Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

Part II

Chapters V to VIII

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github. An overview of the material covered in the book can be found in the separate document "Synge overview.pdf".

Some notation conventions

$$\partial_r \equiv \tfrac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \quad \text{Christoffel symbol of the second kind}$$

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5.1 p153 - Exercise

If μ^α are the contravariant components of a unit vector in a surface S, show that $\mu^\alpha f_\alpha$ is the physical component of acceleration in the direction tangent to S defined by μ^α.

As we are in an Euclidean space we can interpret $a_{mn}\mu^\alpha f^\alpha$ as $|\mu|\,|f|\cos\theta$ with θ the angle between the two vectors. As $|\mu| = 1$ we have

$$a_{mn}\mu^\alpha f^\alpha = \mu^\alpha f_\alpha \tag{1}$$

$$= |f|\cos\theta \tag{2}$$

which is the projection of the vector f on the unit vector μ.

♦

5.2 p154 - Clarification to 5.226.

5.226. $$\mathbf{v}\frac{\mathbf{dv}}{\mathbf{ds}} = \mathbf{0}, \quad \overline{\kappa}\mathbf{v^2} = \mathbf{0}$$

Assuming that the particle is not at rest $v \neq 0$, and therefore $\overline{\kappa} = 0$. ***Since this implies that the curve is a geodesic...***

The assertion in bold is a direct consequence

2.513. $$\frac{\delta \frac{dx^r}{ds}}{\delta s} = 0$$

As in **5.233** we have $\frac{\delta\lambda^\alpha}{\delta s} = \frac{\delta \frac{dx^\alpha}{ds}}{\delta s} = 0$, the considered curve follows the geodesic curve.

♦

5.3 p155 - Exercise

Show that in relativity the force 4-vector X^r lies along the first normal of the trajectory in space-time. Express the first curvature in terms of the proper mass m of the particle and the magnitude X of X^r.

Let us recall the first Frenet formula **2.705** without forgetting that the metric form is not positive-definite,

$$\frac{\delta\lambda^r}{\delta s} = \kappa\nu^r, \quad \epsilon_{(1)}\nu_n\nu^n = 1$$

As **5.299**

$$m\frac{\delta\lambda^r}{\delta s} = X^r$$

it is clear that $X^r = m\kappa\nu^r$ and is collinear with the first normal.

$$X^r = m\kappa\nu^r \tag{1}$$

$$\times \quad a_{mr}X^m \quad \Rightarrow \quad \underbrace{a_{mr}X^m X^r}_{=(X^1)^2+(X^2)^2+(X^3)^2-(X^4)^2} = m\kappa \underbrace{a_{mr}\nu^m\nu^r}_{=\epsilon_{(1)}} \tag{2}$$

$$\Rightarrow \quad \kappa = \epsilon_{(1)}\frac{\left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2 - \left(X^4\right)^2}{m}$$

♦

5.4 p156 - Clarification

Interpretation of

5.231. $M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r$

What do the M_{rs} represent?

