

Tensor Calculus  
J.L. Synge and A.Schild (Dover Publication)  
Solutions to exercises

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## Remarks and warnings

### Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

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# Spaces and Tensors

## 1.1 p5-exercise

The parametric equations of a hypersurface in  $V_n$  are

$$\begin{aligned} x^1 &= a \cos(u^1) \\ x^2 &= a \sin(u^1) \cos(u^2) \\ x^3 &= a \sin(u^1) \sin(u^2) \cos(u^3) \\ &\vdots \\ x^{N-1} &= a \sin(u^1) \sin(u^2) \sin(u^3) \dots \sin(u^{N-2}) \cos(u^{N-1}) \\ x^N &= a \sin(u^1) \sin(u^2) \sin(u^3) \dots \sin(u^{N-2}) \sin(u^{N-1}) \end{aligned}$$

where  $a$  is a constant. Find the single equation of the hyperspace in the form 1.103.

We have:

$$\begin{aligned} (x^N)^2 + (x^{N-1})^2 &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i) (\cos^2(u^{N-1}) + \sin^2(u^{N-1})) \\ &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) \sin^2(u^{N-2}) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) (1 - \cos^2(u^{N-2})) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - a^2 \prod_{i=1}^{N-3} \sin^2(u^i) \cos^2(u^{N-2}) \\ &= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - (x^{N-2})^2 \end{aligned}$$

giving

$$(x^N)^2 + (x^{N-1})^2 + (x^{N-2})^2 = a^2 \prod_{i=1}^{N-3} \sin^2(u^i)$$

In general, by recursion

$$\sum_{i=0}^k (x^{N-i})^2 = a^2 \prod_{i=1}^{N-k-1} \sin^2(u^i) \quad (k \leq N-2)$$

be  $k = N - 2$  ( $N - k - 1 = 1$ ) and in the left term put  $j = N - i$  ( $j$  goes from 2 to  $N$ ), we get

$$\begin{aligned}\sum_{j=2}^N (x^j)^2 &= a^2 \prod_{i=1}^1 \sin^2(u^i) \\ &= a^2(1 - \cos^2(u^1)) \\ &= a^2 - (x^1)^2\end{aligned}$$

and thus the equation of the hyperspace is given by

$$\sum_{j=1}^N (x^j)^2 - a^2 = 0$$

Determine whether the points  $(\frac{1}{2}a, 0, 0, \dots, 0)$ ,  $(0, 0, \dots, 0, 2a)$  lie on the same or opposite sides of the hyperspace.

For  $(\frac{1}{2}a, 0, 0, \dots, 0)$  we have  $\sum_{j=1}^N (x^j)^2 - a^2 = -\frac{3a^2}{4} < 0$  and for  $(0, 0, \dots, 0, 2a)$  we have  $\sum_{j=1}^N (x^j)^2 - a^2 = \frac{3a^2}{4} > 0$ .

So the points lie on opposite sides of the hyperplane.





## 1.2 p6-exercise

Let  $U_2$  and  $W_2$  be subspaces of  $V_N$ . Show that if  $N = 3$  they will in general intersect in a curve; if  $N = 4$  they will in general intersect in a finite number of points; and if  $N > 4$  they will not in general intersect at all.

We have (see 1.102 page 5):  $x^r = f^r(u^1, u^2, \dots, u^M) \quad (r = 1, 2, \dots, N)$

Case  $N=3$ :

For  $U_2$  we have:

$$x^r = \phi^r(u^1, u^2) \quad (r = 1, 2, 3)$$

For  $W_2$  we have:

$$x^r = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

The intersect of the two hyperplanes is given by the  $N$  equations:

$$\phi^r(u^1, u^2) = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

So we have 3 equations in 4 unknown  $u^1, u^2, v^1, v^2$  and can choose (fix) one e.g.  $u^1$  and solve the set of equations for  $u^2, v^1, v^2$  giving

$$x^r = \theta^r(u^1) \quad (r = 1, 2, 3)$$

This is an equation of a curve in space (1 parameter equation)

Case  $N=4$ :

Using the same reasoning as with  $N=3$ , we get 4 equations for 4 unknown  $u^1, u^2, v^1, v^2$ .

Provided that the set of equation does not degenerate, these 4 equations will determine  $u^1, u^2, v^1, v^2$  without any degree of freedom. So we get points as solutions. This solution does not to be unique, e.g. if the  $\phi^r(u^1, u^2)$  are quadratic form, then the solutions

$$(u^1, u^2, v^1, v^2)$$

$$(-u^1, u^2, v^1, v^2)$$

$$(u^1, -u^2, v^1, v^2)$$

$$(-u^1, -u^2, v^1, v^2)$$

are possible.

Case  $N=5$ : There are more equations than variables. If the equations are not linear dependent, no solutions will be found.



### 1.3 p8-exercise

Show that  $(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = 3a_{rst}x^r x^s x^t$

$(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = a_{rst}x^r x^s x^t + a_{rts}x^r x^s x^t + a_{srt}x^r x^s x^t$  so by just renaming the dummy indices e.g. for the second term  $r \mapsto s$  ,  $s \mapsto t$  and  $t \mapsto r$  we get the desired result.



## 1.4 p8-exercise

If  $\phi = a_{rs}x^r x^s$ , show that

$$\frac{\partial \phi}{\partial x^r} = (a_{rs} + a_{sr})x^s \quad , \quad \frac{\partial^2 \phi}{\partial x^r \partial x^s} = a_{rs} + a_{sr}$$

Simplify these expressions in the case where  $a^{rs} = a^{sr}$

We have

$$\frac{\partial \phi}{\partial x^t} = \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs} \frac{\partial x^r}{\partial x^t} x^s + a_{rs} x^r \frac{\partial x^s}{\partial x^t} \quad (1)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs} \delta_t^r x^s + a_{rs} x^r \delta_t^s \quad (2)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts} x^s + a_{rt} x^r \quad (3)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts} x^s + a_{st} x^s \quad (\text{rename dummy variable in third term}) \quad (4)$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + (a_{ts} + a_{st})x^s \quad (5)$$

Replace  $x^t$  by  $x^r$ , we get

$$\frac{\partial \phi}{\partial x^r} = \frac{\partial a_{rs}}{\partial x^r} x^r x^s + (a_{rs} + a_{sr})x^s \quad (6)$$

So the asked expression is only true if  $a_{rs}$  is not a function of the  $x^s$ . Assuming that  $a_{rs}$  is not a function of the  $x^s$ , take the partial derivative of (6) with respect to  $x^t$ , we get

$$\frac{\partial^2 \phi}{\partial x^r \partial x^t} = (a_{rs} + a_{sr}) \frac{\partial x^s}{\partial x^t} \quad (7)$$

$$= (a_{rs} + a_{sr}) \delta_t^s \quad (8)$$

$$= (a_{rt} + a_{tr}) \quad (9)$$

Replace  $x^t$  by  $x^s$ , and we get the proposed expression.



## 1.5 p8-clarification on expression 1.210

$$\frac{\partial^2 x^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial x^p} \frac{\partial x^n}{\partial x^s} \frac{\partial x^q}{\partial x^r} = 0$$

From 1.209:

$$\frac{\partial^2 x^r}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial x^p} \frac{\partial x^n}{\partial x^s} + \frac{\partial x^r}{\partial x^n} \frac{\partial^2 x^n}{\partial x^p \partial x^s} = 0 \quad (1)$$

multiply (1) with  $\frac{\partial x^q}{\partial x^r}$

$$\frac{\partial^2 x^r}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial x^p} \frac{\partial x^n}{\partial x^s} \frac{\partial x^q}{\partial x^r} + \frac{\partial x^r}{\partial x^n} \frac{\partial^2 x^n}{\partial x^p \partial x^s} \frac{\partial x^q}{\partial x^r} = 0 \quad (2)$$

$$\Leftrightarrow \frac{\partial x^r}{\partial x^n} \frac{\partial^2 x^n}{\partial x^p \partial x^s} \frac{\partial x^q}{\partial x^r} + \frac{\partial^2 x^r}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial x^p} \frac{\partial x^n}{\partial x^s} \frac{\partial x^q}{\partial x^r} = 0 \quad (3)$$

$$\text{in the first term we get} \quad \frac{\partial x^q}{\partial x^r} \frac{\partial x^r}{\partial x^n} = \frac{\partial x^q}{\partial x^n} = \delta_n^q \quad (4)$$

(3) becomes

$$\frac{\partial^2 x^n}{\partial x^p \partial x^s} \delta_n^q + \frac{\partial^2 x^r}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial x^p} \frac{\partial x^n}{\partial x^s} \frac{\partial x^q}{\partial x^r} = 0 \quad (5)$$

$$\Leftrightarrow \frac{\partial^2 x^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial x^p} \frac{\partial x^n}{\partial x^s} \frac{\partial x^q}{\partial x^r} = 0 \quad (6)$$



## 1.6 p9-exercise

If  $A_s^r$  are the elements of a determinant A, and  $B_s^r$  the elements of a determinant B, show that the element of the product determinant is  $A_n^r B_s^n$ . Hence show that the product of the two jacobians

$$J = \left| \frac{\partial x^r}{\partial x'^s} \right|, \quad J' = \left| \frac{\partial x'^r}{\partial x^s} \right|$$

is unity.

Remark: Some nitpick about the formulation:  $A_s^r$  are not the elements of a determinant A, but elements of the matrix A which gives  $\det\{A\}$  provided that A is square (which is not explicitly mentioned.). The same remark for B and  $A_n^r B_s^n$ .

Be  $A_k^i$  the elements of matrix A and  $B_j^k$  the elements of matrix B and  $C = A.B$  the resulting matrix of the multiplication of A and B, then

$$C_j^i = A_k^i B_j^k$$

are the elements of matrix C. Now, put  $A_k^i = \frac{\partial x^i}{\partial x'^k}$  and  $B_j^k = \frac{\partial x'^k}{\partial x^j}$  then,

$$C_j^i = A_k^i B_j^k \tag{1}$$

$$= \frac{\partial x^i}{\partial x'^k} \frac{\partial x'^k}{\partial x^j} \tag{2}$$

$$= \delta_k^i \tag{3}$$

So  $C = JJ'$  becomes the unity matrix.



## 1.7 p11-exercise

Show that a finite contravariant vector determines the ratios of the components of an infinitesimal displacement. (Consider the transformation of the equation  $dx^r = \theta T^r$ , where  $\theta$  is an arbitrary factor which does not change under the transformation. Alternatively, show that the equations  $T^r dx^s - T^s x^r = 0$  remain true when we transform the coordinates.)

Be  $T^q$  a contravariant vector.

$$T^q = T^r \frac{\partial x^q}{\partial x^r} \quad (\text{by definition}) \quad (1)$$

Be  $\theta$  a small infinitesimal factor invariant for a coordinate transformation, define

$$dx^r = \theta T^r \quad (2)$$

$$(3)$$

then

$$\frac{dx^r}{dx^s} = \frac{\theta T^r}{\theta T^s} \quad (4)$$

$$\Leftrightarrow T^s dx^r - T^r dx^s = 0 \quad (5)$$

Alternatively, multiply (5) with  $\partial_{x^r} x^q$ , then

$$\frac{\partial x^q}{\partial x^r} dx^r T^s - \frac{\partial x^q}{\partial x^r} dx^s T^r = 0 \quad (6)$$

$$\Leftrightarrow \frac{\partial x^q}{\partial x^r} dx^r T^s - dx^s T^q = 0 \quad (\text{use (1) in the second term}) \quad (7)$$

$$\Leftrightarrow dx^q T^s - dx^s T^q = 0 \quad (8)$$

$$(9)$$

Multiply (8) with  $\partial_{x^s} x^p$ , then

$$dx^q T^s \partial_{x^s} x^p - dx^s T^q \partial_{x^s} x^p = 0 \quad (10)$$

$$\Leftrightarrow T^p dx^q - T^q dx^p = 0 \quad (\text{use (1) in the first term}) \quad (11)$$

and thus

$$\frac{dx^q}{dx^p} = \frac{T^q}{T^p}$$



## 1.8 p12-exercise

Write down the equation of transformation, analogous to 1.305, of a contravariant tensor of the third order. Solve the equation so as to express the unprimed components in terms of the primed components.

Be

$$T^{,uvw} = T^{rst} \frac{\partial x^{,u}}{\partial x^r} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (\text{by definition}) \quad (1)$$

a contravariant vector.

Multiply (1) by  $\frac{\partial x^n}{\partial x^{,u}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{rst} \frac{\partial x^{,u}}{\partial x^r} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (2)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{rst} \delta_r^n \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (3)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{nst} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad (4)$$

Multiply (4) by  $\frac{\partial x^m}{\partial x^{,v}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nst} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^{,w}}{\partial x^t} \quad (5)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nst} \delta_s^m \frac{\partial x^{,w}}{\partial x^t} \quad (6)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nmt} \frac{\partial x^{,w}}{\partial x^t} \quad (7)$$

Multiply (7) by  $\frac{\partial x^p}{\partial x^{,w}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmt} \frac{\partial x^{,w}}{\partial x^t} \frac{\partial x^p}{\partial x^{,w}} \quad (8)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmt} \delta_t^p \quad (9)$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmp} \quad (10)$$

Giving

$$T^{nmp} = T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}}$$



## 1.9 p14-exercise

For a transformation from on set of rectangular Cartesian coordinates to another in Euclidean 3-space, show that the law of transformation of a contravariant vector is precisely the same as that of a covariant vector. Can this statements be extended to cover tensor of higher orders?

We have to prove that, given that,

$$T^{,i} = T^j \frac{\partial x^{,i}}{\partial x^j} \quad T_i = T_j \frac{\partial x^j}{\partial x^{,i}}$$

that also

$$T^{,i} = T^j \frac{\partial x^j}{\partial x^{,i}} \quad T_i = T_j \frac{\partial x^{,i}}{\partial x^j} \quad (1)$$

$$\Leftrightarrow \frac{\partial x^j}{\partial x^{,i}} = \frac{\partial x^{,i}}{\partial x^j} \quad (2)$$

Be

$$\hat{e}^{,i} = g_k^i \hat{e}^k \quad \text{and} \quad \hat{e}^i = h_k^i \hat{e}^{,k} \quad (3)$$

the transformation rules from one set of (rectangular Cartesian) basis vectors to another set of (rectangular Cartesian) basis vectors. Then,

$$\langle \hat{e}^{,i}, \hat{e}^{,j} \rangle = \langle g_k^i \hat{e}^k, g_k^j \hat{e}^k \rangle \quad \text{and} \quad \langle \hat{e}^i, \hat{e}^j \rangle = \langle h_k^i \hat{e}^{,k}, h_k^j \hat{e}^{,k} \rangle \quad (4)$$

$$\Leftrightarrow \delta_j^p = g_k^p g_k^j \quad \text{and} \quad \delta_j^p = h_k^p h_k^j \quad (5)$$

$$(6)$$

Be  $\vec{v}$  a random vector in the Euclidean space,

$$\vec{v} = x^j \hat{e}^j = x^{,j} \hat{e}^{,j} \quad (7)$$

then

$$(3) \Rightarrow x^j \hat{e}^j = x^j h_k^j \hat{e}^{,k} \quad \text{and} \quad x^{,j} \hat{e}^{,j} = x^{,j} g_k^j \hat{e}^k \quad (8)$$

$$\Rightarrow x^{,j} = x^m h_j^m \quad \text{and} \quad x^m = x^{,j} g_m^j \quad (9)$$

$$\Rightarrow x^{,j} = x^{,i} g_m^i h_j^m \quad \text{and} \quad x^m = x^k h_j^k g_m^j \quad (10)$$

$$\Rightarrow \delta_j^p = g_k^p h_j^k \quad \text{and} \quad \delta_j^p = g_j^k h_k^p \quad (11)$$

$$(5) \Rightarrow g_k^p g_k^j = g_k^p h_j^k \quad \text{and} \quad h_k^p h_k^j = g_j^k h_k^p \quad (12)$$

$$\Rightarrow g_k^j = h_j^k \quad \text{and} \quad h_k^j = g_j^k \quad (13)$$



From (9)

$$x^j = x^m g_j^m \text{ and } x^{,k} = x^n h_k^n \quad (14)$$

$$\Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^n}{\partial x^j} h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \frac{\partial x^m}{\partial x^{,k}} g_j^m \quad (15)$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = \delta_j^n h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \delta_k^m g_j^m \quad (16)$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = h_k^j \text{ and } \frac{\partial x^j}{\partial x^{,k}} = g_j^k \quad (17)$$

$$(13) \Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^j}{\partial x^{,k}} \quad (18)$$

So (13) matches (2), proving the assertion.

Can this statements be extended to cover tensor of higher orders? Consider

$$T^{,i,j,\dots,n} = T^{r,s,\dots,w} \frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \dots \frac{\partial x^{,n}}{\partial x^w} \text{ and } T^{r,s,\dots,w} = T^{,i,j,\dots,n} \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \dots \frac{\partial x^w}{\partial x^{,n}}$$

Using the same reasoning as in (1) to (2) we need

$$\frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \dots \frac{\partial x^{,n}}{\partial x^w} = \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \dots \frac{\partial x^w}{\partial x^{,n}}$$

As the conclusion (18) is independent of the order of the tensor, it is obvious that the above equality yields. Hence, the answer is YES.



## 1.10 p16-exercise

In a space of 4 dimensions, the tensor  $A_{rst}$  is skew-symmetric in the last pair of suffixes. Show that only 24 of the 64 components may be chosen arbitrarily. If the further condition  $A_{rst} + A_{str} + A_{trs} = 0$  is imposed, show that that only 20 components may be chosen arbitrarily.

We have, as  $A$  is skew-symmetric in the last pair of suffixes

$$A_{rst} = -A_{rts} \Rightarrow s = t: A_{rst} = 0$$

So, for each  $r$  (4 possible choices as  $N = 4$ ) we have  $4 \times 4 / 2 - 4 = 6$  degrees of freedom. [we have the term  $4 \times 4 / 2$  as the tensor is (skew-)symmetric, e.g. once we choose element  $a_{12}$ , then  $a_{21}$  is also known. The term  $-4$  takes into account the diagonal element which are 0 and thus cannot be chosen.] So, we have  $4 \times 6 = 24$  degrees of freedom.

What about the supplementary constraint  $A_{rst} + A_{str} + A_{trs} = 0$  :

Consider the two possible excluding cases:

$$\text{i) } r = s \neq t \text{ (} \Leftrightarrow r = t \neq s \text{)}$$

This case gives - without the additional constraint (1) -  $4 \times (4 \times 3 / 2 - 4) = 8$  degrees of freedom. Does the constraint (1) reduces this degree of freedom?

We have,

$$A_{rst} + A_{str} + A_{trs} = 0 \tag{1}$$

$$\begin{aligned} \Rightarrow \underbrace{A_{rrt} + A_{rtr}}_{= 0 \text{ (non-diagonal terms)}} + \underbrace{A_{trr}}_{= 0 \text{ (diagonal terms)}} &= 0 \end{aligned} \tag{2}$$

So, no additional constraints are added by (1) to the restriction i) and the DOF remains 8.

$$\text{ii) } t \neq r \neq s \neq t$$

This case means that we have to choose a set of 3 elements out of 4 elements without repetition. This a *variation* of 3 elements out of 4.

$$V_k^n = \frac{n!}{(n-k)!} \text{ giving } V_3^4 = \frac{4!}{(4-3)!} = 24$$

The constraint (1) gives us 24 equations but as  $A_{rst} = -A_{rts}$  only 12 equations have to be considered. So, with the additional constraints the DOF becomes  $24 - 12 = 12$ .

As i) and ii) are independent and excluding events we can add the DOF of both events and we get  $8 + 12 = 20$  DOF.



## 1.11 p16-exercise

If  $A^{rs}$  is skew-symmetric and  $B_{rs}$  is symmetric, prove that  $A^{rs}B_{rs} = 0$ . Hence show that the quadratic form  $a_{ij}x^ix^j$  is unchanged if  $a_{ij}$  is replaced by its symmetric part.

We can split the summation  $A^{rs}B_{rs}$  in three subsummations:

$$A^{rs}B_{rs} = A^{rs}B_{rs}|_{r=s} \tag{1}$$

$$+ A^{rs}B_{rs}|_{r>s} \tag{2}$$

$$+ A^{rs}B_{rs}|_{r<s} \tag{3}$$

We have:

$$(1) = 0 \text{ as } A^{kk} = 0 \text{ (skew-symmetric)}$$

$$(2)+(3) = A^{rs}B_{rs}|_{r>s} + A^{rs}B_{rs}|_{r<s}$$

As  $A^{rs} = -A^{sr}$  and  $B^{rs} = B^{sr}$  we can write (2)+(3) as :

$$A^{rs}B_{rs}|_{r>s} + (-A^{sr})B_{sr}|_{r>s} = 0$$

So,  $A^{rs}B_{rs} = 0$

Consider the quadratic form  $\phi = a_{ij}x^ix^j$

Be  $A_{ij} = (a_{ij})$  and  $B_{ij} = (x^ix^j)$ , then it is obvious that  $B_{ij}$  is symmetric and that  $C_{ij} = -A_{ij}$  is the form where  $-a_{ij}$  is replaced by its symmetric part (skew-symmetric). Hence  $\phi = a_{ij}x^ix^j = a_{ij}b^{ij} = 0$  and so is  $\phi = c_{ij}b^{ij} = 0$



## 1.12 p18-exercise

What are the values (in a space of  $N$  dimensions) of the following contractions formed from the Kronecker delta?

$$\delta_m^m, \delta_n^m \delta_m^n, \delta_n^m \delta_r^n \delta_m^r$$

$$\delta_m^m = N \tag{1}$$

$$\delta_n^m \delta_m^n = \delta_m^m = N \tag{2}$$

$$\delta_n^m \delta_r^n \delta_m^r = \delta_n^m \delta_m^n = \delta_m^m = N \tag{3}$$



### 1.13 p19-exercise

If  $X^r, Y^r$  are arbitrary contravariant vectors and  $a_{rs}X^rY^s$  is an invariant, then  $a_{rs}$  are the components of a covariant tensor of the second order.

We have to prove that

$$a_{rs} = a_{ij} \frac{\partial x^i}{\partial x^{,r}} \frac{\partial x^j}{\partial x^{,s}} \text{ or } a_{ij} = a_{rs} \frac{\partial x^{,r}}{\partial x^i} \frac{\partial x^{,s}}{\partial x^j} \quad (1)$$

$a_{rs}X^rY^s$  is an invariant, means

$$a_{rs}X^{,r}Y^{,s} = a_{rs}X^rY^s \quad (2)$$

As  $X^r, Y^r$  are arbitrary contravariant vectors, we have

$$X^{,r} = X^i \frac{\partial x^{,r}}{\partial x^i} \text{ and } Y^{,s} = Y^j \frac{\partial x^{,s}}{\partial x^j} \quad (3)$$

(3) in (2) gives

$$a_{rs}X^i \frac{\partial x^{,r}}{\partial x^i} Y^j \frac{\partial x^{,s}}{\partial x^j} = a_{rs}X^rY^s \quad (4)$$

$$\Leftrightarrow a_{rs} \frac{\partial x^{,r}}{\partial x^i} \frac{\partial x^{,s}}{\partial x^j} X^i Y^j = a_{ij} X^i Y^j \quad (5)$$

$$\Leftrightarrow (a_{rs} \frac{\partial x^{,r}}{\partial x^i} \frac{\partial x^{,s}}{\partial x^j} - a_{ij}) X^i Y^j = 0 \quad (6)$$

As  $X^r, Y^r$  are arbitrary contravariant vectors, we conclude that

$$a_{rs} \frac{\partial x^{,r}}{\partial x^i} \frac{\partial x^{,s}}{\partial x^j} - a_{ij} = 0 \quad (7)$$

$$\Leftrightarrow a_{ij} = a_{rs} \frac{\partial x^{,r}}{\partial x^i} \frac{\partial x^{,s}}{\partial x^j} \quad (8)$$

(8) = (1): OK



## 1.14 p19-exercise

If  $X_{rs}$  is an arbitrary covariant tensor of the second order, and  $A_r^{mn} X_{mn}$  is a covariant vector, then  $A_r^{mn}$  has the mixed tensor character indicated by the positions of its suffixes

We have to prove that

$$A_r^{vw} = A_k^{mn} \frac{\partial x^k}{\partial x^{,r}} \frac{\partial x^{,v}}{\partial x^m} \frac{\partial x^{,w}}{\partial x^n} \quad (1)$$

We have

$$P_r = A_r^{mn} X_{mn} \quad (2)$$

is a covariant vector

$$\Rightarrow P_r = A_k^{mn} X_{mn} \frac{\partial x^k}{\partial x^{,r}} \quad (3)$$

but  $X_{mn}$  is a covariant tensor

$$\Rightarrow X_{mn} = X_{ps} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \quad (4)$$

So (4) in (3) gives

$$P_r = A_k^{mn} X_{ps} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}} \quad (5)$$

$$\Leftrightarrow P_r = A_k^{mn} \underbrace{\frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}}_{(*)} X_{ps} \quad (6)$$

Putting (\*) as  $A_r^{ps} = A_k^{mn} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}$  we see that (6) has the form (2) and that  $A_r^{ps}$  obeys the rule of a mixed tensor (1).



## 1.15 p21-exercise

If  $A_{rs}$  is a skew-symmetric covariant tensor, prove that  $B_{rst}$  defined as

$$B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$$

is a covariant tensor, and that it is skew-symmetric in all pairs of suffixes.

We have  $A_{rs}$  is a covariant tensor

$$A_{ij} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^i} \frac{\partial x^\beta}{\partial x^j} \quad (1)$$

$$\Rightarrow B_{rst} = \partial_r (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}) + \partial_s (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r}) + \partial_t (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s}) \quad (2)$$

Note that

$$\partial_k (A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}) = \partial_k (A_{\alpha\beta}) \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \partial_k (\frac{\partial x^\alpha}{\partial x^s}) \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \partial_k (\frac{\partial x^\beta}{\partial x^t}) \quad (3)$$

$$(4)$$

so,

$$\begin{aligned} B_{rst} = & \partial_r A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \underbrace{A_{\alpha\beta} \partial_r \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}}_{*} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \partial_r \frac{\partial x^\beta}{\partial x^t}}_{**} \\ & + \partial_s A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \underbrace{A_{\alpha\beta} \partial_s \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r}}_{***} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \partial_s \frac{\partial x^\beta}{\partial x^r}}_{*} \\ & + \partial_t A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} + \underbrace{A_{\alpha\beta} \partial_t \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s}}_{**} + \underbrace{A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \partial_t \frac{\partial x^\beta}{\partial x^s}}_{***} \end{aligned} \quad (5)$$

In (5) consider the two terms with (\*)

$$T = A_{\alpha\beta} \partial_r \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \partial_s \frac{\partial x^\beta}{\partial x^r} \quad (6)$$

$$= A_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x^s \partial x^r} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial^2 x^\beta}{\partial x^r \partial x^s} \quad (7)$$

$$= A_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x^s \partial x^r} \frac{\partial x^\beta}{\partial x^t} + A_{\beta\alpha} \frac{\partial x^\beta}{\partial x^t} \frac{\partial^2 x^\alpha}{\partial x^r \partial x^s} \text{ (by renaming dummy variables)} \quad (8)$$

As  $A_{ij} = -A_{ji}$  (skew-symmetric tensor), we get  $T = 0$ . The same yields for the (\*\*) and (\*\*\*) terms. So,  $B_{rst}$  reduces to

$$B_{rst} = \partial_r A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \partial_s A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \partial_t A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \quad (9)$$

$$\Leftrightarrow B_{rst} = \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^r} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^s} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^t} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \quad (10)$$

By adequate renaming of the dummy variable in the 3 terms:

$$\begin{bmatrix} 1^{st}term \\ 2^{nd}term \\ 3^{rd}term \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma \rightarrow \alpha & \alpha \rightarrow \beta & \beta \rightarrow \gamma \\ \beta \rightarrow \alpha & \gamma \rightarrow \beta & \alpha \rightarrow \gamma \\ \alpha \rightarrow \alpha & \beta \rightarrow \beta & \gamma \rightarrow \gamma \end{bmatrix}$$

we get

$$B_{rst} = \left( \frac{\partial A_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial A_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \right) \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \frac{\partial x^\gamma}{\partial x^t} \quad (11)$$

$$\Leftrightarrow B_{rst} = \underbrace{\left( \partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta} \right)}_{(****)} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \frac{\partial x^\gamma}{\partial x^t} \quad (12)$$

The expression (\*\*\*\*) has exactly the required form  $B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$  and is transformed (12) according the rules of a covariant tensor.

Let's prove now that it is skew-symmetric in all pairs of suffixes. We have to consider the following permutations

$$\begin{bmatrix} rst \\ rts \\ srt \\ str \\ trs \\ tsr \end{bmatrix}$$

E.g.  $srt$

$$B_{rts} = \partial_r A_{ts} + \partial_t A_{sr} + \partial_s A_{rt} \quad (13)$$

$$= -\partial_r A_{st} - \partial_t A_{rs} - \partial_s A_{tr} \quad (14)$$

$$= -B_{rst} \quad (15)$$

The same calculations can be done for the other permutations.





## 1.16 p23-exercise 1.

In a  $V_4$  there are two 2-spaces with equations

$$x^r = f^r(u^1, u^2), \quad x^r = g^r(u^3, u^4)$$

Prove that if these 2-spaces have a curve of intersection, then the determinantal equation

$$\left| \frac{\partial x^r}{\partial u^s} \right| = 0$$

is satisfied along the curve.

Having a curve means that one of the parameters  $u^i$  can be freely chosen while the other 3 are determined by the chosen parameter.

We have,

$$\left| \frac{\partial x^r}{\partial u^s} \right| = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} & \frac{\partial x^1}{\partial u^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^4}{\partial u^1} & \frac{\partial x^4}{\partial u^2} & \frac{\partial x^4}{\partial u^3} & \frac{\partial x^4}{\partial u^4} \end{vmatrix} \quad (1)$$

Suppose we choose  $u^4$  as parameter. This means  $u^i = \phi^i(u^4)$  for  $i=1,2,3$  and thus we can write

$$\frac{\partial x^i}{\partial u^4} = \frac{\partial x^i}{\partial u^j} \frac{d\phi^j}{du^4} + \frac{\partial x^i}{\partial u^4} \quad \text{with } j=1,2,3 \quad i = 1,2,3,4 \quad (2)$$

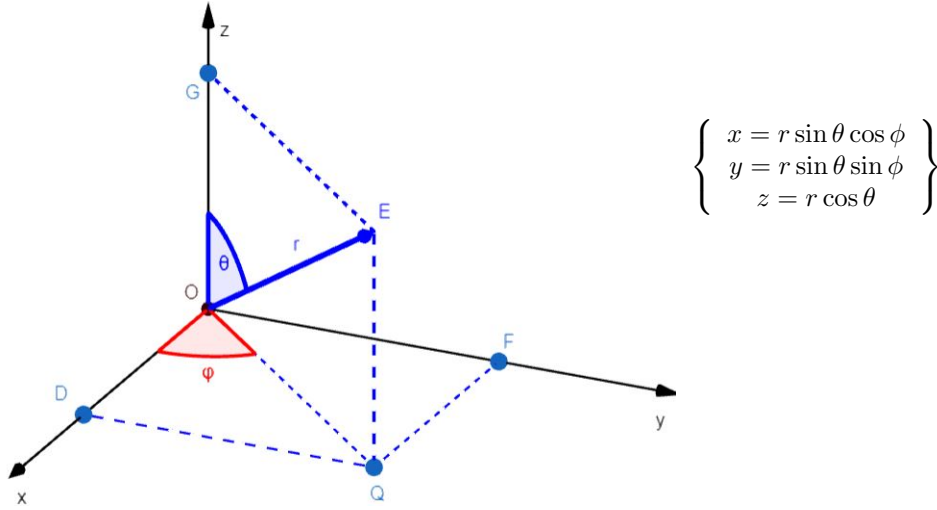
$$\Rightarrow \frac{\partial x^i}{\partial u^j} \frac{d\phi^j}{du^4} = 0 \quad (3)$$

This means that in (1) the three first columns are not linearly independent and thus have  $\left| \frac{\partial x^r}{\partial u^s} \right| = 0$



## 1.17 p23-exercise 2.

In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates  $x, y, z$  and spherical polar coordinates  $r, \theta, \phi$ . Find the Jacobian of the transformation. Where is it zero or infinite?



Partial differentiating of  $(x, y, z)$  with respect to  $(r, \theta, \phi)$  gives the Jacobian

$$J = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} \quad (1)$$

$$= r^2 \sin \theta (\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi) \quad (2)$$

$$= r^2 \sin \theta \quad (3)$$

$J=0$ : for  $r = 0$  or  $\theta = 0$   $r \in (-\infty, +\infty)$  and  $J \rightarrow \pm\infty$  or  $\mp\infty$  for  $r \rightarrow \pm\infty | \theta \neq 0$ . But what about the case  $r \rightarrow \pm\infty | \theta \rightarrow 0$ ? This case is not determined as long as no path is chosen in the  $(r, \theta)$  configuration space.



### 1.18 p23-exercise 3.

If  $X, Y, Z$  are the components of a contravariant vector for rectangular Cartesian coordinates in Euclidean 3-space, find its components for spherical polar coordinates.

