

Tensor Calculus
J.L. Synge and A.Schild (Dover Publication)
Solutions to exercices

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Remarks and warnings

Some notation conventions

$$\partial_r a_{mn} \equiv \frac{\partial a_{mn}}{\partial x^r}$$

$$\Gamma_{mn}^r \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

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Spaces and Tensors

1.1 p5-exercise

The parametric equations of a hypersurface in V_n are

$$\begin{aligned} x^1 &= a \cos(u^1) \\ x^2 &= a \sin(u^1) \cos(u^2) \\ x^3 &= a \sin(u^1) \sin(u^2) \cos(u^3) \\ &\vdots \\ x^{N-1} &= a \sin(u^1) \sin(u^2) \sin(u^3) \dots \sin(u^{N-2}) \cos(u^{N-1}) \\ x^N &= a \sin(u^1) \sin(u^2) \sin(u^3) \dots \sin(u^{N-2}) \sin(u^{N-1}) \end{aligned}$$

where a is a constant. Find the single equation of the hyperspace in the form 1.103.

We have:

$$\begin{aligned} (x^N)^2 + (x^{N-1})^2 &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i) (\cos^2(u^{N-1}) + \sin^2(u^{N-1})) \\ &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) \sin^2(u^{N-2}) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) (1 - \cos^2(u^{N-2})) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - a^2 \prod_{i=1}^{N-3} \sin^2(u^i) \cos^2(u^{N-2}) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - (x^{N-2})^2 \end{aligned}$$

giving

$$(x^N)^2 + (x^{N-1})^2 + (x^{N-2})^2 = a^2 \prod_{i=1}^{N-3} \sin^2(u^i)$$

In general, by recursion

$$\sum_{i=0}^k (x^{N-i})^2 = a^2 \prod_{i=1}^{N-k-1} \sin^2(u^i) \quad (k \leq N-2)$$

be $k = N - 2$ ($N - k - 1 = 1$) and in the left term put $j = N - i$ (j goes from 2 to N), we get

$$\begin{aligned}\sum_{j=2}^N (x^j)^2 &= a^2 \prod_{i=1}^1 \sin^2(u^i) \\ &= a^2 (1 - \cos^2(u^1)) \\ &= a^2 - (x^1)^2\end{aligned}$$

and thus the equation of the hyperspace is given by

$$\sum_{j=1}^N (x^j)^2 - a^2 = 0$$

Determine whether the points $(\frac{1}{2}a, 0, 0, \dots, 0)$, $(0, 0, \dots, 0, 2a)$ lie on the same or opposite sides of the hyperspace.

For $(\frac{1}{2}a, 0, 0, \dots, 0)$ we have $\sum_{j=1}^N (x^j)^2 - a^2 = -\frac{3a^2}{4} < 0$ and for $(0, 0, \dots, 0, 2a)$ we have $\sum_{j=1}^N (x^j)^2 - a^2 = \frac{3a^2}{4} > 0$.

So the points lie on opposite sides of the hyperplane.



1.2 p6-exercise

Let U_2 and W_2 be subspaces of V_N . Show that if $N = 3$ they will in general intersect in a curve; if $N = 4$ they will in general intersect in a finite number of points; and if $N > 4$ they will not in general intersect at all.

We have (see 1.102 page 5): $x^r = f^r(u^1, u^2, \dots, u^M) \quad (r = 1, 2, \dots, N)$

Case $N=3$:

For U_2 we have:

$$x^r = \phi^r(u^1, u^2) \quad (r = 1, 2, 3)$$

For W_2 we have:

$$x^r = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

The intersect of the two hyperplanes is given by the N equations:

$$\phi^r(u^1, u^2) = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

So we have 3 equations in 4 unknown u^1, u^2, v^1, v^2 and can choose (fix) one e.g. u^1 and solve the set of equations for u^2, v^1, v^2 giving

$$x^r = \theta^r(u^1) \quad (r = 1, 2, 3)$$

This is an equation of a curve in space (1 parameter equation)

Case $N=4$:

Using the same reasoning as with $N=3$, we get 4 equations for 4 unknown u^1, u^2, v^1, v^2 .

Provided that the set of equation does not degenerate, these 4 equations will determine u^1, u^2, v^1, v^2 without any degree of freedom. So we get points as solutions. This solution does not to be unique, e.g. if the $\phi^r(u^1, u^2)$ are quadratic form, then the solutions

$$(u^1, u^2, v^1, v^2)$$

$$(-u^1, u^2, v^1, v^2)$$

$$(u^1, -u^2, v^1, v^2)$$

$$(-u^1, -u^2, v^1, v^2)$$

are possible.

Case $N=5$: There are more equations than variables. If the equations are not linear dependent, no solutions will be found.