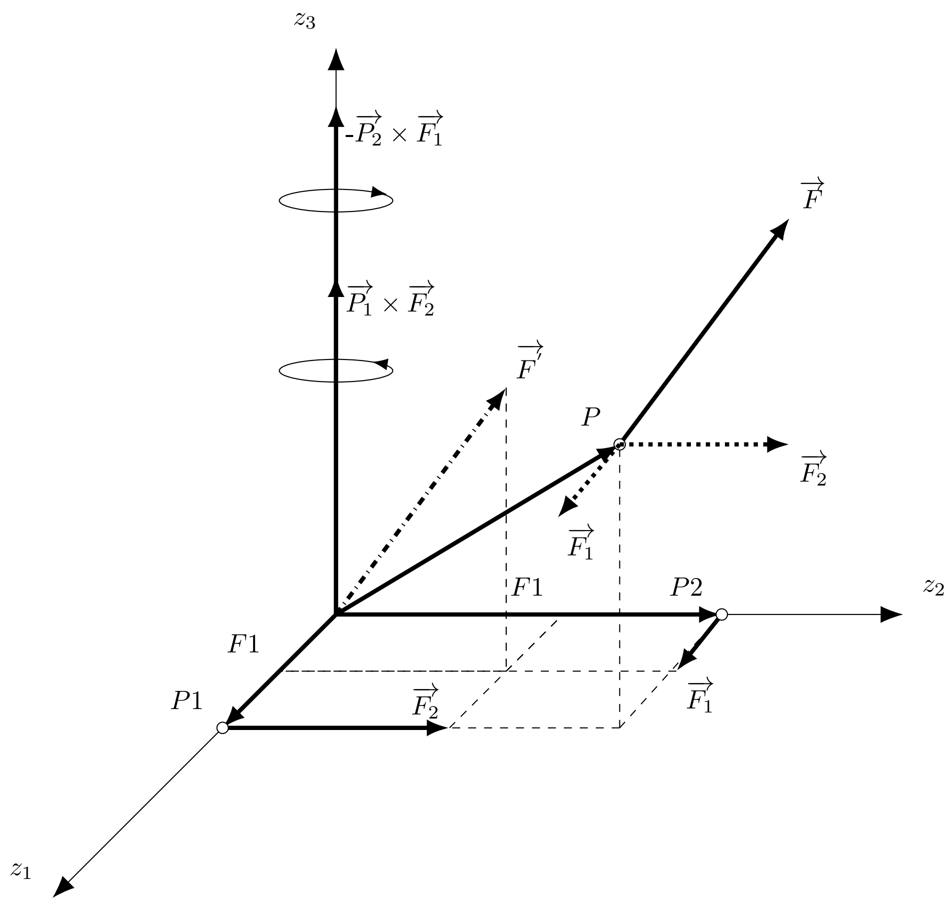

Figure 5.1: Interpretation of the tensor moment M_{12}

Let's consider a mass point P on which a force $\overrightarrow{F}$ is acting. The force has components (F_x, F_y, F_z) in the space $V_3^{'}$ (which is by the way not the space V_3 of the considered mass point).
Let's investigate the element M_{12} of the *tensor moment*.
$P_1F_2\overrightarrow{e_3}$ is the vector product $\overrightarrow{P_1} \times \overrightarrow{F_2}$ and is as such the torque of the component F_2 of $\overrightarrow{F}$ acting on the mass point situated at P_1. The origin being fixed, $\overrightarrow{F_2}$ tries to move P_1, clockwise along the z_3 axis. The same is true for the component $\overrightarrow{F_1}$ acting on the mass point situated at P_2, and is represented here by the vector $-\overrightarrow{P_2} \times \overrightarrow{F_1}$ ($\overrightarrow{F_1}$ tries to move P_2, counter clockwise along the z_3 axis). Hence, $P_1F_2 - P_2F_1$ is the net force trying to move the point P along the z_3 axis (i.e. in the plane $\|$ with the $z_3 = 0$ plane).

♦

5.5 p156 - Clarification

5.234. $$\frac{dh_r}{dt} = M_r$$

$$
\begin{aligned}
& & h_r &= m\epsilon_{rmn} z_m v_n && (1)\\
&\Rightarrow & \frac{dh_r}{dt} &= m\epsilon_{rmn} \frac{dz_m}{dt} v_n + m\epsilon_{rmn} z_m \frac{dv_n}{dt} && (2)\\
& & &= m \underbrace{\epsilon_{rmn} v_m v_n}_{=0} + \underbrace{\epsilon_{rmn} z_m F_n}_{=M_r} && (3)\\
& & &= M_r && (4)
\end{aligned}
$$

♦

5.6 p158-159 - Clarification

> **5.313.** $\omega_{rs} = -\omega_{sr}$
>
> From 5.310 and the vector character of v_r and z_r (for transformations which do not change the origin), **it follows that ω_{rs} is a Cartesian tensor of second order**.

