kappa\nu^r + 2\kappa X\nu^r = \frac{dX}{ds}\lambda^{'r} + X\nu^{'r} \tag{6}$$

$$\times \quad \nu^r \quad \Rightarrow \quad 3\kappa X = \frac{dX}{ds}\underbrace{\lambda^{'r}\nu^r}_{=0} + X\kappa^{'}\nu^{'r}\nu^r \tag{7}$$

$$\Rightarrow 3\kappa = \kappa^{'}\nu^{'r}\nu^r \tag{8}$$

Note that ${\lambda'}^r \nu^r = 0$ as ${\lambda'}^r$ coincides with $\lambda^r$. On the other hand we still have to prove that ${\nu'}^r$ coincides with $\nu^r$ at $O$.

$$(6) \times {\nu'}^r \quad \Rightarrow \quad X\kappa\nu^r{\nu'}^r + 2\kappa X\nu^r{\nu'}^r = X\kappa' \tag{9}$$

$$3\kappa\nu^r{\nu'}^r = \kappa' \tag{10}$$

From (8) and (10) we see that $\nu^r{\nu'}^r = 1$ and so

$$\mathbf{3}\kappa = \kappa'$$

♦

## 5.18 p181-p182 - Clarification Figures 13., 14. and 15.

There are several ways to perform a map of the configuration space of a rigid body with fixed point.

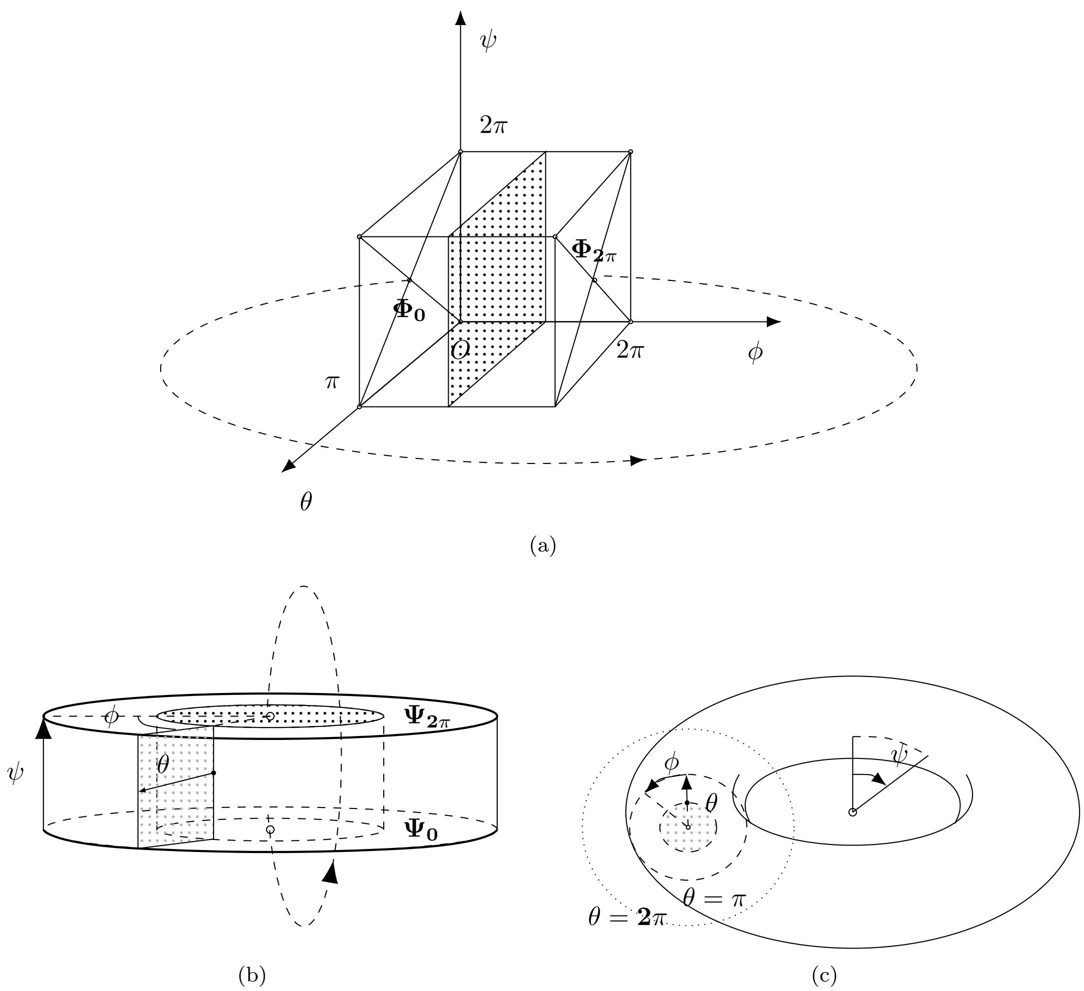



Figure 5.4: Map of the configuration space of a rigid body with fixed point.

Consider figure 5.2($a$). We can stretch like an accordion the cuboid along the $\phi$ axis and bent it so that the planes $\phi = 0$ and $\phi = 2\pi$ join. We get ($b$), a torus with square sections. The dimension $\phi$ is dealt with as a point $P(\theta, \phi, \psi)$ in the configuration space returns to the same point when varying $\phi$ to $\phi + 2k\pi$.

We can apply the same procedure of stretching and bending for the $\psi$ dimension so that the planes $\Psi = 0$ and $\Psi = 2\pi$ join. We get ($c$), a torus-like object.

The only dimension left is $\theta$ which our multi-dimensional crippled mind can't find a way to reshape this pseudo-torus so that when varying $\theta$ we can come back to the same point as started.

♦

## 5.19 p183 - Clarification for 5.561

The kinetic energy is

$$\textbf{5.561}. \qquad T = \frac{1}{2} I \left( \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta \right)$$

We first determine the general form of the kinetic energy for a rigid body rotating around a fixed point. From $\mathbf{5,310}$ we have

$$v_r = -\omega_{rm} z_m = -\epsilon_{rst}\omega_s z_t \tag{1}$$

$$T = \frac{1}{2}\sum m v_r v_r \tag{2}$$

$$\Rightarrow \qquad T = \frac{1}{2}\sum m\epsilon_{rst}\omega_s z_t \epsilon_{ruv}\omega_u z_v \tag{3}$$

$$= \frac{1}{2}\sum m \left(\delta_{su}\delta_{tv}\omega_s\omega_u z_t z_v - \delta_{sv}\delta_{tu}\omega_s\omega_u z_t z_v\right) \tag{4}$$

For the case $N = 3$ we get from (4):

$$T = \frac{1}{2}\sum m \left[\omega_1^2\left(z_2^2 + z_3^2\right) + \omega_2^2\left(z_1^2 + z_3^2\right) + \omega_3^2\left(z_1^2 + z_2^2\right) - 2\omega_1\omega_2 z_1 z_2 - 2\omega_1\omega_3 z_1 z_3 - 2\omega_2\omega_3 z_2 z_3\right] \tag{5}$$

Using the result from $\mathbf{5.336}$ this can be written as

$$T = \frac{1}{2}\left[I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2 + 2I_{12}\omega_1\omega_2 + 2I_{13}\omega_1\omega_3 + 2I_{23}\omega_2\omega_3\right] \tag{6}$$

Considering that the matrix $I_{ij}$ is symmetric, one can always find an appropriate basis so that the matrix becomes diagonal. Hence (6) can be simplified to

$$T = \frac{1}{2}\left[I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2\right] \tag{7}$$

Of course the $\omega_i$ in (7) are not the Euler angles and we have to express the $\omega_i$ as functions of the Euler angles.

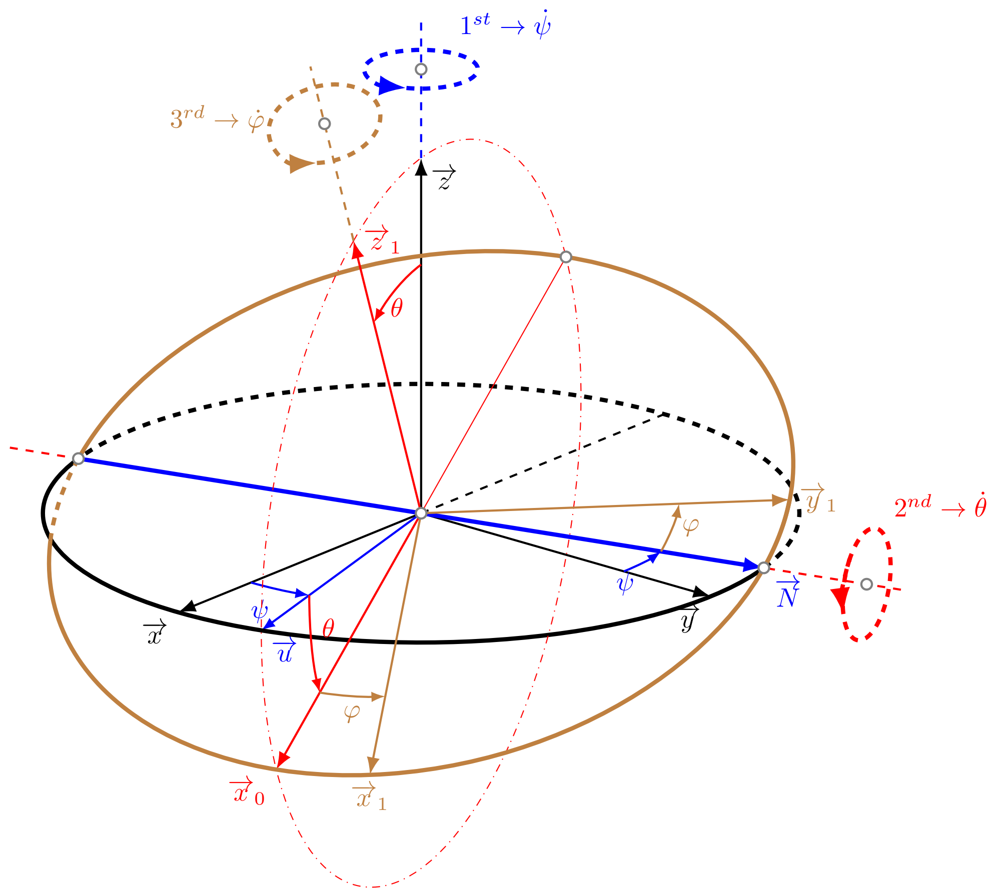



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Figure 5.5: Euler angles

Consider the Euler angles as in figure 5.5. The resulting angular velocity of the rigid body can be expressed as

$$\overline{\omega} = \dot{\psi}\overline{z} + \dot{\theta}\overline{N} + \dot{\phi}\overline{z}_1 \tag{8}$$

The projection of $\overline{\omega}$ on the basis $\overline{x}_1, \overline{y}_1, \overline{z}_1$ (which we choose fixed to the rigid body) will then coincide with the $\omega_i$.
We determine the components of $\overline{z}, \overline{N}, \overline{z}_1$ with $\overline{x}_1, \overline{y}_1, \overline{z}_1$ as basis.
We have

$$\begin{cases} \overline{N} = \cos\phi\ \overline{y}_1 + \sin\phi\ \overline{x}_1 \\ \overline{z} = \cos\theta\ \overline{z}_1 - \sin\theta\ \overline{x}_0 \\ \overline{x}_0 = \cos\phi\ \overline{x}_1 - \sin\phi\ \overline{y}_1 \end{cases} \tag{9}$$

$$\Rightarrow \quad \begin{cases} \overline{N} = \cos\phi\ \overline{y}_1 + \sin\phi\ \overline{x}_1 \\ \overline{z} = \cos\theta\ \overline{z}_1 - \sin\theta\ \cos\phi\ \overline{x}_1 + \sin\theta\ \sin\phi\ \overline{y}_1 \end{cases} \tag{10}$$

Hence,

$$\overline{\omega} = \dot{\psi}\cos\theta \; \overline{z}_1 - \dot{\psi}\sin\theta \;\; \cos\phi \; \overline{x}_1 + \dot{\psi}\sin\theta \;\; \sin\phi \; \overline{y}_1 + \dot{\theta}\cos\phi \; \overline{y}_1 + \dot{\theta}\sin\phi \; \overline{x}_1 + \dot{\phi}\overline{z}_1 \tag{11}$$

giving

$$\begin{cases} \omega_1 = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta \;\; \cos\phi \\ \omega_2 = \dot{\psi}\sin\theta \;\; \sin\phi + \dot{\theta}\cos\phi \\ \omega_3 = \dot{\psi}\cos\theta + \dot{\phi} \end{cases} \tag{12}$$

In the case considered $I_{11} = I_{22} = I_{33} = I$. Plugging (12) in (7) gives indeed

$$T = \frac{1}{2} I \left( \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta \right)$$

♦

## 5.20 p186 - Exercise 1

If a vector at the point with coordinates $(1, 1, 1)$ in Euclidean 3-space has components $(3, -1, 2)$, find the contra-variant, covariant and physical components in spherical polar co-ordinates.

The tensor $T_n$ to consider is $(3, -1, 2) - (1, 1, 1) = (2, -2, 1)$.
The Jacobian matrix for the transformation $z^n \to x^k$, evaluated at the point $(1, 1, 1)$ is

$$J_{(1,1,1)} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{2}$$

$$\Rightarrow \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'^n} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \tag{4}$$

We have the metric tensor evaluated at $(1,1,1)$

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{5}$$

$$\Rightarrow \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T_n'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \tag{7}$$

And the physical components

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T_{ph.}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \tag{9}$$

Another way to find the physical components is to project orthogonally the tensor on the unit vectors of a local Cartesian coordinate system, oriented along the unit vectors $\overline{e}_r, \overline{e}_\theta, \overline{e}_\phi$ corresponding to the vector $P\,(1,1,1)$ with modulus $|P| = \sqrt{3}$. We have for the tensor $T_n(2,-2,1)$ with modulus $|T_n| = 3$ as component along $\overline{e}_r$:

$$|T_n| \cos\alpha = |T_n| \frac{\langle T_n, P\rangle}{|T_n|\,|P|} \tag{10}$$

$$= |T_n| \frac{2-2+1}{|T_n|\,|P|} \tag{11}$$

$$= \frac{1}{\sqrt{3}} \tag{12}$$

For the component along $\overline{e}_\theta$ we first have to determine the vector $\overline{e}_\theta$. As first equation we have the orthogonality condition with $\overline{e}_r$ and putting $\overline{e}_\theta = (a, b, c)$, get $\langle \overline{e}_r, \overline{e}_\theta \rangle = a + b + c = 0$. As $\overline{e}_\theta$ lies in the plane $(1, 1, 0) - (0, 0, 0) - (0, 0, 1)$ we can put $a = b$ and get $\overline{e}_\theta = \frac{1}{\sqrt{6}}(1, 1, -2)$ and get for the tensor $T_n(2, -2, 1)$ as component along $\overline{e}_\theta$:

$$\begin{aligned} |T_n| \cos \beta &= |T_n| \frac{\langle T_n, \overline{e}_\theta \rangle}{|T_n|} && (13) \\ &= |T_n| \frac{2 - 2 - 2}{|T_n| \sqrt{6}} && (14) \\ &= -\frac{\sqrt{2}}{\sqrt{3}} && (15) \end{aligned}$$

For the component along $\overline{e}_\phi$ we first have to determine the vector $\overline{e}_\phi$. As first equation we have the orthogonality condition with the pair $\overline{e}_r, \overline{e}_\theta$ and get $\overline{e}_\phi = \overline{e}_r \times \overline{e}_\theta = \frac{1}{\sqrt{3}\sqrt{6}}(-3, 3, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$. For the tensor $T_n(2, -2, 1)$ as component along $\overline{e}_\phi$:

$$\begin{aligned} |T_n| \cos \gamma &= |T_n| \frac{\langle T_n, \overline{e}_\phi \rangle}{|T_n|} && (16) \\ &= |T_n| \frac{-2 - 2}{|T_n| \sqrt{2}} && (17) \\ &= -\frac{4}{\sqrt{2}} && (18) \\ &= -2\sqrt{2} && (19) \end{aligned}$$

giving

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \tag{20}$$

as in (9).

♦

## 5.21 p186 - Exercise 2

In cylindrical coordinates $(r, \phi, z)$ in Euclidean 3-space, a vector field is such that the vector at each point points along the parametric line of $\phi$, in the sense of $\phi$ increasing, and its magnitude is $kr$, where $k$ is a constant. Find the contra-variant, covariant and physical components of this vector field.

We can work backwards, with the physical components as starting point. Indeed, at a point $P\,(r, \phi, z)$ the tensor of this vector field will have $(0, kr, 0)$ as physical components in the cylindrical coordinates $(r, \phi, z)$ system.
We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Giving

$$\begin{cases} X_1 = & h_1 X_1^{phys.} & = 0 \\ X_2 = & h_2 X_2^{phys.} & = kr^2 \\ X_3 = & h_3 X_3^{phys.} & = 0 \end{cases} \tag{2}$$

and

$$\begin{cases} X^1 = & \frac{X_1^{phys.}}{h_1} & = 0 \\ X^2 = & \frac{X_2^{phys.}}{h_2} & = k \\ X^3 = & \frac{X_3^{phys.}}{h_3} & = 0 \end{cases} \tag{3}$$

♦

## 5.22 p186 - Exercise 3

Find the physical components of velocity and acceleration along the parametric lines of cylindrical coordinates in terms of the and their derivatives with respect to time.

We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

and the contravariant velocities

$$\begin{cases} v^1 = & \frac{dr}{dt} \\ v^2 = & \frac{d\phi}{dt} \\ v^3 = & \frac{dz}{dt} \end{cases} \tag{2}$$

giving by $v_K^{phys.} = h_K v^K$

$$\begin{cases} v_r = & \frac{dr}{dt} \\ v_\phi = & r\frac{d\phi}{dt} \\ v_z = & \frac{dz}{dt} \end{cases} \tag{3}$$

For the acceleration using $f^r = \frac{\delta v^r}{\delta t}$ and the Christoffel symbols being

$$\begin{cases} \Gamma^m_{nk} = 0 \quad \forall \quad (nk) \neq (r,\theta),(\theta,\theta) \\ \Gamma^\theta_{r\theta} = \frac{1}{r} \quad \text{and} \quad \Gamma^r_{\theta\theta} = -r \end{cases} \tag{4}$$

we have

$$\left\{\begin{array}{ll} f^1 = & \frac{dv^1}{dt} - r\,\underbrace{v^2\frac{dx^2}{dt}}_{=(v^2)^2} \\ \\ f^2 = & \frac{dv^2}{dt} + \underbrace{\frac{1}{r}v^1\frac{dx^2}{dt} + \frac{1}{r}v^2\frac{dx^2 1}{dt}}_{=\frac{2}{r}v^1v^2} \\ \\ f^3 = & \frac{dv^3}{dt} \end{array}\right. \tag{5}$$

giving by $f_K^{phys.} = h_K f^K$

$$\left\{\begin{array}{ll} f_r = & \frac{dv^1}{dt} - r\left(v^2\right)^2 \\ \\ f_{phi} = & r\frac{dv^2}{dt} + r\frac{2}{r}v^1v^2 \\ \\ f_z = & \frac{dv^3}{dt} \end{array}\right. \tag{6}$$

$$\Rightarrow \left\{\begin{array}{ll} f_r = & \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \\ \\ f_{phi} = & r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} \\ \\ f_z = & \frac{d^2z}{dt^2} \end{array}\right. \tag{7}$$

♦

## 5.23 p186 - Exercise 4

A particle moves on a sphere under the action of gravity. Find the contra-variant an co-variant components of the force, using colatitude and azimuth, and write down the equation of motion.

We determine first the physical components of the force.

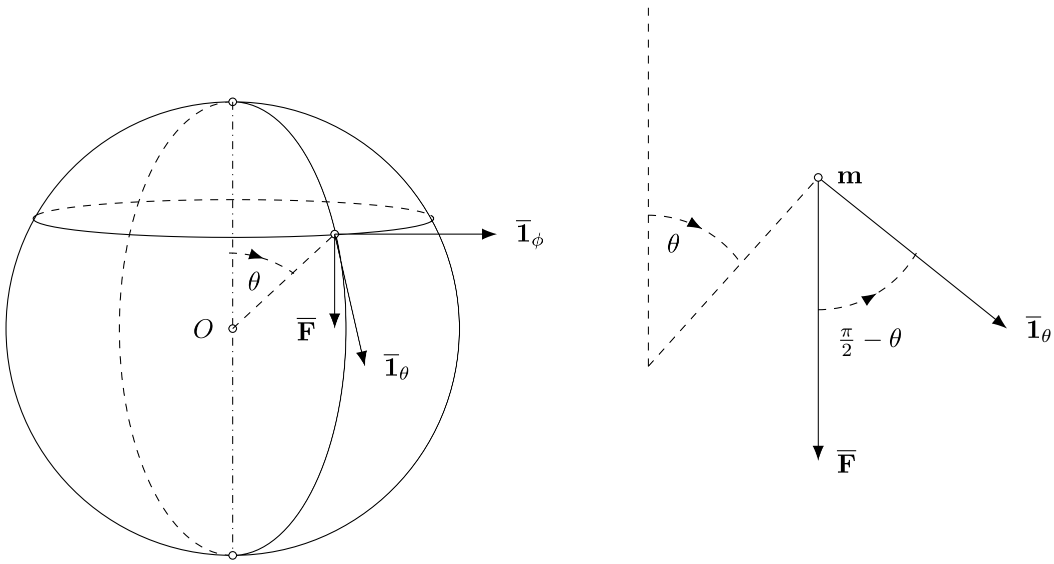



Figure 5.6: Physical components of the gravitational force tensor acting on a mass **m** on a sphere

We note first that the unit vector $\overline{1}_\phi$ is perpendicular to the place formed by the vectors $\overline{1}_\theta, \overline{F}$ and s the force has no components projected on this vector. The vector $\overline{F}$ is parallel with the axis of reference of the sphere with radius $R$ and so the physical components become

$$\begin{cases} F_\phi^{phys} = 0 \\ \\ F_\theta^{phys} = mg\sin\theta \\ \end{cases} \tag{1}$$

$$\Rightarrow \quad \begin{cases} F^\phi = 0 & F_\phi = 0 \\ \\ F^\theta = \frac{1}{R}mg\sin\theta & F_\theta = Rmg\sin\theta \\ \end{cases} \tag{2}$$

We use equation **5.212**.

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ \\ T = \frac{1}{2}ma_{pq}\dot{x}^p\dot{x}^q,\ \dot{x}^s = \frac{dx^s}{dt} \\ \end{cases} \tag{3}$$

with for our case

$$T = \frac{1}{2} m R^2 \left( \dot{\theta}^2 + \sin^2\theta \; \dot{\phi}^2 \right) \tag{4}$$

and get the set of equation of motion (the second column gives the dimensional analysis as a check for consistency)

$$\begin{cases} \frac{\ddot{\phi}}{\dot{\phi}} = -2\cot\theta \; \dot{\theta} & : \quad \frac{[T]^{-2}}{[T]^{-1}} \cong [T]^{-1} \\ \ddot{\theta} - \left(\dot{\phi}\right)^2 \sin\theta\cos\theta = \frac{g}{R}\sin\theta & : \quad [T]^{-2} + \left([T]^{-1}\right)^2 \cong \frac{[L][T]^{-2}}{[L]} \end{cases} \tag{5}$$

Let's check the special case when $\dot{\phi} = 0$.
The first equation can be rewritten and gives of course $\phi = C$ while the second equation becomes

$$\ddot{\theta} = \frac{g}{R}\sin\theta$$

which is similar to the equation of the simple gravity pendulum.

♦

## 5.24 p186 - Exercise 5

Consider the motion of a particle on a smooth torus under no forces except normal reaction. The geometrical line element may be written

$$ds^2 = (a - b\cos\theta)^2 \, d\phi^2 + b^2 d\theta^2$$

where $\phi$ is an azimuthal angle and $\theta$ an angular displacement from the equatorial plane. Show that the path of a particle satisfies the following two differential equations in which $h$ is a constant

$$(a) \qquad (a - b\cos\theta)^2 \frac{d\phi}{ds} = h$$

$$(b) \qquad b^2 \left(\frac{d\theta}{d\phi}\right)^2 = \frac{(a - b\cos\theta)^4}{h^2} - (a - b\cos\theta)^2$$

We use equation **5.212**. and **5.212**.

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ \\ T = \frac{1}{2} m a_{pq} \dot{x}^p \dot{x}^q, \ \dot{x}^s = \frac{dx^s}{dt} \end{cases} \tag{1}$$

with for our case

$$T = \frac{1}{2} m \left( b^2 \dot{\theta}^2 + (a - b\cos\theta)^2 \ \dot{\phi}^2 \right) \tag{2}$$

$$\begin{cases} \frac{\partial T}{\partial \dot{\phi}} = m (a - b\cos\theta)^2 \dot{\phi} & \frac{\partial T}{\partial \phi} = 0 \\ \\ \frac{\partial T}{\partial \dot{\theta}} = m b^2 \dot{\theta} & \frac{\partial T}{\partial \theta} = m b (a - b\cos\theta) \dot{\phi}^2 \sin\theta \end{cases} \tag{3}$$

giving

$$\begin{cases} (a - b\cos\theta)^2 \ddot{\phi} + 2b (a - b\cos\theta) \dot{\theta}\dot{\phi} \sin\theta = 0 \\ \\ b^2 \ddot{\theta} - b (a - b\cos\theta) \dot{\phi}^2 \sin\theta = 0 \end{cases} \tag{4}$$

$$\Rightarrow \qquad \begin{cases} (a - b\cos\theta) \ddot{\phi} = -2b\dot{\theta}\dot{\phi} \sin\theta \\ \\ b^2 \ddot{\theta} - b (a - b\cos\theta) \dot{\phi}^2 \sin\theta = 0 \end{cases} \tag{5}$$

In the first equation, put $y \equiv \dot{\phi}$ giving for the first equation:

$$\frac{dy}{y} = -2b\frac{\sin\theta d\theta}{(a - b\cos\theta)} \tag{6}$$

$$\Leftrightarrow \qquad \frac{dy}{y} = -2\frac{d\,(a - b\cos\theta)}{(a - b\cos\theta)} \tag{7}$$

$$\Rightarrow \qquad \log y = -2\log\,(a - b\cos\theta) + \log C \tag{8}$$

$$\Rightarrow \qquad \dot{\phi} = C\,(a - b\cos\theta)^{-2} \tag{9}$$

Note that $\dot{\phi}$ is a time derivative. But as we are on a geodesic, **5.226**. stands and so $v$ is constant as $\frac{dv}{ds} = 0$. Using $v = \frac{ds}{dt}$, (9) can be written as

$$(a - b\cos\theta)^2\,\frac{d\phi}{dt} = C \tag{10}$$

$$\Leftrightarrow \qquad (a - b\cos\theta)^2\,\frac{d\phi}{ds}\underbrace{\frac{ds}{dt}}_{=v} = C \tag{11}$$

$$\Leftrightarrow \qquad (a - b\cos\theta)^2\,\frac{d\phi}{ds} = h \quad \text{with } h = \frac{C}{v} \tag{12}$$

Next, we don't use the second equation in (5) but the line element equation instead

$$ds^2 = (a - b\cos\theta)^2\,d\phi^2 + b^2 d\theta^2 \tag{13}$$

$$\Rightarrow \qquad \left(\frac{ds}{d\phi}\right)^2 = (a - b\cos\theta)^2 + b^2\left(\frac{d\theta}{d\phi}\right)^2 \tag{14}$$

$$\Rightarrow \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = \left(\frac{d\phi}{ds}\right)^{-2} - (a - b\cos\theta)^2 \tag{15}$$

$$(12) \quad : \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = \frac{(a - b\cos\theta)^4}{h^2} - (a - b\cos\theta)^2 \tag{16}$$

♦

## 5.25 p186 - Exercise 6

Consider the motion of a particle under gravity on the smooth torus o the previous problem, the equatorial plane of the torus being horizontal. Taking the mass of the particle to unity, so that $V = bg\sin\theta$, show that the path of the particle satisfies the following two differential equations.

$$(a) \qquad (E - V)\,(a - b\cos\theta)^2\,\frac{d\phi}{ds} = h$$

$$(b) \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = (E - V)\,\frac{(a - b\cos\theta)^4}{h^2} - (a - b\cos\theta)^2$$

where $E$ is the total energy, $h$ is a constant and $d\sigma$ is the action line element.

The line of reasoning is quite the same as problem (5). We use equation **5.212**. and **5.212**.

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ \\ T = \frac{1}{2}ma_{pq}\dot{x}^p\dot{x}^q,\ \dot{x}^s = \frac{dx^s}{dt} \end{cases} \tag{1}$$

with for our case

$$T = \frac{1}{2}m\left(b^2\dot{\theta}^2 + (a - b\cos\theta)^2\ \dot{\phi}^2\right) \tag{2}$$

$$\begin{cases} \frac{\partial T}{\partial \dot{\phi}} = m\,(a - b\cos\theta)^2\,\dot{\phi} & \frac{\partial T}{\partial \phi} = 0 \\ \\ \frac{\partial T}{\partial \dot{\theta}} = mb^2\dot{\theta} & \frac{\partial T}{\partial \theta} = mb\,(a - b\cos\theta)\,\dot{\phi}^2\sin\theta \end{cases} \tag{3}$$

giving (as $F_\phi = -\partial_\phi V = 0$ and $F_\theta = -\partial_\theta V = -bg\cos\theta$)

$$\begin{cases} (a - b\cos\theta)^2\,\ddot{\phi} + 2b\,(a - b\cos\theta)\,\dot{\theta}\dot{\phi}\sin\theta = 0 \\ \\ b^2\ddot{\theta} - b\,(a - b\cos\theta)\,\dot{\phi}^2\sin\theta = -bg\cos\theta \end{cases} \tag{4}$$

$$\Rightarrow \qquad \begin{cases} (a - b\cos\theta)\,\ddot{\phi} = -2b\dot{\theta}\dot{\phi}\sin\theta \\ \\ b^2\ddot{\theta} - b\,(a - b\cos\theta)\,\dot{\phi}^2\sin\theta = -bg\cos\theta \end{cases} \tag{5}$$

In the first equation, put $y \equiv \dot{\phi}$ giving for the first equation:

$$\frac{dy}{y} = -2b\frac{\sin\theta d\theta}{(a-b\cos\theta)} \tag{6}$$

$$\Leftrightarrow \qquad \frac{dy}{y} = -2\frac{d\,(a-b\cos\theta)}{(a-b\cos\theta)} \tag{7}$$

$$\Rightarrow \qquad \log y = -2\log\,(a-b\cos\theta) + \log C \tag{8}$$

$$\Rightarrow \qquad \dot{\phi} = C\,(a-b\cos\theta)^{-2} \tag{9}$$

Note that $\dot{\phi}$ is a time derivative. Using $\frac{ds}{dt} = v = \sqrt{2T} = \sqrt{2}\sqrt{E-V}$, (9) can be written as

$$(a-b\cos\theta)^2\,\frac{d\phi}{dt} = C \tag{10}$$

$$\Leftrightarrow \qquad (a-b\cos\theta)^2\,\frac{d\phi}{d\sigma}\underbrace{\frac{d\sigma}{ds}}_{=\sqrt{E-V}}\underbrace{\frac{ds}{dt}}_{=\sqrt{2}\sqrt{E-V}} = C \tag{11}$$

$$\Leftrightarrow \qquad (E-V)\,(a-b\cos\theta)^2\,\frac{d\phi}{d\sigma} = h \tag{12}$$

with $h = \frac{C}{\sqrt{2}}$.
Next, we don't use the second equation in (5) but the line element equation instead

$$ds^2 = (a-b\cos\theta)^2\,d\phi^2 + b^2 d\theta^2 \tag{13}$$

$$\Rightarrow \qquad \left(\frac{ds}{d\phi}\right)^2 = (a-b\cos\theta)^2 + b^2\left(\frac{d\theta}{d\phi}\right)^2 \tag{14}$$

$$\Rightarrow \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = \left(\frac{d\phi}{ds}\right)^{-2} - (a-b\cos\theta)^2 \tag{15}$$

$$\Leftrightarrow \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = \left(\frac{d\phi}{d\sigma}\right)^{-2}\left(\frac{d\sigma}{ds}\right)^{-2} - (a-b\cos\theta)^2 \tag{16}$$

$$(12) \quad : \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = (E-V)^2\,\frac{1}{(E-V)}\,\frac{(a-b\cos\theta)^4}{h^2} - (a-b\cos\theta)^2 \tag{17}$$

$$\Rightarrow \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = (E-V)\,\frac{(a-b\cos\theta)^4}{h^2} - (a-b\cos\theta)^2 \tag{18}$$

♦

## 5.26 p187 - Exercise 7

A dynamical system consists of a thin straight smooth tube which can rotate in a horizontal plan about one end $O$, together with a bead$B$ inside the tube connected to $O$ by a spring. Taking as coordinates $r = OB$ and $\theta =$ angle of rotation of the tube about $O$, the potential energy $V$ is a function of $r$ only. Show that in configuration space , all the lines of force are geodesics for the kinematical line element.

Well understanding the question is of course paramount:

- The tube mentioned plays only a functional role to hold the spring "stiff" along the line $OB$ as its mass can be neglected.It will play no further role in the dynamics of the system.
- Nothing is said that the system contains any force that keeps the angular velocity at a constant speed $\omega$.

That being clarified, one can expect that the system will behave as a harmonic oscillator along the line $OB$ and that ,given an initial rotational momentum, the angular momentum will be a constant during the trajectory of the bead. This means that the bead will oscillate along $OB$ but as the angular momentum is a constant and given $m\omega r^2 = C(m =$ mass of the bead), the instant radial speed will vary.
The only conservative force acting on the bead will be that of the spring and will be $V = \frac{1}{2}k\left(r - r_0\right)^2$, $r_0$ being the point along $OB$ where the spring is not stretched. The generalized forces are $F_r = -k\left(r - r_0\right)$ and $F_\theta = 0$ meaning the lines of force are straight lines pointing to the origin $O$.
About the geodesics. Clearly the instantaneous velocity of the bead is $\overrightarrow{v} = \dot{r}\overrightarrow{1}_r + \dot{\theta}r\overrightarrow{1}_\theta$ giving as kinetic energy $T = \frac{1}{2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right)$ giving as kinematic line element

$$ds^2 = 2Tdt = dr^2 + r^2 d\theta^2$$

Referring to **3.101**, the configuration space is flat and the geodesics are straight lines. As the line forces are straight lines towards the origin $O$, these line of force are also geodesics in the configuration space equipped with the kinematical line element.

♦

## 5.27 p187 - Exercise 8

Show that if a line of force is a geodesic for the kinematical line element, it is also a geodesic for the action line element.

From **5.516** and **5.529** we have

$$X^r = v\frac{dv}{ds}\lambda^r + \kappa v^2 \nu^r \tag{1}$$

As the line of force is a geodesic, we can start with a velocity tangent to the line of force, ensuring that the trajectory of the dynamical system will lie on the geodesic line of force (see page 175) and thus $\kappa = 0$ for the trajectory. Hence,

$$X^r = v\frac{dv}{ds}\lambda^r \tag{2}$$

expressing now the function of the action line element $d\sigma = \sqrt{E-V}ds$ we have

$$\begin{aligned} X^r &= v\frac{dv}{ds}\lambda^r &(3)\\ &= v\frac{dv}{ds}\frac{dx^r}{d\sigma}\frac{d\sigma}{ds} &(4)\\ &= v\frac{dv}{ds}\frac{dx^r}{d\sigma}\sqrt{E-V} &(5)\\ &= \sqrt{E-V}v\frac{dv}{ds}{\lambda'}^r &(6) \end{aligned}$$

As stated page 177, this dynamical system will describe in configuration space a geodesic for the action metric, meaning that ${\lambda'}^r$ is tangent to this geodesic and that $X^r$, being collinear with ${\lambda'}^r$ (with the factor $\sqrt{E-V}v\frac{dv}{ds}$), is also tangent to this geodesic. Hence, this line of force is also a geodesic for the action line element.