1.3 p8-exercise

Show that $(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = 3a_{rst}x^r x^s x^t$

$(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = a_{rst}x^r x^s x^t + a_{rts}x^r x^s x^t + a_{srt}x^r x^s x^t$ so by just renaming the dummy indices e.g. for the second term $r \mapsto s$, $s \mapsto t$ and $t \mapsto r$ we get the desired result.



1.4 p8-exercise

If $\phi = a_{rs}x^r x^s$, show that

$$\frac{\partial \phi}{\partial x^r} = (a_{rs} + a_{sr})x^s \quad , \quad \frac{\partial^2 \phi}{\partial x^r \partial x^s} = a_{rs} + a_{sr}$$

Simplify these expressions in the case where $a^{rs} = a^{sr}$

We have

$$\frac{\partial \phi}{\partial x^t} = \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs} \frac{\partial x^r}{\partial x^t} x^s + a_{rs} x^r \frac{\partial x^s}{\partial x^t} \quad (1)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs} \delta_t^r x^s + a_{rs} x^r \delta_t^s \quad (2)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts} x^s + a_{rt} x^r \quad (3)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts} x^s + a_{st} x^s \quad (\text{rename dummy variable in third term}) \quad (4)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + (a_{ts} + a_{st})x^s \quad (5)$$

Replace x^t by x^r , we get

$$\frac{\partial \phi}{\partial x^r} = \frac{\partial a_{rs}}{\partial x^r} x^r x^s + (a_{rs} + a_{sr})x^s \quad (6)$$

So the asked expression is only true if a_{rs} is not a function of the x^s . Assuming that a_{rs} is not a function of the x^s , take the partial derivative of (6) with respect to x^t , we get

$$\frac{\partial^2 \phi}{\partial x^r \partial x^t} = (a_{rs} + a_{sr}) \frac{\partial x^s}{\partial x^t} \quad (7)$$

$$= (a_{rs} + a_{sr}) \delta_t^s \quad (8)$$

$$= (a_{rt} + a_{tr}) \quad (9)$$

Replace x^t by x^s , and we get the proposed expression.



1.5 p8-clarification on expression 1.210

$$\frac{\partial^2 x^{,q}}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0$$

From 1.209:

$$\frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} + \frac{\partial x^r}{\partial x^{,n}} \frac{\partial^2 x^{,n}}{\partial x^p \partial x^s} = 0 \quad (1)$$

multiply (1) with $\frac{\partial x^{,q}}{\partial x^r}$

$$\frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} + \frac{\partial x^r}{\partial x^{,n}} \frac{\partial^2 x^{,n}}{\partial x^p \partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0 \quad (2)$$

$$\Leftrightarrow \frac{\partial x^r}{\partial x^{,n}} \frac{\partial^2 x^{,n}}{\partial x^p \partial x^s} \frac{\partial x^{,q}}{\partial x^r} + \frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0 \quad (3)$$

$$\text{in the first term we get} \quad \frac{\partial x^{,q}}{\partial x^r} \frac{\partial x^r}{\partial x^{,n}} = \frac{\partial x^{,q}}{\partial x^{,n}} = \delta_n^q \quad (4)$$

(3) becomes

$$\frac{\partial^2 x^{,n}}{\partial x^p \partial x^s} \delta_n^q + \frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0 \quad (5)$$

$$\Leftrightarrow \frac{\partial^2 x^{,q}}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x^{,m} \partial x^{,n}} \frac{\partial x^{,m}}{\partial x^p} \frac{\partial x^{,n}}{\partial x^s} \frac{\partial x^{,q}}{\partial x^r} = 0 \quad (6)$$



1.6 p9-exercise

If A_s^r are the elements of a determinant A, and B_s^r the elements of a determinant B, show that the element of the product determinant is $A_n^r B_s^n$. Hence show that the product of the two jacobians

$$J = \left| \frac{\partial x^r}{\partial x^s} \right|, \quad J' = \left| \frac{\partial x'^r}{\partial x^s} \right|$$

is unity.

Remark: Some nitpick about the formulation: A_s^r are not the elements of a determinant A, but elements of the matrix A which gives $\det\{A\}$ provided that A is square (which is not explicitly mentioned.). The same remark for B and $A_n^r B_s^n$.

Be A_k^i the elements of matrix A and B_j^k the elements of matrix B and $C = A.B$ the resulting matrix of the multiplication of A and B, then

$$C_j^i = A_k^i B_j^k$$

are the elements of matrix C. Now, put $A_k^i = \frac{\partial x^i}{\partial x'^k}$ and $B_j^k = \frac{\partial x'^k}{\partial x^j}$ then,

$$C_j^i = A_k^i B_j^k \tag{1}$$

$$= \frac{\partial x^i}{\partial x'^k} \frac{\partial x'^k}{\partial x^j} \tag{2}$$

$$= \delta_k^i \tag{3}$$

So $C = JJ'$ becomes the unity matrix.



