Tensor Calculus
J.L. Synge and A.Shild (Dover Publication)
Solutions to exercises

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Remarks and warnings

Some notation conventions

$$\partial_r a_{mn} \equiv \frac{\partial a_{mn}}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \qquad \text{Christoffel symbol of the second kind}$$

Contents

Spaces and Tensors

1.1 p5-exercise

The parametric equations of a hypersurface in V_n are

$$
\begin{aligned}
x^1 &= a\cos\left(u^1\right)\\
x^2 &= a\sin\left(u^1\right)\cos\left(u^2\right)\\
x^3 &= a\sin\left(u^1\right)\sin\left(u^2\right)\cos\left(u^3\right)\\
&\vdots\\
x^{N-1} &= a\sin\left(u^1\right)\sin\left(u^2\right)\sin\left(u^3\right)\dots\sin\left(u^{N-2}\right)\cos\left(u^{N-1}\right)\\
x^N &= a\sin\left(u^1\right)\sin\left(u^2\right)\sin\left(u^3\right)\dots\sin\left(u^{N-2}\right)\sin\left(u^{N-1}\right)
\end{aligned}
$$

where a is a constant. Find the single equation of the hyperspace in the form 1.103.

We have:

$$
\begin{aligned}
(x^N)^2 + (x^{N-1})^2 &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i)(\cos^2(u^{N-1}) + \sin^2(u^{N-1}))\\
&= a^2 \prod_{i=1}^{N-2} \sin^2(u^i)\\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i)\sin^2(u^{N-2})\\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i)(1-\cos^2(u^{N-2})\\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - a^2 \prod_{i=1}^{N-3} \sin^2(u^i)\cos^2(u^{N-2})\\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - (x^{N-2})^2
\end{aligned}
$$

giving

$$(x^N)^2 + (x^{N-1})^2 + (x^{N-2})^2 = a^2 \prod_{i=1}^{N-3} \sin^2(u^i)$$

In general, by recursion

$$\sum_{i=0}^{k} (x^{N-i})^2 = a^2 \prod_{i=1}^{N-k-1} \sin^2(u^i) \quad (k \le N-2)$$

be k = N - 2 (N - k - 1 = 1) and in the left term put j = N - i (j goes from 2 to N), we get

$$\begin{aligned}\sum_{j=2}^{N}(x^j)^2 &= a^2\prod_{i=1}^{1}\sin^2(u^i)\\ &= a^2(1-\cos^2(u^1))\\ &= a^2-(x^1)^2\end{aligned}$$

and thus the equation of the hyperspace is given by

$$\sum_{j=1}^{N}(x^j)^2 - a^2 = 0$$

Determine whether the points $(\frac{1}{2}a, 0, 0, ...0)$, $(0, 0, ..., 0, 2a)$ lie on the same or opposite sides of the hyperspace.

For $(\frac{1}{2}a, 0, 0, ...0)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = -\frac{3a^2}{4} < 0$ and for $(0, 0, ..., 0, 2a)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = \frac{3a^2}{4} > 0$.
So the points lie on opposite sides of the hyperplane.

◆

1.2 p6-exercise

Let U_2 and W_2 be subspaces of V_N. Show that if N = 3 they will in general intersect in a curve; if N = 4 they will in general intersect in a finite number of points; and if $N > 4$ they will not in general intersect at all.

We have (see 1.102 page 5): $x^r = f^r(u^1, u^2, ..., u^M) \quad (r = 1, 2, ..., N)$
Case N=3:

For U_2 we have:

$$x^r = \phi^r(u^1, u^2) \quad (r = 1, 2, 3)$$

For W_2 we have:

$$x^r = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

The intersect of the two hyperplanes is given by the N equations:

$$\phi^r(u^1, u^2) = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

So we have 3 equations in 4 unknown u^1, u^2, v^1, v^2 and can choose (fix) one e.g. u^1 and solve the set of equations for u^2, v^1, v^2 giving

$$x^r = \theta^r(u^1) \quad (r = 1, 2, 3)$$

This is an equation of a curve in space (1 parameter equation)
Case N=4:
Using the same reasoning as with N=3, we get 4 equations for 4 unknown u^1, u^2, v^1, v^2.
Provided that the set of equation does not degenerate, these 4 equations will determine u^1, u^2, v^1, v^2 without any degree of freedom. So we get points as solutions. This solution does not to be unique, e.g. if the $\phi^r(u^1, u^2)$ are quadratic form, then the solutions

$$(u^1, u^2, v^1, v^2)$$

$$(-u^1, u^2, v^1, v^2)$$

$$(u^1, -u^2, v^1, v^2)$$

$$(-u^1, -u^2, v^1, v^2)$$

are possible.
Case N=5: There are more equations than variables. If the equations are not linear dependent, no solutions will be found.

◆

1.3 p8-exercise

Show that $(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = 3a_{rst}x^r x^s x^t$

$(a_{rst}+a_{str}+a_{srt})x^r x^s x^t = a_{rst}x^r x^s x^t + a_{rts}x^r x^s x^t + a_{srt}x^r x^s x^t$ so by just renaming the dummy indices e.g. for the second term $r \mapsto s$, $s \mapsto t$ and $t \mapsto r$ we get the desired result.

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1.4 p8-exercise

If $\phi = a_{rs}x^r x^s$, show that

$$\frac{\partial \phi}{\partial x^r} = (a_{rs} + a_{sr})x^s \quad , \quad \frac{\partial^2 \phi}{\partial x^r \partial x^s} = a_{rs} + a_{sr}$$

Simplify these expressions in the case where $a^{rs} = a^{sr}$

We have

$$\begin{aligned}
\frac{\partial \phi}{\partial x^t} &= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs}\frac{\partial x^r}{\partial x^t} x^s + a_{rs}x^r \frac{\partial x^s}{\partial x^t} && (1)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs}\delta^r_t x^s + a_{rs}x^r \delta^s_t && (2)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts}x^s + a_{rt}x^r && (3)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts}x^s + a_{st}x^s \quad \text{(rename dummy variable in third term)} && (4)\\
&= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + (a_{ts} + a_{st})x^s && (5)
\end{aligned}$$

Replace x^t by x^r, we get

$$\frac{\partial \phi}{\partial x^r} = \frac{\partial a_{rs}}{\partial x^r} x^r x^s + (a_{rs} + a_{sr})x^s \tag{6}$$

So the asked expression is only true if a_{rs} is not a function of the x^s. Assuming that a_{rs} is not a function of the x^s, take the partial derivative of (6) with respect to x^t, we get

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x^r \partial x^t} &= (a_{rs} + a_{sr})\frac{\partial x^s}{\partial x^t} && (7)\\
&= (a_{rs} + a_{sr})\delta^s_t && (8)\\
&= (a_{rt} + a_{tr}) && (9)
\end{aligned}$$

Replace x^t by x^s, and we get the proposed expression.

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1.5 p8-clarification on expression 1.210

$$\boxed{\frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} = 0}$$

From 1.209:

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s} + \frac{\partial x^r}{\partial x'^n}\frac{\partial^2 x'^n}{\partial x^p \partial x^s} = 0 \tag{1}$$

multiply (1) with $\frac{\partial x'^q}{\partial x^r}$

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} + \frac{\partial x^r}{\partial x'^n}\frac{\partial^2 x'^n}{\partial x^p \partial x^s}\frac{\partial x'^q}{\partial x^r} = 0 \tag{2}$$

$$\Leftrightarrow \frac{\partial x^r}{\partial x'^n}\frac{\partial^2 x'^n}{\partial x^p \partial x^s}\frac{\partial x'^q}{\partial x^r} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} = 0 \tag{3}$$

$$\text{in the first term we get} \qquad \frac{\partial x'^q}{\partial x^r}\frac{\partial x^r}{\partial x'^n} = \frac{\partial x'^q}{\partial x'^n} = \delta^q_n \tag{4}$$

(3) becomes

$$\frac{\partial^2 x'^n}{\partial x^p \partial x^s}\delta^q_n + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} = 0 \tag{5}$$

$$\Leftrightarrow \frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^p}\frac{\partial x'^n}{\partial x^s}\frac{\partial x'^q}{\partial x^r} = 0 \tag{6}$$

◆

1.6 p9-exercise

If A_s^r are the elements of a determinant A, and B_s^r the elements of a determinant B, show that the element of the product determinant is $A_n^r B_s^n$. Hence show that the product of the two jacobians

$$J = \left|\frac{\partial x^r}{\partial x'^s}\right|, \quad J' = \left|\frac{\partial x'^r}{\partial x^s}\right|$$

is unity.

Remark: Some nitpick about the formulation: A_s^r are not the elements of a determinant A, but elements of the matrix A which gives $\det\{A\}$ provided that A is square (which is not explicitly mentioned.). The same remark for B and $A_n^r B_s^n$.

Be A_k^i the elements of matrix A and B_j^k the elements of matrix B and C = A.B the resulting matrix of the multiplication of A and B, then

$$C_j^i = A_k^i B_j^k$$

are the elements of matrix C. Now, put $A_k^i = \frac{\partial x^i}{\partial x'^k}$ and $B_j^k = \frac{\partial x'^k}{\partial x^j}$ then,

$$C_j^i = A_k^i B_j^k \tag{1}$$

$$= \frac{\partial x^i}{\partial x'^k}\frac{\partial x'^k}{\partial x^j} \tag{2}$$

$$= \delta_k^i \tag{3}$$

So $C = JJ'$ becomes the unity matrix.

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1.7 p11-exercise

Show that a finite contravariant vector determines the ratios of the components of an infinitesimal displacement. (Consider the transformation of the equation $dx^r = \theta T^r$, where θ is an arbitrary factor which does not change under the transformation. Alternatively, show that the equations $T^r dx^s - T^s x^r = 0$ remain true when we transform the coordinates.)

Be T^q a contravariant vector.

$$T'^q = T^r \frac{\partial x'^q}{\partial x^r} \quad \text{(by definition)} \tag{1}$$

Be θ a small infinitesimal factor invariant for a coordinate transformation, define

$$dx^r = \theta T^r \tag{2}$$

$$\tag{3}$$

then

$$\frac{dx^r}{dx^s} = \frac{\theta T^r}{\theta T^s} \tag{4}$$

$$\Leftrightarrow T^s dx^r - T^r dx^s = 0 \tag{5}$$

Alternatively, multiply (5) with $\partial_{x^r} x'^q$, then

$$\frac{\partial x'^q}{\partial x^r} dx^r T^s - \frac{\partial x'^q}{\partial x^r} dx^s T^r = 0 \tag{6}$$

$$\Leftrightarrow \frac{\partial x'^q}{\partial x^r} dx^r T^s - dx^s T'^q = 0 \quad \text{(use (1) in the second term)} \tag{7}$$

$$\Leftrightarrow dx'^q T^s - dx^s T'^q = 0 \tag{8}$$

$$\tag{9}$$

Multiply (8) with $\partial_{x^s} x'^p$, then

$$dx'^q T^s \partial_{x^s} x'^p - dx^s T'^q \partial_{x^s} x'^p = 0 \tag{10}$$

$$\Leftrightarrow \quad T'^p dx^q - T'^q dx'^p = 0 \quad \text{(use (1) in the first term)} \tag{11}$$

and thus

$$\frac{dx'^q}{dx'^p} = \frac{T'^q}{T'^p}$$

◆

1.8 p12-exercise

Write down the equation of transformation, analogous to 1.305, of a contravariant tensor of the third order. Solve the equation so as to express the unprimed components in terms of the primed components.

Be

$$T'^{uvw} = T^{rst}\frac{\partial x'^u}{\partial x^r}\frac{\partial x'^v}{\partial x^s}\frac{\partial x'^w}{\partial x^t} \quad \text{(by definition)} \tag{1}$$

a contravariant vector.
Multiply (1) by $\frac{\partial x^n}{\partial x'^u}$

$$T'^{uvw}\frac{\partial x^n}{\partial x'^u} = T^{rst}\frac{\partial x'^u}{\partial x^r}\frac{\partial x^n}{\partial x'^u}\frac{\partial x'^v}{\partial x^s}\frac{\partial x'^w}{\partial x^t} \tag{2}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u} = T^{rst}\delta^n_r\frac{\partial x'^v}{\partial x^s}\frac{\partial x'^w}{\partial x^t} \tag{3}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u} = T^{nst}\frac{\partial x'^v}{\partial x^s}\frac{\partial x'^w}{\partial x^t} \tag{4}$$

Multiply (4) by $\frac{\partial x^m}{\partial x'^v}$

$$T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v} = T^{nst}\frac{\partial x'^v}{\partial x^s}\frac{\partial x^m}{\partial x'^v}\frac{\partial x'^w}{\partial x^t} \tag{5}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v} = T^{nst}\delta^m_s\frac{\partial x'^w}{\partial x^t} \tag{6}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v} = T^{nmt}\frac{\partial x'^w}{\partial x^t} \tag{7}$$

Multiply (7) by $\frac{\partial x^p}{\partial x'^w}$

$$T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v}\frac{\partial x^p}{\partial x'^w} = T^{nmt}\frac{\partial x'^w}{\partial x^t}\frac{\partial x^p}{\partial x'^w} \tag{8}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v}\frac{\partial x^p}{\partial x'^w} = T^{nmt}\delta^p_t \tag{9}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v}\frac{\partial x^p}{\partial x'^w} = T^{nmp} \tag{10}$$

Giving

$$T^{nmp} = T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v}\frac{\partial x^p}{\partial x'^w}$$

◆

1.9 p14-exercise

For a transformation from on set of rectangular Cartesian coordinates to another in Euclidean 3-space, show that the law of transformation of a contravariant vector is precisely the same as that of a covariant vector. Can this statements be extended to cover tensor of higher orders?

We have to prove that, given that,

$$T^{,i} = T^j \frac{\partial x^{,i}}{\partial x^j} \quad T^{,}_i = T_j \frac{\partial x^j}{\partial x^{,i}}$$

that also

$$T^{,i} = T^j \frac{\partial x^j}{\partial x^{,i}} \quad T^{,}_i = T_j \frac{\partial x^{,i}}{\partial x^j} \tag{1}$$

$$\Leftrightarrow \frac{\partial x^j}{\partial x^{,i}} = \frac{\partial x^{,i}}{\partial x^j} \tag{2}$$

Be

$$\hat{e^{,i}} = g^i_k \hat{e^k} \quad \text{and } \hat{e^i} = h^i_k \hat{e^{,k}} \tag{3}$$

the transformation rules from one set of (rectangular Cartesian) basis vectors to another set of (rectangular Cartesian) basis vectors. Then,

$$\langle \hat{e^{,i}}, \hat{e^{,j}} \rangle = \langle g^i_k \hat{e^k}, g^j_k \hat{e^k} \rangle \text{ and } \langle \hat{e^i}, \hat{e^j} \rangle = \langle h^i_k \hat{e^{,k}}, h^j_k \hat{e^{,k}} \rangle \tag{4}$$

$$\Leftrightarrow \delta^p_j = g^p_k g^j_k \text{ and } \delta^p_j = h^p_k h^j_k \tag{5}$$

$$\tag{6}$$

Be $\vec{v}$ a random vector in the Euclidean space,

$$\vec{v} = x^j \hat{e^j} = x^{,j} \hat{e^{,j}} \tag{7}$$

then

$$(3) \;\Rightarrow x^j \hat{e^j} = x^j h^j_k \hat{e^{,k}} \text{ and } x^{,j} \hat{e^{,j}} = x^{,j} g^j_k \hat{e^k} \tag{8}$$

$$\Rightarrow x^{,j} = x^m h^m_j \text{ and } x^m = x^{,j} g^j_m \tag{9}$$

$$\Rightarrow x^{,j} = x^{,i} g^i_m h^m_j \text{ and } x^m = x^k h^k_j g^j_m \tag{10}$$

$$\Rightarrow \delta^p_j = g^p_k h^k_j \text{ and } \delta^p_j = g^k_j h^p_k \tag{11}$$

$$(5) \;\Rightarrow g^p_k g^j_k = g^p_k h^k_j \text{ and } h^p_k h^j_k = g^k_j h^p_k \tag{12}$$

$$\Rightarrow g^j_k = h^k_j \text{ and } h^j_k = g^k_j \tag{13}$$

From (9)

$$x^j = x^{,m} g_j^m \text{ and } x^{,k} = x^n h_k^n \tag{14}$$

$$\Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^n}{\partial x^j} h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \frac{\partial x^{,m}}{\partial x^{,k}} g_j^m \tag{15}$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = \delta_j^n h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \delta_k^m g_j^m \tag{16}$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = h_k^j \text{ and } \frac{\partial x^j}{\partial x^{,k}} = g_j^k \tag{17}$$

$$(13) \;\Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^j}{\partial x^{,k}} \tag{18}$$

So (13) matches (2), proving the assertion.

Can this statements be extended to cover tensor of higher orders? Consider

$$T^{,i,j,\ldots,n} = T^{r,s,\ldots w} \frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \cdots \frac{\partial x^{,n}}{\partial x^w} \text{ and } T^{r,s,\ldots w} = T^{,i,j,\ldots,n} \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \cdots \frac{\partial x^w}{\partial x^{,n}}$$

Using the same reasoning as in (1) to (2) we need

$$\frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \cdots \frac{\partial x^{,n}}{\partial x^w} = \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \cdots \frac{\partial x^w}{\partial x^{,n}}$$

As the conclusion (18) is independent of the order of the tensor, it is obvious that the above equality yields. Hence, the answer is YES.

◆

1.10 p16-exercise

In a space of 4 dimensions, the tensor A_{rst} is skew-symmetric in the last pair of suffixes. Show that only 24 of the 64 components may be chosen arbitrarily. If the further condition $A_{rst} + A_{str} + A_{trs} = 0$ is imposed, show that that only 20 components may be chosen arbitrarily.

We have, as A is skew-symmetric in the last pair of suffixes

$$A_{rst} = -A_{rts} \Rightarrow s = t \colon A_{rst} = 0$$

So, for each r (4 possible choices as N = 4) we have 4x4/2 - 4 = 6 degrees of freedom. [we have the term 4x4/2 as the tensor is (skew-)symmetric, e.g. once we choose element a_{12}, then a_{21} is also known. The term -4 takes into account the diagonal element which are 0 and thus cannot be chosen.]
So, we have 4x6 = 24 degrees of freedom.
What about the supplementary constraint $A_{rst} + A_{str} + A_{trs} = 0$:
Consider the two possible excluding cases:

i) $r = s \neq t$ ($\Leftrightarrow r = t \neq s$)
This case gives - without the additional constraint (1) - 4x(4x3/2-4) = 8 degrees of freedom. Does the constraint (1) reduces this degree of freedom?
We have,

$$A_{rst} + A_{str} + A_{trs} = 0 \tag{1}$$

$$\Rightarrow \underbrace{A_{rrt} + A_{rtr}}_{= 0 \text{ (non-diagonal terms)}} + \underbrace{A_{trr}}_{= 0 \text{ (diagonal terms)}} = 0 \tag{2}$$

So, no additional constraints are added by (1) to the restriction i) and the DOF remains 8.

ii) $t \neq r \neq s \neq t$
This case means that we have to choose a set of 3 elements out of 4 elements without repetition. This a *variation* of 3 elements out of 4.

$$V_k^n = \frac{n!}{(n-k)!} \text{ giving } V_3^4 = \frac{4!}{(4-3)!} = 24$$

The constraint (1) gives us 24 equations but as $A_{rst} = -A_{rts}$ only 12 equations have to be considered.
So, with the additional constraints the DOF becomes 24-12 = 12.
As i) and ii) are independent and excluding events we can add the DOF of both events and we get 8+12 = 20 DOF.

◆

1.11 p16-exercise

If A^{rs} is skew-symmetric and B_{rs} is symmetric, prove that $A^{rs}B_{rs} = 0$. Hence show that the quadratic form $a_{ij}x^i x^j$ is unchanged if a_{ij} is replaced by its symmetric part.

