Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github.

Some notation conventions

$$\partial_r \equiv \tfrac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \quad \text{Christoffel symbol of the second kind}$$

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Curvature of space

1.1 p98 - Clarification

> ... it is easy to see that the expansion takes the form
>
> **3.425.** $$\eta = \theta\left(s - \frac{1}{6}\epsilon K s^3 + \dots\right)$$

Expanding η in a power series gives

$$\eta = \underbrace{\eta\big|_0}_{=0} + \underbrace{\left.\frac{d\eta}{ds}\right|_0}_{=\theta} s - \frac{1}{2}\underbrace{\left.\frac{d^2\eta}{ds^2}\right|_0}_{=0} s^2 + \frac{1}{6}\left.\frac{d^3\eta}{ds^3}\right|_0 s^3 + \dots \tag{1}$$

$$\frac{d^2(1)}{ds^2} \quad \Rightarrow \quad \frac{d^2\eta}{ds^2} = \left.\frac{d^3\eta}{ds^3}\right|_0 s + \dots \tag{2}$$

$$\text{for } \lim_{s\to 0} \quad \text{we have } \eta \approx \theta s \quad \text{so (2)} \quad \Rightarrow \quad \frac{d^2\eta}{ds^2} = \left.\frac{d^3\eta}{ds^3}\right|_0 \frac{\eta}{\theta} + \dots \tag{3}$$

$$\lim_{s\to 0} \quad \Rightarrow \quad \left.\frac{d^3\eta}{ds^3}\right|_0 = \theta \underbrace{\lim_{s\to 0}\frac{1}{\eta}\frac{d^2\eta}{ds^2}}_{=-\epsilon K} \tag{4}$$

$$\Rightarrow \quad \eta = \theta\left(s - \frac{1}{6}\epsilon K s^3 + \dots\right) \tag{5}$$

♦

1.2 p109 - Exercise 6

For an orthogonal coordinates system in a V_2 we have

$$ds^2 = a_{11}\left(dx^1\right)^2 + a_{22}\left(dx^2\right)^2$$

Show that

$$\frac{1}{a}R_{1212} = -\frac{1}{2}\frac{1}{\sqrt{a}}\left[\partial_1\left(\frac{1}{\sqrt{a}}\partial_1 a_{22}\right) + \partial_2\left(\frac{1}{\sqrt{a}}\partial_2 a_{11}\right)\right]$$

We have

$$(a_{mn}) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \qquad (a^{mn}) = \frac{1}{a}\begin{pmatrix} a_{22} & 0 \\ 0 & a_{11} \end{pmatrix} \qquad a = a_{11}a_{22} \tag{1}$$

We have also

$$R = -\frac{2}{a}R_{1212} \tag{2}$$

$$R = a^{mn}R_{mn} \quad \Rightarrow \quad R = a^{11}R_{11} + a^{22}R_{22} \tag{3}$$

Looking at the pattern generated by equations (2) and (3) suggests that using these equations could lead to the proposed equation. Let's have a try ...

$$\begin{cases} \Gamma^1_{11} = \frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11} & \Gamma^1_{22} = -\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22} \\ \\ \Gamma^2_{11} = -\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11} & \Gamma^2_{22} = \frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22} \\ \\ \Gamma^1_{12} = \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} & \Gamma^2_{12} = \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} \end{cases} \tag{4}$$

$$\text{3.205.} \quad \Rightarrow \quad R_{rm} = \frac{1}{2}\partial_{rm}\log a - \frac{1}{2}\Gamma^p_{rm}\partial_p \log a - \partial_n\Gamma^n_{rm} + \Gamma^p_{rn}\Gamma^n_{pm} \tag{5}$$

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\Gamma^1_{11}\partial_1\log a - \frac{1}{2}\Gamma^2_{11}\partial_2\log a \\ -\partial_1\Gamma^1_{11} - \partial_2\Gamma^2_{11} + \\ \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^2_{21} \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a - \frac{1}{2}\Gamma^1_{22}\partial_1\log a - \frac{1}{2}\Gamma^2_{22}\partial_2\log a \\ -\partial_1\Gamma^1_{22} - \partial_2\Gamma^2_{22} + \\ \Gamma^1_{21}\Gamma^1_{12} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{21}\Gamma^1_{22} + \Gamma^2_{22}\Gamma^2_{22} \end{cases} \tag{6}$$

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}\partial_1 \log a - \frac{1}{2}(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11})\partial_2 \log a \\ -\partial_1(\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}) - \partial_2(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11}) + \\ \frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11}\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{11} + \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11}(-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11}) + \\ (-\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{11})\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} + \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22}\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a - \frac{1}{2}(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22})\partial_1 \log a - \frac{1}{2}\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}\partial_2 \log a \\ -\partial_1(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22}) - \partial_2(\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}) + \\ \frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11}\frac{1}{2}\frac{a_{22}}{a}\partial_2 a_{11} + (-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22})\frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22} + \\ \frac{1}{2}\frac{a_{11}}{a}\partial_1 a_{22}(-\frac{1}{2}\frac{a_{22}}{a}\partial_1 a_{22}) + \frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22}\frac{1}{2}\frac{a_{11}}{a}\partial_2 a_{22} \\ \end{cases} \tag{7}$$

Simplifying the notational burden by replacing a_{11} by γ and a_{22} by η:

$$\Rightarrow \begin{cases} R_{11} = \frac{1}{2}\partial_{11}\log a - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\partial_1\gamma\partial_1 \log a + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\partial_2\gamma\partial_2 \log a \\ -\frac{1}{2}\partial_1(\frac{1}{\gamma}\partial_1\gamma) + \frac{1}{2}\partial_2(\frac{1}{\eta}\partial_2\gamma) \\ +\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_1\gamma\partial_1\gamma - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\gamma \\ -\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\gamma + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_1\eta\partial_1\eta \\ \\ R_{22} = \frac{1}{2}\partial_{22}\log a + \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\partial_1\eta\partial_1 \log a - \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\partial_2\eta\partial_2 \log a \\ +\frac{1}{2}\partial_1(\frac{1}{\gamma}\partial_1\eta) - \frac{1}{2}\partial_2(\frac{1}{\eta}\partial_2\eta) \\ +\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_2\gamma\partial_2\gamma - \frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\eta\partial_1\eta \\ -\frac{1}{2}\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\eta\partial_1\eta + \frac{1}{2}\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_2\eta\partial_2\eta \\ \end{cases} \tag{8}$$

Noting that $\partial_{ii}\log a = \partial_i\left(\frac{1}{a_{11}}\partial_i a_{11}\right) + \partial_i\left(\frac{1}{a_{22}}\partial_i a_{22}\right)$ and $\partial_i \log a = \frac{1}{a_{11}}\partial_i a_{11} + \frac{1}{a_{22}}\partial_i a_{22} \quad (i = 1, 2)$, we get:

$$
\left|
\begin{array}{c|c}
2R_{11} = & 2R_{22} = \\[2ex]
\underbrace{\partial_1\left(\frac{1}{\gamma}\partial_1\gamma\right)}_{*} + \partial_1\left(\frac{1}{\eta}\partial_1\eta\right) & \partial_2\left(\frac{1}{\gamma}\partial_2\gamma\right) + \underbrace{\partial_2\left(\frac{1}{\eta}\partial_2\eta\right)}_{*} \\[2ex]
-\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_1\gamma)^2}_{-} - \frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\gamma\partial_1\eta & +\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_1\gamma\partial_1\eta + \frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2 \\[2ex]
+\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2 + \frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_2\gamma\partial_2\eta & -\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\eta - \underbrace{\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_2\eta)^2}_{-} \\[2ex]
-\underbrace{\partial_1(\frac{1}{\gamma}\partial_1\gamma)}_{*} + \partial_2(\frac{1}{\eta}\partial_2\gamma) & +\partial_1(\frac{1}{\gamma}\partial_1\eta) - \underbrace{\partial_2(\frac{1}{\eta}\partial_2\eta)}_{*} \\[2ex]
+\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_1\gamma)^2}_{-} - \underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2}_{+} & +\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_2\gamma)^2 - \underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2}_{+} \\[2ex]
-\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2}_{+} + \frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_1\eta)^2 & -\underbrace{\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2}_{+} + \underbrace{\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_2\eta)^2}_{-}
\end{array}
\right|
\tag{9}
$$

$$
\Rightarrow
\left|
\begin{array}{c|c}
2R_{11} = & 2R_{22} = \\[2ex]
\partial_1\left(\frac{1}{\eta}\partial_1\eta\right) + \partial_2(\frac{1}{\eta}\partial_2\gamma) & \partial_1(\frac{1}{\gamma}\partial_1\eta) + \partial_2\left(\frac{1}{\gamma}\partial_2\gamma\right) \\[2ex]
+\frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}(\partial_1\eta)^2 - \frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_2\gamma)^2 & -\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}(\partial_1\eta)^2 + \frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}(\partial_2\gamma)^2 \\[2ex]
-\frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_1\gamma\partial_1\eta + \frac{1}{2}\frac{1}{\eta}\frac{1}{\eta}\partial_2\gamma\partial_2\eta & +\frac{1}{2}\frac{1}{\gamma}\frac{1}{\gamma}\partial_1\gamma\partial_1\eta - \frac{1}{2}\frac{1}{\gamma}\frac{1}{\eta}\partial_2\gamma\partial_2\eta
\end{array}
\right|
\tag{10}
$$

Be $R = \frac{1}{\gamma}R_{11} + \frac{1}{\eta}R_{22}$, all first order derivatives vanish and we get,

$$
\frac{1}{\gamma}R_{11} + \frac{1}{\eta}R_{22} = \frac{1}{2}\left[\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)\right] + \frac{1}{2}\left[\frac{1}{\gamma}\partial_2(\frac{1}{\gamma}\partial_2\gamma) + \frac{1}{\eta}\partial_2(\frac{1}{\eta}\partial_2\gamma)\right]
\tag{11}
$$

We further simplify this expression. Considering the symmetry of (11) we only explicit the calculations for the first terms in ∂_1.

$$\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta) = \frac{1}{\eta}\partial_1(\frac{1}{\sqrt{\gamma}}\frac{1}{\sqrt{\gamma}}\frac{\sqrt{\eta}}{\sqrt{\eta}}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\sqrt{\eta}}\frac{1}{\sqrt{\eta}}\frac{\sqrt{\gamma}}{\sqrt{\gamma}}\partial_1\eta) \tag{12}$$

$$= \frac{1}{\eta}\partial_1\left[\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}\frac{1}{\sqrt{a}}\partial_1\eta\right] + \frac{1}{\gamma}\partial_1\left[\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{a}}\partial_1\eta\right] \tag{13}$$

$$= \begin{cases} \underbrace{\frac{1}{\eta}\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}}}_{=\frac{1}{\sqrt{a}}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1\eta\right] + \underbrace{\frac{1}{\gamma}\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}}_{=\frac{1}{\sqrt{a}}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1\eta\right] \\ \\ +\frac{1}{\sqrt{a}}\partial_1\eta\underbrace{\left[\frac{1}{\eta}\partial_1\left(\frac{\eta}{\gamma}\right)^{\frac{1}{2}} + \frac{1}{\gamma}\partial_1\left(\frac{\eta}{\gamma}\right)^{-\frac{1}{2}}\right]}_{=0} \end{cases} \tag{14}$$

$$= 2\frac{1}{\sqrt{a}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1 a_{22}\right] \tag{15}$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{1}{\eta}\partial_1(\frac{1}{\gamma}\partial_1\eta) + \frac{1}{\gamma}\partial_1(\frac{1}{\eta}\partial_1\eta)\right] = \frac{1}{\sqrt{a}}\partial_1\left[\frac{1}{\sqrt{a}}\partial_1 a_{22}\right] \tag{16}$$

Using (16) and the same calculations for the terms in ∂_2 and using (2) and (3) we get

$$\frac{1}{a}R_{1212} = -\frac{1}{2}\frac{1}{\sqrt{a}}\left[\partial_1\left(\frac{1}{\sqrt{a}}\partial_1 a_{22}\right) + \partial_2\left(\frac{1}{\sqrt{a}}\partial_2 a_{11}\right)\right]$$

♦

1.3 p109 - Exercise 7

Suppose that in a V_3 the metric is :

$$ds^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2$$

whre h_1, h_2, h_3 are functions of the three coordinates. Calculate the curvature tensor in terms of the $h^{'}s$ and their derivatives. Check your result by nothing that the curvature tensor will vanish if h_1 is a function of x^1 only, h_2 a function of x^2 only, and h_3 a function of x^3 only.