1.7 p11-exercise

Show that a finite contravariant vector determines the ratios of the components of an infinitesimal displacement. (Consider the transformation of the equation $dx^r = \theta T^r$, where θ is an arbitrary factor which does not change under the transformation. Alternatively, show that the equations $T^r dx^s - T^s dx^r = 0$ remain true when we transform the coordinates.)

Be T^q a contravariant vector.

$$T^{,q} = T^r \frac{\partial x^{,q}}{\partial x^r} \quad (\text{by definition}) \quad (1)$$

Be θ a small infinitesimal factor invariant for a coordinate transformation, define

$$dx^r = \theta T^r \quad (2)$$

$$(3)$$

then

$$\frac{dx^r}{dx^s} = \frac{\theta T^r}{\theta T^s} \quad (4)$$

$$\Leftrightarrow T^s dx^r - T^r dx^s = 0 \quad (5)$$

Alternatively, multiply (5) with $\partial_{x^r} x^{,q}$, then

$$\frac{\partial x^{,q}}{\partial x^r} dx^r T^s - \frac{\partial x^{,q}}{\partial x^r} dx^s T^r = 0 \quad (6)$$

$$\Leftrightarrow \frac{\partial x^{,q}}{\partial x^r} dx^r T^s - dx^s T^{,q} = 0 \quad (\text{use (1) in the second term}) \quad (7)$$

$$\Leftrightarrow dx^{,q} T^s - dx^s T^{,q} = 0 \quad (8)$$

$$(9)$$

Multiply (8) with $\partial_{x^s} x^{,p}$, then

$$dx^{,q} T^s \partial_{x^s} x^{,p} - dx^s T^{,q} \partial_{x^s} x^{,p} = 0 \quad (10)$$

$$\Leftrightarrow T^{,p} dx^{,q} - T^{,q} dx^{,p} = 0 \quad (\text{use (1) in the first term}) \quad (11)$$

and thus

$$\frac{dx^{,q}}{dx^{,p}} = \frac{T^{,q}}{T^{,p}}$$



1.8 p12-exercise

Write down the equation of transformation, analogous to 1.305, of a contravariant tensor of the third order. Solve the equation so as to express the unprimed components in terms of the primed components.

Be

$$T^{,uvw} = T^{rst} \frac{\partial x^{,u}}{\partial x^r} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (\text{by definition}) \quad (1)$$

a contravariant vector.

Multiply (1) by $\frac{\partial x^n}{\partial x^{,u}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{rst} \frac{\partial x^{,u}}{\partial x^r} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (2)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{rst} \delta_r^n \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (3)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{nst} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (4)$$

Multiply (4) by $\frac{\partial x^m}{\partial x^{,v}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nst} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^{,w}}{\partial x^t} \quad (5)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nst} \delta_s^m \frac{\partial x^{,w}}{\partial x^t} \quad (6)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nmt} \frac{\partial x^{,w}}{\partial x^t} \quad (7)$$

Multiply (7) by $\frac{\partial x^p}{\partial x^{,w}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmt} \frac{\partial x^{,w}}{\partial x^t} \frac{\partial x^p}{\partial x^{,w}} \quad (8)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmt} \delta_t^p \quad (9)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmp} \quad (10)$$

Giving

$$T^{nmp} = T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}}$$



1.9 p14-exercise

For a transformation from on set of rectangular Cartesian coordinates to another in Euclidean 3-space, show that the law of transformation of a contravariant vector is precisely the same as that of a covariant vector. Can this statements be extended to cover tensor of higher orders?

We have to prove that, given that,

$$T^{,i} = T^j \frac{\partial x^{,i}}{\partial x^j} \quad T_i = T_j \frac{\partial x^j}{\partial x^{,i}}$$

that also

$$T^{,i} = T^j \frac{\partial x^j}{\partial x^{,i}} \quad T_i = T_j \frac{\partial x^{,i}}{\partial x^j} \quad (1)$$

$$\Leftrightarrow \frac{\partial x^j}{\partial x^{,i}} = \frac{\partial x^{,i}}{\partial x^j} \quad (2)$$

Be

$$\hat{e}^{,i} = g_k^i \hat{e}^k \quad \text{and} \quad \hat{e}^i = h_k^i \hat{e}^{,k} \quad (3)$$

the transformation rules from one set of (rectangular Cartesian) basis vectors to another set of (rectangular Cartesian) basis vectors. Then,

$$\langle \hat{e}^{,i}, \hat{e}^{,j} \rangle = \langle g_k^i \hat{e}^k, g_k^j \hat{e}^k \rangle \quad \text{and} \quad \langle \hat{e}^i, \hat{e}^j \rangle = \langle h_k^i \hat{e}^{,k}, h_k^j \hat{e}^{,k} \rangle \quad (4)$$

$$\Leftrightarrow \delta_j^p = g_k^p g_k^j \quad \text{and} \quad \delta_j^p = h_k^p h_k^j \quad (5)$$

