

TENSOR CALCULUS
J.L. SYNGE AND A.SCHILD (DOVER PUBLICATION)
SOLUTIONS TO EXERCISES
PART II
CHAPTERS V TO VIII

by

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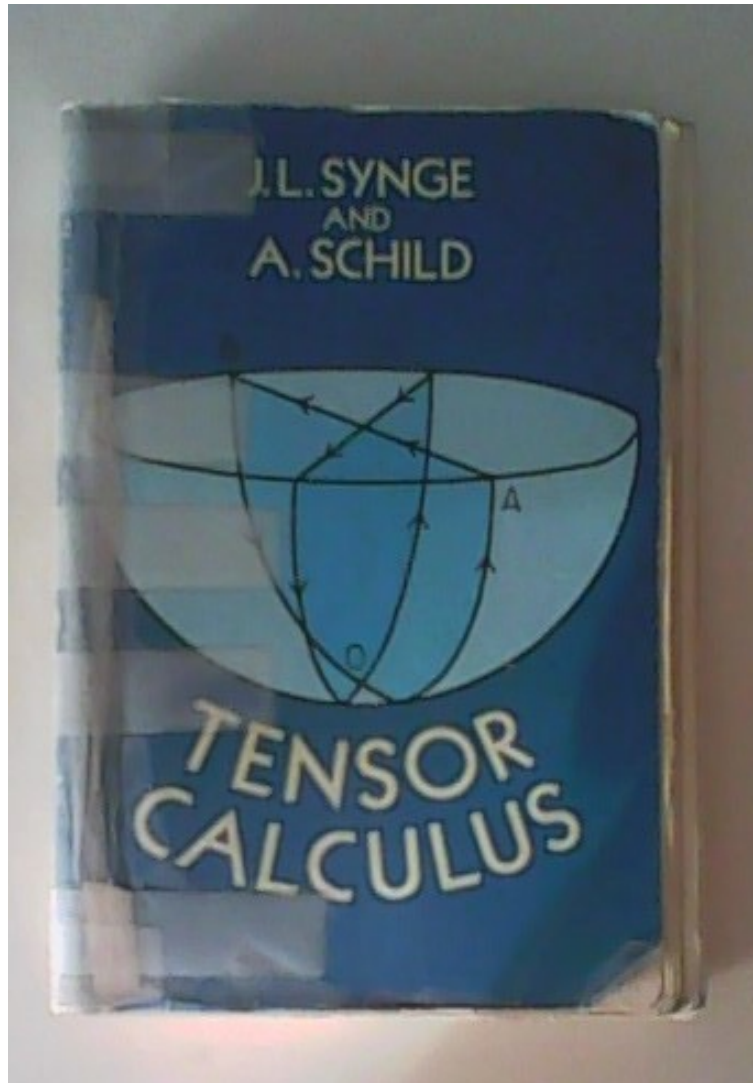


Figure 1: My copy, falling apart...

Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution : I graduated in 1979, went straight in the industry (where I didn't have to use fancy maths), and picked mathematics and physics again after I retired, so my mathematics got rusty for sure. If you do find an error, typo's , I'd be happy to receive bug reports, suggestions, and the like, through Github. An overview of the material covered in the book can be found in the separate document "Synge overview.pdf".

Some notation conventions

† means that the exercise has only been solved partially or contains i.m.o. a doubtful step

†† means that the exercise has not been solved as it should.

◆ end of an exercise or proof.

◇ end of Lemma or sub-task of an exercise.

**As a rule, I followed the notation used in the book,
except some which where easier to type in Latex.**

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\partial_{rs}^2 \equiv \frac{\partial^2}{\partial x^r \partial x^s}$$

$$\Gamma_{mn}^r \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

Contents

7	Relative tensors, ideas of volume, Green-Stokes theorems.	6
7.1	p241 - Exercise	7
7.2	p242 - Exercise	9
7.3	p243 - Exercise	10
7.4	p243 - Exercise	11
7.5	p245 - Exercise	12
7.6	p245 - Clarification to 7.113	13
7.7	p245 - Exercice	14
7.8	p245 - Exercice	15
7.9	p255 - Exercice	16
7.10	p257 - Exercice	18
7.11	p263 - Exercise	22
7.12	p265 - Exercise	23

List of Figures

1	My copy, falling apart...	2
7.1	Permutations	14
7.2	Permutations	17
7.3	Projections of extensions	20
7.4	A disk defined as $S : \left\{ \mathbb{R}^2 \rightarrow \mathbb{R}^3, S(u, v) = \left(\frac{u}{\sqrt{u^2+v^2+C}}, \frac{v}{\sqrt{u^2+v^2+C}}, 1 \right) \right\}$	21
7.5	Manifold with $ds^2 = R^2 [dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)]$ metric, embedded in a 4-space	25

Relative tensors, ideas of volume,
Green-Stokes theorems.

7.1 p241 - Exercise

If b_{rs} is an absolute tensor, show that the determinant $|b_{rs}|$ is a relative invariant of weight 2. What are the tensor characters of $|c^{rs}|$ and $|f_s^r|$?