From **3.115**. and **3.115**. we get for the non vanishing components of the covariant curvature tensor(6 independent components to calculate):

$$R_{1212} = \begin{cases} -R_{1221} \\ -R_{2112} \\ R_{2121} \end{cases} \quad R_{2323} = \begin{cases} -R_{2332} \\ -R_{3223} \\ R_{3232} \end{cases} \quad R_{1313} = \begin{cases} -R_{1331} \\ -R_{3113} \\ R_{3131} \end{cases} \tag{1}$$

$$R_{1213} = \begin{cases} -R_{1231} \\ R_{1312} \\ -R_{1321} \\ -R_{2113} \\ R_{2131} \\ -R_{3112} \\ R_{3121} \end{cases} \quad R_{1223} = \begin{cases} -R_{1232} \\ -R_{2123} \\ R_{2132} \\ R_{2312} \\ -R_{2321} \\ R_{3212} \\ -R_{3221} \end{cases} \quad R_{1323} = \begin{cases} -R_{1332} \\ R_{2313} \\ -R_{2331} \\ -R_{3123} \\ R_{3132} \\ -R_{3213} \\ R_{3231} \end{cases} \tag{2}$$

The metrix tensors:

$$(a_{mn}) = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \quad (a_{mn}) = \begin{pmatrix} \frac{1}{h_1^2} & 0 & 0 \\ 0 & \frac{1}{h_2^2} & 0 \\ 0 & 0 & \frac{1}{h_3^2} \end{pmatrix} \tag{3}$$

The Christoffel symbols:

$$\begin{array}{lll} [11,1] = h_1 \partial_1 h_1 & [11,2] = -h_1 \partial_2 h_1 & [11,3] = -h_1 \partial_3 h_1 \\ [12,1] = h_1 \partial_2 h_1 & [12,2] = h_2 \partial_1 h_2 & [12,3] = 0 \\ [22,1] = -h_2 \partial_1 h_2 & [22,2] = h_2 \partial_2 h_2 & [22,3] = -h_2 \partial_3 h_2 \\ [23,1] = 0 & [23,2] = h_2 \partial_3 h_2 & [23,3] = h_3 \partial_2 h_3 \\ [33,1] = -h_3 \partial_1 h_3 & [33,2] = -h_3 \partial_2 h_3 & [33,3] = -h_3 \partial_3 h_3 \\ [31,1] = h_1 \partial_3 h_1 & [31,2] = 0 & [31,3] = h_3 \partial_1 h_3 \end{array} \tag{4}$$

$$\begin{array}{lll}
\Gamma^1_{11} = \frac{1}{h_1}\partial_1 h_1 & \Gamma^2_{11} = -\frac{h_1}{h_2^2}\partial_2 h_1 & \Gamma^3_{11} = -\frac{h_1}{h_3^2}\partial_3 h_1 \\
\Gamma^1_{12} = \frac{1}{h_1}\partial_2 h_1 & \Gamma^2_{12} = \frac{1}{h_2}\partial_1 h_2 & \Gamma^3_{12} = 0 \\
\Gamma^1_{22} = -\frac{h_2}{h_1^2}\partial_1 h_2 & \Gamma^2_{22} = \frac{1}{h_2}\partial_2 h_2 & \Gamma^3_{22} = -\frac{h_2}{h_3^2}\partial_3 h_2 \\
\Gamma^1_{23} = 0 & \Gamma^2_{23} = \frac{1}{h_2}\partial_3 h_2 & \Gamma^3_{23} = \frac{1}{h_3}\partial_2 h_3 \\
\Gamma^1_{33} = -\frac{h_3}{h_1^2}\partial_1 h_3 & \Gamma^2_{33} = -\frac{h_3}{h_2^2}\partial_2 h_3 & \Gamma^3_{33} = \frac{1}{h_3}\partial_3 h_3 \\
\Gamma^1_{31} = \frac{1}{h_1}\partial_3 h_1 & \Gamma^2_{31} = 0 & \Gamma^3_{31} = \frac{1}{h_3}\partial_1 h_3
\end{array} \tag{5}$$

We use 3.113.

$$R_{rsmn} = \partial_m[sn,r] - \partial_n[sm,r] + \Gamma^p_{sm}[rn,p] - \Gamma^p_{sn}[rm,p]$$

Note that we only have to perform the full calculation for two curvature tensors e.g. R_{1212} and R_{1213} as the others can be retrieved by using adequate indices renaming and use of the identities 3.115.

$$R_{1212} = -h_2\partial^2_{11}(h_2) - h_1\partial^2_{22}(h_1) + \frac{h_2}{h_1}\partial_1 h_1\partial_1 h_2 + \frac{h_1}{h_2}\partial_2 h_1\partial_2 h_2 - \frac{h_1 h_2}{h_3^2}\partial_3 h_1\partial_3 h_2 \tag{6}$$

$$R_{2323} = -h_3\partial^2_{22}(h_3) - h_2\partial^2_{33}(h_2) + \frac{h_3}{h_2}\partial_2 h_2\partial_2 h_3 + \frac{h_2}{h_3}\partial_3 h_2\partial_3 h_3 - \frac{h_2 h_3}{h_1^2}\partial_1 h_2\partial_1 h_3 \tag{7}$$

$$R_{1313} = -h_3\partial^2_{11}(h_3) - h_1\partial^2_{33}(h_1) + \frac{h_3}{h_1}\partial_1 h_1\partial_1 h_3 + \frac{h_1}{h_3}\partial_3 h_1\partial_3 h_3 - \frac{h_1 h_3}{h_2^2}\partial_2 h_1\partial_2 h_3 \tag{8}$$

$$R_{1213} = -h_1\partial^2_{32}(h_1) + \frac{h_1}{h_3}\partial_2 h_3\partial_3 h_1 + \frac{h_1}{h_2}\partial_2 h_1\partial_3 h_2 \tag{9}$$

$$R_{1223} = h_2\partial^2_{31}(h_2) - \frac{h_2}{h_1}\partial_1 h_2\partial_3 h_1 - \frac{h_2}{h_3}\partial_3 h_2\partial_1 h_3 \tag{10}$$

$$R_{1323} = -h_3\partial^2_{21}(h_3) + \frac{h_3}{h_1}\partial_1 h_3\partial_3 h_1 + \frac{h_3}{h_2}\partial_2 h_3\partial_1 h_2 \tag{11}$$

And, indeed, all curvature tensors vanish when the h_i are only a function of the indices' dimension.

♦

1.4 p109 - Exercise 8

In relativity we encounter the metric form

$$\Phi = e^{\alpha} + e^{x^1}\left[\left(dx^2\right)^2 + \sin^2 x^2\left(dx^3\right)^2\right] - e^{\gamma}\left(dx^4\right)^2$$

where α and γ are functions of x^1 and x^4 only.
Show that the complete set of non-zero components of the Einstein tensor (see equation (3.214)) for the form given above are as follows

$$\begin{aligned}
G^1_{.1} &= e^{-\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_1\right) + e^{-x^1} \\
G^2_{.2} &= e^{\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_1\gamma_1\right) \\
&\quad + e^{\gamma}\left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4\right) \\
G^3_{.3} &= G^2_{.2} \\
G^4_{.4} &= e^{-\alpha}\left(-\frac{3}{4} - \frac{1}{2}\alpha_1\right) + e^{-x^1} \\
e^{\alpha}G^4_{.1} &= -e^{\gamma}G^4_{.1} = -\frac{1}{2}\alpha_4
\end{aligned}$$

The subscript on α and γ indicate partial derivatives with respect to x^1 and x^4.

We have

$$(a_{mn}) = \begin{pmatrix} e^{\alpha} & 0 & 0 & 0 \\ 0 & e^{x^1} & 0 & 0 \\ 0 & 0 & e^{x^1}\sin^2 x^2 & 0 \\ 0 & 0 & 0 & -e^{\gamma} \end{pmatrix} \quad (a^{mn}) = \begin{pmatrix} e^{-\alpha} & 0 & 0 & 0 \\ 0 & e^{-x^1} & 0 & 0 \\ 0 & 0 & \frac{e^{-x^1}}{\sin^2 x^2} & 0 \\ 0 & 0 & 0 & -e^{-\gamma} \end{pmatrix} \tag{1}$$

And will use the following definitions:

$$G^n_{.t} = R^n_{.t} - \frac{1}{2}\delta^n_t R \tag{2}$$
$$R^n_{.t} = a^{nk}R_{kt} \tag{3}$$
$$R_{kt} = a^{sn}R_{sktn} \tag{4}$$
$$R = a^{kt}R_{kt} \tag{5}$$

Considering that the non-diagonal components of a_{mn} vanish and as $R_{sktn} = 0$ when $s = k$ or $t = n$, we can write :

$$\begin{pmatrix} R_{11} \\ R_{22} \\ R_{33} \\ R_{44} \end{pmatrix} = \begin{pmatrix} 0 & R_{2112} & R_{3113} & R_{4114} \\ R_{1221} & 0 & R_{3223} & R_{4224} \\ R_{1331} & R_{2332} & 0 & R_{4334} \\ R_{1441} & R_{2442} & R_{3443} & 0 \end{pmatrix} \begin{pmatrix} a^{11} \\ a^{22} \\ a^{33} \\ a^{44} \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} R_{12} \\ R_{13} \\ R_{14} \\ R_{23} \\ R_{24} \\ R_{34} \end{pmatrix} = \begin{pmatrix} 0 & 0 & R_{3123} & R_{4124} \\ 0 & R_{2132} & 0 & R_{4134} \\ 0 & R_{2142} & R_{3143} & 0 \\ R_{1231} & 0 & 0 & R_{4234} \\ R_{1241} & 0 & R_{3243} & 0 \\ R_{1341} & R_{2342} & 0 & 0 \end{pmatrix} \begin{pmatrix} a^{11} \\ a^{22} \\ a^{33} \\ a^{44} \end{pmatrix} \tag{7}$$

$$\tag{8}$$

The Christoffel symbols of the first kind are:

$$\begin{array}{llll}
[11,1] = \frac{1}{2}\alpha_1 e^{\alpha} & [11,2] = 0 & [11,3] = 0 & [11,4] = -\frac{1}{2}\alpha_4 e^{\alpha} \\
[12,1] = 0 & [12,2] = \frac{1}{2}e^{x^1} & [12,3] = 0 & [12,4] = 0 \\
[13,1] = 0 & [13,2] = 0 & [13,3] = \frac{1}{2}e^{x^1}\sin^2 x^2 & [13,4] = 0 \\
[14,1] = \frac{1}{2}\alpha_4 e^{\alpha} & [14,2] = 0 & [14,3] = 0 & [14,4] = -\frac{1}{2}\gamma_1 e^{\gamma} \\
[22,1] = -\frac{1}{2}e^{x^1} & [22,2] = 0 & [22,3] = 0 & [22,4] = 0 \\
[23,1] = 0 & [23,2] = 0 & [23,3] = \frac{1}{2}e^{x^1}\sin 2x^2 & [23,4] = 0 \\
[24,1] = 0 & [24,2] = 0 & [24,3] = 0 & [24,4] = 0 \\
[33,1] = -\frac{1}{2}e^{x^1} sin^2 x^2 & [33,2] = -\frac{1}{2}e^{x^1}\sin 2x^2 & [33,3] = 0 & [33,4] = 0 \\
[34,1] = 0 & [34,2] = 0 & [34,3] = 0 & [34,4] = 0 \\
[44,1] = \frac{1}{2}\gamma_1 e^{\gamma} & [44,2] = 0 & [44,3] = 0 & [44,4] = -\frac{1}{2}\gamma_4 e^{\gamma}
\end{array} \tag{9}$$

We use 3.114. and considering that $a_{mn} = a^{mn} = 0$ for $m \neq n$:

$$R_{rsmn} = \begin{cases} \frac{1}{2}\left(\partial^2_{sm} a_{rn} + \partial^2_{rn} a_{sm}\right) \\ +\frac{1}{e^{\alpha}}\left([rn,1][sm,1] - [rm,1][sn,1]\right) \\ +\frac{1}{e^{x^1}}\left([rn,2][sm,2] - [rm,2][sn,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([rn,3][sm,3] - [rm,3][sn,3]\right) \\ -\frac{1}{e^{\gamma}}\left([rn,4][sm,4] - [rm,4][sn,4]\right) \end{cases}$$