$$(6)$$

Be \vec{v} a random vector in the Euclidean space,

$$\vec{v} = x^j \hat{e}^j = x^{,j} \hat{e}^{,j} \quad (7)$$

then

$$(3) \Rightarrow x^j \hat{e}^j = x^j h_k^j \hat{e}^{,k} \quad \text{and} \quad x^{,j} \hat{e}^{,j} = x^{,j} g_k^j \hat{e}^k \quad (8)$$

$$\Rightarrow x^{,j} = x^m h_j^m \quad \text{and} \quad x^m = x^{,j} g_m^j \quad (9)$$

$$\Rightarrow x^{,j} = x^{,i} g_m^i h_j^m \quad \text{and} \quad x^m = x^k h_j^k g_m^j \quad (10)$$

$$\Rightarrow \delta_j^p = g_k^p h_j^k \quad \text{and} \quad \delta_j^p = g_j^k h_k^p \quad (11)$$

$$(5) \Rightarrow g_k^p g_k^j = g_k^p h_j^k \quad \text{and} \quad h_k^p h_k^j = g_j^k h_k^p \quad (12)$$

$$\Rightarrow g_k^j = h_j^k \quad \text{and} \quad h_k^j = g_j^k \quad (13)$$

From (9)

$$x^j = x^{,m} g_j^m \text{ and } x^{,k} = x^n h_k^n \quad (14)$$

$$\Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^n}{\partial x^j} h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \frac{\partial x^{,m}}{\partial x^{,k}} g_j^m \quad (15)$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = \delta_j^n h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \delta_k^m g_j^m \quad (16)$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = h_k^j \text{ and } \frac{\partial x^j}{\partial x^{,k}} = g_j^k \quad (17)$$

$$(13) \Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^j}{\partial x^{,k}} \quad (18)$$

So (13) matches (2), proving the assertion.

Can this statements be extended to cover tensor of higher orders? Consider

$$T^{i,j,\dots,n} = T^{r,s,\dots,w} \frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \dots \frac{\partial x^{,n}}{\partial x^w} \text{ and } T^{r,s,\dots,w} = T^{i,j,\dots,n} \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \dots \frac{\partial x^w}{\partial x^{,n}}$$

Using the same reasoning as in (1) to (2) we need

$$\frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \dots \frac{\partial x^{,n}}{\partial x^w} = \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \dots \frac{\partial x^w}{\partial x^{,n}}$$

As the conclusion (18) is independent of the order of the tensor, it is obvious that the above equality yields. Hence, the answer is YES.



1.10 p16-exercise

In a space of 4 dimensions, the tensor A_{rst} is skew-symmetric in the last pair of suffixes. Show that only 24 of the 64 components may be chosen arbitrarily. If the further condition $A_{rst} + A_{str} + A_{trs} = 0$ is imposed, show that that only 20 components may be chosen arbitrarily.

We have, as A is skew-symmetric in the last pair of suffixes

$$A_{rst} = -A_{rts} \Rightarrow s = t: A_{rst} = 0$$

So, for each r (4 possible choices as $N = 4$) we have $4 \times 4 / 2 - 4 = 6$ degrees of freedom. [we have the term $4 \times 4 / 2$ as the tensor is (skew-)symmetric, e.g. once we choose element a_{12} , then a_{21} is also known. The term -4 takes into account the diagonal element which are 0 and thus cannot be chosen.] So, we have $4 \times 6 = 24$ degrees of freedom.

What about the supplementary constraint $A_{rst} + A_{str} + A_{trs} = 0$:

Consider the two possible excluding cases:

$$i) r = s \neq t \quad (\Leftrightarrow r = t \neq s)$$

This case gives - without the additional constraint (1) - $4 \times (4 \times 3 / 2 - 4) = 8$ degrees of freedom. Does the constraint (1) reduces this degree of freedom?

We have,

$$A_{rst} + A_{str} + A_{trs} = 0 \quad (1)$$

$$\Rightarrow \underbrace{A_{rrt} + A_{rtt}}_{= 0 \text{ (non-diagonal terms)}} + \underbrace{A_{trr}}_{= 0 \text{ (diagonal terms)}} = 0 \quad (2)$$

So, no additional constraints are added by (1) to the restriction i) and the DOF remains 8.

$$ii) t \neq r \neq s \neq t$$

This case means that we have to choose a set of 3 elements out of 4 elements without repetition. This a *variation* of 3 elements out of 4.

$$V_k^n = \frac{n!}{(n-k)!} \text{ giving } V_3^4 = \frac{4!}{(4-3)!} = 24$$

The constraint (1) gives us 24 equations but as $A_{rst} = -A_{rts}$ only 12 equations have to be considered. So, with the additional constraints the DOF becomes $24 - 12 = 12$.