As b_{rs} is an absolute tensor, we have

$$b'_{uv} = b_{rs} \frac{\partial x^r}{\partial x'^u} \frac{\partial x^s}{\partial x'^v} \quad (1)$$

Hence,

$$|b'_{uv}| = |b_{rs}| \left| \frac{\partial x^r}{\partial x'^u} \right| \left| \frac{\partial x^s}{\partial x'^v} \right| \quad (2)$$

and as $J = \left| \frac{\partial x^k}{\partial x'^s} \right|$ we get

$$|b'_{uv}| = J^2 |b_{rs}| \quad (3)$$

Conclusion, $|b_{rs}|$ is a relative invariant of weight 2.

◇

As c^{rs} is an absolute tensor, we have

$$c'^{uv} = c^{rs} \frac{\partial x'^u}{\partial x^r} \frac{\partial x'^v}{\partial x^s} \quad (4)$$

Hence,

$$|c'^{uv}| = |c^{rs}| \left| \frac{\partial x'^u}{\partial x^r} \right| \left| \frac{\partial x'^v}{\partial x^s} \right| \quad (5)$$

and as $J^{-1} = \left| \frac{\partial x'^s}{\partial x^k} \right|$ we get

$$|c'^{uv}| = J^{-2} |c^{rs}| \quad (6)$$

Conclusion, $|c^{rs}|$ is a relative invariant of weight -2 .

◇

As f_s^r is an absolute tensor, we have

$$f_v'^u = f_s^r \frac{\partial x'^u}{\partial x^r} \frac{\partial x^s}{\partial x'^v} \quad (7)$$

Hence,

$$\left| f_v'^u \right| = \left| f_s^r \right| \left| \frac{\partial x'^u}{\partial x^r} \right| \left| \frac{\partial x^s}{\partial x'^v} \right| \quad (8)$$

and we get

$$\left| f_v'^u \right| = J J^{-1} \left| f_s^r \right| \quad (9)$$

Conclusion, $|f_s^r|$ is an absolute invariant tensor .



7.2 p242 - Exercise

Show that, in three dimensions, the only non-vanishing components of δ_{rs}^{kl} are

$$\delta_{23}^{23} = \delta_{32}^{32} = \delta_{31}^{31} = \delta_{13}^{13} = \delta_{12}^{12} = \delta_{21}^{21} = 1$$

$$\delta_{32}^{23} = \delta_{23}^{32} = \delta_{13}^{31} = \delta_{31}^{13} = \delta_{21}^{12} = \delta_{12}^{21} = -1$$

This is easily seen. If $(k, l), (r, s)$ are considered as sets, then $\delta_{rs}^{kl} \neq 0 \Leftrightarrow (k, l) \neq (r, s)$. And $\delta_{rs}^{kl} = 1 \Leftrightarrow k = r \wedge l = s$ and on the opposite $\delta_{rs}^{kl} = -1 \Leftrightarrow k = s \wedge l = r$



7.3 p243 - Exercise

Show that equations **5.231** and **6.128** can be written as follows:

$$M_{rs} = \delta_{rs}^{kl} z_k F_l$$

$$\omega_{rs} = \frac{1}{2} \delta_{rs}^{kl} v_{l,k}$$

$$(5.231) \quad M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r \quad (1)$$

In this expression $M_{rs} = 0$ when $r = s$, but this is also the case with δ_{rs}^{kl} .

In $M_{rs} = \delta_{rs}^{kl} z_k F_l$ we see that there is no contribution in the summation when $k = l$. The only contribution being those for which $k = r \wedge l = s$ (positive contribution) \vee $k = s \wedge l = r$ (negative contribution), hence

$$\delta_{rs}^{kl} z_k F_l \Leftrightarrow z_r F_s - z_s F_r$$

◇

$$(6.128) \quad \omega_{rs} = \frac{1}{2} (v_{s,r} - v_{r,s}) \quad (2)$$

The same arguments of the previous case apply to this case (a way to see this is to represent symbolically, $z_r F_s$ and $v_{s,r}$ by T_{rs})

◆

7.4 p243 - Exercise

If $T_{k_1 k_2 \dots k_M}$ is completely skew-symmetric, determine

$$\delta_{s_1 s_2 \dots s_M}^{k_1 k_2 \dots k_M} T_{k_1 k_2 \dots k_M}$$

$\delta_{s_1 s_2 \dots s_M}^{k_1 k_2 \dots k_M} T_{k_1 k_2 \dots k_M}$ is a sum of $M!$ terms: the first of these is $T_{s_1 s_2 \dots s_M}$; the other terms are obtained from it by permuting the subscripts and a minus sign is attached if the permutation is odd. Since $T_{s_1 s_2 \dots s_M}$ is completely skew-symmetric, each of the $M!$ terms equals $\pm T_{s_1 s_2 \dots s_M}$. Hence,

$$\delta_{s_1 s_2 \dots s_M}^{k_1 k_2 \dots k_M} T_{k_1 k_2 \dots k_M} = M! T_{s_1 s_2 \dots s_M}$$



7.5 p245 - Exercise

Show that $\epsilon^{r_1 r_2 \dots r_N} \epsilon_{r_1 r_2 \dots r_N} = N!$.

First note that $\text{sign}(\epsilon^{r_1 r_2 \dots r_N}) = \text{sign}(\epsilon_{r_1 r_2 \dots r_N})$ so that each term in the summation is always +1.