We can split the summation $A^{rs}B_{rs}$ in three subsummations:

$$
\begin{aligned}
A^{rs}B_{rs} =& A^{rs}B_{rs}|_{r=s} & (1)\\
&+ A^{rs}B_{rs}|_{r>s} & (2)\\
&+ A^{rs}B_{rs}|_{r<s} & (3)
\end{aligned}
$$

We have:
(1) = 0 as $A^{kk} = 0$ (skew-symmetric)
(2)+(3) = $A^{rs}B_{rs}|_{r>s} \ + A^{rs}B_{rs}|_{r<s}$
As $A^{rs} = -A^{sr}$ and $B^{rs} = B^{sr}$ we can write (2)+(3) as :

$$A^{rs}B_{rs}|_{r>s} \ + (-A^{sr})B_{sr}|_{r>s} = 0$$

So, $A^{rs}B_{rs} = 0$

Consider the quadratic form $\phi = a_{ij}x^i x^j$
Be $A_{ij} = (a_{ij})$ and $B_{ij} = (x^i x^j)$, then it is obvious that B_{ij} is symmetric and that $C_{ij} = -A_{ij}$ is the form where $-a_{ij}$ is replaced by its symmetric part (skew-symmetric). Hence $\phi = a_{ij}x^i x^j = a_{ij}b^{ij} = 0$ and so is $\phi = c_{ij}b^{ij} = 0$

◆

1.12 p18-exercise

What are the values (in a space of N dimensions) of the folllowing contractions formed from the Kronecker delta?

$$\delta^m_m, \delta^m_n \delta^n_m, \delta^m_n \delta^n_r \delta^r_m$$

$$\delta^m_m = N \tag{1}$$

$$\delta^m_n \delta^n_m = \delta^m_m = N \tag{2}$$

$$\delta^m_n \delta^n_r \delta^r_m = \delta^m_n \delta^n_m = \delta^m_m = N \tag{3}$$

◆

1.13 p19-exercise

If X^r, Y^r are arbitrary contravariant vectors and $a_{rs}X^rY^s$ is an invariant, then a_{rs} are the components of a covariant tensor of the second order.

We have to prove that

$$a'_{rs} = a_{ij}\frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^s} \text{ or } a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{1}$$

$a_{rs}X^rY^s$ is an invariant, means

$$a'_{rs}X'^rY'^s = a_{rs}X^rY^s \tag{2}$$

As X^r, Y^r are arbitrary contravariant vectors, we have

$$X'^r = X^i\frac{\partial x'^r}{\partial x^i} \text{ and } Y'^s = Y^j\frac{\partial x'^s}{\partial x^j} \tag{3}$$

(3) in (2) gives

$$a'_{rs}X^i\frac{\partial x'^r}{\partial x^i}Y^j\frac{\partial x'^s}{\partial x^j} = a_{rs}X^rY^s \tag{4}$$

$$\Leftrightarrow a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j}X^iY^j = a_{ij}X^iY^j \tag{5}$$

$$\Leftrightarrow (a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij})X^iY^j = 0 \tag{6}$$

As X^r, Y^r are arbitrary contravariant vectors, we conclude that

$$a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij} = 0 \tag{7}$$

$$\Leftrightarrow a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{8}$$

(8) = (1): OK

◆

1.14 p19-exercise

If X_{rs} is an arbitrary covariant tensor of the second order, and $A_r^{mn} X_{mn}$ is a covariant vector, then A_r^{mn} has the mixed tensor character indicated by the positions of its suffixes

We have to prove that

$$A'^{vw}_r = A^{mn}_k \frac{\partial x^k}{\partial x'^r} \frac{\partial x'^v}{\partial x^m} \frac{\partial x'^w}{\partial x^n} \tag{1}$$

We have

$$P_r = A^{mn}_r X_{mn} \tag{2}$$

is a covariant vector

$$\Rightarrow P'_r = A^{mn}_k X_{mn} \frac{\partial x^k}{\partial x'^r} \tag{3}$$

but X_{mn} is a covariant tensor

$$\Rightarrow X_{mn} = X'_{ps} \frac{\partial x'^p}{\partial x^m} \frac{\partial x'^s}{\partial x^n} \tag{4}$$

So (4) in (3) gives

$$P'_r = A^{mn}_k X'_{ps} \frac{\partial x'^p}{\partial x^m} \frac{\partial x'^s}{\partial x^n} \frac{\partial x^k}{\partial x'^r} \tag{5}$$

$$\Leftrightarrow P'_r = \underbrace{A^{mn}_k \frac{\partial x'^p}{\partial x^m} \frac{\partial x'^s}{\partial x^n} \frac{\partial x^k}{\partial x'^r}}_{(*)} X'_{ps} \tag{6}$$

Putting (*) as $A'^{ps}_r = A^{mn}_k \frac{\partial x'^p}{\partial x^m} \frac{\partial x'^s}{\partial x^n} \frac{\partial x^k}{\partial x'^r}$ we see that (6) has the form (2) and that A'^{ps}_r obeys the rule of a mixed tensor (1).

◆

1.15 p21-exercise

If A_{rs} is a skew-symmetric covariant tensor, prove that B_{rst} defined as

$$B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$$

is a covariant tensor, and that it is skew-symmetric in all pairs of suffixes.

We have A_{rs} is a covariant tensor

$$A_{ij} = A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^i}\frac{\partial x^\beta}{\partial x^j} \tag{1}$$

$$\Rightarrow B_{rst} = \partial_r(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}) + \partial_s(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r}) + \partial_t(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}) \tag{2}$$

Note that

$$\partial_k(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}) = \partial_k(A_{\alpha\beta})\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\partial_k(\frac{\partial x^\alpha}{\partial x^s})\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\partial_k(\frac{\partial x^\beta}{\partial x^t}) \tag{3}$$

$$\tag{4}$$

so,

$$\begin{aligned}B_{rst} = {} & \partial_r A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^s}\tfrac{\partial x^\beta}{\partial x^t} + \underbrace{A_{\alpha\beta}\partial_r\tfrac{\partial x^\alpha}{\partial x^s}\tfrac{\partial x^\beta}{\partial x^t}}_{*} + \underbrace{A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^s}\partial_r\tfrac{\partial x^\beta}{\partial x^t}}_{**}\\ & +\partial_s A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^t}\tfrac{\partial x^\beta}{\partial x^r} + \underbrace{A_{\alpha\beta}\partial_s\tfrac{\partial x^\alpha}{\partial x^t}\tfrac{\partial x^\beta}{\partial x^r}}_{***} + \underbrace{A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^t}\partial_s\tfrac{\partial x^\beta}{\partial x^r}}_{*}\\ & +\partial_t A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^r}\tfrac{\partial x^\beta}{\partial x^s} + A_{\alpha\beta}\underbrace{\partial_t\tfrac{\partial x^\alpha}{\partial x^r}\tfrac{\partial x^\beta}{\partial x^s}}_{**} + \underbrace{A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^r}\partial_t\tfrac{\partial x^\beta}{\partial x^s}}_{***}\end{aligned} \tag{5}$$

In (5) consider the two terms with (*)

$$\begin{aligned}T &= A_{\alpha\beta}\partial_r\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\partial_s\frac{\partial x^\beta}{\partial x^r} & (6)\\ &= A_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x^s\partial x^r}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial^2 x^\beta}{\partial x^r\partial x^s} & (7)\\ &= A_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x^s\partial x^r}\frac{\partial x^\beta}{\partial x^t} + A_{\beta\alpha}\frac{\partial x^\beta}{\partial x^t}\frac{\partial^2 x^\alpha}{\partial x^r\partial x^s} \text{ (by renaming dummy variables)} & (8)\end{aligned}$$

As $A_{ij} = -A_{ji}$ (skew-symmetric tensor), we get $T = 0$. The same yields for the (**) and (***) terms. So, B_{rst} reduces to

$$B_{rst} = \partial_r A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \partial_s A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r} + \partial_t A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s} \tag{9}$$

$$\Leftrightarrow B_{rst} = \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^r}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^s}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^t}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s} \tag{10}$$

By adequate renaming of the dummy variable in the 3 terms:

$$\begin{bmatrix} 1^{st} term \\ 2^{nd} term \\ 3^{rd} term \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma \to \alpha & \alpha \to \beta & \beta \to \gamma \\ \beta \to \alpha & \gamma \to \beta & \alpha \to \gamma \\ \alpha \to \alpha & \beta \to \beta & \gamma \to \gamma \end{bmatrix}$$

we get

$$B_{rst} = \left(\frac{\partial A_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial A_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \right) \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \frac{\partial x^\gamma}{\partial x^t} \tag{11}$$

$$\Leftrightarrow B_{rst} = \underbrace{(\partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta})}_{(****)} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \frac{\partial x^\gamma}{\partial x^t} \tag{12}$$

The expression (****) has exactly the required form $B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$ and is transformed (12) according the rules of a covariant tensor.

Let's prove now that it is skew-symmetric in all pairs of suffixes. We have to consider the following permutations

$$\begin{bmatrix} rst \\ rts \\ srt \\ str \\ trs \\ tsr \end{bmatrix}$$

E.g. srt

$$B_{rts} = \partial_r A_{ts} + \partial_t A_{sr} + \partial_s A_{rt} \tag{13}$$

$$= -\partial_r A_{st} - \partial_t A_{rs} - \partial_s A_{tr} \tag{14}$$

$$= -B_{rst} \tag{15}$$

The same calculations can be done for the other permutations.

◆

1.16 p23-exercise 1.

In a V_4 there are two 2-spaces with equations

$$x^r = f^r(u^1, u^2),\ x^r = g^r(u^3, u^4)$$

Prove that if these 2-spaces have a curve of intersection, then the determinal equation

$$\left|\frac{\partial x^r}{\partial u^s}\right| = 0$$

is satisfied along the curve.

Having a curve means that one of the parameters u^i can be freely chosen while the other 3 are determined by the chosen parameter.
We have,

$$\left|\frac{\partial x^r}{\partial u^s}\right| = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} & \frac{\partial x^1}{\partial u^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^4}{\partial u^1} & \frac{\partial x^4}{\partial u^2} & \frac{\partial x^4}{\partial u^3} & \frac{\partial x^4}{\partial u^4} \end{vmatrix} \tag{1}$$

Suppose we choose u^4 as parameter. This means $u^i = \phi^i(u^4)$ for i=1,2,3 and thus we can write

$$\frac{\partial x^i}{\partial u^4} = \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} + \frac{\partial x^i}{\partial u^4} \quad \text{with j=1,2,3} \quad \text{i = 1,2,3,4} \tag{2}$$

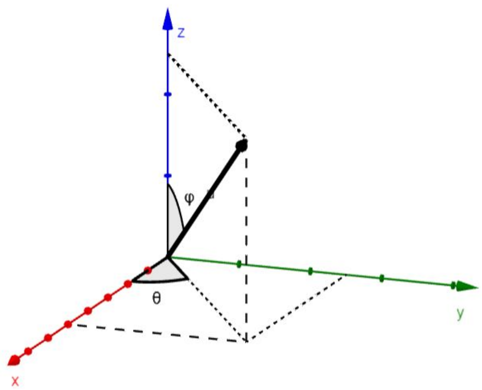
$$\Rightarrow \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} = 0 \tag{3}$$

This means that in (1) the three first columns a not linearly independent and thus have $\left|\frac{\partial x^r}{\partial u^s}\right| = 0$

◆

1.17 p23-exercise 2.

In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates x, y, z and spherical polar coordinates r,θ,ϕ.
Find the Jacobian of the transformation. Where is it zero or infinite?



We use the latitude ψ instead of the co-latitude ϕ.

$$\left\{ \begin{array}{c} x = r\cos(\psi)\cos(\theta) \\ y = r\cos(\psi)\sin(\theta) \\ z = r\sin(\psi) \end{array} \right\}$$

Partial differentiating of (x,y,z) with respect to (r,ψ,θ) gives the Jacobian

$$J = \begin{vmatrix} \cos(\psi)\cos(\theta) & -r\sin(\psi)\cos(\theta) & -r\cos(\psi)\sin(\theta) \\ \cos(\psi)\sin(\theta) & -r\sin(\psi)\sin(\theta) & r\cos(\psi)\cos(\theta) \\ \sin(\psi) & r\cos(\psi) & 0 \end{vmatrix} \tag{1}$$

$$\begin{aligned} J &= \cos(\psi)\cos(\theta)(-r^2)\cos^2(\psi)\cos(\theta)) &(2)\\ &+ r\sin(\psi)\cos(\theta)(-r\cos(\psi)\cos(\theta)\sin(\psi)) &(3)\\ &- r\cos(\psi)\sin(\theta)(r\cos^2(\psi)\sin(\theta) + r\sin^2(\psi)\sin(\theta)) &(4)\\ &= -r^2\cos^3(\psi)\cos^2(\theta) - r^2\sin^2(\psi)\cos^2(\theta)\cos(\psi) - r^2\cos(\psi)\sin^2(\theta) &(5) \end{aligned}$$

Noting that the 2^{nd} term in (5) can be written as $-r^2\cos^2(\theta)\cos(\psi) + r^2\cos^2(\theta)\cos^3(\psi)$, we get

$$\begin{aligned} J &= -r^2(\cos^3(\psi)\cos^2(\theta) + \cos^2(\theta)\cos(\psi) - \cos^3(\psi)\cos^2(\theta) + \cos(\psi)\sin^2(\theta)) &(6)\\ &= -r^2\cos(\psi) &(7) \end{aligned}$$

J=0: for $r = 0$ or $\psi = \frac{\pi}{2}|_{r\epsilon(-\infty,+\infty)}$ and $J \to \pm\infty$ or $\mp\infty$ for $r \to \pm\infty|_{\psi\neq 0}$. But what about the case $r \to \pm\infty|_{\psi\to 0}$? This case is not determined as long as no path is chosen in the (r,ψ) configuration space.

◆

1.18 p23-exercise 3.

If X, Y, Z are the components of a contravariant vector for rectangular Cartesian coordinates in Euclidean 3-space, find it's components for spherical polar coordinates.

Be x^α the components of a contravariant vector in spherical polar coordinates and x^i it's components in rectangular Cartesian coordinates. As we have

$$\begin{aligned} x^\rho &= \sqrt{x^j x^j} \\ x^\theta &= \operatorname{atan}\frac{x^2}{x^1} \\ x^\phi &= \operatorname{asin}\frac{x^3}{\sqrt{x^j x^j}} \end{aligned} \qquad \text{and} \qquad A^\alpha = A^i \frac{\partial x^\alpha}{\partial x^i} \tag{1}$$

$$\Rightarrow \left[A^\alpha\right] = \left[A^i \frac{\partial x^\alpha}{\partial x^i}\right] = \begin{bmatrix} \frac{x^1}{\sqrt{x^j x^j}} & \frac{x^2}{\sqrt{x^j x^j}} & \frac{x^3}{\sqrt{x^j x^j}} \\ -\frac{x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} & 0 \\ -\frac{x^3 x^1}{(x^j x^j)\sqrt{(x^1)^2+(x^2)^2}} & -\frac{x^3 x^2}{(x^j x^j)\sqrt{(x^1)^2+(x^2)^2}} & \frac{\sqrt{(x^1)^2+(x^2)^2}}{(x^j x^j)} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} \tag{2}$$

◆

1.19 p23-exercise 4.

In a space of three dimensions, how many different expressions are represented by the product $A^m_{np}B^{pq}_{rs}C^s_{tu}$? How many terms occur in each such expression, when written out explicitly?

As we have V_3 and considering that in $A^m_{np}B^{pq}_{rs}C^s_{tu}$ the six indices m, n, q, r, t, u are not dummy indices, we get 3^6 different expressions (first choose m: you have three choices, then n: also three choices giving 3x3 possibilities, etc for q, r, t, u).
For the second question, as in $A^m_{np}B^{pq}_{rs}$ there is only summation on over index (p) we get three terms for this part. As the summation with $A^m_{np}B^{pq}_{rs}$ and C^s_{tu} occurs only on one index also (s) we get 3x3 terms in the expression.

◆

1.20 p23-exercise 5.

If A is an invariant in V_n, are the second derivatives $\frac{\partial^2 A}{\partial x^r \partial x^s}$ the components of a tensor?

As A is invariant (note: different alphabets in the indices indicates different coordinate systems):

$$A(x^\rho) = A(x^i) \tag{1}$$

$$\Rightarrow \frac{\partial A(x^\rho)}{\partial x^i} = \frac{\partial A(x^j)}{\partial x^i} \tag{2}$$

To simplify the notation, we put $A(x^\rho) = A$' and $A(x^j) = A$' then (2) can be written as

$$\frac{\partial A'}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^i} = \frac{\partial A}{\partial x^i} \tag{3}$$

Conclusion: $\frac{\partial A}{\partial x^i}$ is a covariant tensor.

Consider now $\frac{\partial A}{\partial x^i} = \frac{\partial A'}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^i}$. Then,

$$\frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A'}{\partial x^\rho \partial x^j}\frac{\partial x^\rho}{\partial x^i} + \frac{\partial A'}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \tag{4}$$

$$\Leftrightarrow \frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A'}{\partial x^\rho \partial x^\gamma}\frac{\partial x^\gamma}{\partial x^j}\frac{\partial x^\rho}{\partial x^i} + \frac{\partial A'}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \tag{5}$$

The first term on the right side, behaves as covariant tensor but the presence of the second term makes that generally, $\frac{\partial^2 A}{\partial x^i \partial x^j}$ has not a tensor character. This is only when $\frac{\partial A'}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} = 0$, which means that x^ρ, x^i are a linear map of each other.

◆

1.21 p23-exercise 6.

Suppose that in V_2 the components of a contravariant tensor field T^{mn} in a coordinate system x^r are

$$T^{11} = 1 \quad T^{12} = 0$$

$$T^{21} = 1 \quad T^{22} = 0$$

Find the components $T^{,mn}$ in a coordinate system $x^{,r}$, where

$$x^{,1} = (x^1)^2 \quad x^{,2} = (x^2)^2$$

Write down the values of these components in particular at the point $x^1 = 1, x^2 0 =$.

As we have a contravariant tensor field :

$$T^{,mn} = T^{ij} \frac{\partial x^{,m}}{\partial x^i} \frac{\partial x^{,n}}{\partial x^j} \tag{1}$$

$$\begin{array}{ll} x^{,1} = (x^1)^2 \Rightarrow & \frac{\partial x^{,1}}{\partial x^1} = 2x^1 \qquad \frac{\partial x^{,1}}{\partial x^2} = 0 \\ x^{,2} = (x^2)^2 \Rightarrow & \frac{\partial x^{,2}}{\partial x^1} = 0 \qquad \frac{\partial x^{,2}}{\partial x^2} = 2x^2 \end{array} \tag{2}$$

$$\tag{3}$$

$$\Rightarrow T^{,11} = 4(x^1)^2 + 4(x^2)^2 \tag{4}$$

$$\Rightarrow T^{,12} = T^{,21} = 0 \tag{5}$$

$$\Rightarrow T^{,22} = 4(x^1)^2 + 4(x^2)^2 \tag{6}$$

The components in at the point $x^1 = 1, x^2 = 0$ are

$$T^{,}(1,0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

◆

1.22 p24-exercise 7.

Given that if T_{mnrs} is a covariant tensor, and

$$T_{mnrs} + T_{mnsr} = 0$$

in a coordinate system x^p, establish directly that

$$T'_{mnrs} + T'_{mnsr} = 0$$

in any other coordinate system x, q.

Note: in the following, different alphabets in the indices indicates different coordinate systems. As we T_{mnrs} is a covariant tensor :

$$T_{\alpha\beta\gamma\delta} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} \tag{1}$$

$$\Rightarrow T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} + T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\delta}\frac{\partial x^s}{\partial x^\gamma} \tag{2}$$

Now, swap the dummy indices r and s in the second term on the right and as $T_{mnrs} = -T_{mnsr}$:

$$\begin{aligned} T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} &= T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} + T_{mnsr}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^s}{\partial x^\delta}\frac{\partial x^r}{\partial x^\gamma} && (3)\\ &= (T_{mnrs} + T_{mnsr})\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^s}{\partial x^\delta}\frac{\partial x^r}{\partial x^\gamma} && (4)\\ &= 0 && (5)\end{aligned}$$

◆

1.23 p24-exercise 8.

Prove that if A_r is a covariant vector, then $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ is a skew-symmetric covariant tensor of the second order (use the notation of 1.7).

Be $B_{rs} = \frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$.

i) B_{rs} is skew-symmetric: It is obvious that:

$$-B_{rs} = -\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r} = \frac{\partial A_s}{\partial x^r} - \frac{\partial A_r}{\partial x^s} \equiv B_{sr}$$

ii) B_{rs} is covariant:

Note: in the following, different alphabets in the indices indicates different coordinate systems.