As i) and ii) are independent and excluding events we can add the DOF of both events and we get $8 + 12 = 20$ DOF.



1.11 p16-exercise

If A^{rs} is skew-symmetric and B_{rs} is symmetric, prove that $A^{rs}B_{rs} = 0$. Hence show that the quadratic form $a_{ij}x^i x^j$ is unchanged if a_{ij} is replaced by its symmetric part.

We can split the summation $A^{rs}B_{rs}$ in three subsummations:

$$A^{rs}B_{rs} = A^{rs}B_{rs}|_{r=s} \tag{1}$$

$$+ A^{rs}B_{rs}|_{r>s} \tag{2}$$

$$+ A^{rs}B_{rs}|_{r<s} \tag{3}$$

We have:

$$(1) = 0 \text{ as } A^{kk} = 0 \text{ (skew-symmetric)}$$

$$(2)+(3) = A^{rs}B_{rs}|_{r>s} + A^{rs}B_{rs}|_{r<s}$$

As $A^{rs} = -A^{sr}$ and $B^{rs} = B^{sr}$ we can write (2)+(3) as :

$$A^{rs}B_{rs}|_{r>s} + (-A^{sr})B_{sr}|_{r>s} = 0$$

So, $A^{rs}B_{rs} = 0$

Consider the quadratic form $\phi = a_{ij}x^i x^j$

Be $A_{ij} = (a_{ij})$ and $B_{ij} = (x^i x^j)$, then it is obvious that B_{ij} is symmetric and that $C_{ij} = -A_{ij}$ is the form where $-a_{ij}$ is replaced by its symmetric part (skew-symmetric). Hence $\phi = a_{ij}x^i x^j = a_{ij}b^{ij} = 0$ and so is $\phi = c_{ij}b^{ij} = 0$



1.12 p18-exercise

What are the values (in a space of N dimensions) of the following contractions formed from the Kronecker delta?

$$\delta_m^m, \delta_n^m \delta_m^n, \delta_n^m \delta_r^n \delta_m^r$$

We can split the summation $A^{rs} B_{rs}$ in three subsummations:

$$\delta_m^m = N \tag{1}$$

$$\delta_n^m \delta_m^n = \delta_m^m = N \tag{2}$$

$$\delta_n^m \delta_r^n \delta_m^r = \delta_n^m \delta_m^n = \delta_m^m = N \tag{3}$$



1.13 p19-exercise

If X^r, Y^r are arbitrary contravariant vectors and $a_{rs}X^rY^s$ is an invariant, then a_{rs} are the components of a covariant tensor of the second order.

We have to prove that

$$a'_{rs} = a_{ij} \frac{\partial x^i}{\partial x'^r} \frac{\partial x^j}{\partial x'^s} \text{ or } a_{ij} = a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} \quad (1)$$

$a_{rs}X^rY^s$ is an invariant, means

$$a'_{rs}X'^rY'^s = a_{rs}X^rY^s \quad (2)$$

As X^r, Y^r are arbitrary contravariant vectors, we have

$$X'^r = X^i \frac{\partial x'^r}{\partial x^i} \text{ and } Y'^s = Y^j \frac{\partial x'^s}{\partial x^j} \quad (3)$$

(3) in (2) gives

$$a'_{rs}X^i \frac{\partial x'^r}{\partial x^i} Y^j \frac{\partial x'^s}{\partial x^j} = a_{rs}X^rY^s \quad (4)$$

$$\Leftrightarrow a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} X^i Y^j = a_{ij} X^i Y^j \quad (5)$$

$$\Leftrightarrow (a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} - a_{ij}) X^i Y^j = 0 \quad (6)$$

As X^r, Y^r are arbitrary contravariant vectors, we conclude that

$$a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} - a_{ij} = 0 \quad (7)$$

$$\Leftrightarrow a_{ij} = a'_{rs} \frac{\partial x'^r}{\partial x^i} \frac{\partial x'^s}{\partial x^j} \quad (8)$$

(8) = (1): OK



1.14 p19-exercise

If X_{rs} is an arbitrary covariant tensor of the second order, and $A_r^{mn} X_{mn}$ is a covariant vector, then A_r^{mn} has the mixed tensor character indicated by the positions of its suffixes

We have to prove that

$$A_r^{vw} = A_k^{mn} \frac{\partial x^k}{\partial x^{,r}} \frac{\partial x^{,v}}{\partial x^m} \frac{\partial x^{,w}}{\partial x^n} \quad (1)$$

We have

$$P_r = A_r^{mn} X_{mn} \quad (2)$$

is a covariant vector

$$\Rightarrow P_r^{,} = A_k^{mn} X_{mn} \frac{\partial x^k}{\partial x^{,r}} \quad (3)$$

but X_{mn} is a covariant tensor

$$\Rightarrow X_{mn} = X_{ps} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \quad (4)$$

So (4) in (3) gives

$$P_r^{,} = A_k^{mn} X_{ps} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}} \quad (5)$$

$$\Leftrightarrow P_r^{,} = \underbrace{A_k^{mn} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}}_{(*)} X_{ps} \quad (6)$$

Putting (*) as $A_r^{ps} = A_k^{mn} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}$ we see that (6) has the form (2) and that A_r^{ps} obeys the rule of a mixed tensor (1).