Be  $x^\alpha$  the components of a contravariant vector in spherical polar coordinates and  $x^i$  its components in rectangular Cartesian coordinates. As we have

$$\begin{aligned} x^\rho &= \sqrt{x^j x^j} \\ x^\theta &= \text{atan} \frac{x^2}{x^1} \\ x^\phi &= \text{asin} \frac{x^3}{\sqrt{x^j x^j}} \end{aligned} \quad \text{and} \quad A^\alpha = A^i \frac{\partial x^\alpha}{\partial x^i} \quad (1)$$

$$\Rightarrow [A^\alpha] = \left[ A^i \frac{\partial x^\alpha}{\partial x^i} \right] = \begin{bmatrix} \frac{x^1}{\sqrt{x^j x^j}} & \frac{x^2}{\sqrt{x^j x^j}} & \frac{x^3}{\sqrt{x^j x^j}} \\ -\frac{x^2}{(x^1)^2 + (x^2)^2} & \frac{x^1}{(x^1)^2 + (x^2)^2} & 0 \\ -\frac{x^3 x^1}{(x^j x^j) \sqrt{(x^1)^2 + (x^2)^2}} & -\frac{x^3 x^2}{(x^j x^j) \sqrt{(x^1)^2 + (x^2)^2}} & \frac{\sqrt{(x^1)^2 + (x^2)^2}}{(x^j x^j)} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} \quad (2)$$

◆

### 1.19 p23-exercise 4.

In a space of three dimensions, how many different expressions are represented by the product  $A_{np}^m B_{rs}^{pq} C_{tu}^s$ ? How many terms occur in each such expression, when written out explicitly?

As we have  $V_3$  and considering that in  $A_{np}^m B_{rs}^{pq} C_{tu}^s$  the six indices  $m, n, q, r, t, u$  are not dummy indices, we get  $3^6$  different expressions (first choose  $m$ : you have three choices, then  $n$ : also three choices giving  $3 \times 3$  possibilities, etc for  $q, r, t, u$ ).

For the second question, as in  $A_{np}^m B_{rs}^{pq}$  there is only summation on over index ( $p$ ) we get three terms for this part. As the summation with  $A_{np}^m B_{rs}^{pq}$  and  $C_{tu}^s$  occurs only on one index also ( $s$ ) we get  $3 \times 3$  terms in the expression.



## 1.20 p23-exercise 5.

If  $A$  is an invariant in  $V_n$ , are the second derivatives  $\frac{\partial^2 A}{\partial x^r \partial x^s}$  the components of a tensor?

As  $A$  is invariant (note: different alphabets in the indices indicates different coordinate systems):

$$A(x^\rho) = A(x^i) \quad (1)$$

$$\Rightarrow \frac{\partial A(x^\rho)}{\partial x^i} = \frac{\partial A(x^j)}{\partial x^i} \quad (2)$$

To simplify the notation, we put  $A(x^\rho) = A_\cdot$  and  $A(x^j) = A_\cdot$  then (2) can be written as

$$\frac{\partial A_\cdot}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^i} = \frac{\partial A_\cdot}{\partial x^i} \quad (3)$$

Conclusion:  $\frac{\partial A_\cdot}{\partial x^i}$  is a covariant tensor.

Consider now  $\frac{\partial^2 A}{\partial x^i} = \frac{\partial A_\cdot}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^i}$ . Then,

$$\frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A_\cdot}{\partial x^\rho \partial x^j} \frac{\partial x^\rho}{\partial x^i} + \frac{\partial A_\cdot}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \quad (4)$$

$$\Leftrightarrow \frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A_\cdot}{\partial x^\rho \partial x^\gamma} \frac{\partial x^\gamma}{\partial x^j} \frac{\partial x^\rho}{\partial x^i} + \frac{\partial A_\cdot}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \quad (5)$$

The first term on the right side, behaves as covariant tensor but the presence of the second term makes that generally,  $\frac{\partial^2 A}{\partial x^i \partial x^j}$  has not a tensor character. This is only when  $\frac{\partial A_\cdot}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^i \partial x^j} = 0$ , which means that  $x^\rho, x^i$  are a linear map of each other.



## 1.21 p23-exercise 6.

Suppose that in  $V_2$  the components of a contravariant tensor field  $T^{mn}$  in a coordinate system  $x^r$  are

$$T^{11} = 1 \quad T^{12} = 0$$

$$T^{21} = 1 \quad T^{22} = 0$$

Find the components  $T^{mn}$  in a coordinate system  $x'^r$ , where

$$x'^1 = (x^1)^2 \quad x'^2 = (x^2)^2$$

Write down the values of these components in particular at the point  $x^1 = 1, x^2 = 0$ .

As we have a contravariant tensor field :

$$T^{mn} = T^{ij} \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^n}{\partial x^j} \quad (1)$$

$$\begin{aligned} x'^1 = (x^1)^2 &\Rightarrow \frac{\partial x'^1}{\partial x^1} = 2x^1 & \frac{\partial x'^1}{\partial x^2} &= 0 \\ x'^2 = (x^2)^2 &\Rightarrow \frac{\partial x'^2}{\partial x^1} &= 0 & \frac{\partial x'^2}{\partial x^2} = 2x^2 \end{aligned} \quad (2)$$

$$(3)$$

$$\Rightarrow T'^{11} = 4(x^1)^2 + 4(x^2)^2 \quad (4)$$

$$\Rightarrow T'^{12} = T'^{21} = 0 \quad (5)$$

$$\Rightarrow T'^{22} = 4(x^1)^2 + 4(x^2)^2 \quad (6)$$

The components in at the point  $x^1 = 1, x^2 = 0$  are

$$T'(1, 0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$



## 1.22 p24-exercise 7.

Given that if  $T_{mnrs}$  is a covariant tensor, and

$$T_{mnrs} + T_{mnsr} = 0$$

in a coordinate system  $x^p$ , establish directly that

$$T_{mnrs} + T_{mnsr} = 0$$

in any other coordinate system  $x, q$ .

Note: in the following, different alphabets in the indices indicates different coordinate systems.  
As we  $T_{mnrs}$  is a covariant tensor :

$$T_{\alpha\beta\gamma\delta} = T_{mnrs} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^r}{\partial x^\gamma} \frac{\partial x^s}{\partial x^\delta} \quad (1)$$

$$\Rightarrow T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnrs} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^r}{\partial x^\gamma} \frac{\partial x^s}{\partial x^\delta} + T_{mnrs} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^r}{\partial x^\delta} \frac{\partial x^s}{\partial x^\gamma} \quad (2)$$

Now, swap the dummy indices r and s in the second term on the right and as  $T_{mnrs} = -T_{mnsr}$ :

$$T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnrs} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^r}{\partial x^\gamma} \frac{\partial x^s}{\partial x^\delta} + T_{mnsr} \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^s}{\partial x^\delta} \frac{\partial x^r}{\partial x^\gamma} \quad (3)$$

$$= (T_{mnrs} + T_{mnsr}) \frac{\partial x^m}{\partial x^\alpha} \frac{\partial x^n}{\partial x^\beta} \frac{\partial x^s}{\partial x^\delta} \frac{\partial x^r}{\partial x^\gamma} \quad (4)$$

$$= 0 \quad (5)$$



## 1.23 p24-exercise 8.

Prove that if  $A_r$  is a covariant vector, then  $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$  is a skew-symmetric covariant tensor of the second order (use the notation of 1.7).

Be  $B_{rs} = \frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ .

i)  $B_{rs}$  is skew-symmetric: It is obvious that:

$$-B_{rs} = -\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r} = \frac{\partial A_s}{\partial x^r} - \frac{\partial A_r}{\partial x^s} \equiv B_{sr}$$

ii)  $B_{rs}$  is covariant:

*Note: in the following, different alphabets in the indices indicates different coordinate systems.*

Let

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X_\alpha^r X_\beta^s. \quad (1)$$

We know that  $A_i = A_\gamma X_i^\gamma$  as  $A_i$  is covariant. Hence,

$$\partial_j A_i = \partial_j A_\gamma X_i^\gamma + A_\gamma \partial_j X_i^\gamma \quad (2)$$

$$= \partial_\alpha A_\gamma X_j^\alpha X_i^\gamma + A_\gamma \partial_j X_i^\gamma \quad (3)$$

Using (3), we compute the first term in (1)

$$\partial_s A_r X_\alpha^r X_\beta^s = \partial_\rho A_\gamma X_s^\rho X_r^\gamma X_\alpha^r X_\beta^s + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s \quad (4)$$

$$= \partial_\rho A_\gamma X_\beta^\rho X_\alpha^\gamma + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s \quad (5)$$

$$= \partial_\rho A_\gamma \delta_\beta^\rho \delta_\alpha^\gamma + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s \quad (6)$$

$$= \partial_\beta A_\alpha + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s \quad (7)$$

In the same way, we get for the second term in (1)

$$\partial_r A_s X_\alpha^s X_\beta^r = \partial_\alpha A_\beta + A_\gamma \partial_r X_s^\gamma X_\alpha^r X_\beta^s \quad (8)$$

And thus,

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X_\alpha^r X_\beta^s = \partial_\beta A_\alpha + A_\gamma \partial_s X_r^\gamma X_\alpha^r X_\beta^s - \partial_\alpha A_\beta - A_\gamma \partial_r X_s^\gamma X_\alpha^r X_\beta^s \quad (9)$$

$$\Rightarrow \partial_\beta A_\alpha - \partial_\alpha A_\beta = (\partial_s A_r - \partial_r A_s) X_\alpha^r X_\beta^s \quad (10)$$

So, i) and (10) proves that  $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$  is skew-symmetric tensor of the second order.





## 1.24 p24-exercise 9.

Let  $x^r, \bar{x}^r, y^r, \bar{y}^r$  be four systems of coordinates. Examine the tensor character of  $\frac{\partial x^r}{\partial y^s}$  with respect to the following transformations:

- i) A transformation  $x^r = f^r(\bar{x}^1, \dots, \bar{x}^N)$ , with  $y^r$  unchanged;
- ii) A transformation  $y^r = g^r(\bar{y}^1, \dots, \bar{y}^N)$ , with  $x^r$  unchanged;

*Note: in the following, different alphabets in the indices indicates different coordinate systems.*

i) Let's compute the expression  $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta}$ . Obviously, the right side is an expression of a (possible) mixed tensor of the second order ( $\frac{\partial x^r}{\partial y^s}$ ) under transformation from the (r) coordinate system to the ( $\alpha$ ) coordinate system. Then,

$$A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta} \quad (1)$$

$$= \frac{\partial x^\alpha}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \quad (2)$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \quad (3)$$

If we consider the  $\bar{y}^r$  coordinate system as the  $y^\rho$  coordinate system and as  $\bar{y}^r = y^r$  then  $\frac{\partial y^\rho}{\partial y^s} = \delta_s^\rho$  and we get from (3)

$$A(\alpha, \beta) = \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \quad (4)$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \delta_s^\rho \frac{\partial x^s}{\partial x^\beta} \quad (5)$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial x^\rho}{\partial x^\beta} \quad (6)$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \delta_\beta^\rho \quad (7)$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \quad (8)$$

$$(1) \text{ and } (8) \Rightarrow \frac{\partial x^r}{\partial y^s} = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta} \quad (9)$$

So  $A(r, s) = \frac{\partial x^r}{\partial y^s}$  is a mixed tensor of type  $A_s^r$

ii) Let's compute the expression  $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta}$ . Obviously, the right side is an expression of a (possible) mixed tensor of the second order ( $\frac{\partial x^r}{\partial y^s}$ ) under transformation from the (r) coordinate

system to the  $(\alpha)$  coordinate system. Then,

$$A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta} \quad (10)$$

$$= \frac{\partial x^r}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta} \quad (11)$$

$$= \frac{\partial x^r}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (12)$$

$$= \frac{\partial x^r}{\partial y^\rho} \delta_\beta^\rho \frac{\partial y^\alpha}{\partial y^r} \quad (13)$$

$$= \frac{\partial x^r}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (14)$$

$$= \frac{\partial x^r}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (15)$$

If we consider the  $\bar{x}^r$  coordinate system as the  $x^\sigma$  coordinate system and as  $\bar{x}^r = x^r$  then  $\frac{\partial x^\sigma}{\partial x^r} = \delta_r^\sigma$  and we get from (15)

$$A(\alpha, \beta) = \frac{\partial x^r}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (16)$$

$$= \delta_\sigma^r \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^r} \quad (17)$$

$$= \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^\sigma} \quad (18)$$

$$= \frac{\partial x^\sigma}{\partial y^\beta} \delta_\sigma^\alpha \quad (19)$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \quad (20)$$

$$(10) \text{ and } (19) \Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta} \quad (21)$$

So  $A(r, s) = \frac{\partial x^r}{\partial y^s}$  is a mixed tensor of type  $A_s^r$



## 1.25 p24-exercise 10.

If  $x^r, y^r, z^r$  are three systems of coordinates, prove the following rule for the multiplication of Jacobians.

$$\left| \frac{\partial x^m}{\partial y^n} \right| \left| \frac{\partial y^r}{\partial z^s} \right| = \left| \frac{\partial x^t}{\partial z^u} \right|$$

As we have

$$\frac{\partial x^t}{\partial z^u} = \frac{\partial x^t}{\partial y^k} \frac{\partial y^k}{\partial z^u} \quad (1)$$

$$\begin{bmatrix} \frac{\partial x^1}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial z^N} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial y^k} \frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial y^k} \frac{\partial y^k}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^k} \frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial y^k} \frac{\partial y^k}{\partial z^N} \end{bmatrix} \quad (2)$$

$$= \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^1} & \cdots & \frac{\partial x^N}{\partial y^N} \end{bmatrix} \begin{bmatrix} \frac{\partial y^1}{\partial z^1} & \cdots & \frac{\partial y^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial y^N}{\partial z^1} & \cdots & \frac{\partial y^N}{\partial z^N} \end{bmatrix} \quad (3)$$

$$\Rightarrow \left| \frac{\partial x^m}{\partial y^n} \right| \left| \frac{\partial y^r}{\partial z^s} \right| = \left| \frac{\partial x^t}{\partial z^u} \right| \quad (4)$$



## 1.26 p24-exercise 11.

Prove that with respect to transformations

$$x^{,r} = C_{rs}x^s$$

where the coefficients are constants satisfying

$$C_{mr}C_{ms} = \delta_s^r$$

contravariant and covariant vectors have the same formula of transformation

$$A^{,r} = C_{rs}A^s, A_{,r} = C_{rs}A_s$$

i)  $A^{,r} = C_{rs}A^s$

Be  $A^{,r} = A^s \frac{\partial x^{,r}}{\partial x^s}$  and as  $x^{,r} = C_{rs}x^s$  we have  $\frac{\partial x^{,r}}{\partial x^s} = C_{rs}$ . Hence,

$$A^{,r} = C_{rs}A^s$$

.

i)  $A_{,r} = C_{rs}A_s$

Be  $A_{,r} = A_s \frac{\partial x^s}{\partial x^{,r}}$  and as  $x^{,r} = C_{rs}x^s$  we have

$$\frac{\partial x^{,r}}{\partial x^{,t}} = C_{rs} \frac{\partial x^s}{\partial x^{,t}} \quad (1)$$

$$\Rightarrow \delta_t^r = C_{rs} \frac{\partial x^s}{\partial x^{,t}} \quad (2)$$

Now, multiply (2) by  $C_{rq}$ . We get,

$$\delta_t^r C_{rq} = C_{rq} C_{rs} \frac{\partial x^s}{\partial x^{,t}} \quad (3)$$

$$C_{tq} = C_{rq} C_{rs} \frac{\partial x^s}{\partial x^{,t}} \quad (4)$$

$$\text{as } C_{mr}C_{ms} = \delta_s^r \Rightarrow C_{tq} = \delta_s^q \frac{\partial x^s}{\partial x^{,t}} \quad (5)$$

$$\Rightarrow C_{tq} = \frac{\partial x^q}{\partial x^{,t}} \text{ or } C_{rs} = \frac{\partial x^s}{\partial x^{,r}} \quad (6)$$

$$\text{as } A_{,r} = A_s \frac{\partial x^s}{\partial x^{,r}} \Rightarrow A_{,r} = C_{rs} \frac{\partial x^s}{\partial x^{,r}} \quad (7)$$



## 1.27 p25-exercise 12.

Prove that

$$\frac{\partial \ln \left| \frac{\partial x^m}{\partial y^n} \right|}{\partial x^r} = \frac{\partial^2 y^m}{\partial x^r \partial x^n} \frac{\partial x^n}{\partial y^m}$$

**Lemma** Be  $A$  a square matrix  $N \times N$ ; Be  $f$  a  $C^1$  function  $f : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$ . Define  $A'$  as  $(A'_{ij}) = \frac{df}{dA_{ij}}$ . Then,

$$(\ln |A|)' = (A^{-1})^T \text{ with } f = |A|$$

*Proof:*

By definition of the determinant, we have

$$|A| = A_{iK} C_K^i \quad (\text{no summation on } K!) \quad (1)$$

with  $(C_K^i) = (-1)^{i+K} M_K^i$  being the cofactor of element  $A_{iK}$  and  $M_K^i$  the minor  $(N-1) \times (N-1)$  matrix associated with the cofactor  $A_{K^i}$ . Be  $C = (C_{ij})$  the  $N \times N$  matrix formed with all possible cofactor elements  $C_j^i$  ( $i, j = 1 \dots, N$ ).

We have

$$A^{-1} = \frac{C^T}{|A|} \quad (2)$$

$$\Rightarrow (A^{-1})^T = \frac{C}{|A|} \quad (3)$$

$$\text{differentiating (1)} \Rightarrow \frac{\partial |A|}{\partial A_{mn}} = \frac{\partial A_{iK}}{\partial A_{mn}} C_K^i + A_{iK} \frac{\partial C_K^i}{\partial A_{mn}} \quad (4)$$

$$\text{we have for } i = m \quad \begin{aligned} \frac{\partial A_{iK}}{\partial A_{mn}} &= 1 & K &= n \\ \frac{\partial A_{iK}}{\partial A_{mn}} &= 0 & K &\neq n \end{aligned} \quad (5)$$

Also,  $\forall K : \frac{\partial C_K^i}{\partial A_{in}} = 0$  as by definition of the cofactor matrix,  $A_{ij}$  is not contained in  $C_{ij}$ .

Hence, (4) becomes

$$\frac{\partial |A|}{\partial A_{ij}} = C_j^i \quad (6)$$

$$\text{But, } \frac{\partial \ln |A|}{\partial A_{ij}} = \frac{\frac{\partial |A|}{\partial A_{ij}}}{|A|} \quad (7)$$

$$(6) \text{ and } (7) \text{ gives } \frac{\partial \ln |A|}{\partial A_{ij}} = \frac{C_j^i}{|A|} \quad (8)$$

$$(3) \text{ and } (8) \text{ gives } \frac{\partial \ln |A|}{\partial A_{ij}} = \frac{(A_{ij}^{-1})^T |A|}{|A|} = (A_{ij}^{-1})^T \quad (9)$$

$$\Rightarrow (\ln |A|)' = (A^{-1})^T \quad (10)$$

◇

Now the main proof:

Let,

$$A \equiv [a_{mn}] = \left[ \frac{\partial y^m}{\partial x^n} \right] \quad (11)$$

$$\Rightarrow \frac{\partial \ln |A|}{\partial x^r} = \frac{\partial \ln |A|}{\partial a_{mn}} \frac{\partial a_{mn}}{\partial x^r} \quad (12)$$

$$\text{from (10) we get } \frac{\partial \ln |A|}{\partial a_{mn}} = (A^{-1})_{mn}^T \quad (13)$$

$$\text{But A is a Jacobian, so } (A^{-1})_{mn} = \frac{\partial x^m}{\partial y^n} \quad (14)$$

$$\text{and thus } (A^{-1})_{mn}^T = \frac{\partial x^n}{\partial y^m} \quad (15)$$

$$(13) \text{ can be written as } \frac{\partial \ln |A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m} \frac{\partial a_{mn}}{\partial x^r} \quad (16)$$

$$\Rightarrow \frac{\partial \ln |A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m} \frac{\partial^2 y^m}{\partial x^r \partial x^n} \quad (17)$$

◆

## 1.28 p25-exercise 13.

Consider the quantities  $\frac{dx^r}{dt}$  for a particle moving in the plane. If  $x^r$  are the rectangular Cartesian coordinates, are these quantities the components of a contravariant or covariant vector with respect to rotation of the axes? Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?

Note: we suppose that by a rotation of the axes, the problem means a fixed rotation and not a rotation varying in time.

i) Be  $v^r = \frac{dx^r}{dt}$  and consider  $v^\alpha$  the same object but in another the coordinate system. A rotation of the axes implies the linear form

$$x^\alpha = R^\alpha_k x^k \quad \text{with } R^\alpha_k \neq R^\alpha_k(x^k) \quad (1)$$

$$\Rightarrow \frac{\partial x^\alpha}{\partial x^r} = R^\alpha_k \delta_r^k \quad (2)$$

$$\Rightarrow R^\alpha_r = \frac{\partial x^\alpha}{\partial x^r} \quad (3)$$

Consider  $v^\alpha = \frac{dx^\alpha}{dt}$

$$v^\alpha = \frac{dx^\alpha}{dt} \quad (4)$$

$$(1) \Rightarrow v^\alpha = R^\alpha_k \frac{dx^k}{dt} \quad (5)$$

$$\Rightarrow v^\alpha = R^\alpha_k v^k \quad (6)$$

$$(3) \Rightarrow v^\alpha = v^k \frac{\partial x^\alpha}{\partial x^r} \quad (7)$$

Conclusion:  $v^k$  is a contravariant vector.

ii) Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?

We know that

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^r} dx^r \quad (8)$$

$$\Rightarrow \frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r} \frac{dx^r}{dt} \quad (9)$$

$$\Rightarrow v^\alpha = v^r \frac{\partial x^\alpha}{\partial x^r} \quad (10)$$

So  $v^r$  is a contravariant vector in general. Note that this proof is more straightforward than the prove in i).



## 1.29 p25-exercise 14.

Consider the question raised in No. 13 for the acceleration  $\frac{d^2 x^r}{dt^2}$ .

From exercise 13. we know that

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r} \frac{dx^r}{dt} \quad (1)$$

$$\Rightarrow \frac{d^2 x^\alpha}{dt^2} = \frac{d^2 x^r}{dt^2} \frac{\partial x^\alpha}{\partial x^r} + \frac{d \frac{\partial x^\alpha}{\partial x^r}}{dt} \frac{dx^r}{dt} \quad (2)$$

$$= \frac{d^2 x^r}{dt^2} \frac{\partial x^\alpha}{\partial x^r} + \frac{\partial^2 x^\alpha}{\partial x^r \partial x^m} \frac{dx^m}{dt} \frac{dx^r}{dt} \quad (3)$$

The second term on the right does not vanish in general, hence  $\frac{d^2 x^r}{dt^2}$  has not a tensor character.





### 1.30 p25-exercise 15.

It is well known that the equation of an ellipse may be written

$$ax^2 + 2hxy + by^2 = 1$$

What is the tensor character of  $a, h, b$  with respect to transformation to any Cartesian coordinates (rectangular or oblique) in the plane?

Consider the transformation from a  $(w, z)$  coordinate system to a  $(x, y)$  coordinate system. For the considered type of transformation we have

$$\begin{aligned} x &= \alpha w + \beta z \\ y &= \gamma w + \delta z \end{aligned} \tag{1}$$

$$\text{consider} \quad \begin{aligned} ax^2 + 2hxy + by^2 &= 1 \\ pw^2 + 2q wz + rz^2 &= 1 \end{aligned} \tag{2}$$

the two representations of the same ellipse in the respective coordinate systems. Plugging (1) in (2):

$$a\alpha^2 w^2 + 2a\alpha\beta wz + \alpha\beta^2 z^2 + 2h\alpha\gamma w^2 + \beta\delta z^2 + 2h(\alpha\delta + \gamma\beta)wz + b\gamma^2 w^2 + 2b\gamma\delta wz + \delta^2 z^2 = 1 \tag{3}$$

$$\tag{4}$$

Rearranging and equating the terms in  $w^2, wz, z^2$  in (2) gives

$$p = a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \tag{5}$$

$$q = a\alpha\beta + h(\alpha\delta + \gamma\beta) + \gamma\delta \tag{6}$$

$$r = a\beta^2 + 2h\beta\delta + b\delta^2 \tag{7}$$

Consider the following objects

$$(A_{ij}) = \begin{bmatrix} a & h \\ h & b \end{bmatrix} \tag{8}$$

$$(A_{ij})' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix} \tag{9}$$

$$\text{we calculate} \quad A_{ij}' = A_{km} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} \tag{10}$$

with  $(x^1, x^2) = (w, z)$  and  $(x^1, x^2) = (x, y)$ . We have,

$$\frac{\partial x^1}{\partial x^1} = \alpha, \frac{\partial x^1}{\partial x^2} = \beta, \frac{\partial x^2}{\partial x^1} = \gamma, \frac{\partial x^2}{\partial x^2} = \delta \quad (11)$$

$$\begin{aligned} (10) \text{ and } (11) \quad \Rightarrow \quad & a_{11} = a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \\ & a_{22} = a\beta^2 + 2h\delta\beta + b\delta^2 \\ & a_{12} = a_{21} = a\alpha\beta + h(\alpha\delta + \gamma\beta) + b\gamma\delta \end{aligned} \quad (12)$$

Combining (5), (6), (7) and (12) we get

$$p = a_{11}, r = a_{22}, q = a_{12} = a_{21}$$

and so (9) becomes

$$(A_{ij})' = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$$

Considering (10) we see that  $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$  transforms to  $\begin{bmatrix} p & q \\ q & r \end{bmatrix}$  according the rules of a covariant tensor of order two.



### 1.31 p25-exercise 16.

Matter is distributed in a plane and  $A, B, H$  are the moments and product of inertia with respect to rectangular axes  $Oxy$  in a plane. Examine the tensor character of the set of quantities  $A, B, H$  under rotation of the axes. What notation would you suggest for moments and product of inertia in order to exhibit the tensor character? What simple invariant can be formed from  $A, B, H$  ?

Consider the transformation from a  $(x^1, x^2)$  coordinate system to a  $(y^1, y^2)$  coordinate system. For the considered type of transformation we have

$$\begin{aligned} y^1 &= \alpha x^1 + \beta x^2 \\ y^2 &= \gamma x^1 + \delta x^2 \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Be} \quad A &= \sum_{\rho} m_{\rho} (x^{2,\rho})^2 & A' &= \sum_{\rho} m_{\rho} (y^{2,\rho})^2 \\ B &= \sum_{\rho} m_{\rho} (x^{1,\rho})^2 & B' &= \sum_{\rho} m_{\rho} (y^{1,\rho})^2 \\ H &= \sum_{\rho} m_{\rho} x^{1,\rho} x^{2,\rho} & H' &= \sum_{\rho} m_{\rho} y^{1,\rho} y^{2,\rho} \end{aligned} \quad (2)$$

the moments and product of inertia,  $\rho$  being the index of summation over all the points with mass  $m_{\rho}$ .

For the sake of notational simplicity we consider only one point of mass as the linearity of  $A, B, H$  related to  $\rho$  ensures the validity of the next steps for all points in the plane.

From (1) and (2) we have:

$$\frac{A'}{m_{\rho}} = \gamma^2 (x^1)^2 + 2\gamma\delta x^1 x^2 + \delta^2 (x^2)^2 \quad (3)$$

$$\frac{B'}{m_{\rho}} = \alpha^2 (x^1)^2 + 2\alpha\beta x^1 x^2 + \beta^2 (x^2)^2 \quad (4)$$

$$\frac{H'}{m_{\rho}} = \alpha\gamma (x^1)^2 + (\gamma\beta + \alpha\delta) x^1 x^2 + \beta\delta (x^2)^2 \quad (5)$$

$$\text{Note that} \quad \begin{aligned} \frac{\partial y^1}{\partial x^1} &= \alpha & \frac{\partial y^1}{\partial x^2} &= \beta \\ \frac{\partial y^2}{\partial x^1} &= \gamma & \frac{\partial y^2}{\partial x^2} &= \delta \end{aligned} \quad (6)$$

$$(6) \text{ in } (4): \quad \frac{B'}{m_{\rho}} = (x^1)^2 \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + 2(x^1)(x^2) \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \quad (7)$$

$$= (x^1)^2 \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + (x^1)(x^2) \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + (x^2)(x^1) \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^1} + (x^2)^2 \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \quad (8)$$

Repeating the same calculations for  $\frac{A'}{m_{\rho}}$  and  $\frac{H'}{m_{\rho}}$  gives:

$$\begin{aligned} \frac{A'}{m_{\rho}} &= (x^1)^2 \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^1} + (x^1)(x^2) \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^2} + (x^2)(x^1) \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^1} + (x^2)^2 \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^2} \\ \frac{B'}{m_{\rho}} &= (x^1)^2 \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + (x^1)(x^2) \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + (x^2)(x^1) \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^1} + (x^2)^2 \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \\ \frac{H'}{m_{\rho}} &= (x^1)^2 \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^1} + (x^1)(x^2) \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^1} + (x^2)(x^1) \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^2} \end{aligned} \quad (9)$$

Now, define

$$(K_{ij}) = \begin{bmatrix} (x^1)^2 & (x^1)(x^2) \\ (x^2)(x^1) & (x^2)^2 \end{bmatrix} \quad (K_{ij})' = \begin{bmatrix} (y^1)^2 & (y^1)(y^2) \\ (y^2)(y^1) & (y^2)^2 \end{bmatrix} \quad (10)$$

Then (9) can be written as

$$\begin{aligned} \frac{A'}{m_\rho} &= (y^1)^2 = K^{11} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^1} + K^{12} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^2} + K^{21} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^1} + K^{22} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^2} \\ \frac{B'}{m_\rho} &= (y^2)^2 = K^{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + K^{12} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + K^{21} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^1} + K^{22} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \\ \frac{H'}{m_\rho} &= (y^1)(y^2) = K^{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^1} + K^{12} \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^1} + K^{21} \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^2} + K^{22} \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^2} \end{aligned} \quad (11)$$

Hence,

$$\begin{aligned} K^{,11} &= K^{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1} + K^{12} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^2} + K^{21} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^1} + K^{22} \frac{\partial y^1}{\partial x^2} \frac{\partial y^1}{\partial x^2} \\ K^{,22} &= K^{11} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^1} + K^{12} \frac{\partial y^2}{\partial x^1} \frac{\partial y^2}{\partial x^2} + K^{21} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^1} + K^{22} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^2} \\ K^{,12} &= K^{,21} = K^{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^1} + K^{12} \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^1} + K^{21} \frac{\partial y^1}{\partial x^1} \frac{\partial y^2}{\partial x^2} + K^{22} \frac{\partial y^1}{\partial x^2} \frac{\partial y^2}{\partial x^2} \end{aligned} \quad (12)$$

So the object  $(K_{ij}) \equiv K^{ij}$  transforms according (12) like a contravariant second order tensor.

Now, consider  $|K^{ij}|$ , obviously  $|K^{ij}| = (x^1)^2(x^2)^2 - (x^1)(x^2)(x^2)(x^1) = 0$ , but so is also  $|K^{,ij}|$ .

$\Rightarrow |K^{ij}|$  is an invariant under the considered transformation

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## 1.32 p25-exercise 17.

$S_{nmr}$  is a skew-symmetric tensor in the first two indices.  $-f_{mnr} + f_{nmr} = S_{mnr}$ .