Let

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta. \tag{1}$$

We know that $A_i = A_\gamma X^\gamma_i$ as A_i is covariant. Hence,

$$\partial_j A_i = \partial_j A_\gamma X^\gamma_i + A_\gamma \partial_j X^\gamma_i \tag{2}$$

$$= \partial_\alpha A_\gamma X^\alpha_j X^\gamma_i + A_\gamma \partial_j X^\gamma_i \tag{3}$$

Using (3), we compute the first term in (1)

$$\partial_s A_r X^r_\alpha X^s_\beta = \partial_\rho A_\gamma X^\rho_s X^\gamma_r X^r_\alpha X^s_\beta + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{4}$$

$$= \partial_\rho A_\gamma X^\rho_\beta X^\gamma_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{5}$$

$$= \partial_\rho A_\gamma \delta^\rho_\beta \delta^\gamma_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{6}$$

$$= \partial_\beta A_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta \tag{7}$$

In the same way, we get for the second term in (1)

$$\partial_r A_s X^s_\alpha X^r_\beta = \partial_\alpha A_\beta + A_\gamma \partial_r X^\gamma_s X^r_\alpha X^s_\beta \tag{8}$$

And thus,

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta = \partial_\beta A_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta - \partial_\alpha A_\beta - A_\gamma \partial_r X^\gamma_s X^r_\alpha X^s_\beta \tag{9}$$

$$\Rightarrow \partial_\beta A_\alpha - \partial_\alpha A_\beta = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta \tag{10}$$

So, i) and (10) proves that $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ is skew-symmetric tensor of the second order.

◆

1.24 p24-exercise 9.

Let $x^r, \overline{x}^r, y^r, \overline{y}^r$ be four systems of coordinates. Examine the tensor character of $\frac{\partial x^r}{\partial y^s}$ with respect to the following transformations:
i) A transformation $x^r = f^r(\overline{x}^1, \ldots, \overline{x}^N)$, with y^r unchanged;
ii) A transformation $y^r = g^r(\overline{y}^1, \ldots, \overline{y}^N)$, with x^r unchanged;

Note: in the following, different alphabets in the indices indicates different coordinate systems.

i) Let's compute the expression $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta}$. Obviously, the right side is an expression of a (possible) mixed tensor of the second order ($\frac{\partial x^r}{\partial y^s}$) under transformation from the (r) coordinate system to the (α) coordinate system. Then,

$$A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta} \tag{1}$$

$$= \frac{\partial x^\alpha}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \tag{2}$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \tag{3}$$

If we consider the $\overline{y}^r$ coordinate system as the y^ρ coordinate system and as $\overline{y}^r = y^r$ then $\frac{\partial y^\rho}{\partial y^s} = \delta^\rho_s$ and we get from (3)

$$A(\alpha, \beta) = \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial y^\rho}{\partial y^s} \frac{\partial x^s}{\partial x^\beta} \tag{4}$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \delta^\rho_s \frac{\partial x^s}{\partial x^\beta} \tag{5}$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial x^\rho}{\partial x^\beta} \tag{6}$$

$$= \frac{\partial x^\alpha}{\partial y^\rho} \delta^\rho_\beta \tag{7}$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \tag{8}$$

$$\text{(1) and (8)} \implies \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^s}{\partial x^\beta} \tag{9}$$

So $A(r, s) = \frac{\partial x^r}{\partial y^s}$ is a mixed tensor of type A^r_s

ii) Let's compute the expression $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s} \frac{\partial y^\alpha}{\partial y^r} \frac{\partial y^s}{\partial y^\beta}$. Obviously, the right side is an expression of a (possible) mixed tensor of the second order ($\frac{\partial x^r}{\partial y^s}$) under transformation from the (r) coordinate system to the (α) coordinate system. Then,

$$A(\alpha,\beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{10}$$

$$= \frac{\partial x^r}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{11}$$

$$= \frac{\partial x^r}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{12}$$

$$= \frac{\partial x^r}{\partial y^\rho}\delta^\rho_\beta\frac{\partial y^\alpha}{\partial y^r} \tag{13}$$

$$= \frac{\partial x^r}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{14}$$

$$= \frac{\partial x^r}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{15}$$

If we consider the $\overline{x}^r$ coordinate system as the x^σ coordinate system and as $\overline{x}^r = x^r$ then $\frac{\partial x^\sigma}{\partial x^r} = \delta^\sigma_r$ and we get from (15)

$$A(\alpha,\beta) = \frac{\partial x^r}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{16}$$

$$= \delta^r_\sigma\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{17}$$

$$= \frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^\sigma} \tag{18}$$

$$= \frac{\partial x^\sigma}{\partial y^\beta}\delta^\alpha_\sigma \tag{19}$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \tag{20}$$

$$\text{(10) and (19)} \Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{21}$$

So $A(r,s) = \frac{\partial x^r}{\partial y^s}$ is a mixed tensor of type A^r_s

◆

1.25 p24-exercise 10.

If x^r, y^r, z^r are three systems of coordinates, prove the follwoing rule for the multiplication of Jacobians.

$$\left|\frac{\partial x^m}{\partial y^n}\right|\left|\frac{\partial y^r}{\partial z^s}\right| = \left|\frac{\partial x^t}{\partial z^u}\right|$$

As we have

$$\frac{\partial x^t}{\partial z^u} = \frac{\partial x^t}{\partial y^k}\frac{\partial y^k}{\partial z^u} \tag{1}$$

$$\begin{bmatrix} \frac{\partial x^1}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial z^N} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial y^k}\frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial y^k}\frac{\partial y^k}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^k}\frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial y^k}\frac{\partial y^k}{\partial z^N} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^1} & \cdots & \frac{\partial x^N}{\partial y^k} \end{bmatrix}\begin{bmatrix} \frac{\partial y^1}{\partial z^1} & \cdots & \frac{\partial y^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial y^N}{\partial z^1} & \cdots & \frac{\partial y^N}{\partial z^N} \end{bmatrix} \tag{3}$$

$$\Rightarrow \left|\frac{\partial x^m}{\partial y^n}\right|\left|\frac{\partial y^r}{\partial z^s}\right| = \left|\frac{\partial x^t}{\partial z^u}\right| \tag{4}$$

◆

1.26 p24-exercise 11.

Prove that with respect to transformations

$$x'^{r} = C_{rs}x^{s}$$

where the coefficients are constants satisfying

$$C_{mr}C_{ms} = \delta^{r}_{s}$$

contravariant and covariant vectors have the same formula of transformation

$$A'^{r} = C_{rs}A^{s},\ A_{,r} = C_{rs}A_{s}$$

i) $A'^{r} = C_{rs}A^{s}$

Be $A'^{r} = A^{s}\frac{\partial x'^{r}}{\partial x^{s}}$ and as $x'^{r} = C_{rs}x^{s}$ we have $\frac{\partial x'^{r}}{\partial x^{s}} = C_{rs}$. Hence,

$$A'^{r} = C_{rs}A^{s}$$

.

i) $A_{,r} = C_{rs}A_{s}$
Be $A_{,r} = A_{s}\frac{\partial x^{s}}{\partial x'^{r}}$ and as $x'^{r} = C_{rs}x^{s}$ we have

$$\frac{\partial x'^{r}}{\partial x'^{t}} = C_{rs}\frac{\partial x^{s}}{\partial x'^{t}} \tag{1}$$

$$\Rightarrow \delta^{r}_{t} = C_{rs}\frac{\partial x^{s}}{\partial x'^{t}} \tag{2}$$

Now, multiply (2) by C_{rq}. We get,

$$\delta^{r}_{t}C_{rq} = C_{rq}C_{rs}\frac{\partial x^{s}}{\partial x'^{t}} \tag{3}$$

$$C_{tq} = C_{rq}C_{rs}\frac{\partial x^{s}}{\partial x'^{t}} \tag{4}$$

$$\text{as } C_{mr}C_{ms} = \delta^{r}_{s} \Rightarrow \qquad C_{tq} = \delta^{q}_{s}\frac{\partial x^{s}}{\partial x'^{t}} \tag{5}$$

$$\Rightarrow C_{tq} = \frac{\partial x^{q}}{\partial x'^{t}} \text{ or } C_{rs} = \frac{\partial x^{s}}{\partial x'^{r}} \tag{6}$$

$$\text{as } A_{,r} = A_{s}\frac{\partial x^{s}}{\partial x'^{r}} \Rightarrow A_{,r} = C_{rs}\frac{\partial x^{s}}{\partial x'^{r}} \tag{7}$$

◆

1.27 p25-exercise 12.

Prove that

$$\frac{\partial ln\left|\frac{\partial x^m}{\partial y^n}\right|}{\partial x^r} = \frac{\partial^2 y^m}{\partial x^r \partial x^n}\frac{\partial x^n}{\partial y^m}$$

.

Lemma *Be A a square matrix NxN; Be f a C^1 function $f : \mathbb{R}^{NxN} \to \mathbb{R}$. Define A' as $(A'_{ij}) = \frac{df}{dA_{ij}}$. Then,*

$$(ln\,|A|)' = (A^{-1})^T \;\; wih\; f = |A|$$

Proof:
By definition of the determinant, we have

$$|A| = A_{iK}C^i_K \qquad \textit{(no summation on K!)} \tag{1}$$

with $(C^i_K) = (-1)^{i+K}M^i_K$ being the cofactor of element A_{iK} and M^i_K the minor (N-1)x(N-1) matrix associated with the cofactor A_{K^i}. Be $C = (C_{ij})$ the NxN matrix formed with all possible cofactor elements C^i_j $(i, j = 1\ldots, N)$.
We have

$$A^{-1} = \frac{C^T}{|A|} \tag{2}$$

$$\Rightarrow (A^{-1})^T = \frac{C}{|A|} \tag{3}$$

$$\textit{differentiating (1)} \;\Rightarrow \frac{\partial\,|A|}{\partial A_{mn}} = \frac{\partial A_{iK}}{\partial A_{mn}}C^i_K + A_{iK}\frac{\partial C^i_K}{\partial A_{mn}} \tag{4}$$

$$\textit{we have for } i = m \quad \begin{array}{l} \frac{\partial A_{iK}}{\partial A_{mn}} = 1 \quad K = n \\ \frac{\partial A_{iK}}{\partial A_{mn}} = 0 \quad K \neq n \end{array} \tag{5}$$

Also, $\forall K : \frac{\partial C^i_K}{\partial A_{in}} = 0$ as by definition of the cofactor matrix, A_{ij} is not contained in C_{ij}.
Hence, (4) becomes

$$\frac{\partial\,|A|}{\partial A_{ij}} = C^i_j \tag{6}$$

$$\textit{But,} \quad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{\frac{\partial |A|}{\partial A_{ij}}}{|A|} \tag{7}$$

$$\textit{(6) and (7) gives} \quad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{C^i_j}{|A|} \tag{8}$$

$$\textit{(3) and (8) gives} \quad \frac{\partial ln\,|A|}{\partial A_{ij}} = \frac{(A^{-1}_{ij})^T\,|A|}{|A|} = (A^{-1}_{ij})^T \tag{9}$$

$$\Rightarrow (ln\,|A|)' = (A^{-1})^T \tag{10}$$

◇

Now the main proof:
Let,

$$A \equiv [a_{mn}] = \left[\frac{\partial y^m}{\partial x^n}\right] \tag{11}$$

$$\Rightarrow \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial ln\,|A|}{\partial a_{mn}}\frac{\partial a_{mn}}{\partial x^r} \tag{12}$$

$$\text{from (10) we get} \quad \frac{\partial ln\,|A|}{\partial a_{mn}} = (A^{-1})^T_{mn} \tag{13}$$

$$\text{But A is a Jacobian, so } (A^{-1})_{mn} = \frac{\partial x^m}{\partial y^n} \tag{14}$$

$$\text{and thus } (A^{-1})^T_{mn} = \frac{\partial x^n}{\partial y^m} \tag{15}$$

$$\text{(13) can be written as } \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m}\frac{\partial a_{mn}}{\partial x^r} \tag{16}$$

$$\Rightarrow \frac{\partial ln\,|A|}{\partial x^r} = \frac{\partial x^n}{\partial y^m}\frac{\partial^2 y^m}{\partial x^r \partial x^n} \tag{17}$$

◆

1.28 p25-exercise 13.

Consider the quantities $\frac{dx^r}{dt}$ for a particle moving in the plane. If x^r are the rectangular Cartesian coordinates, are these quantities the components of a contravariant or covariant vector with respect to rotation of the axes? Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?

Note: we suppose that by a rotation of the axes, the problem means a fixed rotation and not a rotation varying in time.

i) Be $v^r = \frac{dx^r}{dt}$ and consider v^α the same object but in another the coordinate system. A rotation of the axes implies the linear form

$$x^\alpha = R^\alpha_k x^k \quad \text{with } R^\alpha_k \neq R^\alpha_k(x^k) \tag{1}$$
$$\Rightarrow \quad \frac{\partial x^\alpha}{\partial x^r} = R^\alpha_k \delta^k_r \tag{2}$$
$$\Rightarrow \quad R^\alpha_r = \frac{\partial x^\alpha}{\partial x^r} \tag{3}$$

Consider $v^\alpha = \frac{dx^\alpha}{dt}$

$$v^\alpha = \frac{dx^\alpha}{dt} \tag{4}$$
$$(1) \quad \Rightarrow \quad v^\alpha = R^\alpha_k \frac{dx^k}{dt} \tag{5}$$
$$\Rightarrow \quad v^\alpha = R^\alpha_k v^k \tag{6}$$
$$(3) \quad \Rightarrow \quad v^\alpha = v^k \frac{\partial x^\alpha}{\partial x^r} \tag{7}$$

Conclusion: v^k is a contravariant vector.

ii) Are they components of a vector with respect to transformation to any curvilinear coordinates (e.g. polar coordinates)?
We know that

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^r} dx^r \tag{8}$$
$$\Rightarrow \quad \frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r} \frac{dx^r}{dt} \tag{9}$$
$$\Rightarrow \quad v^\alpha = v^r \frac{\partial x^\alpha}{\partial x^r} \tag{10}$$

So v^r is a contravariant vector in general. Note that this proof is more straightforward than the prove in i).

◆

1.29 p25-exercise 14.

Consider the question raised in No. 13 for the acceleration $\frac{d^2x^r}{dt^2}$.

From exercise 13. we know that

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial x^r}\frac{dx^r}{dt} \tag{1}$$

$$\Rightarrow \quad \frac{d^2x^\alpha}{dt^2} = \frac{d^2x^r}{dt^2}\frac{\partial x^\alpha}{\partial x^r} + \frac{d\frac{\partial x^\alpha}{\partial x^r}}{dt}\frac{dx^r}{dt} \tag{2}$$

$$= \frac{d^2x^r}{dt^2}\frac{\partial x^\alpha}{\partial x^r} + \frac{\partial^2 x^\alpha}{\partial x^r \partial x^m}\frac{dx^m}{dt}\frac{dx^r}{dt} \tag{3}$$

The second term on the right does not vanish in general, hence $\frac{d^2x^r}{dt^2}$ has not a tensor character.

◆

1.30 p25-exercise 15.

It is well known that the equation of an ellipse may be written

$$ax^2 + 2hxy + by^2 = 1$$

What is the tensor character of a, h, b with respect to transformation to any Cartesian coordinates (rectangular or oblique) in the plane?

Consider the transformation from a (w, z) coordinate system to a (x, y) coordinate system. For the considered type of transformation we have

$$\begin{aligned} x &= \alpha w + \beta z \\ y &= \gamma w + \delta z \end{aligned} \tag{1}$$

$$\text{consider} \qquad \begin{aligned} ax^2 + 2hxy + by^2 &= 1 \\ pw^2 + 2qwz + rz^2 &= 1 \end{aligned} \tag{2}$$

the two representations of the same ellipse in the respective coordinate systems. Plugging (1) in (2):

$$a\alpha^2 w^2 + a2\alpha\beta wz + \alpha\beta^2 z^2 + 2h\alpha\gamma w^2 + \beta\delta z^2 + 2h(\alpha\delta + \gamma\beta)wz + b\gamma^2 w^2 + 2b\gamma\delta wz + \delta^2 z^2 b = 1 \tag{3}$$

$$\tag{4}$$

Rearranging and equating the terms in w^2, wz, z^2 in (2) gives

$$p = a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \tag{5}$$

$$q = a\alpha\beta + h(\alpha\delta + \gamma\beta) + \gamma d \tag{6}$$

$$r = a\beta^2 + 2h\beta\delta + b\delta^2 \tag{7}$$

Consider the following objects

$$(A_{ij}) = \begin{bmatrix} a & h \\ h & b \end{bmatrix} \tag{8}$$

$$(A_{ij})' = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{21} \end{bmatrix} \tag{9}$$

$$\text{we calculate} \quad A'_{ij} = A_{km} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} \tag{10}$$

with $(x'^1, x'^2) = (w, z)$ and $(x^1, x^2) = (x, y)$. We have,

$$\frac{\partial x^1}{\partial x'^1} = \alpha, \frac{\partial x^1}{\partial x'^2} = \beta, \frac{\partial x^2}{\partial x'^1} = \gamma, \frac{\partial x^2}{\partial x'^2} = \delta \tag{11}$$

$$\text{(10) and (11)} \quad \Rightarrow \quad \begin{aligned} a'_{11} &= a\alpha^2 + 2h\alpha\gamma + b\gamma^2 \\ a'_{22} &= a\beta^2 + 2h\delta\beta + b\delta^2 \\ a'_{12} = a'_{21} &= a\alpha\beta + h(\alpha\delta + \gamma\beta) + b\gamma\delta \end{aligned} \tag{12}$$

Combining (5), (6), (7) and (12) we get

$$p = a'_{11}, r = a'_{22}, q = a'_{12} = a'_{21}$$

and so (9) becomes

$$(A_{ij})' = \begin{bmatrix} p & q \\ q & r \end{bmatrix}$$

Considering (10) we see that $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ transforms to $\begin{bmatrix} p & q \\ q & r \end{bmatrix}$ according the rules of a covariant tensor of order two.

◆

1.31 p25-exercise 16.

Matter is distributed in a plane and A, B, H are the moments and product of inertia with respect to rectangular aces $0xy$ in a plane. Examine the tensor character of the set of quantities A, B, H under rotation of the axes. What notation would you suggest for moments and product of inertia in order to exhibit the tensor character? What simple invariant can be formed from A, B, H ?