There are N choices to chose from for r_1 , $N - 1$ for r_2 , etc. and only one for r_N . And so $\epsilon^{r_1 r_2 \dots r_N} \epsilon_{r_1 r_2 \dots r_N} = N!$



7.6 p245 - Clarification to 7.113

$$\epsilon^{k_1 \dots k_M r_1 \dots r_{N-M}} \epsilon_{s_1 \dots s_M r_1 \dots r_{N-M}} = (N-M)! \delta_{s_1 \dots s_M}^{k_1 \dots k_M}$$

This can be seen as followed.

As the permutation $(r_1 \dots r_{N-M})$ is the same for both covariant and contravariant permutation symbols, the product $\epsilon^{k_1 \dots k_M r_1 \dots r_{N-M}} \epsilon_{s_1 \dots s_M r_1 \dots r_{N-M}}$ for a fixed permutation $(r_1 \dots r_{N-M})$ (i.e. no summation on repeated indexes) will be determined by $\delta_{s_1 \dots s_M}^{k_1 \dots k_M}$. Indeed, the difference in "oddness" between $(k_1 \dots k_M r_1 \dots r_{N-M})$ and $(s_1 \dots s_M r_1 \dots r_{N-M})$ is only determined by the difference in "oddness" between $(k_1 \dots k_M r_1)$ and $(s_1 \dots s_M)$. So each term in the summation has the same contribution, i.e.; $\delta_{s_1 \dots s_M}^{k_1 \dots k_M}$.

There are M choices to chose from for r_1 , $M-1$ for r_2 , etc. and only one for r_M . And so

$$\epsilon^{k_1 \dots k_M r_1 \dots r_{N-M}} \epsilon_{s_1 \dots s_M r_1 \dots r_{N-M}} = (N-M)! \delta_{s_1 \dots s_M}^{k_1 \dots k_M}$$



7.7 p245 - Exercice

If T_{rs} is an absolute skew-symmetric tensor in a 4-space, show that

$$T_{14}T_{23} + T_{24}T_{31} + T_{34}T_{12}$$

is a tensor density

Be $P = T_{14}T_{23} + T_{24}T_{31} + T_{34}T_{12}$, we can write this as $P = \frac{1}{8}\delta_{1234}^{ijmn}T_{ij}T_{mn}$

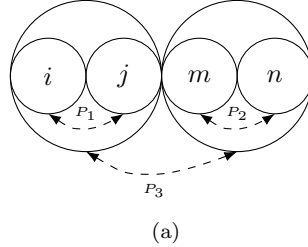


Figure 7.1: Permutations

The factor $\frac{1}{8}$ is explained by the fact that there are 2^3 possible permutations in the i, j, m, n indexes i.e. 2×2 for the permutations P_1 and P_2 and again 2 for the permutation P_3 . Note that a single permutation P_1 or P_2 changes the sign of $T_{ij}T_{mn}$ but also changes the sign of δ_{1234}^{ijmn} , so the combined sign doesn't change. A double permutation P_1 and P_2 changes the sign of T_{ij} and T_{mn} resulting in an unchanged sign of $T_{ij}T_{mn}$ but also δ_{1234}^{ijmn} is unchanged because of the double permutation. Finally P_3 has no effect, nor on $T_{ij}T_{mn}$ nor on δ_{1234}^{ijmn} . So we have 8 repetitions for the same set i, j, m, n . So we have,

$$P = \frac{1}{8}\delta_{1234}^{ijmn}T_{ij}T_{mn} \quad (1)$$

$$= \frac{1}{8}\epsilon^{ijmn}T_{ij}T_{mn} \quad (2)$$

From this follows immediately as ϵ^{ijmn} is a relative tensor of weight 1 and T_{ij} an absolute tensor (i.e. a relative tensor of weight 0) that P is a relative tensor of weight 1 i.e. a density.



7.8 p245 - Exercice

Show that, for rectangular Cartesian coordinates, the vorticity tensor and the vorticity vector of a fluid are duals (cf. **6.130**).

6.130:

$$\omega_r = \frac{1}{2} \epsilon_{rmn} \omega_{mn}, \quad \omega_{mn} = \epsilon_{rmn} \omega_r \quad (1)$$

Put $\hat{T}^r = \omega_r$ and $T_{mn} = \omega_{mn}$ then the expressions in (1) can be expressed as (considering that the covariant and contravariant expressions are identical in rectangular Cartesian coordinates)

$$\hat{T}^r = \frac{1}{(3-2)!} \epsilon^{mnr} T_{mn}, \quad T_{mn} = \epsilon_{rmn} \hat{T}^r \quad (2)$$

which are exactly the general definitions **7.121** and **7.122** (with $N = 3$ and $M = 2$) for dual tensors.



7.9 p255 - Exercice

Show that **7.305** may be written in the equivalent form

$$d\tau_{(M)}^{k_1 \dots k_m} = \epsilon^{\beta_1 \dots \beta_M} d_{(\beta_1)} x^{k_1} \dots d_{(\beta_M)} x^{k_M}$$

The determinant of a matrix and its transpose are equal.

