

TENSOR CALCULUS
J.L. SYNGE AND A.SCHILD (DOVER PUBLICATION)
SOLUTIONS TO EXERCISES
PART II
CHAPTERS V TO VIII

by

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Relative tensors, ideas of volume,
Green-Stokes theorems.

7.1 p241 - Exercise

If b_{rs} is an absolute tensor, show that the determinant $|b_{rs}|$ is a relative invariant of weight 2. What are the tensor characters of $|c^{rs}|$ and $|f_s^r|$?

As b_{rs} is an absolute tensor, we have

$$b'_{uv} = b_{rs} \frac{\partial x^r}{\partial x'^u} \frac{\partial x^s}{\partial x'^v} \quad (1)$$

Hence,

$$|b'_{uv}| = |b_{rs}| \left| \frac{\partial x^r}{\partial x'^u} \right| \left| \frac{\partial x^s}{\partial x'^v} \right| \quad (2)$$

and as $J = \left| \frac{\partial x^k}{\partial x'^s} \right|$ we get

$$|b'_{uv}| = J^2 |b_{rs}| \quad (3)$$

Conclusion, $|b_{rs}|$ is a relative invariant of weight 2.

◇

As c^{rs} is an absolute tensor, we have

$$c'^{uv} = c^{rs} \frac{\partial x'^u}{\partial x^r} \frac{\partial x'^v}{\partial x^s} \quad (4)$$

Hence,

$$|c'^{uv}| = |c^{rs}| \left| \frac{\partial x'^u}{\partial x^r} \right| \left| \frac{\partial x'^v}{\partial x^s} \right| \quad (5)$$

and as $J^{-1} = \left| \frac{\partial x'^s}{\partial x^k} \right|$ we get

$$|c'^{uv}| = J^{-2} |c^{rs}| \quad (6)$$

Conclusion, $|c^{rs}|$ is a relative invariant of weight -2 .

◇

As f_s^r is an absolute tensor, we have

$$f_v'^u = f_s^r \frac{\partial x'^u}{\partial x^r} \frac{\partial x^s}{\partial x'^v} \quad (7)$$

Hence,

$$\left| f_v'^u \right| = \left| f_s^r \right| \left| \frac{\partial x'^u}{\partial x^r} \right| \left| \frac{\partial x^s}{\partial x'^v} \right| \quad (8)$$

and we get

$$\left| f_v'^u \right| = J J^{-1} \left| f_s^r \right| \quad (9)$$

Conclusion, $|f_s^r|$ is an absolute invariant tensor .



7.2 p242 - Exercise

Show that, in three dimensions, the only non-vanishing components of δ_{rs}^{kl} are

$$\delta_{23}^{23} = \delta_{32}^{32} = \delta_{31}^{31} = \delta_{13}^{13} = \delta_{12}^{12} = \delta_{21}^{21} = 1$$

$$\delta_{32}^{23} = \delta_{23}^{32} = \delta_{13}^{31} = \delta_{31}^{13} = \delta_{21}^{12} = \delta_{12}^{21} = -1$$

This is easily seen. If $(k, l), (r, s)$ are considered as sets, then $\delta_{rs}^{kl} \neq 0 \Leftrightarrow (k, l) \neq (r, s)$. And $\delta_{rs}^{kl} = 1 \Leftrightarrow k = r \wedge l = s$ and on the opposite $\delta_{rs}^{kl} = -1 \Leftrightarrow k = s \wedge l = r$



7.3 p243 - Exercise

Show that equations **5.231** and **6.128** can be written as follows:

$$M_{rs} = \delta_{rs}^{kl} z_k F_l$$

$$\omega_{rs} = \frac{1}{2} \delta_{rs}^{kl} v_{l,k}$$

$$(5.231) \quad M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r \quad (1)$$

In this expression $M_{rs} = 0$ when $r = s$, but this is also the case with δ_{rs}^{kl} .

In $M_{rs} = \delta_{rs}^{kl} z_k F_l$ we see that there is no contribution in the summation when $k = l$. The only contribution being those for which $k = r \wedge l = s$ (positive contribution) \vee $k = s \wedge l = r$ (negative contribution), hence

$$\delta_{rs}^{kl} z_k F_l \Leftrightarrow z_r F_s - z_s F_r$$

◇

$$(6.128) \quad \omega_{rs} = \frac{1}{2} (v_{s,r} - v_{r,s}) \quad (2)$$

The same arguments of the previous case apply to this case (a way to see this is to represent symbolically, $z_r F_s$ and $v_{s,r}$ by T_{rs})

◆

7.4 p243 - Exercise

If $T_{k_1 k_2 \dots k_M}$ is completely skew-symmetric, determine

$$\delta_{s_1 s_2 \dots s_M}^{k_1 k_2 \dots k_M} T_{k_1 k_2 \dots k_M}$$

$\delta_{s_1 s_2 \dots s_M}^{k_1 k_2 \dots k_M} T_{k_1 k_2 \dots k_M}$ is a sum of $M!$ terms: the first of these is $T_{s_1 s_2 \dots s_M}$; the other terms are obtained from it by permuting the subscripts and a minus sign is attached if the permutation is odd. Since $T_{s_1 s_2 \dots s_M}$ is completely skew-symmetric, each of the $M!$ terms equals $\pm T_{s_1 s_2 \dots s_M}$. Hence,

$$\delta_{s_1 s_2 \dots s_M}^{k_1 k_2 \dots k_M} T_{k_1 k_2 \dots k_M} = M! T_{s_1 s_2 \dots s_M}$$



7.5 p245 - Exercise

Show that $\epsilon^{r_1 r_2 \dots r_N} \epsilon_{r_1 r_2 \dots r_N} = N!$.

First note that $\text{sign}(\epsilon^{r_1 r_2 \dots r_N}) = \text{sign}(\epsilon_{r_1 r_2 \dots r_N})$ so that each term in the summation is always +1.

There are N choices to chose from for r_1 , $N - 1$ for r_2 , etc. and only one for r_N . And so $\epsilon^{r_1 r_2 \dots r_N} \epsilon_{r_1 r_2 \dots r_N} = N!$



7.6 p245 - Clarification to 7.113

$$\epsilon^{k_1 \dots k_M r_1 \dots r_{N-M}} \epsilon_{s_1 \dots s_M r_1 \dots r_{N-M}} = (N - M)! \delta_{s_1 \dots s_M}^{k_1 \dots k_M}$$

This can be seen as followed.

As the permutation $(r_1 \dots r_{N-M})$ is the same for both covariant and contravariant permutation symbols, the product $\epsilon^{k_1 \dots k_M r_1 \dots r_{N-M}} \epsilon_{s_1 \dots s_M r_1 \dots r_{N-M}}$ for a fixed permutation $(r_1 \dots r_{N-M})$ (i.e. no summation on repeated indexes) will be determined by $\delta_{s_1 \dots s_M}^{k_1 \dots k_M}$. Indeed, the difference in "oddness" between $(k_1 \dots k_M r_1 \dots r_{N-M})$ and $(s_1 \dots s_M r_1 \dots r_{N-M})$ is only determined by the difference in "oddness" between $(k_1 \dots k_M r_1)$ and $(s_1 \dots s_M)$. So each term in the summation has the same contribution, i.e.; $\delta_{s_1 \dots s_M}^{k_1 \dots k_M}$.

There are M choices to chose from for r_1 , $M - 1$ for r_2 , etc. and only one for r_M . And so

$$\epsilon^{k_1 \dots k_M r_1 \dots r_{N-M}} \epsilon_{s_1 \dots s_M r_1 \dots r_{N-M}} = (N - M)! \delta_{s_1 \dots s_M}^{k_1 \dots k_M}$$



7.7 p247 - Exercise

If T_{rs} is an absolute skew-symmetric tensor in a 4-space, show that

$$T_{14}T_{23} + T_{24}T_{31} + T_{34}T_{12}$$

is a tensor density

Be $P = T_{14}T_{23} + T_{24}T_{31} + T_{34}T_{12}$, we can write this as $P = \frac{1}{8}\delta_{1234}^{ijmn}T_{ij}T_{mn}$

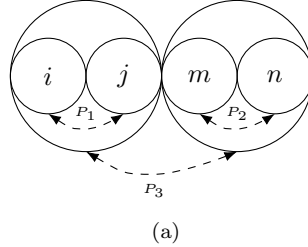


Figure 7.1: Permutations

The factor $\frac{1}{8}$ is explained by the fact that there are 2^3 possible permutations in the i, j, m, n indexes i.e. 2×2 for the permutations P_1 and P_2 and again 2 for the permutation P_3 . Note that a single permutation P_1 or P_2 changes the sign of $T_{ij}T_{mn}$ but also changes the sign of δ_{1234}^{ijmn} , so the combined sign doesn't change. A double permutation P_1 and P_2 changes the sign of T_{ij} and T_{mn} resulting in an unchanged sign of $T_{ij}T_{mn}$ but also δ_{1234}^{ijmn} is unchanged because of the double permutation. Finally P_3 has no effect, nor on $T_{ij}T_{mn}$ nor on δ_{1234}^{ijmn} . So we have 8 repetitions for the same set i, j, m, n . So we have,

$$P = \frac{1}{8}\delta_{1234}^{ijmn}T_{ij}T_{mn} \quad (1)$$

$$= \frac{1}{8}\epsilon^{ijmn}T_{ij}T_{mn} \quad (2)$$

From this follows immediately as ϵ^{ijmn} is a relative tensor of weight 1 and T_{ij} an absolute tensor (i.e. a relative tensor of weight 0) that P is a relative tensor of weight 1 i.e. a density.



7.8 p247 - Exercise

Show that, for rectangular Cartesian coordinates, the vorticity tensor and the vorticity vector of a fluid are duals (cf. **6.130**).