♦

## 5.28 p187 - Exercise 9

Using the methods of Chapter II and **5.532**, show that the trajectories of a dynamical system with kinetic energy $T$ and potential energy $V$ satisfy the variational equation

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

Let's start with a function $L$ defined by

$$dL = (T - V)du \tag{1}$$

$$\tag{2}$$

As in figure 2 page 38 we will make $L$ a function of two parameters, $u$ and $v$ , the latter defining a family of curves between the begin point $u_1$ and the endpoint $u_2$.

$$L = L(u, v) \tag{3}$$

with

$$(T - V)(u_1, v) = (T - V)_1 \quad (T - V)(u_2, v) = (T - V)_2 \quad \forall v \tag{4}$$

We will try to minimize (with respect to $v$) the following functional

$$L == \int_{u_1}^{u_2} (T - V)(u, v)du \tag{5}$$

It's derivative with respect to $v$

$$\frac{dL}{dv} = \int_{u_1}^{u_2} \frac{\partial (T - V)(u, v)}{\partial v} du \tag{6}$$

We express $(T-V)(u,v)$ as a function of the generalized coordinates $x^r$ and their derivatives. Then,

$$\frac{\partial (T - V)(u, v)}{\partial v} = \frac{\partial (T - V)(u, v)}{\partial \dot{x^r}} \frac{\partial \dot{x^r}}{\partial v} + \frac{\partial (T - V)(u, v)}{\partial x^r} \frac{\partial x^r}{\partial v} \tag{7}$$

where $\dot{x^r} = \frac{\partial x^r}{\partial u}$.
We have

$$\frac{\partial \dot{x^r}}{\partial v} = \frac{\partial}{\partial v} \frac{\partial x^r}{\partial u} = \frac{\partial}{\partial u} \frac{\partial x^r}{\partial v} \tag{8}$$

So,

$$\frac{\partial (T - V)(u, v)}{\partial v} = \frac{\partial (T - V)(u, v)}{\partial \dot{x^r}} \frac{\partial}{\partial u} \frac{\partial x^r}{\partial v} + \frac{\partial (T - V)(u, v)}{\partial x^r} \frac{\partial x^r}{\partial v} \tag{9}$$

Consider the expression

$$\int_{u_1}^{u_2} d(AB) = \int_{u_1}^{u_2} Ad(B) + \int_{u_1}^{u_2} Bd(A) \tag{10}$$

$$\Rightarrow \qquad \int_{u_1}^{u_2} Ad(B) = \int_{u_1}^{u_2} d(AB) - \int_{u_1}^{u_2} Bd(A) \tag{11}$$

Put $B = \frac{\partial x^r}{\partial v}$ and $A = \frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}$ and putting this inside (9) and (6):

$$\frac{dL}{dv} = \int_{u_1}^{u_2} d\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\frac{\partial x^r}{\partial v}\right) - \int_{u_1}^{u_2} \frac{\partial x^r}{\partial v} d\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) + \int_{u_1}^{u_2} \frac{\partial (T-V)(u,v)}{\partial x^r}\frac{\partial x^r}{\partial v} du \tag{12}$$

$$= \frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\frac{\partial x^r}{\partial v}\Bigg|_{u_1}^{u_2} - \left[\int_{u_1}^{u_2}\left(\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u,v)}{\partial x^r}\right)\frac{\partial x^r}{\partial v} du\right] \tag{13}$$

We express now the results in term of infinitesimals. A change in "length" $\delta L$ when we pas from a curve $v$ to a curve $v + dv$ is

$$\delta L = \frac{dL}{dv}\delta v \tag{14}$$

$$= \frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\frac{\partial x^r}{\partial v}\delta v\Bigg|_{u_1}^{u_2} - \int_{u_1}^{u_2}\left(\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u,v)}{\partial x^r}\right)\frac{\partial x^r}{\partial v}\delta v du \tag{15}$$

$$= \frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\delta x^r\Bigg|_{u_1}^{u_2} - \int_{u_1}^{u_2}\left(\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u,v)}{\partial x^r}\right)\delta x^r du \tag{16}$$

The first term vanish as at the endpoints the $\delta x^r$ are zero and hence we get

$$\delta L = -\int_{u_1}^{u_2}\left(\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u,v)}{\partial x^r}\right)\delta x^r du \tag{17}$$

As the $\delta x^r$ are arbitrary, we must have for $\delta L = 0$

$$\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u)}{\partial x^r} = 0 \tag{18}$$

This is the same equation as **5.532** which describe the motion of a system with a conservative force.

♦

## 5.29 p188 - Exercise 10

Using the definition **5.5335** for $I_{rs}$, prove that if $X_r$ is any non-zero vector, then $I_{rs}X_rX_s \geq 0$, and that the equality occurs only if all particles of the system are distributed on a single line.

By **5.335**

$$I_{rs} = \delta_{rs} \sum m z_q z_q - \sum m z_r z_s \tag{1}$$

Multiplying by $X_r X_s$:

$$I_{rs}X_rX_s = \underbrace{X_rX_s\delta_{rs}}_{=X_rX_r} \sum m z_q z_q - \sum m \underbrace{z_rX_r}_{=|z|_{(m)}|X|\cos\theta_m} \underbrace{z_sX_s}_{=|z|_{(m)}|X|\cos\theta_m} \tag{2}$$

with $\theta_m$ the angle between the vector $X_r$ and the position vector $z_m$ of a particle.

$$\begin{aligned} I_{rs}X_rX_s &= |X|^2 \sum m\,|z|^2_{(m)} - |X|^2 \sum m\,|z|^2_{(m)} \cos^2\theta_m && (3)\\ &= |X|^2 \sum m\,|z|^2_{(m)} \left(1 - \cos^2\theta_m\right) && (4) \end{aligned}$$

As we have $\left(1-\cos^2\theta_m\right) \in [0,1]$ it is clear that $I_{rs}X_rX_s \geq 0$ and that it only will be zero when $\theta_m = 0 \quad \forall m$ which means that all position vectors are collinear wit $X_r$ and are on a line.

♦

## 5.30 p188 - Exercise 11

Let $Oz_1z_2z_3$ and $O^{'}z_1^{'}z_2^{'}z_3^{'}$ be two sets of Cartesian axes parallel to one another. Consider a mass distribution and let $I_{rs}, I_{rs}^{'}$be its moment of inertia tensors calculated for these two axes in accordance with **5.335**. Writing $I_{rs}^{'} = I_{rs} + K_{rs}$, evaluate $K_{rs}$.

By **5.335**

$$I_{rs} = \delta_{rs} \sum m z_q z_q - \sum m z_r z_s \tag{1}$$

As the axes of both coordinate systems are parallel, we can write

$$z_q^{'} = z_q + b_q \tag{2}$$

which gives for (1):

$$I_{rs}^{'} = \delta_{rs} \sum m \left(z_q + b_q\right)\left(z_q + b_q\right) - \sum m \left(z_r + b_r\right)\left(z_s + b_s\right) \tag{3}$$

$$= \begin{cases} \delta_{rs} \sum m z_q z_q - \sum m z_r z_s \\ +\delta_{rs} \sum m b_q z_q - \sum m b_r z_s \\ +\delta_{rs} \sum m b_q z_q - \sum m b_s z_r \\ +\delta_{rs} \sum m b_q b_q - \sum m b_r b_s \end{cases} \tag{4}$$

$$= \begin{cases} I_{rs} \\ +\delta_{rs} \sum m b_q z_q - \sum m b_r z_s \\ +\delta_{rs} \sum m b_q z_q - \sum m b_s z_r \\ +\delta_{rs} \sum m b_q b_q - \sum m b_r b_s \end{cases} \tag{5}$$

$$\tag{6}$$

The last term $\delta_{rs} \sum m b_q b_q - \sum m b_r b_s$ can be interpreted as a moment of inertia tensor for a single virtual mass $M = \sum m$ situated at the point $b_q$ seen from the axes $Oz_1z_2z_3$. Let's denote it with $\tilde{I}_{rs} = \sum m \left(\delta_{rs} b_q b_q - b_r b_s\right)$.

The other two terms can also be seen as a rigid body of particles distributed in a plane perpendicular to one of the axis i.e. all particles are transported perpendicularity to a plane. We note that $\delta_{rs} \sum m b_q z_q - \sum m b_r z_s = \delta_{rs} \sum m b_q z_q - \sum m b_s z_r$. This follows immediately from the symmetric character of $I_{rs}^{'}, I_{rs}, \tilde{I}_{rs}$.

Denoting $\overline{I}_{rs} = \delta_{rs} \sum m b_q z_q - \sum m b_r z_s + \delta_{rs} \sum m b_q z_q - \sum m b_s z_r$ giving

$$K_{rs} = I_{rs} + \overline{I}_{rs} + \tilde{I}_{rs}$$

♦

## 5.31 p188 - Exercise 12

A rigid body is turning about a fixed point. Referred to right-handed axes $Oz_1z_2z_3$, its angular velocity tensor has components

$$\omega_{23} = 1, \quad \omega_{31} = 2, \quad \omega_{12} = 3$$

If we refer the same motion to the axis $O^{'}z_1^{'}z_2^{'}z_3^{'}$ , such that the axis $O^{'}z_1^{'}$ is $Oz_1^{'}$ reversed, while $z_2z_3$ coincide with $O^{'}z_2^{'}z_3^{'}$, what are the $\omega_{rs}^{'}$ and $\omega_{rs}^{'}$?

We use the following identities

$$\begin{cases} \mathbf{5.312} & \omega_{rm} = -\omega_{mr} \\ \mathbf{5.316} & \omega_{rs} = \epsilon_{rsn}\omega_n \\ \mathbf{5.317} & \omega_1 = \omega_{23} \quad \omega_2 = \omega_{31} \quad \omega_3 = \omega_{12} \end{cases} \tag{1}$$

The angular velocity tensor is

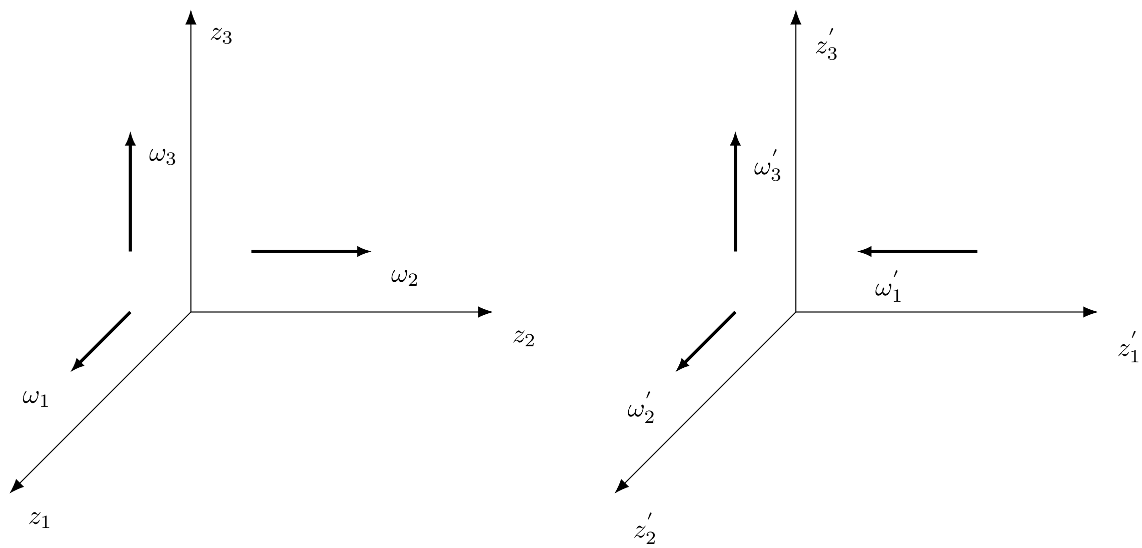
$$\Omega = \begin{pmatrix} 0 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \tag{2}$$

giving by **5.317**

$$\omega_1 = \omega_{23} \quad \omega_2 = \omega_{31} \quad \omega_3 = \omega_{12} \tag{3}$$

From pure geometrical consideration we can conclude that

$$\omega_1^{'} = -\omega_1 \quad \omega_2^{'} = \omega_2 \quad \omega_3^{'} = \omega_3 \tag{4}$$




(a)

Figure 5.7: Angular velocity vectors in mirrored axis

Indeed, the $\omega_i$ can be considered as vectors, objects independent from the chosen coordinate system. Reversing the direction of the first axis, will for the observer looking along the positive direction, look as if the $\omega_1$ is reversed. We now use $\omega_{rs} = \epsilon_{rsn}\omega_n$ but here we have to be careful with $\epsilon_{rsn}$ when using the equation in the transformed coordinate system.

Looking at **4.312** $\epsilon_{stu}^{'} = \epsilon_{mnr}\frac{\partial z_m}{\partial z_s^{'}}\frac{\partial z_n}{\partial z_t^{'}}\frac{\partial z_r}{\partial z_u^{'}}$ and noting that $\frac{\partial z_1}{\partial z_1^{'}} = -1$ and 1 or 0 for the others, we have $\epsilon_{stu}^{'} = -\epsilon_{mnr}$. Now with **5.316** we get

$$\omega_{rs}^{'} = -\epsilon_{rsn}\omega_n^{'} \tag{5}$$

giving

$$\omega_{12}^{'} = -\omega_{12} \quad \omega_{13}^{'} = -\omega_{13} \quad \omega_{23}^{'} = \omega_{23} \tag{6}$$

Giving

$$\Omega^{'} = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \tag{7}$$

♦

## 5.32 p188 - Exercise 13

Consider three rigid bodies, $S, S^{'}, S^{"}$, turning about a common point. If all angular velocities are referred to common axes, show that the angular velocity tensors of $S^{"}$ relative to $S$ is the sum of the angular velocity tensors of $S^{'}$ relative to $S$ and of $S^{"}$ relative to $S^{'}$.

Consider the following three transformation from one axes system to another

$$\begin{cases} z_r^{'} = A_{rm} z_m & z_r = A_{mr} z_m^{'} & A_{mp} A_{mq} = \delta_{pq} & A_{pm} A_{qm} = \delta_{pq} \\ z_r^{"} = B_{rm} z_m^{'} & z_r^{'} = B_{mr} z_m^{"} & B_{mp} B_{mq} = \delta_{pq} & B_{pm} B_{qm} = \delta_{pq} \\ z_r^{"} = C_{rm} z_m & z_r = C_{mr} z_m^{"} & C_{mp} C_{mq} = \delta_{pq} & C_{pm} C_{qm} = \delta_{pq} \end{cases} \tag{1}$$

We then have,

$$\begin{cases} \omega_{pq}^{'} \left(S^{'}, S\right) = -A_{pm} \dot{A_{qm}} \\ \omega_{pq}^{"} \left(S^{"}, S^{'}\right) = -B_{pm} \dot{B_{qm}} \\ \omega_{pq}^{"} \left(S^{"}, S\right) = -C_{pm} \dot{C_{qm}} \end{cases} \tag{2}$$

From (1) we see that

$$C_{rq} = B_{rm} A_{mq} \tag{3}$$

And thus

$$\omega_{pq}^{"} \left(S^{"}, S\right) = -B_{pk} A_{km} \dot{(B_{qn} A_{nm})} \tag{4}$$

$$\Rightarrow \qquad = \underbrace{-A_{km} \dot{A_{nm}}}_{=\omega_{kn}^{'}\left(S^{'},S\right)} B_{pk} B_{qn} - \underbrace{A_{km} A_{nm}}_{=\delta_{kn}} B_{pk} \dot{B_{qn}} \tag{5}$$

$$\Rightarrow \qquad = \omega_{kn}^{'} \left(S^{'}, S\right) B_{pk} B_{qn} - \underbrace{B_{pn} \dot{B_{qn}}}_{=-\omega_{pq}^{"}\left(S^{"},S^{'}\right)} \tag{6}$$

The fist term of the right side expression is a bilinear map of the tensor $\omega_{kn}^{'} \left(S^{'}, S\right)$ from the reference axis $S^{'}$ to $S^{"}$. Hence we get

$$\omega_{pq}^{"} \left(S^{"}, S\right) = \omega_{pq}^{"} \left(S^{"}, S^{'}\right) + \omega_{pq}^{"} \left(S^{'}, S\right) \tag{7}$$

♦

## 5.33 p188 - Exercise 14

A freely moving particle is observed from a platform which rotates with angular velocity $\omega_r = n\delta_{r3}$, where $n$ is constant, relative to a Newtonian frame $S$ in which $z_r$ are rectangular Cartesians. Use **5.421** to find the equations of motion relative to $S^{'}$ in terms of coordinates $z_r^{'}$ in $S^{'}$, such that the axis of $z_3^{'}$ coincides permanently with the axis of $z_3$.

**5.421** gives (where the equation is expressed in term of the $z_r^{'}$

$$\begin{cases} mf_s = F_s^{'} + C_s^{'} + G_s^{'} \\ C_s^{'} = m\left[\dot{\omega}_{sn}^{'}\left(S^{'},S\right) + \omega_{sm}^{'}\left(S^{'},S\right)\omega_{nm}^{'}\left(S^{'},S\right)\right]z_n^{'} \\ C_s^{'} = 2m\omega_{sm}^{'}v_m^{'}\left(S^{'}\right) \end{cases} \tag{1}$$

We note the particle is free, so $F_s^{'} = 0$ and the angular velocity is a constant, so $\dot{\omega}_{sn}^{'}\left(S^{'},S\right) = 0$, and the equation simplify to

$$\begin{cases} f_s^{'} = K_s^{'} + J_s^{'} \\ K_s^{'} = \left[\omega_{sm}^{'}\left(S^{'},S\right)\omega_{nm}^{'}\left(S^{'},S\right)\right]z_n^{'} \\ J_s^{'} = 2\omega_{sm}^{'}v_m^{'}\left(S^{'}\right) \end{cases} \tag{2}$$

As $\omega_s = n\delta_{s3}$ and by the requirement that the axis of $z_3^{'}$ coincides permanently with the axis of $z_3$, it is not hard to see that

$$\begin{cases} \omega_{12}\left(S^{'},S\right) = n \\ \omega_{12}^{'}\left(S^{'},S\right) = n \\ \omega_{12}\left(S,S^{'}\right) = -n \\ \omega_{12}^{'}\left(S,S^{'}\right) = -n \end{cases} \tag{3}$$

while all other elements vanish.
We get

$$\begin{cases} K_1^{'} = \omega_{12}^{'}\left(S^{'},S\right)\omega_{12}^{'}\left(S^{'},S\right)z_1^{'} = n^2 z_1^{'} \\ K_1^{'} = \omega_{21}^{'}\left(S^{'},S\right)\omega_{21}^{'}\left(S^{'},S\right)z_1^{'} = n^2 z_1^{'} \\ K_3^{'} = 0 \end{cases} \tag{4}$$

$$\begin{cases} J_1^{'} = 2\omega_{12}^{'}\left(S^{'},S\right)v_2^{'}\left(S^{'}\right) = 2nv_2^{'}\left(S^{'}\right) \\ J_2^{'} = 2\omega_{21}^{'}\left(S^{'},S\right)v_1^{'}\left(S^{'}\right) = -2nv_1^{'}\left(S^{'}\right) \\ J_3^{'} = 0 \end{cases} \tag{5}$$

and get as equations of motion

$$\begin{cases} f_1' = n^2 z_1' + 2nv_2' \left(S'\right) \\ f_2' = n^2 z_1' - 2nv_1' \left(S'\right) \\ f_3' = 0 \end{cases} \tag{6}$$

♦

## 5.34 p188 - Exercise 15

If the tensor $I_{st}$ is defined by **5.335** for N dimensions, and $J_{nprq}$ is defined by **5.330**, establish the following relations:

$$J_{nprq} = (N-1)^{-1} I_{ss} (\delta_{nr}\delta_{pq} - \delta_{nq}\delta_{pr}) - \delta_{nr} I_{pq} + \delta_{pr} I_{nq}$$

$$J_{nppq} = I_{ss}$$

$$I_{nq} = (N-1)^{-1} (J_{nprq} - \delta_{nq} J_{nprq})$$

**5.421** and **5.421**:

$$\begin{cases} I_{st} = \delta_{st} \sum m z_q z_q - \sum m z_s z_t \\ J_{nprq} = \sum m (\delta_{nr} z_p z_q - \delta_{pr} z_n z_q) \end{cases} \tag{1}$$

The first equation can be expressed as $\sum m z_p z_q = \delta_{pq} \sum m z_k z_k - I_{pq}$ and $\sum m z_n z_q = \delta_{st} \sum m z_k z_k - I_{nq}$

giving

$$J_{nprq} = \delta_{nr}\delta_{pq} \sum m z_k z_k - \delta_{nr} I_{pq} - \delta_{pr}\delta_{st} \sum m z_k z_k + \delta_{pr} I_{nq} \tag{2}$$

$$= \sum m z_k z_k (\delta_{nr}\delta_{pq} - \delta_{nr} I_{pq}) - \delta_{nr} I_{pq} + \delta_{pr} I_{nq} \tag{3}$$

Now, consider the expressions

$$\begin{cases} I_{11} = \sum m z_q z_q - \sum m z_1 z_1 \\ I_{11} = \sum m z_q z_q - \sum m z_1 z_1 \\ \vdots \\ I_{NN} = \sum m z_q z_q - \sum m z_N z_N \end{cases} \tag{4}$$

Summing up these $N$ expressions we have

$$I_{ss} = N \left( \sum m z_q z_q \right) - \sum m z_q z_q \tag{5}$$

$$= (N-1) \sum m z_q z_q \tag{6}$$

$$\Rightarrow \qquad \sum m z_q z_q = I_{ss} (N-1)^{-1} \tag{7}$$

Plugging this in (3) we get

$$J_{nprq} = I_{ss} (N-1)^{-1} (\delta_{nr}\delta_{pq} - \delta_{nr} I_{pq}) - \delta_{nr} I_{pq} + \delta_{pr} I_{nq} \tag{8}$$

♦

## 5.35 p188 - Exercise 16 †

The motion of a dynamical system is represented by a curve in configuration-space. Using the kinematical line element, express the curvature as a function of its total energy $E$, and deduce that as $E$ tends to infinity, the trajectory tends to become a geodesic . Illustrate by considering a particle moving under gravity on a smooth sphere.

We have **5.512** and **5.533**:

$$\begin{cases} v^2 = a_{mn} v^m v^n = 2T \\ \kappa v^2 = X_r \nu^r \end{cases} \quad \Rightarrow \quad \kappa = \frac{X_r \nu^r}{2T} \tag{1}$$

First, we have to note that nothing is said about the nature of the generalized forces (conservative or not) and therefore we use **5.517**

$$dW = X_r dx^r \tag{2}$$

From this we can express $T$ as

$$T(s) = T_0 + \int_0^s dW \tag{3}$$

$$= T_0 + \int_0^s X_r dx^r \tag{4}$$

where $T_0$ is the kinetic energy at the initial configuration $s = 0$.

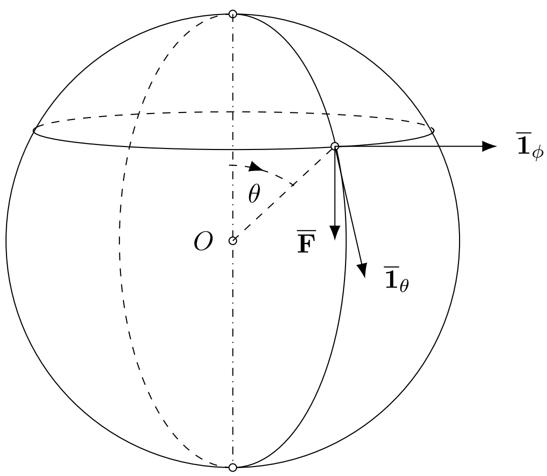
†

*Suppose now that the for $s \to +\infty$, $\int_0^s dW \to +\infty$. In that case, the kinematical energy will represent the total energy of the system, $T \to E$ and $\kappa \to 0$ for $E \to +\infty$ provided that $X_r \nu^r \not\to \pm\infty$ which we will assume .*

†

◊

Let's illustrate this with a particle on a smooth sphere moving under gravity.



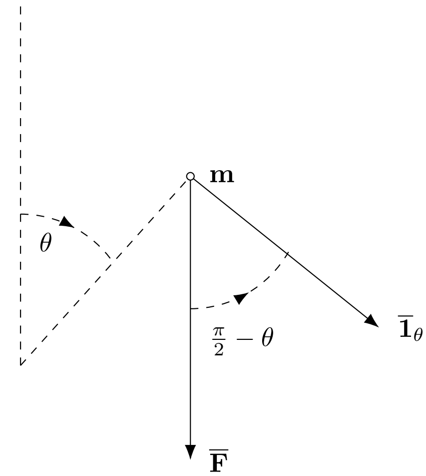




Figure 5.8: Physical components of the gravitational force tensor acting on a mass **m** on a sphere

**INTERMEZZO**:
The next calculation are showed how careful we have to be when applying blindly some formulas. The aim is to find the equation of motion in the configuration space equipped with the kinematical fundamental form, starting from the physical components of the force field.

We have as physical components:

$$\begin{cases} F_\phi^{phys} = 0 \\ \\ F_\theta^{phys} = mg\sin\theta \\ \end{cases} \tag{5}$$

Using **5.109** and **5.110** combined with the kinematical fundamental form, $ds^2 = 2Tdt^2 = \left(\sqrt{m}R\right)^2 d\theta^2 + \left(\sqrt{m}R\sin\theta\right)^2 d\phi^2$

$$(a_{mn}) = \begin{pmatrix} mR^2 & 0 \\ 0 & mR^2\sin^2\theta \end{pmatrix} \tag{6}$$

$$\begin{cases} h_1 = \sqrt{m}R & [M]^{\frac{1}{2}}[L] \\ h_2 = \sqrt{m}R\sin\theta & [M]^{\frac{1}{2}}[L] \\ F_\phi^{phys} = 0 & [M][L][T]^{-2} \\ F_\theta^{phys} = mg\sin\theta & [M][L][T]^{-2} \\ v_\phi^{phys} = R\sin\theta\dot{\phi} & [L][T]^{-1} \\ v_\theta^{phys} = R\dot{\theta} & [L][T]^{-1} \end{cases} \tag{7}$$

we have

$$\begin{cases} X^{\phi} = 0 & X_{\phi} = 0 \\ X_{\theta} = F_{\theta}^{phys} h_1 = m^{\frac{3}{2}} g R \sin\theta & [M]^{\frac{3}{2}}[L]^2[T]^{-2} \\ X^{\theta} = \frac{F_{\theta}^{phys}}{h_1} = \frac{mg\sin\theta}{\sqrt{m}R} = \frac{\sqrt{m}g\sin\theta}{R} & [M]^{\frac{1}{2}}[T]^{-2} \\ v_{\theta} = v_{\theta}^{phys} h_1 = \sqrt{m}R^2\dot{\theta} & [M]^{\frac{1}{2}}[L]^2[T]^{-1} \\ v^{\theta} = \frac{v_{\theta}^{phys}}{h_1} = \frac{R\dot{\theta}}{\sqrt{m}R} = \frac{1}{\sqrt{m}}\dot{\theta} & [M]^{-\frac{1}{2}}[T]^{-1} \\ v_{\phi} = v_{\phi}^{phys} h_1 = \sqrt{m}R^2 \sin\theta\ \dot{\phi} & [M]^{\frac{1}{2}}[L]^2[T]^{-1} \\ v^{\phi} = \frac{v_{\phi}^{phys}}{h_1} = \frac{R\sin\theta\ \dot{\phi}}{\sqrt{m}R} = \frac{1}{\sqrt{m}}\dot{\phi} & [M]^{-\frac{1}{2}}[T]^{-1} \end{cases} \tag{8}$$

Check 1:

$$X_{\theta} = a_{11} X^{\theta} \tag{9}$$
$$= mR^2 \frac{\sqrt{m}g\sin\theta}{R} \tag{10}$$
$$= m^{\frac{3}{2}} g R \sin\theta \tag{11}$$

Check 2:

$$v_{\theta} = a_{11} v^{\theta} + a_{12} v^{\phi} \tag{12}$$
$$= mR^2 \frac{1}{\sqrt{m}}\dot{\theta} \tag{13}$$
$$= \sqrt{m}R^2\dot{\theta} \tag{14}$$

Check 3:

$$\underbrace{f^{\theta} = X^{\theta}}_{\sim [M]^{\frac{1}{2}}[T]^{-2}} = \underbrace{\frac{\delta v^{\theta}}{\delta t}}_{\sim [M]^{-\frac{1}{2}}[T]^{-2}} \tag{15}$$
$$[M]^{\frac{1}{2}}[T]^{-2} \neq [M]^{-\frac{1}{2}}[T]^{-2} \tag{16}$$

So, obviously, we ran into a problem.

The reason is due to the fact that the physical components and the generalized coordinates are not equipped with the same metric. Indeed, consider a system with only one mass. Then there is no fundamental difference in the geometry of the two spaces ('physical' and 'generalized coordinates space' )but , if $ds$ is a distance in the physical space and $ds^{'}$ a distance in the 'generalized coordinates space' (abuse of language), then $ds^{'} \neq ds$ as $ds^{'} = \sqrt{m}ds$. So, the same physical object will be 'seen' stretched by a factor $\sqrt{m}$ when observed in the generalized configuration space. So let's begin again but with adapted conversion factors $h_i$.

$$\begin{cases} h_1 = R & [M]^{\frac{1}{2}}[L] \\ h_2 = R\sin\theta & [M]^{\frac{1}{2}}[L] \\ F_{\phi}^{phys} = 0 & [M][L][T]^{-2} \\ F_{\theta}^{phys} = mg\sin\theta & [M][L][T]^{-2} \\ v_{\phi}^{phys} = R\sin\theta\dot{\phi} & [L][T]^{-1} \\ v_{\theta}^{phys} = R\dot{\theta} & [L][T]^{-1} \end{cases} \tag{17}$$

we have

$$\begin{cases} X^{\phi} = 0 & X_{\phi} = 0 \\ X_{\theta} = F_{\theta}^{phys} h_1 = mgR\sin\theta & [M][L]^2[T]^{-2} \\ X^{\theta} = \frac{F_{\theta}^{phys}}{h_1} = \frac{mg\sin\theta}{R} = \frac{mg\sin\theta}{R} & [M][T]^{-2} \\ v_{\theta} = v_{\theta}^{phys} h_1 = R^2\dot{\theta} & [L]^2[T]^{-1} \\ v^{\theta} = \frac{v_{\theta}^{phys}}{h_1} = \frac{R\dot{\theta}}{R} = \dot{\theta} & [T]^{-1} \\ v_{\phi} = v_{\phi}^{phys} h_2 = R^2\sin^2\theta\ \dot{\phi} & [L]^2[T]^{-1} \\ v^{\phi} = \frac{v_{\phi}^{phys}}{h_2} = \frac{R\sin\theta\ \dot{\phi}}{R\sin\theta} = \dot{\phi} & [T]^{-1} \end{cases} \tag{18}$$

Check 1:

$$X_{\theta} = a_{11} X^{\theta} \tag{19}$$
$$= mR^2 \frac{mg\sin\theta}{R} \tag{20}$$
$$= m^2 gR\sin\theta \tag{21}$$
$$\neq mgR\sin\theta \tag{22}$$

Check 2:

$$v_{\theta} = a_{11} v^{\theta} + a_{12} v^{\phi} \tag{23}$$
$$= mR^2\dot{\theta} \tag{24}$$
$$\neq R^2\dot{\theta} \tag{25}$$

Check 3:

$$\underbrace{f^\theta = X^\theta}_{\sim[M][T]^{-2}} = \underbrace{\frac{\delta v^\theta}{\delta t}}_{\sim[T]^{-2}} \tag{26}$$

$$[M][T]^{-2} \neq [T]^{-2} \tag{27}$$

Again, we ran into problems.
In fact, the idea of going from the physical space into the configuration space, by some transformation rule is wrong because, in configuration space equipped with the kinematical fundamental form, velocity and generalized forces have their very own, specific definition:

$$\begin{cases} \text{velocity:} & v^r = \frac{dx^r}{dt} \\ \text{generalized forces:} & dW = X_r dx^r \end{cases} \tag{28}$$

and get for the velocities

$$v^\theta = \dot{\theta} \tag{29}$$

$$v^\phi = \dot{\phi} \tag{30}$$

and for the generalized forces

$$\begin{cases} dW = F_\theta^{phys} R d\theta & \text{in the physical space} \\ dW = X_\theta d\theta & \text{in the configurartion space} \end{cases} \tag{31}$$

and by the invariance of $dW$ we get

$$X_\theta = mgR\sin\theta \tag{32}$$

$$\Rightarrow \quad X^\theta = a^{11} mgR\sin\theta \tag{33}$$

$$= \frac{mgR\sin\theta}{mR^2} \tag{34}$$

$$= \frac{g\sin\theta}{R} \tag{35}$$

and get for check 3:

$$\underbrace{f^\theta = X^\theta}_{\sim[T]^{-2}} = \underbrace{\frac{\delta v^\theta}{\delta t}}_{\sim[T]^{-2}} \tag{36}$$

$$[T]^{-2} = [T]^{-2} \tag{37}$$

and get what we wanted.

Let's go on with the equations of motion in the configuration space: assuming that the gravitational force is conservative, we get as potential energy:

$$V = -mgR\cos\theta \tag{38}$$

We use equation **5.531**.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^s} - \frac{\partial L}{\partial x^s} = 0 \tag{39}$$

with for our case

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\sin^2\theta\dot{\phi}^2 + mgR\cos\theta \tag{40}$$

and get the set of equations of motion (the second column gives the dimensional analysis as a check for consistency)

$$\begin{cases} \frac{\ddot{\phi}}{\dot{\phi}} = -2\cot\theta\,\dot{\theta} & : \quad \frac{[T]^{-2}}{[T]^{-1}} \cong [T]^{-1} \\ \\ \ddot{\theta} - \left(\dot{\phi}\right)^2 \sin\theta\cos\theta = -\frac{g}{R}\sin\theta & : \quad [T]^{-2} + \left([T]^{-1}\right)^2 \cong \frac{[L][T]^{-2}}{[L]} \\ \end{cases} \tag{41}$$

♦

## 5.36 p189 - Exercise 17

A particle moves on a smooth sphere under action of gravity. Using he action line element, calculate the Gaussian curvature of configuration-space as a function of total energy $E$ and height $z$ above the centre of the sphere. Show that if the total energy is not sufficient to raise the particle to the top of the sphere, but only to a level $z = h$, then the Gaussian curvature tends to infinity as $z$ approaches $h$ from below.