1.15 p21-exercise

If A_{rs} is a skew-symmetric covariant tensor, prove that B_{rst} defined as

$$B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$$

is a covariant tensor, and that it is skew-symmetric in all pairs of suffixes.

We have A_{rs} is a covariant tensor

$$A_{ij} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^i} \frac{\partial x^\beta}{\partial x^j} \quad (1)$$

$$\Rightarrow B_{rst} = \partial_r (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}) + \partial_s (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r}) + \partial_t (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s}) \quad (2)$$

Note that

$$\partial_k (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}) = \partial_k (A_{\alpha\beta}) \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \partial_k (\frac{\partial x^\alpha}{\partial x^s}) \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \partial_k (\frac{\partial x^\beta}{\partial x^t}) \quad (3)$$

$$(4)$$

so,

$$\begin{aligned} B_{rst} &= \partial_r A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \underbrace{A_{\alpha\beta} \partial_r \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}}_{*} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \partial_r \frac{\partial x^\beta}{\partial x^t}}_{**} \\ &\quad + \partial_s A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \underbrace{A_{\alpha\beta} \partial_s \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r}}_{***} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \partial_s \frac{\partial x^\beta}{\partial x^r}}_{*} \\ &\quad + \partial_t A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} + \underbrace{A_{\alpha\beta} \partial_t \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s}}_{**} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \partial_t \frac{\partial x^\beta}{\partial x^s}}_{***} \end{aligned} \quad (5)$$

In (5) consider the two terms with (*)

$$T = A_{\alpha\beta} \partial_r \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \partial_s \frac{\partial x^\beta}{\partial x^r} \quad (6)$$

$$= A_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x^s \partial x^r} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial^2 x^\beta}{\partial x^r \partial x^s} \quad (7)$$

$$= A_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x^s \partial x^r} \frac{\partial x^\beta}{\partial x^t} + A_{\beta\alpha} \frac{\partial x^\beta}{\partial x^t} \frac{\partial^2 x^\alpha}{\partial x^r \partial x^s} \text{ (by renaming dummy variables)} \quad (8)$$

As $A_{ij} = -A_{ji}$ (skew-symmetric tensor), we get $T = 0$. The same yields for the (**) and (***) terms. So, B_{rst} reduces to

$$B_{rst} = \partial_r A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \partial_s A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \partial_t A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \quad (9)$$

$$\Leftrightarrow B_{rst} = \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^r} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^s} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^t} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \quad (10)$$

By adequate renaming of the dummy variable in the 3 terms:

$$\begin{bmatrix} 1^{st} term \\ 2^{nd} term \\ 3^{rd} term \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma \rightarrow \alpha & \alpha \rightarrow \beta & \beta \rightarrow \gamma \\ \beta \rightarrow \alpha & \gamma \rightarrow \beta & \alpha \rightarrow \gamma \\ \alpha \rightarrow \alpha & \beta \rightarrow \beta & \gamma \rightarrow \gamma \end{bmatrix}$$

we get

$$B_{rst} = \left(\frac{\partial A_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial A_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \right) \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \frac{\partial x^\gamma}{\partial x^t} \quad (11)$$

$$\Leftrightarrow B_{rst} = \underbrace{(\partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta})}_{(****)} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \frac{\partial x^\gamma}{\partial x^t} \quad (12)$$

The expression (****) has exactly the required form $B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$ and is transformed (12) according the rules of a covariant tensor.

Let's prove now that it is skew-symmetric in all pairs of suffixes. We have to consider the following permutations

$$\begin{bmatrix} rst \\ rts \\ srt \\ str \\ trs \\ tsr \end{bmatrix}$$

E.g. *srt*

$$B_{rts} = \partial_r A_{ts} + \partial_t A_{sr} + \partial_s A_{rt} \quad (13)$$

$$= -\partial_r A_{st} - \partial_t A_{rs} - \partial_s A_{tr} \quad (14)$$

$$= -B_{rst} \quad (15)$$

The same calculations can be done for the other permutations.



1.16 p23-exercise 1.

In a V_4 there are two 2-spaces with equations

$$x^r = f^r(u^1, u^2), \quad x^r = g^r(u^3, u^4)$$

Prove that if these 2-spaces have a curve of intersection, then the determinantal equation

$$\left| \frac{\partial x^r}{\partial u^s} \right| = 0$$

is satisfied along the curve.

Having a curve means that one of the parameters u^i can be freely chosen while the other 3 are determined by the chosen parameter.