Hence we can rewrite **7.305** $\delta_{s_1 \dots s_M}^{k_1 \dots k_M} d_{(1)} x^{s_1} \dots d_{(M)} x^{s_M}$ as

$$d\tau_{(M)}^{k_1 \dots k_m} = \delta_{k_1 \dots k_M}^{s_1 \dots s_M} d_{(s_1)} x^1 \dots d_{(s_M)} x^M \quad (1)$$

In order to be consistent with the notation we replace the s_i by α_i as the summation occurs along the constants $c^{(i)}$

$$d\tau_{(M)}^{k_1 \dots k_m} = \delta_{k_1 \dots k_M}^{\alpha_1 \dots \alpha_M} d_{(\alpha_1)} x^1 \dots d_{(\alpha_M)} x^M \quad (2)$$

Given the set $\{k_1, k_2, \dots, k_M\}$ we can represent the sequence $\{1, 2, \dots, M\}$ by $\{k_j, k_m, \dots, k_M, \dots, k_n\}$ (imagine that $k_j = 1, k_m = 2, \dots$ etc.). We rewrite (2) as

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = (\theta_\alpha) \delta_{12 \dots M}^{\alpha_1 \dots \alpha_M} d_{(\alpha_1)} x^{k_j} d_{(\alpha_2)} x^{k_m} \dots d_{(\alpha_M)} x^{k_n} \quad (3)$$

where

$$\theta_\alpha = \epsilon_{k_1 k_2 \dots k_M} \quad (4)$$

(a permutation in the lower indexes of the generalized Kronecker deltas symbol will invert the sign depending on the 'oddness' of the permutation). Let's rearrange the product $d_{(\alpha_1)} x^{k_j} d_{(\alpha_2)} x^{k_m} \dots d_{(\alpha_M)} x^{k_n}$ so that the indexes k_i are naturally ordered

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = (\theta_\alpha) \delta_{12 \dots M}^{\alpha_1 \dots \alpha_M} d_{(\alpha_r)} x^{k_1} d_{(\alpha_n)} x^{k_2} \dots d_{(\alpha_1)} x^{k_n} \dots d_{(\alpha_s)} x^{k_M} \quad (5)$$

and changing the order in the upper indexes of the general Kroneckers delta's:

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = (\theta_\alpha)(\theta_k) \delta_{12 \dots M}^{\alpha_r \alpha_n \dots \alpha_s} d_{(\alpha_r)} x^{k_1} d_{(\alpha_n)} x^{k_2} \dots d_{(\alpha_1)} x^{k_n} \dots d_{(\alpha_s)} x^{k_M} \quad (6)$$

where $\theta_k = \pm 1$ depending on the 'oddness' of the permutation needed to go from $\{\alpha_1 \dots \alpha_M\}$ to $\{\alpha_r \alpha_n \dots \alpha_s\}$.

As we can see in figure 7.2, it's no hard to see that

$$\theta_k = \theta_\alpha \quad (7)$$

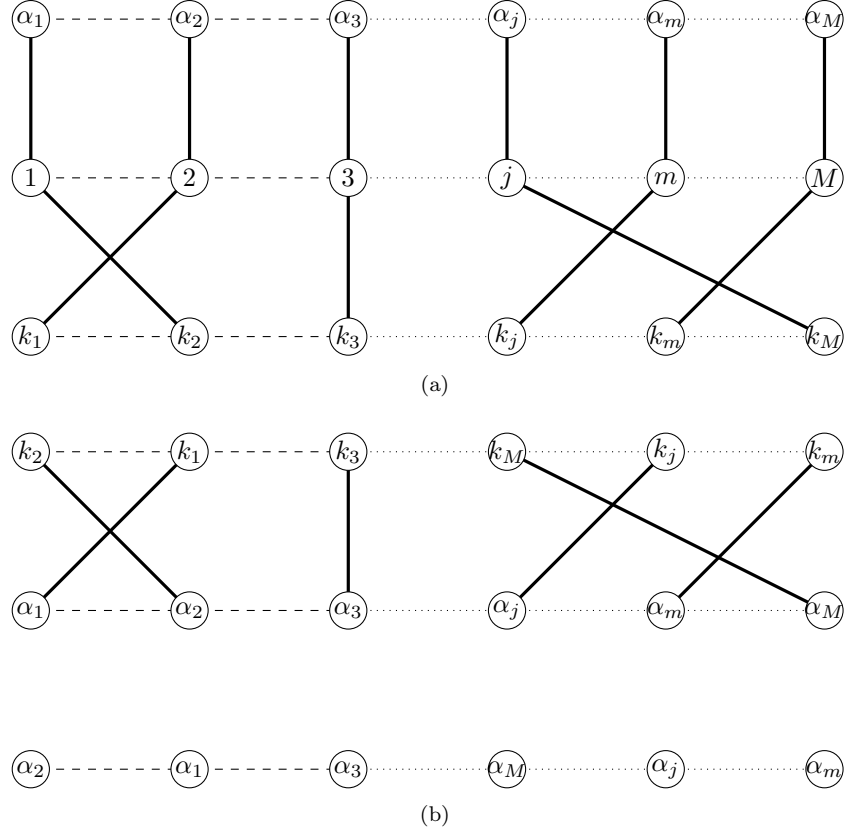


Figure 7.2: Permutations

Indeed suppose, as in the example (a), $k_1 = 2, k_2 = 1, k_3 = 3, \dots, k_j = m, \dots, k_m = j, \dots$ etc., so we get a sequence $\{k_2, k_1, k_3, \dots, k_m, \dots, k_j, \dots\}$ as illustrated in (b). But to have - with this sequence - an equivalent expression of $d\tau_{(M)}^{k_1 k_2 \dots k_m} = (\theta_\alpha) \delta_{12 \dots M}^{\alpha_1 \dots \alpha_M} d_{(\alpha_1)} x^{k_j} d_{(\alpha_2)} x^{k_m} \dots d_{(\alpha_M)} x^{k_n}$, we need to make an equivalent permutation so that α_r gets in the same position as k_r , resulting in a new sequence $\{\alpha_2, \alpha_1, \alpha_3, \dots, \alpha_M, \dots, \alpha_j, \alpha_m\}$.