From exercise 13. we know that

$$-f_{mnr} + f_{nmr} = S_{mnr} \quad (1)$$

Swap the indices three times

$$\text{i) } n \leftrightarrow r : (1) \Rightarrow -f_{mrn} + f_{rmn} = S_{mrn} \quad (2)$$

$$\Leftrightarrow \underbrace{f_{mnr}}_* + \underbrace{f_{rmn}}_{**} = -S_{rmn} \quad (3)$$

$$\text{ii) } m \leftrightarrow r : (1) \Rightarrow -f_{rnm} + f_{nrn} = S_{rnm} \quad (4)$$

$$\Leftrightarrow \underbrace{f_{rmn}}_{**} + \underbrace{f_{nrn}}_{***} = -S_{nrn} \quad (5)$$

$$\text{iii) } m \leftrightarrow n : (1) \Rightarrow -f_{nmr} + f_{mnr} = S_{nmr} \quad (6)$$

$$\Leftrightarrow \underbrace{f_{nrm}}_{***} + \underbrace{f_{mnr}}_* = -S_{mnr} \quad (7)$$

$$(3) - (5) + (7): \quad 2 \underbrace{f_{mnr}}_* = -S_{rmn} + S_{nrn} - S_{mnr} \quad (8)$$

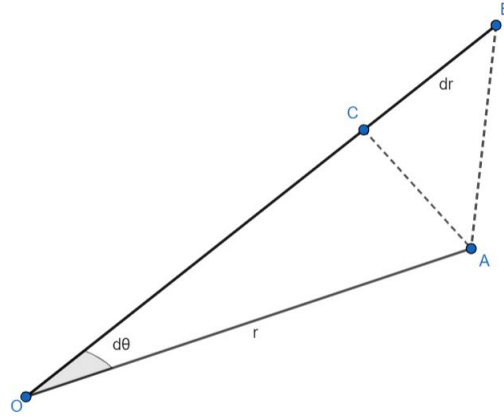
$$\Leftrightarrow f_{mnr} = \frac{-S_{rmn} + S_{nrn} - S_{mnr}}{2} \quad (9)$$



# Basic Operations in Riemannian Space

## 2.1 p27-exercise

Take polar coordinates  $r, \theta$  in a plane. Draw the infinitesimal triangle with vertices  $(r, \theta)$ ,  $(r + dr, \theta)$ ,  $(r, \theta + d\theta)$ . Evaluate the square on the hypotenuse of this infinitesimal triangle, and so obtain the metric tensor for the plan for the coordinates  $(r, \theta)$ .



$$ds^2 = |AB|^2 \quad (1)$$

$$= dr^2 + |CA|^2 \quad (2)$$

$$|CA| = r \sin(d\theta) \approx r d\theta \quad (3)$$

$$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 \quad (4)$$

$$\Rightarrow (a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad (5)$$



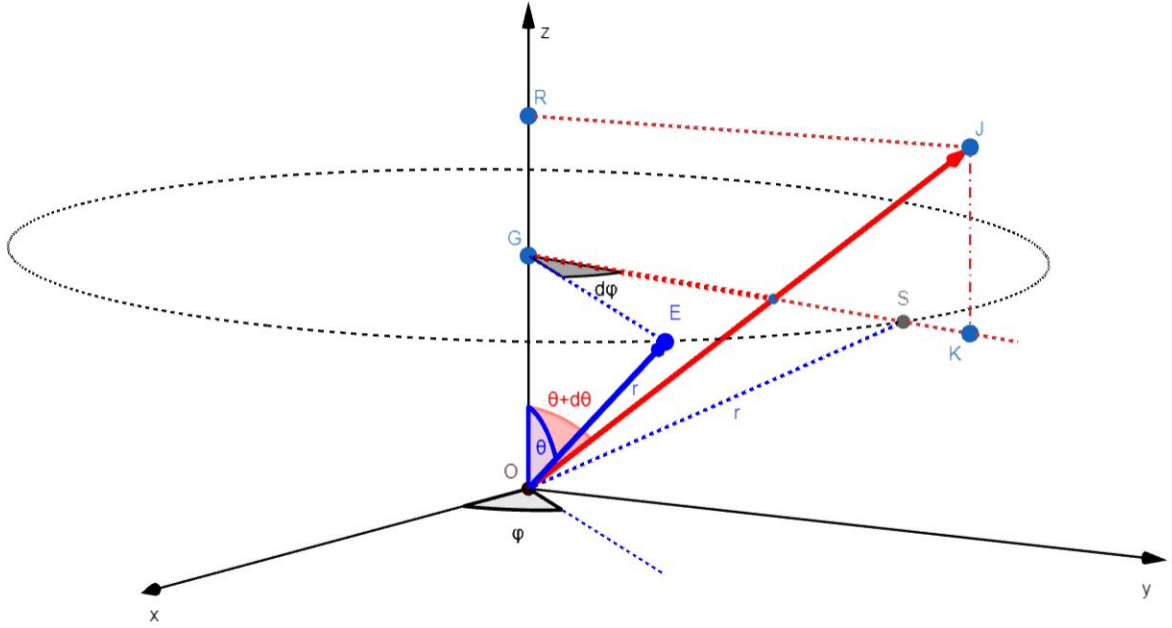
## 2.2 p27-exercise

Show that if  $x^1 = r, x^2 = \theta, x^3 = \phi$ , in the usual notation of spherical polar coordinates, then

$$a_{11} = 1, a_{22} = r^2, a_{33} = r^2 \sin^2 \theta$$

and the other components vanish.

One can choose to start from  $ds^2 = dx^2 + dy^2 + dz^2$  and then expanding the  $dx^i$  along  $(r, \theta, \phi)$  but this a rather tedious way. So we use a more geometrical way of deriving the metric



Consider an infinitesimal displacement of point E to J with  $(dr, d\theta, d\phi)$ .

$$ds^2 = |EJ|^2 \quad (1)$$

As we use infinitesimal displacements we can assume that

$$|ES| \perp |GK| \perp |JK| \perp |ES|$$

. Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \quad (2)$$



We have the following relationships

$$\left. \begin{aligned}
 |GE| &= |GS| = r \sin \theta \\
 |ES| &= |GE| d\phi = r \sin \theta d\phi \\
 |GK| &= |RJ| = (r + dr) \sin(\theta + d\theta) \\
 &= (r + dr)(\cos(\theta) \sin(d\theta) + \sin(\theta) \cos(d\theta)) \\
 &= (r + dr)(\cos(\theta) d\theta + \sin(\theta)) \\
 &= r \cos(\theta) d\theta + r \sin(\theta) + \sin(\theta) dr \\
 |OR| &= (r + dr) \cos(\theta + d\theta) \\
 &= (r + dr)(\cos(\theta) \cos(d\theta) - \sin(\theta) \sin(d\theta)) \\
 &= (r + dr)(\cos(\theta) - \sin(\theta) d\theta) \\
 &= r \cos(\theta) - r \sin(\theta) d\theta + \cos(\theta) dr \\
 |OG| &= r \cos(\theta) \\
 |JK| &= |OR| - |OG| = \cos(\theta) dr - r \sin(\theta) d\theta \\
 |SK| &= |GK| - |GS| = r \cos(\theta) d\theta + \sin(\theta) dr
 \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned}
 |ES|^2 &= r^2 \sin^2(\theta) d\phi^2 \\
 |SK|^2 &= r^2 \cos^2(\theta) d\theta^2 + \sin^2(\theta) dr^2 + 2r \cos(\theta) \sin(\theta) dr d\theta \\
 |JK|^2 &= \cos^2(\theta) dr^2 + r^2 \sin^2(\theta) d\theta^2 - 2r \cos(\theta) \sin(\theta) dr d\theta
 \end{aligned} \right\} \quad (4)$$

Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \quad (5)$$

$$= \begin{cases} r^2 \sin^2(\theta) d\phi^2 \\ + r^2 \cos^2(\theta) d\theta^2 + \sin^2(\theta) dr^2 + 2r \cos(\theta) \sin(\theta) dr d\theta \\ + r^2 \sin^2(\theta) d\theta^2 + \cos^2(\theta) dr^2 - 2r \cos(\theta) \sin(\theta) dr d\theta \end{cases} \quad (6)$$

$$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \quad (7)$$

$$\Rightarrow (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (8)$$

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## 2.3 p27-exercise

Starting from 3.103, show that

$$a_{mn} = \frac{\partial y^1}{\partial x^m} \frac{\partial y^1}{\partial x^n} + \frac{\partial y^2}{\partial x^m} \frac{\partial y^2}{\partial x^n} + \frac{\partial y^3}{\partial x^m} \frac{\partial y^3}{\partial x^n}$$

and calculate the quantities for a sphere, taking as curvilinear coordinates on the sphere

$$x^1 = y^1, x^2 = y^2$$

We have

$$(2.103) \Rightarrow y^1 = x^1, y^2 = x^2, y^3 = f^3(x^1, x^2) \quad (1)$$

$$\text{surface} = \text{sphere} \Rightarrow y^3 = \pm \sqrt{R^2 - (x^1)^2 - (x^2)^2} \quad (2)$$

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad (3)$$

$$(1) \text{ and } (2) \Rightarrow \begin{cases} dy^1 = dx^1 \\ dy^2 = dx^2 \\ dy^3 = \pm \frac{1}{2} \frac{-2x^1 dx^1 - 2x^2 dx^2}{\sqrt{R^2 - (x^1)^2 - (x^2)^2}} \end{cases} \quad (4)$$

$$\Rightarrow ds^2 = (dx^1)^2 + (dx^2)^2 + \frac{(x^1)^2 (dx^1)^2 + (x^2)^2 (dx^2)^2 + 2x^1 x^2 dx^1 dx^2}{R^2 - (x^1)^2 - (x^2)^2} \quad (5)$$

$$\Leftrightarrow ds^2 = \frac{(R^2 - (x^2)^2)(dx^1)^2 + (R^2 - (x^1)^2)(dx^2)^2 + 2x^1 x^2 dx^1 dx^2}{R^2 - (x^1)^2 - (x^2)^2} \quad (6)$$

$$\Rightarrow (a_{mn}) = \frac{1}{R^2 - (x^1)^2 - (x^2)^2} \begin{pmatrix} R^2 - (x^2)^2 & x^1 x^2 \\ x^1 x^2 & R^2 - (x^1)^2 \end{pmatrix} \quad (7)$$



## 2.4 p30-clarification 2.202

$$a_{mr}\Delta^{ms} = a_{rm}\Delta^{sm} = \delta_r^s a$$

Case 1:  $r = s$

We have,  $a_{Rm}\Delta^{Rm}$  (no summation on  $R$ ) is the definition of the determinant of  $A$  developed along the row  $R$ : OK.

Case 2:  $r \neq s$

Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \quad (1)$$

and consider the matrix  $A'$

$$A' = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{R1} & a_{R2} & \dots & a_{RN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{R1} & a_{R2} & \dots & a_{RN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \begin{matrix} \vdots \\ \vdots \\ \leftarrow S^{th} \text{ row} \\ \vdots \\ \leftarrow R^{th} \text{ row} \\ \vdots \\ \vdots \end{matrix} \quad (2)$$

This matrix corresponds to the way  $a_{Rm}\Delta^{Sm}$  is computed. Indeed with the factor  $a_{Rm}$  is not associated it's own cofactor  $\Delta^{Rm}$  but the cofactor of the  $m^{th}$  column in row  $S$ . Replacing the  $S^{th}$  row with the row  $R$  and calculating it's determinant is the same as calculating  $a_{Rm}\Delta^{Sm}$

But,  $|A'| = 0$  as we have two identical rows. So,  $a_{Rm}\Delta^{Sm} = 0$

Conclusion : The same reasoning can be applied when expanding the determinant along the columns instead of the rows we have indeed  $a_{mr}\Delta^{ms} = a_{rm}\Delta^{sm} = \delta_r^s a$ .



## 2.5 p31-exercise

Show that if  $am_n = 0$  for  $m \neq n$ , then

$$a^{11} = \frac{1}{a_{11}}, a^{22} = \frac{1}{a_{22}}, \dots, a^{12} = 0, \dots$$

We have to prove that:

$$a^{ij} = \begin{cases} \frac{1}{a_{ij}} & : i = j \\ 0 & : i \neq j \end{cases} \quad (1)$$

From 2.204:

$$a_{mR}a_{mS} = \delta_R^S \quad (2)$$

i) Be  $R \neq S$

$$(2) \Rightarrow a_{mR}a_{mS} = 0 \quad (3)$$

$$\text{but } a_{mR} = 0 \quad \forall m \neq R \quad (4)$$

$$\Rightarrow a_{RR}a^{RS} = 0 \quad (5)$$

but  $a_{RR} \neq 0$  ( $a_{RR}$  can't be 0 as the metric tensor would degenerate if  $a_{mn} = 0 \quad \forall m \neq n$ )

$$\Rightarrow a^{RS} = 0 \quad (6)$$

$$(7)$$

$$(8)$$

i) Be  $R = S$

$$(2) \Rightarrow a_{mR}a_{mR} = 1 \quad (9)$$

$$\text{but } a_{mR} = 0 \quad \forall m \neq R \quad (10)$$

$$\Rightarrow a_{RR}a^{RR} = 1 \quad (11)$$

$$\Rightarrow a^{RR} = \frac{1}{a_{RR}} \quad (12)$$



## 2.6 p31-exercise

Find the components of  $a^{mn}$  for spherical polar coordinates in Eulidean 3-space.

We have (see exercise page 27):

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (1)$$

As  $a_{mn} = 0 \quad \forall m \neq n$  we deduce (see previous exercise p31)

$$(a^{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix} \quad (2)$$



## 2.7 p32-exercise

Find the mixed metric tensor  $a_m^{\cdot n}$  obtained from  $a_{mn}$  by raising the second subscript

We have :

$$a_i^{\cdot j} = a_{in} a^{nj} \quad (1)$$

$$= a_{in} a^{jn} \quad a^{jn} \text{ is symmetric} \quad (2)$$

$$= \delta_i^j \quad (\text{see 2.205 pg. 30}) \quad (3)$$

$$\Rightarrow a_i^{\cdot j} = \delta_i^j \quad (4)$$



## 2.8 p32-clarification 2.214

$$\frac{\partial a}{\partial a_{mn}} = aa^{mn}$$

Put  $a_{MN} \equiv a_{mn}$ . By definition, we have

$$a \equiv |a_{mn}| = a_{Mk} \Delta^{Mk} \quad (\text{develop determinant along row M}) \quad (1)$$

$$\Rightarrow \frac{\partial a}{\partial a_{mn}} = \frac{\partial a_{Mk}}{\partial a_{mn}} \Delta^{Mk} + a_{Mk} \frac{\partial \Delta^{Mk}}{\partial a_{mn}} \quad (2)$$

$$\text{but } \frac{\partial a_{Mk}}{\partial a_{mn}} = \begin{cases} 1 & \text{if } k = N \\ 0 & \text{if } k \neq N \end{cases} \quad (3)$$

$$\text{and } \frac{\partial \Delta^{Mk}}{\partial a_{mn}} = 0 \quad \forall k \text{ as } \Delta^{Mk} \text{ does not contain the row with } a_{mn} \text{ as element.} \quad (4)$$

$$(3) \text{ and } (4) \Rightarrow \frac{\partial a}{\partial a_{mn}} = \Delta^{MN} \quad (5)$$

$$a^{mn} = \frac{\Delta^{mn}}{a} \quad \text{by definition (see 2.203 page 30)} \quad (6)$$

$$\Rightarrow \frac{\partial a}{\partial a_{mn}} = aa^{mn} \quad (7)$$



## 2.9 p32-exercise

Prove that  $a_{mn}a^{mn} = N$ .

From 2.204, we have

$$a_{mr}a^{ms} = \delta_r^s \quad (1)$$

$$\text{Consider } a_{mR}a^{mR} = 1 \quad (2)$$

$$\text{We can repeat (2) for } R = 1, 2, \dots, N \Rightarrow a_{mr}a^{mr} = N \quad (3)$$

$$(4)$$





## 2.10 p33-exercise

Show that in Euclidean 3-space with rectangular Cartesian coordinates, the definition 2.301 coincides with the usual definition of the magnitude of a vector.

The length of an arbitrary vector in Euclidean 3-space with rectangular Cartesian coordinates, is

$$ds^2 = (dy^1)^2 + (dy^2)^2 + (dy^3)^2 \quad (1)$$

From 2.301, it is obvious that the metric tensor can be expressed as,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2)$$



## 2.11 p34-exercise

A curve in Euclidean 3-space has the equations

$$x^1 = a \cos(u), x^2 = a \sin(u), x^3 = bu$$

where  $x^1, x^2, x^3$  are rectangular Cartesian coordinates,  $u$  is a parameter, and  $a, b$  are positive constants. Find the length of this curve between the point  $u = 0$  and  $u = 2\pi$ .

The metric tensor has the following form,

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

$$\text{and (2.306)} \quad s = \int_0^{2\pi} [\epsilon a_{mn} p^m p^n]^{\frac{1}{2}} du \quad (2)$$

with

$$p^1 = \frac{dx^1}{du} = -a \sin(u), \quad p^2 = \frac{dx^2}{du} = a \cos(u), \quad p^3 = \frac{dx^3}{du} = b \quad (3)$$

Hence (2) becomes

$$s = \int_0^{2\pi} \epsilon [a^2 \sin^2(u) + a^2 \cos^2(u) + b^2]^{\frac{1}{2}} du \quad (4)$$

$$= \int_0^{2\pi} \epsilon [a^2 + b^2]^{\frac{1}{2}} du \quad (5)$$

$$= [a^2 + b^2]^{\frac{1}{2}} u \Big|_0^{2\pi} \quad (6)$$

$$= 2\pi [a^2 + b^2]^{\frac{1}{2}} \quad (7)$$



## 2.12 p36-clarification 2.314

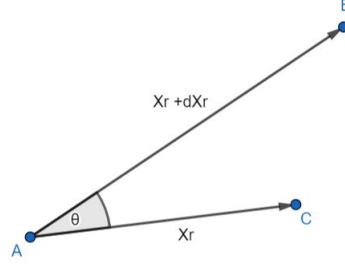
Going from 2.313 to 2.314 yields because both  $X^m$  and  $Y^m$  are unit vectors and by definition of the magnitude (see 2.301) both  $a_{mn}X^mX^n$  and  $a_{mn}Y^mY^n$  are 1 (also due to the fact that only a positive definite metric tensor is considered,  $\epsilon = 1$ ).



## 2.13 p37-exercise

Show that the small angle between unit vectors  $X^r$  and  $X^r + dX^r$  (these increments being infinitesimal) is given by

$$\theta^2 = a_{mn} X^m X^n$$



By definition (2.302 page 33)

$$|BC|^2 = \epsilon a_{mn} dX^m dX^n \quad (1)$$

(2)

We can drop  $\epsilon = 1$  as the considered space is positive definite.

As  $\theta$  is infinitesimal, we can state

$$|BC| \approx |AC|\theta \quad (3)$$

$$\text{and } |AC| = X^r = 1 \quad (\text{as } X^r \text{ is a unit vector}) \quad (4)$$

$$\Rightarrow \theta^2 = a_{mn} dX^m dX^n \quad (5)$$



## 2.14 p39-clarification 2.409

We clarify the integration by parts in the derivation of the general geodesic equation.

We have

$$\int d(A.B) = \int AdB + \int BdA \quad (1)$$

$$\Rightarrow \int AdB = \int BdA - \int d(A.B) \quad (2)$$

Now, substitute 2.407 in 2.406, we get

$$\frac{dL}{dv} = \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r} \frac{\partial x^r}{\partial v} du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} \frac{\partial p^r}{\partial v} du \quad (3)$$

$$= \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r} \frac{\partial x^r}{\partial v} du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} \frac{\partial \frac{\partial x^r}{\partial v}}{\partial u} du \quad (4)$$

$$= \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r} \frac{\partial x^r}{\partial v} du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} d\left(\frac{\partial x^r}{\partial v}\right) \quad (5)$$

To integrate by parts the second term in (5) we put in (2)

$$A = \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} \quad \text{and} \quad B = \frac{\partial x^r}{\partial v}$$

$$\int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} d\left(\frac{\partial x^r}{\partial v}\right) = \int AdB \quad (6)$$

$$= \int BdA - \int d(A.B) \quad (7)$$

$$= \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} \frac{\partial x^r}{\partial v} \Big|_{u_0}^{u_1} - \int \frac{\partial x^r}{\partial v} d\left(\frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\right) \quad (8)$$

$$= \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} \frac{\partial x^r}{\partial v} \Big|_{u_0}^{u_1} - \int \frac{\partial x^r}{\partial v} \frac{\partial \left(\frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\right)}{\partial u} du \quad (9)$$

Replacing (9) in (5) gives the formula 2.409.



## 2.15 p41-exercise

Prove the following identities:

$$[mn, r] = [nm, r], \quad [rm, n] + [rn, m] = \partial_r a_{mn}$$

$$[mn, r] = \frac{1}{2}(\partial_n a_{mr} + \partial_m a_{nr} - \partial_r a_{mn}) \quad (1)$$

$$= \frac{1}{2}(\partial_m a_{nr} + \partial_n a_{mr} - \partial_r a_{nm}) \quad (2)$$

$$= [nm, r] \quad (3)$$

and

$$[rm, n] + [rn, m] = \frac{1}{2}(\partial_r a_{mn} + \partial_m a_{rn} - \partial_n a_{rm} + \partial_n a_{rm} + \partial_r a_{mn} - \partial_m a_{rn}) \quad (4)$$

$$= \frac{1}{2}(\partial_r a_{mn} + \partial_r a_{mn}) \quad (5)$$

$$= \partial_r a_{mn} \quad (6)$$



## 2.16 p42-exercise

Prove that

$$[mn, r] = a_{rs}\Gamma_{mn}^s$$

$$a_{rs}\Gamma_{mn}^s = a_{rs}a^{sk}[mn, k] \tag{1}$$

$$= \delta_r^k[mn, k] \tag{2}$$

$$= [mn, r] \tag{3}$$



## 2.17 p42-clarification on 2.430

... This may be proved without difficulty by starting with 2.427, in which  $\lambda$  is a known function of  $u$ , and defining  $s$  by the relation

$$s = \int_{u_0}^u (\exp \int_{v_0}^v \lambda(w) dw) dv$$

$u_0, v_0$  being constants....

$$\text{see 2.428 : } \lambda(u) = -\frac{\frac{d^2 u}{ds^2}}{\left(\frac{du}{ds}\right)^2} \quad (1)$$

$$(2)$$

We suppose  $u(s)$  continuous by parts with continuous inverse.

$$\Rightarrow \frac{ds}{du} = \frac{1}{\frac{du}{ds}} \quad (3)$$

$$\Rightarrow \frac{d^2 s}{du^2} = \frac{d\left(\frac{1}{\frac{du}{ds}}\right)}{ds} \frac{ds}{du} \quad (4)$$

$$= -\frac{\frac{d^2 u}{ds^2}}{\left(\frac{du}{ds}\right)^2} \frac{ds}{du} \quad (5)$$

$$\Rightarrow \frac{\frac{d^2 s}{du^2}}{\frac{ds}{du}} = -\frac{\frac{d^2 u}{ds^2}}{\left(\frac{du}{ds}\right)^2} \quad (6)$$

By definition (2.428)

$$\lambda(u) = -\frac{\frac{d^2 u}{ds^2}}{\left(\frac{du}{ds}\right)^2} \quad (7)$$

$$\text{hence by (6) and (7): } \lambda(u) = \frac{\frac{d^2 s}{du^2}}{\frac{ds}{du}} \quad (8)$$

$$\text{in (8) put } y = \frac{ds}{du} \quad (9)$$

$$\text{and so } \lambda(u) = \frac{y'}{y} \quad (10)$$

$$\Rightarrow \int \frac{y'}{y} dw = \int \lambda(w) dw \quad (11)$$

$$\Leftrightarrow \int d(\ln y) = \int \lambda(w) dw \quad (12)$$

$$\Rightarrow \ln(y)|_{v_0}^v = \int_{v_0}^v \lambda(w) dw \quad (13)$$

$$\Rightarrow y = \exp\left(\int_{v_0}^v \lambda(w) dw\right) + C \quad (14)$$



Taking into account (9), we get:

$$\frac{ds}{dv} = \exp\left(\int_{v_0}^v \lambda(w)dw\right) + C \quad (15)$$

$$\Rightarrow s|_{u_0}^u = \int_{u_0}^u \exp\left(\int_{v_0}^v \lambda(w)dw\right)dv + Cu + B \quad (16)$$

$$\Leftrightarrow s = \int_{u_0}^u \exp\left(\int_{v_0}^v \lambda(w)dw\right)dv + Cu + B \quad (17)$$

We show tha we have to put  $C = 0$  and can drop the constant  $B$ . Remember by (8)

$$\lambda(u) = \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} \quad (18)$$

$$\text{by (17)} \quad \frac{ds}{du} = \exp\left(\int_{u_0}^u \lambda(w)dw\right) + C \quad (19)$$

$$\text{and} \quad \frac{d^2s}{du^2} = \lambda(u) \exp\left(\int_{u_0}^u \lambda(w)dw\right) \quad (20)$$

$$\text{hence by (18), (19) and (20):} \quad \lambda(u) = \frac{\lambda(u) \exp\left(\int_{u_0}^u \lambda(w)dw\right)}{\exp\left(\int_{u_0}^u \lambda(w)dw\right) + C} \quad (21)$$

So, whatever the constant B, the relation (18) is correct on the condition that C=0. So, indeed, we can choose the independent variable  $s$  as

$$s = \int_{u_0}^u \exp\left(\int_{v_0}^v \lambda(w)dw\right)dv$$



## 2.18 p42-clarification on 2.430

After 2.430 it is stated:

"No matter what values these constants have, 2.424 is satisfied, and by adjusting the constant  $v_0$ , we can ensure that  $a_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = \pm 1$  along  $C$ , so that  $s$  is actually the arc length."

We first prove that 2.4.24 is satisfied, no matter what values the constants take. We have

$$(2.430) \quad s = \int_{u_0}^u (\exp \int_{v_0}^v \lambda(w) dw) dv \quad (1)$$

$$\text{and (2.427)} \quad \frac{d^2 x^r}{du^2} + \Gamma_{mn}^r \frac{dx^m}{du} \frac{dx^n}{du} = \lambda \frac{dx^r}{du} \quad (2)$$

In (2) we can write the first term as

$$\frac{d^2 x^r}{du^2} = \frac{d(\frac{dx^r}{du})}{ds} \frac{ds}{du} \quad (3)$$

$$\text{with} \quad \frac{d(\frac{dx^r}{du})}{ds} = \frac{d(\frac{dx^r}{ds} \frac{ds}{du})}{ds} = \frac{d^2 x^r}{ds^2} \frac{ds}{du} + \frac{dx^r}{ds} \frac{d(\frac{ds}{du})}{ds} \quad (4)$$

Assuming the curve smooth, we have

$$\frac{d(\frac{ds}{du})}{ds} = \frac{d(\frac{1}{\frac{du}{ds}})}{ds} = -\frac{\frac{d^2 u}{ds^2}}{(\frac{du}{ds})^2} = \lambda \quad (5)$$

Putting (4) and (5) in (3) we get

$$\frac{d^2 x^r}{du^2} = \frac{d^2 x^r}{ds^2} \left( \frac{ds}{du} \right)^2 + \lambda \frac{dx^r}{ds} \frac{ds}{du} \quad (6)$$

Plugging (6) in 2.427 gives:

$$\frac{d^2 x^r}{ds^2} \left( \frac{ds}{du} \right)^2 + \lambda \frac{dx^r}{ds} \frac{ds}{du} + \Gamma_{mn}^r \frac{dx^m}{ds} \frac{dx^n}{ds} \left( \frac{ds}{du} \right)^2 = \lambda \frac{dx^r}{ds} \frac{ds}{du} \quad (7)$$

$$\Leftrightarrow \frac{d^2 x^r}{ds^2} \left( \frac{ds}{du} \right)^2 + \Gamma_{mn}^r \frac{dx^m}{ds} \frac{dx^n}{ds} \left( \frac{ds}{du} \right)^2 = 0 \quad (8)$$

We can assume that  $\frac{ds}{du}$  does not become 0 or  $\pm\infty$  along the curve by choosing an adequate constant  $v_0$ . Indeed, from (2.430) we get

$$\frac{ds}{du} = \exp\left(\int_{u_0}^u \lambda(w) dw\right) \quad (9)$$

$$= \frac{\phi(u)}{\phi(u_0)} \quad (10)$$

with  $\phi(u_0) = e^{\theta(u_0)}$ ,  $\theta(u)$  being the indefinite integral  $\int \lambda(w) dw$ .

So, it is sufficient to choose  $v_0$  so that  $\theta(u_0)$  does not become  $\pm\infty$  to ensure that  $\frac{ds}{du} \neq 0$  or  $\neq \pm\infty$

along the curve and so we have from (8)

$$\frac{d^2 x^r}{ds^2} + \Gamma_{mn}^r \frac{dx^m}{ds} \frac{dx^n}{ds} = 0 \quad (11)$$

which is the definition (2.424) of a geodesic.

The same reasoning about  $\frac{ds}{du} \neq 0$  can be made to prove that  $a_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = \pm 1$  along  $C$ . Indeed, by definition (2.305):

$$ds = \left[ \epsilon a_{mn} \frac{dx^m}{du} \frac{dx^n}{du} \right]^{\frac{1}{2}} \quad (12)$$

$$\text{equating with (9)} \quad \left[ \epsilon a_{mn} \frac{dx^m}{du} \frac{dx^n}{du} \right]^{\frac{1}{2}} = \exp\left(\int_{u_0}^u \lambda(w) dw\right) \quad (13)$$

$$\Rightarrow \epsilon a_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} \left(\frac{ds}{du}\right)^2 = \left[\exp\left(\int_{u_0}^u \lambda(w) dw\right)\right]^2 \quad (14)$$

$$\text{but } \frac{ds}{du} = \exp\left(\int_{u_0}^u \lambda(w) dw\right) \quad (15)$$

$$\text{and so, (9) becomes } \epsilon a_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} \left(\frac{ds}{du}\right)^2 = \left(\frac{ds}{du}\right)^2 \quad (16)$$

$$\Rightarrow a_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = \epsilon \quad (17)$$



## 2.19 p43-clarification

$$\lambda = \Gamma_{\mu\nu}^N \frac{dx^\mu}{dx^N} \frac{dx^\nu}{dx^N} + 2\Gamma_{\mu N}^N \frac{dx^\mu}{dx^N} + \Gamma_{NN}^N$$

We start with 2.427 with  $r = N$

$$\frac{d^2 x^N}{dx^{N^2}} + \Gamma_{\mu\nu}^N \frac{dx^\mu}{dx^N} \frac{dx^\nu}{dx^N} = \lambda \frac{dx^N}{dx^N} \quad (1)$$

$$\Rightarrow \Gamma_{\mu\nu}^N \frac{dx^\mu}{dx^N} \frac{dx^\nu}{dx^N} = \lambda \quad (\text{as } \frac{d^2 x^N}{dx^{N^2}} = 0 \quad \frac{dx^N}{dx^N} = 1) \quad (2)$$

But (2) is only valid with the dummy indices  $\mu$  and  $\nu$  spanning the whole dimension  $(1, 2, \dots, N)$ , but by choice  $\mu, \nu \in (1, 2, \dots, N-1)$ . We have thus to add in the left term of (2) the cases

$$\left\{ \begin{array}{l} \Gamma_{N\nu}^N \quad \nu = (1, 2, \dots, N-1) \\ \Gamma_{\mu N}^N \quad \mu = (1, 2, \dots, N-1) \\ \Gamma_{NN}^N \end{array} \right. \quad (3)$$

$$(2) \text{ becomes } \Gamma_{\mu\nu}^N \frac{dx^\mu}{dx^N} \frac{dx^\nu}{dx^N} + \Gamma_{N\nu}^N \frac{dx^N}{dx^N} \frac{dx^\nu}{dx^N} + \Gamma_{\mu N}^N \frac{dx^\mu}{dx^N} \frac{dx^N}{dx^N} + \Gamma_{NN}^N \frac{dx^N}{dx^N} \frac{dx^N}{dx^N} = \lambda \quad (4)$$

$$\Rightarrow \Gamma_{\mu\nu}^N \frac{dx^\mu}{dx^N} \frac{dx^\nu}{dx^N} + \Gamma_{N\nu}^N \frac{dx^\nu}{dx^N} + \Gamma_{\mu N}^N \frac{dx^\mu}{dx^N} + \Gamma_{NN}^N = \lambda \quad (5)$$

As  $\Gamma_{\mu N}^N$  is symmetric on the lower indices and  $\mu, \nu$  being dummy indices:

$$\lambda = \Gamma_{\mu\nu}^N \frac{dx^\mu}{dx^N} \frac{dx^\nu}{dx^N} + 2\Gamma_{N\nu}^N \frac{dx^\nu}{dx^N} + \Gamma_{NN}^N \quad (6)$$

The other  $N-1$  equation for  $r = 1, \dots, N-1$  can be deduced following the same reasoning.



## 2.20 p45-clarification

$f_r \equiv a_{rm} \frac{d^2 x^m}{du^2} + [mn, r] \frac{dx^m}{du} \frac{dx^n}{du}$  is covariant and  $f^r \equiv \frac{d^2 x^m}{du^2} + \Gamma_{mn}^r \frac{dx^m}{du} \frac{dx^n}{du}$  is contravariant.

$$f_r \equiv a_{rm} \frac{d^2 x^m}{du^2} + [mn, r] \frac{dx^m}{du} \frac{dx^n}{du} \quad (1)$$

$$\text{multiply with } a^{sr} \Rightarrow f_r a^{sr} = a^{sr} a_{rm} \frac{d^2 x^m}{du^2} + a^{sr} [mn, r] \frac{dx^m}{du} \frac{dx^n}{du} \quad (2)$$

$$\Rightarrow f^s = \delta_m^s \frac{d^2 x^m}{du^2} + \underbrace{a^{sr} [mn, r]}_{\Gamma_{mn}^s} \frac{dx^m}{du} \frac{dx^n}{du} \quad (3)$$

$$\Rightarrow f^s = \frac{d^2 x^s}{du^2} + \Gamma_{mn}^s \frac{dx^m}{du} \frac{dx^n}{du} \quad (4)$$

By lifting the index of  $f_r$  we get a contravariant vector confirming that (4) is contravariant.