6.130:

$$\omega_r = \frac{1}{2} \epsilon_{rmn} \omega_{mn}, \quad \omega_{mn} = \epsilon_{rmn} \omega_r \quad (1)$$

Put $\hat{T}^r = \omega_r$ and $T_{mn} = \omega_{mn}$ then the expressions in (1) can be expressed as (considering that the covariant and contravariant expressions are identical in rectangular Cartesian coordinates)

$$\hat{T}^r = \frac{1}{(3-2)!} \epsilon^{mnr} T_{mn}, \quad T_{mn} = \epsilon_{rmn} \hat{T}^r \quad (2)$$

which are exactly the general definitions **7.121** and **7.122** (with $N = 3$ and $M = 2$) for dual tensors.



7.9 p250 - Exercise

Show that

$$\eta_{r_1 \dots r_N} = \epsilon(a) a_{r_1 s_1} \dots a_{r_N s_N} \eta^{s_1 \dots s_N}$$

$$\eta^{r_1 \dots r_N} = \epsilon(a) a^{r_1 s_1} \dots a^{r_N s_N} \eta_{s_1 \dots s_N}$$



7.10 p252 - Exercise

Using Riemannian coordinates, prove that **7.216** $\epsilon_{r_1 \dots r_N | k} = \epsilon^{r_1 \dots r_N}_{|k} = 0$ $\eta_{r_1 \dots r_N | k} = \eta^{r_1 \dots r_N}_{|k} = 0$



7.11 p252 - Exercise

Prove that if T^n is a relative vector of weight W then,

$$\mathbf{7.220} \quad T^n|_n = (\epsilon(a)a)^{\frac{1}{2}(W-1)} \frac{\partial}{\partial x^n} \left[(\epsilon(a)a)^{\frac{1}{2}(1-W)} T^n \right]$$



7.12 p255 - Clarification

$$\mathbf{7.304} \quad \Delta^{k_1 \dots k_M} = \delta_{s_1 \dots s_M}^{k_1 \dots k_M} \Delta_{(1)} x^{s_1} \dots \Delta_{(M)} x^{s_M}$$

Using **7.303** and noting that a permutation of columns in the matrix changes or not the sign of its determinant depending on the sign of the permutation, we can write

$$\Delta^{k_1 \dots k_M} = \begin{vmatrix} \Delta_{(1)} x^{k_1} & \Delta_{(1)} x^{k_2} & \dots & \Delta_{(1)} x^{k_M} \\ \Delta_{(2)} x^{k_1} & \Delta_{(2)} x^{k_2} & \dots & \Delta_{(2)} x^{k_M} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{(M)} x^{k_1} & \Delta_{(M)} x^{k_2} & \dots & \Delta_{(M)} x^{k_M} \end{vmatrix} \quad (1)$$

$$= \epsilon^{k_1 k_2 \dots k_M} \begin{vmatrix} \Delta_{(1)} x^1 & \Delta_{(1)} x^2 & \dots & \Delta_{(1)} x^M \\ \Delta_{(2)} x^1 & \Delta_{(2)} x^2 & \dots & \Delta_{(2)} x^M \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{(M)} x^1 & \Delta_{(M)} x^2 & \dots & \Delta_{(M)} x^M \end{vmatrix} \quad (2)$$

And using **4.313** : $|A_{pq}| = \epsilon_{s_1 s_2 \dots s_M} A_{1s_1} A_{2s_2} \dots A_{Ms_M}$:

$$\Delta^{k_1 \dots k_M} = \underbrace{\epsilon^{k_1 k_2 \dots k_M} \epsilon_{s_1 s_2 \dots s_M}}_{\text{see (7.114)}} \Delta_{(1)} x^{s_1} \Delta_{(2)} x^{s_2} \dots \Delta_{(M)} x^{s_M} \quad (3)$$

$$= \delta_{s_1 \dots s_M}^{k_1 \dots k_M} \Delta_{(1)} x^{s_1} \Delta_{(2)} x^{s_2} \dots \Delta_{(M)} x^{s_M} \quad (4)$$



7.13 p255 - Exercise

Show that **7.305** may be written in the equivalent form

$$d\tau_{(M)}^{k_1 \dots k_m} = \epsilon^{\beta_1 \dots \beta_M} d_{(\beta_1)} x^{k_1} \dots d_{(\beta_M)} x^{k_M}$$

The determinant of a matrix and its transpose are equal.

Hence we can rewrite **7.305** $\delta_{s_1 \dots s_M}^{k_1 \dots k_M} d_{(1)} x^{s_1} \dots d_{(M)} x^{s_M}$ as

$$d\tau_{(M)}^{k_1 \dots k_m} = \delta_{k_1 \dots k_M}^{s_1 \dots s_M} d_{(s_1)} x^1 \dots d_{(s_M)} x^M \quad (1)$$

In order to be consistent with the notation we replace the s_i by α_i as the summation occurs along the constants $c^{(i)}$

$$d\tau_{(M)}^{k_1 \dots k_m} = \delta_{k_1 \dots k_M}^{\alpha_1 \dots \alpha_M} d_{(\alpha_1)} x^1 \dots d_{(\alpha_M)} x^M \quad (2)$$

Given the set $\{k_1, k_2, \dots, k_M\}$ we can represent the sequence $\{1, 2, \dots, M\}$ by $\{k_j, k_m, \dots, k_M, \dots, k_n\}$ (imagine that $k_j = 1, k_m = 2, \dots$ etc.). We rewrite (2) as

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = (\theta_\alpha) \delta_{12 \dots M}^{\alpha_1 \dots \alpha_M} d_{(\alpha_1)} x^{k_j} d_{(\alpha_2)} x^{k_m} \dots d_{(\alpha_M)} x^{k_n} \quad (3)$$

where

$$\theta_\alpha = \epsilon_{k_1 k_2 \dots k_M} \quad (4)$$

(a permutation in the lower indexes of the generalized Kronecker deltas symbol will invert the sign depending on the 'oddness' of the permutation). Let's rearrange the product $d_{(\alpha_1)} x^{k_j} d_{(\alpha_2)} x^{k_m} \dots d_{(\alpha_M)} x^{k_n}$ so that the indexes k_i are naturally ordered

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = (\theta_\alpha) \delta_{12 \dots M}^{\alpha_1 \dots \alpha_M} d_{(\alpha_r)} x^{k_1} d_{(\alpha_n)} x^{k_2} \dots d_{(\alpha_1)} x^{k_n} \dots d_{(\alpha_s)} x^{k_M} \quad (5)$$

and changing the order in the upper indexes of the general Kroneckers delta's:

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = (\theta_\alpha)(\theta_k) \delta_{12 \dots M}^{\alpha_r \alpha_n \dots \alpha_s} d_{(\alpha_r)} x^{k_1} d_{(\alpha_n)} x^{k_2} \dots d_{(\alpha_1)} x^{k_n} \dots d_{(\alpha_s)} x^{k_M} \quad (6)$$

where $\theta_k = \pm 1$ depending on the 'oddness' of the permutation needed to go from $\{\alpha_1 \dots \alpha_M\}$ to $\{\alpha_r \alpha_n \dots \alpha_s\}$.

As we can see in figure 7.2, it's no hard to see that

$$\theta_k = \theta_\alpha \quad (7)$$

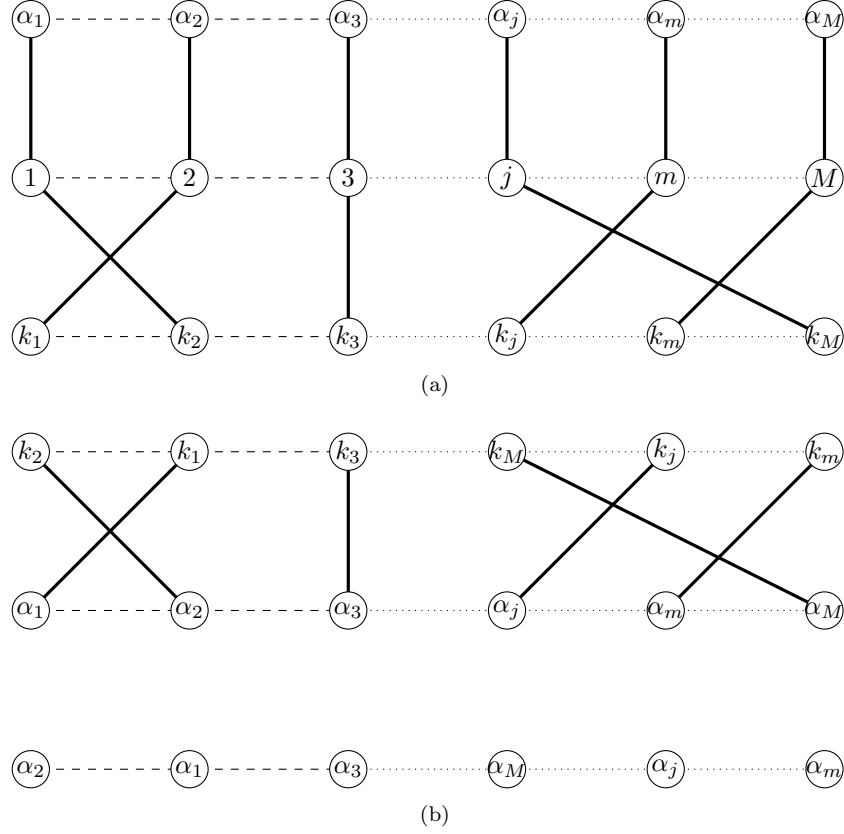


Figure 7.2: Permutations

Indeed suppose, as in the example (a), $k_1 = 2, k_2 = 1, k_3 = 3, \dots, k_j = m, \dots, k_m = j, \dots$ etc., so we get a sequence $\{k_2, k_1, k_3, \dots, k_m, \dots, k_j, \dots\}$ as illustrated in (b). But to have - with this sequence - an equivalent expression of $d\tau_{(M)}^{k_1 k_2 \dots k_m} = (\theta_\alpha) \delta_{12 \dots M}^{\alpha_1 \dots \alpha_M} d_{(\alpha_1)} x^{k_j} d_{(\alpha_2)} x^{k_m} \dots d_{(\alpha_M)} x^{k_n}$, we need to make an equivalent permutation so that α_r gets in the same position as k_r , resulting in a new sequence $\{\alpha_2, \alpha_1, \alpha_3, \dots, \alpha_M, \dots, \alpha_j, \alpha_m\}$.