Be

$$v_r = -\omega_{rn} z_n \tag{1}$$

Considering orthogonal transformation in a flat space $z'_m = A_{mr} z_r + B_m$ with $B_m = 0$ as we consider only transformations which do not change the origin. Differentiation with the parameter t gives

$$\begin{aligned} v'_m &= A_{mr} v_r && (2)\\ &= -\omega_{rn} A_{mr} z_n && (3)\\ & && (4) \end{aligned}$$

But $z'_q = A_{qr} z_r \quad \Rightarrow \quad A_{qn} z'_q = A_{qn} A_{qr} z_r \quad \Rightarrow \quad A_{qn} z'_q = z_n$ Hence

$$\begin{aligned} v'_m &= -\omega_{rn} A_{mr} z_n && (5)\\ &= -\underbrace{\omega_{rn} A_{mr} A_{qn}}_{\stackrel{\text{def}}{=} \omega'_{mq}} z'_q && (6)\\ v'_m &= -\omega'_{mq} z'_q && (7) \end{aligned}$$

♦

5.7 p159 - Exercise

Show that if a rigid body rotates about the point $z_r = b_r$ as fixed point, the velociy of a general point of the body is given by

$$v_r = -\omega_{rm} \left(z_m - b_m \right)$$

By **5.302.**:

$$\left(z_m^{(1)} - z_m^{(2)} \right) \left(dz_m^{(1)} - dz_m^{(2)} \right) = 0 \tag{1}$$

At the fixed point we have $z_m^{(2)} = b_m$ and $dz_m^{(2)} = 0$, hence

$$\left(z_m^{(1)} - b_m \right) \left(dz_m^{(1)} \right) = 0 \tag{2}$$

$$\Rightarrow \qquad z_m^{(1)} dz_m^{(1)} = b_m dz_m^{(1)} \tag{3}$$

As this is true for any point of the rigid mass, expanding (1) and using (3) we get when dividing by dt

$$\left(z_m^{(2)} - b_m \right) v_m^{(1)} + \left(z_m^{(1)} - b_m \right) v_m^{(2)} = 0 \tag{4}$$

Taking twice the partial derivative $\frac{\partial^2}{\partial z_p^{(1)} \partial z_q^{(1)}}$ we get

$$\left(z_m^{(2)} - b_m \right) \frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \tag{5}$$

As this is true for any arbitrary point in the rigid body we get

$$\frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \tag{6}$$

$$\Rightarrow \qquad v_m = K_{mr} z_r + B_m \tag{7}$$

At the fixed point we have

$$K_{mr} b_r + B_m = 0 \tag{8}$$

Plugging this in (7)

$$v_m = K_{mr} \left(z_r - b_m \right) \tag{9}$$

Putting $K_{mr} = -\omega_{mr}$ gives us indeed the asked expression.

♦

5.8 p161 - Clarification

5.325. $$\Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p$$

and hence, since Ω_{np} is arbitrary,

5.326. $$\sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

To be complete the following step should be inserted

$$\Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p \tag{1}$$

As Ω_{np} is skew-symmetric: $$-\Omega_{np} \sum (m f_p z_n) = -\Omega_{np} \sum F_p z_n \tag{2}$$

(1)+(2) $$\Omega_{np} \sum m (f_n z_p - f_p z_n) = \Omega_{np} \sum (F_n z_p - F_p z_n) \tag{3}$$

and hence, since Ω_{np} is arbitrary,

5.326. $$\sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

♦

5.9 p161 - Clarification

5.329.
$$h_{np} = \sum m\left(\omega_{nq} z_q z_p - \omega_{pq} z_q z_n\right)$$
$$= J_{npqr}\omega_{rq}$$

where

5.330.
$$J_{npqr} = \sum m\left(\delta_{nr} z_q z_p - \delta_{pr} z_n z_q\right)$$

$$\begin{aligned}
h_{np} &= \sum m\left(\omega_{nq} z_q z_p - \omega_{pq} z_q z_n\right) && (1)\\
&= \sum m\left(\omega_{rq}\delta_{rn} z_q z_p - \omega_{rq}\delta_{rp} z_q z_n\right) && (2)\\
&= \omega_{rq}\sum m\left(\delta_{rn} z_q z_p - \delta_{rp} z_q z_n\right) && (3)\\
&= J_{npqr}\omega_{rq} && (4)
\end{aligned}$$

♦

$$J^{-1} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}$$

5.10 p186 - Exercise 1

If a vector at the point with coordinates $(1,1,1)$ in Euclidean 3-space has components $(3,-1,2)$, find the contravariant, covariant and physical components in spherical polar coordinates.