Using polar spherical coordinates, the line element on the sphere is

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{1}$$

For the potential energy, we use the lowest point (along the axis of the gravitational filed) as reference. Hence the potential is given by

$$\begin{aligned} V &= mgR + mgz & (2)\\ &= R\left(1 + mg\cos\theta\right) & (3) \end{aligned}$$

Giving for the action line element (with a total energy of the system $E_0$)

$$d\sigma^2 = \left(E_0 - mgR - mgR\cos\theta\right) ds^2 \tag{4}$$

Be $E = E_0 - mgR$

$$(a_{mn}) = \begin{pmatrix} R^2\left(E - mgR\cos\theta\right) & 0 \\ 0 & R^2\left(E - mgR\cos\theta\right)\sin^2\theta \end{pmatrix} \tag{5}$$

**3.114** and the exercise (Riemann curvature of a 2-space) on page 112 gives:

$$\begin{cases} G = \frac{R_{1212}}{a_{11}a_{22}} \\ \\ R_{1212} = -\frac{1}{2}\partial^2_{11}a_{22} - \frac{1}{4}a^{11}\partial_1 a_{11}\partial_1 a_{22} + \frac{1}{4}a^{22}\partial_1 a_{22}\partial_1 a_{22} \end{cases} \tag{6}$$

With

$$\begin{cases} \partial_1 a_{11} = mgR^3 \sin\theta \\ \\ \partial_1 a_{22} = 2ER^2 \sin\theta\cos\theta + R^3 mg\left(\sin^3\theta - 2\sin\theta\cos^2\theta\right) \\ \\ \partial^2_{11} a_{22} = 2ER^2\left(\cos^2\theta - \sin^2\theta\right) + R^3 mg\left(3\cos\theta\sin^2\theta - 2\left(\cos^3\theta - 2\sin^2\theta\cos\theta\right)\right) \\ \quad = 2ER^2\left(\cos^2\theta - \sin^2\theta\right) + R^3 mg\left(7\cos\theta\sin^2\theta - 2\cos^3\theta\right) \\ \\ a^{11} = \frac{1}{R^2(E - mgR\cos\theta)} \\ \\ a^{22} = \frac{1}{R^2(E - mgR\cos\theta)\sin^2\theta} \end{cases} \tag{7}$$

We first try now to replace the expressions in $\theta$ with expressions in $R\cos\theta = z$ and $R\sin\theta = \sqrt{R^2 - z^2}$

$$\begin{cases} \partial_1 a_{11} = mgR^2\sqrt{R^2 - z^2} \\ \\ \partial_1 a_{22} = 2Ez\sqrt{R^2 - z^2} + mg\left(\left(R^2 - z^2\right)\sqrt{R^2 - z^2} - 2z^2\sqrt{R^2 - z^2}\right) \\ \\ \partial^2_{11} a_{22} = 2E\left(2z^2 - R^2\right) + mg\left(7z\left(R^2 - z^2\right) - 2z^3\right) \\ \\ a^{11} = \frac{1}{R^2(E - mgz)} \\ \\ a^{22} = \frac{1}{(E - mgz)(R^2 - z^2)} \end{cases} \tag{8}$$

$$\Rightarrow \begin{cases} \partial_1 a_{11} = mgR^2\sqrt{R^2 - z^2} & ML^4T^{-2} \\ \\ \partial_1 a_{22} = \left(mgR^2 + 2Ez - 3mgz^2\right)\sqrt{R^2 - z^2} & ML^4T^{-2} \\ \\ \partial^2_{11} a_{22} = 2E\left(2z^2 - R^2\right) + mgz\left(7R^2 - 9z^2\right) & ML^4T^{-2} \\ \\ a^{11} = \frac{1}{R^2(E - mgz)} & M^{-1}L^{-4}T^2 \\ \\ a^{22} = \frac{1}{(E - mgz)(R^2 - z^2)} & M^{-1}L^{-4}T^2 \\ \\ a_{11} = R^2\left(E - mgz\right) & ML^4T^{-2} \\ \\ a_{22} = \left(E - mgz\right)\left(R^2 - z^2\right) & ML^4T^{-2} \end{cases} \tag{9}$$

giving

$$R_{1212} = -\frac{1}{2}\partial_{11}^2 a_{22} - \frac{1}{4}a^{11}\partial_1 a_{11}\partial_1 a_{22} + \frac{1}{4}a^{22}\partial_1 a_{22}\partial_1 a_{22} \tag{10}$$

$$= \begin{cases} -E\left(2z^2 - R^2\right) - \frac{1}{2}mgz\left(7R^2 - 9z^2\right) \\ \\ -\frac{1}{4}\frac{1}{(E-mgz)}mg\left(R^2 - z^2\right)\left(mgR^2 + 2Ez - 3mgz^2\right) \\ \\ +\frac{1}{4}\frac{1}{(E-mgz)}\left(mgR^2 + 2Ez - 3mgz^2\right)^2 \end{cases} \tag{11}$$

$$= \begin{cases} \frac{1}{4}\frac{1}{(E-mgz)}\left[(E-mgz)\left(-E\left(2z^2 - R^2\right) - \frac{1}{2}mgz\left(7R^2 - 9z^2\right)\right)\right. \\ \\ -mg\left(R^2 - z^2\right)\left(mgR^2 + 2Ez - 3mgz^2\right) \\ \\ \left. + \left(mgR^2 + 2Ez - 3mgz^2\right)^2\right] \end{cases} \tag{12}$$

$$= \frac{R^2 - z^2}{(E - mgz)}\left[3m^2g^2z^2 - 4Emgz + E^2\right] \tag{13}$$

$$= \frac{R^2 - z^2}{(E - mgz)}(E - mgz)\left[E - 3mgz\right] \tag{14}$$

$$= \left(R^2 - z^2\right)\left[E - 3mgz\right] \tag{15}$$

For the Gauss curvature we get then

$$G = \frac{R_{1212}}{R^2\left(E - mgz\right)^2\left(R^2 - z^2\right)} \tag{16}$$

and so

$$G = \frac{E - 3mgz}{R^2\left(E - mgz\right)^2} \tag{17}$$

Be $h = \frac{E}{mg}$. From (17) we see that as long $z < \frac{h}{3}$, G is defined and positive. It becomes 0 for $z = \frac{h}{3}$ and negative for $z > \frac{h}{3}$ to become $-\infty$ for $z \to h$.
Remember that $E = E_0 - mgR$ with $E_0$ the total energy of the system and that the maximum potential energy is $V_{max} = 2mgR$. In order to reach the top, a particle starting from the bottom of the sphere ($V = 0$), should have at least a total energy $E_0 = 2mgR$.
Suppose now, that we configure the system so that the particle starts from the bottom and gets zero velocity at a point $z = h$. Then $E_0 = mg\left(R + h\right) < 2mgR$ and so $E = E_0 - mgR = mgh$.
(17) becomes

$$G = \frac{h - 3z}{mgR^2\left(h - z\right)^2} \tag{18}$$

$$\Rightarrow \qquad \lim_{z \to h} G = -\infty \tag{19}$$

♦

## 5.37 p189 - Exercise 18

Show that the equations of motion of a rigid body with a fixed point may be written in either of the forms

$(a)$ $\qquad \dot{h}'_r + \omega'_{mr}\left(S', S\right) h'_m = M'_{rs},$

$(b)$ $\qquad \dot{h}'_r - K'_{rmn} h'_m h'_n = M'_{rs},$

where $h'_r$ are the components on $z'$-axes (moving with the body) of angular moment as given in **5.338** and $K'_{rmn}$ is a certain moment of inertia tensor. Evaluate the components $K'_{rmn}$ in terms of the moments and products of inertia.

We use **5.329** ,**5.231**,**5.233** and **5.424**:

$$M'_{rs} = \epsilon_{rsn} M'_n \tag{1}$$

$$h'_{rs} = \epsilon_{rsn} h'_n \tag{2}$$

$$h'_{np} = J'_{nprq} \omega'_{rq} \tag{3}$$

$$M'_{ab} = J'_{abrq} \dot{\omega}'_{rq}\left(S', S\right) + J'_{cdrq}\left(\delta_{ac}\delta_{du}\delta_{bv} + \delta_{bd}\delta_{cu}\delta_{av}\right)\omega'_{uv}\left(S', S\right)\omega'_{rq}\left(S', S\right) \tag{4}$$

Then, using $(1),(2),(3)$ in $(4)$ and contracting the terms in $\delta_{ij}$

$$M'_{ab} = \dot{h_{ab}}' + h'_{au}\omega'_{ub}\left(S', S\right) + h'_{ub}\omega'_{ua}\left(S', S\right) \tag{5}$$

$$\Leftrightarrow \qquad \epsilon_{abn} M'_n = \epsilon_{abn}\dot{h_n}' + \epsilon_{aun} h'_n \omega'_{ub}\left(S', S\right) + \epsilon_{ubn} h'_n \omega'_{ua}\left(S', S\right) \tag{6}$$

$$\times\epsilon_{abt} \quad \Rightarrow \quad \epsilon_{abt}\epsilon_{abn} M'_n = \epsilon_{abt}\epsilon_{abn}\dot{h_n}' + \epsilon_{abt}\epsilon_{aun} h'_n \omega'_{ub}\left(S', S\right) + \epsilon_{bat}\epsilon_{bun} h'_n \omega'_{ua}\left(S', S\right) \tag{7}$$

$$\Rightarrow \quad 2M'_t = 2\dot{h_t}' + \left(\delta_{bu}\delta_{tn} - \delta_{bn}\delta_{tu}\right) h'_n \omega'_{ub}\left(S', S\right) + \left(\delta_{au}\delta_{tn} - \delta_{an}\delta_{tu}\right) h'_n \omega'_{ua}\left(S', S\right) \tag{8}$$

$$= 2\dot{h_t}' + h'_t \omega'_{bb}\left(S', S\right) - h'_b \omega'_{tb}\left(S', S\right) + h'_t \omega'_{uu}\left(S', S\right) - h'_a \omega'_{ta}\left(S', S\right) \tag{9}$$

And so,

$$M'_r = \dot{h_r}' + \omega'_{mr}\left(S', S\right) h'_m \tag{10}$$

$\Diamond$

Let's try to express equation (4) but with the inertia tensor $I_{ij}$ as parameter. We have **5.332**:

$$\frac{d\left(I_{st}\omega_t\left(S', S\right)\right)}{dt} = M_s \tag{11}$$

Let's express this in the coordinate system $S'$ so that $I'_{sr}$ will not depend of the time. Be $A_{ij}$ the map from $S^{'}$ to $S$. Then:

$$\frac{d\left(A_{ks}I^{'}_{st}\omega^{'}_{t}\left(S^{'},S\right)\right)}{dt}=M_{s} \tag{12}$$

$$\times \quad A_{ps} \qquad A_{ps}\frac{d\left(A_{ks}I^{'}_{st}\omega^{'}_{t}\left(S^{'},S\right)\right)}{dt}=M^{'}_{p} \tag{13}$$

$$A_{ps}\dot{A_{ks}}I^{'}_{kt}\omega^{'}_{t}\left(S^{'},S\right)+A_{ps}A_{ks}I^{'}_{kt}\dot{\omega}^{'}_{t}\left(S^{'},S\right)=M^{'}_{p} \tag{14}$$

We have

$$\begin{cases} \mathbf{5.408} \quad \omega^{'}_{ts}\left(S^{'},S\right)=A_{tm}\dot{A}_{sm} \\ \mathbf{5.401} \quad A_{mp}A_{mq}=\delta_{pq} \qquad\qquad A_{pm}A_{qm}=\delta_{pq} \end{cases} \tag{15}$$

So, (14) becomes

$$\omega^{'}_{pk}\left(S^{'},S\right)I^{'}_{kt}\omega^{'}_{t}\left(S^{'},S\right)+I^{'}_{pt}\dot{\omega}^{'}_{t}\left(S^{'},S\right)=M^{'}_{p} \tag{16}$$

We use

$$\omega^{'}_{pk}\left(S^{'},S\right)=\epsilon_{pkm}\omega^{'}_{m}\left(S^{'},S\right) \tag{17}$$

and get for (16):

$$I^{'}_{pt}\dot{\omega}^{'}_{t}\left(S^{'},S\right)+\epsilon_{pkm}\omega^{'}_{m}\left(S^{'},S\right)I^{'}_{kt}\omega^{'}_{t}\left(S^{'},S\right)=M^{'}_{p} \tag{18}$$

Note also that $h^{'}_{s}=I_{sr}\omega^{'}_{r}\left(S^{'},S\right)$. Indeed,

$$h^{'}_{np}=J^{'}_{npqr}\omega^{'}_{rq}\left(S^{'},S\right) \tag{19}$$

$$=J^{'}_{npqr}\epsilon_{rqm}\omega^{'}_{m}\left(S^{'},S\right) \tag{20}$$

$$\times \quad \frac{1}{2}\epsilon_{snp} \qquad h^{'}_{s}=\underbrace{\frac{1}{2}J^{'}_{npqr}\epsilon_{snp}\epsilon_{rqm}}_{=I^{'}_{sm}}\omega^{'}_{m}\left(S^{'},S\right) \tag{21}$$

$$\Rightarrow \qquad h^{'}_{s}=I^{'}_{sm}\omega^{'}_{m}\left(S^{'},S\right) \tag{22}$$

and (18) becomes

$$\dot{h}^{'}_{p}\left(S^{'},S\right)+\epsilon_{pkm}I^{'}_{kt}\omega^{'}_{m}\left(S^{'},S\right)\omega^{'}_{t}\left(S^{'},S\right)=M^{'}_{p} \tag{23}$$

Let's examine the term $\epsilon_{pkm}I^{'}_{kt}\omega^{'}_{m}\left(S^{'},S\right)\omega^{'}_{t}\left(S^{'},S\right)$ and let's write tentatively

$$K'_{pqn}h^{'}_{q}h^{'}_{n}=\epsilon_{pkm}I^{'}_{kt}\omega^{'}_{m}\left(S^{'},S\right)\omega^{'}_{t}\left(S^{'},S\right) \tag{24}$$

using $h_r^{'} = I_{rv}^{'}\omega_v^{'}\left(S^{'},S\right)$ in (24) we get

$$K_{pkn}^{'}I_{kt}^{'}I_{nm}^{'}\omega_m^{'}\left(S^{'},S\right)\omega_t^{'}\left(S^{'},S\right) = \epsilon_{pkm}I_{kt}^{'}\omega_m^{'}\left(S^{'},S\right)\omega_t^{'}\left(S^{'},S\right) \tag{25}$$

$$\Rightarrow \qquad K_{pkn}^{'}I_{kt}^{'}I_{nm}^{'} = \epsilon_{pkm}I_{kt}^{'} \tag{26}$$

$$\Rightarrow \qquad K_{pkn}^{'}I_{nm}^{'} = \epsilon_{pkm} \tag{27}$$

$$\times \quad \epsilon_{pkt} \Rightarrow \qquad \epsilon_{pkt}K_{pkn}^{'}I_{nm}^{'} = \delta_{mt} \tag{28}$$

Let's write

$$I_{tn}^{'-1} = \epsilon_{pkt}K_{pkn}^{'}$$

We can truly consider $I_{tn}^{'-1}$ as the inverse of $I_{tn}^{'}$ due to (28) and the fact that $I_{tm}^{'}$ is represented as a symmetric square matrix with real numbers as elements and hence has a non-zero determinant and has indeed an inverse. Multiplying (27) by $I_{mt}^{'-1}$ gives us finally

$$\mathbf{K_{pkt}^{'} = \epsilon_{pkm}I_{mt}^{'-1}}$$

**Q: Why the minus sign in the question?**
Let's now calculate $I_{tn}^{'-1}$

$$I_{tn}^{'} = \begin{pmatrix} I_{11}^{'} & I_{12}^{'} & I_{13}^{'} \\ I_{12}^{'} & I_{22}^{'} & I_{23}^{'} \\ I_{13}^{'} & I_{23}^{'} & I_{33}^{'} \end{pmatrix} \tag{29}$$

The determinant

$$\Delta = I_{12}^{'}I_{22}^{'}I_{33}^{'} + 2I_{12}^{'}I_{13}^{'}I_{23}^{'} - I_{11}^{'}I_{23}^{'2} - I_{22}^{'}I_{13}^{'2} - I_{33}^{'}I_{12}^{'2} \tag{30}$$

giving

$$I_{tn}^{'-1} = \frac{1}{\Delta}\begin{pmatrix} I_{22}^{'}I_{33}^{'} - I_{23}^{'2} & -I_{12}^{'}I_{33}^{'} + I_{13}^{'}I_{23}^{'} & I_{12}^{'}I_{23}^{'} - I_{22}^{'}I_{13}^{'} \\ -I_{12}^{'}I_{33}^{'} + I_{13}^{'}I_{23}^{'} & I_{11}^{'}I_{33}^{'} - I_{13}^{'2} & -I_{11}^{'}I_{23}^{'} + I_{12}^{'}I_{13}^{'} \\ I_{12}^{'}I_{23}^{'} - I_{22}^{'}I_{13}^{'} & -I_{11}^{'}I_{23}^{'} + I_{12}^{'}I_{13}^{'} & I_{11}^{'}I_{22}^{'} - I_{12}^{'2} \end{pmatrix} \tag{31}$$

giving for $\mathbf{K}^{'}_{\mathbf{pkt}} = \epsilon_{\mathbf{pkm}} \mathbf{I}^{'-1}_{\mathbf{mt}}$

$$\begin{cases} K^{'}_{121} = \frac{1}{\Delta}\left(I^{'}_{12}I^{'}_{23} - I^{'}_{22}I^{'}_{13}\right) \\ K^{'}_{122} = \frac{1}{\Delta}\left(-I^{'}_{11}I^{'}_{23} + I^{'}_{12}I^{'}_{13}\right) \\ K^{'}_{123} = \frac{1}{\Delta}\left(I^{'}_{11}I^{'}_{22} - I^{'2}_{12}\right) \\ K^{'}_{131} = -\frac{1}{\Delta}\left(-I^{'}_{12}I^{'}_{33} + I^{'}_{13}I^{'}_{23}\right) \\ K^{'}_{132} = \frac{1}{\Delta}\left(I^{'}_{11}I^{'}_{33} - I^{'2}_{13}\right) \\ K^{'}_{133} = -\frac{1}{\Delta}\left(-I^{'}_{11}I^{'}_{23} + I^{'}_{12}I^{'}_{13}\right) \\ K^{'}_{231} = -\frac{1}{\Delta}\left(I^{'}_{22}I^{'}_{33} - I^{'2}_{23}\right) \\ K^{'}_{232} = \frac{1}{\Delta}\left(-I^{'}_{11}I^{'}_{23} + I^{'}_{12}I^{'}_{13}\right) \\ K^{'}_{233} = \frac{1}{\Delta}\left(I^{'}_{12}I^{'}_{23} - I^{'}_{22}I^{'}_{13}\right) \end{cases} \tag{32}$$

all others can be found by symmetry considerations.

♦

## 5.38 p189 - Exercise 19

A rigid body turns about a fixed point $O$in a flat space of $N$ dimensions. prove that if $N$ is odd, there exists at any instant a line $OP$ of particles instantaneously at rest, but that, if $N$ is even, no point other than $O$ is, in general, instantaneously at rest. Show that if $N = 4$, there are are points other than $O$ instantaneously at rest if, and only if,

$$\omega_{23}\omega_{14} + \omega_{31}\omega_{24} + \omega_{12}\omega_{34} = 0$$

Consider **5.310**

$$v_p = -\omega_{pr} z_r \tag{1}$$

What we seek, is a vector $z_r$ so that

$$v_p = -\theta\omega_{pr} z_r = 0 \quad \theta \in \mathbb{R} \tag{2}$$

which means that we have to solve the homogeneous system of linear equations

$$\Omega\mathbf{z} = 0 \tag{3}$$

with $\Omega$ the skew-symmetric matrix containing the elements of the tensor $\omega_{pr}$. From algebra, we know that when the dimension of a skew-symmetric matrix is odd, then it's determinant is zero, and hence the homogeneous system will have an infinity of solutions that can be of the form $z_r = a_r Z_N \quad r = \{1, 2, \ldots, N-1\}$, (we take the last coordinate as free parameter). This represents a line along which, all velocities are zero.
On the contrary if $N$ is even, the determinant might be non-zero and the system will not have any solution except the trivial solution $z_r = 0$.

$\Diamond$

Let's investigate this for $N = 4$. We have for (3):

$$\Omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix} \tag{4}$$

giving

$$\det\{\Omega\} = -\omega_{12}\begin{vmatrix} -\omega_{12} & \omega_{23} & \omega_{24} \\ -\omega_{13} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{34} & 0 \end{vmatrix} + \omega_{13}\begin{vmatrix} -\omega_{12} & 0 & \omega_{24} \\ -\omega_{13} & -\omega_{23} & \omega_{34} \\ -\omega_{14} & -\omega_{24} & 0 \end{vmatrix} - \omega_{14}\begin{vmatrix} -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \\ -\omega_{14} & -\omega_{24} & -\omega_{34} \end{vmatrix} \tag{5}$$

$$= \begin{cases} -\omega_{12}\left(-\omega_{12}\omega_{34}\omega_{34} - \omega_{23}\omega_{14}\omega_{34} + \omega_{24}\omega_{13}\omega_{34}\right) \\ +\omega_{13}\left(-\omega_{12}\omega_{14}\omega_{34} + \omega_{24}\left(\omega_{13}\omega_{24} - \omega_{23}\omega_{14}\right)\right) \\ -\omega_{14}\left(-\omega_{12}\omega_{23}\omega_{34} + \omega_{23}\left(\omega_{13}\omega_{24} - \omega_{23}\omega_{14}\right)\right) \end{cases} \tag{6}$$

$$= \begin{cases} +\omega_{12}\omega_{12}\omega_{34}\omega_{34} + \omega_{12}\omega_{14}\omega_{23}\omega_{34} - \omega_{12}\omega_{13}\omega_{24}\omega_{34} \\ -\omega_{12}\omega_{13}\omega_{14}\omega_{34} + \omega_{13}\omega_{13}\omega_{24}\omega_{24} - \omega_{13}\omega_{14}\omega_{23}\omega_{24} \\ +\omega_{12}\omega_{14}\omega_{23}\omega_{34} - \omega_{13}\omega_{14}\omega_{23}\omega_{24} + \omega_{14}\omega_{14}\omega_{23}\omega_{23} \end{cases} \tag{7}$$

Define

$$\begin{cases} A = \omega_{12}\omega_{34} \\ B = \omega_{13}\omega_{24} \\ C = \omega_{14}\omega_{23} \end{cases} \tag{8}$$

then we can write (7) as

$$\det\{\Omega\} = A^2 + B^2 + C^2 - 2AB - 2BC + 2AB \tag{9}$$

$$= (A - B + C)^2 \tag{10}$$

So, in the space of even dimensions,the system of homogeneous linear equations will have non-trivial solutions, only if

$$\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0$$

$\Diamond$

**Let's find now some possible instantaneous lines of rotation.**

Suppose $N$=4.

Let's define a line with

$$(z_r) = \theta\begin{pmatrix} \omega_{23} - \omega_{24} + \omega_{34} \\ -\omega_{13} + \omega_{14} + \omega_{34} \\ \omega_{12} - \omega_{14} - \omega_{24} \\ -\omega_{12} + \omega_{13} + \omega_{23} \end{pmatrix} \tag{11}$$

Then calculating the velocities with (1) we get

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = -\theta \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix} \begin{pmatrix} \omega_{23} - \omega_{24} + \omega_{34} \\ -\omega_{13} + \omega_{14} + \omega_{34} \\ \omega_{12} - \omega_{14} - \omega_{24} \\ -\omega_{12} + \omega_{13} + \omega_{23} \end{pmatrix} \tag{12}$$

$$= -\theta \begin{pmatrix} \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} \\ -\omega_{12}\omega_{34} + \omega_{13}\omega_{24} - \omega_{14}\omega_{23} \\ -\omega_{12}\omega_{34} + \omega_{13}\omega_{24} - \omega_{14}\omega_{23} \\ -\omega_{12}\omega_{34} + \omega_{13}\omega_{24} - \omega_{14}\omega_{23} \end{pmatrix} \tag{13}$$

So the velocities will vanish when

$$\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0$$

$\Diamond$

Suppose $N$ is odd. Let's define the following vector

$$\omega_{i_1} = \frac{1}{2^{\frac{N-1}{2}} \frac{N-1}{2}!} \epsilon_{i_1 i_2 \ldots i_N} \prod_{k=1}^{\frac{N-1}{2}} \omega_{i_{2k} i_{2k+1}} \tag{14}$$

and a line

$$z_{i_1} = \theta \omega_{i_1} \quad (\theta \in \mathbb{R}) \tag{15}$$

First we note that $\omega_{i_1}$ (and hence $z_{i_1}$) is not a null-vector:
Let's consider in (14) the terms consisting of the permutation of the sequence of pairs

$$\{(i_2, i_3), (i_4, i_5), (i_6, i_7), \ldots, (i_{N-1}, i_N)\}$$

This sequence contains $\frac{N-1}{2}$ pairs and so can be arranged in $\frac{N-1}{2}!$ ways. As for each pair we have two valid possibilities e.g. $(i_2, i_3)$ and $(i_3, i_2)$ and as a sequence contains $\frac{N-1}{2}$ pairs, we will have for a given order of pairs $2^{\frac{N-1}{2}}$ possibilities. So in (1) there will be $2^{\frac{N-1}{2}} \frac{N-1}{2}!$ terms consisting of the permutation of the sequence of pairs $\{(i_2, i_3), (i_4, i_5), (i_6, i_7), \ldots, (i_{N-1}, i_N)\}$.
Without loss of generality, suppose that $\epsilon_{i_1 i_2 \ldots i_N}$ is positive and also all $\omega_{i_{2k} i_{2k+1}}$ are positive. Let's first consider a permutation of two pairs in the sequence $\{(i_2, i_3), (i_4, i_5), (i_6, i_7), \ldots, (i_{N-1}, i_N)\}$. Obviously, this does not change the product of the $\omega_{i_{2k} i_{2k+1}}$. Also $\epsilon_{i_1 i_2 \ldots i_N}$ will hold it's initial sign as the considered permutation needs two permutation of indices.
Next consider a permutation in one of the pairs of the sequence. Obviously $\epsilon_{i_1 i_2 \ldots i_N}$ will change sign but also the picked $\omega_{i_{2k} i_{2k+1}}$(skew-symmetric).
Conclusion, all $2^{\frac{N-1}{2}} \frac{N-1}{2}!$ terms can be reduced to the sum of $2^{\frac{N-1}{2}} \frac{N-1}{2}!$ of a same quantity and the $\omega_{i_1}$ will not trivially be zero.

Let's consider now **5.310**

$$v_p = -\omega_{pi_1} z_{i_1} \tag{16}$$

$$(14): \qquad v_p = -\frac{1}{2^{\frac{N-1}{2}} \frac{N-1}{2}!} \theta \epsilon_{i_1 i_2 \dots i_N} \omega_{pi_1} \prod_{k=1}^{\frac{N-1}{2}} \omega_{i_{2k} i_{2k+1}} \tag{17}$$

On the right side, for having a non-zero term, we need that $p \neq i_1$ ( $\omega_{st}$ skew-symmetric ). This leaves us with only $N-1$ possible choices in the indices but as $\epsilon_{i_1 i_2 \dots i_N}$ needs $N$ mutual different indices it is obvious that each term in (17) will have a $\epsilon_{i_1 i_2 \dots i_N} = 0$
Conclusion, all $v_p$ are zero and hence the defined line in (15) is an instantaneous line of rotation.

♦

## 5.39 p189 - Exercise 20

The equations **5.329** do not determine $J_{nprq}$ uniquely. Why? As an alternative to **5.330**, we can require $J_{npqr}$ to be skew-symmetric in the last two suffixes. Show that this defines $J_{nprq}$ uniquely as follows:

$$J_{nprq} = \frac{1}{2}\sum m\left(\delta_{nr}z_p z_q + \delta_{pq}z_n z_r - \delta_{nq}z_p z_r - \delta_{pr}z_n z_q\right)$$

Prove that$J_{nprq}$, as defined here, has the same symmetries as the covariant curvature tensor (see **3.115**, **3.116**) and that, for $N = 3$, we have

$$I_{st} = \frac{1}{2}\epsilon_{snp}\epsilon_{trq}J_{nprq}, \quad J_{nprq} = \frac{1}{2}\epsilon_{snp}\epsilon_{trq}I_{st}$$

The equations **5.329** , $h_{np} = J_{nprq}\omega_{rq}$ do not determine $J_{nprq}$ uniquely because $\omega_{rq}$ is skew symmetric, so all elements at the positions $J_{np(rr)}$ can be chosen arbitrarily and still comply with the equation.
Consider now the expression

$$\begin{aligned}
J'_{nprq} &= \frac{1}{2}\left(J_{nprq} - J'_{npqr}\right) && (1)\\
\Rightarrow \qquad h'_{np} &= J'_{nprq}\omega_{rq} && (2)\\
&= \frac{1}{2}\left(J_{nprq}\omega_{rq} - J_{npqr}\omega_{rq}\right) && (3)\\
&= \frac{1}{2}\left(J_{nprq}\omega_{rq} + J_{npqr}\omega_{qr}\right) && (4)\\
&= \frac{1}{2}\left(J_{nprq}\omega_{rq} + J_{nprq}\omega_{rq}\right) && (5)\\
&= h_{np} && (6)
\end{aligned}$$

So this expression $J_{nprq} = \frac{1}{2}\sum m\left(\delta_{nr}z_p z_q + \delta_{pq}z_n z_r - \delta_{nq}z_p z_r - \delta_{pr}z_n z_q\right)$ still describes the dynamical system and we note that this expression is skew-symmetric in the two last suffixes:

$$\begin{aligned}
J_{np(rr)} &= \frac{1}{2}\sum m\left(\delta_{nr}z_p z_r + \delta_{pr}z_n z_r - \delta_{nr}z_p z_r - \delta_{pr}z_n z_r\right) = 0 && (7)\\
J_{npqr} &= \frac{1}{2}\sum m\left(\delta_{nq}z_p z_r + \delta_{pr}z_n z_q - \delta_{nr}z_p z_q - \delta_{pq}z_n z_r\right) && (8)\\
&= -\frac{1}{2}\sum m\left(-\delta_{nq}z_p z_r - \delta_{pr}z_n z_q + \delta_{nr}z_p z_q + \delta_{pq}z_n z_r\right) && (9)\\
&= -J_{nprq} && (10)
\end{aligned}$$

**Symmetries to prove:**

$$\begin{cases} J_{nprq} = -J_{pnrq}, \quad J_{nprq} = -J_{npqr}, \quad J_{nprq} = J_{rqnp} \\ J_{nprq} + J_{nrqp} + J_{nqpr} = 0 \end{cases} \qquad (11)$$

The second identity of (11) is already proven as $J_{nprq}$ is skew-symmetric in the last two suffixes.

For the rest:

$$\begin{aligned}
J_{nprq} &= \frac{1}{2}\sum m\left(\delta_{nr}z_pz_q + \delta_{pq}z_nz_r - \delta_{nq}z_pz_r - \delta_{pr}z_nz_q\right) && (12)\\
&= -\frac{1}{2}\sum m\left(-\delta_{nr}z_pz_q - \delta_{pq}z_nz_r + \delta_{nq}z_pz_r + \delta_{pr}z_nz_q\right) && (13)\\
&= J_{pnrq} && (14)
\end{aligned}$$

and

$$\begin{aligned}
J_{nprq} &= \frac{1}{2}\sum m\left(\delta_{nr}z_pz_q + \delta_{pq}z_nz_r - \delta_{nq}z_pz_r - \delta_{pr}z_nz_q\right) && (15)\\
J_{rqnp} &= \frac{1}{2}\sum m\left(\delta_{rn}z_qz_p + \delta_{qp}z_rz_n - \delta_{rp}z_qz_n - \delta_{qn}z_rz_p\right) && (16)\\
&= J_{pnrq} && (17)
\end{aligned}$$

and

$$J_{nprq} + J_{nrqp} + J_{nqpr} = \begin{cases} +\frac{1}{2}\sum m\left(\delta_{nr}z_pz_q + \delta_{pq}z_nz_r - \delta_{nq}z_pz_r - \delta_{pr}z_nz_q\right) \\ +\frac{1}{2}\sum m\left(\delta_{nq}z_pz_r + \delta_{rp}z_nz_q - \delta_{np}z_qz_r - \delta_{rq}z_nz_p\right) \\ +\frac{1}{2}\sum m\left(\delta_{np}z_rz_q + \delta_{qr}z_nz_p - \delta_{nr}z_qz_p - \delta_{qp}z_nz_r\right) \end{cases} = 0 \qquad (18)$$

For the last part: From **5.33**, ($J^{'}_{nprq}$ being a not necessarily skew-symmetric tensor)

$$I_{st} = \frac{1}{2}J^{'}_{nprq}\epsilon_{rqt}\epsilon_{snp} \qquad (19)$$

or

$$\begin{aligned}
I_{st} &= \frac{1}{2}J^{'}_{npqr}\epsilon_{qrt}\epsilon_{snp} && (20)\\
&= -\frac{1}{2}J'_{npqr}\epsilon_{rqt}\epsilon_{snp} && (21)
\end{aligned}$$

Adding (19) and (21) gives

$$\begin{aligned}
2I_{st} &= \underbrace{\frac{1}{2}\left(J^{'}_{npqr} - J'_{npqr}\right)}_{=J_{npqr}}\epsilon_{rqt}\epsilon_{snp} && (22)\\
\Rightarrow \qquad I_{st} &= \frac{1}{2}J_{npqr}\epsilon_{rqt}\epsilon_{snp} && (23)
\end{aligned}$$

And

$$\begin{aligned}
I_{st}\epsilon_{trq}\epsilon_{snp} &= \frac{1}{2}J^{'}_{kjuv}\underbrace{\epsilon_{tuv}\epsilon_{trq}}_{=\delta_{ur}\delta_{vq}-\delta_{uq}\delta_{vr}}\underbrace{\epsilon_{skj}\epsilon_{snp}}_{=\delta_{kn}\delta_{jp}-\delta_{kp}\delta_{jn}} && (24)\\
&= -\frac{1}{2}J'_{npqr}\epsilon_{rqt}\epsilon_{snp} && (25)
\end{aligned}$$

expanding the right product we get

$$I_{st}\epsilon_{trq}\epsilon_{snp} = \frac{1}{2}\left(J_{nprq} + J_{pnqr} - J_{npqr} - J_{pnrq}\right) \tag{26}$$

And considering the symmetries described previously we get

$$\begin{aligned} I_{st}\epsilon_{trq}\epsilon_{snp} &= \frac{1}{2}4J_{nprq} && (27)\\ &= 2J_{nprq} && (28)\\ \Rightarrow \qquad J_{nprq} &= \frac{1}{2}I_{st}\epsilon_{trq}\epsilon_{snp} && (29) \end{aligned}$$

♦

# Applications to Hydrodynamics, Elasticity, and Electromagnetic radiation

## 5.1 p191 - Exercise

A fluid rotates as a rigid body about the axis of $z_3$ with variable angular velocity $\omega(t)$. Write out explicitly the three Lagrangian equations **6.101** and the three Eulerian equations **6.103**.

The motion described reduces to a motion in a $V_2$ plane with $z_3$ a constant for a definite particle.

**Lagrangian**

A particular particle with starting coordinates $\left(z_1^{(*)}, z_2^{(*)}, z_3^{(*)}\right)$ will describe a circle with radius $\sqrt{z_1^{(*)2} + z_2^{(*)2}}$ in the plane $V_2$ parallel with the $1, 2$ axes. Taking axis 1 as reference to determine the instantaneous angle $\theta$ of the vertex $OP$ (origin and particle) we get

$$\begin{cases} z_1 = \sqrt{z_1^{(*)2} + z_2^{(*)2}} \cos\left(\theta(t) + \phi_0\right) \\ z_2 = \sqrt{z_1^{(*)2} + z_2^{(*)2}} \sin\left(\theta(t) + \phi_0\right) \\ z_3 = z_3^{(*)} \end{cases} \tag{1}$$

with

$$\phi_0 = \arctan \frac{z_2^{(*)}}{z_1^{(*)}} \tag{2}$$

Note that $\omega(t)$ is not a constant, so

$$\theta(t) = \int_0^t \omega(\tau) d\tau \tag{3}$$

and get

$$\begin{cases} z_1 = \sqrt{z_1^{(*)2} + z_2^{(*)2}} \cos\left(\int_0^t \omega(\tau) d\tau + \phi_0\right) \\ \\ z_2 = \sqrt{z_1^{(*)2} + z_2^{(*)2}} \sin\left(\int_0^t \omega(\tau) d\tau + \phi_0\right) \\ \\ z_3 = z_3^{(*)} \\ \\ \phi_0 = \arctan \frac{z_2^{(*)}}{z_1^{(*)}} \end{cases} \tag{4}$$

$\Diamond$

**Eulerian**

The equations get simplified and reduce to a motion of a particle on a circle.

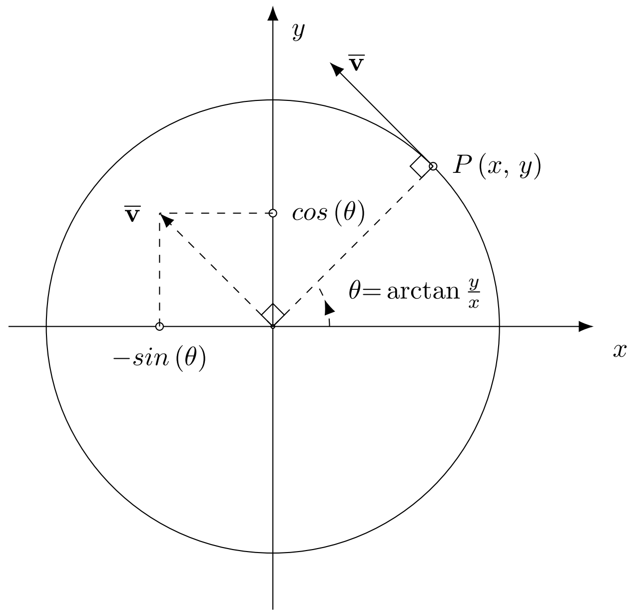



Figure 5.1: Eulerian viewpoint of a spinning fluid

$$\begin{cases} v_1 = -\sqrt{z_1^2 + z_2^2}\omega(t) \sin\left(\arctan \frac{z_2}{z_1}\right) \\ v_2 = \sqrt{z_1^2 + z_2^2}\omega(t) \cos\left(\arctan \frac{z_2}{z_1}\right) \\ v_3 = 0 \end{cases} \tag{5}$$

♦

## 5.2 p191 - Exercise

Compute the components of acceleration for the motion described in the preceding exercise.