Consider the transformation from a (x^1, x^2) coordinate system to a (y^1, y^2) coordinate system. For the considered type of transformation we have

$$\begin{aligned} y^1 &= \alpha x^1 + \beta x^2 \\ y^2 &= \gamma x^1 + \delta x^2 \end{aligned} \tag{1}$$

$$\text{Be} \quad \begin{aligned} A &= \textstyle\sum_\rho m_\rho (x^{2,\rho})^2 \\ B &= \textstyle\sum_\rho m_\rho (x^{1,\rho})^2 \\ H &= \textstyle\sum_\rho m_\rho x^{1,\rho} x^{2,\rho} \end{aligned} \quad \text{and} \quad \begin{aligned} A' &= \textstyle\sum_\rho m_\rho (y^{2,\rho})^2 \\ B' &= \textstyle\sum_\rho m_\rho (y^{1,\rho})^2 \\ H' &= \textstyle\sum_\rho m_\rho y^{1,\rho} y^{2,\rho} \end{aligned} \tag{2}$$

the moments and product of inertia, ρ being the index of summation over all the points with mass m_ρ.
For the sake of notational simplicty we consider only one point of mass as the linearity of A, B, H related to ρ ensures the validity of the nexts steps for all points in the plane.
From (1) and (2) we have:

$$\frac{A'}{m_\rho} = \gamma^2 (x^1)^2 + 2\gamma\delta x^1 x^2 + \delta^2 (x^2)^2 \tag{3}$$

$$\frac{B'}{m_\rho} = \alpha^2 (x^1)^2 + 2\alpha\beta x^1 x^2 + \beta^2 (x^2)^2 \tag{4}$$

$$\frac{H'}{m_\rho} = \alpha\gamma (x^1)^2 + (\gamma\beta + \alpha\delta) x^1 x^2 + \beta\delta (x^2)^2 \tag{5}$$

$$\text{Note that} \quad \begin{aligned} \tfrac{\partial y^1}{\partial x^1} &= \alpha & \tfrac{\partial y^1}{\partial x^2} &= \beta \\ \tfrac{\partial y^2}{\partial x^1} &= \gamma & \tfrac{\partial y^2}{\partial x^2} &= \delta \end{aligned} \tag{6}$$

$$\text{(6) in (4):} \quad \frac{B'}{m_\rho} = (x^1)^2 \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + 2(x^1)(x^2)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} \tag{7}$$

$$= (x^1)^2 \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + (x^1)(x^2)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)(x^1)\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + (x^2)^2 \frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} \tag{8}$$

Repeating the same calculations for $\frac{A'}{m_\rho}$ and $\frac{H'}{m_\rho}$ gives:

$$\begin{aligned} \tfrac{A'}{m_\rho} &= (x^1)^2 \tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + (x^1)(x^2)\tfrac{\partial y^2}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + (x^2)(x^1)\tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + (x^2)^2 \tfrac{\partial y^2}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \\ \tfrac{B'}{m_\rho} &= (x^1)^2 \tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^1} + (x^1)(x^2)\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + (x^2)(x^1)\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^1}{\partial x^2} + (x^2)^2 \tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^1}{\partial x^2} \\ \tfrac{H'}{m_\rho} &= (x^1)^2 \tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^1} + (x^1)(x^2)\tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^1} + (x^2)(x^1)\tfrac{\partial y^1}{\partial x^1}\tfrac{\partial y^2}{\partial x^2} + (x^2)^2 \tfrac{\partial y^1}{\partial x^2}\tfrac{\partial y^2}{\partial x^2} \end{aligned} \tag{9}$$

Now, define

$$(K_{ij}) = \begin{bmatrix} (x^1)^2 & (x^1)(x^2) \\ (x^2)(x^1) & (x^2)^2 \end{bmatrix} \qquad (K_{ij})' = \begin{bmatrix} (y^1)^2 & (y^1)(y^2) \\ (y^2)(y^1) & (y^2)^2 \end{bmatrix} \tag{10}$$

Then (9) can be written as

$$\frac{A'}{m_\rho} = (y^1)^2 = K^{11}\frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^1} + K^{12}\frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^2} + K^{21}\frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^1} + K^{22}\frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^2}$$

$$\frac{B'}{m_\rho} = (y^2)^2 = K^{11}\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + K^{12}\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + K^{21}\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + K^{22}\frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2} \tag{11}$$

$$\frac{H'}{m_\rho} = (y^1)(y^2) = K^{11}\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^1} + K^{12}\frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1} + K^{21}\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} + K^{22}\frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^2}$$

Hence,

$$K'^{11} = K^{11}\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + K^{12}\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + K^{21}\frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^2} + K^{22}\frac{\partial y^1}{\partial x^2}\frac{\partial y^1}{\partial x^2}$$

$$K'^{22} = K^{11}\frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^1} + K^{12}\frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^2} + K^{21}\frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^1} + K^{22}\frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^2} \tag{12}$$

$$K'^{12} = K'^{21} = K^{11}\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^1} + K^{12}\frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1} + K^{21}\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} + K^{22}\frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^2}$$

So the object $(K_{ij}) \equiv K^{ij}$ transforms according (12) like a contravariant second order tensor.

Now, consider $|K^{ij}|$, obviously $|K^{ij}| = (x^1)^2(x^2)^2 - (x^1)(x^2)(x^2)(x^1) = 0$, but so is also $|K'^{ij}|$.

$$\Rightarrow \quad |K^{ij}| \text{ is an invariant under the considered transformation}$$

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1.32 p25-exercise 17.

S_{nmr} is a skew-symmetric tensor in the first two indices. $-f_{mnr} + f_{nmr} = S_{mnr}$.

From exercise 13. we know that

$$-f_{mnr} + f_{nmr} = S_{mnr} \tag{1}$$

Swap the indices three times

$$\text{i)} \quad n \leftrightarrow r : (1) \Rightarrow -f_{mrn} + f_{rmn} = S_{mrn} \tag{2}$$

$$\Leftrightarrow \underbrace{f_{mnr}}_{*} + \underbrace{f_{rmn}}_{**} = -S_{rmn} \tag{3}$$

$$\text{ii)} \quad m \leftrightarrow r : (1) \Rightarrow -f_{rnm} + f_{nrm} = S_{rnm} \tag{4}$$

$$\Leftrightarrow \underbrace{f_{rmn}}_{**} + \underbrace{f_{nrm}}_{***} = -S_{nrm} \tag{5}$$

$$\text{iii)} \quad m \leftrightarrow n : (1) \Rightarrow -f_{nmr} + f_{mnr} = S_{nmr} \tag{6}$$

$$\Leftrightarrow \underbrace{f_{nrm}}_{***} + \underbrace{f_{mnr}}_{*} = -S_{mnr} \tag{7}$$

$$\text{(3) - (5) + (7):} \quad 2\underbrace{f_{mnr}}_{*} = -S_{rmn} + S_{nrm} - S_{mnr} \tag{8}$$

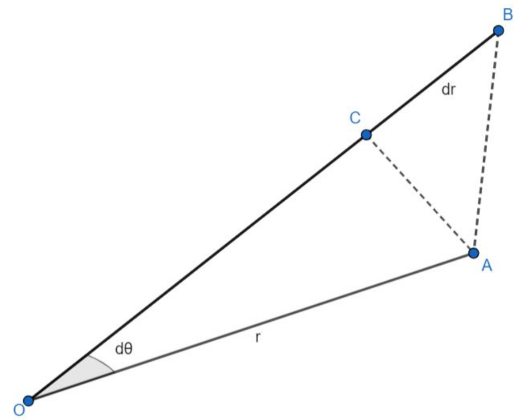
$$\Leftrightarrow f_{mnr} = \frac{-S_{rmn} + S_{nrm} - S_{mnr}}{2} \tag{9}$$

◆

Basic Operations in Riemannian Space

2.1 p27-exercise

Take polar coordinates r, θ in a plane. Draw the infinitesimal triangle with vertices (r, θ), $(r + dr, \theta)$, $(r, \theta + d\theta)$. Evaluate the square on the hypotenuse of this infinitisimal triangle, and so obtain the metric tensor for the plan for the coordinates(r, θ).



$$ds^2 = |AB|^2 \tag{1}$$
$$= dr^2 + |CA|^2 \tag{2}$$
$$|CA| = r\sin(d\theta) \approx r d\theta \tag{3}$$
$$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 \tag{4}$$
$$\Rightarrow (a_{mn}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{5}$$

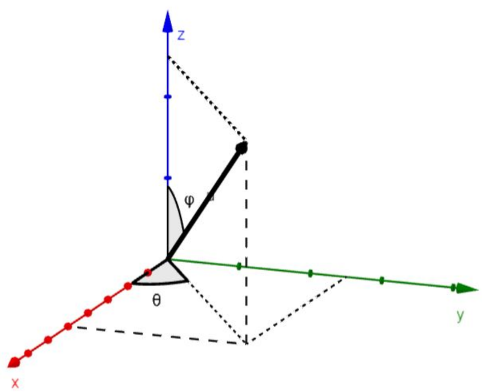
◆

2.2 p27-exercise

Show that if $x^1 = r, x^2 = \theta, x^3 = \phi$, in the usual notation of spherical polar coordinates, then

$$a_{11} = 1, a_{22} = r^2, a_{33} = r^2 \sin^2 \theta$$

and the other components vanish.



We use the latitude ψ instead of the co-latitude ϕ.

$$\begin{cases} x = r\cos(\psi)\cos(\theta) \\ y = r\cos(\psi)\sin(\theta) \\ z = r\sin(\psi) \end{cases}$$

$$ds^2 = dx^2 + dy^2 + dz^2 \quad \text{with} \tag{1}$$

$$\left.\begin{aligned} dx &= dr\cos(\psi)\cos(\theta) - r\sin(\psi)d\psi\cos(\theta) - r\cos(\psi)\sin(\theta)d\theta \\ dy &= dr\cos(\psi)\sin(\theta) - r\sin(\psi)d\psi\sin(\theta) + r\cos(\psi)\cos(\theta)d\theta \\ dz &= dr\sin(\psi) + r\cos(\psi)d\psi \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}
dx^2 &= \cos^2(\psi)\cos^2(\theta)dr^2 \\
&\quad -r^2\sin^2(\psi)\cos^2(\theta)d\psi^2 \\
&\quad -r^2\cos^2(\psi)\sin^2(\theta)d\theta^2 \\
&\quad -\cos(\psi)\cos(\theta)r\sin(\psi)\cos(\theta)drd\psi \\
&\quad -\cos(\psi)\cos(\theta)r\cos(\psi)\sin(\theta)drd\theta \\
&\quad +r\sin(\psi)\cos(\theta)r\cos(\psi)\sin(\theta)d\psi d\theta \\
\\
dy^2 &= \cos^2(\psi)\sin^2(\theta)dr^2 \\
&\quad +r^2\sin^2(\psi)\sin^2(\theta)d\psi^2 \\
&\quad +r^2\cos^2(\psi)\cos^2(\theta)d\theta^2 \\
&\quad -\cos(\psi)\sin(\theta)r\sin(\psi)\sin(\theta)drd\psi \\
&\quad -\cos(\psi)\sin(\theta)r\cos(\psi)\cos(\theta)drd\theta \\
&\quad -r\sin(\psi)\sin(\theta)r\cos(\psi)\cos(\theta)d\psi d\theta \\
\\
dz^2 &= \sin^2(\psi)dr^2 + r^2\cos^2(\psi)d\psi^2 + r\sin(\psi)\cos(\psi)drd\psi
\end{aligned}\right\} \tag{3}$$

Rearrange terms:

$$\left.\begin{aligned}
dx^2 &= \cos^2(\psi)\cos^2(\theta)dr^2 \\
&\quad +r^2\sin^2(\psi)\cos^2(\theta)d\psi^2 \\
&\quad +r^2\cos^2(\psi)\sin^2(\theta)d\theta^2 \\
&\quad -r\cos(\psi)\sin(\psi)\cos^2(\theta)drd\psi \\
&\quad -r\cos^2(\psi)\cos(\theta)\sin(\theta)drd\theta \\
&\quad +r^2\sin(\psi)\cos(\theta)\cos(\psi)\sin(\theta)d\psi d\theta \\
\\
dy^2 &= \cos^2(\psi)\sin^2(\theta)dr^2 \\
&\quad +r^2\sin^2(\psi)\sin^2(\theta)d\psi^2 \\
&\quad +r^2\cos^2(\psi)\cos^2(\theta)d\theta^2 \\
&\quad -r\cos(\psi)\sin(\psi)\sin^2(\theta)drd\psi \\
&\quad -r\cos^2(\psi)\sin(\theta)\cos(\theta)drd\theta \\
&\quad -r^2\sin(\psi)\sin(\theta)\cos(\psi)\cos(\theta)d\psi d\theta \\
\\
dz^2 &= \sin^2(\psi)dr^2 + r^2\cos^2(\psi)d\psi^2 + r\sin(\psi)\cos(\psi)drd\psi
\end{aligned}\right\} \tag{4}$$

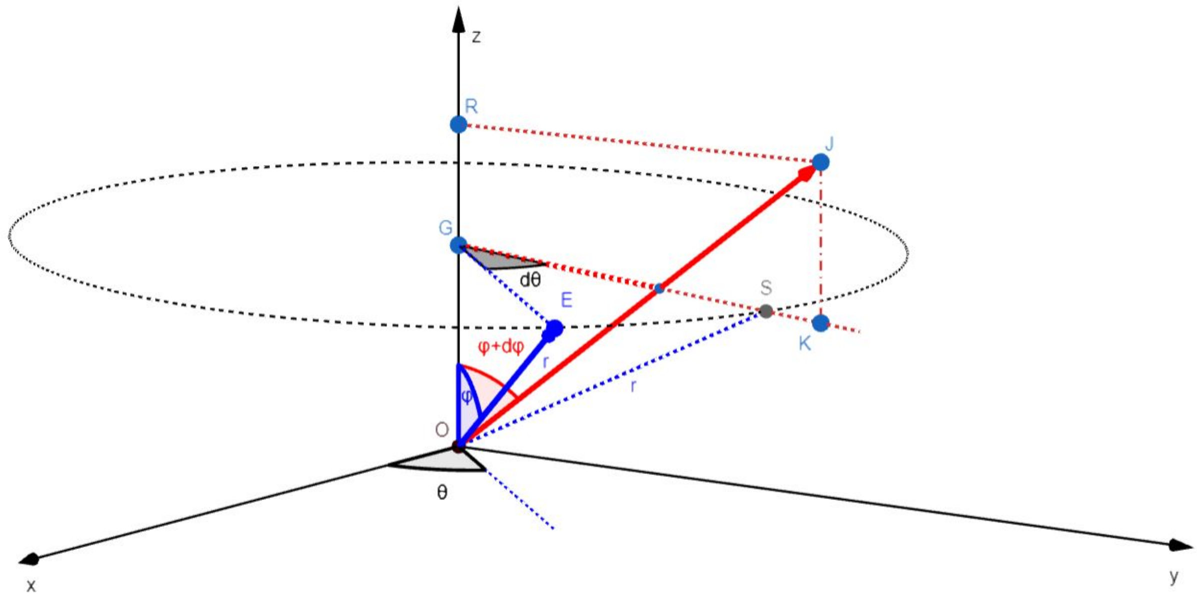
Grouping similar infinitesimal components and using basic trigonometric identities gives:

$$ds^2 = dr^2 + r^2 d\psi^2 + r^2\cos^2(\psi)d\theta^2 \tag{5}$$

$$\text{replace } \psi \text{ with } \frac{\pi}{2} - \phi \Rightarrow ds^2 = dr^2 + r^2 d\phi^2 + r^2\sin^2(\phi)d\theta^2 \tag{6}$$

$$\Rightarrow (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\phi) \end{pmatrix} \tag{7}$$

A more geometrical way of deriving the metric



Consider an infinitesimal displacement of point E to J with $(dr, d\phi, d\theta)$.

$$ds^2 = |EJ|^2 \tag{8}$$

As we use infinitesimal displacements we can assume that

$$|ES| \perp |GK| \perp |JK| \perp |ES|$$

. Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \tag{9}$$

We have the following relationships

$$\left.\begin{array}{c}
|ES| = r\sin(\phi)d\theta \\[1em]
|GE| = |GS| = r\sin(\phi) \\[1em]
|GK| = |RJ| = (r+dr)\sin(\phi+d\phi) \\
= (r+dr)(\cos(\phi)\sin(d\phi)+\sin(\phi)\cos(d\phi)) \\
= (r+dr)(\cos(\phi)d\phi+\sin(\phi)) \\
= r\cos(\phi)d\phi + r\sin(\phi) + \sin(\phi)dr \\[1em]
|OR| = (r+dr)\cos(\phi+d\phi) \\
= (r+dr)(\cos(\phi)\cos(d\phi)-\sin(\phi)\sin(d\phi)) \\
= (r+dr)(\cos(\phi)-\sin(\phi)d\phi) \\
= r\cos(\phi) - r\sin(\phi)d\phi + \cos(\phi)dr \\[1em]
|OG| = r\cos(\phi) \\[1em]
|JK| = |OR| - |OG| = \cos(\phi)dr - r\sin(\phi)d\phi \\[1em]
|SK| = |GK| - |GS| = r\cos(\phi)d\phi + \sin(\phi)dr
\end{array}\right\} \tag{10}$$

$$\left.\begin{array}{c}
|ES|^2 = r^2\sin^2(\phi)d\theta^2 \\
|SK|^2 = r^2\cos^2(\phi)d\phi^2 + \sin^2(\phi)dr^2 + 2r\cos(\phi)\sin(\phi)drd\phi \\
|JK|^2 = \cos^2(\phi)dr^2 + r^2\sin^2(\phi)d\phi^2 - 2r\cos(\phi)\sin(\phi)drd\phi
\end{array}\right\} \tag{11}$$

Hence,

$$ds^2 = |ES|^2 + |SK|^2 + |KJ|^2 \tag{12}$$

$$= \left\{\begin{array}{c}
r^2\sin^2(\phi)d\theta^2 \\
+r^2\cos^2(\phi)d\phi^2 + \sin^2(\phi)dr^2 + 2r\cos(\phi)\sin(\phi)drd\phi \\
+r^2\sin^2(\phi)d\phi^2 + \cos^2(\phi)dr^2 - 2r\cos(\phi)\sin(\phi)drd\phi
\end{array}\right. \tag{13}$$

$$\Rightarrow ds^2 = dr^2 + r^2d\phi^2 + r^2\sin^2(\phi)d\theta^2 \tag{14}$$

$$\Rightarrow (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\phi) \end{pmatrix} \tag{15}$$

◆

2.3 p27-exercise

Starting from 3.103, show that

$$a_{mn} = \frac{\partial y^1}{\partial x^m}\frac{\partial y^1}{\partial x^n} + \frac{\partial y^2}{\partial x^m}\frac{\partial y^2}{\partial x^n} + \frac{\partial y^3}{\partial x^m}\frac{\partial y^3}{\partial x^n}$$

and calculate the quantities for a sphere, taking as curvilinear coordinates on he sphere

$$x^1 = y^1, x^2 = y^2$$

We have

$$\text{(2.103)} \quad \Rightarrow y^1 = x^1, y^2 = x^2, y^3 = f^3(x^1, x^2) \tag{1}$$

$$\text{surface = sphere} \quad \Rightarrow y^3 = \pm\sqrt{R^2 - (x^1)^2 - (x^2)^2} \tag{2}$$

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{3}$$

$$\text{(1) and (2)} \quad \Rightarrow \begin{cases} dy^1 = dx^1 \\ dy^2 = dx^2 \\ dy^3 = \pm\frac{1}{2}\frac{-2x^1dx^1 - 2x^2dx^2}{\sqrt{R^2-(x^1)^2-(x^2)^2}} \end{cases} \tag{4}$$

$$\Rightarrow ds^2 = (dx^1)^2 + (dx^2)^2 + \frac{(x^1)^2(dx^1)^2 + (x^2)^2(dx^2)^2 + 2x^1x^2dx^1dx^2}{R^2 - (x^1)^2 - (x^2)^2} \tag{5}$$

$$\Leftrightarrow ds^2 = \frac{(R^2 - (x^2)^2)(dx^1)^2 + (R^2 - (x^1)^2)(dx^2)^2 + 2x^1x^2dx^1dx^2}{R^2 - (x^1)^2 - (x^2)^2} \tag{6}$$

$$\Rightarrow (a_{mn}) = \frac{1}{R^2 - (x^1)^2 - (x^2)^2}\begin{pmatrix} R^2 - (x^2)^2 & x^1x^2 \\ x^1x^2 & R^2 - (x^1)^2 \end{pmatrix} \tag{7}$$

◆

2.4 p30-clarification 2.202

$$a_{mr}\Delta^{ms} = a_{rm}\Delta^{sm} = \delta_r^s a$$

Case 1: $r = s$

We have, $a_{Rm}\Delta^{Rm}$ (no summation on R) is the definition of the determinant of A developed along the row R: OK.

Case 2: $r \neq s$

Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \tag{1}$$

and consider the matrix A'

$$A\text{'} = \begin{array}{cl} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{R1} & a_{R2} & \dots & a_{RN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{R1} & a_{R2} & \dots & a_{RN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} & \begin{array}{c} \vdots \\ \vdots \\ \leftarrow S^{th} \text{ row} \\ \vdots \\ \leftarrow R^{th} \text{ row} \\ \vdots \\ \vdots \end{array} \end{array} \tag{2}$$

This matrix corresponds to the way $a_{Rm}\Delta^{Sm}$ is computed. Indeed with the factor a_{Rm} is not associated it's own cofactor Δ^{Rm} but the cofactor of the m^{th} column in row S. Replacing the S^{th} row with the row R and calculating it's determinant is the same as calculating $a_{Rm}\Delta^{Sm}$

But, $|A\text{'}| = 0$ as we have two identical rows. So, $a_{Rm}\Delta^{Sm} = 0$

Conclusion : The same reasoning can be applied when expanding the determinant along the columns instead of the rows we have indeed $a_{mr}\Delta^{ms} = a_{rm}\Delta^{sm} = \delta_r^s a$.