The number of permutations to generate θ_α and θ_k are identical resulting in $\theta_\alpha \theta_k = 1$. So (6) can be rewritten (noting that the α_r are dummy indexes and that we are free to rename them so that $r = 1, n = 2, \dots$)

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = \delta_{12 \dots M}^{\beta_1 \beta_2 \dots \beta_M} d_{(\beta_1)} x^{k_1} d_{(\beta_2)} x^{k_2} \dots d_{(\beta_n)} x^{k_n} \dots d_{(\beta_M)} x^{k_M} \quad (8)$$

Finally, using **7.114**

$$d\tau_{(M)}^{k_1 k_2 \dots k_m} = \underbrace{\epsilon_{12 \dots M}}_{=1} \epsilon^{\beta_1 \beta_2 \dots \beta_M} d_{(\beta_1)} x^{k_1} d_{(\beta_2)} x^{k_2} \dots d_{(\beta_n)} x^{k_n} \dots d_{(\beta_M)} x^{k_M} \quad (9)$$

$$= \epsilon^{\beta_1 \beta_2 \dots \beta_M} d_{(\beta_1)} x^{k_1} d_{(\beta_2)} x^{k_2} \dots d_{(\beta_n)} x^{k_n} \dots d_{(\beta_M)} x^{k_M} \quad (10)$$



7.14 p257 - Exercise

Let x^k be rectangular Cartesian coordinates in Euclidean 3-space. Introduce polar coordinates r, θ, ϕ and consider the surface of the sphere $r = a$. On this sphere form the infinitesimal 2-cell with corners (θ, ϕ) , $(\theta + d\theta, \phi)$, $(\theta, \phi + d\phi)$, $(\theta + d\theta, \phi + d\phi)$. Determine the extension of this cell and interpret the rectangular components. In particular, show that the three independent components of the extension are (apart from the sign) equal to the areas obtained by normal projection of the cell onto the three rectangular planes. Does this interpretation remain valid if the sphere is replaced by some other surface?

We use **7.312**:

$$d\tau_{(2)}^{k_1 k_2} = \epsilon^{\alpha_1 \alpha_2} \frac{\partial x^{k_1}}{\partial y^{\alpha_1}} \frac{\partial x^{k_2}}{\partial y^{\alpha_2}} |d_{(\beta)} y^\gamma| \quad (1)$$

with $(y^1, y^2) = (\theta, \phi)$ giving if we take $f^{(i)} = c^{(i)}$ as $\theta = c^{(1)}$, $\phi = c^{(2)}$:

$$|d_{(\beta)} y^\gamma| = \begin{vmatrix} d_{(1)} y^1 & d_{(1)} y^2 \\ d_{(2)} y^1 & d_{(2)} y^2 \end{vmatrix} \quad (2)$$

$$= \begin{vmatrix} d\theta & 0 \\ 0 & d\phi \end{vmatrix} \quad (3)$$

$$= d\theta d\phi \quad (4)$$

We also have

$$\begin{cases} x = a \sin \theta \cos \phi \\ y = a \sin \theta \sin \phi \\ z = a \cos \theta \end{cases} \quad (5)$$

$$(6)$$

giving

$$\begin{cases} \frac{\partial x}{\partial \theta} = a \cos \theta \cos \phi \\ \frac{\partial x}{\partial \phi} = -a \sin \theta \sin \phi \\ \frac{\partial y}{\partial \theta} = a \cos \theta \sin \phi \\ \frac{\partial y}{\partial \phi} = a \sin \theta \cos \phi \\ \frac{\partial z}{\partial \theta} = -a \sin \theta \\ \frac{\partial z}{\partial \phi} = 0 \end{cases} \quad (7)$$

$$(8)$$

and get

$$d\tau_{(2)}^{xy} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \phi} d\theta d\phi + \epsilon^{21} \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \phi} d\theta d\phi \quad (9)$$

$$= a^2 \cos \theta \cos \phi \sin \theta \cos \phi d\theta d\phi + a^2 \sin \theta \sin \phi \cos \theta \sin \phi d\theta d\phi \quad (10)$$

$$= a^2 \cos \theta \sin \theta d\theta d\phi \quad (11)$$

$$d\tau_{(2)}^{yx} = -a^2 \cos \theta \sin \theta d\theta d\phi \quad (12)$$

$$d\tau_{(2)}^{xz} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial x}{\partial \theta} \frac{\partial z}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial x}{\partial \theta} \underbrace{\frac{\partial z}{\partial \phi}}_{=0} d\theta d\phi + \epsilon^{21} \frac{\partial x}{\partial \phi} \frac{\partial z}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial x}{\partial \phi} \frac{\partial z}{\partial \phi} d\theta d\phi \quad (13)$$

$$= -a^2 \sin^2 \theta \sin \phi d\theta d\phi \quad (14)$$

$$d\tau_{(2)}^{zx} = a^2 \sin^2 \theta \sin \phi d\theta d\phi \quad (15)$$

$$d\tau_{(2)}^{yz} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial y}{\partial \theta} \frac{\partial z}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial y}{\partial \theta} \underbrace{\frac{\partial z}{\partial \phi}}_{=0} d\theta d\phi + \epsilon^{21} \frac{\partial y}{\partial \phi} \frac{\partial z}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial y}{\partial \phi} \frac{\partial z}{\partial \phi} d\theta d\phi \quad (16)$$

$$= a^2 \sin^2 \theta \cos \phi d\theta d\phi \quad (17)$$

$$d\tau_{(2)}^{zy} = -a^2 \sin^2 \theta \cos \phi d\theta d\phi \quad (18)$$

$$d\tau_{(2)}^{xx} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \phi} d\theta d\phi + \epsilon^{21} \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} d\theta d\phi \quad (19)$$

$$= 0 \quad (20)$$

$$d\tau_{(2)}^{yy} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \phi} d\theta d\phi + \epsilon^{21} \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \phi} d\theta d\phi \quad (21)$$

$$= 0 \quad (22)$$

$$d\tau_{(2)}^{zz} = \underbrace{\epsilon^{11}}_{=0} \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \theta} d\theta d\phi + \epsilon^{12} \frac{\partial z}{\partial \theta} \underbrace{\frac{\partial z}{\partial \phi}}_{=0} d\theta d\phi + \epsilon^{21} \underbrace{\frac{\partial z}{\partial \phi}}_{=0} \frac{\partial z}{\partial \theta} d\theta d\phi + \underbrace{\epsilon^{22}}_{=0} \frac{\partial z}{\partial \phi} \frac{\partial z}{\partial \phi} d\theta d\phi \quad (23)$$

