Tensor Calculus
J.L. Synge and A.Schild (Dover Publication)
Solutions to exercises
Part II
Chapters V to VIII

by

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Relative tensors, ideas of volume, Green-Stokes theorems.

7.1 p279 - Exercise 5

Determine the tensor character of the cofactors in the determinants formed by the components of
(a) a mixed absolute tensor
(b) a relative contravaraint tensor of weight 1.

(a)
Let's put $b \doteq |b^m_n|$. We have (see **2.202**)

$$b^m_r \Delta^{ms} = \delta^s_r b \tag{1}$$

where Δ^{ms} is the cofactor associated with element b^m_s. Note that, for the moment being, the position of the indexes does not imply any tensor characteristics of this object. δ^s_r is a mixed absolute tensor of order 2 (see page 243). Also b is a relative invariant of weight 2 (see **7.202**).

So we have the following transformation rules

$$\begin{cases} \delta'^s_r = \delta^u_v \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \\ \\ b' = J^2 b \\ \\ b'^m_r = b^u_v \frac{\partial x'^m}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \end{cases} \tag{2}$$

Using (1)

$$b'^m_r \Delta'^{ms} = \delta'^s_r b' \tag{3}$$

Substituting with (2)

$$b^u_v \frac{\partial x'^m}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \Delta'^{ms} = \delta^u_v \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b \tag{4}$$

$$b^u_v \frac{\partial x'^m}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \Delta'^{ms} = \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b^m_v \Delta^{mu} \tag{5}$$

$$\times \frac{\partial x'^r}{\partial x^p} \qquad b^u_p \frac{\partial x'^m}{\partial x^u} \Delta'^{ms} = J^2 b^m_p \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \tag{6}$$

This is a system of N^2 linear equations in N^2 unknowns Δ'^{ms}. We claim that $\Delta'^{ms} = J^2 \Delta^{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j}$ is a solution of that system.
Substituting this candidate solution in the left side of (6) gives

$$b^u_p \frac{\partial x'^m}{\partial x^u} J^2 \Delta^{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j} = J^2 b^u_p \delta^i_u \Delta^{ij} \frac{\partial x'^s}{\partial x^j} \tag{7}$$

$$= J^2 b^m_p \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \tag{8}$$

proving that our candidate solution is indeed the solution for (6), provided of course the usual conditions on the solvability of a system of linear equations.
We rewrite the cofactor as Δ^s_m which transform as

$$\Delta^s_m = J^2 \Delta^i_j \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j}$$

and conclude that the cofactor is a relative mixed tensor of weight 2.

$\lozenge$

(b) Let's put $b \doteq |b^{mn}|$. We have (see **2.202**)

$$b^{mr} \Delta^{ms} = \delta^s_r b \tag{9}$$

where Δ^{ms} is the cofactor associated with element b^{ms}. Note that, for the moment being, the position of the indexes does not imply any tensor characteristics of this object. δ^s_r is a mixed absolute tensor of order 2 (see page 243). Also b is a relative invariant of weight 2 (see **7.202**).

So we have the following transformation rules

$$\begin{cases} \delta'^s_r = \delta^u_v \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} \\ \\ b' = J^2 b \\ \\ b'^{mr} = J b^{uv} \frac{\partial x'^m}{\partial x^u} \frac{\partial x'^r}{\partial x^v} \end{cases} \tag{10}$$

Using (1)

$$b'^{mr} \Delta'^{ms} = \delta'^s_r b' \tag{11}$$

Substituting with (2)

$$J b^{uv} \frac{\partial x'^m}{\partial x^u} \frac{\partial x'^r}{\partial x^v} \Delta'^{ms} = \delta^u_v \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b \tag{12}$$

$$J b^{uv} \frac{\partial x'^m}{\partial x^u} \frac{\partial x'^r}{\partial x^v} \Delta'^{ms} = \frac{\partial x'^s}{\partial x^u} \frac{\partial x^v}{\partial x'^r} J^2 b^{mv} \Delta^{mu} \tag{13}$$

$$\times \frac{\partial x^p}{\partial x'^r} \qquad b^{up} \frac{\partial x'^m}{\partial x^u} \Delta'^{ms} = J b^{mp} \Delta^{mu} \frac{\partial x'^s}{\partial x^u} \tag{14}$$

This is a system of N^2 linear equations in N^2 unknowns Δ'^{ms}. We claim that $\Delta'^{ms} = J \Delta^{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^s}{\partial x^j}$ is a solution of that system.

Substituting this candidate solution in the left side of (6) gives

$$b^{up}\frac{\partial x'^m}{\partial x^u} J\Delta^{ij}\frac{\partial x^i}{\partial x'^m}\frac{\partial x'^s}{\partial x^j} = Jb^{up}\delta^i_u\Delta^{ij}\frac{\partial x'^s}{\partial x^j} \tag{15}$$

$$= Jb^{mp}\Delta^{mu}\frac{\partial x'^s}{\partial x^u} \tag{16}$$

proving that our candidate solution is indeed the solution for (6), provided of course the usual conditions on the solvability of a system of linear equations.
We rewrite the cofactor as Δ^s_m which transform as

$$\Delta^s_m = J\Delta^i_j\frac{\partial x^i}{\partial x'^m}\frac{\partial x'^s}{\partial x^j}$$

and conclude that the cofactor is a relative mixed tensor of weight 1.

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7.2 p279 - Exercise 6

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