The number of permutations to generate θ_α and θ_k are identical resulting in $\theta_\alpha \theta_k = 1$. So (6) can be rewritten (noting that the α_r are dummy indexes and that we are free to rename them so that $r = 1, n = 2, \dots$)

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = \delta_{12 \dots M}^{\beta_1 \beta_2 \dots \beta_M} d_{(\beta_1)} x^{k_1} d_{(\beta_2)} x^{k_2} \dots d_{(\beta_n)} x^{k_n} \dots d_{(\beta_M)} x^{k_M} \quad (8)$$

Finally, using **7.114**

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = \underbrace{\epsilon_{12 \dots M}}_{=1} \epsilon^{\beta_1 \beta_2 \dots \beta_M} d_{(\beta_1)} x^{k_1} d_{(\beta_2)} x^{k_2} \dots d_{(\beta_n)} x^{k_n} \dots d_{(\beta_M)} x^{k_M} \quad (9)$$

$$= \epsilon^{\beta_1 \beta_2 \dots \beta_M} d_{(\beta_1)} x^{k_1} d_{(\beta_2)} x^{k_2} \dots d_{(\beta_n)} x^{k_n} \dots d_{(\beta_M)} x^{k_M} \quad (10)$$



7.10 p257 - Exercice

Let x^k be rectangular Cartesian coordinates in Euclidean 3-space. Introduce polar coordinates r, θ, ϕ and consider the surface of the sphere $r = a$. On this sphere form the infinitesimal 2-cell with corners (θ, ϕ) , $(\theta + d\theta, \phi)$, $(\theta, \phi + d\phi)$, $(\theta + d\theta, \phi + d\phi)$. Determine the extension of this cell and interpret the rectangular components. In particular, show that the three independent components of the extension are (apart from the sign) equal to the areas obtained by normal projection of the cell onto the three rectangular planes. Does this interpretation remain valid if the sphere is replaced by some other surface?

We use **7.312**:

$$d\tau_{(2)}^{k_1 k_2} = \epsilon^{\alpha_1 \alpha_2} \frac{\partial x^{k_1}}{\partial y^{\alpha_1}} \frac{\partial x^{k_2}}{\partial y^{\alpha_2}} |d_{(\beta)} y^\gamma| \quad (1)$$

with $(y^1, y^2) = (\theta, \phi)$ giving if we take $f^{(i)} = c^{(i)}$ as $\theta = c^{(1)}$, $\phi = c^{(2)}$:

$$|d_{(\beta)} y^\gamma| = \begin{vmatrix} d_{(1)} y^1 & d_{(1)} y^2 \\ d_{(2)} y^1 & d_{(2)} y^2 \end{vmatrix} \quad (2)$$

$$= \begin{vmatrix} d\theta & 0 \\ 0 & d\phi \end{vmatrix} \quad (3)$$

$$= d\theta d\phi \quad (4)$$

We also have

$$\begin{cases} x = a \sin \theta \cos \phi \\ y = a \sin \theta \sin \phi \\ z = a \cos \theta \end{cases} \quad (5)$$

$$(6)$$

giving

$$\begin{cases} \frac{\partial x}{\partial \theta} = a \cos \theta \cos \phi \\ \frac{\partial x}{\partial \phi} = -a \sin \theta \sin \phi \\ \frac{\partial y}{\partial \theta} = a \cos \theta \sin \phi \\ \frac{\partial y}{\partial \phi} = a \sin \theta \cos \phi \\ \frac{\partial z}{\partial \theta} = -a \sin \theta \\ \frac{\partial z}{\partial \phi} = 0 \end{cases} \quad (7)$$

$$(8)$$

and get

$$d\tau_{(2)}^{xy} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \phi} d\theta d\phi + \epsilon^{21} \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \phi} d\theta d\phi \quad (9)$$

$$= a^2 \cos \theta \cos \phi \sin \theta \cos \phi d\theta d\phi + a^2 \sin \theta \sin \phi \cos \theta \sin \phi d\theta d\phi \quad (10)$$

$$= a^2 \cos \theta \sin \theta d\theta d\phi \quad (11)$$

$$d\tau_{(2)}^{yx} = -a^2 \cos \theta \sin \theta d\theta d\phi \quad (12)$$

$$d\tau_{(2)}^{xz} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial x}{\partial \theta} \frac{\partial z}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial x}{\partial \theta} \underbrace{\frac{\partial z}{\partial \phi}}_{=0} d\theta d\phi + \epsilon^{21} \frac{\partial x}{\partial \phi} \frac{\partial z}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial x}{\partial \phi} \frac{\partial z}{\partial \phi} d\theta d\phi \quad (13)$$

$$= -a^2 \sin^2 \theta \sin \phi d\theta d\phi \quad (14)$$

$$d\tau_{(2)}^{zx} = a^2 \sin^2 \theta \sin \phi d\theta d\phi \quad (15)$$

$$d\tau_{(2)}^{yz} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial y}{\partial \theta} \frac{\partial z}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial y}{\partial \theta} \underbrace{\frac{\partial z}{\partial \phi}}_{=0} d\theta d\phi + \epsilon^{21} \frac{\partial y}{\partial \phi} \frac{\partial z}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial y}{\partial \phi} \frac{\partial z}{\partial \phi} d\theta d\phi \quad (16)$$