Giving:

$$R_{2112} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{22} + \partial^2_{22} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][11,1] - [21,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][11,2] - [21,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][11,3] - [21,3][12,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][11,4] - [21,4][12,4]\right) \end{cases} \tag{10}$$

(11)

$$R_{3113} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{33} + \partial^2_{33} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][11,1] - [31,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][11,2] - [31,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][11,3] - [31,3][13,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][11,4] - [31,4][13,4]\right) \end{cases} \tag{12}$$

(13)

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{44} + \partial^2_{44} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][11,1] - [41,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][11,2] - [41,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][11,3] - [41,3][14,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][11,4] - [41,4][14,4]\right) \end{cases} \tag{14}$$

(15)

$$R_{3223} = \begin{cases} \frac{1}{2}\left(\partial^2_{22}a_{33} + \partial^2_{33}a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][22,1] - [32,1][23,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][22,2] - [32,2][23,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][22,3] - [32,3][23,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][22,4] - [32,4][23,4]\right) \end{cases} \tag{16}$$

$$\tag{17}$$

$$R_{4224} = \begin{cases} \frac{1}{2}\left(\partial^2_{22}a_{44} + \partial^2_{44}a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][22,1] - [42,1][24,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][22,2] - [42,2][24,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][22,3] - [42,3][24,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][22,4] - [42,4][24,4]\right) \end{cases} \tag{18}$$

$$\tag{19}$$

$$R_{4334} = \begin{cases} \frac{1}{2}\left(\partial^2_{33}a_{44} + \partial^2_{44}a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][33,1] - [43,1][34,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][33,2] - [43,2][34,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][33,3] - [43,3][34,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][33,4] - [43,4][34,4]\right) \end{cases} \tag{20}$$

$$\tag{21}$$

$$R_{3123} = \begin{cases} \frac{1}{2}\left(\partial^2_{12} a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][12,1]-[32,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][12,2]-[32,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][12,3]-[32,3][13,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][12,4]-[32,4][13,4]\right) \end{cases} \tag{22}$$

(23)

$$R_{4124} = \begin{cases} \frac{1}{2}\left(\partial^2_{12} a_{44}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][12,1]-[42,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][12,2]-[42,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][12,3]-[42,3][14,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][12,4]-[42,4][14,4]\right) \end{cases} \tag{24}$$

(25)

$$R_{2132} = \begin{cases} \frac{1}{2}\left(\partial^2_{13} a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][13,1]-[23,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][13,2]-[23,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][13,3]-[23,3][12,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][13,4]-[23,4][12,4]\right) \end{cases} \tag{26}$$

(27)

$$R_{4134} = \begin{cases} \frac{1}{2}\left(\partial^2_{13} a_{44}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][13,1] - [43,1][14,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][13,2] - [43,2][14,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][13,3] - [43,3][14,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][13,4] - [43,4][14,4]\right) \end{cases} \tag{28}$$

(29)

$$R_{2142} = \begin{cases} \frac{1}{2}\left(\partial^2_{14} a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][14,1] - [24,1][12,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][14,2] - [24,2][12,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][14,3] - [24,3][12,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][14,4] - [24,4][12,4]\right) \end{cases} \tag{30}$$

(31)

$$R_{3143} = \begin{cases} \frac{1}{2}\left(\partial^2_{14} a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][14,1] - [34,1][13,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][14,2] - [34,2][13,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][14,3] - [34,3][13,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][14,4] - [34,4][13,4]\right) \end{cases} \tag{32}$$

(33)

$$R_{1231} = \begin{cases} \frac{1}{2}\left(\partial^2_{23} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][23,1]-[13,1][21,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][23,2]-[13,2][21,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][23,3]-[13,3][21,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][23,4]-[13,4][21,4]\right) \end{cases} \tag{34}$$

(35)

$$R_{4234} = \begin{cases} \frac{1}{2}\left(\partial^2_{23} a_{44}\right) \\ +\frac{1}{e^{\alpha}}\left([44,1][23,1]-[43,1][24,1]\right) \\ +\frac{1}{e^{x^1}}\left([44,2][23,2]-[43,2][24,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([44,3][23,3]-[43,3][24,3]\right) \\ -\frac{1}{e^{\gamma}}\left([44,4][23,4]-[43,4][24,4]\right) \end{cases} \tag{36}$$

(37)

$$R_{1241} = \begin{cases} \frac{1}{2}\left(\partial^2_{24} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][24,1]-[14,1][21,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][24,2]-[14,2][21,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][24,3]-[14,3][21,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][24,4]-[14,4][21,4]\right) \end{cases} \tag{38}$$

(39)

$$R_{3243} = \begin{cases} \frac{1}{2}\left(\partial^2_{24} a_{33}\right) \\ +\frac{1}{e^{\alpha}}\left([33,1][24,1]-[34,1][23,1]\right) \\ +\frac{1}{e^{x^1}}\left([33,2][24,2]-[34,2][23,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([33,3][24,3]-[34,3][23,3]\right) \\ -\frac{1}{e^{\gamma}}\left([33,4][24,4]-[34,4][23,4]\right) \end{cases} \tag{40}$$

(41)

$$R_{1341} = \begin{cases} \frac{1}{2}\left(\partial^2_{34} a_{11}\right) \\ +\frac{1}{e^{\alpha}}\left([11,1][34,1]-[14,1][31,1]\right) \\ +\frac{1}{e^{x^1}}\left([11,2][34,2]-[14,2][31,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([11,3][34,3]-[14,3][31,3]\right) \\ -\frac{1}{e^{\gamma}}\left([11,4][34,4]-[14,4][31,4]\right) \end{cases} \tag{42}$$

(43)

$$R_{2342} = \begin{cases} \frac{1}{2}\left(\partial^2_{34} a_{22}\right) \\ +\frac{1}{e^{\alpha}}\left([22,1][34,1]-[24,1][32,1]\right) \\ +\frac{1}{e^{x^1}}\left([22,2][34,2]-[24,2][32,2]\right) \\ +\frac{1}{e^{x^1}\sin^2 x^2}\left([22,3][34,3]-[24,3][32,3]\right) \\ -\frac{1}{e^{\gamma}}\left([22,4][34,4]-[24,4][32,4]\right) \end{cases} \tag{44}$$

(45)

Considering that $[mn,q]=0$ for $m \neq n \neq q \neq m$ and replacing the remaining Christoffels symbols:

$$R_{2112} = \begin{cases} \frac{1}{2}\left(\partial^2_{11} a_{22} + \partial^2_{22} a_{11}\right) \\ -\frac{1}{4}\alpha_1 e^{x^1} \end{cases} \tag{46}$$

(47)

$$R_{3113} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{33} + \partial_{33}^2 a_{11}\right) \\ \\ -\frac{1}{4}\left(1+\alpha_1\right) e^{x^1} sin^2 x^2 \end{cases} \tag{48}$$

(49)

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{44} + \partial_{44}^2 a_{11}\right) \\ \\ +\frac{1}{4}\left(\alpha_1\gamma_1 e^{\gamma} - \alpha_4^2 e^{\alpha}\right) + \frac{1}{4}\left(\alpha_4\gamma_4 e^{\alpha} - \gamma_1^2 e^{\gamma}\right) \end{cases} \tag{50}$$

(51)

$$R_{4114} = \begin{cases} \frac{1}{2}\left(\partial_{11}^2 a_{44} + \partial_{44}^2 a_{11}\right) \\ \\ +\frac{1}{4}\left(\alpha_1\gamma_1 e^{\gamma} - \alpha_4^2 e^{\alpha}\right) - \frac{1}{4}\left(\alpha_4\gamma_4 e^{\alpha} - \gamma_1^2 e^{\gamma}\right) \end{cases} \tag{52}$$

(53)

$$R_{3223} = \begin{cases} \frac{1}{2}\left(\partial_{22}^2 a_{33} + \partial_{33}^2 a_{22}\right) \\ \\ +\frac{1}{4}\frac{e^{x^1}}{e^{2\alpha}} sin^2 x^2 - e^{x^1}\cos^2 x^2 \end{cases} \tag{54}$$

(55)

$$R_{4224} = \begin{cases} \frac{1}{2}\left(\partial_{22}^2 a_{44} + \partial_{44}^2 a_{22}\right) \\ \\ -\frac{1}{4}\gamma_1 \frac{e^{\gamma}e^{x^1}}{e^{\alpha}} \end{cases} \tag{56}$$

(57)

$$R_{4334} = \begin{cases} \frac{1}{2}\left(\partial_{33}^2 a_{44} + \partial_{44}^2 a_{33}\right) \\ \\ -\frac{1}{4}\gamma_1 \frac{e^{\gamma}e^{x^1}}{e^{\alpha}} \sin^2 x^2 \end{cases} \tag{58}$$

(59)

$$R_{3123} = \frac{1}{2}\left(\partial^2_{12} a_{33}\right) = 0 \tag{60}$$

$$R_{4124} = \frac{1}{2}\left(\partial^2_{12} a_{44}\right) = 0 \tag{61}$$

$$R_{2132} = \frac{1}{2}\left(\partial^2_{13} a_{22}\right) = 0 \tag{62}$$

$$R_{4134} = \frac{1}{2}\left(\partial^2_{13} a_{44}\right) = 0 \tag{63}$$

$$R_{2142} = e^{-\alpha}\left([22,1][14,1]\right) \tag{64}$$

$$R_{3143} = e^{-\alpha}\left([33,1][14,1]\right) \tag{65}$$

$$R_{1231} = \frac{1}{2}\left(\partial^2_{23} a_{11}\right) = 0 \tag{66}$$

$$R_{4234} = \frac{1}{2}\left(\partial^2_{23} a_{44}\right) = 0 \tag{67}$$

$$R_{1241} = \frac{1}{2}\left(\partial^2_{24} a_{11}\right) = 0 \tag{68}$$

$$R_{3243} = \frac{1}{2}\left(\partial^2_{24} a_{33}\right) = 0 \tag{69}$$

$$R_{1341} = \frac{1}{2}\left(\partial^2_{34} a_{11}\right) = 0 \tag{70}$$

$$R_{2342} = \frac{1}{2}\left(\partial^2_{34} a_{22}\right) = 0 \tag{71}$$

Giving:

$$R_{2112} = \frac{1}{4}\left(1 - \alpha_1\right) e^{x^1} \tag{72}$$

$$R_{3113} = \frac{1}{4}\left(1 - \alpha_1\right) e^{x^1} \sin^2 x^2 \tag{73}$$

$$R_{4114} = \begin{cases} \frac{1}{2} e^{\alpha}\left(\alpha_{44} + \frac{1}{2}\alpha_4^2 - \frac{1}{2}\alpha_4\gamma_4\right) \\ -\frac{1}{2} e^{\gamma}\left(\gamma_{11} + \frac{1}{2}\gamma_1^2 - \frac{1}{2}\alpha_1\gamma_1\right) \end{cases} \tag{74}$$

$$R_{3223} = \left(\frac{1}{4}\frac{e^{x^1}}{e^{\alpha}} - 1\right) e^{x^1} \sin^2 x^2 \tag{75}$$

$$R_{4224} = -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} \tag{76}$$

$$R_{4334} = -\frac{1}{4}\gamma_1 \frac{e^{\gamma} e^{x^1}}{e^{\alpha}} sin^2 x^2 \tag{77}$$

$$R_{2142} = -\frac{1}{4}\alpha_4 e^{x^1} \tag{78}$$

$$R_{3143} = -\frac{1}{4}\alpha_4 e^{x^1} \sin^2 x^2 \tag{79}$$

As all other curvature components vanish, we get

$$\begin{pmatrix} R_{11} \\ \\ R_{22} \\ \\ R_{33} \\ \\ R_{44} \\ \end{pmatrix} = P \begin{pmatrix} e^{-\alpha} \\ \\ e^{-x^1} \\ \\ \frac{e^{-x^1}}{\sin^2 x^2} \\ \\ -e^{-\gamma} \\ \end{pmatrix} \tag{80}$$