$$= 0 \quad (24)$$

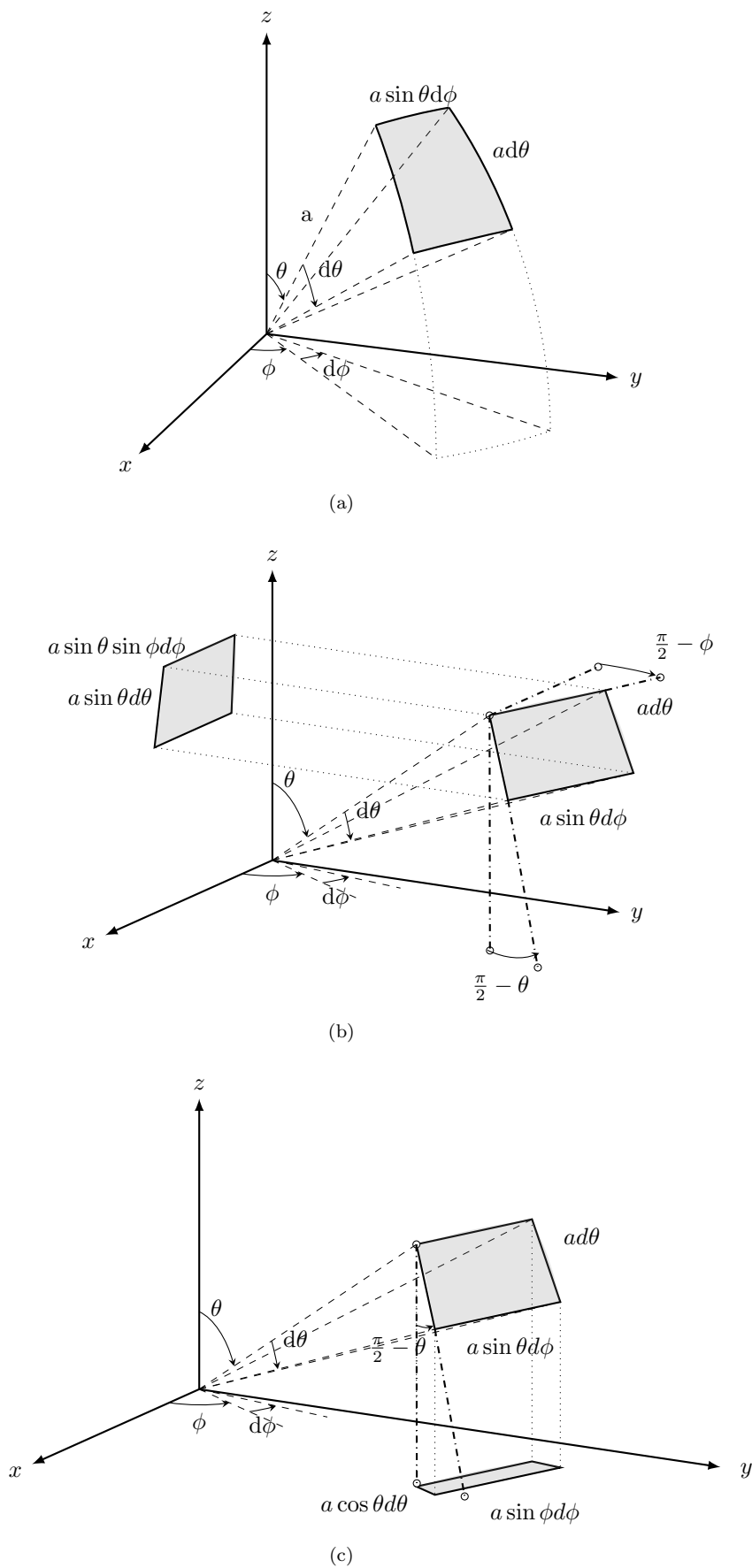


Figure 7.3: Projections of extensions

The quantities $d\tau_{(2)}^{xy}$, $d\tau_{(2)}^{xz}$, $d\tau_{(2)}^{yz}$ are the projections of the extension on the respective Cartesian coordinates planes as can be seen in figure 7.3 where figure (a) depicts the extension (area = $a^2 \sin \theta d\theta d\phi$) when choosing θ, ϕ as the parameters y^k , while figure (b) represents the projection of this extension on the xz -plane and figure (c) represents the projection of this extension on the xy -plane.

◇

Does this interpretation remain valid if the sphere is replaced by some other surface?

The answer is no. For a two-space in Cartesian coordinates system and with surface with parameters (u, v) , equation (2) reduces to

$$d\tau_{(2)}^{xy} = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du dv \quad (25)$$

So $d\tau_{(2)}^{xy} = 0$ if $\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 0$. Consider the disk defined by the following parametric function

$$S : \left\{ \mathbb{R}^2 \rightarrow \mathbb{R}^3 : S(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2 + C}}, \frac{v}{\sqrt{u^2 + v^2 + C}}, 1 \right) \right\}$$

(the constant C is there just to avoid the undefinedness of the surface for $(u, v) = (0, 0)$).

It is easy to see that $\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 0$, yet the surface is parallel with the xy -plane, which implies that the projection on the xy -plane of an elementary cell on S will not have a zero area as can be seen in the figure hereunder.

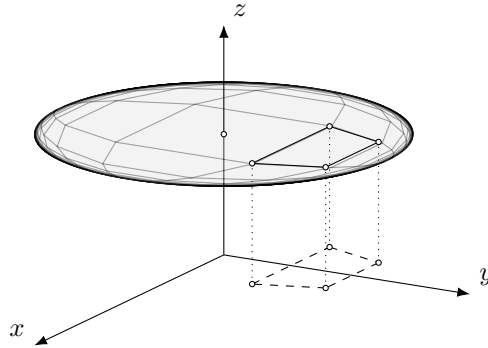


Figure 7.4: A disk defined as $S : \left\{ \mathbb{R}^2 \rightarrow \mathbb{R}^3 : S(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2 + C}}, \frac{v}{\sqrt{u^2 + v^2 + C}}, 1 \right) \right\}$

Conclusion: The interpretation of $d\tau_{(2)}^{k_1 k_2}$ as the projection of a cell on a axis-plane, does not hold for every surface.

◆

7.15 p263 - Exercise

Using polar coordinates in Euclidean 3-space find the volume of an infinitesimal cell whose edges are tangent to the coordinate curves. Obtain the volume of a sphere by integration.

For polar spherical coordinates we have

$$(a_{mn}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (1)$$

giving

$$|a_{mn}| = r^4 \sin^2 \theta \quad (2)$$

and using as parameters the $x^k \equiv (r, \theta, \phi)$ as parameters for the parametric surface we get

$$|d_{(s)}x^k| = \begin{vmatrix} dr & 0 & 0 \\ 0 & d\theta & 0 \\ 0 & 0 & d\phi \end{vmatrix} \quad (3)$$

$$= dr d\theta d\phi \quad (4)$$

Using **7.405**:

$$dv_{(N)}^2 = \epsilon(a) |a_{mn}| |d_{(s)}x^k|^2 \quad (5)$$

$$= r^4 \sin^2 \theta (dr d\theta d\phi)^2 \quad (6)$$

getting for the volume of a sphere with radius R :

$$V = 8 \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} dv_{(N)} \quad (7)$$

$$= 8 \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r^2 \sin \theta dr d\theta d\phi \quad (8)$$

$$= \frac{4}{3} \pi R^3 \quad (9)$$



7.16 p265 - Exercise

In the relativistic theory of finite, expanding universe, the following line element is adopted:

$$ds^2 = R^2 [dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)] - dt^2$$

where $R = R(t)$ is a function of the "time" t . the ranges of the coordinates may be taken to be $0 \leq r \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, $-\infty < t < +\infty$.

Find the total volume of "space", i.e., of the surface $t = \text{constant}$, and show that it varies with the "time" t as $R^3(t)$.

For the considered metric, we have

$$(a_{mn}) = \begin{pmatrix} R^2 & 0 & 0 & 0 \\ 0 & R^2 \sin^2 r & 0 & 0 \\ 0 & 0 & R^2 \sin^2 r \sin^2 \theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1)$$

Using as parameters the $x^k \equiv (r, \theta, \phi)$ as parameters for the parametric surface and using **7.409** : $b_{\alpha\beta} = a_{ks} \frac{\partial x^k}{\partial y^\alpha} \frac{\partial x^s}{\partial y^\beta}$, we get for the 3-space $t = \text{constant}$

$$(b_{mn}) = \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2 \sin^2 r & 0 \\ 0 & 0 & R^2 \sin^2 r \sin^2 \theta \end{pmatrix} \quad (2)$$

giving

$$|b_{mn}| = R^6 \sin^4 r \sin^2 \theta \quad (3)$$

Using **7.413**:

$$dv_{(M)}^2 = \frac{\epsilon(b)}{M!} a_{k_1 s_1} \dots a_{k_M s_M} d\tau_{(M)}^{k_1 \dots k_M} d\tau_{(M)}^{s_1 \dots s_M} \quad (4)$$

$$= -\frac{1}{6} a_{k_1 s_1} \dots a_{k_M s_M} d\tau_{(M)}^{k_1 \dots k_M} d\tau_{(M)}^{s_1 \dots s_M} \quad (5)$$

$$= -\frac{6}{6} a_{11} a_{22} a_{33} \left(\underbrace{d\tau_{(M)}^{123}}_{=drd\theta d\phi} \right)^2 \quad (6)$$

$$= -R^6 \sin^4 r \sin^2 \theta (drd\theta d\phi)^2 \quad (7)$$

$$\Rightarrow dv_{(M)} = R^3 \sin^2 r \sin \theta drd\theta d\phi \quad (8)$$

getting for the volume of "space" with "radius" R :

$$V = \int_0^\pi \int_0^\pi \int_0^{2\pi} R^3 \sin^2 r \sin \theta dr d\theta d\phi \quad (9)$$

$$= 2R^3 \pi \int_0^R \sin^2 r dr \underbrace{\int_0^\pi \sin \theta d\theta}_{=-\cos \theta|_0^\pi} \quad (10)$$

$$= 4R^3 \pi \underbrace{\int_0^R \sin^2 r dr}_{=\frac{1}{2}(\theta - \frac{1}{2}\sin(2r))|_0^\pi} \quad (11)$$

$$= 2\pi^2 R^3 \quad (12)$$

(the integral in (11) can be found by substituting $\sin^2 r = 1 - \cos^2 r$ and using the cosine sum of angles rule $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ with $\alpha = \beta = r$).

◇

In order to try to understand a little bit better what manifold is represented by the metric, let's use the trick to embed it in an higher dimensional space with an Euclidean metric.

Therefore let's use the following map:

$$\begin{cases} x = R \sin r \sin \theta \cos \phi \\ y = R \sin r \sin \theta \sin \phi \\ z = R \sin r \cos \theta \\ w = R \cos r \end{cases} \quad (13)$$

then it's easy to see that

$$ds_{(4)}^2 = dx^2 + dy^2 + dz^2 + dw^2 \quad (14)$$

$$= R^2 [dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (15)$$

which is exactly the hypersurface we sought. Now let's see what happens when we keep $\phi = 0$ or π . Then $y = 0$ and in fact we take a slice of the 4-space along the xzw subspace, and can visualize this 3-space as a sphere as represented by figure 7.5(a).

We can also keep r at a certain value. We get also a sphere as represented in figure 7.5(b). Here the biggest possible sphere corresponds to $r = (2k+1)\frac{\pi}{2}$, $k = \{\dots, -1, 0, 1, \dots\}$ all other spheres having a smaller radius. When r tends to $r = k\pi$ $k = \{\dots, -1, 0, 1, \dots\}$ the sphere shrinks to a point with coordinates $(0, 0, 0, \pm R)$.

Finally, keeping $\theta = k\pi$, $k = \{\dots, -1, 0, 1, \dots\}$ we get a circle of radius R situated in the xw plane.

Remember that what we do here is just take 'slices' of a space. compare it to an Euclidean 3-space with a sphere: taking a slice e.g. parallel to the xy plane will give us a circle, a point or an empty set while taking a one-dimensional slice along a line will give us 2, 1 or 0 point(s).