## 2.21 p45-clarification

(2.443) and (2.444)

$$p^r \frac{\partial w}{\partial p^r} - w = C^t \quad \Rightarrow \quad w = C^t$$

By definition

$$w = a_{mn} p^m p^n \tag{1}$$

$$\Rightarrow \frac{\partial w}{\partial p^r} = a_{mn} \left( \frac{\partial p^m}{\partial p^r} p^n + p^m \frac{\partial p^n}{\partial p^r} \right) \tag{2}$$

$$= a_{mn} (\delta_r^m p^n + p^m \delta_r^n) \tag{3}$$

$$= a_{rn} p^n + a_{mr} p^m \tag{4}$$

$$= 2a_{mr} p^m \quad (\text{as } a_{mn} \text{ is symmetric}) \tag{5}$$

$$(4) \quad \Rightarrow \quad p^r \frac{\partial w}{\partial p^r} = 2a_{mr} p^r p^m \tag{6}$$

$$= 2w \tag{7}$$

$$(2.443) \quad \Rightarrow \quad p^r \frac{\partial w}{\partial p^r} - w = 2w - w = w = C^t \tag{8}$$

$$\Rightarrow \quad w \equiv a_{mn} p^m p^n = C^t \tag{9}$$



## 2.22 p46-”geodesic null lines”, some personal thoughts

Can we grasp intuitively the notion of ”geodesic null lines”? To understand this, here are two example of ”geodesic null lines”

We consider two examples, a classic one (Minkowski space) and a weird space.

i) on dimensional Minkowski space

For this space the metric is in normalized coordinates

$$ds^2 = dx^2 - dt^2$$

It is easy to see that the equations of a geodesic null line (taking  $t$  as parameter for the curve), reduce to

$$\left. \begin{aligned} \frac{d^2x}{dt^2} &= 0 \\ dx^2 - dt^2 &= 0 \end{aligned} \right\} \quad (1)$$

$$\Rightarrow x = at + b \quad \text{and} \quad x = \pm t + g \quad (2)$$

If we choose  $x = t$  at  $t = 0$  we see that the geodesic null lines are the bisectors of the x-t axes.

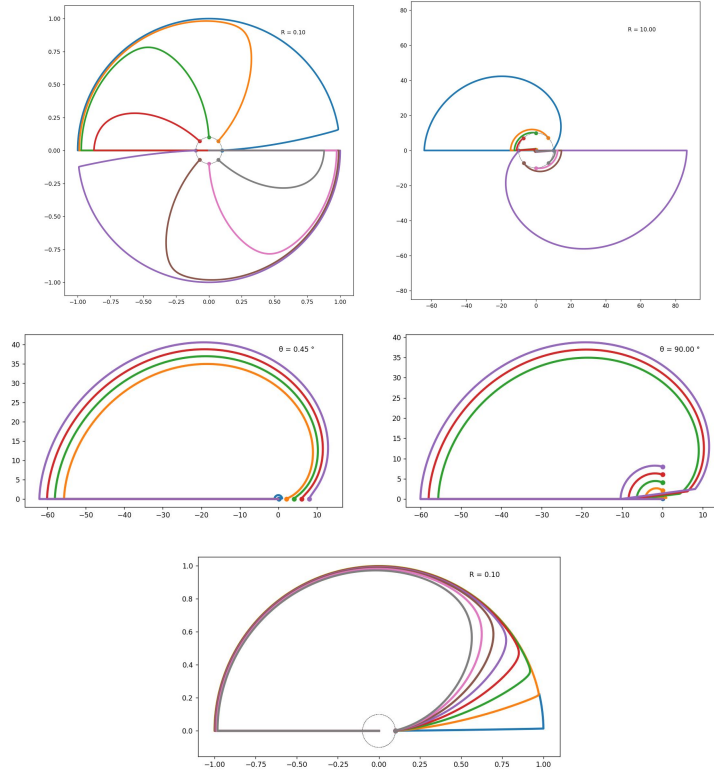
ii) We consider a  $V_2$  space equipped with a polar coordinate system and with a metric tensor defined as

$$ds^2 = (1 - r)dr^2 + \sin(\theta)d\theta^2 \quad (3)$$

Giving as geodesic null line equations

$$\left. \begin{aligned} \frac{d^2r}{du^2} + \frac{1}{2(r-1)}\left(\frac{dr}{du}\right)^2 &= 0 \\ \frac{d^2\theta}{du^2} + \frac{1}{2}\cot(\theta)\left(\frac{d\theta}{du}\right)^2 &= 0 \\ (1 - r)\left(\frac{dr}{du}\right)^2 + \sin(\theta)\left(\frac{d\theta}{du}\right)^2 &= 0 \end{aligned} \right\} \quad (4)$$

We see immediately that for  $r \rightarrow 1$  and  $\theta \rightarrow k\frac{\pi}{2}$  ( $k = 0, \pm 1, \pm 2, \dots$ ), some problems will occur and that no solutions might be found. On the next page some geodesic null lines are showed (equations solved numerically with Python Scipy library)



The first row shows the results for points at 2 different distance from the origin while changing the angle  $\theta$ .

The second row shows the results for points at 2 different angles while changing the distance from the origin.

The second row shows the results for a specific point while changing the the  $\frac{d\theta}{du}$  value in the initial condition when solving the system of differential equations.

Conclusion: with non-trivial metrics, getting an intuitive idea about the geodesic null lines gets very hard.





## 2.23 p47-exercise

The class of all parameters  $u$ , for which the equations of a geodesic null line assume the simple form 2.445, are obtained from any one such parameter by linear transformation

$$\bar{u} = au + b$$

$a$  and  $b$  being constants.

The simple form 2.445 is :

$$\frac{d^2 x^r}{du^2} + \Gamma_{mn}^r \frac{dx^m}{du} \frac{dx^n}{du} = 0 \quad (1)$$

(2)

The general form of a geodesic is (2.447)

$$\frac{d^2 x^r}{d\bar{u}^2} + \Gamma_{mn}^r \frac{dx^m}{d\bar{u}} \frac{dx^n}{d\bar{u}} = \lambda \frac{dx^r}{d\bar{u}} \quad (3)$$

(4)

So (2.447) can only of the form (2.445) if  $\lambda = 0$

$$\lambda = - \frac{\frac{d^2 \bar{u}}{du^2}}{\left(\frac{d\bar{u}}{du}\right)^2} = 0 \quad (5)$$

(6)

We can state that  $\frac{d\bar{u}}{du} \neq 0$  as  $\bar{u}$  can't be a constant (being a parameter of a curve). So,

$$\frac{d^2 \bar{u}}{du^2} = 0 \quad (7)$$

$$\Rightarrow \frac{d\bar{u}}{du} = a \quad (8)$$

$$\Rightarrow \bar{u} = au + b \quad (9)$$



## 2.24 p47-exercise

Consider a 3-space with coordinates  $x, y, z$  and a metric form  $\Phi = (dx)^2 + (dy)^2 - (dz)^2$ .  
prove that the geodesic null lines may be represented by the equations

$$x = au + a' \quad y = bu + b' \quad z = cu + c'$$

where  $u$  is a parameter and  $a, a', b, b', c, c'$  are constants which are arbitrary except for the relation  $a^2 + b^2 - c^2 = 0$ .

Given is

$$\Phi = (dx)^2 + (dy)^2 - (dz)^2 \tag{1}$$

From the previous exercise we have already proven that  $x, y, z$  are of the form

$$x^i = q_i u + q'_i \tag{2}$$

To be a null geodesic null line we need to have (2.448)

$$a_{mn} \frac{dx^m}{du} \frac{dx^n}{du} = 0 \tag{3}$$

$$\text{from (1) we deduce } (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4}$$

$$(3) \Rightarrow (dx)^2 + (dy)^2 - (dz)^2 = 0 \tag{5}$$

$$(2) \Rightarrow (q_1)^2 + (q_2)^2 - (q_3)^2 = 0 \tag{6}$$



## 2.25 p48-exercise

Prove that the Christoffel symbols of the first kind transform according the equation

$$[mn, r]' = [pq, s] \frac{\partial x^p}{\partial x^{\cdot m}} \frac{\partial x^q}{\partial x^{\cdot n}} \frac{\partial x^s}{\partial x^{\cdot r}} + a_{pq} \frac{\partial x^p}{\partial x^{\cdot r}} \frac{\partial^2 x^q}{\partial x^{\cdot m} \partial x^{\cdot n}}$$

From 2.438 page 45, we have

$$f_r \equiv a_{rm} \frac{d^2 x^m}{du^2} + [mn, r] \frac{dx^m}{du} \frac{dx^n}{du} \quad \text{is covariant} \quad (1)$$

$$\Rightarrow f_r' = f_s \frac{\partial x^s}{\partial x^{\cdot r}} \quad (2)$$

$$\text{with } f_r' = a_{rm} \frac{d^2 x^m}{du^2} + [mn, r]' \frac{dx^m}{du} \frac{dx^n}{du} \quad (3)$$

Combining (1), (2) and (3) gives

$$a_{rm} \frac{d^2 x^m}{du^2} + [mn, r]' \frac{dx^m}{du} \frac{dx^n}{du} = (a_{sm} \frac{d^2 x^m}{du^2} + [mn, s] \frac{dx^m}{du} \frac{dx^n}{du}) \frac{\partial x^s}{\partial x^{\cdot r}} \quad (4)$$

We rewrite (4) as

$$[mn, r]' \frac{dx^m}{du} \frac{dx^n}{du} = - \underbrace{a_{rm} \frac{d^2 x^m}{du^2}}_{(*)} + \underbrace{a_{sm} \frac{d^2 x^m}{du^2} \frac{\partial x^s}{\partial x^{\cdot r}}}_{(**)} + \underbrace{[mn, s] \frac{dx^m}{du} \frac{dx^n}{du} \frac{\partial x^s}{\partial x^{\cdot r}}}_{(***)} \quad (5)$$

$$(***) \Leftrightarrow [mn, s] \frac{\partial x^m}{\partial x^{\cdot p}} \frac{dx^p}{du} \frac{\partial x^n}{\partial x^{\cdot q}} \frac{dx^q}{du} \frac{\partial x^s}{\partial x^{\cdot r}} \quad (6)$$

In (6) renaming the dummy indices  $m, n, p, q$  gives

$$(***) \Leftrightarrow [pq, s] \frac{\partial x^p}{\partial x^{\cdot m}} \frac{dx^m}{du} \frac{\partial x^q}{\partial x^{\cdot n}} \frac{dx^n}{du} \frac{\partial x^s}{\partial x^{\cdot r}} \quad (7)$$

$$\Leftrightarrow [pq, s] \frac{\partial x^p}{\partial x^{\cdot m}} \frac{\partial x^q}{\partial x^{\cdot n}} \frac{\partial x^s}{\partial x^{\cdot r}} \left( \frac{dx^m}{du} \frac{dx^n}{du} \right) \quad (8)$$

Also,

$$(**) \Leftrightarrow a_{sm} \frac{d^2 x^m}{du^2} \frac{\partial x^s}{\partial x^{\cdot r}} \quad (9)$$

$$\text{As we have also } \frac{d^2 x^m}{du^2} = \frac{d(\frac{\partial x^m}{\partial x^{\cdot p}} \frac{dx^p}{du})}{du} \quad (10)$$

$$= \frac{\partial x^m}{\partial x^{\cdot p}} \frac{d^2 x^p}{du^2} + \frac{dx^p}{du} \frac{\partial^2 x^m}{\partial x^{\cdot p} \partial x^{\cdot q}} \frac{dx^q}{du} \quad (11)$$

(11) and (9) gives by changing the dummy indices (  $m \rightarrow t, p \rightarrow m, q \rightarrow n$  )

$$(**) = \underbrace{a_{st} \frac{\partial x^t}{\partial x^{\cdot p}} \frac{d^2 x^p}{du^2} \frac{\partial x^s}{\partial x^{\cdot r}}}_{(****)} + a_{pq} \frac{\partial^2 x^q}{\partial x^{\cdot m} \partial x^{\cdot n}} \frac{\partial x^p}{\partial x^{\cdot r}} \left( \frac{dx^{\cdot m}}{du} \frac{dx^{\cdot m}}{du} \right) \quad (12)$$

$$\text{with } (****) = a_{st} \left( \frac{\partial x^t}{\partial x^{\cdot m}} \frac{\partial x^s}{\partial x^{\cdot r}} \right) \frac{d^2 x^{\cdot m}}{du^2} \quad (13)$$

But  $a_{st}$  is a covariant tensor, so

$$a'_{rm} = a_{st} \frac{\partial x^t}{\partial x^{\cdot m}} \frac{\partial x^s}{\partial x^{\cdot r}} \quad (14)$$

$$(13) \text{ becomes } (****) = a'_{rm} \frac{d^2 x^{\cdot m}}{du^2} \quad (15)$$

$$\text{and from (5) we have } (*) = -a'_{rm} \frac{d^2 x^{\cdot m}}{du^2} \quad (16)$$

$$(17)$$

and both terms cancel each other in equation (5). So adding (\*), (\*\*) and (\*\*\*) in (5) , we get

$$[mn, r] \frac{dx^{\cdot m}}{du} \frac{dx^{\cdot n}}{du} = a_{pq} \frac{\partial^2 x^q}{\partial x^{\cdot m} \partial x^{\cdot n}} \frac{\partial x^p}{\partial x^{\cdot r}} \left( \frac{dx^{\cdot m}}{du} \frac{dx^{\cdot m}}{du} \right) + [pq, s] \frac{\partial x^p}{\partial x^{\cdot m}} \frac{\partial x^q}{\partial x^{\cdot n}} \frac{\partial x^s}{\partial x^{\cdot r}} \left( \frac{dx^{\cdot m}}{du} \frac{dx^{\cdot n}}{du} \right) \quad (18)$$

$$\Rightarrow [mn, r] = [pq, s] \frac{\partial x^p}{\partial x^{\cdot m}} \frac{\partial x^q}{\partial x^{\cdot n}} \frac{\partial x^s}{\partial x^{\cdot r}} + a_{pq} \frac{\partial x^p}{\partial x^{\cdot r}} \frac{\partial^2 x^q}{\partial x^{\cdot m} \partial x^{\cdot n}} \quad (19)$$



## 2.26 p50-clarification 2.515

$$\frac{d(T_r S^r)}{du} = \frac{dT_r}{du} S^r + T_r \frac{dS^r}{du} = \left( \frac{dT_r}{du} - \Gamma_{rn}^m T_m \frac{dx^n}{du} \right) S^r$$

with

$$\frac{\delta T_r}{\delta u} = \frac{dT_r}{du} - \Gamma_{rn}^m \frac{dx^n}{du} T_m$$

a covariant vector.

It is given that  $S^r$  is a tensor propagated parallelly along the curve. The by (2.5212) we have

$$\frac{dS^r}{du} + \Gamma_{mn}^r S^m \frac{dx^n}{du} = 0 \quad (1)$$

$$\frac{dS^r}{du} = -\Gamma_{mn}^r S^m \frac{dx^n}{du} \quad (2)$$

$$\text{and } \frac{d(T_r S^r)}{du} = \frac{dT_r}{du} S^r + T_r \frac{dS^r}{du} \quad (3)$$

$$= \frac{dT_r}{du} S^r - \Gamma_{mn}^r S^m \frac{dx^n}{du} T_r \quad (4)$$

$$(5)$$

Swap dummy indices  $r$  and  $m$  in the second term:

$$\frac{d(T_r S^r)}{du} = \frac{dT_r}{du} S^r - \Gamma_{rn}^m S^r \frac{dx^n}{du} T_m \quad (6)$$

$$= \left( \frac{dT_r}{du} - \Gamma_{rn}^m \frac{dx^n}{du} T_m \right) S^r \quad (7)$$

$$(8)$$

As  $T_r S^r$  is an invariant and thus is also  $\frac{d(T_r S^r)}{du}$  and as  $S^r$  can be chosen arbitrarily (as long it is a contravariant tensor), implies that

$$\frac{dT_r}{du} - \Gamma_{rn}^m \frac{dx^n}{du} T_m$$

is covariant and thus also

$$\frac{\delta T_r}{\delta u} = \frac{dT_r}{du} - \Gamma_{rn}^m \frac{dx^n}{du} T_m$$



## 2.27 p50-clarification 2.516

$$\frac{\delta T_{rs}}{\delta u} \equiv \frac{dT_{rs}}{du} - \Gamma_{rn}^m T_{ms} \frac{dx^n}{du} - \Gamma_{sn}^m T_{rm} \frac{dx^n}{du}$$

is a covariant vector.

We build an invariant  $T_{rs}S^rU^s$  with  $S^r$  and  $U^s$  arbitrary contravariant tensors. Then we know that  $\frac{d(T_{rs}S^rU^s)}{du}$  is also an invariant. We have

$$\frac{d(T_{rs}S^rU^s)}{du} = \frac{dT_{rs}}{du}S^rU^s + T_{rs}\frac{dS^r}{du}U^s + T_{rs}S^r\frac{dU^s}{du} \quad (1)$$

with  $S^r$  and  $U^s$  propagated parallelly along the curve. Then,

$$\frac{dS^r}{du} = -\Gamma_{mn}^r S^m \frac{dx^n}{du} \quad (2)$$

$$\frac{dU^s}{du} = -\Gamma_{mn}^s U^m \frac{dx^n}{du} \quad (3)$$

(2), (3) in (1) gives

$$\frac{d(T_{rs}S^rU^s)}{du} = \frac{dT_{rs}}{du}S^rU^s - T_{rs}U^s\Gamma_{mn}^r S^m \frac{dx^n}{du} - T_{rs}S^r\Gamma_{mn}^s U^m \frac{dx^n}{du} \quad (4)$$

Changing the dummy indices in the second and third term gives:

$$\frac{d(T_{rs}S^rU^s)}{du} = \left( \frac{dT_{rs}}{du} - \Gamma_{rn}^m T_{ms} \frac{dx^n}{du} - \Gamma_{sn}^m T_{rm} \frac{dx^n}{du} \right) S^r U^s \quad (5)$$

As the left term is an invariant and  $S^r$  and  $U^s$  are arbitrary contravariant tensors, means that the expression in the brackets in the right part of the equation, is a covariant tensor.



## 2.28 p51-exercise

Find the absolute derivative of  $T_{st}^r$ .

Define the invariant  $I = D_{st}^r R^r S_s T_t$

$$I = D_{st}^r R^r S_s T_t \quad (1)$$

$$\Rightarrow A = \frac{dI}{du} = \frac{d(D_{st}^r)}{du} R^r S_s T_t + D_{st}^r S_s T_t \frac{d(R^r)}{du} + D_{st}^r R_r T_t \frac{d(S^s)}{du} + D_{st}^r R_r S_s \frac{d(T^t)}{du} \quad (2)$$

Reminder, performing a parallel propagation of a covariant and contravariant vector gives as equations

$$\frac{dV^v}{du} = -\Gamma_{mn}^v V^m \frac{dx^n}{du} \quad (3)$$

$$\frac{dW_w}{du} = +\Gamma_{wn}^m W^m \frac{dx^n}{du} \quad (4)$$

So (2) becomes:

$$A = \begin{cases} \frac{d(D_{st}^r)}{du} R^r S_s T_t \\ -D_{st}^r S_s T_t \Gamma_{mn}^r R^m \frac{dx^n}{du} \\ +D_{st}^r R_r T_t \Gamma_{sn}^m S^m \frac{dx^n}{du} \\ +D_{st}^r R_r S_s \Gamma_{tn}^m T^m \frac{dx^n}{du} \end{cases} \quad (5)$$

In (5) apply the following renaming of dummy variables

$$\begin{cases} 2^{nd} line : & r \rightarrow m, m \rightarrow r \\ 3^{rd} line : & s \rightarrow m, m \rightarrow s \\ 4^{th} line : & t \rightarrow m, m \rightarrow t \end{cases}$$

and regrouping terms with  $R^r S_s T_t$ , (5) becomes then

$$A = \left[ \frac{d(D_{st}^r)}{du} + (D_{mt}^r \Gamma_{mn}^s + D_{sm}^r \Gamma_{mn}^t - D_{st}^m \Gamma_{rn}^m) \frac{dx^n}{du} \right] R^r S_s T_t \quad (6)$$

But  $A$  is an invariant, so the expression in the square parenthesis is a tensor of the form  $T_{st}^r$  and we define the absolute derivative of  $T_{st}^r$  as:

$$\frac{\delta T_{st}^r}{\delta u} = \frac{d(T_{st}^r)}{du} + \Gamma_{mn}^s T_{mt}^r \frac{dx^n}{du} + \Gamma_{mn}^t T_{sm}^r \frac{dx^n}{du} - \Gamma_{rn}^m T_{st}^m \frac{dx^n}{du}$$



## 2.29 p53-exercise

Prove that

$$\delta_{s|t}^r = 0, \quad a_{|t}^{rs} = 0$$

i)  $\delta_{s|t}^t = 0$

$$(2.524) \text{ gives: } \delta_{s|t}^r = \underbrace{\frac{\partial \delta_s^r}{\partial x^t}}_{=0} + \Gamma_{mt}^r \delta_s^m - \Gamma_{st}^m \delta_m^r \quad (1)$$

$$= \Gamma_{st}^r - \Gamma_{st}^r = 0 \quad (2)$$

ii)  $a_{|t}^{rs} = 0$  We know that

$$\delta_{s|t}^r = a_{sk} a^{kr}|_t \quad (3)$$

$$\Rightarrow \delta_{s|t}^r = \frac{\partial a_{sk}}{\partial x^t} a^{kr} + a_{sk} \frac{\partial a^{kr}}{\partial x^t} + \Gamma_{mt}^r a_{sk} a^{km} - \Gamma_{st}^m a_{mk} a^{kr} \quad (4)$$

Rearrange (4) and add  $\Gamma_{mt}^k a_{ks} a^{mr}$  and subtract  $\Gamma_{kt}^m a_{ms} a^{kr}$  (as  $\Gamma_{mt}^k a_{ks} a^{mr} - \Gamma_{kt}^m a_{ms} a^{kr} = 0$ )

$$\delta_{s|t}^r = \left( \frac{\partial a_{sk}}{\partial x^t} - \Gamma_{st}^m a_{mk} - \Gamma_{kt}^m a_{ms} \right) a^{kr} + \left( \frac{\partial a^{kr}}{\partial x^t} + \Gamma_{mt}^r a^{km} + \Gamma_{mt}^k a^{mr} \right) a_{sk} \quad (5)$$

$$\text{but } a_{sk|t} = \left( \frac{\partial a_{sk}}{\partial x^t} - \Gamma_{st}^m a_{mk} - \Gamma_{kt}^m a_{ms} \right) \quad (6)$$

$$\text{and as (2.526) } a_{sk|t} = 0 \quad (7)$$

$$(5) \text{ becomes } \delta_{s|t}^r = \underbrace{\left( \frac{\partial a^{kr}}{\partial x^t} + \Gamma_{mt}^r a^{km} + \Gamma_{mt}^k a^{mr} \right)}_{a_{|t}^{kr}} a_{sk} \quad (8)$$

$$= a_{|t}^{kr} a_{sk} \quad (9)$$

$$= 0 \quad \text{as } \delta_{s|t}^r = 0 \quad (\text{see first part of this exercise}) \quad (10)$$

As all  $a_{ks}$  can't be zero and as we didn't choose any special Riemannian space, we can conclude from  $a_{|t}^{kr} a_{sk} = 0$  that

$$a_{|t}^{rs} = 0$$





## 2.30 p54-exercise

Prove that

$$\frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s}$$

$$\frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{d\lambda^n}{ds} \quad (1)$$

By definition of the absolute derivative, we have:

$$\frac{\delta\lambda^n}{\delta s} = \frac{d\lambda^n}{ds} + \Gamma_{pk}^n\lambda^p\frac{dx^k}{ds} \quad (2)$$

$$(2) \text{ in } (1) \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\left(\frac{\delta\lambda^n}{\delta s} - \Gamma_{pk}^n\lambda^p\frac{dx^k}{ds}\right) \quad (3)$$

$$= \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2a_{mn}\lambda^m\Gamma_{pk}^n\lambda^p\frac{dx^k}{ds} \quad (4)$$

$$\text{we have } \Gamma_{pk}^n = a^{ns}[pk, s] \quad (5)$$

$$\text{so, } \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2\underbrace{a_{mn}a^{ns}}_{=\delta_m^s}[pk, s]\lambda^m\lambda^p\frac{dx^k}{ds} \quad (6)$$

$$= \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2[pk, m]\lambda^m\lambda^p\frac{dx^k}{ds} \quad (7)$$

$$\text{but } 2[pk, m]\lambda^m\lambda^p = [pk, m]\lambda^m\lambda^p + [pk, m]\lambda^m\lambda^p \quad (8)$$

$$= [pk, m]\lambda^m\lambda^p + [mk, p]\lambda^m\lambda^p \quad (9)$$

$$\text{we have also } \begin{cases} [pk, m] = \frac{1}{2}(\partial_k a_{pm} + \partial_p a_{km} - \partial_m a_{pk}) \\ [mk, p] = \frac{1}{2}(\partial_k a_{pm} + \partial_m a_{pk} - \partial_p a_{mk}) \end{cases} \quad (10)$$

$$\Rightarrow 2[pk, m] = \partial_k a_{pm} \quad (11)$$

$$\text{so } \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n - \underbrace{\partial_k a_{pm}}_{=\frac{da_{mn}}{ds}}\frac{dx^k}{ds}\lambda^m\lambda^p + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} \quad (12)$$

$$\Rightarrow \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} \quad (13)$$



## 2.31 p54-exercise

Prove that

$$(T^r S_s)|_n = T^r_{|n} S_s + T^r S_{|n}$$

$$(T^r S_s)|_n = \partial_n(T^r S_s) + \Gamma_{nm}^r T^m S_s - \Gamma_{sn}^m T^r S_m \quad (1)$$

$$= \partial_n(T^r) S_s + T^r \partial_n(S_s) + S_s \Gamma_{nm}^r T^m - T^r \Gamma_{sn}^m S_m \quad (2)$$

$$= T^r \underbrace{(\partial_n(S_s) - \Gamma_{sn}^m S_m)}_{S_{s|n}} + S_s \underbrace{(\partial_n(T^r) + \Gamma_{nm}^r T^m)}_{T^r_{|n}} \quad (3)$$

$$= T^r S_{s|n} + S_s T^r_{|n} \quad (4)$$



## 2.32 p57-exercise

Compute the Christoffel symbols in 2.540 directly from the definitions 2.421 and 2.422. Check that all Christoffels symbols not shown explicitly in 2.540 vanish.

*Easy but very tedious, not reproduced yet, later perhaps*



### 2.33 p57-exercise

Show that for the spherical polar metric 2.532, we have  $\ln \sqrt{a} = 2\ln(x^1) + \ln(\sin(x^2))$  and

$$\mathbf{2.544} \quad \Gamma_{1n}^n = \frac{2}{x^1}, \quad \Gamma_{2n}^n = \cot(x^2), \quad \Gamma_{3n}^n = 0$$

The spherical polar metric 2.532 is,

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1 \sin(x^2))^2 \end{pmatrix} \quad (1)$$

$$\Rightarrow |a_{mn}| = [(x^1)^2 \sin(x^2)]^2 \quad (2)$$

$$\Rightarrow \ln(\sqrt{|a_{mn}|}) = 2\ln(x^1) + \ln(\sin(x^2)) \quad (3)$$

$$\Rightarrow \begin{cases} \Gamma_{1n}^n = \partial_1(\ln(\sqrt{a})) = \frac{2}{x^1} \\ \Gamma_{2n}^n = \partial_2(\ln(\sqrt{a})) = \frac{\cos(x^2)}{\sin(x^2)} = \cot(x^2) \\ \Gamma_{3n}^n = 0 \end{cases} \quad (4)$$



## 2.34 p58-exercise

Show that for the spherical polar metric

$$\mathbf{2.546} \quad T^n|_n = \frac{1}{r^2} \partial_r(r^2 T^1) + \frac{1}{\sin \theta} \partial_\theta(\sin \theta T^2) + \partial_\phi T^3$$

Obtain a similar expression for the "Laplacian"  $\Delta V$  of an invariant  $V$  defined as

$$\mathbf{2.547} \quad \Delta V = (a^{mn} \partial_m \partial_n V)|_n$$

We have

$$\mathbf{(2.545)} \quad T^n|_n = \frac{1}{\sqrt{a}} \partial_n(\sqrt{a} T^n) \quad (1)$$

$$\text{and from the previous exercise p.58: } \sqrt{a} = (x^1)^2 \sin(x^2) \quad (2)$$

$$\Rightarrow T^n|_n = \frac{1}{\sqrt{a}} (\sin(x^2) \partial_1[(x^1)^2 T^1] + (x^1)^2 \partial_2[\sin(x^2) T^2] + (x^1)^2 \sin(x^2) \partial_3 T^3) \quad (3)$$

$$= \frac{1}{x^1} \partial_1[(x^1)^2 T^1] + \frac{1}{\sin(x^2)} \partial_2[\sin(x^2) T^2] + \partial_3 T^3 \quad (4)$$

Replace in (4)  $x^1 = r$ ,  $x^2 = \theta$  and  $x^3 = \phi$

$$T^n|_n = \frac{1}{r^2} \partial_r[r^2 T^r] + \frac{1}{\sin \theta} \partial_\theta[\sin \theta T^\theta] + \partial_\phi T^\phi \quad (5)$$

Let's calculate the Laplacian.

$$\Delta V = (a^{mn} \partial_m \partial_n V)|_n \quad (6)$$

$$\text{be} \quad G^n = a^{mn} \partial_m V \quad (7)$$

$$\text{then (see exercise p.32)} \quad \left\{ \begin{array}{l} G^1 = \partial_r V \\ G^2 = \frac{1}{r^2} \partial_\theta V \\ G^3 = \frac{1}{r^2 \sin^2 \theta} \partial_\phi V \end{array} \right. \quad (8)$$

and by the previous result of this exercise

$$\Delta V = G^n|_n = \frac{1}{r^2} \partial_r[r^2 G^1] + \frac{1}{\sin \theta} \partial_\theta[\sin \theta G^2] + \partial_\phi G^3 \quad (9)$$

$$= \frac{1}{r^2} \partial_r[r^2 \partial_r V] + \frac{1}{r^2 \sin \theta} \partial_\theta[\sin \theta \partial_\theta V] + \frac{1}{r^2 \sin^2 \theta} \partial_\phi^2 V \quad (10)$$



## 2.35 p62-exercise

Prove that if a pair of vectors are unit orthogonal vectors at a point on a curve, and if they are both propagated parallelly along the curve, then they remain unit orthogonal vectors along the curve.

Given is, a pair of vectors  $U^m$  and  $V^m$  which are unit orthogonal vectors at a point on a curve. So,

$$\begin{aligned} \text{U is a unit vector (2.302)} \quad & a_{mn}U^mU^n = \epsilon \\ \text{V is a unit vector (2.302)} \quad & a_{mn}V^mV^n = \epsilon \\ \text{U, V are orthogonal (2.317)} \quad & a_{mn}U^mV^n = 0 \end{aligned} \quad (1)$$

at one point on the curve.

We have to prove that the above properties are valid along the curve (i.e.  $\forall$  points on the curve) provided that the vectors are propagated // along the curve, which means

$$(2.512) \quad \frac{\delta U^r}{\delta u} = \frac{dU^r}{du} + \Gamma_{mn}^r U^m \frac{dx^n}{du} = 0 \quad (2)$$

for both vectors  $U, V$ . Thus,

$$\frac{dU^r}{du} = -\Gamma_{mn}^r U^m \frac{dx^n}{du} \quad (3)$$

i) Consider the magnitude  $M$  at a random point on the curve

$$M = a_{mn}U^mU^n \quad (4)$$

$$\Rightarrow \frac{dM}{ds} = \frac{da_{mn}}{du} U^m U^n + 2a_{mn} U^m \frac{dU^n}{du} \quad (5)$$

Obviously  $M$  and  $\frac{dM}{ds}$  are invariants. Also, we can choose at any point on the curve a Riemannian coordinate system (RCS) for which the Christoffel symbols vanish at that point. Hence,  $\frac{dU^r}{du} = 0$  at that point and the second term in the right part of (5) vanish. (5) becomes then,

$$\frac{dM}{ds} = \frac{\partial a_{mn}}{\partial x^{,k}} \frac{dx^{,k}}{ds} U^{,m} U^{,n} \quad (6)$$

We also know (2.425. page 41) that  $[km, n]^{,} + [kn, m]^{,} = \frac{\partial a_{mn}}{\partial x^{,k}}$ . But in the chosen coordinate system,  $[km, n] = 0$  at the origin of this coordinate system. So by (6) we get  $\frac{dM}{ds} = 0$ .

So the magnitude is constant along the curve and as we know that at a certain point  $M = 1$ :

**U, V are unit vectors along the curve**

ii) Consider now the angle between the vectors  $U, V$ . Be  $A = \cos \theta$ . By definition

$$A = a_{mn} U^m V^n \quad (7)$$

$$\Rightarrow \frac{dA}{ds} = \frac{da_{mn}}{du} U^m V^n + a_{mn} (V^m \frac{dU^n}{du} + U^m \frac{dV^n}{du}) \quad (8)$$

We follow the same reasoning as in i) and so

$$\frac{dA}{ds} = 0$$

So, the angle is constant and we know is  $\frac{\pi}{2}$  at a certain point. So,:

**U, V are orthogonal along the curve**



## 2.36 p62-exercise

Given that  $\lambda^r$  is a unit vector field, prove that

$$\lambda^r_{|s} \lambda_r = 0 \quad \text{and} \quad \lambda^r \lambda_{r|s} = 0$$

Is the relation  $\lambda^r_{|s} \lambda_s = 0$  true for a general unit vector field?