With

$$P = \begin{pmatrix} 0 & \frac{1}{4}(1-\alpha_1)\, e^{x^1} & \frac{1}{4}(1-\alpha_1)\, e^{x^1}\sin^2 x^2 & \begin{matrix} \frac{1}{2}e^{\alpha}\left(\alpha_{44}+\frac{1}{2}\alpha_4^2-\frac{1}{2}\alpha_4\gamma_4\right) \\ -\frac{1}{2}e^{\gamma}\left(\gamma_{11}+\frac{1}{2}\gamma_1^2-\frac{1}{2}\alpha_1\gamma_1\right) \end{matrix} \\ \\ . & 0 & \left(\frac{1}{4}\frac{e^{x^1}}{e^{\alpha}}-1\right)e^{x^1}\sin^2 x^2 & -\frac{1}{4}\gamma_1\frac{e^{\gamma}e^{x^1}}{e^{\alpha}} \\ \\ . & . & 0 & -\frac{1}{4}\gamma_1\frac{e^{\gamma}e^{x^1}}{e^{\alpha}} sin^2 x^2 \\ \\ . & . & . & 0 \end{pmatrix} \tag{81}$$

and

$$\begin{pmatrix} R_{12} \\ R_{13} \\ R_{14} \\ R_{23} \\ R_{24} \\ R_{34} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4}\alpha_4 e^{x^1} & -\frac{1}{4}\alpha_4 e^{x^1}\sin^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-\alpha} \\ \\ e^{-x^1} \\ \\ \frac{e^{-x^1}}{\sin^2 x^2} \\ \\ -e^{-\gamma} \\ \end{pmatrix} \tag{82}$$

Finally we can compute the Einstein tensor:

$$R = a^{11}R_{11} + a^{22}R_{22} + a^{33}R_{33} + a^{44}R_{44} \tag{83}$$

$$= \begin{cases} e^{-\alpha}\left(\gamma_{11}+\frac{1}{2}\gamma_1^2-\frac{1}{2}\alpha_1\gamma_1\right) \\ \\ e^{-\gamma}\left(\alpha_{44}+\frac{1}{2}\alpha_4^2-\frac{1}{2}\alpha_4\gamma_4\right) \\ \\ +e^{-\alpha}\left(\frac{3}{2}+\gamma_1-\alpha_1\right) \\ \\ -2e^{-x^1} \end{cases} \tag{84}$$

$$G^1_{.1} = e^{-\alpha} R_{11} - \frac{1}{2} R \tag{85}$$
$$= e^{-\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_1\right) + e^{-x^1} \tag{86}$$

$$G^2_{.2} = e^{-x^1} R_{22} - \frac{1}{2} R \tag{87}$$
$$= \begin{cases} e^{-\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1\gamma_1 + \frac{1}{4}\alpha_1\right) \\ \\ +e^{-\gamma}\left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4\right) \end{cases} \tag{88}$$

$$G^3_{.3} = \frac{e^{-x^1}}{\sin^2 x^2} R_{33} - \frac{1}{2} R \tag{89}$$
$$= \begin{cases} e^{-\alpha}\left(-\frac{1}{4} - \frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^2 - \frac{1}{4}\gamma_1 + \frac{1}{4}\alpha_1\gamma_1 + \frac{1}{4}\alpha_1\right) \\ \\ +e^{-\gamma}\left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^2 - \frac{1}{4}\alpha_4\gamma_4\right) \end{cases} \tag{90}$$

$$G^4_{.4} = -e^{-\gamma} R_{44} - \frac{1}{2} R \tag{91}$$
$$= e^{-\alpha}\left(-\frac{3}{4} + \frac{1}{2}\alpha_1\right) + e^{-x^1} \tag{92}$$

$$G^1_{.4} = e^{-\alpha} R_{14} \tag{93}$$
$$= -\frac{1}{2} e^{-\alpha}\alpha_4 \tag{94}$$

$$G^4_{.1} = -e^{-\gamma} R_{14} \tag{95}$$
$$= \frac{1}{2} e^{-\gamma}\alpha_4 \tag{96}$$

♦

1.5 p110 - Exercise 9

If we change the metric tensor from a_{mn} to $a_{mn}+b_{mn}$ where b_{mn} is small, calculate the principal parts of the increment in the components of the curvature tensor.

Let's start with one form of the curvature tensor

$$R_{rsmn} = \begin{cases} \frac{1}{2}\left(\partial^2_{sm}a_{rn} + \partial^2_{rn}a_{sm} - \partial^2_{sn}a_{rm} - \partial^2_{rm}a_{sn}\right) \\ +a^{pq}\left([rn,p][sm,q] - [rm,p][sn,q]\right) \end{cases} \tag{1}$$

Be $R^{'}_{rsmn}$ and $a^{'}_{rn}$ the components of the tensors after changing the metric form to $a_{mn}+b_{mn}$. Then

$$R^{'}_{rsmn} = \begin{cases} \frac{1}{2}\left(\partial^2_{sm}a^{'}_{rn} + \partial^2_{rn}a^{'}_{sm} - \partial^2_{sn}a^{'}_{rm} - \partial^2_{rm}a^{'}_{sn}\right) \\ +a^{'pq}\left([rn,p]^{'}[sm,q]^{'} - [rm,p]^{'}[sn,q]^{'}\right) \end{cases} \tag{2}$$

At the point where we want to calculate the increment of the curvature tensor, we can choose Riemannian coordinates related to the metric $a^{'}_{mn}$. Then, the Christoffel symbols vanish at that point as origin and (2) becomes

$$R^{'}_{rsmn} = \frac{1}{2}\left(\partial^2_{sm}a^{'}_{rn} + \partial^2_{rn}a^{'}_{sm} - \partial^2_{sn}a^{'}_{rm} - \partial^2_{rm}a^{'}_{sn}\right) \tag{3}$$

$$= \begin{cases} \frac{1}{2}\left(\partial^2_{sm}a_{rn} + \partial^2_{rn}a_{sm} - \partial^2_{sn}a_{rm} - \partial^2_{rm}a_{sn}\right) \\ \\ +\frac{1}{2}\left(\partial^2_{sm}b_{rn} + \partial^2_{rn}b_{sm} - \partial^2_{sn}b_{rm} - \partial^2_{rm}b_{sn}\right) \end{cases} \tag{4}$$

$$= \begin{cases} R_{rsmn} - a^{pq}\left([rn,p][sm,q] - [rm,p][sn,q]\right) \\ \\ +\frac{1}{2}\left(\partial^2_{sm}b_{rn} + \partial^2_{rn}b_{sm} - \partial^2_{sn}b_{rm} - \partial^2_{rm}b_{sn}\right) \end{cases} \tag{5}$$

$$\Rightarrow \quad \Delta R_{rsmn} = \begin{cases} -a^{pq}\left([rn,p][sm,q] - [rm,p][sn,q]\right) \\ \\ +\frac{1}{2}\left(\partial^2_{sm}b_{rn} + \partial^2_{rn}b_{sm} - \partial^2_{sn}b_{rm} - \partial^2_{rm}b_{sn}\right) \end{cases} \tag{6}$$

As $[rn,p]^{'}=0$ we have $[rn,p]^{"} = -[rn,p]$ (the suffix " referring to b_{mn}). So we have for $[rn,s]^{"}$ and $[ms,r]^{"}$

$$\partial_r b_{sn} + \partial_n b_{rs} - \partial_s b_{rn} = -\partial_r a_{sn} - \partial_n a_{rs} + \partial_s a_{rn} \tag{7}$$

$$\partial_m b_{rs} + \partial_s b_{rm} - \partial_r b_{sm} = -\partial_m a_{rs} - \partial_s a_{rm} + \partial_r a_{sm} \tag{8}$$

$$\partial_m(7) \quad \Rightarrow \quad \partial^2_{rm}b_{sn} + \partial^2_{mn}b_{rs} - \partial^2_{sm}b_{rn} = -\partial^2_{rm}a_{sn} - \partial^2_{mn}a_{rs} + \partial^2_{sm}a_{rn} \tag{9}$$

$$\partial_n(8) \quad \Rightarrow \quad \partial^2_{mn}b_{rs} + \partial^2_{sn}b_{rm} - \partial^2_{rn}b_{sm} = -\partial^2_{mn}a_{rs} - \partial^2_{sn}a_{rm} + \partial^2_{rn}a_{sm} \tag{10}$$

Combining (9) and (10) in (6), we get

$$\Delta R_{rsmn} = \begin{cases} -a^{pq}\left([rn,p][sm,q]-[rm,p][sn,q]\right) \\ +\frac{1}{2}\left(\partial^2_{rm}a_{sn}+\partial^2_{mn}a_{rs}-\partial^2_{sm}a_{rn}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}a_{rs}+\partial^2_{sn}a_{rm}-\partial^2_{rn}a_{sm}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}b_{rs}+\partial^2_{mn}b_{rs}\right) \end{cases} \tag{11}$$

$$= \begin{cases} a^{pq}\left([rm,p][sn,q]-[rn,p][sm,q]\right) \\ +\frac{1}{2}\left(\partial^2_{rm}a_{sn}+\partial^2_{sn}a_{rm}-\partial^2_{sm}a_{rn}-\partial^2_{rn}a_{sm}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}a_{rs}+\partial^2_{mn}a_{rs}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}b_{rs}+\partial^2_{mn}b_{rs}\right) \end{cases} \tag{12}$$

$$= \begin{cases} R_{rsnm} \\ +\frac{1}{2}\left(\partial^2_{mn}a_{rs}+\partial^2_{mn}a_{rs}\right) \\ +\frac{1}{2}\left(\partial^2_{mn}b_{rs}+\partial^2_{mn}b_{rs}\right) \end{cases} \tag{13}$$

$$= -R_{rsmn}+\partial^2_{mn}a_{rs}+\partial^2_{mn}b_{rs} \tag{14}$$

Can I simplify further ??

♦

1.6 p110 - Exercise 10

If we use normal coordinates in a Riemannian V_n, the metric form is as in equation 2.630. For this coordinate system, express the curvature tensor, the Ricci tensor, and the curvature invariant in terms of the corresponding quantities for the $(N-1)$ − space $x^N = C^{st}$ and certain additional terms. Check these additional terms by nothing that they must have tensor character with respect to transformations of the coordinates $x^1, x^2, \ldots, x^{N-1}$.

$$R_{rsmn} = \partial_m[sn,r] - \partial_n[sm,r] + \Gamma^p_{sm}[rn,p] - \Gamma^p_{sn}[rm,p]$$

We indicate by $R^{'}_{...}$ the quantity generated by the previous definition, but restricted to the subspace V_{N-1}. Calculating $R_{\alpha\beta\gamma\delta}$ restricted to the subspace V_{N-1} gives:

$$R_{\alpha\beta\gamma\delta} = \underbrace{\partial_\gamma[\beta\delta,\alpha] - \partial_\delta[\beta\gamma,\alpha] + \Gamma^\nu_{\beta\gamma}[\alpha\delta,\nu] - \Gamma^\nu_{\beta\delta}[\alpha\gamma,\nu]}_{=R^{'}_{\alpha\beta\gamma\delta}} + \Gamma^N_{\beta\gamma}[\alpha\delta,N] - \Gamma^N_{\beta\delta}[\alpha\gamma,N] \quad (1)$$

$$= R^{'}_{\alpha\beta\gamma\delta} + \underbrace{\Gamma^N_{\beta\gamma}}_{-\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\beta\gamma}} \underbrace{[\alpha\delta,N]}_{-\frac{1}{2}\partial_N a_{\alpha\delta}} - \underbrace{\Gamma^N_{\beta\delta}}_{-\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\beta\delta}} \underbrace{[\alpha\gamma,N]}_{-\frac{1}{2}\partial_N a_{\alpha\gamma}} \quad \text{(see page 66/67)} \quad (2)$$

$$\Rightarrow \quad R_{\alpha\beta\gamma\delta} = R^{'}_{\alpha\beta\gamma\delta} + \frac{1}{4}\frac{1}{a_{NN}}\left(\partial_N a_{\beta\gamma}\partial_N a_{\alpha\delta} - \partial_N a_{\beta\delta}\partial_N a_{\alpha\gamma}\right) \quad (3)$$