We have

$$\begin{cases} v_1 = -\sqrt{z_1^2+z_2^2}\omega(t)\sin\left(\arctan\frac{z_2}{z_1}\right) \\ v_2 = \sqrt{z_1^2+z_2^2}\omega(t)\cos\left(\arctan\frac{z_2}{z_1}\right) \\ v_3 = 0 \end{cases} \tag{1}$$

and

$$f_r = \partial_t v_r + v_{r,s}v_s \tag{2}$$

$$\begin{cases} \partial_t v_1 = -\sqrt{z_1^2+z_2^2}\dot{\omega}(t)\sin\left(\arctan\frac{z_2}{z_1}\right) \\ \partial_t v_2 = \sqrt{z_1^2+z_2^2}\dot{\omega}(t)\cos\left(\arctan\frac{z_2}{z_1}\right) \\ v_{1,1} = -\omega(t)\left[\frac{z_1}{\sqrt{z_1^2+z_2^2}}\sin\left(\arctan\frac{z_2}{z_1}\right) - \sqrt{z_1^2+z_2^2}\cos\left(\arctan\frac{z_2}{z_1}\right)\frac{z_2}{z_1^2}\frac{1}{1+\frac{z_2^2}{z_1^2}}\right] \\ v_{1,2} = -\omega(t)\left[\frac{z_2}{\sqrt{z_1^2+z_2^2}}\sin\left(\arctan\frac{z_2}{z_1}\right) + \sqrt{z_1^2+z_2^2}\cos\left(\arctan\frac{z_2}{z_1}\right)\frac{1}{z_1}\frac{1}{1+\frac{z_2^2}{z_1^2}}\right] \\ v_{2,1} = \omega(t)\left[\frac{z_1}{\sqrt{z_1^2+z_2^2}}\cos\left(\arctan\frac{z_2}{z_1}\right) + \sqrt{z_1^2+z_2^2}\sin\left(\arctan\frac{z_2}{z_1}\right)\frac{z_2}{z_1^2}\frac{1}{1+\frac{z_2^2}{z_1^2}}\right] \\ v_{2,2} = \omega(t)\left[\frac{z_2}{\sqrt{z_1^2+z_2^2}}\cos\left(\arctan\frac{z_2}{z_1}\right) - \sqrt{z_1^2+z_2^2}\sin\left(\arctan\frac{z_2}{z_1}\right)\frac{1}{z_1}\frac{1}{1+\frac{z_2^2}{z_1^2}}\right] \end{cases} \tag{3}$$

$$= \begin{cases} \partial_t v_1 = -\sqrt{z_1^2+z_2^2}\dot{\omega}(t)\sin\left(\arctan\frac{z_2}{z_1}\right) \\ \\ \partial_t v_2 = \sqrt{z_1^2+z_2^2}\dot{\omega}(t)\cos\left(\arctan\frac{z_2}{z_1}\right) \\ \\ v_{1,1} = -\frac{\omega(t)}{\sqrt{z_1^2+z_2^2}}\left[z_1\sin\left(\arctan\frac{z_2}{z_1}\right) - z_2\cos\left(\arctan\frac{z_2}{z_1}\right)\right] \\ \\ v_{1,2} = -\frac{\omega(t)}{\sqrt{z_1^2+z_2^2}}\left[z_2\sin\left(\arctan\frac{z_2}{z_1}\right) + z_1\cos\left(\arctan\frac{z_2}{z_1}\right)\right] \\ \\ v_{2,1} = \frac{\omega(t)}{\sqrt{z_1^2+z_2^2}}\left[z_1\cos\left(\arctan\frac{z_2}{z_1}\right) + z_2\sin\left(\arctan\frac{z_2}{z_1}\right)\right] \\ \\ v_{2,2} = \frac{\omega(t)}{\sqrt{z_1^2+z_2^2}}\left[z_2\cos\left(\arctan\frac{z_2}{z_1}\right) - z_1\sin\left(\arctan\frac{z_2}{z_1}\right)\right] \\ \end{cases} \tag{4}$$

and get

$$v_{1,s}v_s = \begin{cases} -\sqrt{z_1^2+z_2^2}\omega(t)\sin\left(\arctan\frac{z_2}{z_1}\right)\frac{\omega(t)}{\sqrt{z_1^2+z_2^2}}\left[z_1\sin\left(\arctan\frac{z_2}{z_1}\right) - z_2\cos\left(\arctan\frac{z_2}{z_1}\right)\right] \\ \\ -\sqrt{z_1^2+z_2^2}\omega(t)\cos\left(\arctan\frac{z_2}{z_1}\right)\frac{\omega(t)}{\sqrt{z_1^2+z_2^2}}\left[z_2\sin\left(\arctan\frac{z_2}{z_1}\right) + z_1\cos\left(\arctan\frac{z_2}{z_1}\right)\right] \end{cases} \tag{5}$$

$$= -z_1\omega^2(t) \tag{6}$$

$$v_{2,s}v_s = \begin{cases} \sqrt{z_1^2+z_2^2}\omega(t)\sin\left(\arctan\frac{z_2}{z_1}\right)\frac{\omega(t)}{\sqrt{z_1^2+z_2^2}}\left[z_1\cos\left(\arctan\frac{z_2}{z_1}\right) + z_2\sin\left(\arctan\frac{z_2}{z_1}\right)\right] \\ \\ +\sqrt{z_1^2+z_2^2}\omega(t)\cos\left(\arctan\frac{z_2}{z_1}\right)\frac{\omega(t)}{\sqrt{z_1^2+z_2^2}}\left[z_2\cos\left(\arctan\frac{z_2}{z_1}\right) - z_1\sin\left(\arctan\frac{z_2}{z_1}\right)\right] \\ \end{cases} \tag{7}$$

$$= z_2\omega^2(t) \tag{8}$$

giving with the second derivative term

$$= \begin{cases} f_1 = -\dot{\omega}(t)\sqrt{z_1^2+z_2^2}\sin\left(\arctan\frac{z_2}{z_1}\right) - z_1\omega^2(t) \\ \\ f_2 = \dot{\omega}(t)\sqrt{z_1^2+z_2^2}\cos\left(\arctan\frac{z_2}{z_1}\right) + z_2\omega^2(t) \\ \\ f_3 = 0 \end{cases} \tag{9}$$

♦

## 5.3 p193 - Exercise

Verify that the operator $\frac{\partial}{\partial t}$ does not alter tensor character.

Be $X^r$ and $Y^r$, two tensors so that $I = X_r Y^r$ is an invariant. Obviously, $\frac{\partial I}{\partial t}$ will also be invariant and

$$\frac{\partial I}{\partial t} = \frac{\partial X_r}{\partial t} Y^r + X_r \frac{\partial Y^r}{\partial t} \tag{1}$$

Meaning that the right side is a sum of two invariants, from which we conclude (see page 20, **1.607**) that $\frac{\partial X_r}{\partial t}$ and $\frac{\partial Y^r}{\partial t}$ are tensors.

♦

## 5.4 p193 - Clarification to 6.112

**6.112.** $$\int F n_r dS = \int F_{,r} dV$$

Green's theorem is generally presented in the form

$$\int \overline{F}.\overline{n} dS = \int \overline{\nabla}.\overline{F} dV$$

or

$$\int F_r n_r dS = \int F_{r,r} dV$$

We can define

$$\overline{F} = F\overline{1}_r$$

$\overline{F}.\overline{n}$ will then become $F n_r$ while $\overline{\nabla}.\overline{F}$ will become $\partial_r F$, giving the expression **6.112**..

♦

## 5.5 p196 - Exercise

Write out **6.126** and **6.127b** explicitly for spherical polar coordinates.

For spherical polar coordinates we have

$$(v^r) = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\dot{\phi} \end{pmatrix} \tag{1}$$

and (see **2.546** page 58):

$$\begin{aligned} v^r_{|r} &= \frac{1}{r^2}\partial_r\left(r^2 v^1\right) + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta v^2\right) + \partial_\phi v^3 && (2) \\ &= \frac{1}{r^2}\left(2rv^1 + r^2\partial_r v^1\right) + \frac{1}{\sin\theta}\left(v^2\cos\theta + \sin\theta\partial_\theta v^2\right) && (3) \\ &= \frac{2}{r}\dot{r} + r\dot{\theta}\cot\theta && (4) \end{aligned}$$

and

$$\begin{aligned} & & \partial_t\rho + (\rho v^r)_{|r} = 0 && (5) \\ &\Leftrightarrow & \partial_t\rho + \rho_{|r}v^r + \rho v^r_{|r} = 0 && (6) \\ &\Leftrightarrow & \frac{d\rho}{dt} + \rho\left(\frac{2}{r}\dot{r} + r\dot{\theta}\cot\theta\right) = 0 && (7) \end{aligned}$$

♦

## 5.6 p198 - Exercise

If $\epsilon^{rmn}$ is defined in precisely the same way as $\epsilon_{rmn}$, prove that

$$\epsilon'^{uvw} = J\epsilon^{rmn}\frac{\partial x'^u}{\partial x^r}\frac{\partial x'^v}{\partial x^m}\frac{\partial x'^w}{\partial x^n}$$

We follow the pretty same line of reasoning as for $\epsilon_{rst}$. Going from $x'^r$ to $x^s$ we have (expanding the determinant of the inverse Jacobian along the rows instead of the columns):

$$J^{-1} = \left|\frac{\partial x'^p}{\partial x^q}\right| = \epsilon^{rmn}\frac{\partial x'^1}{\partial x^r}\frac{\partial x'^2}{\partial x^m}\frac{\partial x'^3}{\partial x^n} \tag{1}$$

$$\times\epsilon^{uvw} \qquad J^{-1}\epsilon^{uvw} = \epsilon^{rmn}\frac{\partial x'^u}{\partial x^r}\frac{\partial x'^v}{\partial x^m}\frac{\partial x'^w}{\partial x^n} \tag{2}$$

$$\times J \qquad \epsilon^{uvw} = J\epsilon^{rmn}\frac{\partial x'^u}{\partial x^r}\frac{\partial x'^v}{\partial x^m}\frac{\partial x'^w}{\partial x^n} \tag{3}$$

$$\tag{4}$$

♦

## 5.7 p198 - Exercise

Prove that $\frac{\epsilon^{rmn}}{\sqrt{a}}$ is an (absolute) contravariant tensor of the third order.

$$\sqrt{a'} = J\sqrt{a} \tag{1}$$

$$\epsilon'^{uvw} = J\epsilon^{rmn}\frac{\partial x'^u}{\partial x^r}\frac{\partial x'^v}{\partial x^m}\frac{\partial x'^w}{\partial x^n} \tag{2}$$

(1) in (2)
$$\epsilon'^{uvw} = \frac{\sqrt{a'}}{\sqrt{a}}\epsilon^{rmn}\frac{\partial x'^u}{\partial x^r}\frac{\partial x'^v}{\partial x^m}\frac{\partial x'^w}{\partial x^n} \tag{3}$$

$\Rightarrow$
$$\frac{\epsilon'^{uvw}}{\sqrt{a'}} = \frac{\epsilon^{rmn}}{\sqrt{a}}\frac{\partial x'^u}{\partial x^r}\frac{\partial x'^v}{\partial x^m}\frac{\partial x'^w}{\partial x^n} \tag{4}$$

which is the required transformation rule for a ”normal” (absolute) tensor.

♦

## 5.8 p199 - Clarification to pressure invariance to direction of the surface element.

Pressure is independent of the direction

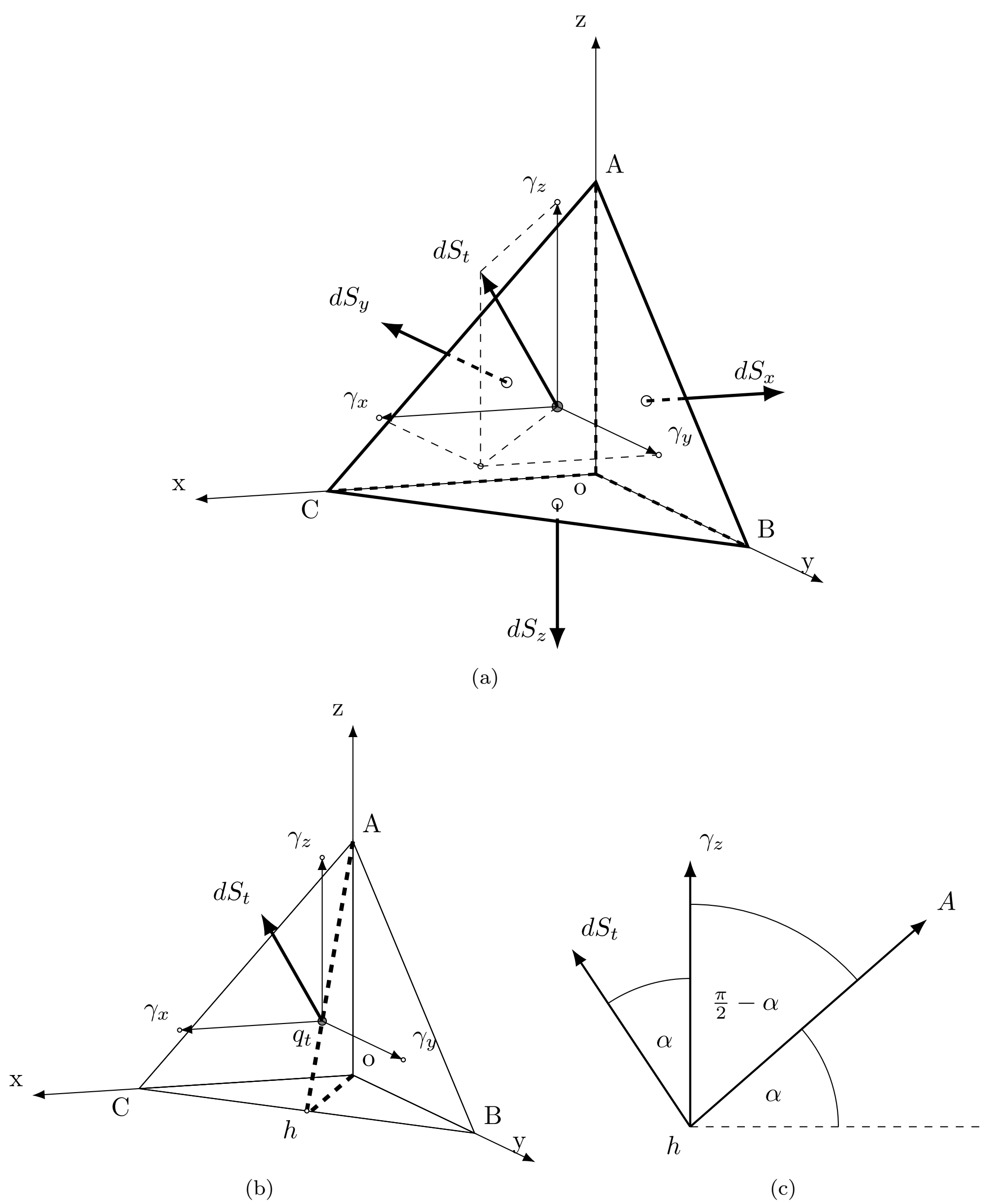



Figure 5.2: Pressure on a trirectangular tetrahedron

To see that the pressure is independent of the direction of the surface element on which we measure it, let's consider a trirectangular tetrahedron $OABC$ as depicted in figure 5.2($a$). Let's define $P_x, P_y, P_z, P_t$ the pressure measured on the 4 surfaces with normal vectors $dS_x, dS_y, dS_z, dS_t$.

Let's neglect second order terms due to acceleration and external forces. For the forces along axis $z$ (the same reasoning is valid for the two others) we will have $P_z dS_z = P_t dS_t \gamma_z$ where $\gamma_z$ is the cosine of the angle formed by normal on $dS_t$ and the z-axis.
Let's investigate the relationship between $dS_t$ and $dS_x, dS_y, dS_z$.
Be $hA$ the line element lying in the plane $ABC$ (see figure 5.2$(b)$) and $hO$ the line element lying in the plane $OBC$. As the area of a triangle $= \frac{1}{2} \times base \times perpendicular\ height$ we get $dS_z = \frac{1}{2}|hO||BC|$. But $|hO| = |hA| \cos\alpha$ (see figure 5.2$(c)$) and so $dS_z = \frac{1}{2}|BC||hA|\cos\alpha = dS_t\gamma_z$ and get

$$P_z dS_z = P_t dS_t \gamma_z \qquad (1)$$

$$\Rightarrow \qquad P_z dS_t \gamma_z = P_t dS_t \gamma_z \qquad (2)$$

$$\Rightarrow \qquad P_z = P_t \qquad (3)$$

♦

## 5.9 p201 - Exercise

Write down the contravariant form of **6.147**

$$\textbf{6.147} \qquad \partial_t v_r + v_s v_{r|s} = X_r - \rho^{-1} p_{,r}$$

$$\textbf{6.147} \qquad \partial_t v_r + v_s v_{r|s} = X_r - \rho^{-1} p_{,r} \tag{1}$$

$$\times a^{mr} \qquad \partial_t a^{mr} v_r + v_s a^{mr} v_{r|s} = a^{mr} X_r - \rho^{-1} a^{mr} p_{,r} \tag{2}$$

By **2.527** page 53 we have $a^{rs}_{|t} = 0$ and thus

$$v^m_{|s} = (a^{mr} v_r)_{|s} = (a^{mr})_{|n} v_r + a^{mr} v_{r|s} = a^{mr} v_{r|s} \tag{3}$$

$$(2) \Rightarrow \qquad \partial_t v^m + v_s v^m{}_{|s} = X^m - \rho^{-1} a^{mr} p_{,r} \tag{4}$$

$$\Rightarrow \qquad \partial_t v^m + v_s v^m{}_{|s} = X^m - \rho^{-1} p^{'}_{,r} \tag{5}$$

Note that $p_{,r}$ in (1) and (5) are not the same vector function as $p_{,r}$ can be written as $\overline{\nabla} p$ which is coordinate system dependent.

♦

## 5.10 p202 - Exercise

Verify by means of **3.204** that **6.157** and **6.156** are the same equation.

$$\begin{cases} \mathbf{3.204} & \Gamma^n_{rn} = \frac{1}{2}\partial_r \log a = \partial_r \log \sqrt{a} \\ \mathbf{6.157} & (\sqrt{a} a^{mn} \phi_{,m})_{,n} = 0 \\ \mathbf{6.156} & a^{mn} \phi_{|mn} = 0 \end{cases} \tag{1}$$

Considering also

$$\begin{cases} a^{mn}_{|k} = 0 \\ T^m_{|n} = \partial_n T_m + \Gamma^m_{kn} T_k \end{cases} \tag{2}$$

So **6.156** can written as

$$\begin{aligned}
& & & a^{mn}\phi_{|mn} = 0 & (3)\\
& & \Leftrightarrow & \left(a^{mn}\phi_{|m}\right)_{|n} = 0 & (4)\\
T^n = a^{mn}\phi_{|m} & & \Rightarrow & \partial_n T^n + \Gamma^n_{kn} T^k = 0 & (5)\\
& & \Rightarrow & \partial_n \left(a^{mn}\phi_{|m}\right) + \Gamma^n_{kn} a^{pk}\phi_{|p} = 0 & (6)\\
& & \Leftrightarrow & \partial_n \left(a^{mn}\right)\phi_{,m} + a^{mn}\phi_{,mn} + \Gamma^n_{kn} a^{pk}\phi_{,p} = 0 & (7)\\
& (2) & \Rightarrow & \partial_n \left(a^{mn}\right)\phi_{,m} + a^{mn}\phi_{,mn} + \partial_k \log\sqrt{a}\, a^{pk}\phi_{,p} = 0 & (8)\\
& & \Leftrightarrow & \left(a^{mn}_{,n}\right)\phi_{,m} + a^{mn}\phi_{,mn} + \frac{1}{\sqrt{a}}\left(\sqrt{a}a^{pk}\right)_{,k}\phi_{,p} = 0 & (9)\\
& & \Leftrightarrow & \sqrt{a}\phi_{,m}\left(a^{mn}\right)_{,n} + \sqrt{a}a^{mn}\phi_{,mn} + a^{mn}\phi_{,m}\left(\sqrt{a}\right)_{,n} = 0 & (10)\\
& & \Leftrightarrow & \left(\sqrt{a}a^{mn}\phi_{,m}\right)_{,n} = 0 & (11)
\end{aligned}$$

♦

## 5.11 p205 - Exercise

Show that a small strain is a rigid body displacement if, and only if, $e_{rs} = 0$. In the case of finite strain, deduce form **6.206** the conditions which must be satisfied by the partial derivatives of the displacement in order that it may b a rigid body displacement.

**Suppose we deal with a rigid body.** Then the position of two particles of the rigid body are given by

$$\begin{cases} p_r = z_r + u_r(z) \\ p_r^{'} = z_r^{'} + u_r(z^{'}) \end{cases} \tag{1}$$

and

$$\begin{cases} L_0 = z_r - z_r^{'} \\ L_1 = z_r^{'} + u_r(z^{'}) - z_r - u_r(z) \end{cases} \tag{2}$$

A rigid body means $L_1 = L_0$, giving $u_r(z^{'}) = u_r(z)$ i.e $u_r(z)$ is a constant and thus $u_{r,s}(z) = 0$. As $e_{rs} = \frac{1}{2}\left(u_{r,s}(z) + u_{s,r}(z)\right)$ we get $e_{rs} = 0$

**Suppose now that** $e_{rs} = 0$
We have $e = e_{rs}\lambda_r\lambda_s = 0$ and $e = u_{r,s}(z)\lambda_r\lambda_s$ As the $\lambda_r$ are arbitrary, in the sense that we are free to choose whatever curve to approach the initial point, we conclude that $u_{r,s}(z) = 0$ . So$u_r(z)$ is a constant, meaning that the mutual distance between two arbitrary points, do not change. The body is a rigid body.

♢

For a finite strain we have **6.206**:

$$\lim \frac{L_1^2 - L_0^2}{L_0^2} = 2u_{r,s}(z)\lambda_r\lambda_s + u_{m,r}(z)u_{m,s}(z)\lambda_r\lambda_s \tag{3}$$

This limit is 0 and so we get as condition

$$\left(2u_{r,s}(z) + u_{m,r}(z)u_{m,s}(z)\right)\lambda_r\lambda_s = 0 \tag{4}$$

As the $\lambda_r$ are arbitrary, in the sense that we are free to choose whatever curve to approach the initial point, we conclude that $2u_{r,s}(z) + u_{m,r}(z)u_{m,s}(z)$ must be zero.

$$2u_{r,s}(z) + u_{m,r}(z)u_{m,s}(z) = 0$$

♦

## 5.12 p207 - Clarification

Then, clearly, since the the volume of the tetrahedron is less than $a^3$,

**6.217** $$\lim_{a\to 0}\frac{1}{a^2}\frac{dM_r}{dt}=0,\quad \lim_{a\to 0}\frac{1}{a^2}\int X_r dV=0$$

$\vdots$

But $\lim_{a\to 0}\frac{S}{a^2}$ is not zero,...

First, note that the volume of a trirectangular tetrahedron is $V=\frac{1}{6}abc$ with $a,b,c$ the bases of the 3 rectangular triangles (see clarification for page 199), so if $a\geq b,c$ we have $V<a^3$.
There is no assurance that $\lim_{a\to 0}\frac{1}{a^3}\frac{dM_r}{dt}=0$. Indeed, consider **5.334**. For a continuous medium, this equation can be written as

$$I_{st}=\int_V \rho\epsilon_{ptq}\epsilon_{psn}z_s z_t dV \tag{1}$$

If $V$ goes to zero, the quantities under the integral can be approximated by constant values and hence, the dynamics of the tetrahedron are govern by

$$\lim_{V\to 0} I_{st}=\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t V \tag{2}$$

**5.332**: $$\frac{dI_{st}\omega_t}{dt}=M_s \tag{3}$$

$$\Rightarrow \quad \lim_{V\to 0}\frac{dM_s}{dt}=\lim_{V\to 0}V\frac{d\left(\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t\omega_t\right)}{dt} \tag{4}$$

$$\Rightarrow \quad \lim_{a\to 0}\frac{1}{a^3}\frac{dM_s}{dt}=\lim_{a\to 0}\frac{1}{a^3}V\frac{d\left(\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t\omega_t\right)}{dt} \tag{5}$$

$$\Rightarrow \quad \lim_{a\to 0}\frac{1}{a^3}\frac{dM_s}{dt}<\lim_{a\to 0}\frac{1}{a^3}a^3\frac{d\left(\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t\omega_t\right)}{dt} \tag{6}$$

$$\Rightarrow \quad \lim_{a\to 0}\frac{1}{a^3}\frac{dM_s}{dt}<\lim_{a\to 0}\frac{d\left(\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t\omega_t\right)}{dt} \tag{7}$$

but there is no reason to admit that $\lim_{V\to 0}\frac{d\left(\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t\omega_t\right)}{dt}=0$.
On the other hand, replacing $a_3$ with $a^2$ in (5) gives

$$\lim_{a\to 0}\frac{1}{a^2}\frac{dM_s}{dt}=\lim_{a\to 0}\frac{1}{a^2}V\frac{d\left(\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t\omega_t\right)}{dt} \tag{8}$$

$$\Rightarrow \quad \lim_{a\to 0}\frac{1}{a^2}\frac{dM_s}{dt}<\lim_{a\to 0}\frac{1}{a^2}a^3\frac{d\left(\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t\omega_t\right)}{dt} \tag{9}$$

$$\Rightarrow \quad \lim_{a\to 0}\frac{1}{a^2}\frac{dM_s}{dt}<\lim_{a\to 0}a\frac{d\left(\rho\epsilon_{ptq}\epsilon_{psn}z_s z_t\omega_t\right)}{dt} \tag{10}$$

$$\Rightarrow \quad \lim_{a\to 0}\frac{1}{a^2}\frac{dM_s}{dt}=0 \tag{11}$$

For $\lim_{a\to 0}\frac{1}{a^2}\int X_r dV = 0$ the reasoning is even simpler as for a volume going to zero , we can consider $X_r$ as constant and thus

$$\lim_{a\to 0}\frac{1}{a^2}\int X_r dV = \lim_{a\to 0}\frac{1}{a^2}X_r\int dV < \lim_{a\to 0}\frac{1}{a^2}X_r a^3 = 0 \tag{12}$$

◊

**But $\lim_{a\to 0}\frac{S}{a^2}$ is not zero,...**

Be $S_t$ the area of the "sloped" triangle in the tetrahedron. Then, the total area of the tetrahedron is:

$$S = \frac{1}{2}(ab+bc+ac) + S_t \tag{13}$$

$$S_t > \frac{1}{2}ab,\ S_t > \frac{1}{2}bc,\ S_t > \frac{1}{2}ac \tag{14}$$

$$\Rightarrow \qquad S > \frac{1}{2}(ab+bc+ac) + \frac{1}{2}ab \tag{15}$$

$$\Rightarrow \qquad \frac{S}{a^2} > \frac{1}{2}\left(\frac{bc}{a^2}+\frac{c}{a}\right) + b \tag{16}$$

If we shrink the tetrahedron uniformly and put $a = \epsilon a_0, b = \epsilon b_0, c = \epsilon c_0$ then (16) can be written as

$$\lim_{\epsilon\to 0}\frac{S}{a^2} > \frac{1}{2}\left(\frac{b_0c_0}{a_0^2}+\frac{c_0}{a_0}\right) + b_0\lim_{\epsilon\to 0}\epsilon \tag{17}$$

which is indeed not zero.

♦

## 5.13 p208 - Exercise

Show that the stress across a plane $z_1 = \text{const.}$ has the components $E_{11},\ E_{21},\ E_{31}$. What are the components across planes $z_2 = \text{const.}$ and $z_3 = \text{const.}$?

$$\textbf{6.223} \qquad T_r = E_{rs} n_s \tag{1}$$

So for the the stress across a plane $z_1 = \text{const.}$, we have $n_1 = 1,\ n_2 = 0,\ n_3 = 0$ and so

$$T_r\left(z_1 = \text{const.}\right) = \begin{pmatrix} E_{11} \\ E_{21} \\ E_{31} \end{pmatrix} \tag{2}$$

For the the stress across a plane $z_2 = \text{const.}$, we have $n_1 = 0,\ n_2 = 1,\ n_3 = 0$ and for the the stress across a plane $z_3 = \text{const.}$, we have $n_1 = 0,\ n_2 = 0,\ n_3 = 1$ and so

$$T_r\left(z_2 = \text{const.}\right) = \begin{pmatrix} E_{12} \\ E_{22} \\ E_{32} \end{pmatrix} \tag{3}$$

$$T_r\left(z_3 = \text{const.}\right) = \begin{pmatrix} E_{13} \\ E_{23} \\ E_{33} \end{pmatrix} \tag{4}$$

♦

## 5.14 p210 - Exercise

Show that the if **6.233** is solved for strain, so as to read

$$\mathbf{6.237} \qquad e_{rs} = C_{rsmn} E_{mn}$$

then the symmetry conditions **6.234**, **6.235**, and **6.236** imply similar conditions on $C_{rsmn}$. (The tensor $C_{rsmn}$ is he second elasticity tensor).

**a)** $C_{rsnm} = C_{rsmn}$ **and** $C_{srmn} = C_{rsmn}$

This is a direct consequence of the symmetries $E_{nm} = E_{mn}$ and $e_{nm} = e_{mn}$.
E.g.:

$$\begin{aligned} & e_{nm} = e_{mn} && (1) \\ \Leftrightarrow \quad & C_{nmrs} E_{rs} = C_{mnrs} E_{rs} && (2) \\ \Rightarrow \quad & C_{nmrs} = C_{mnrs} && (3) \end{aligned}$$

**b)** $C_{rsnm} = C_{mnrs}$

We simplify the notation by using 'compactified' indices (e.g.):

$$e_a = C_{ab} E_b \quad \Leftrightarrow \quad e_{rs} = C_{rsmn} E_{mn}$$

Let's form the invariant $E_a e_a$, then :

$$\begin{aligned} E_a e_a &= c_{au} e_u C_{av} E_v && (4) \\ &= c_{ua} e_a C_{uv} E_v \quad \text{(renaming dummy indices)} && (5) \\ &= C_{uv} E_u E_v && (6) \\ &= C_{vu} E_u E_v \quad \text{(renaming dummy indices)} && (7) \end{aligned}$$

From (6) and (7) we can conclude

$$C_{vu} = C_{uv}$$

♦

## 5.15 p212 - Exercise

Deduce from **6.250** that if an isotropic elastic body is in equilibrium under no body forces, then the expansion $\theta$ is an harmonic function ($\boldsymbol{\Delta}\theta = 0$)

$$\mathbf{6.250}: \qquad \rho f_r = \rho X_r + (\lambda + \mu)\,\theta_{,r} + \mu \boldsymbol{\Delta} u_r$$

In equilibrium under no body forces means $f_r = 0$ and $X_r = 0$. So,

$$(\lambda + \mu)\,\theta_{,r} + \mu \boldsymbol{\Delta} u_r = 0 \tag{1}$$

$$\partial_r \quad \Rightarrow \qquad (\lambda + \mu)\,\underbrace{\theta_{,rr}}_{=\boldsymbol{\Delta}\theta} + \mu\,\underbrace{(\boldsymbol{\Delta} u_r)_{,r}}_{=\boldsymbol{\Delta} u_{r,r}} = 0 \tag{2}$$

$$u_{r,r} = \theta \quad \Rightarrow \qquad (\lambda + \mu)\,\boldsymbol{\Delta}\theta + \mu \boldsymbol{\Delta}\theta = 0 \tag{3}$$

$$\Rightarrow \qquad \boldsymbol{\Delta}\theta = 0 \tag{4}$$

♦

## 5.16 p213 - Exercise

Express the equations of motion **6.250** in curvilinear coordinates.

$$\mathbf{6.250}: \qquad \rho f_r = \rho X_r + (\lambda+\mu)\,\theta_{,r} + \mu\boldsymbol{\Delta} u_r$$

We start from **6.252** : $\qquad \rho f^r = \rho X^r + E^{rs}{}_{|s}$

$$\begin{aligned}
& & \rho f^r &= \rho X^r + E^{rs}{}_{|s} & (1)\\
\mathbf{6.245}: & \Rightarrow & E^{rs} &= \lambda\delta_{rs}\theta + 2\mu e^{rs} & (2)\\
& \Rightarrow & \rho f^r &= \rho X^r + \lambda\delta_{rs}\theta_{|s} + 2\mu e^{rs}{}_{|s} & (3)\\
& \Leftrightarrow & \rho f^r &= \rho X^r + \lambda\theta_{,r} + 2\mu e^{rs}{}_{|s} & (4)\\
\mathbf{6.245}: & & e^{rs} &= \frac{1}{2}\left(u^r{}_{|s} + u^s{}_{|r}\right) & (5)\\
& \Leftrightarrow & \rho f^r &= \rho X^r + \lambda\theta_{,r} + 2\mu\frac{1}{2}\left(u^r{}_{|ss} + u^s{}_{|rs}\right) & (6)\\
\mathbf{6.246}: & & \theta = e^{nn} &= \frac{1}{2}\left(u^n{}_{|n} + u^n{}_{|n}\right) = u^n{}_{|n} & (7)\\
(6) & \Rightarrow & \rho f^r &= \rho X^r + \lambda\theta_{,r} + \mu\left(u^r{}_{|ss} + \theta\ {}_{|r}\right) & (8)\\
& \Rightarrow & \rho f^r &= \rho X^r + (\lambda+\mu)\,\theta_{,r} + \mu u^r{}_{|ss} & (9)
\end{aligned}$$

So,

$$f^r = \rho X^r + (\lambda+\mu)\,\theta_{,r} + \mu\boldsymbol{\Delta} u^r$$

where we define the Laplacian differential operator as

$$\boldsymbol{\Delta}(\cdot) \stackrel{\text{def}}{=} (\cdot)_{|nn}$$

♦

## 5.17 p215 - Exercise

Verify that $E_r = z_r,\ H_r = 0$ satisfy the wave equation but not Maxwell's equations

$$\mathbf{6.306}: \qquad \frac{1}{c^2}\frac{\partial^2 E_r}{\partial t^2} - E_{r,mn} = 0$$

$$\mathbf{6.307}: \qquad \frac{1}{c^2}\frac{\partial^2 H_r}{\partial t^2} - H_{r,mn} = 0$$

Obviously, **6.307** is trivial as $H_r = 0$ is a constant $= 0$, and so are the derivatives.
For **6.306**, $\frac{\partial^2 E_r}{\partial t^2} = 0$ as $E_r$ is no function of time.
So, the defined field satisfy the wave equation.

$$\begin{cases} \mathbf{6.301}: & \frac{1}{c}\frac{\partial E_r}{\partial t} = \epsilon_{rmn} H_{n,m}, \quad \frac{1}{c}\frac{\partial H_r}{\partial t} = -\epsilon_{rmn} E_{n,m} \\ \mathbf{6.302}: & E_{n,n} = 0, \qquad\qquad\quad H_{n,n} = 0 \end{cases} \tag{1}$$

The first equation of **6.302** is not satisfied as $E_{n,n} = N$ with $N$ the space dimension.

♦

## 5.18 p220 - Exercise

Prove a similar statement for the electric and magnetic vectors of the complementary electromagnetic field.