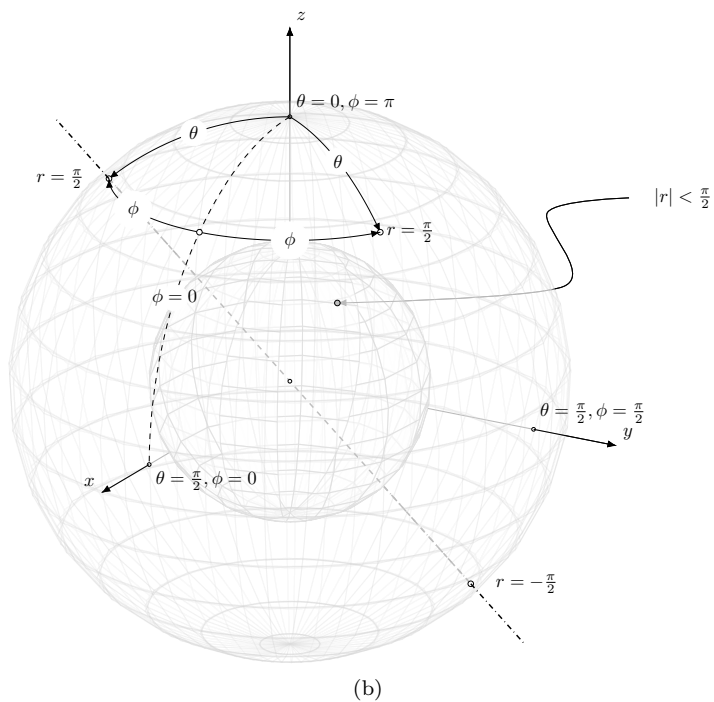
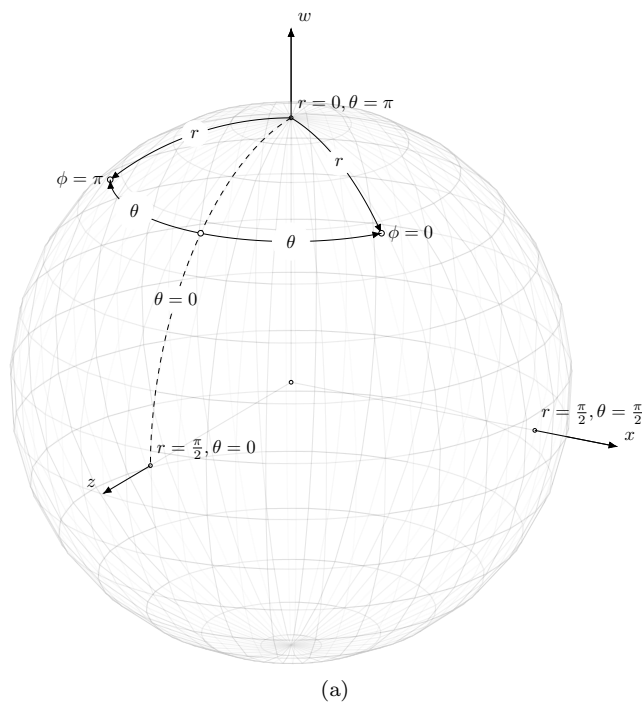


Figure 7.5: Manifold with $ds^2 = R^2 [dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)]$ metric, embedded in an Euclidean 4-space



7.17 p268 - Clarification

Since $\frac{\partial T_k}{\partial x^s}$ differs from the tensor $\frac{1}{2} (T_{k,s} - T_{s,k})$ by an expression symmetric in k, s and $d\tau_{(2)}^{ks}$ is an absolute tensor, skew-symmetric in these suffixes, it follows that the integrand on the left is an invariant.

Let's express **7.504** in curvilinear coordinates, equipped with metric

$$\int_{R_2} T_{k|s} d\tau_{(2)}^{ks} = \int_{R_1} T_r d\tau_{(1)}^r \quad (1)$$

Obviously we have

$$T_{k|s} = \frac{1}{2} (T_{k|s} - T_{s|k}) + \frac{1}{2} (T_{k|s} + T_{s|k}) \quad (2)$$

Let's put $A_{ks} = \frac{1}{2} (T_{k|s} - T_{s|k})$ and $B_{ks} = \frac{1}{2} (T_{k|s} + T_{s|k})$.

A_{ks} is skew-symmetric, while B_{ks} is symmetric. Note also that A_{ks} reduces to $A_{ks} = \frac{1}{2} (T_{k,s} - T_{s,k})$ as the Christoffel symbols vanish in this expression, leaving only the partial differentials and hence, independent of a metric. So the integrand on the left side becomes

$$T_{k|s} d\tau_{(2)}^{ks} = A_{ks} d\tau_{(2)}^{ks} + B_{ks} d\tau_{(2)}^{ks} \quad (3)$$

$$\Rightarrow T_{k|s} d\tau_{(2)}^{ks} = A_{ks} d\tau_{(2)}^{ks} - B_{ks} d\tau_{(2)}^{sk} \quad (4)$$

$$= A_{ks} d\tau_{(2)}^{ks} - B_{sk} d\tau_{(2)}^{sk} \quad (5)$$

$$(3)+(5) \Rightarrow T_{k|s} d\tau_{(2)}^{ks} = A_{ks} d\tau_{(2)}^{ks} \quad (6)$$

We note that both A_{ks} and $d\tau_{(2)}^{ks}$ are tensors, independent of the metric defined, and thus $A_{ks} d\tau_{(2)}^{ks}$ is an invariant of order 0.



7.18 p274 - Exercise

Due to the skew-symmetry of $d\tau_{(M)}^{k_1 \dots k_M}$, the integrand on the left-hand side is also an invariant. This may be proved in a few lines, preferably with the use of the compressed notation of 1.7; the proof is left as an exercise for the reader.

We will use the indices i, j for the transformed tensor components and p, q for the original components. We have

$$T'_{i_1 \dots i_{M-1}} = T_{p_1 \dots p_{M-1}} X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} \quad (1)$$

$$\Rightarrow T'_{i_1 \dots i_{M-1}, i_M} = T_{p_1 \dots p_{M-1}, i_M} X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} + T_{p_1 \dots p_{M-1}} \left(X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} \right)_{, i_M} \quad (2)$$

$$= T_{p_1 \dots p_{M-1}, p_M} X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} X_{i_M}^{p_M} + T_{p_1 \dots p_{M-1}} \left(X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} \right)_{, i_M} \quad (3)$$

and

$$d\tau'_{(M)} = d\tau_{(M)}^{s_1 \dots s_M} X_{s_1}^{i_1} \dots X_{s_M}^{i_M} \quad (4)$$

This gives

$$T'_{i_1 \dots i_{M-1}, i_M} d\tau'_{(M)} = \begin{cases} T_{p_1 \dots p_{M-1}, p_M} X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} X_{i_M}^{p_M} d\tau_{(M)}^{s_1 \dots s_M} X_{s_1}^{i_1} \dots X_{s_M}^{i_M} \\ + T_{p_1 \dots p_{M-1}} \left(X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} \right)_{, i_M} d\tau_{(M)}^{s_1 \dots s_M} X_{s_1}^{i_1} \dots X_{s_M}^{i_M} \end{cases} \quad (5)$$

$$= T_{p_1 \dots p_{M-1}, p_M} X_{s_1}^{p_1} \dots X_{s_{M-1}}^{p_{M-1}} X_{s_M}^{p_M} d\tau_{(M)}^{s_1 \dots s_M} \quad (6)$$

$$= T_{p_1 \dots p_{M-1}, p_M} \delta_{s_1}^{p_1} \dots \delta_{s_{M-1}}^{p_{M-1}} \delta_{s_M}^{p_M} d\tau_{(M)}^{s_1 \dots s_M} \quad (7)$$

$$= T_{s_1 \dots s_{M-1}, s_M} d\tau_{(M)}^{s_1 \dots s_M} \quad (8)$$

The term $T_{p_1 \dots p_{M-1}} \left(X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} \right)_{, i_M} d\tau_{(M)}^{s_1 \dots s_M} X_{s_1}^{i_1} \dots X_{s_M}^{i_M}$ in (5) is zero. Indeed, rewriting this as :

$$T_{p_1 \dots p_{M-1}} \left(X_{i_1}^{p_1} \dots X_{i_{M-1}}^{p_{M-1}} \right)_{, p_M} X_{i_M}^{p_M} d\tau_{(M)}^{s_1 \dots s_M} X_{s_1}^{i_1} \dots X_{s_M}^{i_M} \quad (9)$$

The terms in brackets are zero as they are of the form

$$\dots + X_{i_1}^{p_1} \dots \underbrace{\frac{\partial^2 x^{p_m}}{\partial x^{p_M} \partial x'^{i_m}}}_{= \frac{\partial \delta_{p_M}^{p_m}}{\partial x'^{i_m}} = 0} \dots X_{i_{M-1}}^{p_{M-1}} + \dots \quad (10)$$

From (8) we conclude that the integrand is indeed an invariant.



7.19 p275 - Exercise

The skew-symmetric part of a tensor $T_{k_1 \dots k_M}$ is defined as

$$T_{[k_1 \dots k_M]} = (M!)^{-1} \delta_{k_1 \dots k_M}^{s_1 \dots s_M} T_{s_1 \dots s_M}$$

Show that the left-hand side of equation **7.525** is unchanged if $T_{k_1 \dots k_M}$ is replaced by its skew-symmetric part. Show that the same is true for the right-hand side.