$$= a^2 \sin^2 \theta \cos \phi d\theta d\phi \quad (17)$$

$$d\tau_{(2)}^{zy} = -a^2 \sin^2 \theta \cos \phi d\theta d\phi \quad (18)$$

$$d\tau_{(2)}^{xx} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \phi} d\theta d\phi + \epsilon^{21} \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} d\theta d\phi \quad (19)$$

$$= 0 \quad (20)$$

$$d\tau_{(2)}^{yy} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \phi} d\theta d\phi + \epsilon^{21} \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \phi} d\theta d\phi \quad (21)$$

$$= 0 \quad (22)$$

$$d\tau_{(2)}^{zz} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial z}{\partial \theta} \underbrace{\frac{\partial z}{\partial \phi}}_{=0} d\theta d\phi + \epsilon^{21} \underbrace{\frac{\partial z}{\partial \phi}}_{=0} \frac{\partial z}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial z}{\partial \phi} \frac{\partial z}{\partial \phi} d\theta d\phi \quad (23)$$

$$= 0 \quad (24)$$

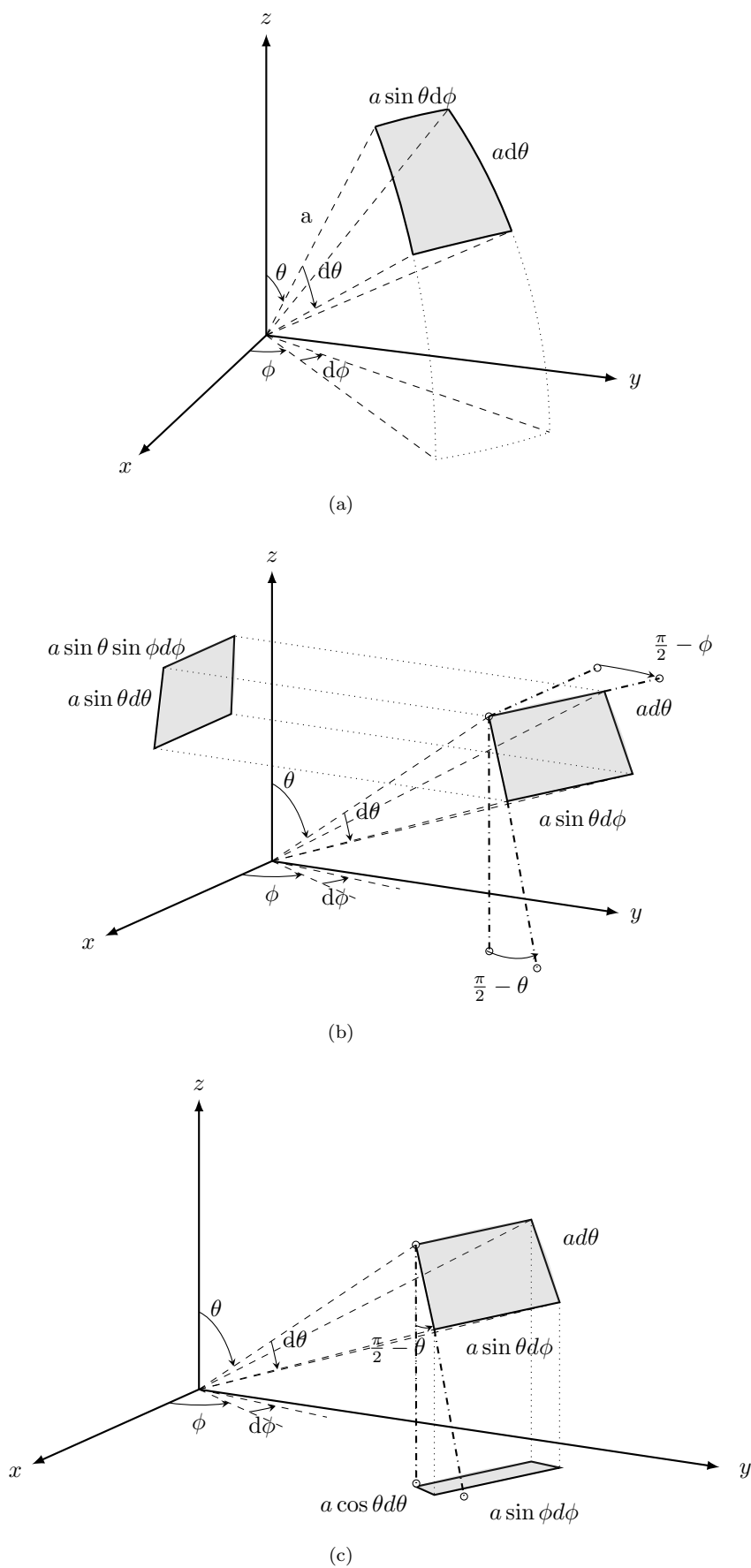


Figure 7.3: Projections of extensions

The quantities $d\tau_{(2)}^{xy}$, $d\tau_{(2)}^{xz}$, $d\tau_{(2)}^{yz}$ are the projections of the extension on the respective Cartesian coordinates planes as can be seen in figure 7.3 where figure (a) depicts the extension (area = $a^2 \sin \theta d\theta d\phi$) when choosing θ, ϕ as the parameters y^k , while figure (b) represents the projection of this extension on the xz -plane and figure (c) represents the projection of this extension on the xy -plane.