We calculate $R_{\alpha\beta}$:

$$R_{\alpha\beta} = a^{sn}R_{s\alpha\beta n} \quad (4)$$

$$= \underbrace{a^{\nu\mu}R_{\nu\alpha\beta\mu}}_{R^{'}_{\alpha\beta}} + \underbrace{a^{N\mu}}_{=0}R_{N\alpha\beta\mu} + \underbrace{a^{\nu N}}_{=0}R_{\nu\alpha\beta N} + \underbrace{a^{NN}}_{=\frac{1}{a_{NN}}} R_{N\alpha\beta N} \quad (5)$$

$$(5) \Rightarrow \quad R_{\alpha\beta} = R^{'}_{\alpha\beta} + \frac{1}{a_{NN}}R_{N\alpha\beta N} \quad (6)$$

$$R_{N\alpha\beta N} = \begin{cases} \partial_\beta[\alpha N,N] - \partial_N[\alpha\beta,N] \\ +\Gamma^\nu_{\alpha\beta}[NN,\nu] - \Gamma^\nu_{\alpha N}[N\beta,\nu] \\ +\Gamma^N_{\alpha\beta}[NN,N] - \Gamma^N_{\alpha N}[N\beta,N] \end{cases} \quad (7)$$

$$= \begin{cases} R^{'}_{N\alpha\beta N} \\ + \underbrace{\Gamma^N_{\alpha\beta}}_{-\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}} \underbrace{[NN,N]}_{\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{NN}} - \underbrace{\Gamma^N_{\alpha N}}_{\frac{1}{2}\frac{1}{a_{NN}}\partial_\alpha a_{NN}} \underbrace{[N\beta,N]}_{\frac{1}{2}\partial_\beta a_{NN}} \end{cases} \quad (8)$$

$$= R^{'}_{N\alpha\beta N} - \frac{1}{4}\frac{1}{a_{NN}}\left(\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}\partial_N a_{NN} + \partial_\alpha a_{NN}\partial_\beta a_{NN}\right) \quad (9)$$

$$\text{(6) and (9)} \Rightarrow \qquad R_{\alpha\beta} = \begin{cases} R'_{\alpha\beta} + \frac{1}{a_{NN}} R'_{N\alpha\beta N} \\ \\ -\frac{1}{4}\frac{1}{a^2_{NN}} \left(\frac{1}{a_{NN}} \partial_N a_{\alpha\beta} \partial_N a_{NN} + \partial_\alpha a_{NN} \partial_\beta a_{NN} \right) \end{cases} \tag{10}$$

And the invariant curvature:

$$R = a^{mn} R_{mn} \tag{11}$$

$$= \underbrace{a^{\mu\nu} R_{\mu\nu}}_{=R'} + \underbrace{a^{N\nu}}_{=0} R_{N\nu} + \underbrace{a^{\mu N}}_{=0} R_{\mu N} + a^{NN} R_{NN} \tag{12}$$

$$= R' + a^{NN} R_{NN} \tag{13}$$

$$R_{NN} = a_{sn} R_{sNNn} \tag{14}$$

$$= \underbrace{a_{\mu\nu} R_{\mu NN\nu}}_{=R'_{NN}} + \underbrace{a_{N\nu} R_{NNN\nu}}_{=0} + \underbrace{a_{\mu N} R_{\mu NNN}}_{=0} + \underbrace{a_{NN} R_{NNNN}}_{=0} \tag{15}$$

$$\text{(13) and (15)} \Rightarrow \qquad R = R' + \frac{1}{a_{NN}} R'_{NN} \tag{16}$$

We now check the tensor character of the residuals in (3), (10) and (16)

♦

1.7 p110 - Exercise 11

Prove that

$$T^{mn}{}_{|mn} = T^{mn}{}_{|nm}$$

whet T^{mn} is not necessarily symmetric.

$$T^{mn}{}_{|s} = \partial_s T^{mn} + \Gamma^m_{ks} T^{kn} + \Gamma^n_{ks} T^{mk} \tag{1}$$

$$T^{mn}{}_{|st} = (\partial_s T^{mn})_{|t} + (\Gamma^m_{ks})_{|t} T^{kn} + (\Gamma^n_{ks})_{|t} T^{mk} + \Gamma^m_{ks} (T^{kn})_{|t} + \Gamma^n_{ks} (T^{mk})_{|t}) \tag{2}$$

We choose Riemannian coordinates and take as origin the point where we want to check the asked identity. At that point the Christoffel symbols vanish and (2) becomes

$$T^{mn}{}_{|st} = (\partial_s T^{mn})_{|t} + (\Gamma^m_{ks})_{|t} T^{kn} + (\Gamma^n_{ks})_{|t} T^{mk} \tag{3}$$

$$= \begin{cases} \partial^2_{st} T^{mn} + \quad \text{terms in} \quad \Gamma\text{'s } (=0) \\ \\ +T^{kn} (\Gamma^m_{ks})_{|t} \\ \\ +T^{mk} (\Gamma^n_{ks})_{|t} \end{cases} \tag{4}$$

$$= \begin{cases} \partial^2_{st} T^{mn} \\ \\ +T^{kn} \left[\underbrace{a^{mp}{}_{|t}}_{=0} [ks,p] + \frac{1}{2} a^{mp} \left((\partial_k a_{sp})_{|t} + (\partial_s a_{kp})_{|t} - (\partial_p a_{ks})_{|t} \right) \right] \\ \\ +T^{mk} \left[\underbrace{a^{np}{}_{|t}}_{=0} [ks,p] + \frac{1}{2} a^{np} \left((\partial_k a_{sp})_{|t} + (\partial_s a_{kp})_{|t} - (\partial_p a_{ks})_{|t} \right) \right] \end{cases} \tag{5}$$

$$= \begin{cases} \partial^2_{st} T^{mn} \\ \\ +\frac{1}{2} a^{mp} \left(\partial^2_{kt} a_{sp} + \partial^2_{st} a_{kp} - \partial^2_{pt} a_{ks} \right) T^{kn} \\ \\ +\frac{1}{2} a^{np} \left(\partial^2_{kt} a_{sp} + \partial^2_{st} a_{kp} - \partial^2_{pt} a_{ks} \right) T^{mk} \end{cases} \tag{6}$$

In the same way we get

$$T^{mn}{}_{|ts} = \begin{cases} \partial^2_{st} T^{mn} \\ \\ +\frac{1}{2} a^{mp} \left(\partial^2_{ks} a_{tp} + \partial^2_{st} a_{kp} - \partial^2_{ps} a_{kt} \right) T^{kn} \\ \\ +\frac{1}{2} a^{np} \left(\partial^2_{ks} a_{tp} + \partial^2_{st} a_{kp} - \partial^2_{ps} a_{kt} \right) T^{mk} \end{cases} \tag{7}$$

Hence

$$2\left(T^{mn}{}_{|st} - T^{mn}{}_{|ts}\right) = \begin{cases} \;\; a^{mp}\left(\partial^2_{kt}a_{sp} + \partial^2_{st}a_{kp} - \partial^2_{pt}a_{ks} - \partial^2_{ks}a_{tp} - \partial^2_{st}a_{kp} + \partial^2_{ps}a_{kt}\right)T^{kn} \\ \\ +a^{np}\left(\partial^2_{kt}a_{sp} + \partial^2_{st}a_{kp} - \partial^2_{pt}a_{ks} - \partial^2_{ks}a_{tp} - \partial^2_{st}a_{kp} + \partial^2_{ps}a_{kt}\right)T^{mk} \end{cases} \tag{8}$$

$$= \begin{cases} \;\; a^{mp}\left(\partial^2_{kt}a_{sp} - \partial^2_{pt}a_{ks} - \partial^2_{ks}a_{tp} + \partial^2_{ps}a_{kt}\right)T^{kn} \\ \\ +a^{np}\left(\partial^2_{kt}a_{sp} - \partial^2_{pt}a_{ks} - \partial^2_{ks}a_{tp} + \partial^2_{ps}a_{kt}\right)T^{mk} \end{cases} \tag{9}$$

Putting $s = m$ and $t = n$:

$$2\left(T^{mn}{}_{|mn} - T^{mn}{}_{|nm}\right) = \begin{cases} \;\; a^{mp}\left(\partial^2_{kn}a_{mp} - \partial^2_{pn}a_{km} - \partial^2_{km}a_{np} + \partial^2_{pm}a_{kn}\right)T^{kn} \\ \\ +a^{np}\left(\partial^2_{kn}a_{mp} - \partial^2_{pn}a_{km} - \partial^2_{km}a_{np} + \partial^2_{pm}a_{kn}\right)T^{mk} \end{cases} \tag{10}$$

In the last term of (10) we can swap the indices m and n giving

$$2\left(T^{mn}{}_{|mn} - T^{mn}{}_{|nm}\right) = \begin{cases} \;\; a^{mp}\left(\partial^2_{kn}a_{mp} - \partial^2_{pn}a_{km} - \partial^2_{km}a_{np} + \partial^2_{pm}a_{kn}\right)T^{kn} \\ \\ +a^{mp}\left(\partial^2_{km}a_{np} - \partial^2_{pm}a_{kn} - \partial^2_{kn}a_{mp} + \partial^2_{pn}a_{km}\right)T^{nk} \end{cases} \tag{11}$$

In the second term of (11) we may again swap the indices n and k giving

$$2\left(T^{mn}{}_{|mn} - T^{mn}{}_{|nm}\right) = \begin{cases} \;\; a^{mp}\left(\cancel{\partial^2_{kn}a_{mp}} - \partial^2_{pn}a_{km} - \partial^2_{km}a_{np} + \cancel{\partial^2_{pm}a_{kn}}\right)T^{kn} \\ \\ +a^{mp}\left(\partial^2_{nm}a_{kp} - \cancel{\partial^2_{pm}a_{kn}} - \cancel{\partial^2_{kn}a_{mp}} + \partial^2_{pk}a_{nm}\right)T^{kn} \end{cases} \tag{12}$$

$$\tag{13}$$

$$= \left(\underbrace{a^{mp}\partial^2_{nm}a_{kp}}_{\text{swap m and p}} + \underbrace{a^{mp}\partial^2_{pk}a_{nm}}_{\text{swap m and p}} - a^{mp}\partial^2_{pn}a_{km} - a^{mp}\partial^2_{km}a_{np}\right)T^{kn} \tag{14}$$

$$= \left(a^{mp}\partial^2_{np}a_{km} + a^{mp}\partial^2_{mk}a_{np} - a^{mp}\partial^2_{pn}a_{km} - a^{mp}\partial^2_{km}a_{np}\right)T^{kn} \tag{15}$$

$$= 0 \tag{16}$$

♦

1.8 p110 - Exercise 12a

Prove that the quantities

$$G^{mn} + \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]$$

can be expressed in terms of the metric tensor and its first derivatives.

Let's define

$$K^{mn} \equiv \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right] \tag{1}$$

$$T^{mn} \equiv G^{mn} + K^{mn} \tag{2}$$

The strategy to proof this, is to separate in both terms of the expression, the parts that can be expressed in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and the parts of higher order differentiation. We begin with the second term.
As the covariant derivatives of the metric tensor vanish, we get

$$K^{mn} = \frac{1}{2a}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|rs} \tag{3}$$

$$= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|r}}_{=0}\right]_{|s} \tag{4}$$

$$= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|s} \tag{5}$$

$$= \frac{1}{2a}\left[\underbrace{a_{|rs}}_{=\partial^2_{rs}a}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a_{|r}\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|s}}_{=0}\right] \tag{6}$$

Considering,

$$\partial^2_{rs}\ln a = \partial_s\left(\frac{1}{a}\partial_r a\right) \tag{7}$$

$$= \frac{1}{a}\partial^2_{rs}a - \frac{1}{a^2}\partial_r a\partial_s a \tag{8}$$

$$\Rightarrow \qquad \frac{1}{a}\partial^2_{rs}a = \partial^2_{rs}\ln a + \frac{1}{a^2}\partial_r a\partial_s a \tag{9}$$

$$\Rightarrow \quad K^{mn} = \frac{1}{2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial^2_{rs}\ln a + \mathcal{E}^{mn} \tag{10}$$

with $\mathcal{E}^{mn}$ being a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.