If we take the complex conjugate of **6.324** and subtract, we obtain after multiplying by $i$:

$$E_r^{**} = -\epsilon_{rmn} H_n^{**} V_{,m}, \quad H_r^{**} = \epsilon_{rmn} E_n^{**} V_{,m}, \tag{1}$$

Hence, the vectors $V_{,r}, E_r^{**}$ and $H_r^{**}$ form a right-handed orthogonal triad, and we obtain the relation

$$E_r^{**} E_r^{**} = H_n^{**} H_n^{**} \tag{2}$$

♦

## 5.19 p221 - Clarification

Some thoughts about polarization

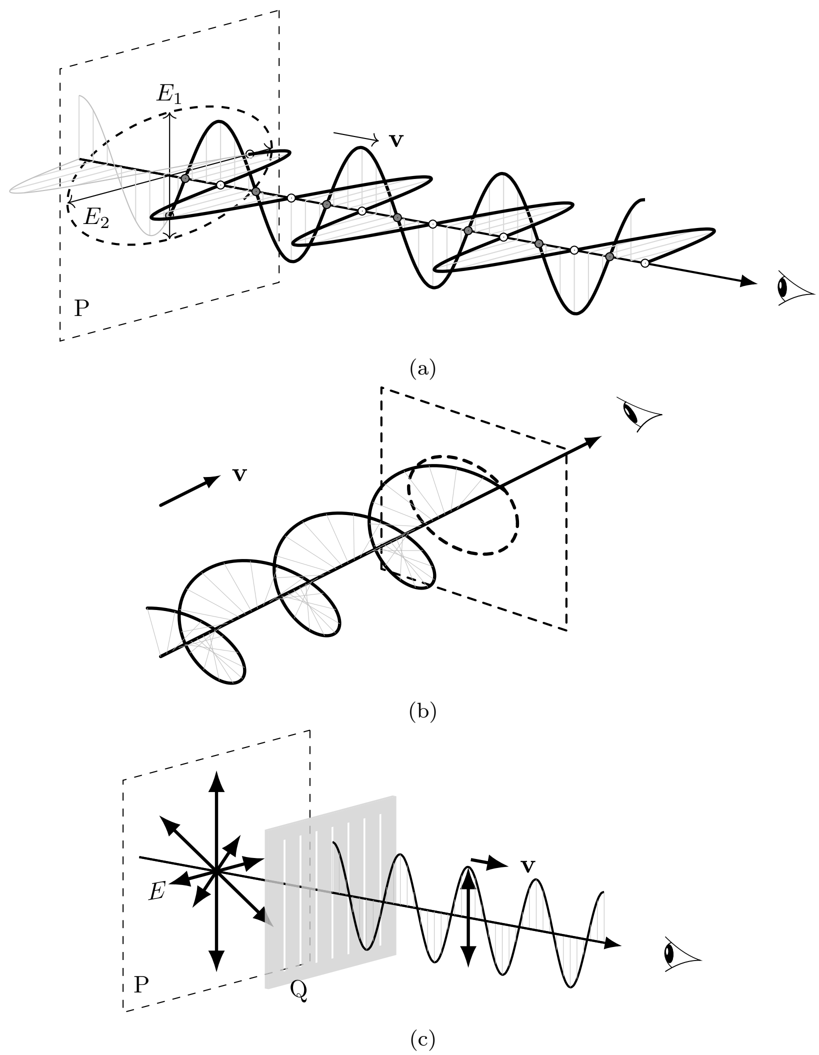



Figure 5.3: Polarization of light

In the above figures, only the electric field is represented (the magnetic field has to be imagined perpendicular to the $E_r$ vector).
In figure (a) we see an elliptical polarization which occurs when the EMW can be split into two perpendicular components. When $|\overline{E_1}| = |\overline{E_2}|$, one can speak about circular polarization. Figure (b) gives a view at a certain time $t$ of the result of $\overline{E_1} + \overline{E_2}$.// In figure (c) an observer 'sees' in the phase wave situated at $P$, an unpolarized EMV. After passing a linear polarizing material at $Q$ the observer will 'see' the EMV oscillating only in the vertical plane along $\mathbf{v}$

♦

## 5.20 p221 - Exercise

What conditions must be imposed on the fixed complex vectors $E_r^{(0)}$ and $H_r^{(0)}$ in order that the wave may be plane-polarized

As stated a plane-polarized wave will have its vectors $E_r^*$ and $E_r^{**}$ have the same directions and moreover the $E_r^*$ vector maintains a fixed directions.
This means that $E_r^*$ can be written as $E_r^* = \alpha(z,t)\,\mathcal{E}_r$ and $E_r^{**} = \beta(z,t)\,\mathcal{E}_r$ with $\alpha,\ \beta$ real valued functions and $\mathcal{E}_r$ a constant. Note that from the definition of $E_r^*$ and $E_r^{**}$ we have

$$E_r = E_r^* + iE_r^{**} \tag{1}$$

and

$$\begin{array}{ll} 6.330 & \left\{\begin{array}{l} \frac{\partial E_r^*}{\partial t} = -\frac{2\pi c}{\lambda} E_r^{**} \\ \\ \frac{\partial E_r^{**}}{\partial t} = \frac{2\pi c}{\lambda} E_r^* \end{array}\right. \\ 6.331 & \end{array} \tag{2}$$

$$\Rightarrow \left\{\begin{array}{l} \frac{\partial \alpha}{\partial t} = -\frac{2\pi c}{\lambda}\beta \\ \\ \frac{\partial \beta}{\partial t} = \frac{2\pi c}{\lambda}\alpha \end{array}\right. \tag{3}$$

$$\Rightarrow \left\{\begin{array}{l} \frac{\partial^2 \alpha}{\partial t^2} = -\left(\frac{2\pi c}{\lambda}\right)^2 \alpha \\ \\ \frac{\partial^2 \beta}{\partial t^2} = -\left(\frac{2\pi c}{\lambda}\right)^2 \beta \end{array}\right. \tag{4}$$

$$\Rightarrow \left\{\begin{array}{l} \alpha = A(z)\cos\frac{2\pi c}{\lambda}t + B(z)\sin\frac{2\pi c}{\lambda}t \\ \\ \beta = C(z)\cos\frac{2\pi c}{\lambda}t + D(z)\sin\frac{2\pi c}{\lambda}t \end{array}\right. \tag{5}$$

$$\Rightarrow \quad \frac{E_r}{\mathcal{E}_r} = A\cos\frac{2\pi c}{\lambda}t + B\sin\frac{2\pi c}{\lambda}t + i\left(C(z)\cos\frac{2\pi c}{\lambda}t + D(z)\sin\frac{2\pi c}{\lambda}t\right) \tag{6}$$

With $A, B, C, D : \mathbb{R}^3 \to \mathbb{R}$.
Let's write $E_r^{(0)}$ as $E_r^{(0)} = a + ib$ with $a, b$ real valued functions depending on the position only. Then, **6.308** can be rewritten as

$$\begin{aligned} E_r &= (a+ib)(\cos S + i\sin S) && (7) \\ &= a\cos S - b\sin S + i(b\cos S + a\sin S) && (8) \end{aligned}$$

We have, with $S=\frac{2\pi V}{\lambda}-\frac{2\pi c}{\lambda}t$

$$\begin{cases}\cos S=\cos\frac{2\pi V}{\lambda}\cos\frac{2\pi c}{\lambda}t+\sin\frac{2\pi V}{\lambda}\sin\frac{2\pi c}{\lambda}t\\ \\ \sin S=\sin\frac{2\pi V}{\lambda}\cos\frac{2\pi c}{\lambda}t-\cos\frac{2\pi V}{\lambda}\sin\frac{2\pi c}{\lambda}t\end{cases}\tag{9}$$

$$(8)\Rightarrow\quad E_r=\begin{cases}a\cos\frac{2\pi V}{\lambda}\cos\frac{2\pi c}{\lambda}t+a\sin\frac{2\pi V}{\lambda}\sin\frac{2\pi c}{\lambda}t+b\cos\frac{2\pi V}{\lambda}\sin\frac{2\pi c}{\lambda}t-b\sin\frac{2\pi V}{\lambda}\cos\frac{2\pi c}{\lambda}t\\ \\ +i\left(b\cos\frac{2\pi V}{\lambda}\cos\frac{2\pi c}{\lambda}t+b\sin\frac{2\pi V}{\lambda}\sin\frac{2\pi c}{\lambda}t-a\cos\frac{2\pi V}{\lambda}\sin\frac{2\pi c}{\lambda}t+a\sin\frac{2\pi V}{\lambda}\cos\frac{2\pi c}{\lambda}t\right)\end{cases}\tag{10}$$

$$=\begin{cases}\left(a\cos\frac{2\pi V}{\lambda}-b\sin\frac{2\pi V}{\lambda}\right)\cos\frac{2\pi c}{\lambda}t+\left(a\sin\frac{2\pi V}{\lambda}+b\cos\frac{2\pi V}{\lambda}\right)\sin\frac{2\pi c}{\lambda}t\\ \\ +i\left[\left(b\cos\frac{2\pi V}{\lambda}+a\sin\frac{2\pi V}{\lambda}\right)\cos\frac{2\pi c}{\lambda}t+\left(b\sin\frac{2\pi V}{\lambda}-a\cos\frac{2\pi V}{\lambda}\right)\sin\frac{2\pi c}{\lambda}t\right]\\ \end{cases}\tag{11}$$

Let's now compare equations (6) and (11)

$$E_r=A\mathcal{E}_r\cos\frac{2\pi c}{\lambda}t+B\mathcal{E}_r\sin\frac{2\pi c}{\lambda}t+i\left(C\mathcal{E}_r\cos\frac{2\pi c}{\lambda}t+D\mathcal{E}_r\sin\frac{2\pi c}{\lambda}t\right)\tag{12}$$

$$E_r=\begin{cases}\left(a\cos\frac{2\pi V}{\lambda}-b\sin\frac{2\pi V}{\lambda}\right)\cos\frac{2\pi c}{\lambda}t+\left(a\sin\frac{2\pi V}{\lambda}+b\cos\frac{2\pi V}{\lambda}\right)\sin\frac{2\pi c}{\lambda}t\\ \\ +i\left[\left(a\sin\frac{2\pi V}{\lambda}+b\cos\frac{2\pi V}{\lambda}\right)\cos\frac{2\pi c}{\lambda}t-\left(a\cos\frac{2\pi V}{\lambda}-b\sin\frac{2\pi V}{\lambda}\right)\sin\frac{2\pi c}{\lambda}t\right]\\ \end{cases}\tag{13}$$

We see that

$$A\mathcal{E}_r=a\cos\frac{2\pi V}{\lambda}-b\sin\frac{2\pi V}{\lambda}\tag{14}$$

$$B\mathcal{E}_r=a\sin\frac{2\pi V}{\lambda}+b\cos\frac{2\pi V}{\lambda}\tag{15}$$

$$C=B\tag{16}$$

$$D=-A\tag{17}$$

$$(14),\ (15)\quad\Rightarrow\quad\begin{cases}a=A\mathcal{E}_r\cos\frac{2\pi V}{\lambda}+B\mathcal{E}_r\sin\frac{2\pi V}{\lambda}\\ b=-A\mathcal{E}_r\sin\frac{2\pi V}{\lambda}+B\mathcal{E}_r\cos\frac{2\pi V}{\lambda}\end{cases}\tag{18}$$

$$\Rightarrow\quad E_r^{(0)}=a+ib\tag{19}$$

$$=\mathcal{E}_r\left(A\left(z\right)+iB\left(z\right)\right)e^{-\frac{2\pi V}{\lambda}}\tag{20}$$

So, the direction of $E_r^{(0)}$ does not change as $\frac{E_r^{(0)}}{E_s^{(0)}}=\frac{\mathcal{E}_r}{\mathcal{E}_s}$ and the magnitude varies with the position but in such a way that the effect of $V$ is annihilated.

♦

## 5.21 p223 - Clarification

Interrelationship between the identities **6.337** to **6.340**

$$\begin{array}{rll}
\mathbf{6.337} & \dfrac{1}{c^2}\dfrac{\partial^2\phi_r}{\partial t^2} + \dfrac{1}{c}\dfrac{\partial\psi_{,r}}{\partial t} = \phi_{r,mm} - \phi_{m,mr} & (1) \\
\mathbf{6.338} & \dfrac{1}{c}\dfrac{\partial}{\partial t}\phi_{m,m} + \psi_{,mm} = 0 & (2) \\
\mathbf{6.339(a)} & \dfrac{1}{c^2}\dfrac{\partial^2\phi_r}{\partial t^2} - \phi_{r,mm} = 0 & (3) \\
\mathbf{6.339(b)} & \dfrac{1}{c^2}\dfrac{\partial^2\psi}{\partial t^2} - \psi_{,mm} = 0 & (4) \\
\mathbf{6.340} & \dfrac{1}{c}\dfrac{\partial\psi}{\partial t} + \phi_{m,m} = 0 & (5)
\end{array}$$

then

$$\begin{array}{rll}
(3)\text{ in }(1) & \dfrac{1}{c}\dfrac{\partial\psi_{,r}}{\partial t} = -\phi_{m,mr} & (6) \\
\dfrac{\partial(5)}{\partial z_r} & \dfrac{1}{c}\dfrac{\partial\psi_{,r}}{\partial t} + \phi_{m,mr} = 0 \quad \Leftrightarrow \quad (6) & (7)
\end{array}$$

What about (4) ?

$$\begin{array}{rll}
(4)\text{ in }(2) & \dfrac{1}{c}\dfrac{\partial}{\partial t}\phi_{m,m} = -\dfrac{1}{c^2}\dfrac{\partial^2\psi}{\partial t^2} & (8) \\
\displaystyle\int_t (8) \Rightarrow & \dfrac{1}{c}\dfrac{\partial\psi}{\partial t} + \phi_{m,m} = C & (9)
\end{array}$$

In (9), $C$ is a function, constant in $t$. Imposing $C = 0$ gives still a valid solution and is equivalent with (5).

♦

## 5.22 p223 - Clarification

$$\mathbf{6.342} \qquad \frac{1}{c^2}\frac{\partial^2 \Pi_r}{\partial t^2} - \Pi_{r,mm} = 0$$

$$\mathbf{6.341(a)} \qquad \mathbf{E_n = \Pi_{m,mn} - \frac{1}{c^2}\frac{\partial^2 \Pi_n}{\partial t^2}} \tag{1}$$

$$\mathbf{6.341(b)} \qquad \mathbf{H_n = \frac{1}{c}\epsilon_{npq}\frac{\partial}{\partial t}\Pi_{q,p}} \tag{2}$$

We first check, under which conditions (1) and (2) satisfy the Maxwell equations

$$\begin{cases} 6.301\text{(a),(b)} & E_{m,m} = 0, \qquad H_{m,m} = 0 \\ 6.302\text{(a)} & \frac{1}{c}\frac{\partial E_r}{\partial t} = \epsilon_{rmn} H_{n,m} \\ 6.302\text{(b)} & \frac{1}{c}\frac{\partial H_r}{\partial t} = -\epsilon_{rmn} E_{n,m} \end{cases} \tag{3}$$

with the wave equations:

$$\begin{cases} 6.306 & \frac{1}{c^2}\frac{\partial^2 E_r}{\partial t^2} - E_{r,mm} = 0 \\ 6.307 & \frac{1}{c^2}\frac{\partial^2 H_r}{\partial t^2} - H_{r,mm} = 0 \end{cases} \tag{4}$$

We have for **6.301(b)**:

$$(2)_{,n} \qquad H_{n,n} = \frac{1}{c}\epsilon_{npq}\frac{\partial \Pi_{q,pn}}{\partial t} \tag{5}$$

$$= 0 \quad (\Pi_{q,pn} = \Pi_{q,np} \text{ and } \epsilon_{npq} = -\epsilon_{pnq}) \tag{6}$$

So, **6.301(b)** is satisfied.

For **6.302(b)** we have:

$$6.302\text{(b)} \qquad \frac{1}{c}\frac{\partial H_r}{\partial t} = -\epsilon_{rmn} E_{n,m} \tag{7}$$

$$(2)_{,t} \quad \Rightarrow \qquad \frac{1}{c}\epsilon_{npq}\frac{\partial^2}{\partial t^2}\Pi_{q,p} = -\epsilon_{rmn} E_{n,m} \tag{8}$$

$$(1) \text{ in } (8) \quad \Rightarrow \qquad \frac{1}{c}\epsilon_{npq}\frac{\partial^2}{\partial t^2}\Pi_{q,p} = -\underbrace{\epsilon_{rmn}\Pi_{q,qnm}}_{=0} + \epsilon_{rmn}\frac{1}{c^2}\frac{\partial^2 \Pi_{n,m}}{\partial t^2} \tag{9}$$

$$\Rightarrow \qquad \frac{1}{c}\epsilon_{npq}\frac{\partial^2}{\partial t^2}\Pi_{q,p} = \epsilon_{rmn}\frac{1}{c^2}\frac{\partial^2 \Pi_{n,m}}{\partial t^2} \tag{10}$$

So, **6.302(b)** is satisfied.

We still have to prove that the expression (1) and (2) are consistent with 6.301(a) and 6.302(a) also for **6.301(a)**:

$$(1)_{,n} \qquad E_{n,n} = \Pi_{m,mnn} - \frac{1}{c^2}\frac{\partial^2 \Pi_{n,n}}{\partial t^2} \tag{11}$$

and for **6.302(a)**:

$$\begin{aligned}
\text{6.302(a)} \qquad \frac{1}{c}\frac{\partial E_r}{\partial t} &= \epsilon_{rmn} H_{n,m} && (12)\\
(2) \quad \Rightarrow \qquad &= \epsilon_{rmn}\frac{1}{c}\epsilon_{npq}\frac{\partial}{\partial t}\Pi_{q,pm} && (13)\\
\Rightarrow \qquad \frac{\partial E_r}{\partial t} &= \epsilon_{rmn}\epsilon_{npq}\frac{\partial}{\partial t}\Pi_{q,pm} && (14)\\
&= \left(\delta_{rp}\delta_{mq} - \delta_{rq}\delta_{mp}\right)\frac{\partial}{\partial t}\Pi_{q,pm} && (15)\\
&= \partial_t \Pi_{m,rm} - \partial_t \Pi_{r,mm} && (16)
\end{aligned}$$

Let's impose the condition on $\Pi_n$:

$$\mathbf{6.342} \qquad \boldsymbol{\Pi_{r,mm} = \frac{1}{c^2}\frac{\partial^2 \Pi_r}{\partial t^2}} \tag{17}$$

From (1) we have

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 \Pi_n}{\partial t^2} &= \Pi_{m,mn} - E_n && (18)\\
\text{(17) becomes} \qquad \Pi_{r,mm} &= \Pi_{m,mn} - E_n && (19)\\
\Rightarrow \qquad E_n &= \Pi_{m,mn} - \Pi_{r,mm} && (20)\\
\Rightarrow \qquad \partial_t E_n &= \partial_t \Pi_{m,mn} - \partial_t \Pi_{r,mm} && (21)
\end{aligned}$$

Which is consistent with **6.302(a)** following (16).

For **6.301(a)** we have

$$\begin{aligned}
(16)_{,n} \quad \Rightarrow \qquad \frac{1}{c^2}\frac{\partial^2 \Pi_{n,n}}{\partial t^2} &= \Pi_{n,mmn} && (22)\\
(11) \quad \Rightarrow \qquad E_{n,n} &= \Pi_{m,mnn} - \Pi_{n,mmn} && (23)\\
&= 0 && (24)
\end{aligned}$$

Which is consistent with **6.301(a)**

♦

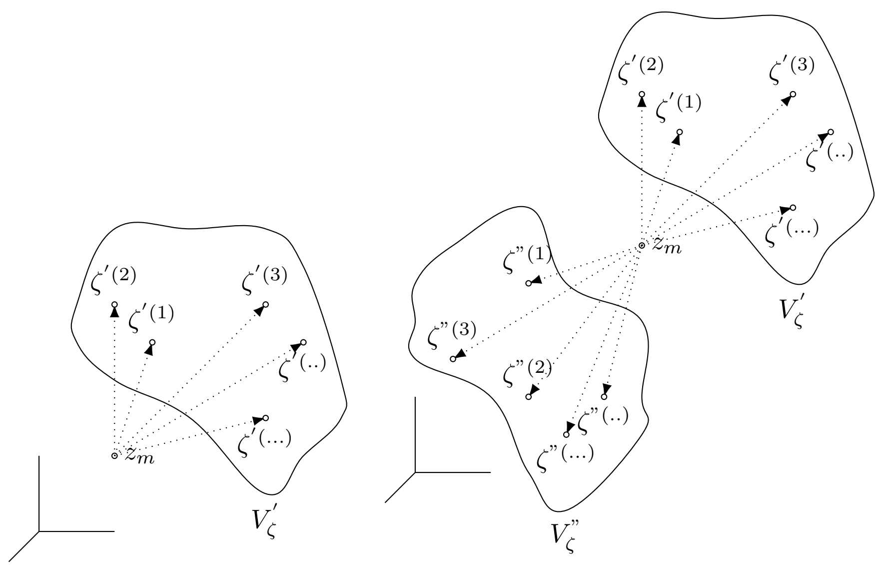
## 5.23 p226 - Clarification

**6.361** $$\Pi_r^{(0)}(z) = \int_{V_\zeta} P_r(\zeta)\, F(z,\zeta)\, dV_\zeta$$

The reason why we can find a solution in the form **6.361** is because the condition **6.358** : $\Pi^{(0)}_{r,mm} + k^2\Pi_r^{(0)} = 0$ is a linear homogeneous differential equation. So any linear combination of solutions of this equation will also be a solution:
Be

$$\Pi_r^{(0)} = \sum^{N} C_n \overset{n}{\Pi}{}_r^{(0)} \tag{1}$$

where the $\overset{n}{\Pi}{}_r^{(0)}$ satisfy the condition $\overset{n}{\Pi}{}^{(0)}_{r,mm} + k^2\overset{n}{\Pi}{}_r^{(0)} = 0$ an are evaluated at the same point $z_r$ but at different $\zeta_n$.
This is the situation as illustrated in the figure $(a)$ below.




(a) (b)

Figure 5.4: Integral form of Hertz vectors

If we take more and more points $\zeta_r$ and take the $C_k$ as a weight factor, then in the limit, we get $\Pi_r^{(0)}(z) = \int_{V_\zeta} P_r(\zeta)\, F(z,\zeta)\, dV_\zeta$ where $P_r(\zeta)$ is a kind of density vector field. On page 227, it is mentioned that the vector field $P_r(\zeta)$ does not need to be continuous. This situation is illustrated in the figure $(b)$ where $V_\zeta = V'_\zeta \oplus V''_\zeta$

♦

## 5.24 p227 - Exercise

Show that Maxwell's equation in the form **6.356** are satisfied by

$$\textbf{6.363} \qquad \begin{aligned} E_r^{(0)} &= ik\epsilon_{rpq} \int_{V_\zeta} Q_q\left(\zeta\right) F_{,p} dV_\zeta \\ H_r^{(0)} &= \int_{V_\zeta} Q_m\left(\zeta\right)\left(F_{,mr} + k^2 F \delta_{mr}\right) dV_\zeta \end{aligned}$$

where $Q_r\left(\zeta\right)$ is an arbitrary vector field, and $F$ is as in **6.344**

Let's first look at this with a discrete point and define

$$E_r^{(0)} = ik\epsilon_{rpq}\Pi_{q,p}^{(0)} \tag{1}$$
$$H_r^{(0)} = \Pi_{m,mr}^{(0)} + k^2\Pi_r^{(0)} \tag{2}$$

We check whether that Maxwell's equation in the form **6.356** are satisfied:

$$-ikE_r^{(0)} = \epsilon_{rmn}H_{n,m}^{(0)} \tag{3}$$
$$(1) \text{ and } (2) \quad \Rightarrow \quad -ikik\epsilon_{rpq}\Pi_{q,p}^{(0)} = \underbrace{\epsilon_{rmn}\Pi_{q,qmn}^{(0)}}_{=0} + \epsilon_{rmn}k^2\Pi_{n,m}^{(0)} \tag{4}$$
$$\Rightarrow \quad k^2\epsilon_{rpq}\Pi_{q,p}^{(0)} = \epsilon_{rmn}k^2\Pi_{n,m}^{(0)} \tag{5}$$

So the first Maxwell equation is satisfied.
For the second

$$ikH_r^{(0)} = \epsilon_{rmn}E_{n,m}^{(0)} \tag{6}$$
$$(1) \text{ and } (2) \quad \Rightarrow \quad ik\Pi_{m,mr}^{(0)} + ik^3\Pi_r^{(0)} = \epsilon_{rmn}ik\epsilon_{npq}\Pi_{q,pm}^{(0)} \tag{7}$$
$$\Rightarrow \quad \Pi_{m,mr}^{(0)} + k^2\Pi_r^{(0)} = \left(\delta_{rp}\delta_{mq} - \delta_{rq}\delta_{mp}\right)\Pi_{q,pm}^{(0)} \tag{8}$$
$$\Rightarrow \quad \Pi_{m,mr}^{(0)} + k^2\Pi_r^{(0)} = \Pi_{m,mr}^{(0)} - \Pi_{r,mm}^{(0)} \tag{9}$$
$$\Rightarrow \quad \Pi_{r,mm}^{(0)} + k^2\Pi_r^{(0)} = \Pi_{m,mr}^{(0)} - \Pi_{m,mr}^{(0)} = 0 \tag{10}$$

Conclusion, (1) and (2) are valid expressions of a EMW provided that the condition (10) is respected. The rest of the reasoning is identical as for the previous form for $E_r^{(0)}$ and $H_r^{(0)}$ when we take a linear combination of $\overset{n}{\Pi}{}_r^{(0)}$, each satisfying (10) and taking the limit on volume $V_\zeta$ while expressing $\Pi_r^{(0)}$ as $Q_r\left(\zeta\right)F\left(z,\zeta\right)$.

♦

## 5.25 p228 - Exercise

Write out Maxwell's equations in terms of a magnetic vector and a skew-symmetric electric tensor.

Let's define

$$E_{rm} = \epsilon_{rmn} E_n \tag{1}$$

$$\Rightarrow \qquad E_r = \frac{1}{2}\epsilon_{rmn} E_{mn} \tag{2}$$

Maxwell's equation :

$$\begin{cases} 6.301(a),(b) & E_{m,m} = 0, \qquad H_{m,m} = 0 \\ 6.302(a) & \frac{1}{c}\frac{\partial E_r}{\partial t} = \epsilon_{rmn} H_{n,m} \\ 6.302(b) & \frac{1}{c}\frac{\partial H_r}{\partial t} = -\epsilon_{rmn} E_{n,m} \end{cases} \tag{3}$$

**6.302**(**a**):

$$6.302(a) \times \epsilon_{rmn} \Rightarrow \qquad \frac{1}{c}\frac{\partial E_{rm}}{\partial t} = -\epsilon_{rmn}\epsilon_{npq} H_{q,p} \tag{4}$$

$$= (\delta mp \delta rq - \delta mq - \delta rp)\, H_{q,p} \tag{5}$$

$$= H_{r,m} - H_{m,r} \tag{6}$$

So,

$$\frac{\partial \mathbf{E_{rm}}}{\partial \mathbf{t}} = \mathbf{H_{r,m}} - \mathbf{H_{m,r}}$$

**6.302**(**b**):

$$(2) \text{ in } 6.302(b) \Rightarrow \qquad \frac{1}{c}\frac{\partial H_r}{\partial t} = \frac{1}{2}\epsilon_{rmn}\epsilon_{npq} E_{pq,m} \tag{7}$$

$$= \frac{1}{2}\left(\delta_{rp}\delta_{mq} - \delta_{rq}\delta_{pm}\right) E_{pq,m} \tag{8}$$

$$= \frac{1}{2}\left(E_{rm,m} - E_{mr,m}\right) \tag{9}$$

$$= E_{rm,m} \quad (E_{rm} \text{ is skew-symmetric}) \tag{10}$$

So,

$$\frac{\mathbf{1}}{\mathbf{c}}\frac{\partial \mathbf{H_r}}{\partial \mathbf{t}} = \mathbf{E_{rm,m}}$$

**6.301(a)**:

$$\text{(2) in 6.301(a)} \quad \Rightarrow \qquad \epsilon_{rmn} E_{mn,r} = 0 \tag{11}$$

As this equation is homogeneous, we can permute the indices in $E_{mn,r}$ and write $\epsilon_{rmn} E_{rn,m} = 0$ and $\epsilon_{rmn} E_{mr,n} = 0$.
Adding this three equation together we get

$$\mathbf{E_{mn,r} + E_{rn,m} + E_{mr,n} = 0}$$

**6.302(b)**:
As in this case, nothing changes for $H_r$ we have

$$\mathbf{H_{n,n} = 0}$$

♦

## 5.26 p229 - Clarification

**6.371** $F_{\alpha\beta} = H_{\alpha\beta},\; = -F_{4\alpha} = E_{\alpha},\; F_{44} = 0$

**6.374** $g_{\alpha\beta} = a_{\alpha\beta},\; g_{\alpha 4} = 0,\; g_{44} = -1$

For clarity this give in matrix form

$$(F_{mn}) = \begin{pmatrix} 0 & H_3 & H_2 & E_1 \\ -H_3 & 0 & H_1 & E_2 \\ H_2 & -H_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix} \tag{1}$$

$$(g_{mn}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2}$$

♦

## 5.27 p231 - Exercise

Show that with homogeneous coordinates $z-r$ ($z_1, z_2, z_3$ being rectangular Cartesians in space and $z_4 = ict = ix^4$ ) Maxwell's equations read

$$F_{rm,n} + F_{mn,r} + F_{nr,m} = 0, \; F_{rm,m} = 0$$

Write out the components of $F_{mn}$ in terms of the real electric and magnetic vectors, noting which components are real and which are imaginary.

We use the same convention as in the book: Greek indices are restricted to the space manifold. Extending this manifold to a 4 dimensional manifold, Latin suffixes will be used. We have **6.369**:

$$\begin{cases} \text{(a)} \quad \frac{1}{c}\frac{\partial E_r}{\partial t} = a^{mn} H_{rm|n} \\ \text{(b)} \quad \frac{1}{c}\frac{\partial H_{rm}}{\partial t} = E_{r,m} - E_{m,r} \\ \text{(c)} \quad a^{mn} E_{n|m} = 0 \\ \text{(d)} \quad H_{rm,n} + H_{mn,r} + H_{nr,m} = 0 \end{cases} \tag{1}$$

We rewrite (1):

$$\begin{cases} \text{(a)} \quad \frac{\partial E_\alpha}{\partial (ict)} = -a^{\beta\gamma} H_{\alpha\beta|\gamma} \\ \text{(b)} \quad i\frac{\partial H_{\alpha\beta}}{\partial (ict)} = E_{\alpha,\beta} - E_{\beta,\alpha} \\ \text{(c)} \quad a^{\alpha\beta} E_{\beta|\alpha} = 0 \\ \text{(d)} \quad H_{\alpha\beta,\gamma} + H_{\beta\gamma,\alpha} + H_{\gamma\alpha,\beta} = 0 \end{cases} \tag{2}$$

instead of **6.371** let's define:

$$F_{\alpha\beta} = H_{\alpha\beta}, \; F_{\alpha 4} = -F_{4\alpha} = -iE_\alpha, \; F_{44} = 0 \tag{3}$$

$$\Rightarrow \qquad E_\alpha = iF_{\alpha 4} = -iF_{4\alpha} \tag{4}$$

Using (4), (2) becomes

$$\begin{cases} (a) & -\frac{\partial F_{\alpha 4}}{\partial (ict)} = a^{\beta\gamma} F_{\alpha\beta|\gamma} \\ \\ (b) & i\frac{\partial F_{\alpha\beta}}{\partial (ict)} = iF_{\alpha 4,\beta} - iF_{\beta 4,\alpha} \\ \\ (c) & a^{\alpha\beta} F_{\beta 4,\alpha} = 0 \\ \\ (d) & F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \end{cases} \tag{5}$$

For homogeneous coordinates we have $a^{\alpha\beta} = 1$ and $a^{\alpha 4} = 1$ so:

$$\begin{cases} (a) & F_{\alpha 4,4} = -F_{\alpha\beta|\beta} \\ \\ (b) & F_{\alpha\beta,4} = F_{\alpha 4,\beta} - F_{\beta 4,\alpha} \\ \\ (c) & F_{\beta 4,\beta} = 0 \\ \\ (d) & F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \end{cases} \tag{6}$$

Let's look what happens when we extend the range $\alpha, \beta, \gamma$ to 4. First let's extend $\gamma$ to 4. The left part of equation (6d) can be written as

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} + F_{\alpha\beta,4} + F_{\beta 4,\alpha} + F_{4\alpha,\beta} \tag{7}$$

Consider

$$P = F_{\alpha\beta,4} + F_{\beta 4,\alpha} + F_{4\alpha,\beta} \tag{8}$$

Let's extend $\alpha$ to 4:

$$P' = F_{4\beta,4} + F_{\beta 4,4} + \underbrace{F_{44,\beta}}_{=0} \tag{9}$$

$$= \underbrace{F_{4\beta,4} + F_{\beta 4,4}}_{=0} \quad (F_{mn} \text{ is skew-symmetric}) \tag{10}$$

Extend $\beta$ to 4:

$$P'' = F_{44,4} + F_{44,4} + F_{44,4} \tag{11}$$

$$= 0 \quad (F_{mn} \text{ is skew-symmetric}) \tag{12}$$

So, we only have to prove that $P = F_{\alpha\beta,4} + F_{\beta 4,\alpha} + F_{4\alpha,\beta} = 0$.

From $(6b)$ we get:

$$P = F_{\alpha\beta,4} + F_{\beta 4,\alpha} + F_{4\alpha,\beta} \tag{13}$$
$$= F_{\alpha 4,\beta} - F_{\beta 4,\alpha} + F_{\beta 4,\alpha} - F_{\alpha 4,\beta} \tag{14}$$
$$= 0 \tag{15}$$

We get so,

$$F_{rm,n} + F_{mn,r} + F_{nr,m} = 0 \tag{16}$$

Consider now equation $(6c)$ and extend the suffixes from 3 to 4.
What is the value of the following expression?

$$Q = F_{4m,m} = F_{4\beta,\beta} + F_{44,4} \tag{17}$$

Obviously, $F_{mn}$ being skew-symmetric we have $F_{44} = 0 \quad \Rightarrow \quad F_{44,4} = 0 \quad \Rightarrow \quad Q = 0$.
So, the Maxwell equations reduce to

$$\begin{cases} F_{rm,n} + F_{mn,r} + F_{nr,m} = 0 \\ F_{rm,m} = 0 \end{cases} \tag{18}$$

For the explicit expression of $F_{mn}$ we get from (3) and (4):

$$F_{mn} = \begin{pmatrix} 0 & H_3 & -H_1 & -iE_1 \\ -H_3 & 0 & H_2 & -iE_2 \\ H_1 & -H_2 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \tag{19}$$

♦

## 5.28 p234 - Exercise 1

For a fluid in motion referred to curvilinear coordinates, the kinetic energy of the fluid in any region $R$ is

$$T = \frac{1}{2}\int_R \rho v_r v^r dV$$

Use the equation of motion **6.147** to show that, if we follow the particles which compose $R$, we have

$$\frac{dT}{dt} = -\int_S p n_r v^r dS + \int_R \theta p dV + \int_R \rho v_r X^r dV$$

where $S$ is the bounding surface to $R$, and $n_r$ the unit vector normal to $S$ and drawn outward. Show further that if, instead of following the particles, we calculate the rate of change of $T$ for a fixed portion of space, we get the above expression with the following additional term

$$-\frac{1}{2}\int_S \rho n_r v^r v_s v^s dS$$

Let's recall that

$$\theta = {v^r}_{|r} \qquad (\mathbf{6.126} \text{ page 196.}) \tag{1}$$

and

$$\begin{cases} \text{(a)} \quad \frac{\partial v_r}{\partial t} + v^s v_{r|s} = X_r - \rho^{-1} p_{,r} & \text{see (6.147)} \\ \\ \text{(b)} \quad \frac{\partial v^r}{\partial t} + v_s {v^r}_{|s} = X^r - \rho^{-1} a^{rm} p_{,m} & \text{see exercise page 201} \end{cases} \tag{2}$$

First let us note that, for the first part of the question, we move with the particles which means that for the considered region the mass contained in this region will remain unchanged and hence $\rho dV$ can be considered as a constant when bringing the derivation operator inside the volume integral. So,

$$T = \frac{1}{2}\int_R \rho v_r v^r dV \tag{3}$$

$$\Rightarrow \qquad \frac{dT}{dt} = \frac{1}{2}\int_R \left(\frac{\delta v_r}{\delta t}v^r + \frac{\delta v_r}{\delta t}v^r\right)\rho dV \tag{4}$$
$$= \frac{1}{2}\int_R \left[\left(\partial_t v_r + v^s v_{r|s}\right)v^r + v_r\left(\partial_t v^r + v_s v^r{}_{|s}\right)\right]\rho dV \tag{5}$$
$$= \frac{1}{2}\int_R \left[\left(X_r - \rho^{-1}p_{,r}\right)v^r + v_r\left(X^r - \rho^{-1}a^{rm}p_{,m}\right)\right]\rho dV \tag{6}$$
$$= \frac{1}{2}\int_R \left(\underbrace{X_r v^r}_{=X^r v_r} - \rho^{-1}p_{,r}v^r + X^r v_r - \rho^{-1}a^{rm}v_r p_{,m}\right)\rho dV \tag{7}$$
$$= -\frac{1}{2}\int_R \left(p_{,r}v^r + \underbrace{a^{rm}v_r}_{=v^m}p_{,m}\right)dV + \int_R \rho X^r v_r dV \tag{8}$$
$$= -\int_R p_{,r}v^r dV + \int_R \rho X^r v_r dV \tag{9}$$

Let's look at the expression $p_{,r}v^r$ in the first integral in (9).
Obviously:

$$(pv^r)_{,r} = p_{,r}v^r + pv^r_{,r} \tag{10}$$
$$\Rightarrow \qquad p_{,r}v^r = (pv^r)_{,r} - pv^r_{,r} \tag{11}$$

Let's note also that

$$(pv^r)_{|r} - pv^r{}_{|r} = (pv^r)_{,r} + \Gamma^r_{mr}(pv^m) - pv^r_{,r} - p\Gamma^r_{mr}v^m \tag{12}$$
$$= (pv^r)_{,r} - pv^r_{,r} \tag{13}$$
$$\text{(11) becomes} \qquad p_{,r}v^r = (pv^r)_{|r} - pv^r{}_{|r} \tag{14}$$

Substituting in (9):

$$\frac{dT}{dt} = -\int_R (pv^r)_{|r}\, dV + \int_R pv^r{}_{|r} dV + \int_R \rho X^r v_r dV \tag{15}$$

Using Green's theorem $\int F_r n^r dS = \int F^r{}_{|r} dV$, and putting $F^r = pv^r$:

$$\frac{dT}{dt} = -\int_S p\,\underbrace{v_r n^r}_{=v^r n_r}\, dS + \int_R p\,\underbrace{v^r{}_{|r}}_{=\theta}\, dV + \int_R \rho X^r v_r dV \tag{16}$$

giving

$$\mathbf{\frac{dT}{dt} = -\int_S pv^r n_r dS + \int_R p\theta dV + \int_R \rho X^r v_r dV} \tag{17}$$

◊

What if we look at the rate of change of $T$ in a fixed region?
Then $\rho dV$ can't be considered as a constant and bringing the derivative operator under the integral

will generate an additional term

$$\frac{1}{2}\int_R v_r v^r \frac{\delta\rho}{\delta t} dV \tag{18}$$

As we fix the spatial coordinates, we have $\frac{\delta\rho}{\delta t} = \partial_t \rho$ and by **6.127b** we have

$$\partial_t \rho = -\left(\rho v^r\right)_{|r} \tag{19}$$

So (18) becomes

$$(18) = -\frac{1}{2}\int_R v_s v^s \left(\rho v^r\right)_{|r} dV \tag{20}$$

Using again Green's theorem $\int F_r n^r dS = \int F^r{}_{|r} dV$ with $F^r = \rho v^r$ and nothing that $v_s v^s$ is an invariant:

$$(18) = -\frac{1}{2}\int_S v_s v^s \rho v^r n_r dV \tag{21}$$

♦

## 5.29 p235 - Exercise 2

Consider a fluid in which $\rho$ is a function of $p$, moving under a conservative body force. Show that if the motion is steady, but not necessarily irrotational, then the following quantity is constant along each stream line:

$$\frac{1}{2}v_r v^r + P + U$$

(a stream line is a curve which, at each point , has the direction of the velocity vector $v^r$). Compare and contrast this result with **6.154**.