◆

2.5 p31-exercise

Show that if $amn = 0$ for $m \neq n$, then

$$a^{11} = \frac{1}{a_{11}}, a^{22} = \frac{1}{a_{22}}, \ldots, a12 = 0, \ldots$$

We have to prove that:

$$a^{ij} = \begin{cases} \frac{1}{a_{ij}} & : i = j \\ 0 & : i \neq j \end{cases} \tag{1}$$

From 2.204:

$$a_{mR} a_{mS} = \delta^S_R \tag{2}$$

i) Be $R \neq S$

$$\begin{aligned} (2) \qquad & \Rightarrow a_{mR} a_{mS} = 0 && (3) \\ \text{but} \quad & a_{mR} = 0 \quad \forall m \neq R && (4) \\ & \Rightarrow a_{RR} a^{RS} = 0 && (5) \end{aligned}$$

but $a_{RR} \neq 0$ (a_{RR} can't be 0 as the metric tensor would degenerate if $a_{mn} = 0 \quad \forall m \neq n$

$$\Rightarrow a^{Rs} = 0 \tag{6}$$

(7)

(8)

i) Be $R = S$

$$\begin{aligned} (2) \qquad & \Rightarrow a_{mR} a_{mR} = 1 && (9) \\ \text{but} \quad & a_{mR} = 0 \quad \forall m \neq R && (10) \\ & \Rightarrow a_{RR} a^{RR} = 1 && (11) \\ & \Rightarrow a^{RR} = \frac{1}{a_{RR}} && (12) \end{aligned}$$

◆

2.6 p31-exercise

Find the components of a^{mn} for spherical polar coordinates in Eulidean 3-space.

We have (see exercise page 27):

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\phi) \end{pmatrix} \tag{1}$$

As $a_{mn} = 0 \quad \forall m \neq n$ we deduce (see previous exercise p31)

$$(a^{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2(\phi)} \end{pmatrix} \tag{2}$$

◆

2.7 p32-exercise

Find the mixed metric tensor $a_m^{\cdot n}$ obtained from a_{mn} by raising the second subscript

We have :

$$\begin{aligned}
a_i^{\cdot j} &= a_{in}a^{nj} && (1)\\
&= a_{in}a^{jn} \quad a^{jn} \text{ is symmetric} && (2)\\
&= \delta_i^j \quad \text{(see 2.205 pg. 30)} && (3)\\
&\Rightarrow a_i^{\cdot j} = \delta_i^j && (4)
\end{aligned}$$

◆

2.8 p32-clarification 2.214

$$\frac{\partial a}{\partial a_{mn}} = a a^{mn}$$

Put $a_{MN} \equiv a_{mn}$. By definition, we have

$$a \equiv |a_{mn}| = a_{Mk}\Delta^{Mk} \quad \text{(develop determinant along row M)} \tag{1}$$

$$\Rightarrow \frac{\partial a}{\partial a_{mn}} = \frac{\partial a_{Mk}}{\partial a_{mn}}\Delta^{Mk} + a_{Mk}\frac{\partial \Delta^{Mk}}{\partial a_{mn}} \tag{2}$$

$$\text{but} \quad \frac{\partial a_{Mk}}{\partial a_{mn}} = \begin{cases} 1 & \text{if} \quad k = N \\ 0 & \text{if} \quad k \neq N \end{cases} \tag{3}$$

$$\text{and} \quad \frac{\partial \Delta^{Mk}}{\partial a_{mn}} = 0 \quad \forall k \text{ as } \Delta^{Mk} \text{does not contain the row with } a_{mn} \text{ as element.} \tag{4}$$

$$\text{(3) and (4)} \quad \Rightarrow \frac{\partial a}{\partial a_{mn}} = \Delta^{MN} \tag{5}$$

$$a^{mn} = \frac{\Delta^{mn}}{a} \quad \text{by definition (see 2.203 page 30)} \tag{6}$$

$$\Rightarrow \frac{\partial a}{\partial a_{mn}} = a a^{mn} \tag{7}$$

◆

2.9 p32-exercise

Prove that $a_{mn}a^{mn} = N$.

From 2.204, we have

$$a_{mr}a^{ms} = \delta_r^s \tag{1}$$

$$\text{Consider} \quad a_{mR}a^{mR} = 1 \tag{2}$$

$$\text{We can repeat (2) for R = 1,2,...,N} \quad \Rightarrow a_{mr}a^{mr} = N \tag{3}$$

$$\tag{4}$$

◆

2.10 p33-exercise

Show that in Euclidean 3-space with rectangular Cartesian coordinates, the definition 2.301 coincides with the usual definition of the magnitude of a vector.

The length of an arbitrary vector in Euclidean 3-space with rectangular Cartesian coordinates, is

$$ds^2 = (dy^1)^2 + (dy^2)^2(dy^2)^2 \tag{1}$$

From 2.301, it is obvious that the metrixc tensor can be expressed as,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

◆

2.11 p34-exercise

A curve in Euclidean 3-space has the equations

$$x^1 = a\cos(u), x^2 = a\sin(u), x^3 = bu$$

where x^1, x^2, x^3 are rectangular Cartesian coordinates,u is a parameter, and a, b are positive constants. Find the length of this curve between the point $u = 0$ and $u = 2\pi$.

The metric tensor has the following form,

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$\text{and (2.306)} \qquad s = \int_0^{2\pi} [\epsilon a_{mn} p^m p^n]^{\frac{1}{2}} du \tag{2}$$

with

$$p^1 = \frac{dx^1}{du} = -a\sin(u), \quad p^2 = \frac{dx^2}{du} = a\cos(u), \quad p^3 = \frac{dx^3}{du} = b \tag{3}$$

Hence (2) becomes

$$s = \int_0^{2\pi} \epsilon[a^2\sin^2(u) + a^2\cos^2(u) + b^2]^{\frac{1}{2}} du \tag{4}$$

$$= \int_0^{2\pi} \epsilon[a^2 + b^2]^{\frac{1}{2}} du \tag{5}$$

$$= [a^2 + b^2]^{\frac{1}{2}}\ u\big|_0^{2\pi} \tag{6}$$

$$= 2\pi[a^2 + b^2]^{\frac{1}{2}} \tag{7}$$

◆

2.12 p36-clarification 2.314

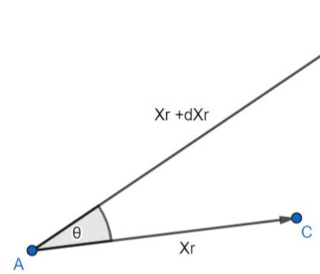
Going from 2.313 to 2.314 yields because both X^m and Y^m are unit vectors and by definition of the magnitude (see 2.301) both $a_{mn}X^mX^n$ and $a_{mn}Y^mY^n$ are 1 (also due to the fact that only a positive definite metric tensor is considered, $\epsilon = 1$).

◆

2.13 p37-exercise

Show that the small angle between unit vectors X^r and $X^r + dX^r$ (these increments being infenitesimal) is given by

$$\theta^2 = a_{mn} X^m X^n$$



By definition (2.302 page 33)

$$|BC|^2 = \epsilon a_{mn} dX^m dX^n \tag{1}$$

$$\tag{2}$$

We can drop $\epsilon = 1$ as the considered space is positive definite.
As θ is infinitesimal, we can state

$$|BC| \approx |AC|\theta \tag{3}$$

$$\text{and} \quad |AC| = X^r = 1 \quad (\text{as } X^r \text{ is a unit vector}) \tag{4}$$

$$\Rightarrow \theta^2 = a_{mn} dX^m dX^n \tag{5}$$

◆

2.14 p39-clarification 2.409

We clarify the integration by parts in the derivation of the general geodesic equation.

We have

$$\int d(A.B) = \int AdB + \int BdA \tag{1}$$

$$\Rightarrow \int AdB = \int BdA - \int d(A.B) \tag{2}$$

Now, substitute 2.407 in 2.406, we get

$$\frac{dL}{dv} = \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial p^r}{\partial v}du \tag{3}$$

$$= \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\frac{\partial \frac{\partial x^r}{\partial v}}{\partial u}du \tag{4}$$

$$= \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial x^r}\frac{\partial x^r}{\partial v}du + \int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}d(\frac{\partial x^r}{\partial v}) \tag{5}$$

To integrate by parts the second term in (5) we put in (2)

$$A = \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r} \quad \text{and } B = \frac{\partial x^r}{\partial v}$$

$$\int \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}d(\frac{\partial x^r}{\partial v}) = \int AdB \tag{6}$$

$$= \int BdA - \int d(A.B) \tag{7}$$

$$= \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\ \frac{\partial x^r}{\partial v}\Big|_{u_0}^{u_1} - \int \frac{\partial x^r}{\partial v}d(\frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}) \tag{8}$$

$$= \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r}\ \frac{\partial x^r}{\partial v}\Big|_{u_0}^{u_1} - \int \frac{\partial x^r}{\partial v}\frac{\partial \frac{\partial(\epsilon w)^{\frac{1}{2}}}{\partial p^r})}{\partial u}du \tag{9}$$

Replacing (9) in (5) gives the formulea 2.409.

◆

2.15 p41-exercise

Prove the following identities:

$$[mn,r] = [nm,r], \quad [rm,n] + [rn,m] = \partial_r a_{mn}$$

$$[mn,r] = \frac{1}{2}(\partial_n a_{mr} + \partial_m a_{nr} - \partial_r a_{mn}) \tag{1}$$
$$= \frac{1}{2}(\partial_m a_{nr} + \partial_n a_{mr} - \partial_r a_{nm}) \tag{2}$$
$$= [nm,r] \tag{3}$$

and

$$[rm,n] + [rn,m] = \frac{1}{2}(\partial_r a_{mn} + \partial_m a_{rn} - \partial_n a_{rm} + \partial_n a_{rm} + \partial_r a_{mn} - \partial_m a_{rn}) \tag{4}$$
$$= \frac{1}{2}(\partial_r a_{mn} + \partial_r a_{mn}) \tag{5}$$
$$= \partial_r a_{mn} \tag{6}$$

◆

2.16 p42-exercise

Prove that

$$[mn, r] = a_{rs}\Gamma^s_{mn}$$

$$
\begin{aligned}
a_{rs}\Gamma^s_{mn} &= a_{rs}a^{sk}[mn, k] && (1)\\
&= \delta^k_r[mn, k] && (2)\\
&= [mn, r] && (3)
\end{aligned}
$$

◆

2.17 p42-clarification on 2.430

... This may proved without difficulty by starting with 2.427, in which λ is a known function of u , and defining s by the relation

$$s = \int_{u_0}^{u} (exp \int_{v_0}^{v} \lambda(w)dw)dv$$

u_0, v_0 being constants....

$$\text{see 2.428 :} \quad \lambda(u) = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{1}$$

$$\tag{2}$$

We suppose $u(s)$ continuous by parts with continuous inverse.

$$\Rightarrow \frac{ds}{du} = \frac{1}{\frac{du}{ds}} \tag{3}$$

$$\Rightarrow \frac{d^2s}{du^2} = \frac{d(\frac{1}{\frac{du}{ds}})}{ds}\frac{ds}{du} \tag{4}$$

$$= -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2}\frac{ds}{du} \tag{5}$$

$$\Rightarrow \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{6}$$

By definition (2.428)

$$\lambda(u) = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} \tag{7}$$

$$\text{hence by (6) and (7):} \quad \lambda(u) = \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} \tag{8}$$

$$\text{in (8) put} \quad y = \frac{ds}{du} \tag{9}$$

$$\text{and so} \quad \lambda(w) = \frac{y'}{y} \tag{10}$$

$$\Rightarrow \int \frac{y'}{y} dw = \int \lambda(w) dw \tag{11}$$

$$\Leftrightarrow \int d(lny) = \int \lambda(w) dw \tag{12}$$

$$\Rightarrow ln(y)|_{v_0}^{v} = \int_{v_0}^{v} \lambda(w) dw \tag{13}$$

$$\Rightarrow y = exp(\int_{v_0}^{v} \lambda(w)dw) + C \tag{14}$$

Taking into account (9), we get:

$$\frac{ds}{dv} = exp(\int_{v_0}^{v} \lambda(w)dw) + C \tag{15}$$

$$\Rightarrow s|_{u_0}^{u} = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv + Cu + B' \tag{16}$$

$$\Leftrightarrow s = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv + Cu + B \tag{17}$$

We show tha we have to put $C = 0$ and can drop the constant B. Remember by (8)

$$\lambda(u) = \frac{\frac{d^2s}{du^2}}{\frac{ds}{du}} \tag{18}$$

$$\text{by (17)} \quad \frac{ds}{du} = exp(\int_{u_0}^{u} \lambda(w)dw) + C \tag{19}$$

$$\text{and} \quad \frac{d^2s}{du^2} = \lambda(u) exp(\int_{u_0}^{u} \lambda(w)dw) \tag{20}$$

$$\text{hence by (18), (19) and (20):} \quad \lambda(u) = \frac{\lambda(u)exp(\int_{u_0}^{u} \lambda(w)dw)}{exp(\int_{u_0}^{u} \lambda(w)dw) + C} \tag{21}$$

So, whatever the constant B, the relation (18) is correct on the condition that C=0. So, indeed, we can choose the independent variable s as

$$s = \int_{u_0}^{u} exp(\int_{v_0}^{v} \lambda(w)dw)dv$$

◆

2.18 p42-clarification on 2.430

After 2.430 it is stated:
"No matter what values these constants have, 2.424 is satisfied, and by adjusting the constant v_0, we can ensure that $a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \pm 1$ along C, so that s is actually the arc length."

We first prove that 2.4.24 is satisfied, no matter what values the constants take. We have

$$(2.430) \quad s = \int_{u_0}^{u} (exp \int_{v_0}^{v} \lambda(w)dw)dv \tag{1}$$

$$\text{and } (2.427) \quad \frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = \lambda\frac{dx^r}{du} \tag{2}$$

In (2) we can write the first term as

$$\frac{d^2x^r}{du^2} = \frac{d(\frac{dx^r}{du})}{ds}\frac{ds}{du} \tag{3}$$

$$\text{with} \quad \frac{d(\frac{dx^r}{du})}{ds} = \frac{d(\frac{dx^r}{ds}\frac{ds}{du})}{ds} = \frac{d^2x^r}{ds^2}\frac{ds}{du} + \frac{dx^r}{ds}\frac{d(\frac{ds}{du})}{ds} \tag{4}$$

Assuming the curve smooth, we have

$$\frac{d(\frac{ds}{du})}{ds} = \frac{d(\frac{1}{\frac{du}{ds}})}{ds} = -\frac{\frac{d^2u}{ds^2}}{(\frac{du}{ds})^2} = \lambda \tag{5}$$

Putting (4) and (5) in (3) we get

$$\frac{d^2x^r}{du^2} = \frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \lambda\frac{dx^r}{ds}\frac{ds}{du} \tag{6}$$

Plugging (6) in 2.427 gives:

$$\frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \lambda\frac{dx^r}{ds}\frac{ds}{du} + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \lambda\frac{dx^r}{ds}\frac{ds}{du} \tag{7}$$

$$\Leftrightarrow \frac{d^2x^r}{ds^2}\left(\frac{ds}{du}\right)^2 + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = 0 \tag{8}$$

We can assume that $\frac{ds}{du}$ does not become 0 or $\pm\infty$ along the curve by choosing an adequate constant v_0. Indeed, from (2.430) we get

$$\frac{ds}{du} = exp(\int_{u_0}^{u} \lambda(w)dw) \tag{9}$$

$$= \frac{\phi(u)}{\phi(u_0)} \tag{10}$$

with $\phi(u_0) = e^{\theta(u_0)}$, $\theta(u)$ being the indefinite integral $\int \lambda(w)dw$.
So, it is sufficient to choose v_0 so that $\theta(u_0)$ does not become $\pm\infty$ to ensure that $\frac{ds}{du} \neq 0$ or $\neq \pm\infty$

along the curve and so we have from (8)

$$\frac{d^2x^r}{ds^2} + \Gamma^r_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = 0 \tag{11}$$

which is the definition (2.424) of a geodesic.

The same reasoning about $\frac{ds}{du} \neq 0$ can be made to prove that $a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \pm 1$ along C. Indeed, by definition (2.305):

$$ds = \left[\epsilon a_{mn}\frac{dx^m}{du}\frac{dx^n}{du}\right]^{\frac{1}{2}} \tag{12}$$

$$\text{equating with (9)} \quad \left[\epsilon a_{mn}\frac{dx^m}{du}\frac{dx^n}{du}\right]^{\frac{1}{2}} = exp\left(\int_{u_0}^{u} \lambda(w)dw\right) \tag{13}$$

$$\Rightarrow \quad \epsilon a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \left[exp\left(\int_{u_0}^{u} \lambda(w)dw\right)\right]^2 \tag{14}$$

$$\text{but} \quad \frac{ds}{du} = exp\left(\int_{u_0}^{u} \lambda(w)dw\right) \tag{15}$$

$$\text{and so, (9) becomes} \quad \epsilon a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\left(\frac{ds}{du}\right)^2 = \left(\frac{ds}{du}\right)^2 \tag{16}$$

$$\Rightarrow \quad a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = \epsilon \tag{17}$$

◆

2.19 p43-clarification

$$\lambda = \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + 2\Gamma^N_{\mu N}\frac{dx^\mu}{dx^N} + \Gamma^N_{NN}$$

We start with 2.427 with $r = N$

$$\frac{d^2x^N}{dx^{N2}} + \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} = \lambda\frac{dx^N}{dx^N} \tag{1}$$

$$\Rightarrow \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} = \lambda \quad \left(\text{as} \quad \frac{d^2x^N}{dx^{N2}} = 0 \quad \frac{dx^N}{dx^N} = 1\right) \tag{2}$$

But (2) is only valid with the dummy indices μ and ν spanning the whole dimension $(1, 2, \ldots N)$, but by choice $\mu, \nu \quad \epsilon \quad (1, 2, \ldots, N-1)$. We have thus to add in the left term of (2) the cases

$$\begin{cases} \Gamma^N_{N\nu} \quad \nu = (1, 2, \ldots, N-1) \\ \\ \Gamma^N_{\mu N} \quad \mu = (1, 2, \ldots, N-1) \\ \\ \Gamma^N_{NN} \end{cases} \tag{3}$$

$$\text{(2) becomes} \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{N\nu}\frac{dx^N}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{\mu N}\frac{dx^\mu}{dx^N}\frac{dx^N}{dx^N} + \Gamma^N_{NN}\frac{dx^N}{dx^N}\frac{dx^N}{dx^N} = \lambda \tag{4}$$

$$\Rightarrow \quad \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + \Gamma^N_{N\nu}\frac{dx^\nu}{dx^N} + \Gamma^N_{\mu N}\frac{dx^\mu}{dx^N} + \Gamma^N_{NN} = \lambda \tag{5}$$

As $\Gamma^N_{\mu N}$ is symmetric on the lower indices and μ, ν being dummy indices:

$$\lambda = \Gamma^N_{\mu\nu}\frac{dx^\mu}{dx^N}\frac{dx^\nu}{dx^N} + 2\Gamma^N_{N\nu}\frac{dx^\nu}{dx^N} + \Gamma^N_{NN} \tag{6}$$

The other $N-1$ equation for $r = 1, \ldots, N-1$ can be deduced following the same reasoning.

◆

2.20 p45-clarification

$f_r \equiv a_{rm}\frac{d^2x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du}$ is covariant and $f^r \equiv \frac{d^2x^m}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du}$ is contravariant.

$$f_r \equiv a_{rm}\frac{d^2x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \tag{1}$$

$$\text{multiply with} \quad a^{sr} \quad \Rightarrow \quad f_r a^{sr} = a^{sr}a_{rm}\frac{d^2x^m}{du^2} + a^{sr}[mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \tag{2}$$

$$\Rightarrow \quad f^s = \delta^s_m\frac{d^2x^m}{du^2} + \underbrace{a^{sr}[mn,r]}_{\Gamma^s_{mn}}\frac{dx^m}{du}\frac{dx^n}{du} \tag{3}$$

$$\Rightarrow \quad f^s = \frac{d^2x^s}{du^2} + \Gamma^s_{mn}\frac{dx^m}{du}\frac{dx^n}{du} \tag{4}$$

By lifting the index of f_r we get a contravariant vector confirming that (4) is contravariant.