Note that $\mathcal{E}^{mn}$ is a symmetrical object in m, n. Indeed,

$$\mathcal{E}^{mn} = \frac{1}{2a^2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r a \partial_s a \tag{11}$$

$$\Rightarrow \qquad \mathcal{E}^{nm} = \frac{1}{2a^2}\left(a^{nm}a^{rs} - a^{nr}a^{ms}\right)\partial_r a \partial_s a \tag{12}$$

$$= \frac{1}{2a^2}\left(a^{mn}a^{sr} - a^{ns}a^{mr}\right)\partial_s a \partial_r a \tag{13}$$

$$= \mathcal{E}^{mn} \tag{14}$$

But by (2.541.) : $\partial_i \ln a = 2\Gamma^t_{it}$ hence,

$$\partial_{rs} \ln a = 2\partial_s \Gamma^t_{rt} = 2\partial_r \Gamma^t_{st} \tag{15}$$

$$\Rightarrow \quad K^{mn} = \left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_s \Gamma^t_{rt} + \mathcal{E}^{mn} \tag{16}$$

$$\tag{17}$$

Considering (10) and swapping dummy indices we get the following expressions for K^{mn}

$$K^{mn} = \left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_s \Gamma^t_{rt} + \mathcal{E}^{mn} \tag{18}$$

$$K^{mn} = \left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r \Gamma^t_{st} + \mathcal{E}^{mn} \tag{19}$$

$$K^{mn} = \left(a^{mn}a^{rs} - a^{ms}a^{nr}\right)\partial_s \Gamma^t_{rt} + \mathcal{E}^{mn} \tag{20}$$

$$K^{mn} = \left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r \Gamma^t_{st} + \mathcal{E}^{mn} \tag{21}$$

$$\tag{22}$$

We now rewrite G^{mn}.
We have:

$$G^{mn} = a^{nk} G^{m}_{\cdot\ k} \tag{23}$$

$$G^{m}_{\cdot\ k} = R^{m}_{\cdot\ k} - \frac{1}{2}\delta^m_k R \tag{24}$$

$$R^{m}_{\cdot\ k} = a^{mp} R_{pk} \tag{25}$$

$$R = a^{pk} R_{pk} \tag{26}$$

And by 3.203.

$$R_{pk} = \partial_k \Gamma^t_{pt} - \partial_t \Gamma^t_{pk} + \mathcal{F}^{pk} \tag{27}$$

With $\mathcal{F}^{pk}$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only. Note that $\mathcal{F}^{pk}$ is a symmetrical object in p, k as

R_{pk} is a symmetrical tensor in p, k. Hence by (15) to (19):

$$G^{mn} = a^{nk} R^{m}_{\cdot\, k} - \frac{1}{2} a^{nk} \delta^{m}_{k} R \tag{28}$$
$$= a^{nk} a^{mp} R_{pk} - \frac{1}{2} a^{nm} a^{pk} R_{pk} \tag{29}$$
$$= \left(a^{nk} a^{mp} - \frac{1}{2} a^{nm} a^{pk} \right) R_{pk} \tag{30}$$
$$= \left(a^{nk} a^{mp} - \frac{1}{2} a^{nm} a^{pk} \right) \left(\partial_k \Gamma^t_{pt} - \partial_t \Gamma^t_{pk} + \mathcal{F}^{pk} \right) \tag{31}$$
$$= \left(a^{nk} a^{mp} - \frac{1}{2} a^{nm} a^{pk} \right) \left(\partial_k \Gamma^t_{pt} - \partial_t \Gamma^t_{pk} \right) + \mathcal{H}^{mn} \tag{32}$$
$$= a^{nk} a^{mp} \partial_k \Gamma^t_{pt} - \frac{1}{2} a^{nm} a^{pk} \partial_k \Gamma^t_{pt} - a^{nk} a^{mp} \partial_t \Gamma^t_{pk} + \frac{1}{2} a^{nm} a^{pk} \partial_t \Gamma^t_{pk} + \mathcal{H}^{mn} \tag{33}$$

with $\mathcal{H}^{mn}$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.
Note that $\mathcal{H}^{mn}$ is a symmetrical object in m, n . Indeed,

$$\mathcal{H}^{mn} = \left(a^{nk} a^{mp} - \frac{1}{2} a^{nm} a^{pk} \right) \mathcal{F}^{pk} \tag{34}$$
$$\Rightarrow \qquad \mathcal{H}^{nm} = \left(a^{mk} a^{np} - \frac{1}{2} a^{mn} a^{pk} \right) \mathcal{F}^{pk} \tag{35}$$
$$= \left(a^{mp} a^{nk} - \frac{1}{2} a^{mn} a^{kp} \right) \mathcal{F}^{kp} \tag{36}$$
$$= \mathcal{H}^{mn} \tag{37}$$

Putting (10) and (2) together with $\mathcal{L}^{mn} = \mathcal{E}^{mn} + \mathcal{H}^{mn}$ we get,

$$T^{mn} = Q^{mn} + \mathcal{L}^{mn} \tag{38}$$

with $\mathcal{L}^{mn}$ a symmetrical object in m, n and depending only on terms in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and

$$Q^{mn} \equiv \begin{cases} \cancel{a^{nk} a^{mp} \partial_r \Gamma^t_{st}} - \frac{1}{2} a^{nm} a^{rs} \partial_r \Gamma^t_{st} \\ \\ -a^{nr} a^{ms} \partial_t \Gamma^t_{rs} + \frac{1}{2} a^{nm} a^{rs} \partial_t \Gamma^t_{pk} \\ \\ +a^{mn} a^{rs} \partial_s \Gamma^t_{rt} - \cancel{a^{mr} a^{ns} \partial_s \Gamma^t_{rt}} \end{cases} \tag{39}$$
$$= \frac{1}{2} a^{mn} a^{rs} \partial_s \Gamma^t_{rt} + \frac{1}{2} a^{nm} a^{rs} \partial_t \Gamma^t_{pk} - a^{nr} a^{ms} \partial_t \Gamma^t_{rs} \tag{40}$$

Q^{mn} a symmetrical object in m, n

♦

1.9 p110 - Exercise 12

Prove that the quantities

$$G^{mn} + \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]$$

can be expressed in terms of the metric tensor and its first derivatives.

Let's define

$$K^{mn} \equiv \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right] \tag{1}$$

$$T^{mn} \equiv G^{mn} + K^{mn} \tag{2}$$

The strategy to proof this, is to separate in both terms of the expression, the parts that can be expressed in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and the parts of higher order differentiation. We begin with the second term.
As the covariant derivatives of the metric tensor vanish, we get

$$\begin{aligned}
K^{mn} &= \frac{1}{2a}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|rs} &(3)\\
&= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|r}}_{=0}\right]_{|s} &(4)\\
&= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|s} &(5)\\
&= \frac{1}{2a}\left[\underbrace{a_{|rs}}_{=\partial^2_{rs}a}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a_{|r}\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|s}}_{=0}\right] &(6)
\end{aligned}$$

Considering,

$$\begin{aligned}
\partial^2_{rs}\ln a &= \partial_s\left(\frac{1}{a}\partial_r a\right) &(7)\\
&= \frac{1}{a}\partial^2_{rs}a - \frac{1}{a^2}\partial_r a\partial_s a &(8)\\
\Rightarrow \qquad \frac{1}{a}\partial^2_{rs}a &= \partial^2_{rs}\ln a + \frac{1}{a^2}\partial_r a\partial_s a &(9)
\end{aligned}$$

$$\Rightarrow \quad K^{mn} = \frac{1}{2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial^2_{rs}\ln a + \mathcal{E}^{mn} \tag{10}$$

with $\mathcal{E}^{mn}$ being a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.

Note that $\mathcal{E}^{mn}$ is a symmetrical object in m, n. Indeed,

$$\mathcal{E}^{mn} = \frac{1}{2a^2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r a \partial_s a \tag{11}$$

$$\Rightarrow \qquad \mathcal{E}^{nm} = \frac{1}{2a^2}\left(a^{nm}a^{rs} - a^{nr}a^{ms}\right)\partial_r a \partial_s a \tag{12}$$

$$= \frac{1}{2a^2}\left(a^{mn}a^{sr} - a^{ns}a^{mr}\right)\partial_s a \partial_r a \tag{13}$$

$$= \mathcal{E}^{mn} \tag{14}$$

We now rewrite G^{mn}.
We have:

$$G^{mn} = a^{nk} G^{m}_{\cdot\ k} \tag{15}$$

$$G^{m}_{\cdot\ k} = R^{m}_{\cdot\ k} - \frac{1}{2}\delta^m_k R \tag{16}$$

$$R^{m}_{\cdot\ k} = a^{mp} R_{pk} \tag{17}$$

$$R = a^{pk} R_{pk} \tag{18}$$

And by 3.205.

$$R_{pk} = \frac{1}{2}\partial^2_{pk} \ln a - \partial_t \Gamma^t_{pk} + \mathcal{F}^{pk} \tag{19}$$

With $\mathcal{F}^{pk}$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only. Note that $\mathcal{F}^{pk}$ is a symmetrical object in p, k as R_{pk} is a symmetrical tensor in p, k. Hence by (15) to (19):

$$G^{mn} = a^{nk} R^{m}_{\cdot\ k} - \frac{1}{2} a^{nk} \delta^m_k R \tag{20}$$

$$= a^{nk} a^{mp} R_{pk} - \frac{1}{2} a^{nm} a^{pk} R_{pk} \tag{21}$$

$$= \left(a^{nk} a^{mp} - \frac{1}{2} a^{nm} a^{pk}\right) R_{pk} \tag{22}$$

$$= \left(a^{nk} a^{mp} - \frac{1}{2} a^{nm} a^{pk}\right)\left(\frac{1}{2}\partial^2_{pk} \ln a - \partial_t \Gamma^t_{pk} + \mathcal{F}^{pk}\right) \tag{23}$$

$$= \left(a^{nk} a^{mp} - \frac{1}{2} a^{nm} a^{pk}\right)\left(\frac{1}{2}\partial^2_{pk} \ln a - \partial_t \Gamma^t_{pk}\right) + \mathcal{H}^{mn} \tag{24}$$

$$= \left[\frac{1}{2} a^{nk} a^{mp} \partial^2_{pk} \ln a - a^{nk} a^{mp} \partial_t \Gamma^t_{pk} - \frac{1}{4} a^{nm} a^{pk} \partial^2_{pk} \ln a + \frac{1}{2} a^{nm} a^{pk} \partial_t \Gamma^t_{pk}\right] + \mathcal{H}^{mn} \tag{25}$$

with $\mathcal{H}^{mn}$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.

Note that $\mathcal{H}^{mn}$ is a symmetrical object in m, n . Indeed,

$$\mathcal{H}^{mn} = \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)\mathcal{F}^{pk} \tag{26}$$

$$\Rightarrow \qquad \mathcal{H}^{nm} = \left(a^{mk}a^{np} - \frac{1}{2}a^{mn}a^{pk}\right)\mathcal{F}^{pk} \tag{27}$$

$$= \left(a^{mp}a^{nk} - \frac{1}{2}a^{mn}a^{kp}\right)\mathcal{F}^{kp} \tag{28}$$

$$= \mathcal{H}^{mn} \tag{29}$$

Putting (10) and (2) together with $\mathcal{L}^{mn} = \mathcal{E}^{mn} + \mathcal{H}^{mn}$ we get,

$$T^{mn} = Q^{mn} + \mathcal{L}^{mn} \tag{30}$$

with $\mathcal{L}^{mn}$ a symmetrical object in m, n and depending only on terms in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and

$$Q^{mn} \equiv \begin{cases} \frac{1}{2}a^{nr}a^{ms}\partial^2_{rs}\ln a - a^{nr}a^{ms}\partial_t\Gamma^t_{rs} \\ \\ -\frac{1}{4}a^{nm}a^{rs}\partial^2_{rs}\ln a + \frac{1}{2}a^{nm}a^{rs}\partial_t\Gamma^t_{rs} \\ \\ +\frac{1}{2}a^{mn}a^{rs}\partial^2_{rs}\ln a - \frac{1}{2}a^{mr}a^{ns}\partial^2_{rs}\ln a \\ \end{cases} \tag{31}$$

$$= \frac{1}{4}a^{mn}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{mn}a^{rs} - a^{ms}a^{nr}\right)\partial_t\Gamma^t_{rs} \tag{32}$$

Note that

$$Q^{nm} = \frac{1}{4}a^{nm}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{nm}a^{rs} - a^{ns}a^{mr}\right)\partial_t\Gamma^t_{rs} \tag{33}$$

$$= \frac{1}{4}a^{mn}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{mn}a^{rs} - a^{nr}a^{ms}\right)\partial_t\Gamma^t_{sr} \tag{34}$$

$$= Q^{mn} \tag{35}$$

So Q^{mn} is a symmetrical object in m, n

Given that by (2.541.),

$$\partial^2_{rs}\ln a = \partial_s\partial_r\ln a \tag{36}$$

$$= \partial_s\left(a^{kt}\partial_r a_{kt}\right) \tag{37}$$

$$= \partial_s a^{kt}\partial_r a_{kt} + a^{kt}\partial^2_{rs}a_{kt} \tag{38}$$

(32) can be written as

$$Q^{mn} = \mathcal{P}^{mn} + \mathcal{Z}^{mn} \tag{39}$$

with $\mathcal{Z}^{mn} = \frac{1}{4}a^{mn}a^{rs}\partial_s a^{kt}\partial_r a_{kt}$ a symmetric object in m, n and depending only on terms in