The integrand of the left-hand side of **7.525** is

$$T_{k_1 \dots k_{M-1}, k_M} d\tau_{(M)}^{k_1 \dots k_M} \quad (1)$$

replacing the tensor with its skew-symmetric part gives

$$T_{[k_1 \dots k_{M-1}], k_M} d\tau_{(M)}^{k_1 \dots k_M} = \frac{1}{(M-1)!} \delta_{k_1 \dots k_{M-1}}^{s_1 \dots s_{M-1}} T_{s_1 \dots s_{M-1}, k_M} d\tau_{(M)}^{k_1 \dots k_M} \quad (2)$$

$$= \frac{1}{(M-1)!} \delta_{k_1 \dots k_{M-1}}^{s_1 \dots s_{M-1}} \delta_{k_M}^{s_M} T_{s_1 \dots s_{M-1}, s_M} d\tau_{(M)}^{k_1 \dots k_M} \quad (3)$$

$$= \frac{1}{(M-1)!} \delta_{k_1 \dots k_{M-1}}^{s_1 \dots s_{M-1}} T_{s_1 \dots s_{M-1}, s_M} d\tau_{(M)}^{k_1 \dots k_{M-1} s_M} \quad (4)$$

$$= T_{s_1 \dots s_{M-1}, s_M} \frac{1}{(M-1)!} \left(\delta_{k_1 \dots k_{M-1}}^{s_1 \dots s_{M-1}} d\tau_{(M)}^{k_1 \dots k_{M-1} s_M} \right) \quad (5)$$

$$(6)$$

The terms in the brackets can be reduced to $(M-1)! d\tau_{(M)}^{s_1 \dots s_{M-1} s_M}$. Indeed, the number of choices in choosing the k_1, \dots, k_{M-1} are restricted to $M-1$ choices, due to the skew-symmetry of $d\tau_{(M)}^{k_1 \dots k_{M-1} s_M}$ (choosing a $k_i = s_M$ would result in a zero value for $d\tau_{(M)}^{k_1 \dots k_{M-1} s_M}$). So, in total there will be $(M-1)!$ terms and due to the skew-symmetry of $d\tau_{(M)}^{k_1 \dots k_{M-1} s_M}$ a change in sign of $\delta_{k_1 \dots k_{M-1}}^{s_1 \dots s_{M-1}}$ will also result in a change of sign in $d\tau_{(M)}^{k_1 \dots k_{M-1} s_M}$. Hence, we get

$$T_{[k_1 \dots k_{M-1}], k_M} d\tau_{(M)}^{k_1 \dots k_M} = T_{s_1 \dots s_{M-1}, s_M} \frac{1}{(M-1)!} (M-1)! d\tau_{(M)}^{s_1 \dots s_{M-1} s_M} \quad (7)$$

$$= T_{s_1 \dots s_{M-1}, s_M} d\tau_{(M)}^{s_1 \dots s_{M-1} s_M} \quad (8)$$

◇

The integrand of the right-hand side of **7.525** is

$$T_{k_1 \dots k_{M-1}} d\tau_{(M-1)}^{k_1 \dots k_{M-1}} \quad (9)$$

replacing the tensor with it's skew-symmetric part gives

$$T_{[k_1 \dots k_{M-1}]} d\tau_{(M-1)}^{k_1 \dots k_{M-1}} = \frac{1}{(M-1)!} \delta_{k_1 \dots k_{M-1}}^{s_1 \dots s_{M-1}} T_{s_1 \dots s_{M-1}} d\tau_{(M-1)}^{k_1 \dots k_{M-1}} \quad (10)$$

$$= T_{s_1 \dots s_{M-1}} \frac{1}{(M-1)!} \left(\delta_{k_1 \dots k_{M-1}}^{s_1 \dots s_{M-1}} d\tau_{(M-1)}^{k_1 \dots k_{M-1}} \right) \quad (11)$$

The terms in the brackets can be reduced to $(M-1)! d\tau_{(M-1)}^{s_1 \dots s_{M-1}}$. Indeed, the number of choices in choosing the k_1, \dots, k_{M-1} is $M-1$, giving $(M-1)!$ terms and due to the skew-symmetry of $d\tau_{(M-1)}^{k_1 \dots k_{M-1}}$ a change in sign of $\delta_{k_1 \dots k_{M-1}}^{s_1 \dots s_{M-1}}$ will also result in a change of sign in $d\tau_{(M-1)}^{k_1 \dots k_{M-1}}$. Hence, we get

$$T_{[k_1 \dots k_{M-1}]} d\tau_{(M-1)}^{k_1 \dots k_{M-1}} = T_{s_1 \dots s_{M-1}} \frac{1}{(M-1)!} (M-1)! d\tau_{(M-1)}^{s_1 \dots s_{M-1}} \quad (12)$$

$$= T_{s_1 \dots s_{M-1}} d\tau_{(M-1)}^{s_1 \dots s_{M-1}} \quad (13)$$



7.20 p278 - Exercise 1

Show from **7.312** that the number of independent components of the extension of an M -cell in N -space is

$$\frac{N!}{M!(N-M)!}$$

$$\mathbf{7.312} : \quad d\tau^{k_1 \dots k_M} = \epsilon^{\alpha_1 \dots \alpha_M} \frac{\partial x^{k_1}}{\partial y^{\alpha_1}} \dots \frac{\partial x^{k_M}}{\partial y^{\alpha_M}} |d_{(\beta)} y^\gamma| \quad (1)$$

As all k_i are different, the unconstrained number of ordered arrangement of M elements picked out of a set of N possibilities is $N(N-1) \dots (N-M+1) = \frac{N!}{(N-M)!}$.

As $d\tau^{k_1 \dots k_M}$ is skew-symmetric we have to divide this number by the number of constraints of the form $d\tau^{k_1 \dots k_j \dots k_p \dots k_M} = -d\tau^{k_1 \dots k_p \dots k_j \dots k_M}$. This number just corresponds to the number of permutations of M distinct objects and is equal to $M!$.

Hence,

$$\frac{N!}{M!(N-M)!}$$



7.21 p278 - Exercise 2

Prove that the covariant derivative of a generalized Kronecker delta is zero.

(2.525) gives:

$$\delta_{s_1 \dots s_M | s_\tau}^{k_1 \dots k_M} = \underbrace{\frac{\partial \delta_{s_1 \dots s_M}^{k_1 \dots k_M}}{\partial x^{s_\tau}}}_{=0} + \Gamma_{qs_\tau}^{k_1} \delta_{s_1 \dots s_M}^{q k_2 \dots k_M} + \dots + \Gamma_{qs_\tau}^{k_M} \delta_{s_1 \dots s_M}^{k_1 \dots k_{(M-1)} q} - \Gamma_{s_1 s_\tau}^q \delta_{q s_2 \dots s_M}^{k_1 \dots k_M} - \dots - \Gamma_{s_M s_\tau}^q \delta_{s_1 \dots s_{(M-1)} q}^{k_1 \dots k_M} \quad (1)$$

(2)

Let us now consider the following notation for $\delta_{s_1 \dots s_M}^{k_1 \dots k_M}$

$$\delta_{s_1 \leftrightarrow k_y \leftrightarrow s_M}^{k_1 \leftrightarrow s_x \leftrightarrow k_M}$$

showing the respective position of an element in the two series use in the generalized Kronecker delta.

Consider in this expression (no summations on the indexes) the positive and negative terms like

$$\Gamma_{qs_\tau}^{k_y} \delta_{s_1 \leftrightarrow k_y \leftrightarrow s_M}^{k_1 \leftrightarrow q \leftrightarrow k_M} \Big|_{q \neq s_x}, \dots, - \Gamma_{s_x s_\tau}^q \delta_{s_1 \leftrightarrow q \leftrightarrow s_M}^{k_1 \leftrightarrow s_x \leftrightarrow k_M} \Big|_{q \neq k_y} \quad (3)$$

Obviously these terms are zero as q does not refer to the value necessary to have no repetition in the set $\{s_1 \dots s_M\}$ or $\{k_1 \dots k_M\}$.

Consider now the sum of two terms like (no summations on the indexes).

$$\Gamma_{qs_\tau}^{k_y} \delta_{s_1 \leftrightarrow k_y \leftrightarrow s_M}^{k_1 \leftrightarrow q \leftrightarrow k_M} \Big|_{q=s_x} - \Gamma_{s_x s_\tau}^q \delta_{s_x \leftrightarrow q \leftrightarrow s_M}^{k_1 \leftrightarrow s_x \leftrightarrow k_M} \Big|_{q=k_y} \quad (4)$$

(5)

As each term in this sum occurs only once in (1) and we get

$$\Gamma_{qs_\tau}^{k_y} \delta_{s_1 \leftrightarrow k_y \leftrightarrow s_M}^{k_1 \leftrightarrow q \leftrightarrow k_M} \Big|_{q=s_x} - \Gamma_{s_x s_\tau}^q \delta_{s_x \leftrightarrow q \leftrightarrow s_M}^{k_1 \leftrightarrow s_x \leftrightarrow k_M} \Big|_{q=k_y} \quad (6)$$

$$= \delta_{s_1 \dots s_M}^{k_1 \dots k_M} (\Gamma_{s_x s_\tau}^{k_y} - \Gamma_{s_x s_\tau}^{k_y}) \quad (7)$$

$$= 0 \quad (8)$$

This show that indeed the covariant derivative of the generalize Kronecker delta is zero.

◇

Note: probably this can also be proved by induction, starting with the result in exercise in chapter 2, page 53 (see part I of this solution manual).

◆

7.22 p278 - Exercise 3

Show that

$$\delta_{k_1 \dots k_M}^{s_1 \dots s_M} = \begin{vmatrix} \delta_{s_1}^{k_1} & \delta_{s_2}^{k_1} & \dots & \delta_{s_M}^{k_1} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{s_1}^{k_M} & \delta_{s_2}^{k_M} & \dots & \delta_{s_M}^{k_M} \end{vmatrix}$$

The determinant on the right side can be represented as

$$\Delta = \epsilon_{i_1 i_2 \dots i_M} \delta_{s_1}^{k_{i_1}} \delta_{s_2}^{k_{i_2}} \dots \delta_{s_M}^{k_{i_M}} \quad (1)$$

Be K_M, S_M the (unordered) sets $\{k_1, k_2, \dots, k_M\}$ and $\{s_1, s_2, \dots, s_M\}$. If $K_M \neq S_M$ then, one of the elements will differ and from (1) it follows that $\Delta = 0$ as for all terms in (1) at least one of the $\delta_{s_m}^{k_{i_n}}$ will be zero.