◆

2.21 p45-clarification

(2.443) and (2.444)

$$p^r \frac{\partial w}{\partial p^r} - w = C^t \quad \Rightarrow \quad w = C^t$$

By definition

$$w = a_{mn} p^m p^n \tag{1}$$

$$\Rightarrow \quad \frac{\partial w}{\partial p^r} = a_{mn} \left(\frac{\partial p^m}{\partial p^r} p^n + p^m \frac{\partial p^n}{\partial p^r} \right) \tag{2}$$

$$= a_{mn} (\delta^m_r p^n + p^m \delta^n_r) \tag{3}$$

$$= a_{rn} p^n + a_{mr} p^m \tag{4}$$

$$= 2a_{mr} p^m \quad \text{(as } a_{mn} \text{ is symmetric)} \tag{5}$$

$$(4) \quad \Rightarrow \quad p^r \frac{\partial w}{\partial p^r} = 2a_{mr} p^r p^m \tag{6}$$

$$= 2w \tag{7}$$

$$(2.443) \quad \Rightarrow \quad p^r \frac{\partial w}{\partial p^r} - w = 2w - w = w = C^t \tag{8}$$

$$\Rightarrow \quad w \equiv a_{mn} p^m p^n = C^t \tag{9}$$

◆

2.22 p46-"geodesic null lines", some personal thoughts

Can we grasp intuitively the notion of "geodesic null lines"? To understand this, here are two example of "geodesic null lines"

We consider two examples, a classic one (Minkowski space) and a weird space.

i) on dimensional Minkowski space

For this space the metric is in normalized coordinates

$$ds^2 = dx^2 - dt^2$$

It is easy to see that the equations of a geodesic null line (taking t as parameter for the curve), reduce to

$$\left.\begin{array}{r} \frac{d^2x}{dt^2} = 0 \\ \\ dx^2 - dt^2 = 0 \end{array}\right\} \tag{1}$$

$$\Rightarrow x = at + b \quad \text{and} \quad x = \pm t + g \tag{2}$$

If we choose $x = t$ at $t = 0$ we see that the geodesic null lines are the bisectors of the x-t axes.

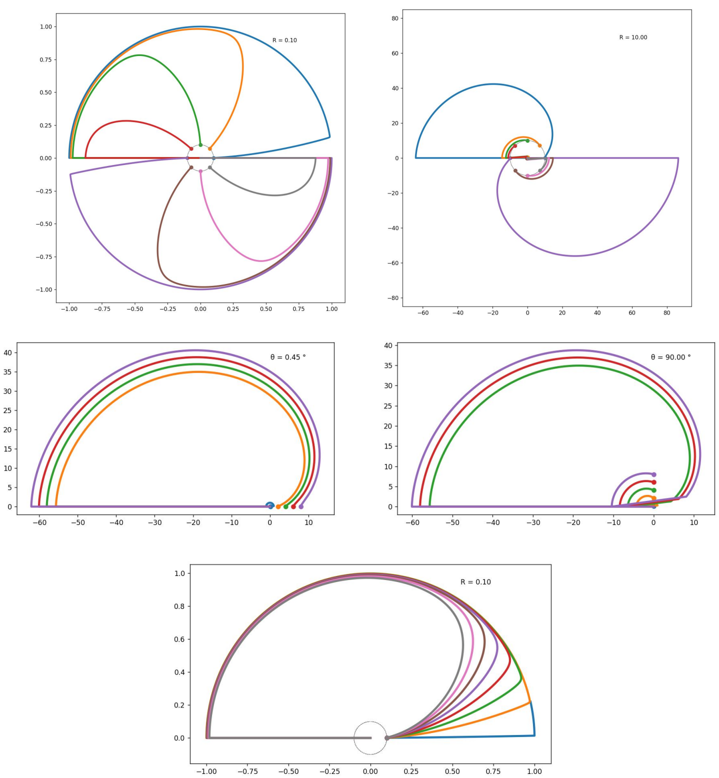
ii) We consider a V_2 space equipped with a polar coordinate system and with a metric tensor defined as

$$ds^2 = (1-r)dr^2 + \sin(\theta)d\theta^2 \tag{3}$$

Giving as geodesic null line equations

$$\left.\begin{array}{r} \frac{d^2r}{du^2} + \frac{1}{2(r-1)}\left(\frac{dr}{du}\right)^2 = 0 \\ \\ \frac{d^2\theta}{du^2} + \frac{1}{2}\cot(\theta)\left(\frac{d\theta}{du}\right)^2 = 0 \\ \\ (1-r)\left(\frac{dr}{du}\right)^2 + \sin(\theta)\left(\frac{d\theta}{du}\right)^2 = 0 \end{array}\right\} \tag{4}$$

We see immediately that for $r \to 1$ and $\theta \to k\frac{\pi}{2} \quad (k = 0, \pm 1, \pm 2, \dots)$, some problems will occur and that no solutions might be found. On the next page some geodesic null lines are showed (equations solved numerically with Python Scipy library)



The first row shows the results for points at 2 different distance from the origin while changing the angle θ.
The second row shows the results for points at 2 different angles while changing the distance from the origin.
The second row shows the results for a specific point while changing the the $\frac{d\theta}{du}$ value in the initial condition when solving the system of differential equations.
Conclusion: with non-trivial metrics, getting an intuitive idea about the geodesic null lines gets very hard.

◆

2.23 p47-exercise

The class of all parameters u, for which the equations of a geodesic null line assume the simple form 2.445, are obtained from any one such parameter by linear transformation

$$\overline{u} = au + b$$

a and b being constants.

The simple form 2.445 is :

$$\frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \tag{1}$$

$$\tag{2}$$

The general form of a geodesic is (2.447)

$$\frac{d^2x^r}{d\overline{u}^2} + \Gamma^r_{mn}\frac{dx^m}{d\overline{u}}\frac{dx^n}{d\overline{u}} = \lambda\frac{dx^r}{d\overline{u}} \tag{3}$$

$$\tag{4}$$

So (2.447) can only of the form (2.445) if $\lambda = 0$

$$\lambda = -\frac{\frac{d^2\overline{u}}{du^2}}{\left(\frac{d\overline{u}}{du}\right)^2} = 0 \tag{5}$$

$$\tag{6}$$

We can state that $\frac{d\overline{u}}{du} \neq 0$ as $\overline{u}$ can't be a constant (being a parameter of a curve). So,

$$\frac{d^2\overline{u}}{du^2} = 0 \tag{7}$$

$$\Rightarrow \quad \frac{d\overline{u}}{du} = a \tag{8}$$

$$\Rightarrow \quad \overline{u} = au + b \tag{9}$$

◆

2.24 p47-exercise

Consider a 3-space with coordinates x, y, z and a metric form $\Phi = (dx)^2 + (dy)^2 - (dz)^2$. prove that the geodesic null lines may be represented by the equations

$$x = au + a' \quad y = bu + b' \quad z = cu + c'$$

where u is a parameter and a, a', b, b', c, c' are constants which are arbitrary except for the relation $a^2 + b^2 - c^2 = 0$.

Given is

$$\Phi = (dx)^2 + (dy)^2 - (dz)^2 \tag{1}$$

From the previous exercise we have already proven that x, y, z are of the form

$$x^i = q_i u + q_i' \tag{2}$$

To be a null geodesic null line we need to have (2.448)

$$a_{mn} \frac{dx^m}{du} \frac{dx^n}{du} = 0 \tag{3}$$

$$\text{from (1) we deduce} \quad (a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4}$$

$$(3) \quad \Rightarrow \quad (dx)^2 + (dy)^2 - (dz)^2 = 0 \tag{5}$$

$$(2) \quad \Rightarrow \quad (q_1)^2 + (q_2)^2 - (q_3)^2 = 0 \tag{6}$$

◆

2.25 p48-exercise

Prove that the Christoffel symbols of the first kind transform according the equation

$$[mn,r]' = [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^s}{\partial x'^r} + a_{pq}\frac{\partial x^p}{\partial x'^r}\frac{\partial^2 x^q}{\partial x'^m \partial x'^n}$$

From 2.438 page 45, we have

$$f_r \equiv a_{rm}\frac{d^2x^m}{du^2} + [mn,r]\frac{dx^m}{du}\frac{dx^n}{du} \quad \text{is covariant} \tag{1}$$

$$\Rightarrow \quad f'_r = f_s\frac{\partial x^s}{\partial x'^r} \tag{2}$$

$$\text{with} \quad f'_r = a'_{rm}\frac{d^2x'^m}{du^2} + [mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} \tag{3}$$

Combining (1), (2) and (3) gives

$$a'_{rm}\frac{d^2x'^m}{du^2} + [mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} = (a_{sm}\frac{d^2x^m}{du^2} + [mn,s]\frac{dx^m}{du}\frac{dx^n}{du})\frac{\partial x^s}{\partial x'^r} \tag{4}$$

We rewrite (4) as

$$[mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} = \underbrace{-a'_{rm}\frac{d^2x'^m}{du^2}}_{(*)} + \underbrace{a_{sm}\frac{d^2x^m}{du^2}\frac{\partial x^s}{\partial x'^r}}_{(**)} + \underbrace{[mn,s]\frac{dx^m}{du}\frac{dx^n}{du}\frac{\partial x^s}{\partial x'^r}}_{(***)} \tag{5}$$

$$(***) \Leftrightarrow [mn,s]\frac{\partial x^m}{\partial x'^p}\frac{dx'^p}{du}\frac{\partial x^n}{\partial x'^q}\frac{dx'^q}{du}\frac{\partial x^s}{\partial x'^r} \tag{6}$$

In (6) renaming the dummy indices m,n,p,q gives

$$(***) \Leftrightarrow [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{dx'^m}{du}\frac{\partial x^q}{\partial x'^n}\frac{dx'^n}{du}\frac{\partial x^s}{\partial x'^r} \tag{7}$$

$$\Leftrightarrow [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^s}{\partial x'^r}\left(\frac{dx'^m}{du}\frac{dx'^n}{du}\right) \tag{8}$$

Also,

$$(**) \Leftrightarrow a_{sm}\frac{d^2x^m}{du^2}\frac{\partial x^s}{\partial x'^r} \tag{9}$$

$$\text{As we have also} \quad \frac{d^2x^m}{du^2} = \frac{d(\frac{\partial x^m}{\partial x'^p}\frac{dx'^p}{du})}{du} \tag{10}$$

$$= \frac{\partial x^m}{\partial x'^p}\frac{d^2x'^p}{du^2} + \frac{dx'^p}{du}\frac{\partial^2 x^m}{\partial x'^p \partial x'^q}\frac{dx'^q}{du} \tag{11}$$

(11) and (9) gives by changing the dummy indices ($m \to t$, $p \to m$, $q \to n$)

$$(**) = \underbrace{a_{st}\frac{\partial x^t}{\partial x'^p}\frac{d^2x'^p}{du^2}\frac{\partial x^s}{\partial x'^r}}_{(****)} + a_{pq}\frac{\partial^2 x^q}{\partial x'^m \partial x'^n}\frac{\partial x^p}{\partial x'^r}\left(\frac{dx'^m}{du}\frac{dx'^m}{du}\right) \tag{12}$$

$$\text{with } (****) \quad = a_{st}\left(\frac{\partial x^t}{\partial x'^m}\frac{\partial x^s}{\partial x'^r}\right)\frac{d^2x'^m}{du^2} \tag{13}$$

But a_{st} is a covariant tensor, so

$$a'_{rm} = a_{st}\frac{\partial x^t}{\partial x'^m}\frac{\partial x^s}{\partial x'^r} \tag{14}$$

$$\text{(13) becomes} \quad (****) = a'_{rm}\frac{d^2x'^m}{du^2} \tag{15}$$

$$\text{and from (5) we have} \quad (*) = -a'_{rm}\frac{d^2x'^m}{du^2} \tag{16}$$

$$\tag{17}$$

and both terms cancel each other in equation (5). So adding $(*), (**)$ and $(***)$ in (5) , we get

$$[mn,r]'\frac{dx'^m}{du}\frac{dx'^n}{du} = a_{pq}\frac{\partial^2 x^q}{\partial x'^m \partial x'^n}\frac{\partial x^p}{\partial x'^r}\left(\frac{dx'^m}{du}\frac{dx'^m}{du}\right) + [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^s}{\partial x'^r}\left(\frac{dx'^m}{du}\frac{dx'^n}{du}\right) \tag{18}$$

$$\Rightarrow [mn,r]' = [pq,s]\frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^s}{\partial x'^r} + a_{pq}\frac{\partial x^p}{\partial x'^r}\frac{\partial^2 x^q}{\partial x'^m \partial x'^n} \tag{19}$$

◆

2.26 p50-clarification 2.515

$$\frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du} S^r + T_r \frac{dS^r}{du} = \left(\frac{dT_r}{du} - \Gamma^m_{rn} T_m \frac{dx^n}{du}\right) S^r$$

with

$$\frac{\delta T_r}{\delta u} = \frac{dT_r}{du} - \Gamma^m_{rn} \frac{dx^n}{du} T_m$$

a covariant vector.

It is given that S^r is a tensor propagated parallelly along the curve. The by (2.5212) we have

$$\frac{dS^r}{du} + \Gamma^r_{mn} S^m \frac{dx^n}{du} = 0 \tag{1}$$

$$\frac{dS^r}{du} = -\Gamma^r_{mn} S^m \frac{dx^n}{du} \tag{2}$$

$$\text{and} \quad \frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du} S^r + T_r \frac{dS^r}{du} \tag{3}$$

$$= \frac{dT_r}{du} S^r - \Gamma^r_{mn} S^m \frac{dx^n}{du} T_r \tag{4}$$

$$\tag{5}$$

Swap dummy indices r and m in the second term:

$$\frac{d\left(T_r S^r\right)}{du} = \frac{dT_r}{du} S^r - \Gamma^m_{rn} S^r \frac{dx^n}{du} T_m \tag{6}$$

$$= \left(\frac{dT_r}{du} - \Gamma^m_{rn} \frac{dx^n}{du} T_m\right) S^r \tag{7}$$

$$\tag{8}$$

As $T_r S^r$ is an invariant and thus is also $\frac{d(T_r S^r)}{du}$ and as S^r can be chosen arbitrarily (as long it is a contravariant tensor), implies that

$$\frac{dT_r}{du} - \Gamma^m_{rn} \frac{dx^n}{du} T_m$$

is covariant and thus also

$$\frac{\delta T_r}{\delta u} = \frac{dT_r}{du} - \Gamma^m_{rn} \frac{dx^n}{du} T_m$$

◆

2.27 p50-clarification 2.516

$$\frac{\delta T_{rs}}{\delta u} \equiv \frac{dT_{rs}}{du} - \Gamma^m_{rn} T_{ms} \frac{dx^n}{du} - \Gamma^m_{sn} T_{rm} \frac{dx^n}{du}$$

is a covariant vector.

We build an invariant $T_{rs}S^rU^s$ with S^r and U^s arbitrary contravariant tensors. The we know that $\frac{d(T_{rs}S^rU^s)}{du}$ is also an invariant. We have

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = \frac{dT_{rs}}{du} S^r U^s + T_{rs} \frac{dS^r}{du} U^s + T_{rs} S^r \frac{dU^s}{du} \tag{1}$$

with with S^r and U^s propagated parallelly along the curve. Then,

$$\frac{dS^r}{du} = -\Gamma^r_{mn} S^m \frac{dx^n}{du} \tag{2}$$

$$\frac{dU^s}{du} = -\Gamma^s_{mn} Y^m \frac{dx^n}{du} \tag{3}$$

(2), (3) in (1) gives

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = \frac{dT_{rs}}{du} S^r U^s - T_{rs} U^s \Gamma^r_{mn} S^m \frac{dx^n}{du} - T_{rs} S^r \Gamma^s_{mn} U^m \frac{dx^n}{du} \tag{4}$$

Changing the dummy indices in the second and third term gives:

$$\frac{d\left(T_{rs}S^rU^s\right)}{du} = \left(\frac{dT_{rs}}{du} - \Gamma^m_{rn} T_{ms} \frac{dx^n}{du} - \Gamma^m_{sn} T_{rm} \frac{dx^n}{du}\right) S^r U^s \tag{5}$$

As the left term is an invariant and S^r and U^s are arbitrary contravariant tensors, means that the expression in the brackets in the right part of the equation, is a covariant tensor.

◆

2.28 p51-exercise

Find the absolute derivative of T^r_{st}.

Define the invariant $I = D^r_{st} R^r S_s T_t$

$$I = D^r_{st} R^r S_s T_t \tag{1}$$

$$\Rightarrow A = \frac{dI}{du} = \frac{d\left(D^r_{st}\right)}{du} R^r S_s T_t + D^r_{st} S_s T_t \frac{d\left(R^r\right)}{du} + D^r_{st} R_r T_t \frac{d\left(S^s\right)}{du} + D^r_{st} R_r S_s \frac{d\left(T^t\right)}{du} \tag{2}$$

Reminder, performing a parallel propagation of a covariant and contravariant vector gives as equations

$$\frac{dV^v}{du} = -\Gamma^v_{mn} V^m \frac{dx^n}{du} \tag{3}$$

$$\frac{dW_w}{du} = +\Gamma^m_{wn} W^m \frac{dx^n}{du} \tag{4}$$

So (2) becomes:

$$A = \begin{cases} \frac{d\left(D^r_{st}\right)}{du} R^r S_s T_t \\ -D^r_{st} S_s T_t \Gamma^r_{mn} R^m \frac{dx^n}{du} \\ +D^r_{st} R_r T_t \Gamma^m_{sn} S^m \frac{dx^n}{du} \\ +D^r_{st} R_r S_s \Gamma^m_{tn} T^m \frac{dx^n}{du} \end{cases} \tag{5}$$

In (5) apply the following renaming of dummy variables

$$\begin{cases} 2^{nd} line: & r \rightarrow m, m \rightarrow r \\ 3^{rd} line: & s \rightarrow m, m \rightarrow s \\ 4^{th} line: & t \rightarrow m, m \rightarrow t \end{cases}$$

and regrouping terms with $R^r S_s T_t$,(5) becomes then

$$A = \left[\frac{d\left(D^r_{st}\right)}{du} + \left(D^r_{mt} \Gamma^s_{mn} + D^r_{sm} \Gamma^t_{mn} - D^m_{st} \Gamma^m_{rn} \right) \frac{dx^n}{du} \right] R^r S_s T_t \tag{6}$$

But A is an invariant, so the expression in the square parenthesis is a tensor of the form T^r_{st} and we define the absolute derivative of T^r_{st} as:

$$\frac{\delta T^r_{st}}{\delta u} = \frac{d\left(T^r_{st}\right)}{du} + \Gamma^s_{mn} T^r_{mt} \frac{dx^n}{du} + \Gamma^t_{mn} T^r_{sm} \frac{dx^n}{du} - \Gamma^m_{rn} T^m_{st} \frac{dx^n}{du}$$

◆

2.29 p53-exercise

Prove that

$$\delta^r_{s|t} = 0, \quad a^{rs}_{|t} = 0$$

i)$\delta^t_{s|t} = 0$

$$\text{(2.524) gives:} \quad \delta^r_{s|t} = \underbrace{\frac{\partial \delta^r_s}{\partial x^t}}_{=0} + \Gamma^r_{mt}\delta^m_s - \Gamma^m_{st}\delta^r_m \tag{1}$$

$$= \Gamma^r_{st} - \Gamma^r_{st} = 0 \tag{2}$$

ii)$a^{rs}_{|t} = 0$ We know that

$$\delta^r_{s|t} = a_{sk}a^{kr}|_t \tag{3}$$

$$\Rightarrow \delta^r_{s|t} = \frac{\partial a_{sk}}{\partial x^t}a^{kr} + a_{sk}\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a_{sk}a^{km} - \Gamma^m_{st}a_{mk}a^{kr} \tag{4}$$

Rearrange (4) and add $\Gamma^k_{mt}a_{ks}a^{mr}$ and subtract $\Gamma^m_{kt}a_{ms}a^{kr}$ (as $\Gamma^k_{mt}a_{ks}a^{mr} - \Gamma^m_{kt}a_{ms}a^{kr} = 0$)

$$\delta^r_{s|t} = (\frac{\partial a_{sk}}{\partial x^t} - \Gamma^m_{st}a_{mk} - \Gamma^m_{kt}a_{ms})a^{kr} + (\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a^{km} + \Gamma^k_{mt}a^{mr})a_{sk} \tag{5}$$

$$\text{but} \quad a_{sk|t} = (\frac{\partial a_{sk}}{\partial x^t} - \Gamma^m_{st}a_{mk} - \Gamma^m_{kt}a_{ms}) \tag{6}$$

$$\text{and as (2.526)} \quad a_{sk|t} = 0 \tag{7}$$

$$\text{(5) becomes} \quad \delta^r_{s|t} = \underbrace{(\frac{\partial a^{kr}}{\partial x^t} + \Gamma^r_{mt}a^{km} + \Gamma^k_{mt}a^{mr})}_{a^{kr}_{|t}} a_{sk} \tag{8}$$

$$= a^{kr}_{|t}a_{sk} \tag{9}$$

$$= 0 \quad \text{as} \quad \delta^r_{s|t} = 0 \quad \text{(see first part of this exercise)} \tag{10}$$