To simplify the calculation, we choose a random element in the unit vector field and use at that point a Riemannian coordinate system (RCS). So, we have

$$\lambda^r_{|s} = \partial_s \lambda^r \quad \text{and} \quad \lambda_{r|s} = \partial_s \lambda_r \quad (1)$$

$$\text{as we have a unit vector fields:} \quad a_{mn} \lambda^m \lambda^n = 1 \quad (2)$$

$$\Leftrightarrow \lambda_n \lambda^n = 1 \quad (\text{by lowering the index } m) \quad (3)$$

$$\Rightarrow \quad \lambda^n \partial_s \lambda_n + \lambda_n \partial_s \lambda^n = 0 \quad (4)$$

We prove that

$$\lambda^n \partial_s \lambda_n = \lambda_n \partial_s \lambda^n \quad \forall \text{ vector fields}$$

We have the trivial identity

$$\partial_s (\lambda^r \lambda_r) = \partial_s (\lambda^r \lambda_r) \quad (5)$$

$$\Leftrightarrow \quad \partial_s (a^{rm} \lambda_m \lambda_r) = \partial_s (a_{rm} \lambda^m \lambda^r) \quad (6)$$

**Lemma** :  $\partial_s a^{rm} = 0$  in a Riemannian coordinate system (i.e. at the origin)

We have

$$a^{rm} a_{ms} = \delta_s^r \quad (7)$$

$$\Rightarrow a_{ms} \partial_k a^{rm} + a^{rm} \partial_k a_{ms} = 0 \quad (8)$$

$$\text{we know (2.618)} \quad \partial_r a_{mn} = 0 \quad \text{at the origin of a RCS} \quad (9)$$

$$\text{so (8) becomes} \quad a_{ms} \partial_k a^{rm} = 0 \quad (10)$$

$$\text{multiply (10) by } a^{ns} \Rightarrow \underbrace{a^{ns} a_{ms}}_{=\delta_m^n} \partial_k a^{rm} = 0 \quad (11)$$

$$\Rightarrow \quad \partial_k a^{rn} = 0 \quad (12)$$

◇



Now, expanding (6) and using **2.618** and the lemma:

$$a^{rm} \lambda_m \partial_s \lambda_r + a^{rm} \lambda_r \partial_s \lambda_m = a_{rm} \lambda^m \partial_s \lambda^r + a_{rm} \lambda^r \partial_s \lambda^m \quad (13)$$

$$\text{renaming dummy indices:} \quad a^{rm} \lambda_r \partial_s \lambda_m = a_{rm} \lambda^r \partial_s \lambda^m \quad (14)$$

$$\Rightarrow \quad \lambda^m \partial_s \lambda_m = \lambda_m \partial_s \lambda^m \quad (15)$$

Considering (5) and (15) we conclude:

$$\lambda^n \partial_s \lambda_n = \lambda_n \partial_s \lambda^n = 0 \quad (16)$$

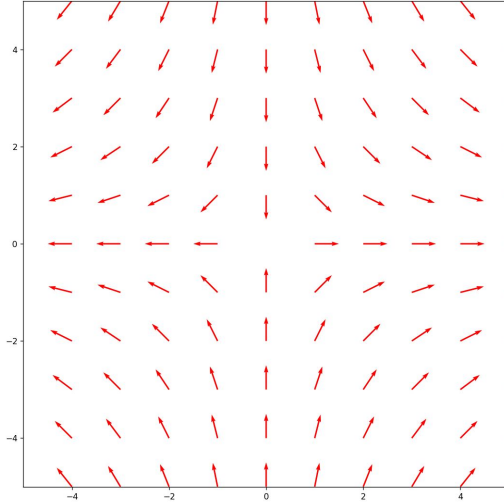
and as  $\partial_s \lambda_n = \lambda_{n|s}$  and  $\partial_s \lambda^n = \lambda^n_{|s}$  at the origin of the considered coordinate system, we have:

$$\lambda^m \lambda_{n|s} = \lambda_m \lambda^n_{|s} = 0 \quad (17)$$

◇

Is the relation  $\lambda^r_{|s} \lambda_s = 0$  true for a general unit vector field?

The answer is NO. Let's consider the following unit vector field in a Cartesian Coordinate system:



$$V : \mathbb{R}^2_* \rightarrow \mathbb{R}^2 | V(x, y) = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle$$

Put  $r = \sqrt{x^2 + y^2}$ , we get (as we have a Cartesian Coordinate system, the calculation simplify as the Christoffel symbols vanish and the covariant components of the vectors are equal to their

contravariant part):

$$\begin{cases} V^1 = V_1 = +\frac{x}{r} \\ V^2 = V_2 = -\frac{y}{r} \end{cases} \quad (18)$$

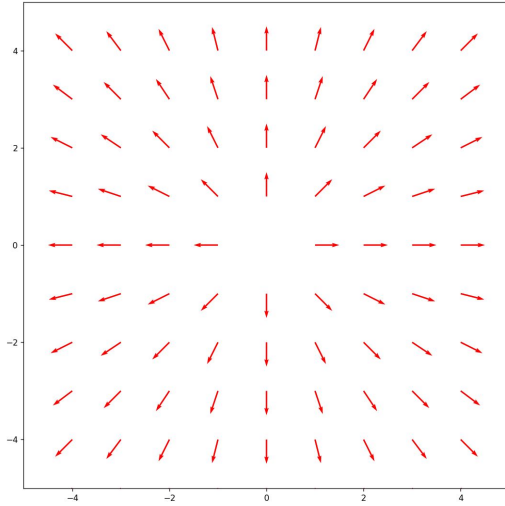
$$\begin{cases} V^1_{|1} = V_{1|1} = \frac{y^2}{r^3} & V^1_{|2} = V_{1|2} = -\frac{xy}{r^3} \\ V^2_{|1} = V_{2|1} = \frac{xy}{r^3} & V^2_{|2} = V_{2|2} = -\frac{x^2}{r^3} \end{cases} \quad (19)$$

$$\Rightarrow \begin{cases} V^1_{|s} V_s = V^1_{|1} V_1 + V^1_{|2} V_2 = \frac{y^2}{r^3} \frac{x}{r} + (-\frac{xy}{r^3})(-\frac{y}{r}) = \frac{xy^2}{r^4} \neq 0 \\ V^2_{|s} V_s = V^2_{|1} V_1 + V^2_{|2} V_2 = \frac{xy}{r^3} \frac{x}{r} + (-\frac{y}{r})(-\frac{y}{r}) = \frac{x^2 y}{r^4} \neq 0 \end{cases} \quad (20)$$

Just as a check, we calculate  $V^r_s V_r$  which should be zero:

$$\Rightarrow \begin{cases} V^s_{|1} V_s = V^1_{|1} V_1 + V^2_{|1} V_2 = (+\frac{y^2}{r^3})\frac{x}{r} + (+\frac{xy}{r^3})(-\frac{y}{r}) = 0 \\ V^s_{|2} V_s = V^1_{|2} V_1 + V^2_{|2} V_2 = (-\frac{xy}{r^3})\frac{x}{r} + (-\frac{y}{r})(-\frac{x^2}{r^3}) = 0 \end{cases} \quad (21)$$

Now, let's consider another unit vector field in a Cartesian Coordinate system:



$$V : \mathbb{R}^2_* \rightarrow \mathbb{R}^2 | V(x, y) = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle$$

$$\begin{cases} V^1 = V_1 = +\frac{x}{r} \\ V^2 = V_2 = +\frac{y}{r} \end{cases} \quad (22)$$

$$\begin{cases} V^1_{|1} = V_{1|1} = \frac{y^2}{r^3} & V^1_{|2} = V_{1|2} = -\frac{xy}{r^3} \\ V^2_{|1} = V_{2|1} = -\frac{xy}{r^3} & V^2_{|2} = V_{2|2} = +\frac{x^2}{r^3} \end{cases} \quad (23)$$

$$\Rightarrow \begin{cases} V^1_{|s} V_s = V^1_{|1} V_1 + V^1_{|2} V_2 = (+\frac{y^2}{r^3})\frac{x}{r} + (-\frac{xy}{r^3})(+\frac{y}{r}) = 0 \\ V^2_{|s} V_s = V^2_{|1} V_1 + V^2_{|2} V_2 = (-\frac{xy}{r^3})\frac{x}{r} + (+\frac{y}{r})(+\frac{y}{r}) = 0 \end{cases} \quad (24)$$

Just as a check, we calculate  $V_{|s}^r V_r$  which should be zero:

$$\Rightarrow \quad \begin{cases} V_{|1}^s V_s = V_{|1}^1 V_1 + V_{|1}^2 V_2 = \left(+\frac{y^2}{r^3}\right)\left(+\frac{x}{r}\right) + \left(-\frac{xy}{r^3}\right)\left(+\frac{y}{r}\right) = 0 \\ V_{|2}^s V_s = V_{|2}^1 V_1 + V_{|2}^2 V_2 = \left(-\frac{xy}{r^3}\right)\left(+\frac{x}{r}\right) + \left(+\frac{y}{r}\right) + \frac{x^2}{r^3} = 0 \end{cases} \quad (25)$$

So, in the second example the relationship  $\lambda_{|s}^r \lambda_s = 0$  holds. Question (to investigate further and later) : does the fact that in the first case  $\nabla \times \bar{V} \neq 0$  and in the second case  $\nabla \times \bar{V} = 0$ , means that there is some relation with this expression?



## 2.37 p64-clarification 2.625

**2.625**

$$\frac{dx^r}{dx^N} = \frac{X^r}{X^N}$$

Be  $C$  a surface defined by the function  $F(x^1, \dots, x^{N-1}) = C$  and  $c_\perp$  the curve intersecting the surface  $C$  perpendicularly at a point  $p$ .

Along the curve at that point  $p$  we have

$$\text{as } \begin{cases} \frac{dx^r}{ds} & \text{is the tangent vector along } c_\perp \\ X^r = a^{rn} \frac{\partial F}{\partial x^n} & \text{is orthogonal on the surface (2.623) } C \end{cases} \quad (1)$$

$$\Rightarrow \frac{dx^r}{ds} = kX^r \quad (2)$$

So,  $\frac{dx^r}{ds}$  is proportional to  $X^r$  (as the curve intersects the surface orthogonally). This means that also all the components (coordinates) of both quantities are proportional. And so,

$$\frac{\frac{dx^r}{ds}}{\frac{dx^N}{ds}} = \frac{kX^r}{kX^N} \quad (3)$$

$$\Rightarrow \frac{dx^r}{dx^N} = \frac{X^r}{X^N} \quad (4)$$



## 2.38 p65-exercise

Deduce from **2.629** that

$$a^{N\rho} = 0 \quad a^{NN} = \frac{1}{a_{NN}}$$

We have (see 2.629):

$$a_{N\rho} = 0 \tag{1}$$

$$\text{and also} \quad a_{Nm}a^{ms} = \delta_N^s \tag{2}$$

In (2) split the  $m$  index in the subspace and the remaining coordinate  $N$

$$a_{N\rho}a^{\rho s} + a_{NN}a^{Ns} = \delta_N^s \tag{3}$$

$$\text{as } a_{N\rho} = 0 \Rightarrow a_{NN}a^{Ns} = \delta_N^s \tag{4}$$

Case 1:  $s \neq N$

$$a_{NN}a^{Ns} = 0 \Leftrightarrow a_{NN}a^{N\rho} = 0 \quad (\text{as } s \neq N) \tag{5}$$

$$\text{as we suppose } a_{NN} \neq 0 \Rightarrow a^{N\rho} = 0 \tag{6}$$

Case 2:  $s = N$

$$a_{NN}a^{NN} = 1 \tag{7}$$

$$\Rightarrow a^{NN} = \frac{1}{a_{NN}} \tag{8}$$



## 2.39 p69-clarification on 2.645

In 2.645 we have

$$T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2} \frac{1}{a_{NN}} \partial_N a_{\alpha\beta} T_N$$

and

$$T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2} \partial_N a_{\alpha\beta} T^N$$

Indeed,

$$T^N = a^{mN} T_m \tag{1}$$

$$= a^{\alpha N} T_\alpha + a^{NN} T_N \tag{2}$$

$$\text{but (2.631)} \quad a^{\alpha N} = 0 \tag{3}$$

$$\Rightarrow T^N = a^{NN} T_N \tag{4}$$

$$\text{as } a^{NN} = \frac{1}{a_{NN}} \quad \Rightarrow \quad T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2} \partial_N a_{\alpha\beta} T^N \tag{5}$$



## 2.40 p69-exercise

Show that

$$\mathbf{2.648} \quad T^\alpha_{|\beta} = T^\alpha_{||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N$$

$$\mathbf{2.649} \quad T^N_{|\alpha} = \partial_\alpha T^N - \frac{1}{2a_{NN}}\partial_N a_{\alpha\mu}T^\mu + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$$

$$\mathbf{2.650} \quad T^\alpha_{|N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$$

i)  $T^\alpha_{|\beta} = T^\alpha_{||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N$

$$\mathbf{(2.520)} \quad \Rightarrow \quad T^\alpha_{|\beta} = \partial_\beta T^\alpha + \Gamma^\alpha_{m\beta}T^m \quad (m = 1, \dots, N) \quad (1)$$

$$\Leftrightarrow \quad T^\alpha_{|\beta} = \partial_\beta T^\alpha + \underbrace{\Gamma^\alpha_{\mu\beta}T^\mu}_{T^\alpha_{||\beta}} + \Gamma^\alpha_{N\beta}T^N \quad (2)$$

$$\mathbf{(2.639)} \quad \Gamma^\alpha_{N\beta} = \frac{1}{2}a^{\alpha\mu}\partial_N a^{\mu\beta} \quad (3)$$

$$(2) \text{ and } (3): \quad T^\alpha_{|\beta} = T^\alpha_{||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N \quad (4)$$

◇

Remark: We also use **(2.639)** for the two other identities.

ii)  $T^N_{|\alpha} = \partial_\alpha T^N - \frac{1}{2a_{NN}}\partial_N a_{\alpha\mu}T^\mu + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$

$$\mathbf{(2.520)} \quad \Rightarrow \quad T^N_{|\alpha} = \partial_\alpha T^N + \Gamma^N_{m\alpha}T^m \quad (m = 1, \dots, N) \quad (5)$$

$$\Leftrightarrow \quad T^N_{|\alpha} = \partial_\beta T^\alpha + \underbrace{\Gamma^N_{\sigma\alpha}T^\sigma}_{-\frac{1}{2a_{NN}}\partial_N a_{\sigma\alpha}} + \underbrace{\Gamma^N_{N\alpha}T^N}_{\frac{1}{2a_{NN}}\partial_\alpha a_{NN}} \quad (6)$$

$$\Rightarrow \quad T^N_{|\alpha} = \partial_\beta T^\alpha - \frac{1}{2a_{NN}}\partial_N a_{\sigma\alpha}T^\sigma + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N \quad (7)$$

iii)  $T^\alpha_{|N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$

$$\mathbf{(2.520)} \quad \Rightarrow \quad T^\alpha_{|N} = \partial_N T^\alpha + \Gamma^\alpha_{mN}T^m \quad (m = 1, \dots, N) \quad (8)$$

$$\Leftrightarrow \quad T^\alpha_{|N} = \partial_N T^\alpha + \underbrace{\Gamma^\alpha_{\sigma N}T^\sigma}_{\frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}} + \underbrace{\Gamma^\alpha_{NN}T^N}_{-\frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}} \quad (9)$$

$$\Rightarrow \quad T^\alpha_{|N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N \quad (10)$$

◆

## 2.41 p71-exercise

Write down equation 2.643 tot 2.650 for the special case of a geodesic normal coordinate system.

$$(2.643) \quad T_{\alpha||\beta} = \partial_\beta T_\alpha - \Gamma_{\alpha\beta}^\gamma T_\gamma \quad (\text{does not change}) \quad (1)$$

$$(2.644) \quad T_{\alpha|\beta} = T_{\alpha||\beta} - \underbrace{\Gamma_{\alpha\beta}^N}_{=\frac{1}{2} \frac{1}{a_{NN}} \partial_N a_{\alpha\beta}} T_N \quad (2)$$

$$= T_{\alpha||\beta} + \frac{1}{2} \epsilon \partial_N a_{\alpha\beta} T_N \quad (3)$$

$$(2.645) \quad T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2} \partial_N a_{\alpha\beta} T^N \quad (\text{does not change}) \quad (4)$$

$$(2.646) \quad T_{N|\alpha} = \partial_\alpha T_N - \frac{1}{2} \partial_N a_{\mu\alpha} T^\mu - \frac{1}{2} \underbrace{\partial_\alpha a_{NN}}_{=0} T^N \quad (5)$$

$$= \partial_\alpha T_N - \frac{1}{2} \partial_N a_{\mu\alpha} T^\mu \quad (6)$$

$$(2.647) \quad T_{\alpha|N} = \partial_N T_\alpha - \frac{1}{2} \partial_N a_{\mu\alpha} T^\mu - \frac{1}{2} \underbrace{\partial_\alpha a_{NN}}_{=0} T^N \quad (7)$$

$$= \partial_N T_\alpha - \frac{1}{2} \partial_N a_{\mu\alpha} T^\mu \quad (8)$$

$$(2.648) \quad T_{|\beta}^\alpha = T_{||\beta}^\alpha + \frac{1}{2} a^{\alpha\mu} \partial_N a_{\mu\beta} T_N \quad (\text{does not change}) \quad (9)$$

$$(2.649) \quad T_{|\alpha}^N = \partial_\alpha T^N - \frac{1}{2} \frac{1}{a_{NN}} \partial_N a_{\mu\alpha} T^\mu - \frac{1}{2} \frac{1}{a_{NN}} \underbrace{\partial_\alpha a_{NN}}_{=0} T^N \quad (10)$$

$$= \partial_\alpha T^N - \frac{1}{2} \epsilon \partial_N a_{\mu\alpha} T^\mu \quad (11)$$

$$(2.650) \quad T_{|N}^\alpha = \partial_N T^\alpha + \frac{1}{2} a^{\alpha\mu} \partial_N a_{\mu\sigma} T^\sigma - \frac{1}{2} a^{\alpha\mu} \underbrace{\partial_\mu a_{NN}}_{=0} T^N \quad (12)$$

$$= \partial_N T^\alpha + \frac{1}{2} a^{\alpha\mu} \partial_N a_{\mu\sigma} T^\sigma \quad (13)$$

To investigate: note (3) and (4) which suggest that  $\epsilon T_N = T^N$ . Prove formally?





## 2.42 p73-Clarification 2.706

... Let us now define a unit vector  $\lambda_{(2)}^r$  and a positive invariant  $\kappa_{(2)}$  by the equation

$$\begin{cases} \frac{\delta \lambda_{(1)}^r}{\delta s} = \kappa_{(2)} \lambda_{(2)}^r - \epsilon \epsilon_{(1)} \kappa_{(1)} \lambda^r \\ \epsilon_{(2)} \lambda_{(2)}^n \lambda_{(2)n} = 1 \end{cases} \quad (1)$$

We can state that  $\kappa_{(2)}$  is an invariant but one has to check whether the expression (1) implies that  $\kappa_{(2)}$  is indeed invariant.

What we know is that  $\lambda^r, \frac{\delta \lambda^r}{\delta s}, \frac{\delta \lambda_{(1)}^r}{\delta s}, \lambda_{(2)}^r$  are contravariant vectors. Also  $\kappa_{(1)}$  is an invariant as  $\frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \lambda_{(1)}^r$  and the magnitude of  $\frac{\delta \lambda^r}{\delta s}$  does not depend on the coordinate system. So,

$$(1) \times \lambda_{(2)r} \Rightarrow \frac{\delta \lambda_{(1)}^r}{\delta s} \lambda_{(2)r} = \kappa_{(2)} \lambda_{(2)}^r \lambda_{(2)r} - \epsilon \epsilon_{(1)} \kappa_{(1)} \lambda^r \lambda_{(2)r} \quad (2)$$

$$\Rightarrow \kappa_{(2)} \underbrace{\lambda_{(2)}^r \lambda_{(2)r}}_{\text{invariant}} = \underbrace{\frac{\delta \lambda_{(1)}^r}{\delta s} \lambda_{(2)r}}_{\text{invariant}} + \underbrace{\epsilon \epsilon_{(1)}}_{\text{invariant}} \underbrace{\kappa_{(1)}}_{\text{invariant}} \underbrace{\lambda^r \lambda_{(2)r}}_{\text{invariant}} \quad (3)$$

$$\Rightarrow \kappa_{(2)} = \text{invariant} \quad (4)$$



## 2.43 p74-Clarification 2.710

$$\mathbf{2.710} \quad \begin{cases} \frac{\delta \lambda_{(M-1)}^r}{\delta s} = \kappa_{(M)} \lambda_{(M)}^r - \epsilon_{(M-2)} \epsilon_{(M-1)} \kappa_{(M-1)} \lambda_{(M-2)}^r \\ \epsilon_{(M-1)} \lambda_{(M-1)}^r \lambda_{(M-1)n} = 1 \quad (M=1,2,\dots,N) \end{cases} \quad (1)$$

... It is easily proved by mathematical induction that the whole sequence of vectors defined by 2.710 are perpendicular to the tangent and to one another ...

We already know from 2.703 to 2.709 that  $\lambda^r, \lambda_{(1)}^r, \lambda_{(2)}^r, \lambda_{(3)}^r$ , satisfying equations (1), are all mutually perpendicular. Let us assume that the orthogonality for the set  $\{\lambda_{(k)}^r : k = 0, 1, 2, 3, \dots, M-1\}$  has been verified. We prove by induction that then,  $\lambda_{(M)}^r$  will be orthogonal to all elements of the set.

i) Consider the set  $\{\lambda_{(k)}^r : k = 0, 1, 2, 3, \dots, M-3\}$  where we already know that  $\lambda_{(k)}^r$  are mutually perpendicular and also  $\lambda_{(k)}^r \perp \lambda_{(M-1)}^r$ ,  $\lambda_{(k)}^r \perp \lambda_{(M-2)}^r$  and  $\lambda_{(M-1)}^r \perp \lambda_{(M-1)}^r \quad \forall k$ .

$$(1) \times \lambda_{(k)r} \Rightarrow \frac{\delta \lambda_{(M-1)}^r}{\delta s} \lambda_{(k)r} = \kappa_{(M)} \lambda_{(M)}^r \lambda_{(k)r} - \epsilon_{(M-2)} \epsilon_{(M-1)} \kappa_{(M-1)} \underbrace{\lambda_{(M-2)}^r \lambda_{(k)r}}_{=0} \quad (2)$$

$$\text{We have} \quad \lambda_{(k)r} \lambda_{(M-1)}^r = 0 \quad (3)$$

$$\Rightarrow \frac{\delta \lambda_{(k)r} \lambda_{(M-1)}^r}{\delta s} = \lambda_{(M-1)}^r \frac{\delta \lambda_{(k)r}}{\delta s} + \lambda_{(k)r} \delta \frac{\lambda_{(M-1)}^r}{\delta s} = 0 \quad (4)$$

$$\Rightarrow \lambda_{(k)r} \delta \frac{\lambda_{(M-1)}^r}{\delta s} = -\lambda_{(M-1)}^r \frac{\delta \lambda_{(k)r}}{\delta s} \quad (5)$$

$$\text{We have} \quad \frac{\delta \lambda_{(k)r}}{\delta s} = \kappa_{(k+1)} \lambda_{(k+1)r} - \epsilon_{(k)} \epsilon_{(k-1)} \kappa_{(k)} \lambda_{(k-1)r} \quad (6)$$

$$(5) \text{ and } (6) \Rightarrow \lambda_{(k)r} \delta \frac{\lambda_{(M-1)}^r}{\delta s} = -\kappa_{(k+1)} \underbrace{\lambda_{(k+1)r} \lambda_{(M-1)}^r}_{=0} - \epsilon_{(k)} \epsilon_{(k-1)} \kappa_{(k)} \underbrace{\lambda_{(k-1)r} \lambda_{(M-1)}^r}_{=0} \quad (7)$$

$$\text{From } (2) \Rightarrow \kappa_{(M)} \lambda_{(M)}^r \lambda_{(k)r} = 0 \quad (8)$$

$$\Rightarrow \lambda_{(M)}^r \perp \lambda_{(k)r} \quad \forall k = 0, 1, 2, 3, \dots, M-3 \quad (9)$$

ii) Consider the case  $k = M-1$

$$(1) \times \lambda_{(M-1)r} \quad \frac{\delta \lambda_{(M-1)}^r}{\delta s} \lambda_{(M-1)r} = \kappa_{(M)} \lambda_{(M)}^r \lambda_{(M-1)r} - \epsilon_{(M-2)} \epsilon_{(M-1)} \kappa_{(M-1)} \underbrace{\lambda_{(M-2)}^r \lambda_{(M-1)r}}_{=0} \quad (10)$$

$$\text{from (2.530):} \quad \frac{\delta \lambda_{(M-1)}^r}{\delta s} \lambda_{(M-1)r} = \frac{1}{2} \underbrace{\frac{\delta \lambda_{(M-1)r} \lambda_{(M-1)}^r}{\delta s}}_{=0 \text{ as } \lambda_{(M-1)r} \lambda_{(M-1)}^r = \epsilon_{(M-1)}} \quad (11)$$

$$\Rightarrow \quad \kappa_{(M)} \lambda_{(M)}^r \lambda_{(M-1)r} = 0 \quad (12)$$

$$\Rightarrow \quad \lambda_{(M-1)r} \perp \lambda_{(M)r} \quad (13)$$

iii) Consider the case  $k = M - 2$

$$(1) \times \lambda_{(M-2)r} \quad \frac{\delta \lambda_{(M-1)}^r}{\delta s} \lambda_{(M-2)r} = \kappa_{(M)} \lambda_{(M)}^r \lambda_{(M-2)r} - \epsilon_{(M-2)} \epsilon_{(M-1)} \kappa_{(M-1)} \underbrace{\lambda_{(M-2)}^r \lambda_{(M-2)r}}_{=\epsilon_{(M-2)}} \quad (14)$$

$$\Rightarrow \quad \frac{\delta \lambda_{(M-1)}^r}{\delta s} \lambda_{(M-2)r} = \kappa_{(M)} \lambda_{(M)}^r \lambda_{(M-2)r} - \epsilon_{(M-1)} \kappa_{(M-1)} \underbrace{\epsilon_{(M-2)} \epsilon_{(M-2)}}_{=1} \quad (15)$$

$$\Rightarrow \quad \frac{\delta \lambda_{(M-1)}^r}{\delta s} \lambda_{(M-2)r} = \kappa_{(M)} \lambda_{(M)}^r \lambda_{(M-2)r} - \epsilon_{(M-1)} \kappa_{(M-1)} \quad (16)$$

$$\text{We have} \quad \lambda_{(M-1)}^r \lambda_{(M-2)r} = 0 \quad (17)$$

$$\Rightarrow \quad \lambda_{(M-2)r} \frac{\delta \lambda_{(M-1)}^r}{\delta s} = -\lambda_{(M-1)}^r \frac{\delta \lambda_{(M-2)r}}{\delta s} \quad (18)$$

$$\text{We have also} \quad \frac{\delta \lambda_{(M-2)r}}{\delta s} = \kappa_{(M-1)} \lambda_{(M-1)r} - \epsilon_{(M-3)} \epsilon_{(M-2)} \kappa_{(M-2)} \lambda_{(M-3)r} \quad (19)$$

$$(19) \times \lambda_{(M-1)}^r \text{ and (18): } \lambda_{(M-2)r} \frac{\delta \lambda_{(M-1)}^r}{\delta s} = -\kappa_{(M-1)} \underbrace{\lambda_{(M-1)}^r \lambda_{(M-1)r}}_{=\epsilon_{(M-1)}} - \epsilon_{(M-3)} \epsilon_{(M-2)} \kappa_{(M-2)} \underbrace{\lambda_{(M-3)}^r \lambda_{(M-1)r}}_{=0} \quad (20)$$

$$\Rightarrow \quad \lambda_{(M-2)r} \frac{\delta \lambda_{(M-1)}^r}{\delta s} = -\kappa_{(M-1)} \epsilon_{(M-1)} \quad (21)$$

$$(16) \text{ and (21):} \quad -\kappa_{(M-1)} \epsilon_{(M-1)} = \kappa_{(M)} \lambda_{(M)}^r \lambda_{(M-2)r} - \epsilon_{(M-1)} \kappa_{(M-1)} \quad (22)$$

$$\Rightarrow \quad \kappa_{(M)} \lambda_{(M)}^r \lambda_{(M-2)r} = 0 \quad (23)$$

$$\Rightarrow \quad \lambda_{(M-2)r} \perp \lambda_{(M)r} \quad (24)$$

With, i), ii), iii) all possible cases are covered which makes the proof complete.

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## 2.44 p75-Clarification 2.714

$$\mathbf{2.714} \quad (\kappa_{(1)})^2 = \epsilon_{(1)} a_{mn} \frac{\delta \lambda^m}{\delta s} \frac{\delta \lambda^n}{\delta s}, \quad \epsilon_{(1)} = \pm 1$$

$$\frac{\delta \lambda^n}{\delta s} = \kappa_{(1)} \lambda_{(1)}^n \quad (1)$$

$$(1) \times (1) \Rightarrow \frac{\delta \lambda^m}{\delta s} \frac{\delta \lambda^n}{\delta s} = (\kappa_{(1)})^2 \lambda_{(1)}^m \lambda_{(1)}^n \quad (2)$$

$$(2) \times a_{mn} \Rightarrow a_{mn} \frac{\delta \lambda^m}{\delta s} \frac{\delta \lambda^n}{\delta s} = a_{mn} (\kappa_{(1)})^2 \lambda_{(1)}^m \lambda_{(1)}^n \quad (3)$$

$$= (\kappa_{(1)})^2 \underbrace{\lambda_{(1)m} \lambda_{(1)}^n}_{=\epsilon_{(1)}} \quad (4)$$

$$= (\kappa_{(1)})^2 \quad (5)$$



## 2.45 p75-exercise

For positive definite metric forms, write out explicitly the Frenet formulae for the case  $N=2$ , 3 and 4.