If $K_M = S_M$ then there will be only 1 term left in (1) having value ± 1 depending on the value of the corresponding $\epsilon_{i_1 i_2 \dots i_M}$ associated with this term. Be \hat{K}_M, \hat{S}_M the (ordered) list generated by K_M, S_M , then the value of the $\epsilon_{i_1 i_2 \dots i_M}$ will be determined by the number of permutation needed to map bijectively \hat{K}_M to \hat{S}_M . The above reasoning shows that the determinant can be expressed as

$$\Delta = \delta_{k_1 \dots k_M}^{s_1 \dots s_M}$$



7.23 p278 - Exercise 4

If T_{rs} is a symmetric tensor density and S_{rs} a skew-symmetric tensor density, show that

$$\begin{aligned} T_{.s|r}^r &= \partial_r T_{.s}^r - \frac{1}{2} T^{rk} \partial_s a_{rk} \\ S_{..|r}^{rs} &= \partial_r S^{rs} \end{aligned}$$

Let's define the absolute oriented tensor.

$$\bar{T}_{.s}^r = (\epsilon(a)a)^{-\frac{1}{2}} T_{.s}^r \quad (1)$$

By **7.214** we have

$$T_{.s|r}^r = (\epsilon(a)a)^{\frac{1}{2}} \left[\bar{T}_{.s}^r \right]_{|r} \quad (2)$$

with

$$\left[\bar{T}_{.s}^r \right]_{|r} = \partial_r \bar{T}_{.s}^r + \Gamma_{mr}^r \bar{T}_{.s}^m - \Gamma_{rs}^m \bar{T}_{.m}^r \quad (3)$$

Using **2.542** : $\Gamma_{mr}^r = (\epsilon(a)a)^{-\frac{1}{2}} \partial_m (\epsilon(a)a)^{\frac{1}{2}}$

$$\left[\bar{T}_{.s}^r \right]_{|r} = \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \partial_r T_{.s}^r + T_{.s}^r \partial_r (\epsilon(a)a)^{-\frac{1}{2}} \\ + (\epsilon(a)a)^{-\frac{1}{2}} \partial_m (\epsilon(a)a)^{\frac{1}{2}} \bar{T}_{.s}^m \\ - \frac{1}{2} (\partial_s a_{rk} + \partial_r a_{sk} - \partial_k a_{rs}) a^{mk} \bar{T}_{.m}^r \end{cases} \quad (4)$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \partial_r T_{.s}^r + T_{.s}^r \partial_r (\epsilon(a)a)^{-\frac{1}{2}} \\ + (\epsilon(a)a)^{-\frac{1}{2}} (\epsilon(a)a)^{-\frac{1}{2}} T_{.s}^m \partial_m (\epsilon(a)a)^{\frac{1}{2}} \\ - \frac{1}{2} (\epsilon(a)a)^{-\frac{1}{2}} \left(T^{rk} \partial_s a_{rk} + \underbrace{T^{rk} \partial_r a_{sk} - T^{rk} \partial_k a_{rs}}_{=0} \right) \end{cases} \quad (5)$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \left(\partial_r T_{.s}^r - \frac{1}{2} T^{rk} \partial_s a_{rk} \right) \\ + T_{.s}^r \left[\partial_r (\epsilon(a)a)^{-\frac{1}{2}} + (\epsilon(a)a)^{-1} \partial_r (\epsilon(a)a)^{\frac{1}{2}} \right] \end{cases} \quad (6)$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \left(\partial_r T_{.s}^r - \frac{1}{2} T^{rk} \partial_s a_{rk} \right) \\ + T_{.s}^r \left[\underbrace{-\frac{1}{2} (\epsilon(a))^{-\frac{3}{2}} \partial_r (\epsilon(a)a) + \frac{1}{2} (\epsilon(a)a)^{-1} (\epsilon(a)a)^{-\frac{1}{2}} \partial_r (\epsilon(a)a)}_{=0} \right] \end{cases} \quad (7)$$

giving

$$\left[\overline{T}^r_{\cdot s} \right]_{|r} = (\epsilon(a)a)^{-\frac{1}{2}} \left(\partial_r T^r_{\cdot s} - \frac{1}{2} T^{rk} \partial_s a_{rk} \right) \quad (8)$$

$$(2) \text{ and } (8): \quad T^r_{\cdot s|r} = \partial_r T^r_{\cdot s} - \frac{1}{2} T^{rk} \partial_s a_{rk} \quad (9)$$

◇

Let's define the absolute oriented tensor.

$$\overline{S}^{rs} = (\epsilon(a)a)^{-\frac{1}{2}} S^{rs} \quad (10)$$

By **7.214** we have

$$S^{rs}_{\cdot|r} = (\epsilon(a)a)^{\frac{1}{2}} \left[\overline{S}^{rs} \right]_{|r} \quad (11)$$

with

$$\left[\overline{S}^{rs} \right]_{|r} = \partial_r \overline{S}^{rs} + \Gamma^r_{mr} \overline{S}^{ms} + \Gamma^s_{mr} \overline{S}^{rm} \quad (12)$$

$$\text{Using } \mathbf{2.542} : \quad \Gamma^r_{mr} = (\epsilon(a)a)^{-\frac{1}{2}} \partial_m (\epsilon(a)a)^{\frac{1}{2}}$$

$$\left[\overline{S}^{rs} \right]_{|r} = \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \partial_r S^{rs} + S^{rs} \partial_r (\epsilon(a)a)^{-\frac{1}{2}} \\ + (\epsilon(a)a)^{-\frac{1}{2}} \partial_m (\epsilon(a)a)^{\frac{1}{2}} \overline{S}^{ms} \\ + \frac{1}{2} (\partial_m a_{rk} + \partial_r a_{mk} - \partial_k a_{mr}) a^{sk} \overline{S}^{rm} \end{cases} \quad (13)$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \partial_r S^{rs} + S^{rs} \partial_r (\epsilon(a)a)^{-\frac{1}{2}} \\ + (\epsilon(a)a)^{-\frac{1}{2}} (\epsilon(a)a)^{-\frac{1}{2}} S^{ms} \partial_m (\epsilon(a)a)^{\frac{1}{2}} \\ + \frac{1}{2} (\epsilon(a)a)^{-\frac{1}{2}} (S^{rm} \partial_m a_{rk} + S^{rm} \partial_r a_{mk} - S^{rm} \partial_k a_{mr}) a^{sk} \end{cases} \quad (14)$$

$$(15)$$

Using the skew-symmetry of S_{rm} we get

$$\left[\bar{S}^{rs} \right]_{|r} = \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \partial_r S^{rs} \\ \underbrace{S^{rs} \partial_r (\epsilon(a)a)^{-\frac{1}{2}} + (\epsilon(a)a)^{-1} S^{ms} \partial_m (\epsilon(a)a)^{\frac{1}{2}}}_{=0} \\ + \frac{1}{2} (\epsilon(a)a)^{-\frac{1}{2}} \left(\underbrace{S^{rm} \partial_m a_{rk} + S^{rm} \partial_r a_{mk} - S^{rm} \partial_k a_{mr}}_{=0} \right) a^{sk} \end{cases} \quad (16)$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \partial_r S^{rs} \\ -\frac{1}{2} (\epsilon(a)a)^{-\frac{1}{2}} (a^{sk} \partial_k a_{mr} S^{rm}) \end{cases} \quad (17)$$

$$(18)$$

In the last term, put $b_{smr} \doteq a^{sk} \partial_k a_{mr}$ which is symmetric in the last two indexes. Due to the skew-symmetry of S^{rm} we have then $b_{srm} S^{rm} = 0$ and get

$$\left[\bar{S}^{rs} \right]_{|r} = (\epsilon(a)a)^{-\frac{1}{2}} \partial_r S^{rs} \quad (19)$$

$$(2) \text{ and } (19): \quad S_{\cdot\cdot|r}^{rs} = \partial_r S^{rs} \quad (20)$$



7.24 p279 - Exercise 5

Let b_{mn} an absolute covariant tensor. Show that the cofactors of the elements b_{mn} in the determinant $|b_{mn}|$ are the components of a relative contravariant tensor of weight 2.

Let's put $b \doteq |b_{mn}|$. We have (see **2.202**)

$$b_{mr}\Delta^{ms} = \delta_r^s b \quad (1)$$

where Δ^{ms} is the cofactor associated with element b_{ms} . Note that, for the moment being, the position of the indexes does not imply any tensor characteristics of this object. δ_r^s is a mixed absolute tensor of order 2 (see page 243). Also b is a relative invariant of weight 2 (see **7.202**).

So we have the following transformation rules

$$\left\{ \begin{array}{l} \delta_r'^s = \delta_v^u \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \\ b' = J^2 b \\ b'_{mr} = b_{uv} \frac{\partial x^u}{\partial x'^m} \frac{\partial x^v}{\partial x'^r} \end{array} \right. \quad (2)$$

Using (1)

$$b'_{mr}\Delta'^{ms} = \delta_r'^s b' \quad (3)$$

Substituting with (2)

$$b_{uv} \frac{\partial x^u}{\partial x'^m} \frac{\partial x^v}{\partial x'^r} \Delta'^{ms} = \delta_v^u \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b \quad (4)$$

$$b_{uv} \frac{\partial x^u}{\partial x'^m} \frac{\partial x^v}{\partial x'^r} \Delta'^{ms} = \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b_{mv} \Delta^{mu} \quad (5)$$

$$\times \frac{\partial x'^r}{\partial x^p} \quad b_{up} \frac{\partial x^u}{\partial x'^m} \Delta'^{ms} = J^2 b_{mp} \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \quad (6)$$

This is a system of N^2 linear equations in N^2 unknowns Δ'^{ms} . We claim that $\Delta'^{ms} = J^2 \Delta^{ij} \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^s}{\partial x^j}$ is a solution of that system.