The tensor T_n to consider is $(3,-1,2)-(1,1,1)=(2,-2,1)$.
The Jacobian matrix for the transformation $z^n \to x^k$, evaluated at the point $(1,1,1)$ is

$$J_{(1,1,1)} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{2}$$

$$\Rightarrow \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'^n} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \tag{4}$$

We have the metric tensor evaluated at $(1,1,1)$

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{5}$$

$$\Rightarrow \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \tag{7}$$

And the physical components

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \tag{9}$$

Another way to find the physical components is to project orthogonally the tensor on the unit vectors of a local Cartesian coordinate system, oriented along the unit vectors $\overline{e}_r, \overline{e}_\theta, \overline{e}_\phi$ corresponding to the vector $P\,(1,1,1)$ with modulus $|P| = \sqrt{3}$. We have for the tensor $T_n(2,-2,1)$ with modulus $|T_n| = 3$ as component along $\overline{e}_r$:

$$|T_n| \cos\alpha = |T_n| \frac{\langle T_n, P \rangle}{|T_n|\,|P|} \tag{10}$$

$$= |T_n| \frac{2 - 2 + 1}{|T_n|\,|P|} \tag{11}$$

$$= \frac{1}{\sqrt{3}} \tag{12}$$

For the component along $\overline{e}_\theta$ we first have to determine the vector $\overline{e}_\theta$. As first equation we have the orthogonality condition with $\overline{e}_r$ and putting $\overline{e}_\theta = (a, b, c)$, get $\langle \overline{e}_r, \overline{e}_\theta \rangle = a + b + c = 0$. As $\overline{e}_\theta$ lies in the plane $(1, 1, 0) - (0, 0, 0) - (0, 0, 1)$ we can put $a = b$ and get $\overline{e}_\theta = \frac{1}{\sqrt{6}}(1, 1, -2)$ and get for the tensor $T_n(2, -2, 1)$ as component along $\overline{e}_\theta$:

$$|T_n| \cos\beta = |T_n| \frac{\langle T_n, \overline{e}_\theta \rangle}{|T_n|} \tag{13}$$

$$= |T_n| \frac{2 - 2 - 2}{|T_n| \sqrt{6}} \tag{14}$$

$$= -\frac{\sqrt{2}}{\sqrt{3}} \tag{15}$$

For the component along $\overline{e}_\phi$ we first have to determine the vector $\overline{e}_\phi$. As first equation we have the orthogonality condition with the pair $\overline{e}_r, \overline{e}_\theta$ and get $\overline{e}_\phi = \overline{e}_r \times \overline{e}_\theta = \frac{1}{\sqrt{3}\sqrt{6}}(-3, 3, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$. For the tensor $T_n(2, -2, 1)$ as component along $\overline{e}_\phi$:

$$|T_n| \cos\gamma = |T_n| \frac{\langle T_n, \overline{e}_\phi \rangle}{|T_n|} \tag{16}$$

$$= |T_n| \frac{-2 - 2}{|T_n| \sqrt{2}} \tag{17}$$

$$= -\frac{4}{\sqrt{2}} \tag{18}$$

$$= -2\sqrt{2} \tag{19}$$

giving

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \tag{20}$$

as in (9).

♦

5.11 p181 and p182 - Clarification Figures 13., 14. and 15.

There are several ways to show the homeomorphism of the configuration space of a rigid body with fixed point.