The general Frenet formulae are

$$\begin{cases} \frac{\delta \lambda_{(M-1)}^r}{\delta s} = \kappa_{(M)} \lambda_{(M)}^r - \epsilon_{(M-2)} \epsilon_{(M-1)} \kappa_{(M-1)} \lambda_{(M-2)}^r \\ \epsilon_{(M-1)} \lambda_{(M-1)}^n \lambda_{(M-1)n} = 1 \end{cases} \quad (M=1,2,\dots,N) \quad (1)$$

As  $\Phi$  is positive definite, we have  $\epsilon_{(k)} = 1 \quad \forall k$

N=2	N=3	N=4
$\frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \lambda_{(1)}^r$ $\frac{\delta \lambda_{(1)}^r}{\delta s} = \kappa_{(2)} \lambda_{(2)}^r - \kappa_{(1)} \lambda^r$ $\lambda^n \lambda_n = 1$ $\lambda_{(1)}^n \lambda_{(1)n} = 1$	$\frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \lambda_{(1)}^r$ $\frac{\delta \lambda_{(1)}^r}{\delta s} = \kappa_{(2)} \lambda_{(2)}^r - \kappa_{(1)} \lambda^r$ $\frac{\delta \lambda_{(2)}^r}{\delta s} = \kappa_{(3)} \lambda_{(3)}^r - \kappa_{(2)} \lambda_{(1)}^r$ $\lambda^n \lambda_n = 1$ $\lambda_{(1)}^n \lambda_{(1)n} = 1$ $\lambda_{(2)}^n \lambda_{(2)n} = 1$	$\frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \lambda_{(1)}^r$ $\frac{\delta \lambda_{(1)}^r}{\delta s} = \kappa_{(2)} \lambda_{(2)}^r - \kappa_{(1)} \lambda^r$ $\frac{\delta \lambda_{(2)}^r}{\delta s} = \kappa_{(3)} \lambda_{(3)}^r - \kappa_{(2)} \lambda_{(1)}^r$ $\frac{\delta \lambda_{(3)}^r}{\delta s} = \kappa_{(4)} \lambda_{(4)}^r - \kappa_{(3)} \lambda_{(2)}^r$ $\lambda^n \lambda_n = 1$ $\lambda_{(1)}^n \lambda_{(1)n} = 1$ $\lambda_{(2)}^n \lambda_{(2)n} = 1$ $\lambda_{(3)}^n \lambda_{(3)n} = 1$

Taking into account that  $\kappa_{(N)} = 0$  for a space  $V_N$ , we get,

N=2	N=3	N=4
$\frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \lambda_{(1)}^r$ $\frac{\delta \lambda_{(1)}^r}{\delta s} = -\kappa_{(1)} \lambda^r$ $\lambda^n \lambda_n = 1$ $\lambda_{(1)}^n \lambda_{(1)n} = 1$	$\frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \lambda_{(1)}^r$ $\frac{\delta \lambda_{(1)}^r}{\delta s} = \kappa_{(2)} \lambda_{(2)}^r - \kappa_{(1)} \lambda^r$ $\frac{\delta \lambda_{(2)}^r}{\delta s} = -\kappa_{(2)} \lambda_{(1)}^r$ $\lambda^n \lambda_n = 1$ $\lambda_{(1)}^n \lambda_{(1)n} = 1$ $\lambda_{(2)}^n \lambda_{(2)n} = 1$	$\frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \lambda_{(1)}^r$ $\frac{\delta \lambda_{(1)}^r}{\delta s} = \kappa_{(2)} \lambda_{(2)}^r - \kappa_{(1)} \lambda^r$ $\frac{\delta \lambda_{(2)}^r}{\delta s} = \kappa_{(3)} \lambda_{(3)}^r - \kappa_{(2)} \lambda_{(1)}^r$ $\frac{\delta \lambda_{(3)}^r}{\delta s} = -\kappa_{(3)} \lambda_{(2)}^r$ $\lambda^n \lambda_n = 1$ $\lambda_{(1)}^n \lambda_{(1)n} = 1$ $\lambda_{(2)}^n \lambda_{(2)n} = 1$ $\lambda_{(3)}^n \lambda_{(3)n} = 1$



## 2.46 p76-exercise

In an Euclidean space  $V_N$ , the fundamental form is given as  $\Phi = dx^n dx^n$ . Show that a curve which has  $\kappa_{(2)} = 0$  and  $\kappa_{(1)} = \text{constant}$  satisfies equations of the form

$$x^r = A^r \cos \kappa_{(1)} s + B^r \sin \kappa_{(1)} s + C^r$$

where  $A^r, B^r, C^r$  are constants satisfying

$$A^r A^r = B^r B^r = \frac{1}{\kappa_{(1)}^2}, \quad A^r B^r = 0$$

so that  $A^r$  and  $B^r$  are vectors of equal magnitude and perpendicular to one another. (This curve is a circle in the N-space)

$$\text{What we know} \quad \Phi = dx^n dx^n \quad (1)$$

$$\Rightarrow \quad (a_{mn}) = (\delta_n^m) \quad (2)$$

$$\text{and given} \quad \kappa_{(1)} = \text{constant} \quad \kappa_{(2)} = 0 \quad \epsilon_{(1)} = \epsilon_{(2)}, \dots = 1 \quad (3)$$

$$\text{we have (2.705)} \quad \frac{\delta \lambda^r}{\delta s} = \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \quad \text{with} \quad \lambda^r = \frac{dx^r}{ds} \quad (4)$$

$$\text{but } (a_{mn}) = (\delta_n^m) \Rightarrow \quad \frac{\delta \lambda^r}{\delta s} = \frac{d\lambda^r}{ds} \quad (5)$$

$$(4) \text{ and } (5) \Rightarrow \quad \frac{d\lambda^r}{ds} = \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \quad (6)$$

$$\text{also} \quad \frac{\delta \lambda_{(1)}^r}{\delta s} = \underbrace{\kappa_{(1)}}_{=0} \frac{\delta \lambda_{(1)}^r}{\delta s} - \kappa_{(1)} \frac{\delta \lambda_{(1)}^r}{\delta s} \quad (7)$$

Hence we get the following set of equations

$$(8) \Rightarrow \left\{ \begin{array}{l} \frac{dx^r}{ds} = \lambda^r \\ \frac{d\lambda^r}{ds} = \kappa_{(1)} \lambda_{(1)}^r \\ \frac{d\lambda_{(1)}^r}{ds} = -\kappa_{(1)} \lambda_{(1)}^r \\ \kappa_{(1)} = \kappa \quad (= \text{constant}) \\ \kappa_{(2)} = 0 \\ \lambda^n \lambda_n = 1 \\ \lambda_{(1)}^n \lambda_{(1)n} = 1 \\ \frac{d^2 \lambda_{(1)}^r}{ds^2} + \kappa^2 \lambda_{(1)}^r = 0 \end{array} \right. \quad (8)$$

$$(8) \Rightarrow \frac{d^2 \lambda_{(1)}^r}{ds^2} + \kappa^2 \lambda_{(1)}^r = 0 \quad (9)$$

Solving the ODE (9). Put  $e^{rs} = \lambda_{(1)}^k$

$$(9): \quad r^2 + \kappa^2 = 0 \quad (10)$$

$$\Rightarrow \quad r = \pm i\kappa \quad (11)$$

$$\text{Hence, a general solution of (9) is of the form:} \quad \lambda_{(1)}^r = p^r e^{i\kappa s} + q^r e^{-i\kappa s} \quad (12)$$

$$\text{put } p^r + q^r = A^{,r} \text{ and } p^r - q^r = B^{,r} \quad (13)$$

$$\Leftrightarrow \quad p^r = \frac{A^{,r} + B^{,r}}{2} \text{ and } q^r = \frac{A^{,r} - B^{,r}}{2} \quad (14)$$

$$(12) \text{ can then be written as } \lambda_{(1)}^r = A^{,r} \frac{e^{i\kappa s} + e^{-i\kappa s}}{2} + B^{,r} \frac{e^{i\kappa s} - e^{-i\kappa s}}{2} \quad (15)$$

$$\text{or } \lambda_{(1)}^r = A^{,r} \cos \kappa s + B^{,r} \sin \kappa s \quad (16)$$

$$\text{We have (8)} \quad \lambda^r = -\kappa_{(1)} \frac{d\lambda_{(1)}^r}{ds} \quad (17)$$

$$\frac{d(16)}{ds} \text{ and (17)} \Rightarrow \quad \lambda^r = A^{,r} \sin \kappa s - B^{,r} \cos \kappa s \quad (18)$$

$$\text{as } \lambda^r = \frac{dx^r}{ds} \text{ with (18)} \Rightarrow \quad x^r = -\frac{A^{,r}}{\kappa} \cos \kappa s - \frac{B^{,r}}{\kappa} \sin \kappa s + C^r \quad (19)$$

$$(20)$$

Replace  $-\frac{A^{,r}}{\kappa}$  with  $A^r$  and  $-\frac{B^{,r}}{\kappa}$  with  $B^r$ , we get then the following set of equations,

$$\begin{cases} x^r = A^r \cos \kappa s + B^r \sin \kappa s + C^r \\ \lambda^r = -\kappa A^r \sin \kappa s + \kappa B^r \cos \kappa s \\ \lambda_{(1)}^r = -\kappa A^r \cos \kappa s - \kappa B^r \sin \kappa s \end{cases} \quad (21)$$

$$\text{with the following constraints} \quad \lambda^n \lambda_n = 1 \quad \lambda_{(1)}^n \lambda_{(1)n} = 1 \quad (22)$$

$$\lambda^n \lambda_n = 1 \Rightarrow \kappa^2 A^r A^r \sin^2 \kappa s + \kappa^2 B^r B^r \cos^2 \kappa s - 2\kappa^2 A^r B^r \sin \kappa s \cos \kappa s = 1 \quad (23)$$

$$\text{or} \quad A^r A^r \sin^2 \kappa s + B^r B^r \cos^2 \kappa s - 2A^r B^r \sin \kappa s \cos \kappa s = \frac{1}{\kappa^2} \quad (24)$$

$$\lambda_{(1)}^n \lambda_{(1)n} = 1 \Rightarrow \kappa^2 A^r A^r \cos^2 \kappa s + \kappa^2 B^r B^r \sin^2 \kappa s + 2\kappa^2 A^r B^r \sin \kappa s \cos \kappa s = 1 \quad (25)$$

$$\text{or} \quad A^r A^r \cos^2 \kappa s + B^r B^r \sin^2 \kappa s + 2A^r B^r \sin \kappa s \cos \kappa s = \frac{1}{\kappa^2} \quad (26)$$

$$\text{Choose} \quad \kappa s = \frac{\pi}{2} \quad \text{and} \quad \kappa s = 0 \quad (27)$$

$$\Rightarrow \quad A^r A^r = \frac{1}{\kappa^2} \quad \text{and} \quad B^r B^r = \frac{1}{\kappa^2} \quad (28)$$

$$\text{Moreover considering (26)-(24) and (28)} \Rightarrow \quad 2A^r B^r \sin \kappa s \cos \kappa s = 0 \quad \forall \kappa s \quad (29)$$

$$\Rightarrow \quad A^r B^r = 0 \quad (30)$$

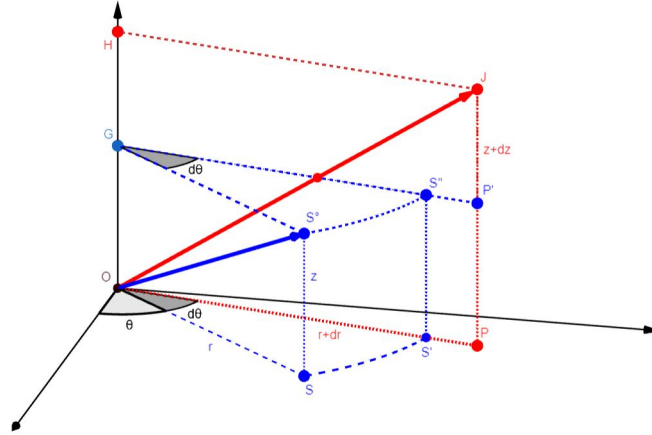
Note: when deriving expressions (23) and (26) we use the fact that  $(a_{mn}) = (a^{mn}) = (\delta_n^m)$





## 2.47 p78-exercise 1

For cylindrical coordinates in Euclidean 3-space, write down the metric form by inspection of a diagram showing a general infinitesimal displacement, and calculate all the Christoffel symbols of both kinds.



From the figure we get (assuming an infinitesimal displacement, we may approximate  $|\overrightarrow{SS'}|$  with the arclength  $r d\theta$  and assume  $|\overrightarrow{SS''}| \perp |\overrightarrow{GS'}|$ ) Hence, the infinitesimal displacement from S

$$ds^2 = |\overrightarrow{SS'}|^2 + |\overrightarrow{S''P'}|^2 + |\overrightarrow{P'J}|^2 \quad (1)$$

$$= dr^2 + ((r + dr)d\theta)^2 + dz^2 \quad (2)$$

$$= dr^2 + r^2 d\theta^2 + dz^2 \quad (3)$$

$$\text{Hence} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4)$$

Note that all  $a_{mn} = 0 \quad \forall m \neq n$ . So,

$$\begin{cases} [r \ r, r] = [\theta \ \theta, \theta] = [z \ z, z] = 0 \\ [r \ \theta, r] = [r \ r, \theta] = [r \ r, z] = 0 \\ [r \ z, \theta] = [z \ \theta, r] = [z \ \theta, z] = 0 \end{cases} \quad (5)$$

$$\text{But:} \quad [\theta \ \theta, r] = -r \quad \text{and} \quad [r \ \theta, \theta] = r \quad (6)$$

$$\text{Hence} \quad \begin{cases} \Gamma_{nk}^m = 0 \quad \forall \quad (nk) \neq (r, \theta), (\theta, \theta) \\ \Gamma_{r\theta}^\theta = \frac{1}{r} \quad \text{and} \quad \Gamma_{\theta\theta}^r = -r \end{cases} \quad (7)$$



## 2.48 p78-exercise 2

If  $a_{rs}$  and  $b_{rs}$  are covariant tensors, show that the roots of the determinant equation

$$|Xa_{rs} - b_{rs}| = 0$$

are invariants.

$$\text{Be} \quad c_{rs} = Xa_{rs} - b_{rs} \quad (1)$$

$$\text{Given} \quad a_{rs} = a_{mk} \frac{\partial x^m}{\partial x^r} \frac{\partial x^k}{\partial x^s} \quad (2)$$

$$\text{and} \quad b_{rs} = b_{mk} \frac{\partial x^m}{\partial x^r} \frac{\partial x^k}{\partial x^s} \quad (3)$$

$$(1) \Rightarrow c_{rs} = \underbrace{(Xa_{mk} - b_{mk})}_{=c_{km}} \frac{\partial x^m}{\partial x^r} \frac{\partial x^k}{\partial x^s} \quad (4)$$

$$= c_{km} \frac{\partial x^m}{\partial x^r} \frac{\partial x^k}{\partial x^s} \quad (5)$$

$$\text{Be} \quad J = \left| \frac{\partial x^m}{\partial x^r} \right| = \left| \frac{\partial x^k}{\partial x^s} \right| \quad (6)$$

$$\text{In (5) put} \quad d_{kr} = c_{km} \frac{\partial x^m}{\partial x^r} \quad (7)$$

$$\Rightarrow c_{rs} = d_{kr} \frac{\partial x^k}{\partial x^s} \quad (8)$$

$$\text{or in matrix form} \quad C = D^T J \quad \text{with} \quad D = C \cdot J \quad (9)$$

$$\Rightarrow |C| = |(C \cdot J)^T J| \quad (10)$$

$$\Leftrightarrow |C| = |C| |J| |J| \quad (11)$$

As  $|J| \neq 0$  ( $J$  is the Jacobian of the transformation, and thus can't be zero), then

$$|C| = 0 \Rightarrow |C \cdot J| = 0$$

So, the root of  $|C| = 0$  is also a root of  $|C \cdot J| = 0$  and is as a consequence, invariant.



## 2.49 p78-exercise 3

Is the form  $dx^2 + 3dxdy + 4dy^2 + dz^2$  positive definite?

$$\Phi = dx^2 + 3dxdy + 4dy^2 + dz^2 \quad (1)$$

Put (1) in the form  $\Phi = X^2 + 3XY + 4Y^2 + Z^2 \quad (2)$

Z has only a positive contribution: so put  $Z = 0 \Rightarrow \Phi = X^2 + 3XY + 4Y^2 \quad (3)$

(3) can only be zero or negative if  $XY < 0$  :put  $Y = -aX \quad (a > 0) \quad (4)$

$$\Rightarrow \Phi = X^2 - 3aX^2 + 4a^2X^2 \quad (5)$$

The roots of (5) are  $a_{1,2} = \frac{3 \pm \sqrt{9 - 16}}{8} \quad (6)$

So, by (6) we can't get a  $a \in \mathbb{R}_*$ , so that (1) can be 0 or negative. Hence,

The form  $\Phi$  is positive definite



## 2.50 p78-exercise 4

If  $X^r, Y^r$  are unit vectors inclined at an angle  $\theta$ , prove that

$$\sin^2 \theta = (a_{rm}a_{sn} - a_{rs}a_{mn})X^rY^sX^mY^n$$

$X^rY^s$  are unit vectors. So,

$$a_{rm}X^rX^m = 1 \quad \text{and} \quad a_{sn}Y^sY^n = 1 \quad (1)$$

$$\text{We have} \quad \sin^2 \theta = 1 - \cos^2 \theta \quad (2)$$

$$\text{and (2.312)} \quad \cos \theta = a_{mn}X^mY^n \quad (3)$$

$$\Rightarrow \quad \sin^2 \theta = a_{rm}X^rX^ma_{sn}Y^sY^n - a_{mn}X^mY^na_{rs}X^rY^s \quad (4)$$

$$= (a_{rm}a_{sn} - a_{mn}a_{rs})X^rY^sX^mY^n \quad (5)$$



## 2.51 p78-exercise 5

Show that, if  $\theta$  is the angle between the normals to the surfaces  $x^1 = C^{st}, x^2 = C^{st}$ , then

$$\cos \theta = \frac{a^{12}}{\sqrt{a^{11}a^{22}}}$$

$$\text{be} \quad \phi_1(x^1, x^2, \dots, x^N) = C^{st} \quad \phi_2(x^1, x^2, \dots, x^N) = C^{st} \quad (1)$$

the two equations representing  $S_1, S_2$  (see page 63). We can rewrite (1) as:

$$x^1 = \phi_1(x^1, x^2, \dots, x^N) = C^{st} \quad (2)$$

$$x^2 = \phi_2(x^1, x^2, \dots, x^N) = C^{st} \quad (3)$$

From (2.622) we know that  $X^m = a^{mn}\partial_n\phi_1$ ,  $Y^m = a^{mn}\partial_n\phi_2$  are  $\perp$  vectors to the surfaces  $\phi_1, \phi_2$  (4)

$$\text{We know also} \quad |X^m|^2 = a^{mk}X^mX^k \quad (5)$$

$$= a_{mk}a^{mn}\partial_n\phi_1a^{kp}\partial_p\phi_1 \quad (6)$$

$$= \delta_k^k a^{kp}\partial_n\phi_1\partial_p\phi_1 \quad (7)$$

$$= a^{np}\partial_n\phi_1\partial_p\phi_1 \quad (8)$$

$$\text{as } \phi_1 = x^1 = C^{st} \Rightarrow = a^{np}\delta_n^1\delta_p^1 \quad (9)$$

$$= \epsilon a^{11} \quad (10)$$

$$\text{Analog, we have} \quad |Y^m|^2 = \epsilon a^{22} \quad (11)$$

$$\text{By definition:} \quad \cos \theta = \frac{a_{mn}X^mY^n}{|X^r||Y^s|} \quad (12)$$

$$\text{and} \quad a_{mn}X^mY^n = a_{mn}a^{mk}\partial_k\phi_1a^{np}\partial_p\phi_2 \quad (13)$$

$$= \delta_n^k a^{np}\partial_k\phi_1\partial_p\phi_2 \quad (14)$$

$$= a^{kp}\partial_k\phi_1\partial_p\phi_2 \quad (15)$$

$$\text{as } \phi_1 = x^1 = C^{st}, \phi_2 = x^2 = C^{st} \Rightarrow = a^{kp}\delta_k^1\delta_p^2 \quad (16)$$

$$= a^{12} \quad (17)$$

$$\text{So (12) becomes with (10), (11) and (17)} \quad \cos \theta = \frac{a_{12}}{\sqrt{\epsilon a^{11}\epsilon a^{22}}} \quad (18)$$

$$= \frac{a_{12}}{\sqrt{a^{11}a^{22}}} \quad (19)$$



## 2.52 p78-exercise 6

Let  $x^1, x^2, x^3$  be rectangular Cartesian coordinates in Euclidean 3-space, and let  $x^1, x^2$  be taken as coordinates on a surface  $x^3 = f(x^1, x^2)$ . Show that the Christoffel symbols of the second kind for the surface are

$$\Gamma_{mn}^r = \frac{f_r f_{mn}}{1 + f_n f_p}$$

the suffixes taking the values 1, 2 and the subscripts indicating partial derivatives.

We have (rectangular Cartesian coordinates in Euclidean 3-space)

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad (1)$$

$$\text{with } x^3 = f(x^1, x^2) \quad (2)$$

$$\text{and thus } dx^3 = \partial_1 f dx^1 + \partial_2 f dx^2 \quad (3)$$

$$\Rightarrow ds^2 = (1 + (\partial_1 f)^2)(dx^1)^2 + (1 + (\partial_2 f)^2)(dx^2)^2 + 2\partial_1 f \partial_2 f dx^1 dx^2 \quad (4)$$

$$\text{put } \begin{cases} f_1 = \partial_1 f \\ f_2 = \partial_2 f \\ f_{11} = \partial_{11} f \\ f_{22} = \partial_{22} f \\ f_{12} = f_{21} = \partial_{12} f \end{cases} \quad (5)$$

$$\text{from (4)} \quad (a_{mn}) = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix} \quad (6)$$

$$\Rightarrow |a_{mn}| = (1 + f_1^2)(1 + f_2^2) - (f_1 f_2)^2 \quad (7)$$

$$= 1 + f_1^2 + f_2^2 \quad (8)$$

$$\text{also } (a^{mn}) = \frac{1}{1 + f_1^2 + f_2^2} \begin{pmatrix} 1 + f_2^2 & -f_1 f_2 \\ -f_1 f_2 & 1 + f_1^2 \end{pmatrix} \quad (9)$$

**Calculating the Christoffels symbols:**  $[mn, k] = \frac{1}{2}(\partial_m a_{nk} + \partial_n a_{mk} - \partial_k a_{mn})$  (10)

$$\Rightarrow \left\{ \begin{array}{l} [11, 1] = f_1 f_{11} \\ [11, 2] = f_2 f_{11} \\ [12, 1] = f_1 f_{12} \\ [12, 2] = f_2 f_{21} \\ [22, 2] = f_2 f_{22} \end{array} \right. \quad (11)$$

From (11) we can see that the general form is:  $[mn, s] = f_{mn} f_s$  (12)

**Calculating the Christoffels symbols:**  $\Gamma_{mn}^r = a_{rs}[mn, s]$  (13)

(13) with (12):  $\Gamma_{mn}^r = a_{rs} f_{mn} f_s = f_{mn} (a^{rs} f_s)$  (14)

put  $\Delta = \frac{1}{1 + f_1^2 + f_2^2} = \frac{1}{1 + f_p f_p}$  (15)

$$\Rightarrow \left\{ \begin{array}{l} \Gamma_{mn}^1 = (a_{11} f_1 + a_{12} f_2) f_{mn} \\ \quad = \Delta (f_1 + f_2^2 f_1 - f_1 f_2^2) f_{mn} \\ \quad = \Delta f_1 f_{mn} \\ \Gamma_{mn}^2 = (a_{21} f_1 + a_{22} f_2) f_{mn} \\ \quad = \Delta (-f_1^2 f_2 + f_2 + f_2 f_1^2) f_{mn} \\ \quad = \Delta f_2 f_{mn} \end{array} \right. \quad (16)$$

$$\Rightarrow \Gamma_{mn}^r = \frac{f_r f_{mn}}{1 + f_p f_p}$$



## 2.53 p78-exercise 7

Write down the differential equations of the geodesics on a sphere, using colatitude  $\theta$  and the azimuth  $\phi$  as coordinates. Integrate the differential equations and obtain a finite equation

$$A \sin \theta \cos \phi + B \sin \theta \sin \phi + C \cos \theta = 0$$

where  $A, B, C$  are arbitrary constants.

We will use two different approaches to determine the relation and finally use a geometrical reasoning allowing us to avoid solving the ODE's resulting from the above mentioned approaches.

We will first find the solution, starting from the variational principle, defining a geodesic. In spherical coordinates we have (see exercise page 27)  $ds^2 = dr^2 + r^2 \sin^2(\theta) d\phi^2 + r^2 d\theta^2$ .

As  $r = R = C^{st}$  this reduces to

$$ds^2 = R^2 \sin^2(\theta) d\phi^2 + R^2 d\theta^2 \quad (1)$$

So the length of a curve on the sphere from a point  $P_1$  to another point  $P_2$ , the curve being determined by  $\theta = \theta(u)$   $\phi = \phi(u)$  is:

$$L = R \int_{P_1}^{P_2} \sqrt{\sin^2(\theta) d\phi^2 + d\theta^2} du \quad (2)$$

$$\text{Be } \theta = u \quad \phi = \phi(\theta) \quad (3)$$

$$\Rightarrow L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2(\theta) \left(\frac{d\phi}{d\theta}\right)^2} d\theta \quad (4)$$



Applying the variational principle on  $\mathcal{L}$  and using the Euler-Langrange equations:

$$\frac{d}{d\theta} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \text{with} \quad \mathcal{L} = \sqrt{1 + \sin^2(\theta)(\dot{\phi})^2} \quad (5)$$

$$\text{as} \quad \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (6)$$

$$(5) \text{ becomes: } \frac{d}{d\theta} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 \quad (7)$$

$$\Leftrightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = C \quad (= \text{constant}) \quad (8)$$

$$\text{with} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \sqrt{1 + \sin^2 \theta \dot{\phi}^2}}{\partial \dot{\phi}} = \frac{\sin^2 \theta \dot{\phi}}{\sqrt{1 + \sin^2 \theta \dot{\phi}^2}} \quad (9)$$

$$(9) \text{ and } (10): \quad C^2 = \frac{\sin^2 \theta \dot{\phi}^2}{1 + \sin^2 \theta \dot{\phi}^2} \quad (10)$$

$$\text{or} \quad \dot{\phi} = \frac{C}{\sin \theta \sqrt{\sin^2 \theta - C^2}} \quad (11)$$

Solving the ODE (12). Put  $u = \cot \theta \Rightarrow du = -\csc^2 \theta d\theta = -\frac{1}{\sin^2 \theta} d\theta$ . So,

$$\phi = -C \int \frac{\sin \theta}{\sqrt{\sin^2 \theta - C^2}} du \quad (12)$$

$$= -C \int \frac{du}{\sqrt{1 - \frac{C^2}{\sin^2 \theta}}} \quad (13)$$

$$= -C \int \frac{du}{\sqrt{1 - C^2 - C^2 \cot^2 \theta}} \quad (14)$$

$$= -C \int \frac{du}{\sqrt{1 - C^2 - C^2 u^2}} \quad (15)$$

$$\text{be} \quad a = \frac{\sqrt{1 - C^2}}{C} \quad (16)$$

$$(15) \text{ becomes} \quad \phi = - \int \frac{1}{\sqrt{a^2 - u^2}} du \quad (17)$$

$$\text{put} \quad u = av \quad (18)$$

$$(18) \text{ becomes} \quad \phi = - \int \frac{1}{\sqrt{1 - v^2}} dv \quad (19)$$

$$= -\arccos v + C^{st} \quad (20)$$

$$= -\arccos \frac{u}{a} + \phi_0 \quad (21)$$

$$\text{or:} \quad \frac{u}{a} = \cos(\phi - \phi_0) \quad (\text{by choosing an adequate } \phi_0) \quad (22)$$

$$\text{so:} \quad \cot \theta = a \cos(\phi - \phi_0) \quad (23)$$

$$\text{expanding } \cos(\phi - \phi_0) \text{ gives: } \frac{\cos \theta}{\sin \theta} = A \cos \phi + B \sin \phi \quad (24)$$

$$\text{or:} \quad A \cos \phi \sin \theta + B \sin \phi \sin \theta - \cos \theta = 0 \quad (25)$$

◇

Finding the geodesics from the tensorial formula's.

Note: For ease of notation we put  $R = 1$  without losing any general solutions.

$$\text{from (1) we get: } (a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix} \quad (26)$$

$$\text{and } (a^{mn}) = \begin{pmatrix} \frac{1}{\sin^2 \theta} & 0 \\ 0 & 1 \end{pmatrix} \quad (27)$$

$$\text{hence: } \begin{cases} \Gamma_{11}^1 = 0 & \Gamma_{11}^2 = 0 \\ \Gamma_{12}^1 = 0 & \Gamma_{12}^2 = \cot \theta \\ \Gamma_{22}^1 = -\cos \theta \sin \theta & \Gamma_{22}^2 = 0 \end{cases} \quad (28)$$

Finding the geodesics from the tensorial formula's, implies solving  $2^{nd}$  order ODE's. In order to find the simplest form to solve, we write down three possible forms of the geodesic equations:

$$\text{arc-length } s \text{ as independent variable } \begin{cases} (a) \quad \frac{d^2 \phi}{ds^2} + 2 \cot \theta \frac{d\phi}{ds} \frac{d\theta}{ds} = 0 \\ (b) \quad \frac{d^2 \theta}{ds^2} - \sin \theta \cos \theta \left( \frac{d\phi}{ds} \right)^2 = 0 \\ (c) \quad \left( \frac{d\theta}{ds} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{ds} \right)^2 = 1 \end{cases} \quad (29)$$

$$\theta \text{ as independent variable } \begin{cases} \lambda = -\sin \theta \cos \theta \left( \frac{d\phi}{d\theta} \right)^2 \\ \frac{d^2 \phi}{d\theta^2} = \lambda \frac{d\phi}{d\theta} - 2 \cot \theta \frac{d\phi}{d\theta} \end{cases} \quad (30)$$

$$\Rightarrow \frac{d^2 \phi}{d\theta^2} = -\sin \theta \cos \theta \left( \frac{d\phi}{d\theta} \right)^3 - 2 \cot \theta \frac{d\phi}{d\theta} \quad (31)$$

$$\phi \text{ as independent variable } \begin{cases} \lambda = 2 \cot \theta \frac{d\theta}{d\phi} \\ \frac{d^2 \theta}{d\phi^2} = \lambda \frac{d\theta}{d\phi} + \sin \theta \cos \theta \left( \frac{d\theta}{d\phi} \right)^2 \end{cases} \quad (32)$$

$$\Rightarrow \frac{d^2 \theta}{d\phi^2} = 2 \cot \theta \left( \frac{d\theta}{d\phi} \right)^2 + \sin \theta \cos \theta \left( \frac{d\theta}{d\phi} \right)^2 \quad (33)$$

$$(34)$$

Inspection shows that the expression (32) and (34) are quite complicated while using (30b) and (30c)

we can get an expression of the form

$$\ddot{\theta} - \sin \theta \cos \theta \left( \frac{1 - \dot{\theta}^2}{\sin^2 \theta} \right) = 0 \quad (35)$$

$$\text{or: } \ddot{\theta} - \cot \theta (1 - \dot{\theta}^2) = 0 \quad (36)$$

$$\text{Put } u(\theta) = \dot{\theta} \Rightarrow \ddot{\theta} = \dot{u}u \quad (37)$$

$$(37) \text{ can be written as: } \dot{u}u - \cot \theta (1 - u^2) = 0 \quad (38)$$

$$\text{or } \frac{\dot{u}u}{(1 - u^2)} = \cot \theta \quad (39)$$

$$\text{or } \frac{u}{(1 - u^2)} du = \frac{\cos \theta}{\sin \theta} d\theta \quad (40)$$

$$\text{or } -\frac{1}{2} \frac{1}{(1 - u^2)} d(1 - u^2) = \frac{1}{\sin \theta} d(\sin \theta) \quad (41)$$

$$\text{hence } -d(\log(\sqrt{1 - u^2})) = d(\log(\sin \theta)) \quad (42)$$

$$\Rightarrow d(\log \sqrt{1 - u^2} + \log(\sin \theta)) = 0 \quad (43)$$

$$\Leftrightarrow d(\log(\sqrt{1 - u^2} \sin \theta)) = 0 \quad (44)$$

$$\Rightarrow (1 - \dot{\theta}^2) \sin^2 \theta = C^2 \quad (45)$$

$$\Rightarrow \dot{\theta}^2 = 1 - \frac{C^2}{\sin^2 \theta} \quad (46)$$

$$\text{we have (30c) } \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta = 1 \quad (47)$$

$$\text{and so (47): } 1 - \frac{C^2}{\sin^2 \theta} + \dot{\phi}^2 \sin^2 \theta = 1 \quad (48)$$

$$\Rightarrow \dot{\phi}^2 = \frac{C^2}{\sin^4 \theta} \quad (49)$$

$$\Rightarrow \dot{\phi} = \frac{C}{\sin^2 \theta} \quad (50)$$

$$\text{we have } \dot{\phi} = \frac{d\phi}{d\theta} \dot{\theta} \quad (51)$$

$$\text{so } d\phi = \frac{C}{\sin^2 \theta \sqrt{1 - \frac{C^2}{\sin^2 \theta}}} d\theta \quad (52)$$

$$\text{so } d\phi = \frac{C}{\sin \theta \sqrt{\sin^2 \theta - C^2}} d\theta \quad (53)$$

Note that expression (54) is exactly the expression (12) we found by applying directly the variational principle to find the general expression. So applying steps (13) to (26) gives us the same expression.

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Instead of solving the ODE's (46) and (53) we can use geometrical considerations to get the asked expression.

Due to the invariance of a sphere regarding rotation of the axes, we can choose a reference axis system  $XYZ$  (from which  $\theta, \phi$  are measured) so that at  $s = 0$  of the geodesic, corresponds the point  $r = 1, \theta = 0$ . As from (45),  $(1 - \dot{\theta}^2) \sin^2 \theta \Big|_{s=0} = C^2$  follows that  $C = 0$  (because  $\theta|_{s=0} = 0$ ). So we get  $(1 - \dot{\theta}^2) \sin^2 \theta = 0 \forall \theta : \Rightarrow \dot{\theta} = 1$  and  $\theta = s$ . Then from (37)  $\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta = 1$  follows that  $\dot{\phi} = 0$  and thus  $\phi = C^{st}$ . Again, considering symmetry we can choose the axis system so that  $\phi = 0$ . The set of equations  $\theta = s, \phi = 0$  represents a circle on the sphere generated by the intersection of the sphere with the  $XZ$  plane. Again, considering symmetry, we can conclude that any circle on the sphere generated by the intersection a plane going through the origin of the axis system, is also a geodesic curve. So, be  $\hat{n} = (A, B, C)$  the normal vector defining a plane going through the origin and  $\hat{p} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$  a point on the sphere, then the intersection of the plane and the sphere is given by

$$\langle \hat{n} | \hat{p} \rangle = A \cos \phi \sin \theta + B \sin \phi \sin \theta + C \cos \theta = 0$$



## 2.54 p79-exercise 8

Find in integrated form the geodesic null lines in a  $V_3$  for which the metric form

$$(dx^1)^2 - R^2[(dx^2)^2 + (dx^3)^2]$$

$R$  being a function of  $x^1$  only.

We have,

$$\Phi = (dx^1)^2 - R^2[(dx^2)^2 + (dx^3)^2] \quad (1)$$

$$\text{Hence, } (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -R^2 & 0 \\ 0 & 0 & -R^2 \end{pmatrix} \quad (2)$$

$$\text{and, } (a^{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{R^2} & 0 \\ 0 & 0 & -\frac{1}{R^2} \end{pmatrix} \quad (3)$$

$$\text{Hence, } \begin{cases} [22, 1] = R \partial_1 R & [12, 2] = -R \partial_1 R \\ [33, 1] = R \partial_1 R & [13, 3] = -R \partial_1 R \end{cases} \quad (4)$$

$$\text{and, } \begin{cases} \Gamma_{22}^1 = R \partial_1 R & \Gamma_{12}^2 = \frac{1}{R} \partial_1 R \\ \Gamma_{33}^1 = R \partial_1 R & \Gamma_{13}^3 = \frac{1}{R} \partial_1 R \end{cases} \quad (5)$$

$$\text{with all other } [mn, s] \text{ and } \Gamma_{mn}^s \text{ being zero.} \quad (6)$$

The equations of null geodesics give:

$$\begin{cases} \frac{d^2 x^r}{du^2} + \Gamma_{mn}^r \frac{dx^m}{du} \frac{dx^n}{du} = 0 \\ a_{mn} \frac{dx^m}{du} \frac{dx^n}{du} = 0 \end{cases} \quad (7)$$

$x^r = x^1$  gives:

$$\frac{d^2 x^1}{du^2} + R \partial_1 R \left( \frac{dx^2}{du} \right)^2 + R \partial_1 R \left( \frac{dx^3}{du} \right)^2 = 0 \quad (8)$$

Put  $u = x^1$

$$\text{from (8): } R \partial_1 R \left[ \left( \frac{dx^2}{du} \right)^2 + \left( \frac{dx^3}{du} \right)^2 \right] = 0 \quad (9)$$

If,  $R = R(x^1) \neq C^{st}$ , the form (9) can only be zero if  $\left( \frac{dx^2}{du} \right)^2 + \left( \frac{dx^3}{du} \right)^2 = 0$  and thus  $\frac{dx^2}{du} = \frac{dx^3}{du} = 0$  and hence  $x^2, x^3$  are constant.