As all a_{ks} can't be zero and as we didn't choose any special Riemannian space, we can conclude from $a^{kr}_{|t}a_{sk} = 0$ that

$$a^{rs}_{|t} = 0$$

◆

2.30 p54-exercise

Prove that

$$\frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s}$$

$$\frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{d\lambda^n}{ds} \tag{1}$$

By definition of the absolute derivative, we have:

$$\frac{\delta\lambda^n}{\delta s} = \frac{d\lambda^n}{ds} + \Gamma^n_{pk}\lambda^p\frac{dx^k}{ds} \tag{2}$$

$$\text{(2) in (1)} \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\left(\frac{\delta\lambda^n}{\delta s} - \Gamma^n_{pk}\lambda^p\frac{dx^k}{ds}\right) \tag{3}$$

$$= \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2a_{mn}\lambda^m\Gamma^n_{pk}\lambda^p\frac{dx^k}{ds} \tag{4}$$

$$\text{we have} \quad \Gamma^n_{pk} = a^{ns}[pk,s] \tag{5}$$

$$\text{so,} \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2\underbrace{a_{mn}a^{ns}}_{=\delta^s_m}[pk,s]\lambda^m\lambda^p\frac{dx^k}{ds} \tag{6}$$

$$= \frac{da_{mn}}{ds}\lambda^m\lambda^n + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} - 2[pk,m]\lambda^m\lambda^p\frac{dx^k}{ds} \tag{7}$$

$$\text{but} \quad 2[pk,m]\lambda^m\lambda^p = [pk,m]\lambda^m\lambda^p + [pk,m]\lambda^m\lambda^p \tag{8}$$

$$= [pk,m]\lambda^m\lambda^p + [mk,p]\lambda^m\lambda^p \tag{9}$$

$$\text{we have also} \quad \begin{cases} [pk,m] = \frac{1}{2}(\partial_k a_{pm} + \partial_p a_{km} - \partial_m a_{pk}) \\ [mk,p] = \frac{1}{2}(\partial_k a_{pm} + \partial_m a_{pk} - \partial_p a_{mk}) \end{cases} \tag{10}$$

$$\Rightarrow 2[pk,m] = \partial_k a_{pm} \tag{11}$$

$$\text{so} \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = \frac{da_{mn}}{ds}\lambda^m\lambda^n - \underbrace{\partial_k a_{pm}\frac{dx^k}{ds}}_{=\frac{da_{mn}}{ds}}\lambda^m\lambda^p + 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} \tag{12}$$

$$\Rightarrow \quad \frac{d(a_{mn}\lambda^m\lambda^n)}{ds} = 2a_{mn}\lambda^m\frac{\delta\lambda^n}{\delta s} \tag{13}$$

◆

2.31 p54-exercise

Prove that

$$(T^r S_s)_{|n} = T^r_{|n} S_s + T^r S_{s|n}$$

$$\begin{aligned}
(T^r S_s)_{|n} &= \partial_n (T^r S_s) + \Gamma^r_{nm} T^m S_s - \Gamma^m_{sn} T^r S_m && (1)\\
&= \partial_n (T^r) S_s + T^r \partial_n (S_s) + S_s \Gamma^r_{nm} T^m - T^r \Gamma^m_{sn} S_m && (2)\\
&= T^r \underbrace{\left(\partial_n (S_s) - \Gamma^m_{sn} S_m\right)}_{S_{s|n}} + S_s \underbrace{\left(\partial_n (T^r) + \Gamma^r_{nm} T^m\right)}_{T^r_{|n}} && (3)\\
&= T^r S_{s|n} + S_s T^r_{|n} && (4)
\end{aligned}$$

◆

2.32 p57-exercise

Compute the Christoffel symbols in 2.540 directly from the definitions 2.421 and 2.422. Check that all Christoffels symbols not shown explicitly in 2.540 vanish.

Easy but very tedious, not reproduced yet, later perhaps

◆

2.33 p57-exercise

Show that for the spherical polar metric 2.532, we have $ln\sqrt{a} = 2ln(x^1) + ln(sin(x^2))$ and

$$\textbf{2.544} \quad \Gamma^n_{1n} = \frac{2}{x^1}, \quad \Gamma^n_{2n} = \cot\left(x^2\right), \quad \Gamma^n_{3n} = 0$$

The spherical polar metric 2.532 is,

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1 \sin\left(x^2\right))^2 \end{pmatrix} \tag{1}$$

$$\Rightarrow |a_{mn}| = \left[(x^1)^2 \sin\left(x^2\right)\right]^2 \tag{2}$$

$$\Rightarrow ln(\sqrt{|a_{mn}|}) = 2ln(x^1) + ln(\sin\left(x^2\right)) \tag{3}$$

$$\Rightarrow \begin{cases} \Gamma^n_{1n} = \partial_1(ln(\sqrt{a})) = \frac{2}{x^1} \\ \\ \Gamma^n_{2n} = \partial_2(ln(\sqrt{a})) = \frac{\cos\left(x^2\right)}{\sin\left(x^2\right)} = \cot\left(x^2\right) \\ \\ \Gamma^n_{3n} = 0 \end{cases} \tag{4}$$

◆

2.34 p58-exercise

Show that for the spherical polar metric

$$\textbf{2.546} \quad T^n_{\ |n} = \frac{1}{r^2}\partial_r(r^2\ T^1) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\ T^2) + \partial_\phi T^3$$

Obtain a similar expression for the "Laplacian" ΔV of an invariant V defined as

$$\textbf{2.547} \quad \Delta V = \left(a^{mn}\partial_m V\right)_{|n}$$

We have

$$\textbf{(2.545)} \quad T^n_{\ |n} = \frac{1}{\sqrt{a}}\partial_n(\sqrt{a}\ T^n) \tag{1}$$

$$\text{and from the previous exercise p.58:} \quad \sqrt{a} = (x^1)^2\sin\left(x^2\right) \tag{2}$$

$$\Rightarrow \quad T^n_{\ |n} = \frac{1}{\sqrt{a}}\left(\sin\left(x^2\right)\partial_1[(x^1)^2T^1] + (x^1)^2\partial_2[\sin\left(x^2\right)T^2] + (x^1)^2\sin\left(x^2\right)\partial_3 T^3\right) \tag{3}$$

$$= \frac{1}{x^1}\partial_1[(x^1)^2T^1] + \frac{1}{\sin\left(x^2\right)}\partial_2[\sin\left(x^2\right)T^2] + \partial_3 T^3 \tag{4}$$

Replace in (4) $\quad x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$

$$T^n_{\ |n} = \frac{1}{r^2}\partial_r[r^2\ T^r] + \frac{1}{\sin\theta}\partial_\theta[\sin\theta\ T^\theta] + \partial_\phi T^\phi \tag{5}$$

Let's calculate the Laplacian.

$$\Delta V = \left(a^{mn}\partial_m V\right)_{|n} \tag{6}$$

$$\text{be} \qquad G^n = a^{mn}\partial_m V \tag{7}$$

$$\text{then (see exercise p.32)} \qquad \begin{cases} G^1 = \partial_r V \\ G^2 = \frac{1}{r^2}\partial_\theta V \\ G^1 = \frac{1}{r^2\sin^2\theta}\partial_\phi V \end{cases} \tag{8}$$

and by the previous result of this exercise

$$\Delta V = G^n_{|n} = \frac{1}{r^2}\partial_r[r^2\ G^1] + \frac{1}{\sin\theta}\partial_\theta[\sin\theta\ G^2] + \partial_\phi G^3 \tag{9}$$

$$= \frac{1}{r^2}\partial_r[r^2\partial_r V] + \frac{1}{r^2\sin\theta}\partial_\theta[\sin\theta\ \partial_\theta V] + \frac{1}{r^2\sin^2\theta}\partial^2_\phi V \tag{10}$$

◆

2.35 p62-exercise

Prove that if a pair of vectors are unit orthogonal vectors at a point on a curve, and if they are both propagated parallelly along the curve, then they remain unit orthogonal vectors along the curve.

Given is, a pair of vectors U^m and V^m which are unit orthogonal vectors at a point on a curve. So,

$$\begin{aligned} &\text{U is a unit vector (2.302)} & a_{mn}U^mU^n = \epsilon \\ &\text{V is a unit vector (2.302)} & a_{mn}V^mV^n = \epsilon \\ &\text{U, V are orthogonal (2.317)} & a_{mn}U^mV^n = 0 \end{aligned} \tag{1}$$

at one point on the curve.
We have to prove that the above properties are valid along the curve (i.e. $\forall$ points on the curve) provided that the vectors are propagated // along the curve, which means

$$\textbf{(2.512)} \qquad \frac{\delta U^r}{\delta u} = \frac{dU^r}{du} + \Gamma^r_{mn}U^m\frac{dx^n}{du} = 0 \tag{2}$$

for both vectors U, V. Thus,

$$\frac{dU^r}{du} = -\Gamma^r_{mn}U^m\frac{dx^n}{du} \tag{3}$$

i) Consider the magnitude M at a random point on the curve

$$M = a_{mn}U^mU^n \tag{4}$$

$$\Rightarrow \qquad \frac{dM}{ds} = \frac{da_{mn}}{du}U^mU^n + 2a_{mn}U^m\frac{dU^n}{du} \tag{5}$$

Obviously M and $\frac{dM}{ds}$ are invariants. Also, we can choose at any point on the curve a Riemaniann coordinate system (RCS) for which the Christoffel symbols vanish at that point. Hence, $\frac{dU^r}{du} = 0$ at that point and the second term in the right part of (5) vanish. (5) becomes then,

$$\frac{dM}{ds} = \frac{\partial a'_{mn}}{\partial x'^k}\frac{dx'^k}{ds}U'^mU'^n \tag{6}$$

We also know **(2.425. page 41)** that $[km, n]' + [kn, m]' = \frac{\partial a'_{mn}}{\partial x'^k}$. But in the chosen coordinate system, $[km, n] = 0$ at the origin of this coordinate system. So by (6) we get $\frac{dM}{ds} = 0$.
So the magnitude is constant along the curve and as we know that a certain point $M = 1$:

U,V are unit vectors along the curve

ii) Consider now the angle between the vectors U, V. Be $A = \cos\ \theta$.By definition

$$A = a_{mn} U^m V^n \tag{7}$$

$$\Rightarrow \qquad \frac{dA}{ds} = \frac{da_{mn}}{du} U^m V^n + a_{mn} (V^m \frac{dU^n}{du} + U^m \frac{dV^n}{du}) \tag{8}$$

We follow the same reasoning as in i) and so

$$\frac{dA}{ds} = 0$$

So, the angle is constant and we know is is $\frac{\pi}{2}$ at a certain point. So,:

U,V are orthogonal along the curve

◆

2.36 p62-exercise

Given that λ^r is a unit vector field, prove that

$$\lambda^r_{|s}\lambda_r = 0 \quad \text{and} \quad \lambda^r\lambda_{r|s} = 0$$

Is the relation $\lambda^r_{|s}\lambda_s = 0$ true for a general unit vector field?

To simplify the calculation, we choose a random element in the unit vector field and use at that point a Riemaniann coordinate system (RCS). So, we have

$$\lambda^r_{|s} = \partial_s\lambda^r \quad \text{and} \quad \lambda_{r|s} = \partial_s\lambda_r \tag{1}$$

$$\text{as we have a unit vector fields:} \qquad a_{mn}\lambda^m\lambda^n = 1 \tag{2}$$

$$\Leftrightarrow \lambda_n\lambda^n = 1 \qquad \text{(by lowering the index m)} \tag{3}$$

$$\Rightarrow \qquad \lambda^n\partial_s\lambda_n + \lambda_n\partial_s\lambda^n = 0 \tag{4}$$

We prove that

$$\lambda^n\partial_s\lambda_n = \lambda_n\partial_s\lambda^n \quad \forall \text{ vector fields}$$

We have the trivial identity

$$\partial_s(\lambda^r\lambda_r) = \partial_s(\lambda^r\lambda_r) \tag{5}$$

$$\Leftrightarrow \qquad \partial_s(a^{rm}\lambda_m\lambda_r) = \partial_s(a_{rm}\lambda^m\lambda^r) \tag{6}$$

Lemma *: $\partial_s a^{rm} = 0$ in a Riemaniann coordinate system (i.e. at the origin)*

We have

$$a^{rm}a_{ms} = \delta^r_s \tag{7}$$

$$\Rightarrow a_{ms}\partial_k a^{rm} + a^{rm}\partial_k a_{ms} = 0 \tag{8}$$

$$\textit{we know (2.618)} \qquad \partial_r a_{mn} = 0 \qquad \textit{at the origin of a RCS} \tag{9}$$

$$\textit{so (8) becomes} \qquad a_{ms}\partial_k a^{rm} = 0 \tag{10}$$

$$\textit{multiply (10) by} \quad a^{ns} \qquad \Rightarrow \underbrace{a^{ns}a_{ms}}_{=\delta^n_m}\partial_k a^{rm} = 0 \tag{11}$$

$$\Rightarrow \qquad \partial_k a^{rn} = 0 \tag{12}$$

◇

Now, expanding (6) and using **2.618** and the lemma:

$$a^{rm}\lambda_m\partial_s\lambda_r + a^{rm}\lambda_r\partial_s\lambda_m = a_{rm}\lambda^m\partial_s\lambda^r + a_{rm}\lambda^r\partial_s\lambda^m \tag{13}$$

$$\text{renaming dummy indices:} \qquad a^{rm}\lambda_r\partial_s\lambda_m = a_{rm}\lambda^r\partial_s\lambda^m \tag{14}$$

$$\Rightarrow \qquad \lambda^m\partial_s\lambda_m = \lambda_m\partial_s\lambda^m \tag{15}$$

Considering (5) and (15) we conclude:

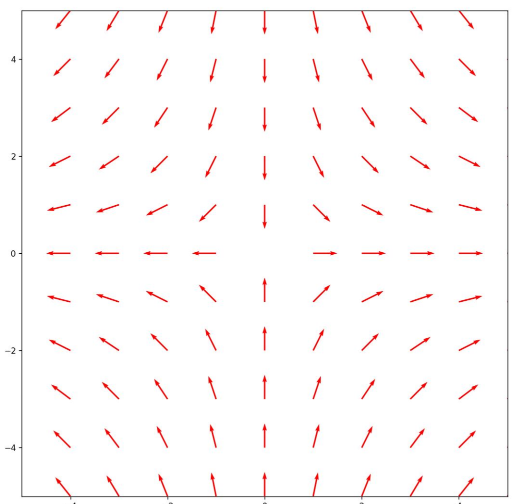
$$\lambda^n\partial_s\lambda_n = \lambda_n\partial_s\lambda^n = 0 \tag{16}$$

and as $\partial_s\lambda_n = \lambda_{n|s}$ and $\partial_s\lambda^n = \lambda^n{}_{|s}$ at the origin of the considered coordinate system, we have:

$$\lambda^m\lambda_{n|s} = \lambda_m\lambda^n{}_{|s} = 0 \tag{17}$$

◇

Is the relation $\lambda^r_{|s}\lambda_s = 0$ true for a general unit vector field?
The answer is NO. Let's consider the following unit vector field in a Cartesian Coordinate system:



$$V : \mathbb{R}^2_* \to \mathbb{R}^2 | V(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

Put $\rho = \sqrt{x^2+y^2}$, we get (as we have a Cartesian Coordinate system, the calculation simplify as the Christoffel symbols vanish and the covariant components of the vectors are equal to their contravariant part):

$$\begin{cases} V^1 = V_1 = +\frac{x}{r} \\ V^2 = V_2 = -\frac{y}{r} \end{cases} \tag{18}$$

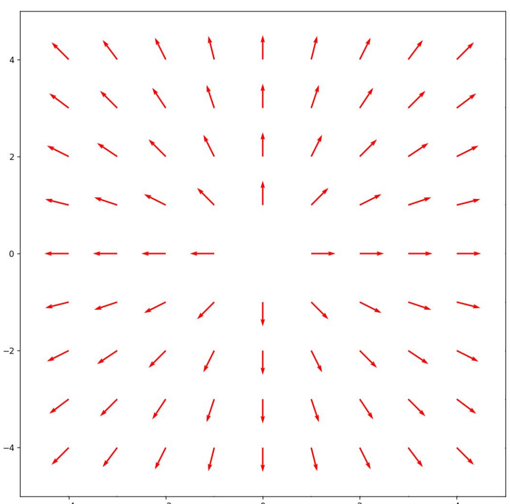
$$\begin{cases} V^1_{\ |1} = V_{1|1} = \frac{y^2}{r^3} & V^1_{\ |2} = V_{1|2} = -\frac{xy}{r^3} \\ V^2_{\ |1} = V_{2|1} = \frac{xy}{r^3} & V^2_{\ |2} = V_{2|2} = -\frac{x^2}{r^3} \end{cases} \tag{19}$$

$$\Rightarrow \begin{cases} V^1_{|s}V_s = V^1_{|1}V_1 + V^1_{|2}V_2 = \frac{y^2}{r^3}\frac{x}{r} + (-\frac{xy}{r^3})(-\frac{y}{r}) = \frac{xy^2}{r^4} \neq 0 \\ V^2_{|s}V_s = V^2_{|1}V_1 + V^2_{|2}V_2 = \frac{xy}{r^3}\frac{x}{r} + (-\frac{y}{r})(-\frac{y}{r}) = \frac{x^2y}{r^4} \neq 0 \end{cases} \tag{20}$$

Just as a check, we calculate $V^r_{|s}V_r$ which should be zero:

$$\Rightarrow \begin{cases} V^s_{|1}V_s = V^1_{|1}V_1 + V^2_{|1}V_2 = (+\frac{y^2}{r^3})\frac{x}{r} + (+\frac{xy}{r^3})(-\frac{y}{r}) = 0 \\ V^s_{|2}V_s = V^1_{|2}V_1 + V^2_{|2}V_2 = (-\frac{xy}{r^3})\frac{x}{r} + (-\frac{y}{r})(-\frac{x^2}{r^3}) = 0 \end{cases} \tag{21}$$

Now, let's consider another unit vector field in a Cartesian Coordinate system:



$$V : \mathbb{R}^2_* \to \mathbb{R}^2 | V(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$\begin{cases} V^1 = V_1 = +\frac{x}{r} \\ V^2 = V_2 = +\frac{y}{r} \end{cases} \tag{22}$$

$$\begin{cases} V^1_{\ |1} = V_{1|1} = \frac{y^2}{r^3} & V^1_{\ |2} = V_{1|2} = -\frac{xy}{r^3} \\ V^2_{\ |1} = V_{2|1} = -\frac{xy}{r^3} & V^2_{\ |2} = V_{2|2} = +\frac{x^2}{r^3} \end{cases} \tag{23}$$

$$\Rightarrow \begin{cases} V^1_{|s}V_s = V^1_{|1}V_1 + V^1_{|2}V_2 = (+\frac{y^2}{r^3})\frac{x}{r} + (-\frac{xy}{r^3})(+\frac{y}{r}) = 0 \\ V^2_{|s}V_s = V^2_{|1}V_1 + V^2_{|2}V_2 = (-\frac{xy}{r^3})\frac{x}{r} + (+\frac{y}{r})(+\frac{y}{r}) = 0 \end{cases} \tag{24}$$

Just as a check, we calculate $V^r_{|s}V_r$ which should be zero:

$$\Rightarrow \quad \begin{cases} V^s_{|1}V_s = V^1_{|1}V_1 + V^2_{|1}V_2 = (+\frac{y^2}{r^3})(+\frac{x}{r}) + (-\frac{xy}{r^3})(+\frac{y}{r}) = 0 \\ V^s_{|2}V_s = V^1_{|2}V_1 + V^2_{|2}V_2 = (-\frac{xy}{r^3})(+\frac{x}{r}) + (+\frac{y}{r}) + \frac{x^2}{r^3}) = 0 \end{cases} \tag{25}$$

So, in the second example the relationship $\lambda^r_{|s}\lambda_s = 0$ holds. Question (to investigate further and later) : does the fact that in the first case $\overline{\nabla} \times \overline{V} \neq 0$ and in the second case $\overline{\nabla} \times \overline{V} = 0$, means there some relation with this expression?