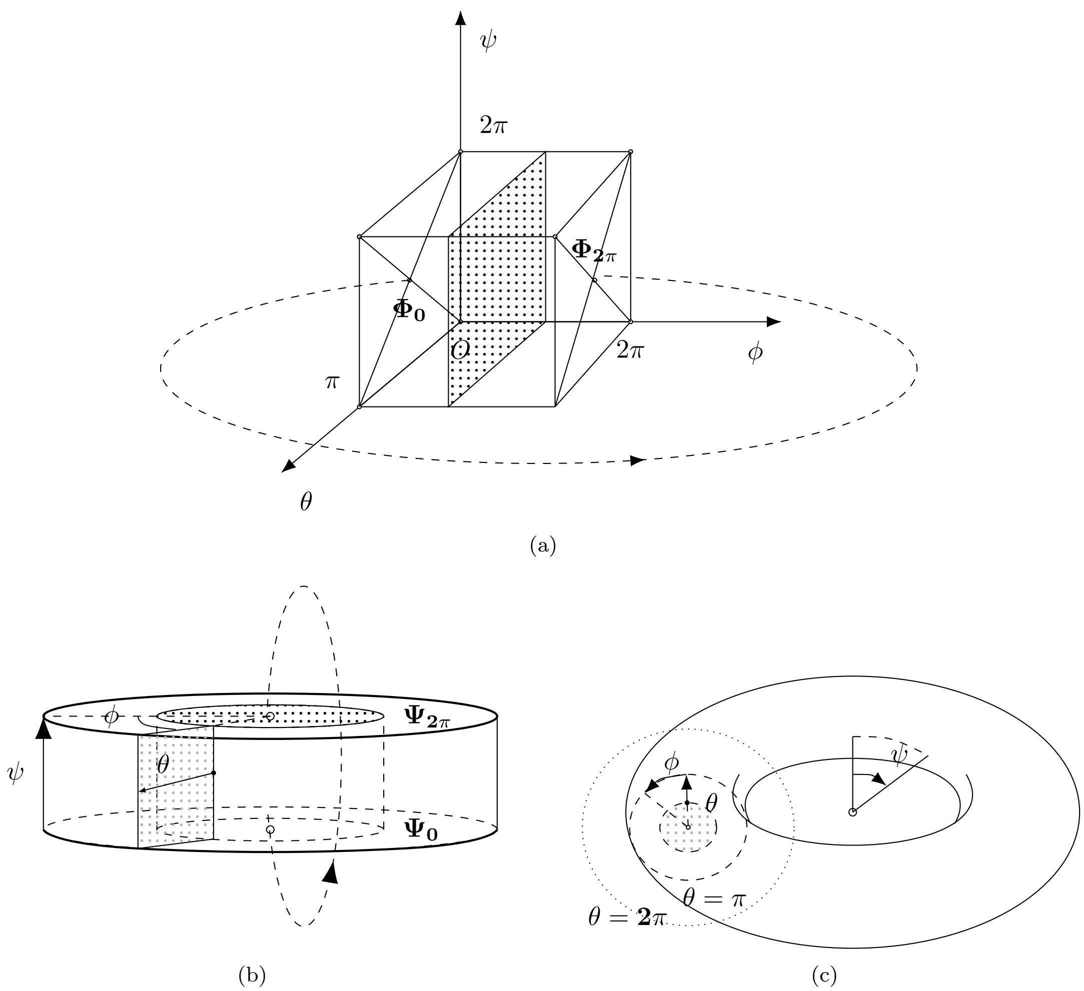

Figure 5.2: Homeomorphism of the configuration space of a rigid body with fixed point.

Consider figure 5.2(a). We can stretch like an accordion the cuboid along the ϕ axis and bent it so that the planes $\phi = 0$ and $\phi = 2\pi$ join. We get (b), a torus with square sections. The dimension ϕ is dealt with as a point $P\,(\theta\phi\psi)$ in the configuration space returns to the same point when varying ϕ to $\phi + 2k\pi$.

We can apply the same procedure of stretching and bending for the *psi* dimension so that the planes $\Psi = 0$ and $\Psi = 2\pi$ join. We get (c), a torus-like object.

The only dimension left is θ which our multi-dimensional crippled mind can't find a way to reshape this pseudo-torus so that when varying θ we can come back to the same point as started.

♦