What is given:

$$\begin{cases} \text{(a)} \quad \partial_t v_r + v^s v_{r|s} = X_r - \rho^{-1} p_{,r} & \text{see (6.147)} \\ \text{(b)} \quad P_{,r} = \rho^{-1} p_{,r} & \text{see (6.150)} \\ \text{(c)} \quad X_r = -U_{,r} & \text{see (6.151)} \end{cases} \tag{1}$$

As the motion is stationary: $\partial_t v_r = 0$ and so .

$$v^s v_{r|s} + U_{,r} + P_{,r} = 0 \tag{2}$$

Let's consider a stream line given by the set of equations $x^r = x^r(u)$, where $u$ is a parameter. Then, by definition of a streamline, and considering a steady flow, we have $\frac{dx^r}{du} = kv^r$. Considering (2) we have

$$v^s v_{r|s} + U_{,r} + P_{,r} = 0 \tag{3}$$

$$\times \frac{dx^r}{du} \qquad v^s v_{r|s} \frac{dx^r}{du} + U_{,r} \frac{dx^r}{du} + P_{,r} \frac{dx^r}{du} = 0 \tag{4}$$

$$v^s v_{r|s} \frac{dx^r}{du} + \frac{dU}{du} + \frac{dP}{du} = 0 \tag{5}$$

Let's look at the first term:

$$v^s v_{r|s} \frac{dx^r}{du} = k v^s v_{r|s} v^r \tag{6}$$

$$= k v^s \left(a_{rm} v^m\right)_{|s} v^r \tag{7}$$

$$= k v^s v^m{}_{|s} a_{rm} v^r \tag{8}$$

$$= k v^s v^m{}_{|s} v_m \tag{9}$$

$$= k v^s v^r{}_{|s} v_r \tag{10}$$

So (5) can be equivalently written as $kv^s v_{r|s} v^r + \frac{dU}{du} + \frac{dP}{du} = 0$ and $kv^s v^r{}_{|s} v_r + \frac{dU}{du} + \frac{dP}{du} = 0$. Summing these two gives

$$kv^s\left(v_{r|s}v^r + v^r{}_{|s}v_r\right) + 2\frac{dU}{du} + 2\frac{dP}{du} = 0 \tag{11}$$

$$\Leftrightarrow \quad kv^s\left(v_r v^r\right)_{|s} + 2\frac{dU}{du} + 2\frac{dP}{du} = 0 \tag{12}$$

$$\Leftrightarrow \quad kv^s\left(v_r v^r\right)_{,s} + 2\frac{dU}{du} + 2\frac{dP}{du} = 0 \qquad v_r v^r \text{ is an invariant} \tag{13}$$

$$\Leftrightarrow \quad \frac{dx^s}{du}\left(v_r v^r\right)_{,s} + 2\frac{dU}{du} + 2\frac{dP}{du} = 0 \tag{14}$$

$$\Leftrightarrow \quad \frac{d\left(v_r v^r\right)}{du} + 2\frac{dU}{du} + 2\frac{dP}{du} = 0 \tag{15}$$

Integrating expression (15) gives

$$\mathbf{\frac{1}{2}v_r v^r + U + P = C}$$

with $C$ a constant along a streamline.

♢

Compare and contrast this result with **6.154** (irrotational motion).

$$-\partial_t\phi + \frac{1}{2}a^{mn}\phi_{,m}\phi_{,n} + P + U = F(t)$$

For a stationary motion ($\partial_t\phi = 0,\ F(t) = constant$) the expression reduces to

$$\mathbf{\frac{1}{2}a^{mn}\phi_{,m}\phi_{,n} + P + U = C}$$

Replacing in the general expression $\frac{1}{2}v_r v^r + U + P = C$, $v_r v^r$ by $v_r = -\phi_{,r}$ and $v^r = -a^{rm}\phi_{,m}$ gives obviously $\frac{1}{2}a^{mn}\phi_{,m}\phi_{,n} + P + U = C$.

♦

## 5.30 p235 - Exercise 3

For the general motion of the fluid described in Exercise 2, prove that

$$\frac{d}{dt}\int_C v_r dx^r = 0$$

where the integral is take round any closed curve, and $\frac{d}{dt}$ is the co-moving time derivative.

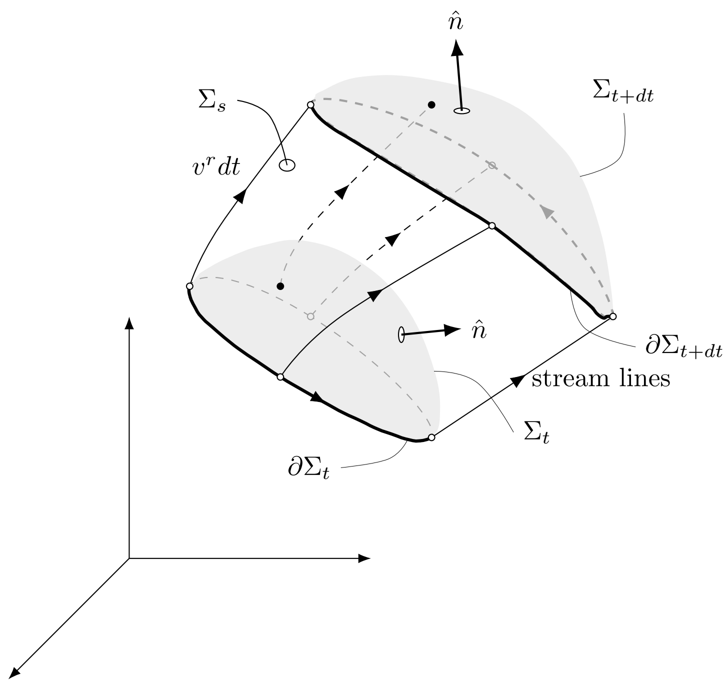



Figure 5.5: Evolution of a closed co-moving curve

Let's see what happens when a closed curve moves during a time $dt$. All the particles along that curve $\partial\Sigma_t$ will move along stream lines and will form a new closed curve $\partial\Sigma_{t+dt}$. Also all particles lying on a surface $\Sigma_t$ enclosed by the closed curve will end on the surface $\Sigma_{t+dt}$. So we can use the Kelvin-Stokes theorem and form

$$\oint_{\partial\Sigma_{t+dt}} v_r dx^r - \oint_{\partial\Sigma_t} v_r dx^r = \iint_{\Sigma_{t+dt}} \epsilon_{rjk} v_{k,j} n^r dS - \iint_{\Sigma_t} \epsilon_{rjk} v_{k,j} (n^r) dS \tag{1}$$

Consider now the surface formed by the envelope of the stream lines starting from the curve $\partial\Sigma_t$.

What we try to achieve is to go from a curve integral to a volume integral using first the Kelvin-Stokes theorem followed by the divergence theorem and adding a term which we have to prove it is equal to zero

$$\oint_{\partial\Sigma_{t+dt}} v_r dx^r - \oint_{\partial\Sigma_t} v_r dx^r = \iint_{\Sigma_{t+dt}} \epsilon_{rjk} v_{k,j} n^r dS - \iint_{\Sigma_t} \epsilon_{rjk} v_{k,j} (-n^r) dS + \underbrace{\iint_{\Sigma_s} \epsilon_{rjk} v_{k,j} n^r dS}_{=0?} \tag{2}$$

$$= \iint_{\Sigma_{t+dt}} \epsilon_{rjk} v_{k,j} n^r dS + \iint_{\Sigma_t} \epsilon_{rjk} v_{k,j} n^r dS + \underbrace{\iint_{\Sigma_s} \epsilon_{rjk} v_{k,j} n^r dS}_{=0?} \tag{3}$$

$$= \iiint_V \epsilon_{rjk} v_{k,jr} dV \tag{4}$$

$$= 0 \tag{5}$$

Note that in the second term of the right expression, we changed the sign of the normal vector $\hat{n}$ as we want the normal vector on the surface point outward of the considered volume.

So we have to prove that indeed,

$$\iint_{\Sigma_s} \epsilon_{rjk} v_{k,j} n^r dS = 0 \tag{6}$$

Nothing that $n^r dS$ can be expressed as the cross product of $\overline{v}dt$ and $d\overline{l}$ (the last being the line segment along the curve $\partial_t\Sigma$), we have

$$n^r dS = \epsilon_{rpq} v^p dx^q dt \tag{7}$$

And get

$$\iint_{\Sigma_s} \epsilon_{rjk} v_{k,j} n^r dS = \iint_{\Sigma_s} \epsilon_{rjk} v_{k,j} \epsilon_{rpq} v^p dx^q dt \tag{8}$$

$$= \iint_{\Sigma_s} (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) v_{k,j} v^p dx^q dt \tag{9}$$

$$= \iint_{\Sigma_s} (v_{q,p} v^p dx^q - v_{p,q} v^p dx^q) dt \tag{10}$$

$$(v_p dt = dx^p) \quad \Rightarrow \quad = \iint_{\Sigma_s} (v_{q,p} dx^p dx^q - v_{p,q} dx^p dx^q) \tag{11}$$

$$= 0 \tag{12}$$

So (5) is proven and get $d\oint_{\partial\Sigma_t} v_r dx^r = 0$ during an infinitesimal time $dt$ giving

$$\frac{d}{dt}\oint_{\partial\Sigma_t} v_r dx^r = 0$$

♦

## 5.31 p235 - Exercise 4

Curves having at each point the direction of the vorticity vector $\omega^r$ are called "vortex lines". Prove that $\int_C v_r dx^r$ has the same value for all closed curves $C$ which lie on the surface of the tube of vortex lines, and go once around the tube in the same sense. (Use Stokes' theorem; cf. 7.502.).

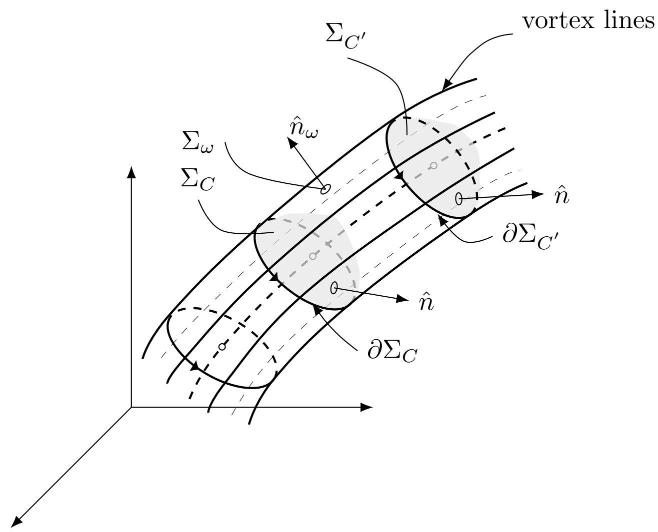



Figure 5.6: Vortex line tube

We have in rectangular Cartesian coordinates

$$\textbf{(6.128)} \qquad \omega_{jk} = \frac{1}{2}\left(v_{k,j} - v_{j,k}\right) \tag{1}$$

$$\textbf{(6.129)} \qquad \omega_r = \frac{1}{2}\epsilon_{rjk}\omega_{jk} \tag{2}$$

So we can use the Kelvin-Stokes theorem and form

$$\oint_{\partial\Sigma_{C'}} v_r dx^r - \oint_{\partial\Sigma_C} v_r dx^r = \iint_{\Sigma_{C'}} \epsilon_{rjk} v_{k,j} n^r dS - \iint_{\Sigma_C} \epsilon_{rjk} v_{k,j} n^r dS \tag{3}$$

Obviously, we have on the surface $\Sigma_\omega$ of the tube, $\hat{\omega} \cdot \hat{n}_\omega = 0$ and thus $\iint_{\Sigma_\omega} \omega_r n^r dS = 0$. Expanding the terms under the integral gives (by (6.129) and (6.128))

$$\iint_{\Sigma_\omega} \omega_r n^r dS = \frac{1}{4}\iint_{\Sigma_\omega} \epsilon_{rjk}\left(v_{k,j} - v_{j,k}\right) n^r dS \tag{4}$$

As $\iint_{\Sigma_\omega} \omega_r n^r dS = 0$ we can put

$$\begin{aligned}
\iint_{\Sigma_\omega} \omega_r n^r dS &= \frac{1}{2}\iint_{\Sigma_\omega} \epsilon_{rjk} v_{k,j} n^r dS - \frac{1}{2}\iint_{\Sigma_\omega} \epsilon_{rjk} v_{j,k} n^r dS &(5)\\
&= \frac{1}{2}\iint_{\Sigma_\omega} \epsilon_{rjk} v_{k,j} n^r dS - \frac{1}{2}\iint_{\Sigma_\omega} \epsilon_{rkj} v_{k,j} n^r dS &(6)\\
&= \frac{1}{2}\iint_{\Sigma_\omega} \epsilon_{rjk} v_{k,j} n^r dS + \frac{1}{2}\iint_{\Sigma_\omega} \epsilon_{rjk} v_{k,j} n^r dS &(7)\\
&= \iint_{\Sigma_\omega} \epsilon_{rjk} v_{k,j} n^r dS \quad (= 0) &(8)
\end{aligned}$$

Adding (8) in (3) we get

$$\oint_{\partial\Sigma_{C'}} v_r dx^r - \oint_{\partial\Sigma_C} v_r dx^r = \iint_{\Sigma_{C'}} \epsilon_{rjk} v_{k,j} n^r dS + \iint_{\Sigma_C} \epsilon_{rjk} v_{k,j} n^r dS + \iint_{\Sigma_\omega} \epsilon_{rjk} v_{k,j} n^r dS \quad (9)$$

Note that in the second term of the right expression, we changed the sign of the normal vector $\hat{n}$ as we want the normal vector on the surface point outward of the considered volume.
Using Green's theorem:

$$\begin{aligned}
\oint_{\partial\Sigma_{C'}} v_r dx^r - \oint_{\partial\Sigma_C} v_r dx^r &= \iint_{\Sigma_{C'}} \epsilon_{rjk} v_{k,j} n^r dS + \iint_{\Sigma_C} \epsilon_{rjk} v_{k,j} n^r dS + \iint_{\Sigma_\omega} \epsilon_{rjk} v_{k,j} n^r dS &(10)\\
&= \iiint_V \epsilon_{rjk} v_{k,jr} dV &(11)\\
&= 0 &(12)
\end{aligned}$$

where $V$ is the volume enclosed by the surfaces $\Sigma_C$, $\Sigma_{C'}$ and $\Sigma_\omega$.
And get $d\oint_{\partial\Sigma_t} v_r dx^r = 0$ along an infinitesimal displacement along the vortex line giving

$$\frac{d}{ds}\oint_{\partial\Sigma_t} v_r dx^r = 0$$

where $s$ is a parameter upon which a vortex line can be expressed.
Hence $\oint_{\partial\Sigma_t} v_r dx^r$ is constant along a vortex line.

♦

## 5.32 p235 - Exercise 5

Prove that for the type of fluid described in Exercise 2, the vorticity tensor satisfies the differential equations

$$\frac{d}{dt}\omega_{rs} = \omega_{pr}v_{p,s} - \omega_{ps}v_{p,r}$$

the coordinates being rectangular Cartesians. Write these equations for curvilinear coordinates.
Deduce from these equations that, if $\omega_{rs} = 0$ initially at some point $P$ in the fluid, these quantities will remain zero permanently for the particle which was initially at $P$.

Let's first write down some useful expressions:

We have in rectangular Cartesian coordinates

$$\begin{aligned}
\textbf{(6.128)} \qquad & \omega_{jk} = \frac{1}{2}\left(v_{k,j} - v_{j,k}\right) && (1)\\
\textbf{(6.129)} \qquad & \omega_{r} = \frac{1}{2}\epsilon_{rjk}\omega_{jk} && (2)\\
\Rightarrow \qquad & \omega_{r} = \frac{1}{2}\frac{1}{2}\epsilon_{rjk}\left(v_{k,j} - v_{j,k}\right) && (3)\\
& = \frac{1}{2}\frac{1}{2}\epsilon_{rjk}v_{k,j} - \frac{1}{2}\frac{1}{2}\epsilon_{rjk}v_{j,k} && (4)\\
& = \frac{1}{2}\frac{1}{2}\epsilon_{rjk}v_{k,j} + \frac{1}{2}\frac{1}{2}\epsilon_{rjk}v_{k,j} && (5)\\
& = \frac{1}{2}\epsilon_{rjk}v_{k,j} && (6)\\
\Rightarrow \qquad & \omega_{r,r} = \frac{1}{2}\epsilon_{rjk}v_{k,jr} && (7)\\
& = 0 && (8)
\end{aligned}$$

So we have, as the flow is steady

$$\begin{aligned}
& \frac{d}{dt}\omega_{rs} = \omega_{rs,p}v_{p} && (9)\\
(1): \qquad & = \frac{1}{2}\left(v_{s,rp} - v_{r,sp}\right)v_{p} && (10)
\end{aligned}$$

We have also

$$\textbf{(6.146)} \qquad \partial_t v_r + v_s v_{r,s} = X_r - \rho^{-1}p_{,r} \qquad (11)$$

In the fluid considered here ($\partial_t v_r = 0$; $\rho = \rho(p) \Rightarrow \rho^{-1}p_{,r} := P_{,r}$; $X_r = -U_{,r}$) and get

$$v_s v_{r,s} + U_{,r} + P_{,r} = 0 \qquad (12)$$

We are now able to deduce the asked expression.

Consider the following expression $\epsilon_{nmr}\epsilon_{nst}v_m v_{t,s}$ This can be expressed as

$$\epsilon_{nrm}\epsilon_{nst}v_m v_{t,s} = (\delta_{rs}\delta_{mt} - \delta_{rt}\delta_{ms})\, v_m v_{t,s} \tag{13}$$
$$= v_m v_{m,r} - v_m v_{r,m} \tag{14}$$

giving

$$v_s v_{r,s} = v_s v_{s,r} - \epsilon_{nrm}\epsilon_{nst}v_m v_{t,s} \tag{15}$$

So (12) can be expressed as

$$v_s v_{s,r} - \epsilon_{nrm}\epsilon_{nst}v_m v_{t,s} + U_{,r} + P_{,r} = 0 \tag{16}$$
$$(6) \quad \Rightarrow \quad v_s v_{s,r} - 2\epsilon_{nrm}\omega_n v_m + U_{,r} + P_{,r} = 0 \tag{17}$$
$$v_s v_{s,r} + 2\epsilon_{rnm}\omega_n v_m + U_{,r} + P_{,r} = 0 \tag{18}$$

Let's take the curl of (18):

$$\epsilon_{pkr}\left(v_s v_{s,r}\right)_{,k} + 2\epsilon_{pkr}\epsilon_{rnm}\left(\omega_n v_m\right)_{,k} + \epsilon_{pkr}U_{,rk} + \epsilon_{pkr}P_{,rk} = 0 \tag{19}$$

The terms with $U$ and $P$ are of the kind $\nabla \times \nabla(G)$ with $G$ a scalar function $G : \mathbb{R}^3 \to \mathbb{R}$. These terms are equal to zero.
So, (19) becomes

$$\underbrace{\epsilon_{pkr}v_{s,k}v_{s,r}}_{=0} + \underbrace{\epsilon_{pkr}v_s v_{s,kr}}_{=0} + 2\epsilon_{pkr}\epsilon_{rnm}\left(\omega_n v_m\right)_{,k} = 0 \tag{20}$$

Nothing that,

$$\epsilon_{pkr}\epsilon_{rnm}\left(\omega_n v_m\right)_{,k} = \epsilon_{rpk}\epsilon_{rnm}\left(\omega_n v_m\right)_{,k} \tag{21}$$
$$= \delta_{pn}\delta_{km}\left(\omega_n v_m\right)_{,k} - \delta_{pm}\delta_{kn}\left(\omega_n v_m\right)_{,k} \tag{22}$$
$$= \left(\omega_p v_k\right)_{,k} - \left(\omega_k v_p\right)_{,k} \tag{23}$$
$$= \omega_{p,k}v_k + \omega_p v_{k,k} - \omega_{k,k}v_p - \omega_k v_{p,k} \tag{24}$$

Note that the divergence of the vorticity is zero, see (8), so

$$\epsilon_{pkr}\epsilon_{rnm}\left(\omega_n v_m\right)_{,k} = \omega_{p,k}v_k + \omega_p v_{k,k} - \omega_k v_{p,k} \tag{25}$$

(25) in (20) gives

$$\omega_{p,k}v_k + \omega_p v_{k,k} - \omega_k v_{p,k} = 0 \tag{26}$$

We have (6): $\omega_p = \frac{1}{2}\epsilon_{pmn}v_{n,m}, \quad \omega_k = \frac{1}{2}\epsilon_{kmn}v_{n,m}$ giving for (26):

$$\epsilon_{pmn}v_{n,mk}v_k + \epsilon_{pmn}v_{n,m}v_{k,k} - \epsilon_{kmn}v_{n,m}v_{p,k} = 0 \tag{27}$$

this can also be expressed as

$$\epsilon_{pnm}v_{m,nk}v_k + \epsilon_{pnm}v_{m,n}v_{k,k} - \epsilon_{knm}v_{m,n}v_{p,k} = 0 \tag{28}$$

adding (27) and (28) gives

$$\epsilon_{pnm}\left(v_{m,nk} - v_{n,mk}\right)v_k + \epsilon_{pnm}\left(v_{m,n} - v_{n,m}\right)v_{k,k} - \epsilon_{knm}\left(v_{m,n} - v_{n,m}\right)v_{p,k} = 0 \tag{29}$$

$$\text{(1),(10):} \quad \epsilon_{pmn}\frac{d\omega_{mn}}{dt} + \epsilon_{pmn}\omega_{mn}v_{k,k} - \epsilon_{kmn}\omega_{mn}v_{p,k} = 0 \tag{30}$$

$$\times\epsilon_{prs} \quad \Rightarrow \quad \left(\delta_{rm}\delta_{sn} - \delta_{rn}\delta_{ms}\right)\frac{d\omega_{mn}}{dt} + \left(\delta_{rm}\delta_{sn} - \delta_{rn}\delta_{ms}\right)\omega_{mn}v_{k,k} - \epsilon_{kmn}\omega_{mn}\epsilon_{prs}v_{p,k} = 0 \tag{31}$$

$$\Rightarrow \quad \frac{d\omega_{rs}}{dt} - \frac{d\omega_{sr}}{dt} + \left(\omega_{rs} - \omega_{sr}\right)v_{k,k} - \epsilon_{kmn}\omega_{mn}\epsilon_{prs}v_{p,k} = 0 \tag{32}$$

and as $\omega_{rs}$ is skew-symmetric

$$\frac{d\omega_{rs}}{dt} + \omega_{rs}v_{k,k} - \frac{1}{2}\epsilon_{kmn}\omega_{mn}\epsilon_{prs}v_{p,k} = 0 \tag{33}$$

Let's look at the term $\epsilon_{kmn}\omega_{mn}\epsilon_{prs}v_{p,k}$
Suppose $k = p$; then in order to have the term in $\epsilon_{(p)mn}\epsilon_{(p)rs}$ not equal to zero we need

$$(m = r \wedge n = s) \vee (m = s \wedge n = r) \quad \text{with } r \neq s$$

and get

$$\frac{1}{2}\sum_{(p)}\epsilon_{(p)mn}\omega_{mn}\epsilon_{(p)rs}v_{(p),(p)} = \frac{1}{2}\sum_{(p)}\epsilon_{(p)rs}\omega_{mn}\epsilon_{(p)rs}v_{(p),(p)} \tag{34}$$

$$= \frac{1}{2}\sum_{(p)}\left(\underbrace{\delta_{(p)(p)}}_{=1}\underbrace{\delta_{ss}}_{=2} - \underbrace{\delta_{(p)s}}_{=0}\delta_{(p)s}\right)\omega_{mn}v_{(p),(p)} \tag{35}$$

$$= \sum_{(p)}\omega_{rs}v_{(p),(p)} = \omega_{rs}v_{k,k} \tag{36}$$

($\delta_{ss} = 2$ as $s$ can not take the value of $(p)$ and so can only span two dimensions.)
Hence, the term $\omega_{rs}v_{k,k}$ in (33) vanishes.

Suppose now that $k \neq p$. We rewrite $\frac{1}{2}\epsilon_{kmn}\omega_{mn}\epsilon_{prs}v_{p,k}$ as

$$\frac{1}{2}\sum_{(p)}^{p\neq k}\sum_{(k)}^{k\neq p}\epsilon_{(k)mn}\omega_{mn}\epsilon_{(p)rs}v_{(p),(k)} \tag{37}$$

nothing that the explicit summation symbol only span two dimensions.
In (37) as $k \neq p$ we need that $m = p$ or $n = p$ and rewrite (37) as

$$\frac{1}{2}\sum_{(p)}^{p\neq k}\sum_{(k)}^{k\neq p}\epsilon_{(k)mn}\omega_{mn}\epsilon_{(p)rs}v_{(p),(k)} = \begin{cases} \frac{1}{2}\sum_{(p)}\sum_{(k)}\epsilon_{(k)(p)n}\omega_{(p)n}\epsilon_{(p)rs}v_{(p),(k)} \\ +\frac{1}{2}\sum_{(p)}\sum_{(k)}\epsilon_{(k)m(p)}\omega_{m(p)}\epsilon_{(p)rs}v_{(p),(k)} \end{cases} \tag{38}$$

$$= \begin{cases} \frac{1}{2}\sum_{(p)}\sum_{(k)}\epsilon_{(p)n(k)}\epsilon_{(p)rs}\omega_{(p)n}v_{(p),(k)} \\ +\frac{1}{2}\sum_{(p)}\sum_{(k)}\epsilon_{(p)m(k)}\epsilon_{(p)rs}\omega_{(p)m}v_{(p),(k)} \end{cases} \tag{39}$$

$$= \begin{cases} \frac{1}{2}\sum_{(p)}\sum_{(k)}\left(\delta_{nr}\delta_{(k)s}-\delta_{ns}\delta_{(k)r}\right)\omega_{(p)n}v_{(p),(k)} \\ +\frac{1}{2}\sum_{(p)}\sum_{(k)}\left(\delta_{mr}\delta_{(k)s}-\delta_{(k)r}\delta_{ms}\right)\omega_{(p)m}v_{(p),(k)} \end{cases} \tag{40}$$

$$= \sum_{(p)}\omega_{(p)r}v_{(p),s} - \sum_{(p)}\omega_{(p)s}v_{(p),r} \tag{41}$$

(41) can now again be written as $\omega_{pr}v_{p,s} - \omega_{ps}v_{p,r}$ (this is possible because, although $p$ only spans two dimensions, the skew-symmetricity of $\omega_{ps}$ permits us to expand $(p)$ to the three dimensions.) Hence, (33) becomes

$$\frac{d\omega_{rs}}{dt} - (\omega_{pr}v_{p,s} - \omega_{ps}v_{p,r}) = 0$$

which is the expected expression.

◊

In curvilinear coordinates the expression becomes

$$\frac{\delta\omega_{rs}}{\delta t} = \omega_{pr}v_{p|s} - \omega_{ps}v_{p|r}$$

◊

Suppose that initially at a certain point $\omega_{rs} = 0$, then obviously $\frac{\delta\omega_{rs}}{\delta t} = \omega_{pr}v_{p|s} - \omega_{ps}v_{p|r} = 0$ and thus $\omega_{rs} = C$, a constant, which must be zero as the initial condiation was that $\omega_{rs}(t = 0) = 0$.

◊

♦

## 5.33 p236 - Exercise 6

By eliminating the three components of displacement $u_r$ from the six equations **6.210**, obtain the following Cartesian equations of compatibility

$$e_{rs,mn} + e_{mn,rs} - e_{rm,sn} - e_{sn,rm} = 0$$

Show that there are only six independent equations here. Write the equations of compatibility in general tensor form.

We have **6.210**

$$e_{rs} = \frac{1}{2}\left(u_{r,s} + u_{s,r}\right) \tag{1}$$

and thus

$$\begin{array}{ll}(a) & e_{rs,mn} = \frac{1}{2}\left(u_{r,smn} + u_{s,rmn}\right) \\ (b) & e_{mn,rs} = \frac{1}{2}\left(u_{m,nrs} + u_{n,mrs}\right) \\ (c) & e_{rm,sn} = \frac{1}{2}\left(u_{r,msn} + u_{m,rsn}\right) \\ (d) & e_{sn,rm} = \frac{1}{2}\left(u_{s,nrm} + u_{n,srm}\right)\end{array} \tag{2}$$

giving $e_{rs,mn} + e_{mn,rs} - e_{rm,sn} - e_{sn,rm} = (a) + (b) - (c) - (d) = 0$

◊

Consider the set $(rs)$. Considering the symmetries induced by the symmetry of $e_{rs}$, the number of possible combinations is one of type "repeated combination", i.e. $\begin{pmatrix} n+m-1 \\ m \end{pmatrix} = \frac{4!}{2!} = 6$. The same is true for the set $(mn)$, so the total of possible combinations is 36. But we notice that half of the combinations make the equation indiscernible of the other half; indeed when $(mn) = (rs)$, $e_{rs,mn} + e_{mn,rs} - e_{rm,sn} - e_{sn,rm} = e_{mn,rs} + e_{rs,mn} - e_{mr,ns} - e_{ns,mr} = 0$. This reduces the number of combinations to $\frac{36}{2} = 18$.
We notice that this is not the total possible number of independent equations yet, as some combinations make the identity trivial. Indeed, consider the case $m = s$. In this case the equation $e_{rs,(s)n} + e_{(s)n,rs} - e_{r(s),sn} - e_{sn,r(s)} = 0$ is trivial. The same is true for $n = r$, but also for $m = r$ and $n = s$. Indeed changing $(rs)$ to $(sr)$ (which is allowed as $e_{rs}$ is symmetric) in $e_{rs,mn} + e_{mn,rs} - e_{rm,sn} - e_{sn,rm} = 0$ we get $e_{rs,mn} + e_{mn,rs} - e_{sm,rn} - e_{rn,sm} = 0$. This expression becomes also trivial when $n = s$ or $m = r$. So the number of independent equations reduces by the number of equations for which yield $m = r \vee m = s \vee n = r \vee n = s$. This number is $4 \times 3 = 12$ (4 independent events with 3 possible outcome, each).

**So the total of independent equations is $18 - 12 = 6$.**

$\Diamond$

In general coordinate system we get,

$$e_{rs|mn} + e_{mn|rs} - e_{rm|sn} - e_{sn|rm} = 0$$

$\blacklozenge$

## 5.34 p236 - Exercise 7

For a rectangular Cartesian coordinates $z_r$, a state of simple tension is represented as $E_{11} = C$ (a constant), all the other components of stress being zero. Find all six covariant components of stress for spherical polar coordinates.