◆

2.37 p64-clarification 2.625

2.625 $\frac{dx^r}{dx^N} = \frac{X^r}{X^N}$

Be C a surface defined by the function $F(x^1, \ldots, x^{N-1}) = C$ and $c_\perp$ the curve intersecting the surface C perpendicularly at a point p.
Along the curve at that point p we have

$$\text{as} \quad \begin{cases} \frac{dx^r}{ds} & \text{is the tangent vector along} \quad c_\perp \\ X^r = a^{rn}\frac{\partial F}{\partial x^n} & \text{is orthogonal on the surface (2.623)} \quad C \end{cases} \tag{1}$$

$$\Rightarrow \quad \frac{dx^r}{ds} = kX^r \tag{2}$$

So, $\frac{dx^r}{ds}$ is proportional to X^r (as the curve intersects the surface orthogonally). This means that also all the components (coordinates) of both quantities are proportional. And so,

$$\frac{\frac{dx^r}{ds}}{\frac{dx^N}{ds}} = \frac{kX^r}{kX^N} \tag{3}$$

$$\Rightarrow \quad \frac{dx^r}{dx^N} = \frac{X^r}{X^N} \tag{4}$$

◆

2.38 p65-exercise

Deduce from **2.629** that

$$a^{N\rho} = 0 \qquad a^{NN} = \frac{1}{a_{NN}}$$

We have (see 2.629):

$$a_{N\rho} = 0 \tag{1}$$

$$\text{and also} \qquad a_{Nm} a^{ms} = \delta^s_N \tag{2}$$

In (2) split the m index in the subspace and the remaining coordinate N

$$a_{N\rho} a^{\rho s} + a_{NN} a^{Ns} = \delta^s_N \tag{3}$$

$$\text{as} \quad a_{N\rho} = 0 \Rightarrow \qquad a_{NN} a^{Ns} = \delta^s_N \tag{4}$$

Case 1: $s \neq N$

$$a_{NN} a^{Ns} = 0 \quad \Longleftrightarrow \quad a_{NN} a^{N\rho} = 0 \qquad (\text{as} \quad s \neq N) \tag{5}$$

$$\text{as we suppose} \quad a_{NN} \neq 0 \quad \Rightarrow \quad a^{N\rho} = 0 \tag{6}$$

Case 2: $s = N$

$$a_{NN} a^{NN} = 1 \tag{7}$$

$$\Rightarrow \quad a^{NN} = \frac{1}{a_{NN}} \tag{8}$$

◆

2.39 p69-clarification on 2.645

In 2.645 we have

$$T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\alpha\beta} T_N$$

and

$$T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N$$

Indeed,

$$\begin{aligned}
&& T^N &= a^{mN} T_m && (1)\\
&& &= a^{\alpha N} T_\alpha + a^{NN} T_N && (2)\\
&\text{but (2.631)} & a^{\alpha N} &= 0 && (3)\\
&\Rightarrow & T^N &= a^{NN} T_N && (4)\\
\text{as}\quad a^{NN} = \frac{1}{a_{NN}} &\Rightarrow & T_{\alpha|\beta} &= T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N && (5)
\end{aligned}$$

◆

2.40 p69-exercise

Show that

$$\textbf{2.648} \qquad T^{\alpha}_{\;|\beta} = T^{\alpha}_{\;||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N$$

$$\textbf{2.649} \qquad T^{N}_{\;|\alpha} = \partial_\alpha T^N - \frac{1}{2a_{NN}}\partial_N a_{\alpha\mu}T^\mu + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$$

$$\textbf{2.650} \qquad T^{\alpha}_{\;|N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$$

i) $T^{\alpha}_{\;|\beta} = T^{\alpha}_{\;||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N$

$$\textbf{(2.520)} \quad \Rightarrow \quad T^{\alpha}_{\;|\beta} = \partial_\beta T^\alpha + \Gamma^{\alpha}_{m\beta}T^m \qquad (m = 1,\dots,N) \tag{1}$$

$$\Leftrightarrow \quad T^{\alpha}_{\;|\beta} = \underbrace{\partial_\beta T^\alpha + \Gamma^{\alpha}_{\mu\beta}T^\mu}_{T^{\alpha}_{\;||\beta}} + \Gamma^{\alpha}_{N\beta}T^N \tag{2}$$

$$\textbf{(2.639)} \quad \Gamma^{\alpha}_{N\beta} = \frac{1}{2}a^{\alpha\mu}\partial_N a^{\mu\beta} \tag{3}$$

$$\text{(2) and (3):} \quad T^{\alpha}_{\;|\beta} = T^{\alpha}_{\;||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta}T^N \tag{4}$$

◇

Remark: We also use **(2.639)** for the two other identities.

ii) $T^{N}_{\;|\alpha} = \partial_\alpha T^N - \frac{1}{2a_{NN}}\partial_N a_{\alpha\mu}T^\mu + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N$

$$\textbf{(2.520)} \quad \Rightarrow \quad T^{N}_{\;|\alpha} = \partial_\alpha T^N + \Gamma^{N}_{m\alpha}T^m \qquad (m = 1,\dots,N) \tag{5}$$

$$\Leftrightarrow \quad T^{N}_{\;|\alpha} = \partial_\beta T^\alpha + \underbrace{\Gamma^{N}_{\sigma\alpha}}_{-\frac{1}{2a_{NN}}\partial_N a_{\sigma\alpha}} T^\sigma + \underbrace{\Gamma^{N}_{N\alpha}}_{\frac{1}{2a_{NN}}\partial_\alpha a_{NN}} T^N \tag{6}$$

$$\Rightarrow \quad T^{N}_{\;|\alpha} = \partial_\beta T^\alpha - \frac{1}{2a_{NN}}\partial_N a_{\sigma\alpha}T^\sigma + \frac{1}{2a_{NN}}\partial_\alpha a_{NN}T^N \tag{7}$$

iii) $T^{\alpha}_{\;|N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N$

$$\textbf{(2.520)} \quad \Rightarrow \quad T^{\alpha}_{\;|N} = \partial_N T^\alpha + \Gamma^{\alpha}_{mN}T^m \qquad (m = 1,\dots,N) \tag{8}$$

$$\Leftrightarrow \quad T^{\alpha}_{\;|N} = \partial_N T^\alpha + \underbrace{\Gamma^{\alpha}_{\sigma N}}_{\frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}} T^\sigma + \underbrace{\Gamma^{\alpha}_{NN}}_{-\frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}} T^N \tag{9}$$

$$\Rightarrow \quad T^{\alpha}_{\;|N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma}T^\sigma - \frac{1}{2}a^{\alpha\mu}\partial_\mu a_{NN}T^N \tag{10}$$

◆

2.41 p71-exercise

Write down equation 2.643 tot 2.650 for the special cae of a geodesic normal coordinate system.

$$
\begin{aligned}
&\textbf{(2.643)} \quad T_{\alpha||\beta} = \partial_\beta T_\alpha - \Gamma^\gamma_{\alpha\beta} T_\gamma \qquad \text{(does not change)} && (1)\\
&\textbf{(2.644)} \quad T_{\alpha|\beta} = T_{\alpha||\beta} - \underbrace{\Gamma^N_{\alpha\beta}}_{=\frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\alpha\beta}} T_N && (2)\\
&\qquad = T_{\alpha||\beta} + \frac{1}{2}\epsilon \partial_N a_{\alpha\beta} T_N && (3)\\
&\textbf{(2.645)} \quad T_{\alpha|\beta} = T_{\alpha||\beta} + \frac{1}{2}\partial_N a_{\alpha\beta} T^N \qquad \text{(does not change)} && (4)\\
&\textbf{(2.646)} \quad T_{N|\alpha} = \partial_\alpha T_N - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N && (5)\\
&\qquad = \partial_\alpha T_N - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu && (6)\\
&\textbf{(2.647)} \quad T_{\alpha|N} = \partial_N T_\alpha - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N && (7)\\
&\qquad = \partial_N T_\alpha - \frac{1}{2}\partial_N a_{\mu\alpha} T^\mu && (8)\\
&\textbf{(2.648)} \quad T^\alpha_{\ |\beta} = T^\alpha_{||\beta} + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\beta} T_N \qquad \text{(does not change)} && (9)\\
&\textbf{(2.649)} \quad T^N_{\ |\alpha} = \partial_\alpha T^N - \frac{1}{2}\frac{1}{a_{NN}}\partial_N a_{\mu\alpha} T^\mu - \frac{1}{2}\frac{1}{a_{NN}}\underbrace{\partial_\alpha a_{NN}}_{=0} T^N && (10)\\
&\qquad = \partial_\alpha T^N - \frac{1}{2}\epsilon \partial_N a_{\mu\alpha} T^\mu && (11)\\
&\textbf{(2.650)} \quad T^\alpha_{\ |N} = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma} T^\sigma - \frac{1}{2}a^{\alpha\mu}\underbrace{\partial_\mu a_{NN}}_{=0} T^N && (12)\\
&\qquad = \partial_N T^\alpha + \frac{1}{2}a^{\alpha\mu}\partial_N a_{\mu\sigma} T^\sigma && (13)
\end{aligned}
$$

To investigate: note (3) and (4) which suggest that $\epsilon T_N = T^N$. Prove formally?

◆

2.42 p73-Clarification 2.706

... Let us now define a unit vector $\lambda^r_{(2)}$ and a positive invariant $\kappa_{(2)}$ by the equation

$$\begin{cases} \frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \epsilon\epsilon_{(1)}\kappa_{(1)}\lambda^r \\ \epsilon_{(2)}\lambda^n_{(2)}\lambda_{(2)n} = 1 \end{cases} \tag{1}$$

We can state that $\kappa_{(2)}$ is an invariant but one has to check whether the expression (1) implies that $\kappa_{(2)}$ is indeed invariant.
What we know is that $\lambda^r, \frac{\delta\lambda^r}{\delta s}, \frac{\delta\lambda^r_{(1)}}{\delta s}, \lambda^r_{(2)}$ are contravariant vectors. Also $\kappa_{(1)}$ is an invariant as $\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ and the magnitude of $\frac{\delta\lambda^r}{\delta s}$ does not depend on the coordinate system. So,

$$(1)\times\lambda_{(2)r} \quad \Rightarrow \quad \frac{\delta\lambda^r_{(1)}}{\delta s}\lambda_{(2)r} = \kappa_{(2)}\lambda^r_{(2)}\lambda_{(2)r} - \epsilon\epsilon_{(1)}\kappa_{(1)}\lambda^r\lambda_{(2)r} \tag{2}$$

$$\Rightarrow \quad \kappa_{(2)}\underbrace{\lambda^r_{(2)}\lambda_{(2)r}}_{\text{invariant}} = \underbrace{\frac{\delta\lambda^r_{(1)}}{\delta s}\lambda_{(2)r}}_{\text{invariant}} + \underbrace{\epsilon\epsilon_{(1)}}_{\text{invariant}}\underbrace{\kappa_{(1)}}_{\text{invariant}}\underbrace{\lambda^r\lambda_{(2)r}}_{\text{invariant}} \tag{3}$$

$$\Rightarrow \quad \kappa_{(2)} = \text{invariant} \tag{4}$$

◆

2.43 p74-Clarification 2.710

$$\textbf{2.710} \qquad \begin{cases} \frac{\delta\lambda^r_{(M-1)}}{\delta s} = \kappa_{(M)}\lambda^r_{(M)} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\lambda^r_{(M-2)} \\ \epsilon_{(M-1)}\lambda^n_{(M-1)}\lambda_{(M-1)n} = 1 \qquad \text{(M=1,2,...,N)} \end{cases} \tag{1}$$

... It is easily proved by mathematical induction that the whole sequence of vectors defined by 2.710 are perpendicular to the tangent and to one another ...

We already know from 2.703 to 2.709 that $\lambda^r, \lambda^r_{(1)}, \lambda^r_{(2)}, \lambda^r_{(3)}$, satisfying equations (1), are all mutually perpendicular. Let us assume that the orthogonality for the set $\{\lambda^r_{(k)} : k = 0, 1, 2, 3, ..., M-1\}$ has been verified. We prove by induction that then, $\lambda^r_{(M)}$ will be orthogonal to all elements of the set.

i) Consider the set $\{\lambda^r_{(k)} : k = 0, 1, 2, 3, ..., M-3\}$ where we already know that $\lambda^r_{(k)}$ are mutually perpendicular and also $\lambda^r_{(k)} \perp \lambda^r_{(M-1)}$, $\lambda^r_{(k)} \perp \lambda^r_{(M-2)}$ and $\lambda^r_{(M-1)} \perp \lambda^r_{(M-1)} \quad \forall k$.

$$(1) \times \lambda_{(k)r} \quad \Rightarrow \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(k)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(k)r} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\lambda^r_{(M-2)}\lambda_{(k)r}}_{=0} \tag{2}$$

$$\text{We have} \qquad \lambda_{(k)r}\lambda^r_{(M-1)} = 0 \tag{3}$$

$$\Rightarrow \qquad \frac{\delta\lambda_{(k)r}\lambda^r_{(M-1)}}{\delta s} = \lambda^r_{(M-1)}\frac{\delta\lambda_{(k)r}}{\delta s} + \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = 0 \tag{4}$$

$$\Rightarrow \qquad \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = -\lambda^r_{(M-1)}\frac{\delta\lambda_{(k)r}}{\delta s} \tag{5}$$

$$\text{We have} \qquad \frac{\delta\lambda_{(k)r}}{\delta s} = \kappa_{(k+1)}\lambda_{(k+1)r} - \epsilon_{(k)}\epsilon_{(k-1)}\kappa_{(k)}\lambda_{(k-1)r} \tag{6}$$

$$(5) \text{ and } (6) \Rightarrow \qquad \lambda_{(k)r}\delta\frac{\lambda^r_{(M-1)}}{\delta s} = -\kappa_{(k+1)}\underbrace{\lambda_{(k+1)r}\lambda^r_{(M-1)}}_{=0} - \epsilon_{(k)}\epsilon_{(k-1)}\kappa_{(k)}\underbrace{\lambda_{(k-1)r}\lambda^r_{(M-1)}}_{=0} \tag{7}$$

$$\text{From } (2) \Rightarrow \qquad \kappa_{(M)}\lambda^r_{(M)}\lambda_{(k)r} = 0 \tag{8}$$

$$\Rightarrow \qquad \lambda^r_{(M)} \perp \lambda_{(k)r} \qquad \forall k = 0, 1, 2, 3, ..., M-3 \tag{9}$$

ii) Consider the case $k = M-1$

$$(1) \times \lambda_{(M-1)r} \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-1)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-1)r} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\lambda^r_{(M-2)}\lambda_{(M-1)r}}_{=0} \tag{10}$$

$$\text{from (2.530):} \qquad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-1)r} = \frac{1}{2}\underbrace{\frac{\delta\lambda_{(M-1)r}\lambda^r_{(M-1)}}{\delta s}}_{=0 \text{ as } \lambda_{(M-1)r}\lambda^r_{(M-1)} = \epsilon_{(M-1)}} \tag{11}$$

$$\Rightarrow \qquad \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-1)r} = 0 \tag{12}$$

$$\Rightarrow \qquad \lambda_{(M-1)r} \perp \lambda_{(M)r} \tag{13}$$

iii) Consider the case $k = M - 2$

$$(1) \times \lambda_{(M-2)r} \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-2)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\lambda^r_{(M-2)}\lambda_{(M-2)r}}_{=\epsilon_{(M-2)}} \tag{14}$$

$$\Rightarrow \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-2)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} - \epsilon_{(M-1)}\kappa_{(M-1)}\underbrace{\epsilon_{(M-2)}\epsilon_{(M-2)}}_{=1} \tag{15}$$

$$\Rightarrow \quad \frac{\delta\lambda^r_{(M-1)}}{\delta s}\lambda_{(M-2)r} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} - \epsilon_{(M-1)}\kappa_{(M-1)} \tag{16}$$

$$\text{We have} \quad \lambda^r_{(M-1)}\lambda_{(M-2)r} = 0 \tag{17}$$

$$\Rightarrow \quad \lambda_{(M-2)r}\frac{\delta\lambda^r_{(M-1)}}{\delta s} = -\lambda^r_{(M-1)}\frac{\delta\lambda_{(M-2)r}}{\delta s} \tag{18}$$

$$\text{We have also} \quad \frac{\delta\lambda_{(M-2)r}}{\delta s} = \kappa_{(M-1)}\lambda_{(M-1)r} - \epsilon_{(M-3)}\epsilon_{(M-2)}\kappa_{(M-2)}\lambda_{(M-3)r} \tag{19}$$

$$(19) \times \lambda^r_{(M-1)} \text{ and (18): } \lambda_{(M-2)r}\frac{\delta\lambda^r_{(M-1)}}{\delta s} = -\kappa_{(M-1)}\underbrace{\lambda^r_{(M-1)}\lambda_{(M-1)r}}_{=\epsilon_{(M-1)}} - \epsilon_{(M-3)}\epsilon_{(M-2)}\kappa_{(M-2)}\underbrace{\lambda^r_{(M-3)}\lambda_{(M-1)r}}_{=0} \tag{20}$$

$$\Rightarrow \quad \lambda_{(M-2)r}\frac{\delta\lambda^r_{(M-1)}}{\delta s} = -\kappa_{(M-1)}\epsilon_{(M-1)} \tag{21}$$

$$\text{(16) and (21):} \quad -\kappa_{(M-1)}\epsilon_{(M-1)} = \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} - \epsilon_{(M-1)}\kappa_{(M-1)} \tag{22}$$

$$\Rightarrow \quad \kappa_{(M)}\lambda^r_{(M)}\lambda_{(M-2)r} = 0 \tag{23}$$

$$\Rightarrow \quad \lambda_{(M-2)r} \perp \lambda_{(M)r} \tag{24}$$

With, i), ii), iii) all possible case are covered which makes the proof complete.

◆

2.44 p75-Clarification 2.714

2.714 $\left(\kappa_{(1)}\right)^2 = \epsilon_{(1)} a_{mn} \frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s}, \qquad \epsilon_{(1)} = \pm 1$

$$\frac{\delta\lambda^n}{\delta s} = \kappa_{(1)}\lambda^n_{(1)} \tag{1}$$

$$(1)\times(1) \quad \Rightarrow \quad \frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s} = \left(\kappa_{(1)}\right)^2 \lambda^m_{(1)}\lambda^n_{(1)} \tag{2}$$

$$(2)\times a_{mn} \Rightarrow \quad a_{mn}\frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s} = a_{mn}\left(\kappa_{(1)}\right)^2 \lambda^m_{(1)}\lambda^n_{(1)} \tag{3}$$

$$= \left(\kappa_{(1)}\right)^2 \underbrace{\lambda_{(1)m}\lambda^n_{(1)}}_{=\epsilon_{(1)}} \tag{4}$$

$$= \left(\kappa_{(1)}\right)^2 \tag{5}$$

◆

2.45 p75-exercise

For positive definite metric forms, write out explicitly the Frenet formulae for the case N=2, 3 and 4.

The general Frenet formulae are

$$\begin{cases} \frac{\delta\lambda^r_{(M-1)}}{\delta s} = \kappa_{(M)}\lambda^r_{(M)} - \epsilon_{(M-2)}\epsilon_{(M-1)}\kappa_{(M-1)}\lambda^r_{(M-2)} \\ \epsilon_{(M-1)}\lambda^n_{(M-1)}\lambda_{(M-1)n} = 1 \qquad \text{(M=1,2,...,N)} \end{cases} \tag{1}$$

As Φ is positive definite, we have $\epsilon_{(k)} = 1 \quad \forall k$

N=2	N=3	N=4
$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(3)}}{\delta s} = \kappa_{(4)}\lambda^r_{(4)} - \kappa_{(3)}\lambda^r_{(2)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$ $\lambda^n_{(3)}\lambda_{(3)n} = 1$

Taking into account that $\kappa_{(N)} = 0$ for a space V_N, we get,

N=2	N=3	N=4
$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = -\kappa_{(1)}\lambda^r$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = -\kappa_{(2)}\lambda^r_{(1)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$	$\frac{\delta\lambda^r}{\delta s} = \kappa_{(1)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(1)}}{\delta s} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r$ $\frac{\delta\lambda^r_{(2)}}{\delta s} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}$ $\frac{\delta\lambda^r_{(3)}}{\delta s} = -\kappa_{(3)}\lambda^r_{(2)}$ $\lambda^n\lambda_n = 1$ $\lambda^n_{(1)}\lambda_{(1)n} = 1$ $\lambda^n_{(2)}\lambda_{(2)n} = 1$ $\lambda^n_{(3)}\lambda_{(3)n} = 1$

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