Substituting this candidate solution in the left side of (6) gives

$$b_{up} \frac{\partial x^u}{\partial x'^m} J^2 \Delta^{ij} \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^s}{\partial x^j} = J^2 b_{up} \delta_i^u \Delta^{ij} \frac{\partial x'^s}{\partial x^j} \quad (7)$$

$$= J^2 b_{mp} \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \quad (8)$$

proving that our candidate solution is indeed the solution for (6), provided of course the usual conditions on the solvability of a system of linear equations.



7.25 p279 - Exercise 6

Determine the tensor character of the cofactors in the determinants formed by the components of

- (a) a mixed absolute tensor
- (b) a relative contravariant tensor of weight 1.

(a)

Let's put $b \doteq |b_n^m|$. We have (see **2.202**)

$$b_r^m \Delta^{ms} = \delta_r^s b \quad (1)$$

where Δ^{ms} is the cofactor associated with element b_s^m . Note that, for the moment being, the position of the indexes does not imply any tensor characteristics of this object. δ_r^s is a mixed absolute tensor of order 2 (see page 243). Also b is a relative invariant of weight 2 (see **7.202**).

So we have the following transformation rules

$$\left\{ \begin{array}{l} \delta_r'^s = \delta_v^u \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \\ b' = J^2 b \\ b_r'^m = b_v^u \frac{\partial x'^m}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \end{array} \right. \quad (2)$$

Using (1)

$$b_r'^m \Delta'^{ms} = \delta_r'^s b' \quad (3)$$

Substituting with (2)

$$b_v^u \frac{\partial x'^m}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \Delta'^{ms} = \delta_v^u \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b \quad (4)$$

$$b_v^u \frac{\partial x'^m}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \Delta'^{ms} = \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b_v^m \Delta^{mu} \quad (5)$$

$$\times \frac{\partial x'^r}{\partial x^p} \quad b_p^u \frac{\partial x'^m}{\partial x^u} \Delta'^{ms} = J^2 b_p^m \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \quad (6)$$

This is a system of N^2 linear equations in N^2 unknowns Δ'^{ms} . We claim that $\Delta'^{ms} = J^2 \Delta^{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j}$ is a solution of that system.

Substituting this candidate solution in the left side of (6) gives

$$b_p^u \frac{\partial x'^m}{\partial x^u} J^2 \Delta^{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j} = J^2 b_p^u \delta_u^i \Delta^{ij} \frac{\partial x'^s}{\partial x^j} \quad (7)$$

$$= J^2 b_p^m \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \quad (8)$$

proving that our candidate solution is indeed the solution for (6), provided of course the usual conditions on the solvability of a system of linear equations.

We rewrite the cofactor as Δ_m^s which transform as

$$\Delta_m^s = J^2 \Delta_j^i \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j}$$

and conclude that the cofactor is a relative mixed tensor of weight 2.

◇

(b) Let's put $b \doteq |b^{mn}|$. We have (see **2.202**)

$$b^{mr} \Delta^{ms} = \delta_r^s b \quad (9)$$

where Δ^{ms} is the cofactor associated with element b^{ms} . Note that, for the moment being, the position of the indexes does not imply any tensor characteristics of this object. δ_r^s is a mixed absolute tensor of order 2 (see page 243). Also b is a relative invariant of weight 2 (see **7.202**).

So we have the following transformation rules

$$\left\{ \begin{array}{l} \delta_r'^s = \delta_v^u \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \\ b' = J^2 b \\ b'^{mr} = J b^{uv} \frac{\partial x'^m}{\partial x^u} \frac{\partial x'^r}{\partial x^v} \end{array} \right. \quad (10)$$

Using (1)

$$b'^{mr} \Delta'^{ms} = \delta_r'^s b' \quad (11)$$

Substituting with (2)

$$J b^{uv} \frac{\partial x'^m}{\partial x^u} \frac{\partial x'^r}{\partial x^v} \Delta'^{ms} = \delta_v^u \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b \quad (12)$$

$$J b^{uv} \frac{\partial x'^m}{\partial x^u} \frac{\partial x'^r}{\partial x^v} \Delta'^{ms} = \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b^{mv} \Delta^{mu} \quad (13)$$

$$\times \frac{\partial x^p}{\partial x'^r} \quad b^{up} \frac{\partial x'^m}{\partial x^u} \Delta'^{ms} = J b^{mp} \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \quad (14)$$

This is a system of N^2 linear equations in N^2 unknowns Δ'^{ms} . We claim that $\Delta'^{ms} = J \Delta^{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j}$ is a solution of that system.

Substituting this candidate solution in the left side of (6) gives

$$b^{up} \frac{\partial x'^m}{\partial x^u} J \Delta^{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j} = J b^{up} \delta_u^i \Delta^{ij} \frac{\partial x'^s}{\partial x^j} \quad (15)$$

$$= J b^{mp} \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \quad (16)$$

proving that our candidate solution is indeed the solution for (6), provided of course the usual conditions on the solvability of a system of linear equations.

We rewrite the cofactor as Δ_m^s which transform as

$$\Delta_m^s = J \Delta_j^i \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j}$$

and conclude that the cofactor is a relative mixed tensor of weight 1.



7.26 p279 - Exercise 7

In the space-time of relativity with metric form

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2$$

the 3-space with equation

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = 0$$

is called a null cone. prove that the 3-volume of any portion of the null cone is zero.

We have **7.414** : $v_{(3)} = \int_{R(3)} dv_{(3)}$ with **7.410** : $dv_{(3)} = (\epsilon(b)b)^{\frac{1}{2}} \epsilon(\tau) |d_{(\beta)} y_\alpha|$ and **(7.409)** : $b_{\alpha\beta} = a_{ks} \frac{\partial x^k}{\partial y_\alpha} \frac{\partial x^s}{\partial y_\beta}$. Let's compute $b = |b_{\alpha\beta}|$. We define the intrinsic parameters (x^1, x^2, x^3) for the (y_α) . We have

$$(a_{mn}) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \quad (1)$$

Let's define $R = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$ and using the equation of the null cone in the form

$$(x^4)^2 = R^2$$

we get

$$|b_{mn}| = \begin{vmatrix} 1 - \left(\frac{x^1}{R}\right)^2 & \frac{x^1 x^2}{R^2} & \frac{x^1 x^3}{R^2} \\ \frac{x^1 x^2}{R^2} & 1 - \left(\frac{x^2}{R}\right)^2 & \frac{x^2 x^3}{R^2} \\ \frac{x^1 x^3}{R^2} & \frac{x^2 x^3}{R^2} & 1 - \left(\frac{x^3}{R}\right)^2 \end{vmatrix} \quad (2)$$

$$= 1 - \left(\frac{x^1}{R}\right)^2 - \left(\frac{x^2}{R}\right)^2 - \left(\frac{x^3}{R}\right)^2 \quad (3)$$

$$= 1 - \frac{R^2}{R^2} \quad (4)$$

$$= 0 \quad (5)$$

So, $dv_{(3)} = 0$ at any point of the null cone giving a zero 3-volume everywhere on that manifold.



7.27 p279 - Exercise 8†

Prove that a polar N -space of constant curvature is oriented if N is odd and unoriented if N is even., but that an antipodal N -space of constant curvature is always oriented.

Let's us first sum up some key-definitions which could be usefull for proving the above assumption.

a) Chapter 4 pages 111-116: *In a space with constant positive curvature, two adjacent geodesics issuing from a point O intersect at a point O' at a distance $s = \frac{\pi}{\sqrt{\epsilon K}}$. If we have $O = O'$, than the space is called **polar** (or elliptic).*

b) The factor $d\tau_{(M)} = |d_{(\beta)}y^\alpha|$ is called the **intrinsic extension of the infinitesimal cell in the subspace V_M** . The following relations exist:

$$d\tau_M^{k_1 \dots k_M} = \nu^{k_1 \dots k_M} d\tau_{(M)}$$

with

$$\nu^{k_1 \dots k_M} = \epsilon^{k_1 \dots k_M} \frac{\partial x^{k_1} \dots x^{k_M}}{\partial y^{\alpha_1} \dots y^{\alpha_M}}$$

$\nu^{k_1 \dots k_M}$ is called the **M-direction of V_M at a point P** .

If $M = N$ then we see that $\nu^{k_1 \dots k_M} = \epsilon^{k_1 \dots k_M}$.

c) page 261: *Comparison of M -cells at two different points A and B in a region R_M of the subspace V_M is achieved as follows: The infinitesimal cell at A , say, is moved in a continuous manner to B along a path C lying in R_M , such that, at each stage of the continuous motion, the intrinsic extension of the cell is non-zero. The orientations of the two cells are then compared at point B .*

d) page 261: *There are, however, regions where the orientations of cells at different points cannot be compared because the procedure adapted above is not unique. Such regions are called unoriented or one-sided.*

Before delving in an attempt, some things are not clear for me. E.g. in c), how can one be sure that such a path exist so that the intrinsic extension does not vanish.



7.28 p279 - Exercise 9

Show that the total volume of a polar 3-space of constant curvature R^{-2} is $\pi^2 R^3$, and that the volume is $2\pi^2 R^3$ if the space is antipodal.



7.29 p279 - Exercise 10

If r_{mn} is an absolute tensor in 4-space, show that L defined by,

$$L = |r_{mn}|^{\frac{1}{2}}$$

is an invariant density. Discuss the tensor character of f^{mn} and g^{mn} , defined by

$$f^{mn} = \frac{\partial L}{\partial r_{mn}}, \quad g^{mn} = f^{mn} |f^{ks}|^{-\frac{1}{2}}$$

and also of g_{mn} , defined by

$$g_{ms} g^{ns} = \delta_m^n$$

prove that $|g_{mn}| = |f^{mn}|$. (Schrödinger)