We have **6.223** : $T_r = E_{rs} n_s$ so in rectangular Cartesian coordinates we have

$$T_1 = C n_1 \quad T_2 = 0 \quad T_3 = 0 \tag{1}$$

In general we have when going from the rectangular Cartesian coordinates to a curvilinear coordinate system

$$T_r^{'} = T_1 {x^1}_{,'r} \tag{2}$$
$$T_r^{'} = E_{rs}^{'} n_s^{'} \tag{3}$$
$$n^{'s} = n^i {x^{'s}}_{,i} \tag{4}$$
$$n^1 = n^{'s} {x^1}_{,'s} \tag{5}$$

Putting this all together we can state

$$E_{rs}^{'} n^{'s} = C n_1 {x^1}_{,'r} \tag{6}$$
$$E_{rs}^{'} n^{'s} = C n^{'s} {x^1}_{,'s} {x^1}_{,'r} \tag{7}$$

From which we deduce

$$E_{uv}^{'} = C {x^1}_{,'v} {x^1}_{,'u} \tag{8}$$

For spherical polar coordinates we have

$$\begin{cases} x^1 = r \sin\theta \cos\phi \\ x^1_{,r} = \sin\theta \cos\phi \\ x^1_{,\theta} = r \cos\theta \cos\phi \\ x^1_{,\phi} = -r \sin\theta \sin\phi \end{cases} \tag{9}$$

giving for (8)

$$\begin{cases} E_{rr} = C \sin^2\theta \cos^2\phi \\ E_{\theta\theta} = C r^2 \cos^2\theta \cos^2\phi \\ E_{\phi\phi} = C r^2 \sin^2\theta \sin^2\phi \\ E_{r\phi} = C r \sin\theta \cos\theta \cos^2\phi \\ E_{r\phi} = C \sin^2\theta \cos\phi \sin\phi \\ E_{\theta\phi} = -C r^2 \cos\theta \sin\theta \cos\phi \sin\phi \end{cases} \tag{10}$$

♦

## 5.35 p236 - Exercise 8 †

By substitution from **6.247** in the Cartesian equations of compatibility in Exercise 6, deduce that in a homogeneous isotropic body in equilibrium under body forces $X_r$, the invariant $\Theta = E_{nn}$ satisfies the following partial differential equation:

$$(1-\sigma)\,\Theta_{,rr} = (1+\sigma)\,\rho X_{r,r}$$

Let's recap what we know

$$\begin{array}{lll} (a) & e_{rs,mn} + e_{mn,rs} - e_{rm,sn} - e_{sn,rm} = 0 & \text{(Exercise 6)} \\ (b) & e_{rs} = \frac{1}{E}\left[(1+\sigma)\,E_{rs} - \sigma\delta_{rs}E_{nn}\right] & (6.247) \\ (c) & E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \sigma = \frac{\lambda}{2(\lambda+\mu)} & (6.248) \\ (d) & \lambda = \frac{\sigma E}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)} & (6.249) \\ (e) & \rho f_r = \rho X_r + (\lambda+\mu)\,\theta_{,r} + \mu\Delta u_r & (6.250) \\ (f) & \theta = e_{nn} & (6.246) \\ (g) & e_{rs} = \frac{1}{2}\left(u_{r,s} + u_{s,r}\right) & (6.210) \end{array} \tag{1}$$

As we are in steady state we have $f_r = \frac{\partial^2 u_r}{\partial t^2} = 0$ and $(e)$ becomes

$$\rho X_r + (\lambda+\mu)\,\theta_{,r} + \mu\Delta u_r = 0 \tag{2}$$

$$\Rightarrow \quad \rho X_{r,r} + (\lambda+\mu)\,\theta_{,rr} + \mu\left(\Delta u_r\right)_{,r} = 0 \tag{3}$$

Put $m = n$ and $s = r$ in $(a)$:

$$\underbrace{e_{rr,nn}}_{=\theta_{,nn}} + \underbrace{e_{nn,rr}}_{=\theta_{,nn}} - e_{rn,rn} - e_{rn,rn} = 0 \tag{4}$$

$$\text{or} \quad \theta_{,nn} = e_{nr,nr} \tag{5}$$

Putting $(b)$ in $(5)$ we get

$$\theta_{,rr} = \frac{1}{E}\left[(1+\sigma)\,E_{nr,nr} - \sigma\delta_{nr}E_{kk,nr}\right] \tag{6}$$

$$= \frac{1}{E}\left[(1+\sigma)\,E_{nr,nr} - \sigma E_{kk,rr}\right] \tag{7}$$

$$= \frac{1}{E}\,(1+\sigma)\,E_{nr,nr} - \frac{\sigma}{E}E_{kk,rr} \tag{8}$$

Put $\Theta = E_{kk}$, then

$$\theta_{,rr} = \frac{1}{E}(1+\sigma)E_{nr,nr} - \frac{\sigma}{E}\Theta_{,rr} \tag{9}$$

Consider now $\Delta u_r = u_{r,kk}$, using $e_{rs} = \frac{1}{2}(u_{r,s} + u_{s,r})$ we have

$$\begin{aligned} e_{rs,rs} &= \frac{1}{2}(u_{r,srs} + u_{s,rrs}) && (10)\\ &= u_{r,ssr} && (11) \end{aligned}$$

but $u_{r,ss} = \Delta u_r$, then

$$(\Delta u_r)_{,r} = e_{rs,rs} \tag{12}$$

So, by (5),

$$(\Delta u_r)_{,r} = \theta_{,rr} \tag{13}$$

Let's recap what we already know. We have (3) and (13)

$$\begin{cases} (3) \quad \rho X_{r,r} + (\lambda+\mu)\theta_{,rr} + \mu\Delta(u_r)_{,r} = 0 \\ \\ (13) \quad (\Delta u_r)_{,r} = \theta_{,rr} \end{cases} \tag{14}$$

and also

$$(1+\sigma)E_{nr,nr} = (1-\sigma)\Theta_{,rr} \tag{15}$$

indeed $(a)$ gives

$$\begin{aligned} & e_{rn,rn} = e_{rr,ss} && (16)\\ \Rightarrow\quad & \frac{1}{E}[(1+\sigma)E_{rn,rn} - \sigma\delta_{rn}E_{kk,rn}] = \frac{1}{E}[(1+\sigma)E_{rr,ss} - \sigma\delta_{rr}E_{kk,ss}] && (17)\\ \Leftrightarrow\quad & (1+\sigma)E_{rn,rn} - \sigma\Theta_{,rr} = (1+\sigma)\Theta_{,rr} - \sigma 3\Theta_{,rr} && (18)\\ \Leftrightarrow\quad & (1+\sigma)E_{rn,rn} = (1-\sigma)\Theta_{,rr} && (19) \end{aligned}$$

So, (9) becomes

$$\begin{aligned} \theta_{,rr} &= \frac{1}{E}(1+\sigma)E_{rn,rn} - \frac{\sigma}{E}\Theta_{,rr} && (20)\\ &= \frac{1-2\sigma}{E}\Theta_{,rr} && (21) \end{aligned}$$

and (14) becomes

$$(\Delta u_r)_{,r} = \frac{1-2\sigma}{E}\Theta_{,rr} \tag{22}$$

Let's put $\xi = (\lambda + \mu)\,\theta_{,rr} + \mu\,(\Delta u_r)_{,r}$ with (22) and (23) we get

$$\xi = \left[(\lambda + \mu)\,\frac{1-2\sigma}{E} + \mu\frac{1-2\sigma}{E}\right]\Theta_{,rr} \tag{23}$$

$$= \frac{1-2\sigma}{E}\,(\lambda + 2\mu)\,\Theta_{,rr} \tag{24}$$

Using $(d)$ in this expression gives

$$\xi = \frac{1-2\sigma}{E}E\left(\frac{\sigma}{(1+\sigma)\,(1-2\sigma)} + \frac{1}{(1+\sigma)}\right)\Theta_{,rr} \tag{25}$$

$$= (1-2\sigma)\,\frac{\sigma + (1-2\sigma)}{(1+\sigma)\,(1-2\sigma)}\Theta_{,rr} \tag{26}$$

$$= \frac{1-\sigma}{1+\sigma}\Theta_{,rr} \tag{27}$$

giving for (3)

$$(1+\sigma)\,\rho X_{r,r} \underbrace{+}_{\text{? should be }-} (1-\sigma)\,\Theta_{,rr} = 0$$

♦

## 5.36 p236 - Exercise 9

In a state of plane stress we have $E_{\alpha 3} = 0, E_{33} = 0$, the coordinates being Cartesian, and Greek suffixes taking the values 1, 2. prove that the equations of equilibrium under no body forces are satisfied if we put

$$E_{\alpha\beta} = \epsilon_{\alpha\rho}\epsilon_{\beta\sigma}\psi_{,\rho\sigma}$$

where $\psi$ is an arbitrary function. Show that this gives

$$E_{11} = \psi_{,22}, \quad E_{12} = -\psi_{,12}, \quad E_{22} = \psi_{,11},$$

Given that we are in equilibrium and no body forces acting we have from $\rho f_r = \rho X_r + E_{rs,s}$ w have

$$E_{rs,s} = 0 \tag{1}$$

Explicitly

$$E_{rs,s} = E_{r1,1} + E_{r2,2} + \underbrace{E_{r3,3}}_{=0} \tag{2}$$

and for $r = 3$

$$E_{3s,s} = \underbrace{E_{31,1}}_{=0} + \underbrace{E_{32,2}}_{=0} + \underbrace{E_{33,3}}_{=0} \tag{3}$$

So (1) can be expressed as

$$E_{\alpha\beta,\beta} = 0 \tag{4}$$

Let's test the expression $E_{\alpha\beta} = \epsilon_{\alpha\rho}\epsilon_{\beta\sigma}\psi_{,\rho\sigma}$

$$E_{\alpha\beta,\beta} = \epsilon_{\alpha\rho}\epsilon_{\beta\sigma}\psi_{,\rho\sigma\beta} \tag{5}$$
$$= 0 \tag{6}$$

The last identity is due to the fact that for a fixed $\alpha$, or $\rho$ will take the value of $\alpha$ (making $\epsilon_{\alpha\rho} = 0$, or for $\rho \neq \alpha$, the only non-zero terms imply that $\beta \neq \sigma$ where the expansion is of the form $\epsilon_{\alpha\rho}\epsilon_{\alpha\rho}\psi_{,\rho\sigma\beta} + \epsilon_{\alpha\rho}\epsilon_{\rho\alpha}\psi_{,\rho\sigma\beta}$ (no summation over dummy indexes) and as the order of partial differentiation is of no importance and $\epsilon_{\rho\alpha}$ and $\epsilon_{\alpha\rho}$ are of opposite sign, the two terms will cancel each other.
So indeed, $E_{\alpha\beta} = \epsilon_{\alpha\rho}\epsilon_{\beta\sigma}\psi_{,\rho\sigma}$ satisfies the differential equation $E_{rs,s} = 0$.

◊

Let's calculate explicitly the $E_{\alpha\beta}$.

$\epsilon_{\alpha\rho} \neq 0$ implies $\alpha \neq \rho$ and $\epsilon_{\beta\sigma} \neq 0$ implies $\beta \neq \sigma$, hence

$$\begin{aligned}
E_{11} &= \epsilon_{12}\epsilon_{12}\psi_{,22} \\
&= (1)(1)\psi_{,22} \\
&= \psi_{,22} \\
\\
E_{22} &= \epsilon_{21}\epsilon_{21}\psi_{,11} \\
&= (-1)(-1)\psi_{,11} \\
&= \psi_{,11} \\
\\
E_{12} &= \epsilon_{12}\epsilon_{21}\psi_{,21} \\
&= (1)(-1)\psi_{,21} \\
&= -\psi_{,21}
\end{aligned} \tag{7}$$

♦

## 5.37 p236 - Exercise 10

An isotropic elastic body is in equilibrium under no body forces. show that, for rectangular Cartesian coordinates, the displacement satisfies the partial differential equations

$$(1-2\sigma)\,\Delta u_r + \theta_{,r} = 0$$

Deduce that $\theta$ is a harmonic function.
Show that the above equations are satisfied if we put

$$u_r = \psi_r - \frac{1}{4\,(1-\sigma)}\,(z_s\psi_s + \phi)_{,r}$$

provided $\Delta\phi = 0$, $\Delta\psi_r = 0$. (Papcovich-Neuber)

As we are in equilibrium (no acceleration) and no body forces, equation

$$\textbf{6.250} \quad \rho f_r = \rho X_r + (\lambda+\mu)\,\theta_{,r} + \mu\Delta u_r$$

becomes

$$(\lambda+\mu)\,\theta_{,r} + \mu\Delta u_r = 0 \tag{1}$$

with

$$\lambda = \frac{\sigma E}{(1+\sigma)\,(1-2\sigma)}, \quad \mu = \frac{E}{2\,(1+\sigma)} \tag{2}$$

so (1) becomes

$$\frac{E}{(1+\sigma)}\left(\frac{\sigma}{1-2\sigma} + \frac{1}{2}\right)\theta_{,r} + \frac{E}{2\,(1+\sigma)}\Delta u_r = 0 \tag{3}$$

$$\Leftrightarrow \quad \left(\frac{2\sigma+1-2\sigma}{2\,(1-2\sigma)}\right)\theta_{,r} + \frac{1}{2}\Delta u_r = 0 \tag{4}$$

$$\Leftrightarrow \quad \theta_{,r} + (1-2\sigma)\,\Delta u_r = 0 \tag{5}$$

$$\Diamond$$

We show now that $\theta$ is an harmonic function .

$$(5)_{,r} \quad \Rightarrow \quad \theta_{,rr} + (1-2\sigma)\,(\Delta u_r)_{,r} = 0 \tag{6}$$

$$\Rightarrow \quad \Delta\theta + (1-2\sigma)\,(\Delta u_r)_{,r} = 0 \tag{7}$$

We have

$$\Delta u_r = u_{r,ss} \tag{8}$$

$$\Rightarrow \quad (\Delta u_r)_{,r} = u_{r,rss} \tag{9}$$

and from exercise 8 (11), (12), (13) we know

$$(\Delta u_r)_{,r} = \theta_{rr} = \Delta\theta \tag{10}$$

and (7) becomes

$$\Delta\theta + (1-2\sigma)\,\Delta\theta = 0 \tag{11}$$
$$\Rightarrow \quad \Delta\theta = 0 \tag{12}$$

$\theta$ is an harmonic function.

$\Diamond$

$$u_r = \psi_r - \frac{1}{4(1-\sigma)}\,(z_s\psi_s+\phi)_{,r} \tag{13}$$
$$\Rightarrow \quad \Delta u_r = \underbrace{\Delta\psi_r}_{=0} - \frac{1}{4(1-\sigma)}\Delta\,(z_s\psi_s+\phi)_{,r} \tag{14}$$
$$= -\frac{1}{4(1-\sigma)}\,(z_s\psi_s+\phi)_{,rkk} \tag{15}$$
$$= -\frac{1}{4(1-\sigma)}\left[(z_{s,r}\psi_s + z_s\psi_{s,r})_{,kk} + \underbrace{\phi_{,rkk}}_{(\Delta\phi)_r=0}\right] \tag{16}$$
$$= -\frac{1}{4(1-\sigma)}\,(\delta_{sr}\psi_s + z_s\psi_{s,r})_{,kk} \tag{17}$$
$$= -\frac{1}{4(1-\sigma)}\,(\psi_r + z_s\psi_{s,r})_{,kk} \tag{18}$$
$$= -\frac{1}{4(1-\sigma)}\left(\underbrace{\psi_{r,kk}}_{=0} + \left(\underbrace{z_{s,k}}_{=\delta_{sk}}\psi_{s,r} + z_s\psi_{s,rk}\right)_{,k}\right) \tag{19}$$
$$= -\frac{1}{4(1-\sigma)}\left(\psi_{k,rk} + z_{s,k}\psi_{s,rk} + z_s\underbrace{\psi_{s,rkk}}_{=0}\right) \tag{20}$$
$$= -\frac{1}{4(1-\sigma)}\,(\psi_{k,rk} + \psi_{k,rk}) \tag{21}$$
$$\Rightarrow \quad \Delta u_r = -\frac{1}{2(1-\sigma)}\psi_{k,kr} \tag{22}$$

Let's look at $\theta = e_{nn}$. As $e_{rs} = \frac{1}{2}\left(u_{r,s} + u_{s,r}\right)$

$$\begin{aligned}
\theta &= u_{k,k} && (23)\\
&= \left(\psi_k - \frac{1}{4\left(1-\sigma\right)}\left(z_s\psi_s + \phi\right)_{,k}\right)_{,k} && (24)\\
&= \psi_{k,k} - \frac{1}{4\left(1-\sigma\right)}\left(\left(z_s\psi_s + \phi\right)_{,k}\right)_{,k} && (25)\\
&= \psi_{k,k} - \frac{1}{4\left(1-\sigma\right)}\left(\left(z_{s,k}\psi_s + z_s\psi_{s,k}\right)_k + \underbrace{\phi_{,kk}}_{=0}\right) && (26)\\
&= \psi_{k,k} - \frac{1}{4\left(1-\sigma\right)}\left(\psi_{k,k} + z_{s,k}\psi_{s,k} + z_s\psi_{s,kk}\right) && (27)\\
&= \psi_{k,k} - \frac{1}{4\left(1-\sigma\right)}\left(\psi_{k,k} + \psi_{k,k} + z_s\underbrace{\psi_{s,kk}}_{=0}\right) && (28)\\
&= \psi_{k,k} - \frac{1}{2\left(1-\sigma\right)}\psi_{k,k} && (29)\\
\Rightarrow\quad \theta &= \frac{1-2\sigma}{2\left(1-\sigma\right)}\psi_{k,k} && (30)\\
\Rightarrow\quad \theta_{,r} &= \frac{1-2\sigma}{2\left(1-\sigma\right)}\psi_{k,kr} && (31)
\end{aligned}$$

Be $\Gamma = \left(1-2\sigma\right)\Delta u_r + \theta_{,r}$ and let's plug into it (22) and (31)

$$\begin{aligned}
\Gamma &= \left(1-2\sigma\right)\Delta u_r + \theta_{,r} && (32)\\
&= -\left(1-2\sigma\right)\frac{1}{2\left(1-\sigma\right)}\psi_{k,kr} + \frac{1-2\sigma}{2\left(1-\sigma\right)}\psi_{k,kr} && (33)\\
&= 0 && (34)
\end{aligned}$$

and indeed

$$u_r = \psi_r - \frac{1}{4\left(1-\sigma\right)}\left(z_s\psi_s + \phi\right)_{,r}$$

is a solution of

$$\left(1-2\sigma\right)\Delta u_r + \theta_{,r} = 0$$

♦

## 5.38 p237 - Exercise 11

If, for rectangular Cartesian coordinates $z_r$, $\chi_{rs}$ is any symmetric tensor, show that the tensor $E_{mn}$ defined by

$$E_{mn} = \epsilon_{mpr}\epsilon_{nqs}\chi_{rs,pq}$$

is symmetric, and satisfies the equations $E_{mn,n} = 0$.
Show that if we choose $\chi_{rs} = z_r z_s$, then $E_{mn} = -2\delta_{mn}$.

$$\begin{aligned}
E_{nm} &= \epsilon_{npr}\epsilon_{mqs}\chi_{rs,pq} && (1)\\
&= \epsilon_{npr}\epsilon_{mqs}\chi_{sr,qp} && (2)\\
&= \epsilon_{nqs}\epsilon_{mpr}\chi_{rs,pq} && (3)\\
&= E_{mn} && (4)
\end{aligned}$$

♢

Consider

$$E_{mn,n} = \epsilon_{mpr}\epsilon_{nqs}\chi_{rs,pqn}$$

As $\epsilon_{nqs}$ is antisymmetric in $(n, q)$ and $\chi_{rs,pqn}$ is symmetric in these indices, $E_{mn,n}$ reduces to a sum of terms of the type $\epsilon_{mpr}\epsilon_{(nq)s}\chi_{rs,p(qn)} - \epsilon_{mpr}\epsilon_{(nq)s}\chi_{rs,p(nq)}$.
So, $E_{mn,n} = 0$

♢

Be $\chi_{rs} = z_r z_s$, then

$$\begin{aligned}
E_{mn} &= \epsilon_{mpr}\epsilon_{nqs}(z_r z_s)_{,pq} && (5)\\
&= \epsilon_{mpr}\epsilon_{nqs}(\delta_{pr} z_s + \delta_{sp} z_r)_{,q} && (6)\\
&= \epsilon_{mpr}\epsilon_{nqs}(\delta_{pr}\delta_{sq} + \delta_{sp}\delta_{rq}) && (7)\\
&= \underbrace{\epsilon_{mpp}\epsilon_{nqq}}_{=0} + \epsilon_{mpq}\epsilon_{nqp} && (8)\\
&= \epsilon_{pmq}\epsilon_{pqn} && (9)\\
&= \delta_{mq}\delta_{qn} - \delta_{mn}\underbrace{\delta_{qq}}_{=3} && (10)\\
&= \delta_{mn} - 3\delta_{mn} && (11)\\
&= -2\delta_{mn} && (12)\\
& && (13)
\end{aligned}$$

♦

## 5.39 p237 - Exercise 12

The determinantal equation $|\lambda\delta_{mn} - E_{mn}| = 0$ is important in elasticity because it gives the three principal stresses at a point. Show that if we introduce the three Cartesian invariants

$$A = E_{mn}, \quad B = E_{mm}E_{mn}, \quad C = E_{mn}E_{np}E_{pm}$$

the cubic equation ma be written in the form

$$\lambda^3 - A\lambda^2 - \frac{1}{2}\left(A^2 - B\right)\lambda - \left(\frac{1}{6}A^3 - \frac{1}{2}AB + \frac{1}{3}C\right) = 0$$

[Hint: Note the Cartesian invariance of this expression, and use coordinates which make $E_{rs} = 0$ for $r \neq s$.]

Be a matrix $M_{pq}$. It's determinant can be expressed as $|M_{pq}| = \epsilon_{mnr}M_{1m}M_{2n}M_{3r}$ or - see (**4.316**)- $\epsilon_{stu}|M_{pq}| = \epsilon_{mnr}M_{sm}M_{tn}M_{ur}$ or $\epsilon_{stu}\epsilon_{stu}|M_{pq}| = 6|M_{pq}| = \epsilon_{stu}\epsilon_{mnr}M_{sm}M_{tn}M_{ur}$.

Put

$$M_{pq} = \lambda\delta_{mn} - E_{mn}$$

and get

$$6|M_{pq}| = \epsilon_{stu}\epsilon_{mnr}\left(\lambda\delta_{sm} - E_{sm}\right)\left(\lambda\delta_{tn} - E_{tn}\right)\left(\lambda\delta_{ur} - E_{ur}\right) \tag{1}$$

$$= \begin{cases} \epsilon_{stu}\epsilon_{mnr}\,\{ \\ \lambda^3\delta_{sm}\delta_{tn}\delta_{ur} \\ -\lambda^2\delta_{sm}\delta_{ur}E_{tn} - \lambda^2\delta_{tn}\delta_{ur}E_{sm} - \lambda^2\delta_{sm}\delta_{tn}E_{ur} \\ +\lambda\delta_{ur}E_{sm}E_{tn} + \lambda\delta_{sm}E_{tn}E_{ur} + \lambda\delta_{tn}E_{sm}E_{ur} \\ -E_{sm}E_{tn}E_{ur} \\ \} \end{cases} \tag{2}$$

$$= \begin{cases} \lambda^3\underbrace{\epsilon_{stu}\epsilon_{stu}}_{=6} \\ -\lambda^2\underbrace{\epsilon_{stu}\epsilon_{snu}}_{=2\delta_{tn}}E_{tn} - \lambda^2\underbrace{\epsilon_{stu}\epsilon_{mtu}}_{=2\delta_{sm}}E_{sm} - \lambda^2\underbrace{\epsilon_{stu}\epsilon_{str}}_{=2\delta_{ur}}E_{ur} \\ +\lambda\underbrace{\epsilon_{stu}\epsilon_{mnu}}_{=\delta_{sm}\delta_{tn}-\delta_{sn}\delta_{tm}}E_{sm}E_{tn} + \lambda\underbrace{\epsilon_{stu}\epsilon_{snr}}_{=\delta_{tn}\delta_{ur}-\delta_{tr}\delta_{un}}E_{tn}E_{ur} + \lambda\underbrace{\epsilon_{stu}\epsilon_{mtr}}_{=\delta_{sm}\delta_{ur}-\delta_{sm}\delta_{ur}}E_{sm}E_{ur} \\ -\epsilon_{stu}\epsilon_{mnr}E_{sm}E_{tn}E_{ur} \end{cases} \tag{3}$$

working out gives

$$6|M_{pq}| = 6\lambda^3 - 6A\lambda^2 + \left(3A^2 - 3B\right)\lambda - \epsilon_{stu}\epsilon_{mnr}E_{sm}E_{tn}E_{ur} \tag{4}$$

but the term $\epsilon_{stu}\epsilon_{mnr}E_{sm}E_{tn}E_{ur} = 6|E_{pq}|$ giving

$$|M_{pq}| = \lambda^3 - A\lambda^2 + \frac{1}{2}\left(A^2 - B\right)\lambda - |E_{pq}| \tag{5}$$

Let's consider now the invariants

$$A = E_{mn}, \quad B = E_{mm}E_{mn}, \quad C = E_{mn}E_{np}E_{pm}$$

As $(E_{pq})$ is symmetric with elements $\in \mathbb{R}$, we know that there is an orthogonal transformations so that $(E_{pq})$ becomes diagonal:

$$(E_{pq}) = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \tag{6}$$

The invariants $A,\ B,\ C$ become

$$\begin{cases} A = E_1 + E_2 + E_3 \\ B = E_1^2 + E_2^2 + E_3^2 \\ A = E_1^3 + E_2^3 + E_3^3 \\ |E_{pq}| = E_1E_2E_3 \end{cases} \tag{7}$$

We will try to express $|E_{pq}|$ as an expression of $A,\ B,\ C$. Obviously as $|E_{pq}| = E_1E_2E_3$ we need at least $A^3$

$$A^3 = \begin{cases} E_1^3 + E_2^3 + E_3^3 \\ +3E_1E_2^2 + 3E_1E_3^2 + 3E_1^2E_2 + 3E_1^2E_3 + 3E_2^2E_3 + 3E_3^2E_2 \\ +6E_1E_2E_3 \end{cases} \tag{8}$$

$$= C + 6\,|E_{pq}| + 3\left(E_1E_2^2 + E_1E_3^2 + E_1^2E_2 + E_1^2E_3 + E_2^2E_3 + E_3^2E_2\right) \tag{9}$$

and $AB$

$$AB = (E_1 + E_2 + E_3)\left(E_1^2 + E_2^2 + E_3^2\right) \tag{10}$$

$$= E_1^3 + E_2^3 + E_3^3 + E_1E_2^2 + E_1E_3^2 + E_1^2E_2 + E_1^2E_3 + E_2^2E_3 + E_3^2E_2 \tag{11}$$

$$= C + E_1E_2^2 + E_1E_3^2 + E_1^2E_2 + E_1^2E_3 + E_2^2E_3 + E_3^2E_2 \tag{12}$$

hence

$$A^3 - 3AB = C + 6\,|E_{pq}| - 3C \tag{13}$$

$$= 6\,|E_{pq}| - 2C \tag{14}$$

$$\Rightarrow \quad 6\,|E_{pq}| = A^3 - 3AB + 2C \tag{15}$$

and (5) becomes

$$|M_{pq}| = \lambda^3 - A\lambda^2 + \frac{1}{2}\left(A^2 - B\right)\lambda - \left(\frac{1}{6}A^3 - \frac{1}{2}AB + \frac{1}{3}C\right) \tag{16}$$

Putting $|M_{pq}| = 0$ gives the required equation.

♦

## 5.40 p237 - Exercise 13

A plane electromagnetic wave in complex form is given,

$$E_\alpha = A_\alpha e^{iS}, \quad E_3 = 0, \quad H_\alpha = -\epsilon_{\alpha\beta} A_\beta e^{iS}, \quad H_3 = 0$$

$$S = \frac{2\pi}{\lambda}\left(z_3 - ct\right)$$

where $A_\alpha$ is a constant complex vector, and Greek suffixes take the values $1, 2$. Verify that Maxwell's equations are satisfied, and that the wave is propagated in the positive $z_3-$direction.
The wave meets a perfectly conducting wall $z_3 = 0$,and is reflected. Given the condition on such wall is that the tangential component of the electric vector for the total field vanishes, show that the reflected wave is given by

$$E_\alpha^{'} = -A_\alpha e^{iS^{'}}, \quad E_3^{'} = 0, \quad H_\alpha^{'} = -\epsilon_{\alpha\beta} A_\beta e^{iS^{'}}, \quad H_3^{'} = 0,$$

$$S^{'} = -\frac{2\pi}{\lambda}\left(z_3 + ct\right)$$

Maxwell's equations read

$$\begin{cases} (6.301) \quad \frac{1}{c}\partial_t E_r = \epsilon_{rmn} H_{n,m} \quad \frac{1}{c}\partial_t H_r = -\epsilon_{rmn} E_{n,m} \\ \\ (6.302) \quad E_{n,n} = 0 \qquad\qquad\quad H_{n,n} = 0 \end{cases} \tag{1}$$

For (**6.302**) this gives for the considered case

$$E_{n,n} = E_{\alpha,\alpha} + \underbrace{E_{3,3}}_{=0} \tag{2}$$
$$= A_\alpha \left(e^{iS}\right)_{,\alpha} \tag{3}$$
$$= iA_\alpha e^{iS} \underbrace{S_{,\alpha}}_{=0} \tag{4}$$
$$= 0 \tag{5}$$

analogously

$$H_{n,n} = H_{\alpha,\alpha} + \underbrace{H_{3,3}}_{=0} \tag{6}$$
$$= -\epsilon_{\alpha\beta} A_\beta \left(e^{iS}\right)_{,\alpha} \tag{7}$$
$$= -i\epsilon_{\alpha\beta} A_\beta \underbrace{S_{,\alpha}}_{=0} \tag{8}$$
$$= 0 \tag{9}$$

So for **6.302** the considered wave meets Maxwell's equations.
And for **6.301**

$$\frac{1}{c}\partial_t E_\alpha = \epsilon_{\alpha mn} H_{n,m} \tag{10}$$

$$\epsilon_{\alpha mn} H_{n,m} = \epsilon_{\alpha\beta\gamma} H_{\gamma,\beta} + \epsilon_{\alpha 3\beta} H_{\beta,3} + \epsilon_{\alpha\beta 3} \underbrace{H_{3,\gamma}}_{=0} \tag{11}$$

$$= -i\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\delta} A_\delta e^{iS} \underbrace{S_{,\beta}}_{=0} - i\epsilon_{\alpha 3\beta}\epsilon_{\beta\gamma} A_\gamma e^{iS} \underbrace{S_{,3}}_{=\frac{2\pi}{\lambda}} \tag{12}$$

$$= -i\frac{2\pi}{\lambda}\epsilon_{\alpha 3\beta}\epsilon_{\beta\gamma} A_\gamma e^{iS} \tag{13}$$

$$= i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta 3}\epsilon_{\beta\gamma} A_\gamma e^{iS} \tag{14}$$

Let's look at (14) and more specifically at the factor $\epsilon_{\alpha\beta 3}$. As the third index is fixed we can replace $\epsilon_{\alpha\beta 3}$ by $\epsilon_{\alpha\beta}$ and get

$$\epsilon_{\alpha mn} H_{n,m} = i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta}\epsilon_{\beta\gamma} A_\gamma e^{iS} \tag{15}$$

$$= -i\frac{2\pi}{\lambda} A_\alpha e^{iS} \tag{16}$$

and

$$\frac{1}{c}\partial_t E_\alpha = \frac{1}{c} i A_\alpha e^{iS}\left(-c\frac{2\pi}{\lambda}\right) \tag{17}$$

$$= -i\frac{2\pi}{\lambda} A_\alpha e^{iS} \tag{18}$$

So for **6.301(a)** the considered wave meets Maxwell's equations.
For the second equation:

$$\frac{1}{c}\partial_t H_\alpha = -\epsilon_{\alpha mn} E_{n,m} \tag{19}$$

$$\epsilon_{\alpha mn} E_{n,m} = -\epsilon_{\alpha\beta\gamma} \underbrace{E_{\gamma,\beta}}_{=0} - \epsilon_{\alpha 3\beta} E_{\beta,3} - \epsilon_{\alpha\beta 3} \underbrace{E_{3,\gamma}}_{=0} \tag{20}$$

$$= -\epsilon_{\alpha 3\beta} E_{\beta,3} \tag{21}$$

$$= -i\frac{2\pi}{\lambda}\epsilon_{\alpha 3\beta} A_\beta e^{iS} \tag{22}$$

$$= i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta} A_\beta e^{iS} \tag{23}$$

$$\frac{1}{c}\partial_t H_\alpha = i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta} A_\beta e^{iS} \tag{24}$$

So for **6.301(b)** the considered wave meets Maxwell's equations and also for all the other equations in Maxwell's equations set.

$\Diamond$

As in the equation of the plane the only varying parameter is $S = \frac{2\pi}{\lambda}(z_3 - ct)$ (depending on space and time), the components of $E_r$, $H_r$ will remain constant only when $S =$ constant and as $S$ is only spatially dependent of $z_3$ the plane can only move in the $z_3-$direction. Indeed, at a certain configuration point $(z_1, z_2, z_3, t)$ the configuration is the same $\forall z_1, \quad z_2$ for a given pair $(z_3, t)$. At a time $t^{'}$ the value of $E_r, \quad H_r$ will be the same if $z_3 - ct = z_3^{'} - ct^{'}$, giving $z_3^{'} = z_3 + c(t^{'} - t)$: a plane with certain configuration $(z_1, z_2, z_3, t_0)$ will move along the $z_3$ axis with speed $c$

$\Diamond$

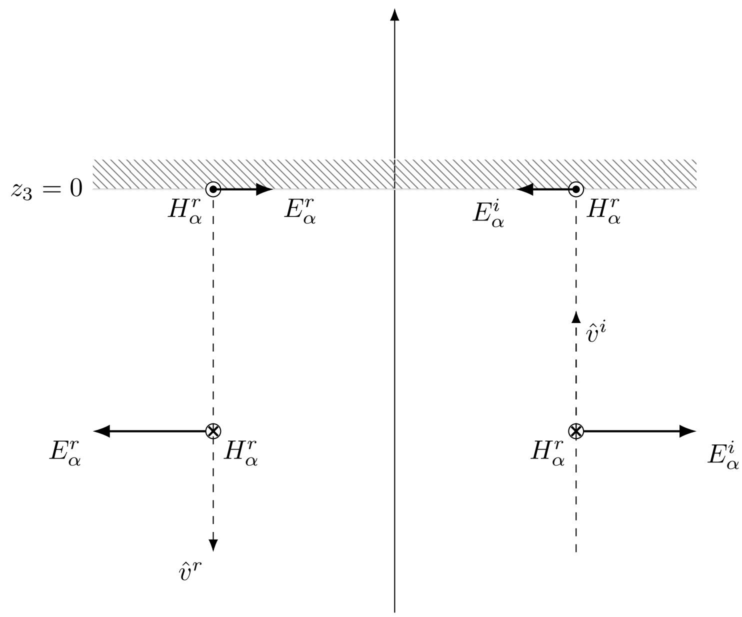



Figure 5.7: Reflection of a plain wave

At the wall we have $E^t_\alpha = E^i_\alpha + E^r_\alpha = 0$, with $E^t_\alpha$ the total field and $E^i_\alpha, \quad E^r_\alpha$ the incoming and reflected field. Without loss of generalization we can put at the wall $t = 0$ and $z_3 = 0$ so we get

$$E^r_\alpha = -A_\alpha$$

so the reflection gives an opposite sign to the electric field of the reflected plane wave.
We note also that the velocity of the wave is reversed, so in the definition of the incoming wave we replace $c$ by $-c$ we get

$$E^r_\alpha = -A_\alpha e^{i\frac{2\pi}{\lambda}(z_3+ct)} \tag{25}$$

It is easily checked, see above, that the wave moves along the $z_3$ axis in the negatives sense, which is what we want.
This expression could be a candidate to describe the reflected wave.
But we note that

$$E^r_\alpha = -A_\alpha e^{-i\frac{2\pi}{\lambda}(z_3+ct)} \tag{26}$$

also represents an electric field moving in the negative sense of $z_3$.

Let's further note that the triad $E^r_\alpha, H^r_\alpha, v^r$ (with $v^r$ the velocity vector), is an oriented triad. As the velocity vector and electric field of the incoming field both are reversed, we conclude that the magnetic vector will maintain it's direction (see figure).
So, let's put tentatively for the magnetic complex vector

$$H^r_\alpha = -\epsilon_{\alpha\beta} A_\beta e^{i\frac{2\pi}{\lambda}(z_3+ct)} \tag{27}$$

or

$$H^r_\alpha = -\epsilon_{\alpha\beta} A_\beta e^{-i\frac{2\pi}{\lambda}(z_3+ct)} \tag{28}$$

We test now these tentative equations with the Maxwell's equations.

**Electric field : (6.301a)**

**Case** $S = \frac{2\pi}{\lambda}(z_3 + ct_0)$

$$\begin{aligned}
\frac{1}{c}\partial_t E_\alpha &= \epsilon_{\alpha mn} H_{n,m} && (29)\\
\epsilon_{\alpha mn} H_{n,m} &= \epsilon_{\alpha\beta\gamma} H_{\gamma,\beta} + \epsilon_{\alpha 3\beta} H_{\beta,3} + \epsilon_{\alpha\beta 3} \underbrace{H_{3,\gamma}}_{=0} && (30)\\
&= -i\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\delta} A_\delta e^{iS} \underbrace{S_{,\beta}}_{=0} - i\epsilon_{\alpha 3\beta}\epsilon_{\beta\gamma} A_\gamma e^{iS} \underbrace{S_{,3}}_{=\frac{2\pi}{\lambda}} && (31)\\
&= -i\frac{2\pi}{\lambda} A_\alpha e^{iS} && (32)\\
\frac{1}{c}\partial_t E_\alpha &= -\frac{1}{c} i A_\alpha e^{iS}\left(c\frac{2\pi}{\lambda}\right) && (33)\\
&= -i\frac{2\pi}{\lambda} A_\alpha e^{iS} && (34)
\end{aligned}$$

**Case** $S = -\frac{2\pi}{\lambda}(z_3 + ct_0)$

$$\begin{aligned}
\frac{1}{c}\partial_t E_\alpha &= \epsilon_{\alpha mn} H_{n,m} && (35)\\
\epsilon_{\alpha mn} H_{n,m} &= \epsilon_{\alpha\beta\gamma} H_{\gamma,\beta} + \epsilon_{\alpha 3\beta} H_{\beta,3} + \epsilon_{\alpha\beta 3} \underbrace{H_{3,\gamma}}_{=0} && (36)\\
&= -i\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\delta} A_\delta e^{iS} \underbrace{S_{,\beta}}_{=0} - i\epsilon_{\alpha 3\beta}\epsilon_{\beta\gamma} A_\gamma e^{iS} \underbrace{S_{,3}}_{=-\frac{2\pi}{\lambda}} && (37)\\
&= i\frac{2\pi}{\lambda} A_\alpha e^{iS} && (38)\\
\frac{1}{c}\partial_t E_\alpha &= -\frac{1}{c} i A_\alpha e^{iS}\left(-c\frac{2\pi}{\lambda}\right) && (39)\\
&= i\frac{2\pi}{\lambda} A_\alpha e^{iS} && (40)
\end{aligned}$$

So, both solutions satisfies the Maxwell equation **6.301a**.

**Magnetic field : (6.301b)**

**Case** $S = \frac{2\pi}{\lambda}\left(z_3 + ct_0\right)$

$$\frac{1}{c}\partial_t H_\alpha = -\epsilon_{\alpha mn} E_{n,m} \tag{41}$$

$$\epsilon_{\alpha mn} E_{n,m} = -\epsilon_{\alpha\beta\gamma} \underbrace{E_{\gamma,\beta}}_{=0} - \epsilon_{\alpha 3\beta} E_{\beta,3} - \epsilon_{\alpha\beta 3} \underbrace{E_{3,\gamma}}_{=0} \tag{42}$$

$$= \epsilon_{\alpha\beta 3} E_{\beta,3} \tag{43}$$

$$= -i\frac{2\pi}{\lambda}\epsilon_{\alpha 3\beta} A_\beta e^{iS} \tag{44}$$

$$= i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta} A_\beta e^{iS} \tag{45}$$

$$\frac{1}{c}\partial_t H_\alpha = i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta}(-A_\beta) e^{iS} \tag{46}$$

$$= -i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta} A_\beta e^{iS} \tag{47}$$

**Case** $S = -\frac{2\pi}{\lambda}\left(z_3 + ct_0\right)$

$$\frac{1}{c}\partial_t H_\alpha = -\epsilon_{\alpha mn} E_{n,m} \tag{48}$$

$$\epsilon_{\alpha mn} E_{n,m} = -\epsilon_{\alpha\beta\gamma} \underbrace{E_{\gamma,\beta}}_{=0} - \epsilon_{\alpha 3\beta} E_{\beta,3} - \epsilon_{\alpha\beta 3} \underbrace{E_{3,\gamma}}_{=0} \tag{49}$$

$$= \epsilon_{\alpha\beta 3} E_{\beta,3} \tag{50}$$

$$= i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta} A_\beta e^{iS} \tag{51}$$

$$\frac{1}{c}\partial_t H_\alpha = -i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta}(-A_\beta) e^{iS} \tag{52}$$

$$= i\frac{2\pi}{\lambda}\epsilon_{\alpha\beta} A_\beta e^{iS} \tag{53}$$

We conclude that $S = \frac{2\pi}{\lambda}\left(z_3 + ct_0\right)$ does not fit with the Maxwell's equations while $S = -\frac{2\pi}{\lambda}\left(z_3 + ct_0\right)$ does.
For the $z_3$ components of the electric and magnetic field, these must remain zero as all the energy the plane components of the incoming electromagnetic wave is transferred to the plane components of the reflected pane wave (the magnitude of these components are the same for the incoming and the reflected plane wave).