We have,

$$\left| \frac{\partial x^r}{\partial u^s} \right| = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} & \frac{\partial x^1}{\partial u^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^4}{\partial u^1} & \frac{\partial x^4}{\partial u^2} & \frac{\partial x^4}{\partial u^3} & \frac{\partial x^4}{\partial u^4} \end{vmatrix} \quad (1)$$

Suppose we choose u^4 as parameter. This means $u^i = \phi^i(u^4)$ for $i=1,2,3$ and thus we can write

$$\frac{\partial x^i}{\partial u^4} = \frac{\partial x^i}{\partial u^j} \frac{d\phi^j}{du^4} + \frac{\partial x^i}{\partial u^4} \quad \text{with } j=1,2,3 \quad i = 1,2,3,4 \quad (2)$$

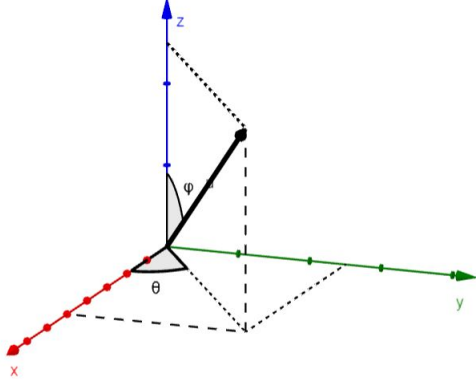
$$\Rightarrow \frac{\partial x^i}{\partial u^j} \frac{d\phi^j}{du^4} = 0 \quad (3)$$

This means that in (1) the three first columns are not linearly independent and thus have $\left| \frac{\partial x^r}{\partial u^s} \right| = 0$



1.17 p23-exercise 2.

In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates x, y, z and spherical polar coordinates r, θ, ϕ . Find the Jacobian of the transformation. Where is it zero or infinite?



We use the latitude ψ instead of the co-latitude ϕ .

$$\begin{cases} x = r \cos(\psi) \cos(\theta) \\ y = r \cos(\psi) \sin(\theta) \\ z = r \sin(\psi) \end{cases}$$

Partial differentiating of (x, y, z) with respect to (r, ψ, θ) gives the Jacobian

$$J = \begin{vmatrix} \cos(\psi) \cos(\theta) & -r \sin(\psi) \cos(\theta) & -r \cos(\psi) \sin(\theta) \\ \cos(\psi) \sin(\theta) & -r \sin(\psi) \sin(\theta) & r \cos(\psi) \cos(\theta) \\ \sin(\psi) & r \cos(\psi) & 0 \end{vmatrix} \quad (1)$$

$$J = \cos(\psi) \cos(\theta) (-r^2) \cos^2(\psi) \cos(\theta) \quad (2)$$

$$+ r \sin(\psi) \cos(\theta) (-r \cos(\psi) \cos(\theta) \sin(\psi)) \quad (3)$$

$$- r \cos(\psi) \sin(\theta) (r \cos^2(\psi) \sin(\theta) + r \sin^2(\psi) \sin(\theta)) \quad (4)$$

$$= -r^2 \cos^3(\psi) \cos^2(\theta) - r^2 \sin^2(\psi) \cos^2(\theta) \cos(\psi) - r^2 \cos(\psi) \sin^2(\theta) \quad (5)$$

Noting that the 2nd term in (5) can be written as $-r^2 \cos^2(\theta) \cos(\psi) + r^2 \cos^2(\theta) \cos^3(\psi)$, we get

$$J = -r^2 (\cos^3(\psi) \cos^2(\theta) + \cos^2(\theta) \cos(\psi) - \cos^3(\psi) \cos^2(\theta) + \cos(\psi) \sin^2(\theta)) \quad (6)$$

$$= -r^2 \cos(\psi) \quad (7)$$

$J=0$: for $r = 0$ or $\psi = \frac{\pi}{2} |_{r \in (-\infty, +\infty)}$ and $J \rightarrow \pm\infty$ or $\mp\infty$ for $r \rightarrow \pm\infty |_{\psi \neq 0}$. But what about the case $r \rightarrow \pm\infty |_{\psi \rightarrow 0}$? This case is not determined as long as no path is chosen in the (r, ψ) configuration space.



1.18 p23-exercise 3.

If X, Y, Z are the components of a contravariant vector for rectangular Cartesian coordinates in Euclidean 3-space, find its components for spherical polar coordinates.