◇

Does this interpretation remain valid if the sphere is replaced by some other surface?

The answer is no. For a two-space in Cartesian coordinates system and with surface with parameters (u, v) , equation (2) reduces to

$$d\tau_{(2)}^{xy} = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du dv \quad (25)$$

So $d\tau_{(2)}^{xy} = 0$ if $\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 0$. Consider the disk defined by the following parametric function

$$S : \left\{ \mathbb{R}^2 \rightarrow \mathbb{R}^3, S(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2 + C}}, \frac{v}{\sqrt{u^2 + v^2 + C}}, 1 \right) \right\}$$

(the constant C is there just to avoid the undefinedness of the surface for $(u, v) = (0, 0)$).

It is easy to see that $\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 0$, yet the surface is parallel with the xy -plane, which implies that the projection on the xy -plane of an elementary cell on S will not have a zero area as can be seen in the figure hereunder.

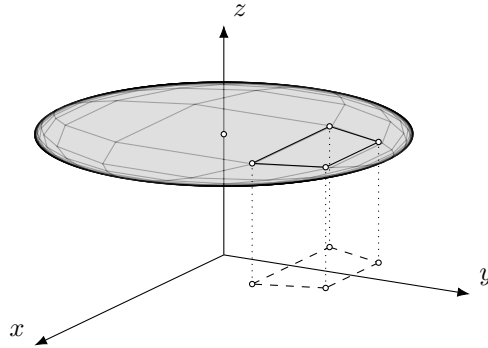


Figure 7.4: A disk defined as $S : \left\{ \mathbb{R}^2 \rightarrow \mathbb{R}^3, S(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2 + C}}, \frac{v}{\sqrt{u^2 + v^2 + C}}, 1 \right) \right\}$

Conclusion: The interpretation of $d\tau_{(2)}^{k_1 k_2}$ as the projection of a cell on a axis-plane, does not hold for every surface.

◆

7.11 p263 - Exercise

Using polar coordinates in Euclidean 3-space find the volume of an infinitesimal cell whose edges are tangent to the coordinate curves. Obtain the volume of a sphere by integration.

For polar spherical coordinates we have

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (1)$$

giving

$$|a_{mn}| = r^4 \sin^2 \theta \quad (2)$$

and using as parameters the $x^k \equiv (r, \theta, \phi)$ as parameters for the parametric surface we get

$$|d_{(s)}x^k| = \begin{vmatrix} dr & 0 & 0 \\ 0 & d\theta & 0 \\ 0 & 0 & d\phi \end{vmatrix} \quad (3)$$

$$= dr d\theta d\phi \quad (4)$$

Using **7.405**:

$$dv_{(N)}^2 = \epsilon(a) |a_{mn}| |d_{(s)}x^k|^2 \quad (5)$$

$$= r^4 \sin^2 \theta (dr d\theta d\phi)^2 \quad (6)$$

getting for the volume of a sphere with radius R :

$$V = 8 \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} dv_{(N)} \quad (7)$$

$$= 8 \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r^2 \sin \theta dr d\theta d\phi \quad (8)$$

$$= \frac{4}{3} \pi R^3 \quad (9)$$



7.12 p265 - Exercise

In the relativistic theory of finite, expanding universe, the following line element is adopted:

$$ds^2 = R^2 [dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)] - dt^2$$

where $R = R(t)$ is a function of the "time" t . the ranges of the coordinates may be taken to be $0 \leq r \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, $-\infty < t < +\infty$.

Find the total volume of "space", i.e., of the surface $t = \text{constant}$, and show that it varies with the "time" t as $R^3(t)$.

For the considered metric, we have

$$(a_{mn}) = \begin{pmatrix} R^2 & 0 & 0 & 0 \\ 0 & R^2 \sin^2 r & 0 & 0 \\ 0 & 0 & R^2 \sin^2 r \sin^2 \theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1)$$

Using as parameters the $x^k \equiv (r, \theta, \phi)$ as parameters for the parametric surface and using **7.409** : $b_{\alpha\beta} = a_{ks} \frac{\partial x^k}{\partial y^\alpha} \frac{\partial x^s}{\partial y^\beta}$, we get for the 3-space $t = \text{constant}$

$$(b_{mn}) = \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2 \sin^2 r & 0 \\ 0 & 0 & R^2 \sin^2 r \sin^2 \theta \end{pmatrix} \quad (2)$$

giving

$$|b_{mn}| = R^6 \sin^4 r \sin^2 \theta \quad (3)$$

Using **7.413**:

$$dv_{(M)}^2 = \frac{\epsilon(b)}{M!} a_{k_1 s_1} \dots a_{k_M s_M} d\tau_{(M)}^{k_1 \dots k_M} d\tau_{(M)}^{s_1 \dots s_M} \quad (4)$$

$$= -\frac{1}{6} a_{k_1 s_1} \dots a_{k_M s_M} d\tau_{(M)}^{k_1 \dots k_M} d\tau_{(M)}^{s_1 \dots s_M} \quad (5)$$

$$= -\frac{6}{6} a_{11} a_{22} a_{33} \left(\underbrace{d\tau_{(M)}^{123}}_{=drd\theta d\phi} \right)^2 \quad (6)$$