**Conclusion:** the null geodesics are the bundle of rays parallel with the  $x^1$  axis with vector equation  $\hat{p} = (s, A, B)$ ,  $s \in (-\infty, +\infty)$  and  $A, B$  arbitrary constants.



## 2.55 p79-exercise 9

Show that, for normal coordinate system, the Christoffel symbols

$$[\rho N, \sigma], [\rho\sigma, N], [\rho N, N], [NN, N]$$

$$\Gamma_{N\sigma}^\rho, \Gamma_{\rho\sigma}^N, \Gamma_{NN}^\rho, \Gamma_{N\rho}^N, \Gamma_{NN}^N$$

have tensor character with respect to the transformation of the coordinates  $x^1, \dots, x^{N-1}$

We know that  $a_{\rho\sigma} = a_{mn} \partial_\rho x^m \partial_\sigma x^n$  with  $\rho, \sigma = 1, \dots, N-1$  and  $m, n = 1, \dots, N$ . We have also  $x^N = x^{,N}$ .

$[\rho N, \sigma]$

$$[\rho N, \sigma] = \frac{1}{2} \partial_N a_{\rho\sigma} \quad \text{see (2.639)} \quad (1)$$

$$= \frac{1}{2} \partial_N (a_{mn} \partial_\rho x^m \partial_\sigma x^n) \quad (2)$$

$$= \begin{cases} \frac{1}{2} (\partial_N a_{mn} \partial_\rho x^m \partial_\sigma x^n \\ + a_{mn} \partial_\sigma x^n \partial_{N\rho} x^m \\ + a_{mn} \partial_\rho x^m \partial_{N\sigma} x^n) \end{cases} \quad (3)$$

$$\text{We have } \partial_N x^m = \delta_N^m \Rightarrow \partial_{N\rho} x^m = \partial_{N\sigma} x^n = 0 \quad (4)$$

$$\Rightarrow [\rho N, \sigma] = \frac{1}{2} \partial_N a_{mn} \partial_\rho x^m \partial_\sigma x^n \quad (5)$$

$$= \begin{cases} \frac{1}{2} (\partial_N a_{\alpha\beta} \partial_\rho x^\alpha \partial_\sigma x^\beta \\ + \partial_N a_{NN} \underbrace{\partial_\rho x^N}_{=0} \underbrace{\partial_\sigma x^N}_{=0} \\ + \partial_N a_{\alpha N} \partial_\rho x^\alpha \underbrace{\partial_\sigma x^N}_{=0} \\ + \partial_N a_{\beta N} \partial_\rho x^\alpha \underbrace{\partial_\sigma x^N}_{=0}) \end{cases} \quad (6)$$

$$\Rightarrow [\rho N, \sigma] = \frac{1}{2} \underbrace{\partial_N a_{\alpha\beta}}_{=[\alpha N, \beta]} \partial_\rho x^\alpha \partial_\sigma x^\beta \quad (7)$$

$$= [\alpha N, \beta] \partial_\rho x^\alpha \partial_\sigma x^\beta \quad (8)$$

This confirms the tensor character of  $[\rho N, \sigma]$

◇

$[\rho\sigma, N]$  this follows immediately from the previous and considering  $[\rho\sigma, N] = -[\rho N, \sigma]$  see(2.639)

◇

$[\rho N, N]$

We prove the case for  $[NN, \rho]$  as  $[\rho N, N] = -[NN, \rho]$

$$[NN, \rho] = \frac{1}{2} \partial_\rho a_{NN} \quad \text{see (2.639)} \quad (9)$$

$$= \frac{1}{2} \partial_\rho (a_{mn} \partial_N x^m \partial_N x^n) \quad (10)$$

$$= \begin{cases} \frac{1}{2} (\partial_\rho a_{mn} \partial_N x^m \partial_N x^n \\ + a_{mn} \partial_\rho x^n \underbrace{\partial_N x^m}_{=0} \\ + a_{mn} \partial_\rho x^m \underbrace{\partial_N x^n}_{=0}) \end{cases} \quad (11)$$

$$= \frac{1}{2} \partial_\rho a_{mn} \partial_N x^m \partial_N x^n \quad (12)$$

$$= \begin{cases} \frac{1}{2} (\partial_\rho a_{\alpha\beta} \underbrace{\partial_N x^\alpha}_{=0} \underbrace{\partial_N x^\beta}_{=0} \\ + \partial_\rho a_{NN} \underbrace{\partial_N x^N}_{=1} \underbrace{\partial_N x^N}_{=1} \\ + \partial_\rho a_{\alpha N} \underbrace{\partial_N x^\alpha}_{=0} \underbrace{\partial_N x^N}_{=1} \\ + \partial_\rho a_{\beta N} \underbrace{\partial_N x^\alpha}_{=0} \underbrace{\partial_N x^N}_{=1}) \end{cases} \quad (13)$$

$$\Rightarrow [NN, \rho] = \frac{1}{2} \partial_\rho a_{NN} \quad (14)$$

$$\Leftrightarrow [NN, \rho] = \underbrace{\frac{1}{2} \partial_\alpha a_{NN} \partial_\rho x^\alpha}_{=[NN, \alpha]} \quad (15)$$

$$\Rightarrow [NN, \rho] = [NN, \alpha] \partial_\rho x^\alpha \quad (16)$$

This confirms the tensor character of  $[NN, \rho]$  and consequently of  $[\rho N, N]$

◇

$$[NN, N]$$

$$[NN, N] = \frac{1}{2} \partial_N a_{NN} \quad \text{see (2.639)} \quad (17)$$

$$= \frac{1}{2} \partial_N (a_{mn} \partial_N x^m \partial_N x^n) \quad (18)$$

$$= \begin{cases} \frac{1}{2} (\partial_N a_{mn} \underbrace{\partial_N x^m}_{\delta_N^m} \underbrace{\partial_N x^n}_{\delta_N^n}) \\ + a_{mn} \partial_N x^n \underbrace{\partial_{NN} x^m}_{=0} \\ + a_{mn} \partial_N x^m \underbrace{\partial_{NN} x^n}_{=0} \end{cases} \quad (19)$$

$$= \frac{1}{2} \partial_N a_{mn} \delta_N^m \delta_N^n \quad (20)$$

$$= \underbrace{\frac{1}{2} \partial_N a_{NN}}_{=[NN, N]} \quad (21)$$

$$\Rightarrow [NN, N] = [NN, N] \quad (22)$$

This confirms the tensor character of  $[NN, N]$  as an invariant under transformation of the coordinates  $x^1, \dots, x^{N-1}$

◇

$$\Gamma_{N\sigma}^\rho, \Gamma_{\rho\sigma}^N, \Gamma_{NN}^\rho, \Gamma_{N\rho}^N, \Gamma_{NN}^N$$

For the Christoffel symbols of the second kind we use:

$$\Gamma_{st}^r = a^{rk} [st, k] \quad (23)$$

$$a^{rk} = a^{mn} \frac{\partial x^r}{\partial x^m} \frac{\partial x^k}{\partial x^n} \quad (24)$$

$$(2.631) \text{ page 65: } a^{N\rho} = 0 \quad (25)$$



$$\Gamma_{N\sigma}^\rho$$

$$\Gamma_{N\sigma}^\rho = a^{\rho k}[N\sigma, k] \quad (26)$$

$$= a^{\rho\tau}[N\sigma, \tau] + \underbrace{a^{\rho N}}_{=0}[N\sigma, N] \quad (27)$$

$$(24): \quad = a^{,mn} \frac{\partial x^\rho}{\partial x^{,m}} \frac{\partial x^\tau}{\partial x^{,n}} [N\sigma, \tau] \quad (28)$$

$$\text{also (see previous results):} \quad [N\sigma, \tau] = [N\mu, \nu] \cdot \partial_\sigma x^{,\mu} \partial_\tau x^{,\nu} \quad (29)$$

And so,

$$\Gamma_{N\sigma}^\rho = a^{,mn} \frac{\partial x^\rho}{\partial x^{,m}} \frac{\partial x^\tau}{\partial x^{,n}} [N\mu, \nu] \cdot \partial_\sigma x^{,\mu} \partial_\tau x^{,\nu} \quad (30)$$

$$= a^{,mn} \frac{\partial x^\rho}{\partial x^{,m}} \underbrace{\frac{\partial x^{,\nu}}{\partial x^{,n}}}_{=\delta_n^\nu} [N\mu, \nu] \cdot \partial_\sigma x^{,\mu} \quad (31)$$

$$= a^{,\theta\nu} [N\mu, \nu] \cdot \frac{\partial x^\rho}{\partial x^{,\theta}} \frac{\partial x^{,\mu}}{\partial x^\sigma} + \underbrace{a^{,N\nu}}_{=0} [N\mu, \nu] \cdot \frac{\partial x^\rho}{\partial x^{,N}} \frac{\partial x^{,\mu}}{\partial x^\sigma} \quad (32)$$

$$= \underbrace{a^{,\theta\nu} [N\mu, \nu]}_{=\Gamma_{N\mu}^\theta} \cdot \frac{\partial x^\rho}{\partial x^{,\theta}} \frac{\partial x^{,\mu}}{\partial x^\sigma} \quad (33)$$

$$\Rightarrow \Gamma_{N\sigma}^\rho = \Gamma_{N\mu}^\theta \frac{\partial x^\rho}{\partial x^{,\theta}} \frac{\partial x^{,\mu}}{\partial x^\sigma} \quad (34)$$

So,  $\Gamma_{N\sigma}^\rho$  is a  $2^{nd}$  order mixed tensor (contravariant in  $\rho$ , covariant in  $\sigma$ .)

◇

$$\Gamma_{\rho\sigma}^N$$

$$\Gamma_{\rho\sigma}^N = a^{Nk}[\rho\sigma, k] \quad (35)$$

$$= \underbrace{a^{N\tau}}_{=0}[\rho\sigma, \tau] + a^{NN}[\rho\sigma, N] \quad (36)$$

$$(8): \quad = a^{,NN}[\alpha\beta, N] \cdot \partial_\rho x^{,\alpha} \partial_\sigma x^{,\beta} \quad (37)$$

$$\text{considering : } a^{,N\tau} = 0 \quad \text{we can write this as:} \quad (38)$$

$$\Gamma_{\rho\sigma}^N = a^{,N\tau}[\alpha\beta, \tau] \cdot \partial_\rho x^{,\alpha} \partial_\sigma x^{,\beta} + a^{,NN}[\alpha\beta, N] \cdot \partial_\rho x^{,\alpha} \partial_\sigma x^{,\beta} \quad (39)$$

$$= \underbrace{a^{,Nk}[\alpha\beta, k]}_{=\Gamma_{\alpha\beta}^{,N}} \cdot \partial_\rho x^{,\alpha} \partial_\sigma x^{,\beta} \quad (40)$$

$$\Rightarrow \Gamma_{\rho\sigma}^N = \Gamma_{\alpha\beta}^{,N} \partial_\rho x^{,\alpha} \partial_\sigma x^{,\beta} \quad (41)$$

So,  $\Gamma_{\rho\sigma}^N$  is a  $2^{nd}$  order mixed tensor (covariant in both indices)

◇

$\Gamma_{NN}^\rho$ 

$$\Gamma_{NN}^\rho = a^{\rho k} [NN, k] \quad (42)$$

$$= a^{\rho\tau} [NN, \tau] + \underbrace{a^{\rho N}}_{=0} [NN, N] \quad (43)$$

$$(16): \quad = a^{,mn} \frac{\partial x^\rho}{\partial x^{,m}} \frac{\partial x^\tau}{\partial x^{,n}} [NN, \alpha] \cdot \partial_\tau x^{,\alpha} \quad (44)$$

$$= a^{,mn} \frac{\partial x^\rho}{\partial x^{,m}} \underbrace{\frac{\partial x^\alpha}{\partial x^{,n}}}_{=\delta_n^\alpha} [NN, \alpha], \quad (45)$$

$$= a^{,\tau\alpha} [NN, \alpha] \cdot \frac{\partial x^\rho}{\partial x^{,\tau}} + \underbrace{a^{,N\alpha}}_{=0} [NN, \alpha] \cdot \frac{\partial x^\rho}{\partial x^{,N}} \quad (46)$$

$$= \underbrace{a^{,\tau\alpha} [NN, \alpha]}_{=\Gamma_{NN}^{,\tau}} \cdot \frac{\partial x^\rho}{\partial x^{,\tau}} \quad (47)$$

$$\Rightarrow \quad \Gamma_{NN}^\rho = \Gamma_{NN}^{,\tau} \frac{\partial x^\rho}{\partial x^{,\tau}} \quad (48)$$

So,  $\Gamma_{NN}^\rho$  is a 1<sup>st</sup> order contravariant tensor.

◇

 $\Gamma_{N\rho}^N$ 

$$\Gamma_{N\rho}^N = a^{Nk} [N\rho, k] \quad (49)$$

$$= a^{NN} [N\rho, N] + \underbrace{a^{N\tau}}_{=0} [N\rho, \tau] \quad (50)$$

$$(16): \quad = a^{,NN} [N\alpha, N] \cdot \partial_\rho x^{,\alpha} \quad (51)$$

$$\text{considering : } a^{,N\tau} = 0 \quad \text{we can write this as:} \quad (52)$$

$$= a^{,N\tau} [N\alpha, \tau] \cdot \partial_\rho x^{,\alpha} + a^{,NN} [N\alpha, N] \cdot \partial_\rho x^{,\alpha} \quad (53)$$

$$= \underbrace{a^{,Nk} [N\alpha, k]}_{=\Gamma_{N\alpha}^{,N}} \cdot \partial_\rho x^{,\alpha} \quad (54)$$

$$\Rightarrow \quad \Gamma_{N\rho}^N = \Gamma_{N\alpha}^{,N} \partial_\rho x^{,\alpha} \quad (55)$$

So,  $\Gamma_{N\rho}^N$  is a 1<sup>st</sup> order covariant tensor.

◇

$$\mathbf{\Gamma}_{NN}^N$$

$$\Gamma_{NN}^N = a^{Nk}[NN, k] \quad (56)$$

$$= a^{NN}[NN, N] + \underbrace{a^{N\tau}}_{=0}[NN, \tau] \quad (57)$$

$$= a^{,NN}[NN, N] \quad (58)$$

$$(22): \quad = a^{,NN}[NN, N], \quad (59)$$

$$\Rightarrow \quad \Gamma_{NN}^N = \Gamma_{NN}^{,N} \quad (60)$$

So,  $\Gamma_{NN}^N$  is an invariant.



## 2.56 p79-exercise 10

If  $\theta, \phi$  are colatitude and azimuth on a sphere, and we take

$$x^1 = \theta \cos \phi, \quad x^2 = \theta \sin \phi$$

Calculate the Christoffels symbols for the coordinate system  $x^1, x^2$  and show that they vanish at the point  $\theta = 0$

We have, page 48 (2.507) :

$$\Gamma_{mn}^r = \Gamma_{pq}^s \frac{\partial x^r}{\partial x^s} \frac{\partial x^p}{\partial x^m} \frac{\partial x^q}{\partial x^n} + \frac{\partial x^r}{\partial x^s} \frac{\partial^2 x^s}{\partial x^m \partial x^n} \quad (1)$$

The calculations are really basic but lengthy and only train your skills in basic calculus. So I will only calculate  $\Gamma_{11}^1$ .

We know that in a spherical coordinate system all  $\Gamma_{pq}^s$  vanish except for  $\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta$  and  $\Gamma_{\phi\theta}^\phi = \cot \theta$ .

$$\Gamma_{11}^1 = \Gamma_{\phi\phi}^\theta \partial_\theta x^1 \partial_1 \phi \partial_1 \phi + 2\Gamma_{\phi\theta}^\phi \partial_\phi x^1 \partial_1 \theta \partial_1 \phi + \partial_\theta x^1 \partial_{11}^2 \theta + \partial_\phi x^1 \partial_{11}^2 \phi \quad (2)$$

we have

$$\begin{cases} x^1 = \theta \cos \phi \\ x^2 = \theta \sin \phi \end{cases} \Rightarrow \begin{cases} \theta = \sqrt{(x^1)^2 + (x^2)^2} \\ \phi = \arctan \frac{x^2}{x^1} \end{cases} \quad (3)$$

so

$$\begin{cases} \partial_1 \theta = \frac{x^1}{\theta} = \cos \phi & \partial_1 \phi = -\frac{x^2}{\theta^2} = -\frac{\sin \phi}{\theta} \\ \partial_\theta x^1 = \cos \phi & \partial_\phi x^1 = -\theta \sin \phi \\ \partial_{11}^2 \theta = \frac{\theta^2 - (x^1)^2}{\theta^3} = \frac{\sin^2 \phi}{\theta} & \partial_{11}^2 \phi = 2 \frac{x^1 x^2}{\theta^4} = 2 \frac{\cos \phi \sin \phi}{\theta^2} \end{cases} \quad (4)$$

$$\Gamma_{11}^1 = \Gamma_{\phi\phi}^\theta \cos \phi \frac{\sin^2 \phi}{\theta^2} + 2\Gamma_{\phi\theta}^\phi (-\theta \sin \phi) \cos \phi \left(-\frac{\sin \phi}{\theta}\right) + \cos \phi \frac{\sin^2 \phi}{\theta} + (-\theta \sin \phi) \left(2 \frac{\cos \phi \sin \phi}{\theta^2}\right) \quad (5)$$

$$= -\sin \theta \cos \theta \frac{\cos \phi \sin^2 \phi}{\theta^2} + 2 \cot \theta \cos \phi \sin^2 \phi + \frac{\cos \phi \sin^2 \phi}{\theta} - 2 \frac{\cos \phi \sin^2 \phi}{\theta} \quad (6)$$

$$= \left(2 \cot \theta - \frac{1}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2}\right) \cos \phi \sin^2 \phi \quad (7)$$

◇

Does  $\Gamma_{11}^1$  vanish for  $\theta \rightarrow 0$  ?

The problematic term in (7) is  $2 \cot \theta - \frac{1}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2}$  for which  $\lim_{\theta \rightarrow 0} = \pm \infty \mp \infty \mp \infty$  is undefined.

$$\text{Consider } L_+ = \lim_{\theta \rightarrow 0_+} 2 \cot \theta - \frac{1}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2} \quad (8)$$

$$\text{We have } \sin \theta, \theta \geq 0 \quad \sin \theta \leq \theta \quad \frac{1}{\theta} \leq \frac{1}{\sin \theta} \quad 0 \geq \cos \theta \leq 1 \quad (9)$$

$$\text{so } L_+ = \lim_{\theta \rightarrow 0_+} 2 \frac{\cos \theta}{\sin \theta} - \frac{1}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2} \quad (10)$$

$$\geq \lim_{\theta \rightarrow 0_+} 2 \frac{\cos \theta}{\theta} - \frac{1}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2} \quad (11)$$

$$\geq \lim_{\theta \rightarrow 0_+} 2 \frac{\cos \theta}{\theta} - \frac{1}{\theta} - \frac{\theta \cos \theta}{\theta^2} \quad (12)$$

$$\geq \lim_{\theta \rightarrow 0_+} \frac{\cos \theta - 1}{\theta} \quad (13)$$

$$(\text{l'Hospitale rule}) \Rightarrow L_+ \geq 0 \quad (14)$$

$$\text{also } L_+ = \lim_{\theta \rightarrow 0_+} 2 \frac{\cos \theta}{\sin \theta} - \frac{1}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2} \quad (15)$$

$$\leq \lim_{\theta \rightarrow 0_+} 2 \frac{\cos \theta}{\sin \theta} - \frac{\cos \theta}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2} \quad (16)$$

$$\leq \lim_{\theta \rightarrow 0_+} \cos \theta \left( \frac{2}{\sin \theta} - \frac{1}{\theta} - \frac{\sin \theta}{\theta^2} \right) \quad (17)$$

$$\frac{\sin \theta}{\theta} \leq 1 \Rightarrow \leq \lim_{\theta \rightarrow 0_+} \cos \theta \left( \frac{2}{\sin \theta} - \frac{1}{\theta} - \frac{1}{\theta} \right) \quad (18)$$

$$\frac{1}{\sin \theta} \geq \frac{1}{\theta} \Rightarrow \leq \lim_{\theta \rightarrow 0_+} \cos \theta \left( \frac{2}{\theta} - \frac{1}{\theta} - \frac{1}{\theta} \right) \quad (19)$$

$$\leq 0 \quad (20)$$

$$(14) \text{ and } ((20) \Rightarrow L_+ = 0 \quad (21)$$

Considering that

$$L_- = \lim_{\theta \rightarrow 0_-} 2 \cot \theta - \frac{1}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2}$$

is equivalent to

$$L_+^\alpha = \lim_{\alpha \rightarrow 0_+} -(2 \cot \alpha - \frac{1}{\alpha} - \frac{\sin \alpha \cos \alpha}{\alpha^2})$$

(substitute  $\theta = -\alpha$ ) we conclude that

$$L_- = -L_+ = 0$$

. And so

$$\Gamma_{11}^1 \Big|_{\theta=0} = \left( 2 \cot \theta - \frac{1}{\theta} - \frac{\sin \theta \cos \theta}{\theta^2} \right) \cos \phi \sin^2 \phi \Big|_{\theta=0} = 0$$



## 2.57 p79-exercise 11

If vectors  $T^r$  and  $S_r$  undergo parallel propagation along a curve, show that  $T^n S_n$  is constant along the curve.

Parallel propagation of  $T^r$  and  $S_r$  along a curve means

$$\frac{\delta T^r}{\delta u} \equiv \frac{dT^r}{du} + \Gamma_{mn}^r T^m \frac{dx^n}{du} = 0 \quad (1)$$

$$\frac{\delta S_r}{\delta u} \equiv \frac{dS_r}{du} - \Gamma_{rn}^m S_m \frac{dx^n}{du} = 0 \quad (2)$$

$$\text{Hence } \frac{dT^r S_r}{du} = S_r \left( -\Gamma_{mn}^r T^m \frac{dx^n}{du} \right) + T^r \left( \Gamma_{rn}^m S_m \frac{dx^n}{du} \right) \quad (3)$$

$$= \frac{dx^n}{du} (\Gamma_{rn}^m S_m T^r - \Gamma_{rn}^m T^r S_m) \quad (4)$$

$$= 0 \quad (5)$$

$$\Rightarrow T^r S_r = C^{st} \quad (6)$$



## 2.58 p79-exercise 12

Deduce from 2.201 that the determinant  $a = |a_{mn}|$  transforms according to

$$a' = aJ^2 \quad , \quad J = \left| \frac{\partial x^r}{\partial x',s} \right|$$

$$a'_{rs} = a_{mk} \frac{\partial x^m}{\partial x',r} \frac{\partial x^k}{\partial x',s} \quad (1)$$

$$\text{Be} \quad (J_{mr}) = \left( \frac{\partial x^m}{\partial x',r} \right) \quad (2)$$

$$\text{In (1) put} \quad c_{kr} = a_{mk} \frac{\partial x^m}{\partial x',r} = a_{km} \frac{\partial x^m}{\partial x',r} \quad (3)$$

$$\Rightarrow \quad a'_{rs} = c_{kr} \frac{\partial x^k}{\partial x',s} \quad (4)$$

$$\text{or in matrix form} \quad A' = C^T J \quad \text{with} \quad C = AJ \quad (5)$$

$$\Rightarrow \quad |A'| = |(AJ)^T J| \quad (6)$$

$$\Leftrightarrow \quad |A'| = |J^T A^T J| \quad (7)$$

$$\Leftrightarrow \quad |A'| = |A| |J| |J| \quad (8)$$

$$\Leftrightarrow \quad |A'| = |A| |J|^2 \quad (9)$$



## 2.59 p79-exercise 13

Using local Cartesians and applying the result of the previous exercise (N° 12), prove that, if the metric form is positive-definite, then the determinant  $a = |a_{mn}|$  is always positive.

Using local Cartesian coordinates we have:

$$\Phi = \epsilon_i (dy^i)^2 \quad \text{with} \quad \Phi > 0 \quad (1)$$

$$\text{so} \quad |a_{mn}| = \begin{vmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & \epsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \epsilon_N \end{vmatrix} = \prod_{i=1}^N \epsilon_i > 0 \quad (2)$$

(3)

Going from  $(a_{mn})$  to any arbitrary coordinate system, we have (see exercise 12 page 79)

$$|a'_{mn}| = \underbrace{|a_{mn}|}_{>0} \underbrace{J^2}_{>0} \quad (4)$$

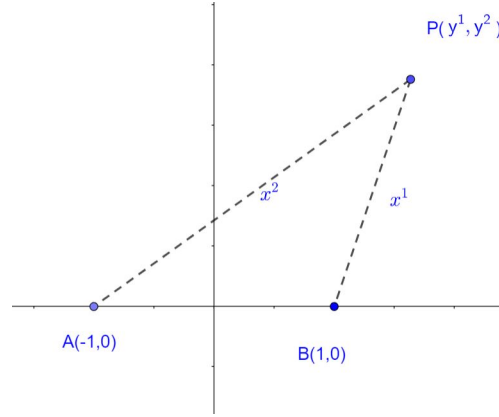
$$\Rightarrow |a'_{mn}| > 0 \quad (5)$$





## 2.60 p79-exercise 14

In a plane, let  $x^1$ ,  $x^2$  be the distances of a general point from the point with rectangular coordinates  $(1, 0)$ ,  $(-1, 0)$ , respectively. (These are bipolar coordinates.) Find the line element for these coordinates, and find the conjugate tensor  $a^{mn}$ .



$$(x^1)^2 = (y^1 - 1)^2 + (y^2)^2 \quad (1)$$

$$(x^2)^2 = (y^1 + 1)^2 + (y^2)^2 \quad (2)$$

$$\partial_1(1) \Rightarrow x^1 = (y^1 - 1) \partial_1 y^1 + (y^2) \partial_1 y^2 \quad (3)$$

$$\partial_2(2) \Rightarrow x^2 = (y^1 + 1) \partial_2 y^1 + (y^2) \partial_2 y^2 \quad (4)$$

$$\partial_2(1) \Rightarrow 0 = (y^1 - 1) \partial_2 y^1 + (y^2) \partial_2 y^2 \quad (5)$$

$$\partial_1(2) \Rightarrow 0 = (y^1 + 1) \partial_1 y^1 + (y^2) \partial_1 y^2 \quad (6)$$

$$\Rightarrow \begin{cases} (3)-(6): & \partial_1 y^1 = -\frac{x^1}{2} \\ (4)-(5): & \partial_2 y^1 = \frac{x^2}{2} \\ (6): & \partial_1 y^2 = \frac{y^1+1}{2y^2} x^1 \\ (5): & \partial_2 y^2 = -\frac{y^1-1}{2y^2} x^2 \end{cases} \quad (7)$$

$$\text{Hence, } J = \left( \frac{\partial y^m}{\partial x^n} \right) = \begin{pmatrix} -\frac{x^1}{2} & \frac{x^2}{2} \\ \frac{y^1+1}{2y^2} x^1 & -\frac{y^1-1}{2y^2} x^2 \end{pmatrix} \quad (8)$$

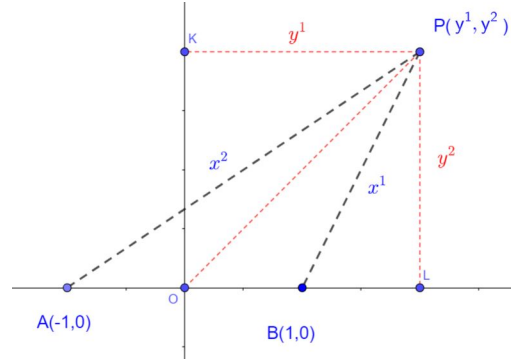
Be  $A$  the metric tensor in Cartesian coordinate system. Then, going to a arbitrary coordinate system gives a metric tensor according to

$$A' = J^T A J = J^T J \quad \text{as} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (9)$$

$$\Rightarrow A' = \begin{pmatrix} -\frac{x^1}{2} & \frac{y^1+1}{2y^2}x^1 \\ \frac{x^2}{2} & -\frac{y^1-1}{2y^2}x^2 \end{pmatrix} \begin{pmatrix} -\frac{x^1}{2} & \frac{x^2}{2} \\ \frac{y^1+1}{2y^2}x^1 & -\frac{y^1-1}{2y^2}x^2 \end{pmatrix} \quad (10)$$

$$\Rightarrow A' = \frac{x^1 x^2}{4(y^2)^2} \begin{pmatrix} x^1 x^2 & -[(y^1)^2 + (y^2)^2 - 1] \\ -[(y^1)^2 + (y^2)^2 - 1] & x^1 x^2 \end{pmatrix} \quad (11)$$

We use plane geometry to express expression (11) as a function of only  $(x^1, x^2)$ . For the triangle  $APB$  in the figure below, we have



$$|OP|^2 = \frac{(x^1)^2 + (x^2)^2}{2} - \underbrace{\frac{|AB|^2}{4}}_{=1} \quad (12)$$

$$\Rightarrow (y^1)^2 + (y^2)^2 = \frac{(x^1)^2 + (x^2)^2}{2} - 1 \quad (13)$$

The area  $K$  of the triangle can be expressed in two ways

$$K = \frac{1}{2} y^2 |AB| = y^2 \quad (14)$$

$$\text{and} \quad K = \sqrt{s(s-x^1)(s-x^2)(s-2)} \quad (15)$$

$$\text{with} \quad s = \frac{1}{2}(x^1 + x^2 + 2) \quad (16)$$

$$\text{hence} \quad (y^2)^2 = \frac{1}{16}(x^1 + x^2 + 2)(x^1 + 2)(x^2 + 2)(x^1 + x^2) \quad (17)$$

So we get from (11):

$$A' = \frac{4x^1x^2}{(x^1+x^2+2)(x^1+2)(x^2+2)(x^1+x^2)} \begin{pmatrix} x^1x^2 & 2 - \frac{(x^1)^2+(x^2)^2}{2} \\ 2 - \frac{(x^1)^2+(x^2)^2}{2} & x^1x^2 \end{pmatrix} \quad (18)$$

For the conjugate metric tensor we start from expression (11) and invert the metric tensor

$$|A'| = \left[ \frac{x^1x^2}{4(y^2)^2} \right]^2 \left[ (x^1x^2)^2 - ((y^1)^2 + (y^2)^2 - 1)^2 \right] \quad (19)$$

$$\text{hence } (a^{,mn}) = \frac{1}{|A'|} \frac{x^1x^2}{4(y^2)^2} \begin{pmatrix} x^1x^2 & [(y^1)^2 + (y^2)^2 - 1] \\ [(y^1)^2 + (y^2)^2 - 1] & x^1x^2 \end{pmatrix} \quad (20)$$

$$= \frac{4(y^2)^2}{x^1x^2 \left[ (x^1x^2)^2 - ((y^1)^2 + (y^2)^2 - 1)^2 \right]} \begin{pmatrix} x^1x^2 & [(y^1)^2 + (y^2)^2 - 1] \\ [(y^1)^2 + (y^2)^2 - 1] & x^1x^2 \end{pmatrix} \quad (21)$$

$$\text{with } x^1x^2 = \sqrt{[(y^1-1)^2 + (y^2)^2][(y^1+1)^2 + (y^2)^2]} \quad (22)$$



## 2.61 p79-exercise 15

Given  $\Phi = a_{mn}dx^m dx^n$ , with  $a_{11} = a_{22} = 0$  but  $a_{12} \neq 0$ , show that  $\Phi$  may be written in the form

$$\Phi = \epsilon \Psi_1^2 - \epsilon \Psi_2^2 + \Phi_2$$

where  $\Phi_2$  is a homogeneous quadratic form in  $dx^3, dx^4, \dots, dx^N$ ,  $\epsilon = \pm 1$ , and where

$$\Psi_1 = \frac{1}{\sqrt{2\epsilon a_{12}}} [a_{12}(dx^1 + dx^2) + (a_{13} + a_{23})dx^3 + \dots + (a_{1N} + a_{2N})dx^N]$$

$$\Psi_2 = \frac{1}{\sqrt{2\epsilon a_{12}}} [a_{12}(-dx^1 + dx^2) + (a_{13} - a_{23})dx^3 + \dots + (a_{1N} - a_{2N})dx^N]$$

Using local Cartesian coordinates we have: Be

$$\Phi = a_{mn}dx^m dx^n \quad (1)$$

$$(2)$$

and consider the following sequences of terms:

$$\Psi_1 = b_{11}dx^1 + b_{12}dx^2 + b_{13}dx^3 + \dots + b_{1N}dx^N \quad (3)$$

$$\Psi_2 = b_{21}dx^1 + b_{22}dx^2 + b_{23}dx^3 + \dots + b_{2N}dx^N \quad (4)$$

The expressions (3) and (4) contain all the terms of  $\Phi$  with  $(dx^1)^2, (dx^2)^2$  and  $dx^1 dx^2$ . So,  $\Phi$  can be expressed as

$$\Phi = \Psi_1^2 - \Psi_2^2 + \Phi_2 \quad (5)$$

With  $\Phi_2$  being a homogeneous form containing only terms in  $dx^i dx^j$   $i, j > 2$ .