Be x^α the components of a contravariant vector in spherical polar coordinates and x^i its components in rectangular Cartesian coordinates. As we have

$$\begin{aligned} x^\rho &= \sqrt{x^j x^j} \\ x^\theta &= \text{atan} \frac{x^2}{x^1} \\ x^\phi &= \text{asin} \frac{x^3}{\sqrt{x^j x^j}} \end{aligned} \quad \text{and} \quad A^\alpha = A^i \frac{\partial x^\alpha}{\partial x^i} \quad (1)$$

$$\Rightarrow [A^\alpha] = \left[A^i \frac{\partial x^\alpha}{\partial x^i} \right] = \begin{bmatrix} \frac{x^1}{\sqrt{x^j x^j}} & \frac{x^2}{\sqrt{x^j x^j}} & \frac{x^3}{\sqrt{x^j x^j}} \\ -\frac{x^2}{(x^1)^2 + (x^2)^2} & \frac{x^1}{(x^1)^2 + (x^2)^2} & 0 \\ -\frac{x^3 x^1}{(x^j x^j) \sqrt{(x^1)^2 + (x^2)^2}} & -\frac{x^3 x^2}{(x^j x^j) \sqrt{(x^1)^2 + (x^2)^2}} & \frac{\sqrt{(x^1)^2 + (x^2)^2}}{(x^j x^j)} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} \quad (2)$$

◆

1.19 p23-exercise 4.

In a space of three dimensions, how many different expressions are represented by the product $A_{np}^m B_{rs}^{pq} C_{tu}^s$? How many terms occur in each such expression, when written out explicitly?

As we have V_3 and considering that in $A_{np}^m B_{rs}^{pq} C_{tu}^s$ the six indices m, n, q, r, t, u are not dummy indices, we get 3^6 different expressions (first choose m : you have three choices, then n : also three choices giving 3×3 possibilities, etc for q, r, t, u).

For the second question, as in $A_{np}^m B_{rs}^{pq}$ there is only summation on over index (p) we get three terms for this part. As the summation with $A_{np}^m B_{rs}^{pq}$ and C_{tu}^s occurs only on one index also (s) we get 3×3 terms in the expression.



1.20 p23-exercise 5.

If A is an invariant in V_n , are the second derivatives $\frac{\partial^2 A}{\partial x^r \partial x^s}$ the components of a tensor?

As A is invariant (note: different alphabets in the indices indicates different coordinate systems):

$$A(x^\rho) = A(x^i) \quad (1)$$

$$\Rightarrow \frac{\partial A(x^\rho)}{\partial x^i} = \frac{\partial A(x^j)}{\partial x^i} \quad (2)$$

To simplify the notation, we put $A(x^\rho) = A'$ and $A(x^j) = A'$ then (2) can be written as

$$\frac{\partial A'}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^i} = \frac{\partial A}{\partial x^i} \quad (3)$$

Conclusion: $\frac{\partial A}{\partial x^i}$ is a covariant tensor.

Consider now $\frac{\partial A}{\partial x^i} = \frac{\partial A'}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^i}$. Then,

$$\frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A'}{\partial x^\rho \partial x^j} \frac{\partial x^\rho}{\partial x^i} + \frac{\partial A'}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \quad (4)$$

$$\Leftrightarrow \frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A'}{\partial x^\rho \partial x^\gamma} \frac{\partial x^\gamma}{\partial x^j} \frac{\partial x^\rho}{\partial x^i} + \frac{\partial A'}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \quad (5)$$

The first term on the right side, behaves as covariant tensor but the presence of the second term makes that generally, $\frac{\partial^2 A}{\partial x^i \partial x^j}$ has not a tensor character. This is only when $\frac{\partial A'}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} = 0$, which means that x^ρ, x^i are a linear map of each other.



1.21 p23-exercise 6.

Suppose that in V_2 the components of a contravariant tensor field T^{mn} in a coordinate system x^r are

$$T^{11} = 1 \quad T^{12} = 0$$

$$T^{21} = 1 \quad T^{22} = 0$$

Find the components T^{mn} in a coordinate system x'^r , where

$$x'^1 = (x^1)^2 \quad x'^2 = (x^2)^2$$

Write down the values of these components in particular at the point $x^1 = 1, x^2 = 0$.

As we have a contravariant tensor field :

$$T'^{mn} = T^{ij} \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^n}{\partial x^j} \quad (1)$$

$$\begin{aligned} x'^1 = (x^1)^2 &\Rightarrow \frac{\partial x'^1}{\partial x^1} = 2x^1 & \frac{\partial x'^1}{\partial x^2} &= 0 \\ x'^2 = (x^2)^2 &\Rightarrow \frac{\partial x'^2}{\partial x^1} &= 0 & \frac{\partial x'^2}{\partial x^2} = 2x^2 \end{aligned} \quad (2)$$

$$(3)$$

$$\Rightarrow T'^{11} = 4(x^1)^2 + 4(x^2)^2 \quad (4)$$

$$\Rightarrow T'^{12} = T'^{21} = 0 \quad (5)$$

$$\Rightarrow T'^{22} = 4(x^1)^2 + 4(x^2)^2 \quad (6)$$

The components in at the point $x^1 = 1, x^2 = 0$ are

$$T'(1, 0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