$$= -R^6 \sin^4 r \sin^2 \theta (drd\theta d\phi)^2 \quad (7)$$

$$\Rightarrow dv_{(M)} = R^3 \sin^2 r \sin \theta drd\theta d\phi \quad (8)$$

getting for the volume of "space" with "radius" R :

$$V = \int_0^\pi \int_0^\pi \int_0^{2\pi} R^3 \sin^2 r \sin \theta dr d\theta d\phi \quad (9)$$

$$= 2R^3 \pi \int_0^R \sin^2 r dr \underbrace{\int_0^\pi \sin \theta d\theta}_{=-\cos \theta|_0^\pi} \quad (10)$$

$$= 4R^3 \pi \underbrace{\int_0^R \sin^2 r dr}_{=\frac{1}{2}(\theta - \frac{1}{2}\sin(2r))|_0^\pi} \quad (11)$$

$$= 2\pi^2 R^3 \quad (12)$$

(the integral in (11) can be found by substituting $\sin^2 r = 1 - \cos^2 r$ and using the cosine sum of angles rule $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ with $\alpha = \beta = r$).

◇

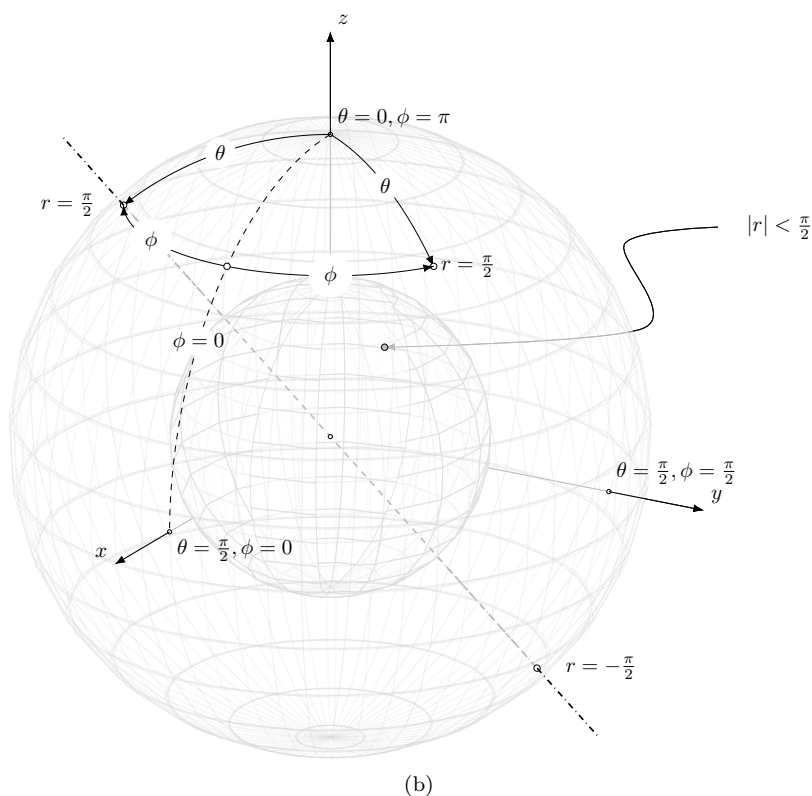
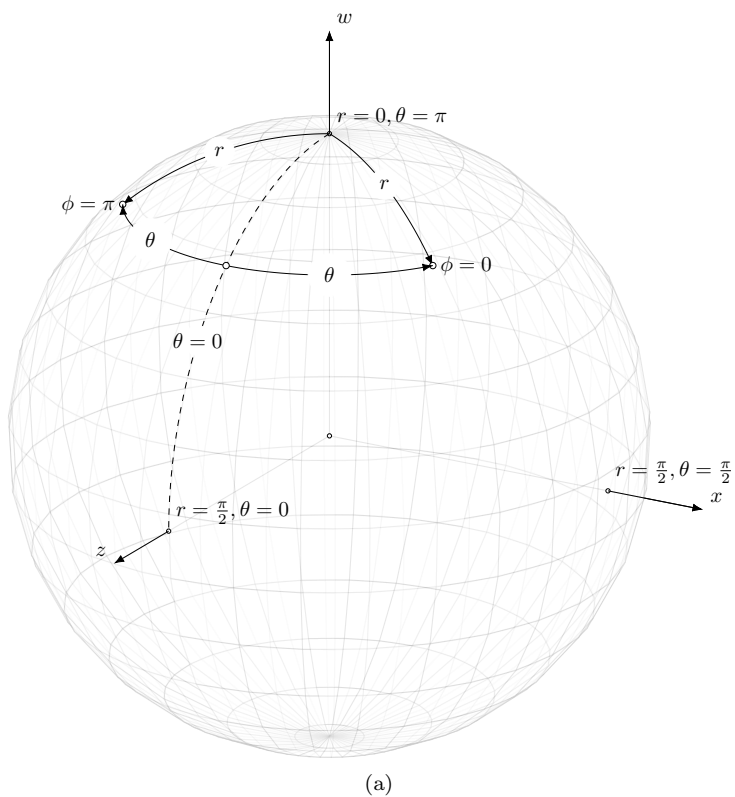


Figure 7.5: Manifold with $ds^2 = R^2 [dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)]$ metric, embedded in a 4- Euclidean space