$a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$

and

$$\mathcal{P}^{mn} = \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{rs} a_{kt} + \left(\frac{1}{2} a^{mn} a^{rs} - a^{ms} a^{nr} \right) \partial_t \Gamma^t_{rs} \tag{40}$$

Expanding the Christoffel symbols and playing with the dummy indices, gives

$$\begin{aligned} \mathcal{P}^{mn} &= \begin{cases} \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{rs} a_{kt} \\ \\ + \left(\frac{1}{2} a^{mn} a^{rs} - a^{ms} a^{nr} \right) \left([rs,k] \partial_t a^{tk} + a^{kt} \partial_t [rs,k] \right) \end{cases} && (41) \\ &= \frac{1}{2} \mathcal{V}^{mn} + \mathcal{A}^{mn} && (42) \end{aligned}$$

with

$$\mathcal{A}^{mn} = \left(\frac{1}{2} a^{mn} a^{rs} - a^{ms} a^{nr} \right) [rs,k] \partial_t a^{tk} \tag{43}$$

and

$$\begin{aligned} \frac{1}{2} \mathcal{V}_{mn} &= \begin{cases} \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{rs} a_{kt} \\ \\ + \left(\frac{1}{4} a^{mn} a^{rs} a^{kt} - \frac{1}{2} a^{ms} a^{nr} a^{kt} \right) \left(\partial^2_{tr} a_{sk} + \partial^2_{ts} a_{rk} - \partial^2_{tk} a_{rs} \right) \end{cases} && (44) \\ &= \begin{cases} \cancel{\frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{rs} a_{kt}} + \underbrace{\frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{tr} a_{sk} + \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{ts} a_{rk}}_{= \frac{1}{2} a^{mn} a^{rs} a^{kt} \partial^2_{tr} a_{sk}} - \cancel{\frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{tk} a_{rs}} \\ \\ - \frac{1}{2} a^{ms} a^{nr} a^{kt} \partial^2_{tr} a_{sk} - \frac{1}{2} a^{ms} a^{nr} a^{kt} \partial^2_{ts} a_{rk} + \frac{1}{2} a^{ms} a^{nr} a^{kt} \partial^2_{tk} a_{rs} \end{cases} && (45) \end{aligned}$$

giving

$$\begin{aligned} \mathcal{V}_{mn} &= a^{mn} a^{rs} a^{kt} \partial^2_{tr} a_{sk} - a^{ms} a^{nr} a^{kt} \partial^2_{tr} a_{sk} - a^{ms} a^{nr} a^{kt} \partial^2_{ts} a_{rk} + a^{ms} a^{nr} a^{kt} \partial^2_{tk} a_{rs} && (46) \\ &= \left(a^{mn} a^{rs} \partial^2_{tr} a_{sk} - a^{ms} a^{nr} \partial^2_{kr} a_{st} - a^{ms} a^{nr} \partial^2_{ts} a_{rk} + a^{ms} a^{nr} \partial^2_{tk} a_{rs} \right) a^{kt} && (47) \\ &= \left(a^{mn} a^{rs} \partial^2_{tr} a_{sk} - a^{ms} a^{nr} \partial^2_{tr} a_{sk} - a^{ms} a^{nr} \partial^2_{tr} a_{sk} + a^{ms} a^{nr} \partial^2_{tk} a_{rs} \right) a^{kt} && (48) \end{aligned}$$

Does $\mathcal{V}_{mn}$ vanish ?!

♦

1.10 p110 - Exercise 12

Prove that the quantities

$$G^{mn} + \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]$$

can be expressed in terms of the metric tensor and its first derivatives.

Let's define

$$K^{mn} \equiv \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right] \tag{1}$$

$$T^{mn} \equiv G^{mn} + K^{mn} \tag{2}$$

The strategy to proof this, is to separate in both terms of the expression, the parts that can be expressed in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and the parts of higher order differentiation. We begin with the second term.
As the covariant derivatives of the metric tensor vanish, we get

$$K^{mn} = \frac{1}{2a}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|rs} \tag{3}$$

$$= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|r}}_{=0}\right]_{|s} \tag{4}$$

$$= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|s} \tag{5}$$

$$= \frac{1}{2a}\left[\underbrace{a_{|rs}}_{=\partial^2_{rs}a+\Gamma^p_{sr}\partial_p a}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a_{|r}\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|s}}_{=0}\right] \tag{6}$$

Considering,

$$\partial^2_{rs}\ln a = \partial_s\left(\frac{1}{a}\partial_r a\right) \tag{7}$$

$$= \frac{1}{a}\partial^2_{rs}a - \frac{1}{a^2}\partial_r a\partial_s a \tag{8}$$

$$\Rightarrow \qquad \frac{1}{a}\partial^2_{rs}a = \partial^2_{rs}\ln a + \frac{1}{a^2}\partial_r a\partial_s a \tag{9}$$

$$\Rightarrow \quad K^{mn} = \frac{1}{2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial^2_{rs}\ln a + \mathcal{E}^{mn} \tag{10}$$

with $\mathcal{E}^{mn}$ being a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.

Note that $\mathcal{E}^{mn}$ is a symmetrical object in m, n. Indeed,

$$\mathcal{E}^{mn} = \frac{1}{2a^2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r a \partial_s a \tag{11}$$

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We now rewrite G^{mn}.
We have:

$$G^{mn} = a^{nk}G^{m}_{\cdot\ k} \tag{15}$$

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$$R^{m}_{\cdot\ k} = a^{mp}R_{pk} \tag{17}$$

$$R = a^{pk}R_{pk} \tag{18}$$

And by 3.205.

$$R_{pk} = \frac{1}{2}\partial^2_{pk}\ln a - \partial_t\Gamma^t_{pk} + \mathcal{F}^{pk} \tag{19}$$

With $\mathcal{F}^{pk} = \Gamma^i_{pj}\Gamma^j_{kj} - \frac{1}{2}\Gamma^i_{pk}\partial_i \ln a$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only. Note that $\mathcal{F}^{pk}$ is a symmetrical object in p, k as R_{pk} is a symmetrical tensor in p, k. Hence by (15) to (19):

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$$= a^{nk}a^{mp}R_{pk} - \frac{1}{2}a^{nm}a^{pk}R_{pk} \tag{21}$$

$$= \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)R_{pk} \tag{22}$$

$$= \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)\left(\frac{1}{2}\partial^2_{pk}\ln a - \partial_t\Gamma^t_{pk} + \mathcal{F}^{pk}\right) \tag{23}$$

$$= \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)\left(\frac{1}{2}\partial^2_{pk}\ln a - \partial_t\Gamma^t_{pk}\right) + \mathcal{H}^{mn} \tag{24}$$

$$= \left[\frac{1}{2}a^{nk}a^{mp}\partial^2_{pk}\ln a - a^{nk}a^{mp}\partial_t\Gamma^t_{pk} - \frac{1}{4}a^{nm}a^{pk}\partial^2_{pk}\ln a + \frac{1}{2}a^{nm}a^{pk}\partial_t\Gamma^t_{pk}\right] + \mathcal{H}^{mn} \tag{25}$$

with $\mathcal{H}^{mn}$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.

Note that $\mathcal{H}^{mn}$ is a symmetrical object in m, n . Indeed,

$$\mathcal{H}^{mn} = \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)\mathcal{F}^{pk} \tag{26}$$

$$\Rightarrow \qquad \mathcal{H}^{nm} = \left(a^{mk}a^{np} - \frac{1}{2}a^{mn}a^{pk}\right)\mathcal{F}^{pk} \tag{27}$$

$$= \left(a^{mp}a^{nk} - \frac{1}{2}a^{mn}a^{kp}\right)\mathcal{F}^{kp} \tag{28}$$

$$= \mathcal{H}^{mn} \tag{29}$$

Putting (10) and (2) together with $\mathcal{L}^{mn} = \mathcal{E}^{mn} + \mathcal{H}^{mn}$ we get,

$$T^{mn} = Q^{mn} + \mathcal{L}^{mn} \tag{30}$$

with $\mathcal{L}^{mn}$ a symmetrical object in m, n and depending only on terms in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and

$$Q^{mn} \equiv \begin{cases} \cancel{\tfrac{1}{2}a^{nr}a^{ms}\partial^2_{rs}\ln a} - a^{nr}a^{ms}\partial_t\Gamma^t_{rs} \\ \\ -\tfrac{1}{4}a^{nm}a^{rs}\partial^2_{rs}\ln a + \tfrac{1}{2}a^{nm}a^{rs}\partial_t\Gamma^t_{rs} \\ \\ +\tfrac{1}{2}a^{mn}a^{rs}\partial^2_{rs}\ln a - \cancel{\tfrac{1}{2}a^{mr}a^{ns}\partial^2_{rs}\ln a} \\ \end{cases} \tag{31}$$

$$= \frac{1}{4}a^{mn}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{mn}a^{rs} - a^{ms}a^{nr}\right)\partial_t\Gamma^t_{rs} \tag{32}$$

Note that

$$Q^{nm} = \frac{1}{4}a^{nm}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{nm}a^{rs} - a^{ns}a^{mr}\right)\partial_t\Gamma^t_{rs} \tag{33}$$

$$= \frac{1}{4}a^{mn}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{mn}a^{rs} - a^{nr}a^{ms}\right)\partial_t\Gamma^t_{sr} \tag{34}$$

$$= Q^{mn} \tag{35}$$

So Q^{mn} is a symmetrical object in m, n

Given that by (2.216.),

$$\partial^2_{rs}\ln a = \partial_s\partial_r\ln a \tag{36}$$

$$= \partial_s\left(a^{kt}\partial_r a_{kt}\right) \tag{37}$$

$$= \partial_s a^{kt}\partial_r a_{kt} + a^{kt}\partial^2_{rs}a_{kt} \tag{38}$$

(32) can be written as

$$Q^{mn} = \mathcal{P}^{mn} + \mathcal{Z}^{mn} \tag{39}$$

with $\mathcal{Z}^{mn} = \frac{1}{4}a^{mn}a^{rs}\partial_s a^{kt}\partial_r a_{kt}$ a symmetric object in m, n and depending only on terms in

$a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$

and

$$\mathcal{P}^{mn} = \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{rs} a_{kt} + \left(\frac{1}{2} a^{mn} a^{rs} - a^{ms} a^{nr} \right) \partial_t \Gamma^t_{rs} \tag{40}$$

Expanding the Christoffel symbols and playing with the dummy indices, gives

$$\mathcal{P}^{mn} = \begin{cases} \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{rs} a_{kt} \\ \\ + \left(\frac{1}{2} a^{mn} a^{rs} - a^{ms} a^{nr} \right) \left([rs,k] \partial_t a^{tk} + a^{kt} \partial_t [rs,k] \right) \end{cases} \tag{41}$$

$$= \frac{1}{2} \mathcal{V}^{mn} + \mathcal{A}^{mn} \tag{42}$$

with

$$\mathcal{A}^{mn} = \left(\frac{1}{2} a^{mn} a^{rs} - a^{ms} a^{nr} \right) [rs,k] \partial_t a^{tk} \tag{43}$$

and

$$\frac{1}{2} \mathcal{V}^{mn} = \begin{cases} \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{rs} a_{kt} \\ \\ + \left(\frac{1}{4} a^{mn} a^{rs} a^{kt} - \frac{1}{2} a^{ms} a^{nr} a^{kt} \right) \left(\partial^2_{tr} a_{sk} + \partial^2_{ts} a_{rk} - \partial^2_{tk} a_{rs} \right) \end{cases} \tag{44}$$

$$= \begin{cases} \cancel{\frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{rs} a_{kt}} \\ \\ + \underbrace{\frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{tr} a_{sk} + \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{ts} a_{rk}}_{= \frac{1}{2} a^{mn} a^{rs} a^{kt} \partial^2_{tr} a_{sk}} \cancel{- \frac{1}{4} a^{mn} a^{rs} a^{kt} \partial^2_{tk} a_{rs}} \\ \\ - \frac{1}{2} a^{ms} a^{nr} a^{kt} \partial^2_{tr} a_{sk} - \frac{1}{2} a^{ms} a^{nr} a^{kt} \partial^2_{ts} a_{rk} + \frac{1}{2} a^{ms} a^{nr} a^{kt} \partial^2_{tk} a_{rs} \end{cases} \tag{45}$$

giving

$$\mathcal{V}^{mn} = a^{mn} a^{rs} a^{kt} \partial^2_{tr} a_{sk} - a^{ms} a^{nr} a^{kt} \partial^2_{tr} a_{sk} - a^{ms} a^{nr} a^{kt} \partial^2_{ts} a_{rk} + a^{ms} a^{nr} a^{kt} \partial^2_{tk} a_{rs} \tag{46}$$

$$= \left(a^{mn} a^{rs} a^{kt} - a^{ms} a^{nr} a^{kt} - a^{mr} a^{ns} a^{kt} + a^{ms} a^{nk} a^{rt} \right) \partial^2_{rt} a_{ks} \tag{47}$$

$$= \left(a^{mn} a^{rs} a^{kt} - a^{mk} a^{nr} a^{st} - a^{mt} a^{ns} a^{kr} + a^{ms} a^{nk} a^{rt} \right) \partial^2_{rt} a_{ks} \tag{48}$$

$$\tag{49}$$

Does $\mathcal{V}_{mn}$ vanish ?!