♦

## 5.41 p238 - Exercise 14a

Taking for the Hertz vector the fundamental solution of the wave equation **6.342**

$$\Pi_r = B_r \frac{e^{-ik(ct-R)}}{R}, \quad R^2 = z_m z_m$$

where $B_r$ is a constant vector, show that, for $R$ much less than $\lambda = \frac{2\pi}{k}$, we have approximatively

$$E_r = -\frac{\partial}{\partial z^r}\left[B_m z_m \frac{e^{-ikct}}{R^3}\right],$$

$$H_r = -ik\epsilon_{rmn} B_m z_n \frac{e^{-ikct}}{R^3}.$$

Show also that for $R$ much greater than $\lambda = \frac{2\pi}{k}$ (wave zone), we have approximatively

$$E_r = k^2 \left(B_r R^2 - B_m z_m z_r\right) \frac{e^{-ik(ct-R)}}{R^3},$$

$$H_r = -k^2 \epsilon_{rmn} B_m z_n \frac{e^{-ik(ct-R)}}{R^2}.$$

(This is the electromagnetic field of the Hertzian dipole oscillation, which is the simplest model of a radio antenna).

We have **6.341**:

$$E_r = \Pi_{m,mr} - \frac{1}{c^2}\partial_t^2 \Pi_r \tag{1}$$

$$H_r = \frac{1}{c}\epsilon_{rpq}\partial_t \Pi_{q,p} \tag{2}$$

and **6.342**:

$$\frac{1}{c^2}\partial_t^2 \Pi_r - \Pi_{r,mm} = 0 \tag{3}$$

**Case $kR << 2\pi$:**

In that case

$$\Pi_r = B_r \frac{e^{-ik(ct-R)}}{R} \approx B_r \frac{e^{-ikct}}{R}$$

and let's express $\Pi_r$ as

$$\Pi_r = B_r F(t) G(z) \tag{4}$$

with

$$F(t) = e^{-ikct} \tag{5}$$

$$G(z) = \frac{1}{R} \tag{6}$$

So (1) can be expressed as

$$E_r = B_m F G_{,mr} - \frac{1}{c^2} B_r G F_{,tt} \tag{7}$$

Let's calculate some necessary derivatives

$$R_{,r} = \frac{z_r}{R} \tag{8}$$

$$F_{,t} = -ikcF \tag{9}$$

$$F_{,tt} = -k^2 c^2 F \tag{10}$$

$$G_{,r} = -\frac{z_r}{R^3} \tag{11}$$

$$\Pi_{m,m} = -B_m z_m \frac{F}{R^3} \tag{12}$$

So, (7) can be expressed as

$$E_r = -\left( B_m z_m \frac{F}{R^3} \right)_{,r} + B_r \frac{1}{R} k^2 F \tag{13}$$

Consider the expression $\frac{1}{c^2} B_m z_m G F_{,tt}$, then

$$\left( \frac{1}{c^2} B_m z_m G F_{,tt} \right)_{,r} = \frac{1}{c^2} B_r G F_{,tt} - \frac{1}{c^2} B_m F_{,tt} z_m z_r \frac{G}{R^2} (1 - ikR) \tag{14}$$

$$-\left( B_m z_m \frac{1}{R} k^2 F \right)_{,r} = -B_r \frac{1}{R} k^2 F + B_m k^2 F z_m z_r \frac{1}{R^3} \tag{15}$$

## 5.42 p238 - Exercise 14b

Taking for the Hertz vector the fundamental solution of the wave equation **6.342**

$$\Pi_r = B_r \frac{e^{-ik(ct-R)}}{R}, \quad R^2 = z_m z_m$$

where $B_r$ is a constant vector, show that, for $R$ much less than $\lambda = \frac{2\pi}{k}$, we have approximatively

$$E_r = -\frac{\partial}{\partial z^r}\left[B_m z_m \frac{e^{-ikct}}{R^3}\right],$$

$$H_r = -ik\epsilon_{rmn} B_m z_n \frac{e^{-ikct}}{R^3}.$$

Show also that for $R$ much greater than $\lambda = \frac{2\pi}{k}$ (wave zone), we have approximatively

$$E_r = k^2\left(B_r R^2 - B_m z_m z_r\right)\frac{e^{-ik(ct-R)}}{R^3},$$

$$H_r = -k^2 \epsilon_{rmn} B_m z_n \frac{e^{-ik(ct-R)}}{R^2}.$$

(This is the electromagnetic field of the Hertzian dipole oscillation, which is the simplest model of a radio antenna).

We have **6.341**:

$$E_r = \Pi_{m,mr} - \frac{1}{c^2}\partial_t^2 \Pi_r \tag{1}$$

$$H_r = \frac{1}{c}\epsilon_{rpq}\partial_t \Pi_{q,p} \tag{2}$$

and **6.342**:

$$\frac{1}{c^2}\partial_t^2 \Pi_r - \Pi_{r,mm} = 0 \tag{3}$$

(3) in (1):

$$E_r = \Pi_{m,mr} - \Pi_{r,mm} \tag{4}$$

$$\tag{5}$$

and let's express $\Pi_r$ as

$$\Pi_r = B_r F(t) G(z) \tag{6}$$

with

$$F(t) = e^{-ikct} \tag{7}$$
$$G(z) = \frac{e^{ikR}}{R} \tag{8}$$

So (1) can be expressed as

$$E_r = B_m F G_{,mr} - \frac{1}{c^2} B_r G F_{,tt} \tag{9}$$

We also know by **6.350**:

$$G_{,mm} = -k^2 G \tag{10}$$

$$\begin{aligned} E_r &= \Pi_{m,mr} - \Pi_{r,mm} && (11) \\ &= \Pi_{m,mr} + k^2 B_r F G && (12) \end{aligned}$$

Let's calculate some necessary derivatives

$$\begin{aligned} & R_{,r} = \frac{z_r}{R} && (13) \\ & F_{,t} = -ikcF && (14) \\ & F_{,tt} = -k^2 c^2 F && (15) \\ & G_{,r} = -\frac{z_r G}{R^2} (1 - ikR) && (16) \\ & \Pi_{m,m} = -B_m z_m F \frac{G}{R^2} (1 - ikR) && (17) \end{aligned}$$

The (12) becomes

$$\begin{aligned} E_r &= \Pi_{m,mr} + k^2 B_r F G && (18) \\ &= -\left[ B_m z_m F \frac{G}{R^2} (1 - ikR) \right]_{,r} + k^2 B_r F G && (19) \\ &= -\left[ B_m z_m F \frac{G}{R^2} \right]_{,r} + \left[ B_m z_m F \frac{G}{R} ik \right]_{,r} + k^2 B_r F G && (20) \\ & && (21) \end{aligned}$$

Consider the expression $\frac{1}{c^2} B_m z_m G F_{,tt}$, then

$$\left( \frac{1}{c^2} B_m z_m G F_{,tt} \right)_{,r} = \frac{1}{c^2} B_r G F_{,tt} - \frac{1}{c^2} B_m F_{,tt} z_m z_r \frac{G}{R^2} (1 - ikR) \tag{22}$$

(16) in (1):

$$E_r = \Pi_{m,mr} + \frac{1}{c^2} B_m F_{,tt} z_m z_r \frac{G}{R^2} (1 - ikR) - \left( \frac{1}{c^2} B_m z_m G F_{,tt} \right)_{,r} \tag{23}$$

$$= \left[ \Pi_{m,m} - \frac{1}{c^2} B_m z_m G F_{,tt} \right]_{,r} + \frac{1}{c^2} B_m F_{,tt} z_m z_r \frac{G}{R^2} (1 - ikR) \tag{24}$$

$$= \left[ -B_m z_m F \frac{G}{R^2} (1 - ikR) + \frac{1}{c^2} B_m z_m G k^2 c^2 F \right]_{,r} - \frac{1}{c^2} B_m k^2 c^2 F z_m z_r \frac{G}{R^2} (1 - ikR) \tag{25}$$

$$= \left[ -B_m z_m F \frac{G}{R^2} (1 - ikR) + B_m z_m G k^2 F \right]_{,r} - B_m k^2 F z_m z_r \frac{G}{R^2} (1 - ikR) \tag{26}$$

$$E_r = \begin{cases} \left[ikB_m z_m F\frac{G}{R}\right]_{,r} \\ -\left[B_m z_m F\frac{G}{R^2}\right]_{,r} \\ +\left[B_m z_m G k^2 F\right]_{,r} \\ -B_m k^2 F z_m z_r \frac{G}{R^2} \\ +iB_m k^3 F z_m z_r \frac{G}{R} \end{cases} \tag{27}$$

$$= \begin{cases} \left[ikB_m z_m F\frac{G}{R}\right]_{,r} \\ -\left[\frac{B_m z_m F}{R^3}(RG)\right]_{,r} \\ +\left[B_m k^2 F z_m G\right]_{,r} \\ -B_m k^2 F z_m z_r \frac{G}{R^2} \\ +iB_m k^3 F z_m z_r \frac{G}{R} \end{cases} \tag{28}$$

$$= \begin{cases} ikB_r F\frac{G}{R} + ikB_m F z_m \left[\frac{G}{R}\right]_{,r} \\ -\left[\frac{B_m z_m F}{R^3}\right]_{,r}(RG) - \frac{B_m z_m F}{R^3}\left[(RG)\right]_{,r} \\ +B_r k^2 F G + B_m k^2 F z_m \left[G\right]_{,r} \\ -B_m k^2 F z_m z_r \frac{G}{R^2} \\ +iB_m k^3 F z_m z_r \frac{G}{R} \end{cases} \tag{29}$$

$$= \begin{cases} ikB_r F\frac{G}{R} - k^2 B_m F z_m z_r \frac{G}{R^2} \\ -\left[\frac{B_m z_m F}{R^3}\right]_{,r}(RG) - i\frac{kB_m z_m F}{R^3} z_r G \\ +B_r k^2 F G - B_m k^2 F z_m \frac{z_r G}{R^2} + ik^3 B_m F z_m \frac{z_r G}{R} \\ -k^2 B_m F z_m z_r \frac{G}{R^2} \\ +ik^3 B_m F z_m z_r \frac{G}{R} \end{cases} \tag{30}$$

$$\tag{31}$$

*******************************************************

$$G_{,rm} = \begin{cases} -\delta_{mr}\frac{G}{R^2}\left(1-ikR\right) \\ -\frac{z_r}{R^2}\left(1-ikR\right)G_{,m} \\ +2\frac{z_r G}{R^3}\left(1-ikR\right)R_{,m} \\ +ik\frac{z_r G}{R^2}R_{,m} \end{cases} \tag{32}$$

$$= \begin{cases} -\delta_{mr}\frac{G}{R^2}\left(1-ikR\right) \\ +\frac{z_r}{R^2}\left(1-ikR\right)\frac{z_r G}{R^2}\left(1-ikR\right) \\ +2\frac{z_r G}{R^3}\left(1-ikR\right)\frac{z_r}{R} \\ +ik\frac{z_r G}{R^2}\frac{z_r}{R} \end{cases} \tag{33}$$

$$= \begin{cases} -\delta_{mr}\frac{G}{R^2}\left(1-ikR\right) \\ +\frac{z_r^2}{R^4}G - 2ik\frac{z_r^2}{R^3}G - k^2\frac{z_r^2}{R^2}G \\ +2\frac{z_r^2 G}{R^4} - 2ik\frac{z_r^2 G}{R^3} \\ +ik\frac{z_r^2 G}{R^3} \end{cases} \tag{34}$$

$$= 3z_r^2\frac{G}{R^4} - k^2 z_r^2\frac{G}{R^2} - \delta_{mr}\frac{G}{R^2} + ik\delta_{mr}\frac{G}{R} - 3ikz_r^2\frac{G}{R^3} \tag{35}$$

$$\Rightarrow \quad G_{,mm} = 3z_r^2\frac{G}{R^4} - k^2 z_r^2\frac{G}{R^2} - 3\frac{G}{R^2} + 3ik\frac{G}{R} - 3ikz_r^2\frac{G}{R^3} \tag{36}$$

$$\Rightarrow \quad \Pi_{r,mm} = 3z_r^2 B_r F\frac{G}{R^4} - k^2 z_r^2 B_r F\frac{G}{R^2} - 3B_r F\frac{G}{R^2} + 3ikB_r F\frac{G}{R} - 3ikz_r^2 B_r F\frac{G}{R^3} \tag{37}$$

(7):

$$E_m = 3z_r^2 B_m F\frac{G}{R^4} - k^2 z_r^2 B_m F\frac{G}{R^2} - \delta_{mr}B_m F\frac{G}{R^2} + ik\delta_{mr}B_m F\frac{G}{R} - 3ikz_m^2 B_m F\frac{G}{R^3} + \frac{1}{c^2}B_m G k^2 c^2 F \tag{38}$$

$$= 3z_r^2 B_m F\frac{G}{R^4} - k^2 z_r^2 B_m F\frac{G}{R^2} - \delta_{mr}B_m F\frac{G}{R^2} + ik\delta_{mr}B_m F\frac{G}{R} - 3ikz_m^2 B_m F\frac{G}{R^3} + B_m G k^2 F \tag{39}$$

♦

## 5.43 p238 - Exercise 14c

Taking for the Hertz vector the fundamental solution of the wave equation **6.342**

$$\Pi_r = B_r \frac{e^{-ik(ct-R)}}{R}, \quad R^2 = z_m z_m$$

where $B_r$ is a constant vector, show that, for $R$ much less than $\lambda = \frac{2\pi}{k}$, we have approximatively

$$E_r = -\frac{\partial}{\partial z^r}\left[B_m z_m \frac{e^{-ikct}}{R^3}\right],$$

$$H_r = -ik\epsilon_{rmn} B_m z_n \frac{e^{-ikct}}{R^3}.$$

Show also that for $R$ much greater than $\lambda = \frac{2\pi}{k}$ (wave zone), we have approximatively

$$E_r = k^2 \left(B_r R^2 - B_m z_m z_r\right) \frac{e^{-ik(ct-R)}}{R^3},$$

$$H_r = -k^2 \epsilon_{rmn} B_m z_n \frac{e^{-ik(ct-R)}}{R^2}.$$

(This is the electromagnetic field of the Hertzian dipole oscillation, which is the simplest model of a radio antenna).

We have **6.341**:

$$E_r = \Pi_{m,mr} - \frac{1}{c^2}\partial_t^2 \Pi_r \tag{1}$$

$$H_r = \frac{1}{c}\epsilon_{rpq}\partial_t \Pi_{q,p} \tag{2}$$

and **6.342**:

$$\frac{1}{c^2}\partial_t^2 \Pi_r - \Pi_{r,mm} = 0 \tag{3}$$

(3) in (1):

$$E_r = \Pi_{m,mr} - \Pi_{r,mm} \tag{4}$$

$$\tag{5}$$

and let's express $\Pi_r$ as

$$\Pi_r = B_r G(t) F(z) \tag{6}$$

with

$$G(t) = e^{-ikct} \tag{7}$$
$$F(z) = \frac{e^{ikR}}{R} \tag{8}$$

So (3) can be expressed as

$$E_r = B_m G F_{,mr} - B_r G F_{,mm} \tag{9}$$

We also know by **6.350**:

$$F_{,mm} = -k^2 F \tag{10}$$

$$E_r = \Pi_{m,mr} - \Pi_{r,mm} \tag{11}$$
$$= \Pi_{m,mr} + k^2 B_r G F \tag{12}$$

Let's calculate some necessary derivatives

$$R_{,r} = \frac{z_r}{R} \tag{13}$$
$$F_{,m} = -\frac{z_m F}{R^2}(1 - ikR) \tag{14}$$
$$\Pi_{m,m} = -B_m z_m G \frac{F}{R^2}(1 - ikR) \tag{15}$$

Consider the case $kR << 2\pi$, then

$$F = \frac{e^i kR}{R} \tag{16}$$
$$\approx \frac{1 + ikR}{R} \tag{17}$$
$$= \frac{1}{R} + ik \tag{18}$$

(15)

$$\Pi_{m,m} = -B_m z_m G \frac{1}{R^3}(1 + ikR)(1 - ikR) \tag{19}$$
$$= -B_m z_m G \frac{1}{R^3}\left(1 + k^2 R^2\right) \tag{20}$$

(12) becomes

$$
\begin{aligned}
E_r &= \Pi_{m,mr} + k^2 B_r G F && (21)\\
&= -\left[B_m z_m G \frac{1}{R^3}\left(1 + k^2 R^2\right)\right]_{,r} + k^2 B_r G F && (22)\\
&= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} - \left[B_m z_m G k^2 \frac{1}{R}\right]_{,r} + k^2 B_r G F && (23)\\
&= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} - B_m z_m G k^2 \left[\frac{1}{R}\right]_{,r} - k^2 B_r G \frac{1}{R} + k^2 B_r G F && (24)\\
&= -\left[B_m z_m \frac{e^{-ikct}}{R^3}\right]_{,r} + k^2 B_m z_m G z_r \frac{1}{R^3} - B_r G k^2 \frac{1}{R} + k^2 B_r G \frac{1}{R} + ik^3 B_r G && (25)\\
&= -\left[B_m z_m \frac{e^{-ikct}}{R^3}\right]_{,r} + k^2 G \left(B_m z_m z_r \frac{1}{R^3} + ikB_r\right) && (26)\\
& && (27)
\end{aligned}
$$

♦

## 5.44 p238 - Exercise 14d

Taking for the Hertz vector the fundamental solution of the wave equation **6.342**

$$\Pi_r = B_r \frac{e^{-ik(ct-R)}}{R}, \quad R^2 = z_m z_m$$

where $B_r$ is a constant vector, show that, for $R$ much less than $\lambda = \frac{2\pi}{k}$, we have approximatively

$$E_r = -\frac{\partial}{\partial z^r}\left[B_m z_m \frac{e^{-ikct}}{R^3}\right],$$

$$H_r = -ik\epsilon_{rmn} B_m z_n \frac{e^{-ikct}}{R^3}.$$

Show also that for $R$ much greater than $\lambda = \frac{2\pi}{k}$ (wave zone), we have approximatively

$$E_r = k^2 \left(B_r R^2 - B_m z_m z_r\right) \frac{e^{-ik(ct-R)}}{R^3},$$

$$H_r = -k^2 \epsilon_{rmn} B_m z_n \frac{e^{-ik(ct-R)}}{R^2}.$$

(This is the electromagnetic field of the Hertzian dipole oscillation, which is the simplest model of a radio antenna).

We have **6.341**:

$$E_r = \Pi_{m,mr} - \frac{1}{c^2}\partial_t^2 \Pi_r \tag{1}$$

$$H_r = \frac{1}{c}\epsilon_{rpq}\partial_t \Pi_{q,p} \tag{2}$$

and **6.342**:

$$\frac{1}{c^2}\partial_t^2 \Pi_r - \Pi_{r,mm} = 0 \tag{3}$$

(3) in (1):

$$E_r = \Pi_{m,mr} - \Pi_{r,mm} \tag{4}$$

$$\tag{5}$$

and let's express $\Pi_r$ as

$$\Pi_r = B_r G(t) F(z) \tag{6}$$

with

$$G(t) = e^{-ikct} \tag{7}$$

$$F(z) = \frac{e^{ikR}}{R} \tag{8}$$

So (3) can be expressed as

$$E_r = B_m G F_{,mr} - B_r G F_{,mm} \tag{9}$$

We also know by **6.350**:

$$F_{,mm} = -k^2 F \tag{10}$$

$$\begin{aligned} E_r &= \Pi_{m,mr} - \Pi_{r,mm} && (11) \\ &= \Pi_{m,mr} + k^2 B_r G F && (12) \end{aligned}$$

Let's calculate some necessary derivatives

$$\begin{aligned} R_{,r} &= \frac{z_r}{R} && (13) \\ F_{,m} &= -\frac{z_m F}{R^2}(1 - ikR) && (14) \\ \Pi_{m,m} &= -B_m z_m G \frac{F}{R^2}(1 - ikR) && (15) \end{aligned}$$

(12) becomes

$$\begin{aligned} E_r &= \Pi_{m,mr} + k^2 B_r G F && (16) \\ &= -\left[B_m z_m G \frac{F}{R^2}(1 - ikR)\right]_{,r} + k^2 B_r G F && (17) \\ &= -\left[B_m z_m G \frac{F}{R^2}\right]_{,r} + \left[ikB_m z_m G \frac{F}{R}\right]_{,r} + k^2 B_r G F && (18) \\ &= -\left[B_m z_m G \frac{F}{R^2}\right]_{,r} + ikB_r G \frac{F}{R} + ikB_m z_m G \left[\frac{F}{R}\right]_{,r} + k^2 B_r G F && (19) \\ &= -\left[B_m z_m G \frac{F}{R^2}\right]_{,r} + ikB_r G \frac{F}{R} + ikB_m z_m G \frac{RF_{,r} - F z_r \frac{1}{R}}{R^2} + k^2 B_r G F && (20) \\ &= -\left[B_m z_m G \frac{F}{R^2}\right]_{,r} + ikB_r G \frac{F}{R} + ikB_m z_m G \frac{1}{R} F_{,r} - ikB_m z_m z_r G F \frac{1}{R^3} + k^2 B_r G F && (21) \\ &= -\left[B_m z_m G \frac{F}{R^2}\right]_{,r} + ikB_r G \frac{F}{R} - ikB_m z_m G \frac{z_r F}{R^2} \frac{1}{R}(1 - ikR) - ikB_m z_m z_r G F \frac{1}{R^3} + k^2 B_r G F && (22) \\ &= -\left[B_m z_m G \frac{F}{R^2}\right]_{,r} + ikB_r G F \frac{1}{R} - 2ikB_m z_m z_r G F \frac{1}{R^3} - k^2 B_m z_m z_r G F \frac{1}{R^2} + k^2 B_r G F && (23) \end{aligned}$$

Suppose $kR << \pi$, then $F \approx \left(\frac{1}{R} + ik\right)$

So,

$$E_r = \begin{cases} -\left[B_m z_m G \frac{\left(\frac{1}{R}+ik\right)}{R^2}\right]_{,r} \\ +ikB_r G\left(\frac{1}{R}+ik\right)\frac{1}{R} \\ -2ikB_m z_m z_r G\left(\frac{1}{R}+ik\right)\frac{1}{R^3} \\ -k^2 B_m z_m z_r G\left(\frac{1}{R}+ik\right)\frac{1}{R^2} \\ +k^2 B_r G\left(\frac{1}{R}+ik\right) \end{cases} \tag{24}$$

$$= \begin{cases} -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} \\ -ikB_r G\frac{1}{R^2} \\ +2ikB_m z_m G\frac{z_r}{R}\frac{1}{R^3} \\ +ikB_r G\frac{1}{R}\left(\frac{1}{R}+ik\right) \\ -2ikB_m z_m z_r G\frac{1}{R^3}\left(\frac{1}{R}+ik\right) \\ -k^2 B_m z_m z_r G\frac{1}{R^2}\left(\frac{1}{R}+ik\right) \\ +k^2 B_r G\left(\frac{1}{R}+ik\right) \end{cases} \tag{25}$$

$$= \begin{cases} -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} \\ -ikB_r G\frac{1}{R^2} \\ +2ikB_m z_m G z_r\frac{1}{R^4} \\ +ikB_r G\frac{1}{R^2} - k^2 B_r G\frac{1}{R} \\ -2ikB_m z_m z_r G\frac{1}{R^4} + 2k^2 B_m z_m z_r G\frac{1}{R^3} \\ -k^2 B_m z_m z_r G\frac{1}{R^3} - ik^3 B_m z_m z_r G\frac{1}{R^2} \\ +k^2 B_r G\frac{1}{R} + ik^3 B_r G \end{cases} \tag{26}$$

$$\tag{27}$$

♦

## 5.45 p238 - Exercise 14e

Taking for the Hertz vector the fundamental solution of the wave equation **6.342**

$$\Pi_r = B_r \frac{e^{-ik(ct-R)}}{R}, \quad R^2 = z_m z_m$$

where $B_r$ is a constant vector, show that, for $R$ much less than $\lambda = \frac{2\pi}{k}$, we have approximatively

$$E_r = -\frac{\partial}{\partial z^r}\left[B_m z_m \frac{e^{-ikct}}{R^3}\right],$$

$$H_r = -ik\epsilon_{rmn} B_m z_n \frac{e^{-ikct}}{R^3}.$$

Show also that for $R$ much greater than $\lambda = \frac{2\pi}{k}$ (wave zone), we have approximatively

$$E_r = k^2\left(B_r R^2 - B_m z_m z_r\right)\frac{e^{-ik(ct-R)}}{R^3},$$

$$H_r = -k^2\epsilon_{rmn} B_m z_n \frac{e^{-ik(ct-R)}}{R^2}.$$

(This is the electromagnetic field of the Hertzian dipole oscillation, which is the simplest model of a radio antenna).

We have **6.341**:

$$E_r = \Pi_{m,mr} - \frac{1}{c^2}\partial_t^2 \Pi_r \tag{1}$$

$$H_r = \frac{1}{c}\epsilon_{rpq}\partial_t \Pi_{q,p} \tag{2}$$

and **6.342**:

$$\frac{1}{c^2}\partial_t^2\Pi_r - \Pi_{r,mm} = 0 \tag{3}$$

(3) in (1):

$$E_r = \Pi_{m,mr} - \Pi_{r,mm} \tag{4}$$

$$\tag{5}$$

and let's express $\Pi_r$ as

$$\Pi_r = B_r G(t) F(z) \tag{6}$$

with

$$G(t) = e^{-ikct} \tag{7}$$

$$F(z) = \frac{e^{ikR}}{R} \tag{8}$$

So (3) can be expressed as

$$E_r = B_m G F_{,mr} - B_r G F_{,mm} \tag{9}$$

We also know by **6.350**:

$$F_{,mm} = -k^2 F \tag{10}$$

$$E_r = \Pi_{m,mr} - \Pi_{r,mm} \tag{11}$$

$$= \Pi_{m,mr} + k^2 B_r G F \tag{12}$$

From **6.348** we have

$$F^{'} = \left( \frac{ik}{R} - \frac{1}{R^2} \right) e^{ikR} \tag{13}$$

Let's calculate some necessary derivatives

$$R_{,r} = \frac{z_r}{R} \tag{14}$$

$$F_{,m} = F^{'} R_{,r} = z_m \left( \frac{ik}{R} - \frac{1}{R^2} \right) F \tag{15}$$

$$\Pi_{m,m} = B_m z_m G \left( \frac{ik}{R} - \frac{1}{R^2} \right) F \tag{16}$$

(12) becomes

$$E_r = \Pi_{m,mr} + k^2 B_r GF \tag{17}$$
$$= -\left[B_m z_m G \frac{1}{R^3}\left(1 - ikR\right)\right]_{,r} + k^2 B_r GF \tag{18}$$
$$= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} + \left[ikB_m z_m G \frac{F}{R}\right]_{,r} + k^2 B_r GF \tag{19}$$
$$= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} + ikB_r G \frac{F}{R} + ikB_m z_m G \left[\frac{F}{R}\right]_{,r} + k^2 B_r GF \tag{20}$$
$$= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} + ikB_r G \frac{F}{R} + ikB_m z_m G \frac{RF_{,r} - F z_r \frac{1}{R}}{R^2} + k^2 B_r GF \tag{21}$$
$$= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} + ikB_r G \frac{F}{R} + ikB_m z_m G \frac{1}{R} F_{,r} - ikB_m z_m z_r GF \frac{1}{R^3} + k^2 B_r GF \tag{22}$$
$$= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} + ikB_r G \frac{F}{R} - ikB_m z_m G \frac{z_r F}{R^2} \frac{1}{R}\left(1 - ikR\right) - ikB_m z_m z_r GF \frac{1}{R^3} + k^2 B_r GF \tag{23}$$
$$= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} + ikB_r G \frac{F}{R} - ikB_m z_m z_r GF \frac{1}{R^3} - k^2 B_m z_m z_r GF \frac{1}{R^2} - ikB_m z_m z_r GF \frac{1}{R^3} + k^2 B_r GF \tag{24}$$
$$= -\left[B_m z_m G \frac{1}{R^3}\right]_{,r} + GF \frac{k}{R^3}\left[kR\left(R^2 B_r - B_m z_m z_r\right) + i\left(R^2 B_r - 2B_m z_m z_r\right)\right] \tag{25}$$
$$\tag{26}$$

♦

## 5.46 p238 - Exercise 14

Taking for the Hertz vector the fundamental solution of the wave equation **6.342**

$$\Pi_r = B_r \frac{e^{-ik(ct-R)}}{R}, \quad R^2 = z_m z_m$$

where $B_r$ is a constant vector, show that, for $R$ much less than $\lambda = \frac{2\pi}{k}$, we have approximatively

$$E_r = -\frac{\partial}{\partial z^r}\left[B_m z_m \frac{e^{-ikct}}{R^3}\right],$$

$$H_r = -ik\epsilon_{rmn} B_m z_n \frac{e^{-ikct}}{R^3}.$$

Show also that for $R$ much greater than $\lambda = \frac{2\pi}{k}$ (wave zone), we have approximatively

$$E_r = k^2\left(B_r R^2 - B_m z_m z_r\right)\frac{e^{-ik(ct-R)}}{R^3},$$

$$H_r = -k^2\epsilon_{rmn} B_m z_n \frac{e^{-ik(ct-R)}}{R^2}.$$

(This is the electromagnetic field of the Hertzian dipole oscillation, which is the simplest model of a radio antenna).

We have **6.341**:

$$E_r = \Pi_{m,mr} - \frac{1}{c^2}\partial_t^2 \Pi_r \tag{1}$$

$$H_r = \frac{1}{c}\epsilon_{rpq}\partial_t \Pi_{q,p} \tag{2}$$

and **6.342**:

$$\frac{1}{c^2}\partial_t^2 \Pi_r - \Pi_{r,mm} = 0 \tag{3}$$

(3) in (1):

$$E_r = \Pi_{m,mr} - \Pi_{r,mm} \tag{4}$$

One is tempted to attack frontally the expression (4) but when you begin, one experiences that this approach requires a lot of lengthy algebraic manipulations. So instead we try first to derive the expression for the magnetic field and then derive the electric field from the Maxwell's equations. Let's express $\Pi_r$ as

$$\Pi_r = B_r G(t) F(z) \tag{5}$$

with

$$G(t) = e^{-ikct} \tag{6}$$

$$F(z) = \frac{e^{ikR}}{R} \tag{7}$$

From **6.348** we have

$$F^{'} = \frac{\partial F}{\partial R} = \left( \frac{ik}{R} - \frac{1}{R^2} \right) e^{ikR} \tag{8}$$

and

$$R_{,r} = \frac{z_r}{R} \tag{9}$$

We have (2)

$$H_r = \frac{1}{c} \epsilon_{rpq} \partial_t \Pi_{q,p} \tag{10}$$

with

$$\Pi_{q,p} = B_q G F^{'} R_{,p} \tag{11}$$

$$= B_q z_p G F \left( \frac{ik}{R} - \frac{1}{R^2} \right) \tag{12}$$

$$\Rightarrow \quad \partial_t \Pi_{q,p} = B_q z_p F \left( \frac{ik}{R} - \frac{1}{R^2} \right) \partial_t G \tag{13}$$

$$= -ikc B_q z_p G F \left( \frac{ik}{R} - \frac{1}{R^2} \right) \tag{14}$$

$$\Rightarrow \quad H_r = -ik \epsilon_{rpq} B_q z_p G F \left( \frac{ik}{R} - \frac{1}{R^2} \right) \tag{15}$$

**For** $kR << 2\pi$ **we have**

$$F = \frac{e^{ikR}}{R} = \frac{\cos kR + i \sin kR}{R} \approx \frac{1 + ikR}{R} = \left( \frac{1}{R} + ik \right) \tag{16}$$

So (15) becomes

$$H_r = -ik\epsilon_{rpq}B_q z_p G\left(\frac{1}{R}+ik\right)\left(\frac{ik}{R}-\frac{1}{R^2}\right) \tag{17}$$
$$= -ik\epsilon_{rpq}B_q z_p G\left(\frac{1}{R}+ik\right)\left(ik-\frac{1}{R}\right)\frac{1}{R} \tag{18}$$
$$= -ik\epsilon_{rpq}B_q z_p G\left(-k^2-\frac{1}{R^2}\right)\frac{1}{R} \tag{19}$$
$$= i\epsilon_{rpq}B_q z_p G\left(k^3+\frac{k}{R^2}\right)\frac{1}{R} \tag{20}$$
$$= i\epsilon_{rpq}B_q z_p G\left(k^3+\frac{k^3}{k^2R^2}\right)\frac{1}{R} \tag{21}$$
$$= ik^3\epsilon_{rpq}B_q z_p G\left(1+\frac{1}{k^2R^2}\right)\frac{1}{R} \tag{22}$$

and as $\frac{1}{k^2R^2} >> 1$

$$H_r = ik^3\epsilon_{rpq}B_q z_p G\frac{1}{k^2R^2}\frac{1}{R} \tag{23}$$
$$= ik\epsilon_{rpq}B_q z_p\frac{1}{R^3}e^{-ikct} \tag{24}$$
$$= -ik\epsilon_{rmn}B_m z_n\frac{1}{R^3}e^{-ikct} \tag{25}$$

so indeed

$$\mathbf{H_r \approx -ik\epsilon_{rmn}B_m z_n\frac{1}{R^3}e^{-ikct}}$$

For the electric field we have the Maxwell equation (for the case $kR << 2\pi$)

$$\frac{1}{c}\partial_t E_r = \epsilon_{rmn}H_{n,m} \tag{26}$$
$$= -ik\epsilon_{rmn}\epsilon_{npq}B_p G\left(z_q\frac{1}{R^3}\right)_{,m} \tag{27}$$
$$= -ik\epsilon_{rmn}\epsilon_{npq}B_p G\left(z_q\right)_{,m}\frac{1}{R^3}+ik\epsilon_{rmn}\epsilon_{npq}B_p G z_q\left(\frac{1}{R^3}\right)_{,m} \tag{28}$$
$$= -ik\underbrace{\epsilon_{rmn}\epsilon_{npm}}_{=2\delta_{rp}}B_p G\frac{1}{R^3}+3ik\underbrace{\epsilon_{rmn}\epsilon_{npq}}_{=-\delta_{mp}\delta_{rq}+\delta_{nq}\delta_{rp}}B_p G z_q\frac{z_m}{R^5} \tag{29}$$
$$= ikG\left(\frac{B_r}{R^3}-3\frac{B_m z_m z_r}{R^5}\right) \tag{30}$$

and note that

$$\left(\frac{B_r}{R^3}-3\frac{B_m z_m z_r}{R^5}\right)=\left[\frac{B_m z_m}{R^3}\right]_{,r}$$

so integration over $t$ of (30)

$$E_r = \left[ ikc \frac{B_m z_m}{R^3} \underbrace{\int G dt}_{-\frac{e^{-ikct}}{ikc}} \right]_{,r} \tag{31}$$

$$= -\left[ iB_m z_m \frac{e^{-ikct}}{R^3} \right]_{,r} \tag{32}$$

so indeed

$$\mathbf{E_r \approx -\left[ iB_m z_m \frac{e^{-ikct}}{R^3} \right]_{,r}}$$

$\Diamond$

**Let's look at the case** $kR >> 2\pi$ **:**

We have (15)

$$H_r = -ik\epsilon_{rpq} B_q z_p GF \left( \frac{ik}{R} - \frac{1}{R^2} \right) \tag{33}$$

$$= -ik^3 \epsilon_{rpq} B_q z_p GF \left( \frac{i}{kR} - \frac{1}{k^2 R^2} \right) \tag{34}$$

as $\frac{1}{kR} >> \frac{1}{k^2 R^2}$ we discard the real part of the complex number and get

$$H_r = k^2 \epsilon_{rpq} B_q z_p GF \frac{1}{R} \tag{35}$$

$$= -k^2 \epsilon_{rmn} B_m z_n \frac{e^{-ik(ct-R)}}{R^2} \tag{36}$$

so indeed

$$\mathbf{H_r \approx -k^2 \epsilon_{rmn} B_m z_n \frac{e^{-ik(ct-R)}}{R^2}}$$

$\blacklozenge$