We can express  $\Psi_1^2 - \Psi_2^2$  as

$$\Psi_1^2 - \Psi_2^2 = \begin{cases} (b_{11}^2 - b_{21}^2)(dx^1)^2 + (b_{12}^2 - b_{22}^2)(dx^2)^2 + (b_{13}^2 - b_{23}^2)(dx^3)^2 + \dots + (b_{1N}^2 - b_{2N}^2)(dx^N)^2 + \\ 2(b_{11}b_{12} - b_{21}b_{22})dx^1 dx^2 + 2(b_{11}b_{13} - b_{21}b_{23})dx^1 dx^3 + \dots + 2(b_{11}b_{1N} - b_{21}b_{2N})dx^1 dx^N + \\ 2(b_{12}b_{13} - b_{22}b_{23})dx^2 dx^3 + 2(b_{12}b_{14} - b_{22}b_{24})dx^2 dx^4 + \dots + 2(b_{12}b_{1N} - b_{21}b_{2N})dx^2 dx^N + \\ + \dots 2(b_{1j}b_{1k} - b_{2j}b_{2k})dx^k dx^j + \dots \quad j > 2, k \neq j \end{cases} \quad (6)$$

Equating the terms in (1) and (6) and taking into account (=given)  $a_{11} = a_{22} = 0$  and  $a_{12} \neq 0$

$$\begin{cases} b_{11}^2 - b_{21}^2 = 0 \\ b_{12}^2 - b_{22}^2 = 0 \\ 2(b_{11}b_{12} - b_{21}b_{22}) = a_{12} + a_{21} \end{cases} \quad (7)$$

$$\Rightarrow \begin{cases} b_{11}^2 = b_{21}^2 \\ b_{12}^2 = b_{22}^2 \\ (b_{11}b_{12} - b_{21}b_{22}) = a_{12} \end{cases} \quad (8)$$

$$\Rightarrow \quad b_{11} = \pm b_{21} \quad b_{22} = \pm b_{12} \quad (9)$$

$$\Rightarrow \begin{cases} \pm b_{11}b_{22} \mp b_{11}b_{22} = a_{12} \\ \text{or} \\ \pm b_{12}b_{21} \mp b_{12}b_{21} = a_{12} \end{cases} \quad (10)$$

$$(11)$$

As  $a_{12} \neq 0$ , the signs in the above expressions must be the same. Hence,

$$\begin{cases} 2b_{11}b_{22} = \pm a_{12} \\ 2b_{12}b_{21} = \pm a_{12} \end{cases} \quad (12)$$

(12) can be satisfied with an infinite combination of  $b_{11}, b_{12}, b_{21}, b_{22}$ . Choose,

$$b_{11} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{22} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{12} = \sqrt{\frac{\epsilon a_{12}}{2}} \quad b_{21} = -\sqrt{\frac{\epsilon a_{12}}{2}} \quad (13)$$

$$\text{with } \epsilon = \pm 1 \quad \text{so that } \epsilon a_{12} \geq 0 \quad (14)$$

Put  $\xi = \sqrt{\frac{\epsilon a_{12}}{2}}$ . Then, (3) and (4) can be expressed as

$$\Psi_1 = \xi dx^1 + \xi dx^2 + b_{13}dx^3 + \cdots + b_{1N}dx^1 dx^N \quad (15)$$

$$\Psi_2 = -\xi dx^1 + \xi dx^2 + b_{23}dx^3 + \cdots + b_{2N}dx^1 dx^N \quad (16)$$

What are the  $b_{.j}$  ( $j > 2$ )?

From (6), e.g. for  $j = 3$ , identifying the terms in  $dx^1 dx^3$  and  $dx^2 dx^3$  we see that

$$\begin{cases} \xi b_{13} + \xi b_{23} = a_{13} \\ \xi b_{13} - \xi b_{23} = a_{23} \end{cases} \quad (17)$$

$$\Rightarrow \begin{cases} b_{13} = \frac{a_{13} + a_{23}}{2\xi} \\ b_{23} = \frac{a_{13} - a_{23}}{2\xi} \end{cases} \quad (18)$$

$$\text{or, in general } \begin{cases} b_{1j} = \frac{a_{1j} + a_{2j}}{2\xi} \\ b_{2j} = \frac{a_{1j} - a_{2j}}{2\xi} \end{cases} \quad (j > 2) \quad (19)$$

Bring  $\frac{1}{\xi}$  out in  $\Psi_1, \Psi_2$ . This gives

$$\Psi_1 = \frac{1}{\xi} \left( \xi^2(dx^1 + dx^2) + \frac{a_{13} + a_{23}}{2} dx^3 + \cdots + \frac{a_{1N} + a_{2N}}{2} dx^N \right) \quad (20)$$

$$\Psi_2 = \frac{1}{\xi} \left( \xi^2(-dx^1 + dx^2) + \frac{a_{13} - a_{23}}{2} dx^3 + \cdots + \frac{a_{1N} - a_{2N}}{2} dx^N \right) \quad (21)$$

$$\Rightarrow \begin{cases} \Psi_1 = \frac{\sqrt{2}}{\sqrt{\epsilon a_{12}}} \left( \frac{\epsilon a_{12}}{2} (dx^1 + dx^2) + \frac{a_{13} + a_{23}}{2} dx^3 + \cdots + \frac{a_{1N} + a_{2N}}{2} dx^N \right) \\ \Psi_2 = \frac{\sqrt{2}}{\epsilon \sqrt{\epsilon a_{12}}} \left( \frac{\epsilon a_{12}}{2} (-dx^1 + dx^2) + \frac{a_{13} - a_{23}}{2} dx^3 + \cdots + \frac{a_{1N} - a_{2N}}{2} dx^N \right) \end{cases} \quad (22)$$

$$\Leftrightarrow \begin{cases} \Psi_1 = \frac{1}{\sqrt{2\epsilon a_{12}}} \left( \epsilon a_{12} (dx^1 + dx^2) + (a_{13} + a_{23}) dx^3 + \cdots + (a_{1N} + a_{2N}) dx^N \right) \\ \Psi_2 = \frac{1}{\sqrt{2\epsilon a_{12}}} \left( \epsilon a_{12} (-dx^1 + dx^2) + (a_{13} - a_{23}) dx^3 + \cdots + (a_{1N} - a_{2N}) dx^N \right) \end{cases} \quad (23)$$

What about the factor  $\epsilon = \pm 1$  in the terms  $\epsilon a_{12}(dx^1 + dx^2)$  and  $\epsilon a_{12}(-dx^1 + dx^2)$  ? Consider the alternate form

$$\begin{cases} \Psi_1 = \frac{1}{\sqrt{2\epsilon a_{12}}} \left( a_{12} (dx^1 + dx^2) + (a_{13} + a_{23}) dx^3 + \cdots + (a_{1N} + a_{2N}) dx^N \right) \\ \Psi_2 = \frac{1}{\sqrt{2\epsilon a_{12}}} \left( a_{12} (-dx^1 + dx^2) + (a_{13} - a_{23}) dx^3 + \cdots + (a_{1N} - a_{2N}) dx^N \right) \end{cases} \quad (24)$$

Let's have a look at the terms in  $dx^i dx^j$  in  $\Psi_1^2 - \Psi_2^2$

$$x_{11}(dx^1)^2 + 2x_{12}dx^1dx^2 + x_{22}(dx^2)^2 = \frac{1}{2\epsilon a_{12}} \left( (a_{12})^2(dx^1 + dx^2)^2 - (a_{12})^2(-dx^1 + dx^2)^2 \right) \quad (25)$$

$$= \frac{\epsilon a_{12}}{2} \left( (dx^1 + dx^2)^2 - (-dx^1 + dx^2)^2 \right) \quad (26)$$

$$= \frac{\epsilon a_{12}}{2} \left( (dx^1)^2 + 2dx^1dx^2 + (dx^2)^2 - (dx^1)^2 + 2dx^1dx^2 - (dx^2)^2 \right) \quad (27)$$

$$= 2\epsilon a_{12}dx^1dx^2 \quad (28)$$

$$2x_{13}dx^1dx^3 = \frac{1}{2\epsilon a_{12}} (2a_{12}(a_{13} + a_{23})dx^1dx^3 - 2a_{12}(a_{13} - a_{23})dx^1dx^3) \quad (29)$$

$$= \frac{1}{2\epsilon a_{12}} (4a_{12}a_{23})dx^1dx^3 \quad (30)$$

$$= 2\epsilon a_{23}dx^1dx^3 \quad (31)$$

$$2x_{23}dx^2dx^3 = \frac{1}{2\epsilon a_{12}} (2a_{12}(a_{13} + a_{23})dx^1dx^3 - 2a_{12}(a_{13} - a_{23})dx^1dx^3) \quad (32)$$

$$= \frac{1}{2\epsilon a_{12}} (4a_{12}a_{23})dx^1dx^3 \quad (33)$$

$$= 2\epsilon a_{23}dx^1dx^3 \quad (34)$$

$$2x_{34}dx^3dx^4 = \frac{1}{2\epsilon a_{12}} \begin{pmatrix} 2(a_{13} + a_{23})(a_{14} + a_{24}) \\ -2(a_{13} - a_{23})(a_{14} - a_{24}) \end{pmatrix} dx^3dx^4 \quad (35)$$

$$= \frac{2\epsilon}{a_{12}} (a_{23}a_{14} + a_{13}a_{24}) \quad (36)$$

We rewrite now the metric form  $\Phi$  as

$$\Phi = \epsilon \Psi_1^2 - \epsilon \Psi_2^2 + \Phi_2$$

From (28), (31) and (34) we see that the terms in  $dx^1, dx^2$  in  $\Phi = \epsilon \Psi_1^2 - \epsilon \Psi_2^2 + \Phi_2$  correspond to the expected metric form and that the other terms in  $dx^j, dx^k$ ,  $j, k \neq 1, 2$  can be corrected in the remaining term  $\Phi_2$ .

## 2.62 p80-exercise 16

Find the null geodesics of a 4-space with line element

$$ds^2 = \epsilon\gamma(dx^2 + dy^2 + dz^2 - dt^2)$$

where  $\gamma$  is an arbitrary function of  $x, y, z, t$ .

We have:

$$(a_{mn}) = \gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (a^{mn}) = \frac{1}{\gamma} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1)$$

(2)

Be  $(x^1, x^2, x^3, x^4) \equiv (x, y, z, t)$ .

The general conditions for a null geodesic are

$$\begin{cases} \frac{d^2 x^r}{dv^2} + \Gamma_{mn}^r \frac{dx^m}{dv} \frac{dx^n}{dv} = 0 \\ a_{mn} dx^m dx^n = 0 \end{cases} \quad (3)$$

When calculating  $[mn, r] = \frac{1}{2}(\partial_n a_{mr} + \partial_m a_{nr} - \partial_r a_{mn})$  we note that  $(a_{mn})$  is a diagonal matrix, and so is also  $(a^{mn})$ . Hence  $\Gamma_{mn}^r$  will contain only one term:

$$\Gamma_{mn}^r = a^{RR}[mn, R] \quad (4)$$

$$\Rightarrow \begin{cases} i) & m, n \neq R \wedge m \neq n & : & [mn, R] = 0 \\ ii) & m \neq n = R \vee n \neq m = R & : & [mn, R] = \frac{1}{2}\partial_m a_{RR} \\ iii) & m = n \neq R & : & [mn, R] = -\frac{1}{2}\partial_R a_{mn} \\ iv) & m = n = R & : & [mn, R] = \frac{1}{2}\partial_R a_{RR} \end{cases} \quad (5)$$

and for the  $\Gamma_{mn}^r$ :



$$\Rightarrow \left\{ \begin{array}{ll} i) & m, n \neq R \wedge m \neq n : \Gamma_{mn}^R = 0 \\ ii) & m \neq n = R \vee n \neq m = R : \Gamma_{nR}^R = \frac{1}{2\gamma} \partial_n \gamma \\ iii) & \begin{array}{ll} m = n \neq R = 1, 2, 3 & : \Gamma_{kk}^R = -\frac{1}{2\gamma} \partial_R \gamma \\ m = n \neq R = 4 & : \Gamma_{kk}^4 = \frac{1}{2\gamma} \partial_R \gamma \end{array} \\ iv) & m = n = R : \Gamma_{RR}^R = \frac{1}{2\gamma} \partial_R \gamma \end{array} \right. \quad (6)$$

Let's compute

$$A_r \equiv \frac{d^2 x^r}{dv^2} + \Gamma_{mn}^r \frac{dx^m}{dv} \frac{dx^n}{dv} \quad (7)$$

and take  $v = x^4$ , so  $x^1, x^2, x^3 = f(x^4)$ . (In the following  $d_k x^j \equiv \frac{dx^j}{d(x^k)}$ ,  $d_k^2 x^j \equiv \frac{d^2 x^j}{d(x^k)^2}$ )

$$\left\{ \begin{array}{l} A_1 = d_4^2 x^1 + \Gamma_{mn}^1 d_4 x^m d_4 x^n \\ A_2 = d_4^2 x^2 + \Gamma_{mn}^2 d_4 x^m d_4 x^n \\ A_3 = d_4^2 x^3 + \Gamma_{mn}^3 d_4 x^m d_4 x^n \\ A_4 = \Gamma_{mn}^4 d_4 x^m d_4 x^n \end{array} \right. \quad (8)$$

and get from (8) and (6):

$$A_1 - d_4^2 x^1 = \left\{ \begin{array}{l} \frac{1}{\gamma} \partial_2 \gamma d_4 x^1 d_4 x^2 + \frac{1}{\gamma} \partial_3 \gamma d_4 x^1 d_4 x^3 + \frac{1}{\gamma} \partial_4 \gamma d_4 x^1 d_4 x^4 \\ + \underbrace{\frac{1}{2\gamma} \partial_1 \gamma (d_4 x^1)^2}_{= \left\{ \begin{array}{l} \frac{1}{\gamma} \partial_1 \gamma (d_4 x^1) (d_4 x^1) \\ - \frac{1}{2\gamma} \partial_1 \gamma (d_4 x^1)^2 \end{array} \right.} - \frac{1}{2\gamma} \partial_1 \gamma (d_4 x^2)^2 - \frac{1}{2\gamma} \partial_1 \gamma (d_4 x^3)^2 + \frac{1}{2\gamma} \partial_1 \gamma (d_4 x^4)^2 \end{array} \right. \quad (9)$$

$$= \left\{ \begin{array}{l} \frac{1}{\gamma} \partial_1 \gamma d_4 x^1 d_4 x^1 + \frac{1}{\gamma} \partial_2 \gamma d_4 x^1 d_4 x^2 + \frac{1}{\gamma} \partial_3 \gamma d_4 x^1 d_4 x^3 + \frac{1}{\gamma} \partial_4 \gamma d_4 x^1 d_4 x^4 \\ - \frac{1}{2\gamma} \partial_1 \gamma (d_4 x^1)^2 - \frac{1}{2\gamma} \partial_1 \gamma (d_4 x^2)^2 - \frac{1}{2\gamma} \partial_1 \gamma (d_4 x^3)^2 + \frac{1}{2\gamma} \partial_1 \gamma (d_4 x^4)^2 \end{array} \right. \quad (10)$$

$$(11)$$

$$= \left\{ \begin{array}{l} \frac{1}{\gamma} d_4 x^1 (\partial_1 \gamma d_4 x^1 + \partial_2 \gamma d_4 x^2 + \partial_3 \gamma d_4 x^3 + \partial_4 \gamma d_4 x^4) \\ - \frac{1}{2\gamma} \partial_1 \gamma \left( \underbrace{(d_4 x^1)^2 + (d_4 x^2)^2 + (d_4 x^3)^2 - (d_4 x^4)^2}_{= a_{mn} dx^m dx^n = 0} \right) \end{array} \right. \quad (12)$$

$$\Rightarrow A_1 = d_4^2 x^1 + \frac{1}{\gamma} d_4 x^1 < \nabla \gamma | \partial_4 \bar{x} > \quad (13)$$

with  $\nabla \gamma = (\partial_x \gamma, \partial_y \gamma, \partial_z \gamma, \partial_t \gamma)$  and  $\partial_4 \bar{x} = (\partial_t x, \partial_t y, \partial_t z, \partial_t t)$ .

So for the geodesic with get for the first coordinate

$$d_4^2 x^1 + \frac{1}{\gamma} d_4 x^1 < \nabla \gamma | \partial_4 \bar{x} > = 0$$

Doing analogous calculations give the following set of equations

$$d_4^2 x^k + \frac{1}{\gamma} d_4 x^k < \nabla \gamma | \partial_4 \bar{x} > = 0 \quad (14)$$

Note that for  $k = 4$  we have as  $d_4 x^4 = 1$  and  $d_4^2 x^4 = 0$ :

$$\frac{1}{\gamma} < \nabla \gamma | \partial_4 \bar{x} > = 0 \quad (15)$$

$$\Rightarrow < \nabla \gamma | \partial_4 \bar{x} > = 0 \quad (16)$$

Note that this means that  $\nabla \gamma$  and  $\partial_4 \bar{x}$  are orthogonal 4-vectors.

So the set of equations in (14) reduce to the following set of  $2^{nd}$  order differential equations:

$$\left\{ \begin{array}{l} x'' = 0 \\ y'' = 0 \\ z'' = 0 \\ t = t \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = x_1 t + x_0 \\ y = y_1 t + y_0 \\ z = z_1 t + z_0 \\ t = t \end{array} \right. \quad (17)$$

This is the equations of 4-space cone, which was expected as the metric is locally a Minkowski-like metric.



## 2.63 p80-exercise 17

In a space  $V_N$  the metric tensor is  $a_{mn}$ . Show that the null geodesics are unchanged if the metric tensor is changed to  $b_{mn} = \gamma a_{mn}$ ,  $\gamma$  being a function of the coordinates.

Null geodesics are determined by the following set of equations

$$\begin{cases} \frac{d^2 x^r}{dv^2} + \Gamma_{mn}^r \frac{dx^m}{dv} \frac{dx^n}{dv} = 0 \\ a_{mn} dx^m dx^n = 0 \end{cases} \quad (1)$$

$$\text{We have } \Gamma_{mn}^r = a^{rs} [mn, s] \quad (2)$$

$$= \frac{1}{\gamma} a^{rs} \gamma [mn, s] \quad (3)$$

$$\text{We have also } b^{rs} = \frac{1}{\gamma} a^{rs} \quad (4)$$

$$\text{Indeed } a^{rs} a_{rs} = \delta_t^t \quad (5)$$

$$\Rightarrow \frac{1}{\gamma} a^{rs} \gamma a_{rs} = \delta_t^t \quad (6)$$

$$\Rightarrow \frac{1}{\gamma} a^{rs} b_{rs} = \delta_t^t \quad (7)$$

$$\Rightarrow \frac{1}{\gamma} a^{rs} = b_{rs} \quad (8)$$

$$\text{So (3) becomes } \Gamma_{mn}^r = b^{rs} \gamma [mn, s] \quad (9)$$

Be  $[mn, s]'$  the Christoffel symbol associated with the metric  $b_{mn} = \gamma a_{mn}$ . Then with

$$[mn, s]' = \frac{1}{2} (\partial_n \gamma a_{ms} + \partial_m \gamma a_{ns} - \partial_s \gamma a_{mn})$$

we get

$$[mn, s]' = \gamma [mn, s] + \frac{1}{2} (a_{ms} \partial_n \gamma + a_{ns} \partial_m \gamma - a_{mn} \partial_s \gamma) \quad (10)$$

Substitute  $\gamma[mn, s]$  from (10) in (9) we get

$$\Gamma_{mn}^r = \underbrace{b^{rs}\gamma[mn, s]}_{=\Gamma_{mn}^{'r}} - \underbrace{\frac{1}{2}b^{rs}(a_{ms}\partial_n\gamma + a_{ns}\partial_m\gamma - a_{mn}\partial_s\gamma)}_{:=Q} \quad (11)$$

$$\text{with } Q = \frac{1}{2}b^{rs}(a_{ms}\partial_n\gamma + a_{ns}\partial_m\gamma - a_{mn}\partial_s\gamma) \quad (12)$$

$$= \frac{1}{2}\gamma \left( \underbrace{a^{rs}a_{ms}}_{=\delta_m^r} \partial_n\gamma + \underbrace{a^{rs}a_{ns}}_{=\delta_n^r} \partial_m\gamma - a^{rs}a_{mn}\partial_s\gamma \right) \quad (13)$$

$$Q \times \frac{dx^m}{dv} \frac{dx^n}{dv} = \frac{1}{2}\gamma \left( \underbrace{\delta_m^r \frac{dx^m}{dv}}_{=\frac{dx^r}{dv}} \underbrace{\partial_n\gamma \frac{dx^n}{dv}}_{=\frac{d\gamma}{dv}} + \underbrace{\delta_n^r \frac{dx^n}{dv}}_{=\frac{dx^r}{dv}} \underbrace{\partial_m\gamma \frac{dx^m}{dv}}_{=\frac{d\gamma}{dv}} - a^{rs}\partial_s\gamma \underbrace{a_{mn} \frac{dx^m}{dv} \frac{dx^n}{dv}}_{=0 \text{ by (1)}} \right) \quad (14)$$

$$= \frac{1}{2}\gamma \left( \frac{dx^r}{dv} \frac{d\gamma}{dv} + \frac{dx^r}{dv} \frac{d\gamma}{dv} \right) \quad (15)$$

$$= \left( \gamma \frac{d\gamma}{dv} \right) \frac{dx^r}{dv} \quad (16)$$

So from (11) and (16), (1) can be expressed as

$$\frac{d^2x^r}{dv^2} + \Gamma_{mn}^{'r} \frac{dx^m}{dv} \frac{dx^n}{dv} = \left( \gamma \frac{d\gamma}{dv} \right) \frac{dx^r}{dv} \quad (17)$$

By (2.449) page 46 we see that the vector  $\frac{d^2x^r}{dv^2} + \Gamma_{mn}^{'r} \frac{dx^m}{dv} \frac{dx^n}{dv}$  is collinear to  $\frac{dx^r}{dv}$ . Also  $b_{mn} \frac{dx^m}{dv} \frac{dx^n}{dv} = 0$ . And hence (17) determines the geodesic null lines. So both expressions (1) and (17) are equivalent to determine the same geodesic null lines in the space  $V_N$  equipped with the metric  $a_{mn}$  or  $b_{mn}$



## 2.64 p80-exercise 18

Are the relations

$$T|_{rs} = T|_{sr}$$

$$T_{r|sk} = T_{r|ks}$$

true (a) in curvilinear coordinates in Euclidean space, (b) in a general Riemannian space?

$$\mathbf{T}|_{rs} \stackrel{?}{=} \mathbf{T}|_{sr}$$

We have  $T|_r = \partial_r T$  (see (2.528) page 53)

$$\begin{array}{ccc} \mathbf{T}|_{rs} & & \mathbf{T}|_{sr} \\ & \leftrightarrow & \\ T|_{rs} = \partial_{rs}^2 T - \Gamma_{rs}^m T|_m & & T|_{sr} = \partial_{sr}^2 T - \Gamma_{sr}^m T|_m \end{array} \quad (1)$$

As  $\partial_{rs}^2 = \partial_{sr}^2$  and  $\Gamma_{rs}^m = \Gamma_{sr}^m$  we can conclude that  $\mathbf{T}|_{rs} = \mathbf{T}|_{sr}$  in both cases (a) and (b).

◇

$$\mathbf{T}_{r|sk} \stackrel{?}{=} \mathbf{T}_{r|ks}$$

$$\begin{array}{ccc} \mathbf{T}_{r|sk} & & \mathbf{T}_{r|ks} \\ \\ A_{rs} := T_{r|s} = \partial_s T_r - \Gamma_{rs}^m T_m & & B_{rk} := T_{r|k} = \partial_k T_r - \Gamma_{rk}^m T_m \\ A_{rs|k} = \partial_k A_{rs} - \Gamma_{rk}^m A_{ms} - \Gamma_{sk}^m A_{rm} & & B_{rk|s} = \partial_s B_{rk} - \Gamma_{rs}^m B_{mk} - \Gamma_{ks}^m B_{rm} \\ A_{ms} = \partial_s T_m - \Gamma_{ms}^n T_n & & B_{mk} = \partial_k T_m - \Gamma_{mk}^n T_n \\ A_{rm} = \partial_m T_r - \Gamma_{rm}^n T_n & & B_{rm} = \partial_m T_r - \Gamma_{rm}^n T_n \\ \\ A_{rs|k} = \left\{ \begin{array}{l} \underbrace{\partial_{sk}^2 T_r}_* - T_m \partial_k \Gamma_{rs}^m - \underbrace{\Gamma_{rs}^m \partial_k T_m}_{**} \\ - \underbrace{\Gamma_{rk}^m \partial_s T_m}_{***} + \Gamma_{rk}^m \Gamma_{ms}^n T_n \\ - \underbrace{\Gamma_{sk}^m \partial_m T_r}_{****} + \underbrace{\Gamma_{sk}^m \Gamma_{rm}^n T_n}_{*****} \end{array} \right. & \leftrightarrow & B_{rk|s} = \left\{ \begin{array}{l} \underbrace{\partial_{ks}^2 T_r}_* - T_m \partial_s \Gamma_{rk}^m - \underbrace{\Gamma_{rk}^m \partial_s T_m}_{***} \\ - \underbrace{\Gamma_{rs}^m \partial_k T_m}_{**} + \Gamma_{rs}^m \Gamma_{mk}^n T_n \\ - \underbrace{\Gamma_{ks}^m \partial_m T_r}_{****} + \underbrace{\Gamma_{ks}^m \Gamma_{rm}^n T_n}_{*****} \end{array} \right. \end{array} \quad (2)$$

$$\Rightarrow A_{rs|k} - B_{rk|s} = T_m \left( \underbrace{\partial_s \Gamma_{rk}^m - \partial_k \Gamma_{rs}^m + \Gamma_{rk}^n \Gamma_{ns}^m - \Gamma_{rs}^n \Gamma_{nk}^m}_{:= R^s{}_{.rmn}} \right) \quad (3)$$

So,  $T_{r|sk} = T_{r|ks}$  only if  $T_m R^s{}_{.rmn} = 0$  and as  $T_m$  is an arbitrary tensor,  $R^s{}_{.rmn}$  must vanish for  $T_{r|sk} = T_{r|ks}$ .

**Note:**

Although  $\Gamma_{rk}^n$  is not a tensor, the quantity  $R^s{}_{.rmn}$  is. Indeed, as both  $A_{rs|k} - B_{rk|s}$  and  $T_m$  have the tensor character, this implies that  $R^s{}_{.rmn}$  is a tensor. Now, for an Euclidean space equipped with Cartesian coordinates all  $\Gamma_{rk}^n$  are constant and vanish. So,  $R^s{}_{.rmn} = 0$ . Let's consider a change of coordinate system from Cartesian to curvilinear coordinate system. Then, by the tensor character of  $R'^a{}_{.bcd}$  we have,

$$R'^a{}_{.bcd} = R^s{}_{.rmn} \partial_s x'^a \partial_{(b} x^r \partial_{c)} x^m \partial_{(d)} x^n \quad (4)$$

but  $R^s{}_{.rmn} = 0$  in the Cartesian coordinate system and so is  $R'^a{}_{.bcd}$

**Conclusion:**

In a general Riemannian space  $T_{r|sk} \neq T_{r|ks}$  but  $T_{r|sk} = T_{r|ks}$  in a curvilinear Euclidean space.



## 2.65 p80-exercise 19

Consider a  $V_N$  with indefinite metric form. For all points  $P$  lying on the cone of geodesic null lines drawn from  $O$ , the definition 2.611 for Riemannian coordinates apparently breaks down. Revise the definition of Riemannian coordinates so as to include such points.

For geodesic null lines we have (2.445 page 46)

$$\begin{cases} \frac{d^2 x^r}{du^2} + \Gamma_{mn}^r \frac{dx^m}{du} \frac{dx^n}{du} = 0 \\ a_{mn} \frac{dx^m}{du} \frac{dx^n}{du} = 0 \end{cases} \quad (1)$$

or (2.448 page 46)

$$\begin{cases} \frac{d^2 x^r}{dv^2} + \Gamma_{mn}^r \frac{dx^m}{dv} \frac{dx^n}{dv} = \lambda(v) \frac{dx^r}{dv} \\ a_{mn} \frac{dx^m}{dv} \frac{dx^n}{dv} = 0 \end{cases} \quad (2)$$

where by suitable choice of the parameter  $v$ ,  $\lambda(v)$  can be made any preassigned function of  $v$ .

$$(2) \Rightarrow \frac{d^2 x^r}{dv^2} = \lambda \frac{dx^r}{dv} - \Gamma_{mn}^r \frac{dx^m}{dv} \frac{dx^n}{dv} \quad (3)$$

$$\Rightarrow \frac{d^3 x^r}{dv^3} = \frac{d\lambda}{dv} \frac{dx^r}{dv} + \lambda \frac{d^2 x^r}{dv^2} + A_{.mns}^r \frac{dx^m}{dv} \frac{dx^n}{dv} \frac{dx^s}{dv} \quad (4)$$

$$\text{with } \Rightarrow A_{.mns}^r = -\partial_s \Gamma_{mn}^r + 2\Gamma_{sp}^r \Gamma_{mn}^p \quad (5)$$

Expanding  $x^r$  in a Taylor series around a point  $O(a^r)$  we get (for ease of notation we put  $p^r := \frac{dx^r}{dv}$ )

$$x^r = a^r + vp^r + \frac{1}{2}v^2\lambda p^r - \frac{1}{2}v^2\Gamma_{mn}^r p^m p^n + \frac{1}{6}v^3\frac{d\lambda}{dv}p^r + \frac{1}{6}v^3\lambda\frac{dp^r}{dv} + \frac{1}{6}v^3A_{.mns}^r p^m p^n p^s + \dots \quad (6)$$

$$= a^r + \left(v + \frac{1}{2}v^2\lambda + \frac{1}{6}v^3\frac{d\lambda}{dv}\right)p^r - \frac{1}{2}v^2\Gamma_{mn}^r p^m p^n + \frac{1}{6}v^3\lambda\frac{dp^r}{dv} + \frac{1}{6}v^3A_{.mns}^r p^m p^n p^s + \dots \quad (7)$$

Put  $x'^r := v\xi(v)p^r$  with  $\xi(v) = (1 + \frac{1}{2}v\lambda + \frac{1}{6}v^2\frac{d\lambda}{dv})$ . Hence (7) becomes

$$x^r = a^r + x'^r - \frac{1}{2\xi^2}\Gamma_{mn}^r x'^m x'^n + \frac{1}{6}v^3\lambda\left(\frac{p'^r}{v\xi} - \frac{\xi + v\xi'}{v^2\xi^2}x'^r\right) + \frac{1}{6\xi^3}A_{.mns}^r x'^m x'^n x'^s + \dots \quad (8)$$

$$= a^r + \tau(v)x'^r + \frac{v^2\lambda}{6\xi}p'^r - \frac{1}{2\xi^2}\Gamma_{mn}^r x'^m x'^n + \frac{1}{6\xi^3}A_{.mns}^r x'^m x'^n x'^s + \dots \quad (9)$$

$$\text{with } \tau(v) := 1 - \frac{v\lambda}{6\xi^2}(\xi + v\xi') \quad (10)$$

Is the Jacobian non-zero?

$$\frac{\partial x^r}{\partial x'^q} = \tau(v)\delta_q^r + \frac{v^2\lambda}{6\xi}\frac{d\delta_q^r}{dv} - \frac{1}{\xi^2}\Gamma_{mq}^r x'^m + \frac{1}{2\xi^3}A_{.mnq}^r x'^m x'^n + \dots \quad (11)$$

In the infinitesimal neighbourhood of  $O$  we have

$$\left\{ \begin{array}{l} v \rightarrow 0 \\ \xi \rightarrow 1 \\ \xi' \rightarrow \frac{\lambda}{2} \\ x'^m \rightarrow 0 \\ \tau \rightarrow 1 \end{array} \right. \quad (12)$$

$$(13)$$

so that the Jacobian determinant becomes

$$\left| \frac{\partial x^r}{\partial x'^q} \right| = |\tau \delta_q^r| = \tau^N = 1 \quad (14)$$

**Conclusion:**

So in order to define a Riemannian coordinates system which is still valid on the null geodesics, it is sufficient to define the Riemannian coordinates around  $O$  as

$$x'^r := v \xi(v) p^r$$

with

$$\xi(v) = \left( 1 + \frac{1}{2} v \lambda + \frac{1}{6} v^2 \frac{d\lambda}{dv} \right)$$

and  $\lambda$ , by suitable choice of  $v$ , being any pre-defined function of  $v$ .