♦

1.11 p110 - Exercise 12

Prove that the quantities

$$G^{mn} + \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]$$

can be expressed in terms of the metric tensor and its first derivatives.

Let's define

$$K^{mn} \equiv \frac{1}{2a}\frac{\delta^2}{\delta x^r \delta x^s}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right] \tag{1}$$

$$T^{mn} \equiv G^{mn} + K^{mn} \tag{2}$$

The strategy to proof this, is to separate in both terms of the expression, the parts that can be expressed in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and the parts of higher order differentiation. We begin with the second term.
As the covariant derivatives of the metric tensor vanish, we get

$$\begin{aligned}
K^{mn} &= \frac{1}{2a}\left[a\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|rs} &(3)\\
&= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|r}}_{=0}\right]_{|s} &(4)\\
&= \frac{1}{2a}\left[a_{|r}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\right]_{|s} &(5)\\
&= \frac{1}{2a}\left[\underbrace{a_{|rs}}_{=\partial^2_{rs}a + \Gamma^p_{sr}\partial_p a}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right) + a_{|r}\underbrace{\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)_{|s}}_{=0}\right] &(6)
\end{aligned}$$

Considering,

$$\begin{aligned}
\partial^2_{rs}\ln a &= \partial_s\left(\frac{1}{a}\partial_r a\right) &(7)\\
&= \frac{1}{a}\partial^2_{rs}a - \frac{1}{a^2}\partial_r a\partial_s a &(8)\\
\Rightarrow \qquad \frac{1}{a}\partial^2_{rs}a &= \partial^2_{rs}\ln a + \frac{1}{a^2}\partial_r a\partial_s a &(9)
\end{aligned}$$

$$\Rightarrow \quad K^{mn} = \frac{1}{2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial^2_{rs}\ln a + \mathcal{E}^{mn} \tag{10}$$

with $\mathcal{E}^{mn}$ being a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.

Note that $\mathcal{E}^{mn}$ is a symmetrical object in m, n. Indeed,

$$\mathcal{E}^{mn} = \frac{1}{2a^2}\left(a^{mn}a^{rs} - a^{mr}a^{ns}\right)\partial_r a \partial_s a \tag{11}$$

$$\Rightarrow \qquad \mathcal{E}^{nm} = \frac{1}{2a^2}\left(a^{nm}a^{rs} - a^{nr}a^{ms}\right)\partial_r a \partial_s a \tag{12}$$

$$= \frac{1}{2a^2}\left(a^{mn}a^{sr} - a^{ns}a^{mr}\right)\partial_s a \partial_r a \tag{13}$$

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We now rewrite G^{mn}.
We have:

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$$R = a^{pk}R_{pk} \tag{17}$$

And by 3.205.

$$R_{pk} = \frac{1}{2}\partial^2_{pk}\ln a - \partial_t\Gamma^t_{pk} + \mathcal{F}^{pk} \tag{18}$$

With $\mathcal{F}^{pk} = \Gamma^i_{pj}\Gamma^j_{kj} - \frac{1}{2}\Gamma^i_{pk}\partial_i \ln a$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only. Note that $\mathcal{F}^{pk}$ is a symmetrical object in p, k as R_{pk} is a symmetrical tensor in p, k. Hence by (15) to (19):

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$$= a^{nk}a^{mp}R_{pk} - \frac{1}{2}a^{nm}a^{pk}R_{pk} \tag{20}$$

$$= \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)R_{pk} \tag{21}$$

$$= \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)\left(\frac{1}{2}\partial^2_{pk}\ln a - \partial_t\Gamma^t_{pk} + \mathcal{F}^{pk}\right) \tag{22}$$

$$= \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)\left(\frac{1}{2}\partial^2_{pk}\ln a - \partial_t\Gamma^t_{pk}\right) + \mathcal{H}^{mn} \tag{23}$$

$$= \left[\frac{1}{2}a^{nk}a^{mp}\partial^2_{pk}\ln a - a^{nk}a^{mp}\partial_t\Gamma^t_{pk} - \frac{1}{4}a^{nm}a^{pk}\partial^2_{pk}\ln a + \frac{1}{2}a^{nm}a^{pk}\partial_t\Gamma^t_{pk}\right] + \mathcal{H}^{mn} \tag{24}$$

with $\mathcal{H}^{mn}$ a function in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ only.
Note that $\mathcal{H}^{mn}$ is a symmetrical object in m, n . Indeed,

$$\mathcal{H}^{mn} = \left(a^{nk}a^{mp} - \frac{1}{2}a^{nm}a^{pk}\right)\mathcal{F}^{pk} \tag{25}$$

$$\Rightarrow \qquad \mathcal{H}^{nm} = \left(a^{mk}a^{np} - \frac{1}{2}a^{mn}a^{pk}\right)\mathcal{F}^{pk} \tag{26}$$

$$= \left(a^{mp}a^{nk} - \frac{1}{2}a^{mn}a^{kp}\right)\mathcal{F}^{kp} \tag{27}$$

$$= \mathcal{H}^{mn} \tag{28}$$

Putting (10) and (2) together with $\mathcal{L}^{mn} = \mathcal{E}^{mn} + \mathcal{H}^{mn}$ we get,

$$T^{mn} = Q^{mn} + \mathcal{L}^{mn} \tag{29}$$

with $\mathcal{L}^{mn}$ a symmetrical object in m, n and depending only on terms in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$ and

$$Q^{mn} \equiv \begin{cases} \cancel{\frac{1}{2}a^{nr}a^{ms}\partial^2_{rs}\ln a} - a^{nr}a^{ms}\partial_t\Gamma^t_{rs} \\ -\frac{1}{4}a^{nm}a^{rs}\partial^2_{rs}\ln a + \frac{1}{2}a^{nm}a^{rs}\partial_t\Gamma^t_{rs} \\ +\frac{1}{2}a^{mn}a^{rs}\partial^2_{rs}\ln a - \cancel{\frac{1}{2}a^{mr}a^{ns}\partial^2_{rs}\ln a} \end{cases} \tag{30}$$

$$= \frac{1}{4}a^{mn}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{mn}a^{rs} - a^{ms}a^{nr}\right)\partial_t\Gamma^t_{rs} \tag{31}$$

Note that

$$Q^{nm} = \frac{1}{4}a^{nm}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{nm}a^{rs} - a^{ns}a^{mr}\right)\partial_t\Gamma^t_{rs} \tag{32}$$

$$= \frac{1}{4}a^{mn}a^{rs}\partial^2_{rs}\ln a + \left(\frac{1}{2}a^{mn}a^{rs} - a^{nr}a^{ms}\right)\partial_t\Gamma^t_{sr} \tag{33}$$

$$= Q^{mn} \tag{34}$$

So Q^{mn} is a symmetrical object in m, n

Given that by (2.216.),

$$\partial^2_{rs}\ln a = \partial_s\partial_r\ln a \tag{35}$$

$$= \partial_s\left(a^{kt}\partial_r a_{kt}\right) \tag{36}$$

$$= \partial_s a^{kt}\partial_r a_{kt} + a^{kt}\partial^2_{rs}a_{kt} \tag{37}$$

(32) can be written as

$$Q^{mn} = \mathcal{P}^{mn} + \mathcal{Z}^{mn} \tag{38}$$

with $\mathcal{Z}^{mn} = \frac{1}{4}a^{mn}a^{rs}\partial_s a^{kt}\partial_r a_{kt}$ a symmetric object in m, n and depending only on terms in $a_{ij}, a^{ij}, \partial_k a_{ij}, \partial_k a^{ij}$

and

$$\mathcal{P}^{mn} = \frac{1}{4}a^{mn}a^{rs}a^{kt}\partial^2_{rs}a_{kt} + \left(\frac{1}{2}a^{mn}a^{rs} - a^{ms}a^{nr}\right)\partial_t\Gamma^t_{rs} \tag{39}$$

Expanding the Christoffel symbols and playing with the dummy indices, gives

$$\mathcal{P}^{mn} = \begin{cases} \frac{1}{4}a^{mn}a^{rs}a^{kt}\partial^2_{rs}a_{kt} \\ \\ +\left(\frac{1}{2}a^{mn}a^{rs} - a^{ms}a^{nr}\right)\left([rs,k]\partial_t a^{tk} + a^{kt}\partial_t[rs,k]\right) \end{cases} \tag{40}$$

$$= \frac{1}{2}\mathcal{V}^{mn} + \mathcal{A}^{mn} \tag{41}$$

with

$$\mathcal{A}^{mn} = \left(\frac{1}{2}a^{mn}a^{rs} - a^{ms}a^{nr}\right)[rs,k]\partial_t a^{tk} \tag{42}$$

and

$$\frac{1}{2}\mathcal{V}^{mn} = \begin{cases} \frac{1}{4}a^{mn}a^{rs}a^{kt}\partial^2_{rs}a_{kt} \\ \\ +\left(\frac{1}{4}a^{mn}a^{rs}a^{kt} - \frac{1}{2}a^{ms}a^{nr}a^{kt}\right)\left(\partial^2_{tr}a_{sk} + \partial^2_{ts}a_{rk} - \partial^2_{tk}a_{rs}\right) \end{cases} \tag{43}$$

$$= \begin{cases} \cancel{\frac{1}{4}a^{mn}a^{rs}a^{kt}\partial^2_{rs}a_{kt}} \\ \\ +\underbrace{\frac{1}{4}a^{mn}a^{rs}a^{kt}\partial^2_{tr}a_{sk} + \frac{1}{4}a^{mn}a^{rs}a^{kt}\partial^2_{ts}a_{rk}}_{=\frac{1}{2}a^{mn}a^{rs}a^{kt}\partial^2_{tr}a_{sk}} \cancel{-\frac{1}{4}a^{mn}a^{rs}a^{kt}\partial^2_{tk}a_{rs}} \\ \\ -\frac{1}{2}a^{ms}a^{nr}a^{kt}\partial^2_{tr}a_{sk} - \frac{1}{2}a^{ms}a^{nr}a^{kt}\partial^2_{ts}a_{rk} + \frac{1}{2}a^{ms}a^{nr}a^{kt}\partial^2_{tk}a_{rs} \end{cases} \tag{44}$$

giving

$$\mathcal{V}^{mn} = a^{mn}a^{rs}a^{kt}\partial^2_{tr}a_{sk} - a^{ms}a^{nr}a^{kt}\partial^2_{tr}a_{sk} - a^{ms}a^{nr}a^{kt}\partial^2_{ts}a_{rk} + a^{ms}a^{nr}a^{kt}\partial^2_{tk}a_{rs} \tag{45}$$

$$= \left(a^{mn}a^{rs}a^{kt} - a^{ms}a^{nr}a^{kt} - a^{mr}a^{ns}a^{kt} + a^{ms}a^{nk}a^{rt}\right)\partial^2_{rt}a_{ks} \tag{46}$$

$$= \left(a^{mn}a^{rs}a^{kt} - a^{mk}a^{nr}a^{st} - a^{mt}a^{ns}a^{kr} + a^{ms}a^{nk}a^{rt}\right)\partial^2_{rt}a_{ks} \tag{47}$$

$$\tag{48}$$

Does $\mathcal{V}_{mn}$ vanish ?!

♦