Tensor Calculus
J.L. Synge and A.Schild (Dover Publication)
Solutions to exercises
Part II
Chapters V to VIII

by

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Relative tensors, ideas of volume, Green-Stokes theorems.

7.1 p276 - Exercise 3

If T_{rs} is a symmetric tensor density and S_{rs} a skew-symmetric tensor density, show that

$$\begin{aligned} T^r_{.s|r} &= \partial_r T^r_{.s} - \frac{1}{2} T^{rk} \partial_s a_{rk} \\ S^{rs}_{..|r} &= \partial_r S^{rs} \end{aligned}$$

Let's define the absolute oriented tensor.

$$\overline{T}^r_{.s} = (\epsilon(a)a)^{-\frac{1}{2}}\, T^r_{.s} \tag{1}$$

By **7.214** we have

$$T^r_{.s|r} = (\epsilon(a)a)^{\frac{1}{2}} \left[\overline{T}^r_{.s}\right]_{|r} \tag{2}$$

with

$$\left[\overline{T}^r_{.s}\right]_{|r} = \partial_r \overline{T}^r_{.s} + \Gamma^r_{mr} \overline{T}^m_{.s} - \Gamma^m_{rs} \overline{T}^r_{.m} \tag{3}$$

Using **2.542** : $\quad \Gamma^r_{mr} = (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_m\, (\epsilon(a)a)^{\frac{1}{2}}$

$$\left[\overline{T}^r_{.s}\right]_{|r} = \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_r T^r_{.s} + T^r_{.s} \partial_r\, (\epsilon(a)a)^{-\frac{1}{2}} \\ \\ + (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_m\, (\epsilon(a)a)^{\frac{1}{2}}\, \overline{T}^m_{.s} \\ \\ -\frac{1}{2} \left(\partial_s a_{rk} + \partial_r a_{sk} - \partial_k a_{rs}\right) a^{mk} \overline{T}^r_{.m} \end{cases} \tag{4}$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_r T^r_{.s} + T^r_{.s} \partial_r\, (\epsilon(a)a)^{-\frac{1}{2}} \\ \\ + (\epsilon(a)a)^{-\frac{1}{2}}\, (\epsilon(a)a)^{-\frac{1}{2}}\, T^m_{.s} \partial_m\, (\epsilon(a)a)^{\frac{1}{2}} \\ \\ -\frac{1}{2}\, (\epsilon(a)a)^{-\frac{1}{2}} \left(T^{rk} \partial_s a_{rk} + \underbrace{T^{rk} \partial_r a_{sk} - T^{rk} \partial_k a_{rs}}_{=0} \right) \end{cases} \tag{5}$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \left(\partial_r T^r_{.s} - \frac{1}{2} T^{rk} \partial_s a_{rk}\right) \\ \\ + T^r_{.s} \left[\partial_r\, (\epsilon(a)a)^{-\frac{1}{2}} + (\epsilon(a)a)^{-1}\, \partial_r\, (\epsilon(a)a)^{\frac{1}{2}}\right] \end{cases} \tag{6}$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}} \left(\partial_r T^r_{.s} - \frac{1}{2} T^{rk} \partial_s a_{rk}\right) \\ \\ + T^r_{.s} \left[\underbrace{-\frac{1}{2}\, (\epsilon(a))^{-\frac{3}{2}}\, \partial_r\, (\epsilon(a)a) + \frac{1}{2}\, (\epsilon(a)a)^{-1}\, (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_r\, (\epsilon(a)a)}_{=0} \right] \end{cases} \tag{7}$$

giving

$$\left[\overline{T}^{r}_{.s}\right]_{|r} = (\epsilon(a)a)^{-\frac{1}{2}} \left(\partial_r T^r_{.s} - \frac{1}{2} T^{rk} \partial_s a_{rk}\right) \tag{8}$$

(2) and (8):
$$T^r_{.s|r} = \partial_r T^r_{.s} - \frac{1}{2} T^{rk} \partial_s a_{rk} \tag{9}$$

$$\Diamond$$

Let's define the absolute oriented tensor.

$$\overline{S}^{rs} = (\epsilon(a)a)^{-\frac{1}{2}}\, S^{rs} \tag{10}$$

By **7.214** we have

$$S^{rs}_{..|r} = (\epsilon(a)a)^{\frac{1}{2}} \left[\overline{S}^{rs}\right]_{|r} \tag{11}$$

with

$$\left[\overline{S}^{rs}\right]_{|r} = \partial_r \overline{S}^{rs} + \Gamma^r_{mr} \overline{S}^{ms} + \Gamma^s_{mr} \overline{S}^{rm} \tag{12}$$

Using **2.542** : $\Gamma^r_{mr} = (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_m\, (\epsilon(a)a)^{\frac{1}{2}}$

$$\left[\overline{S}^{rs}\right]_{|r} = \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_r S^{rs} + S^{rs} \partial_r\, (\epsilon(a)a)^{-\frac{1}{2}} \\ \\ + (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_m\, (\epsilon(a)a)^{\frac{1}{2}}\, \overline{S}^{ms} \\ \\ + \frac{1}{2}\left(\partial_m a_{rk} + \partial_r a_{mk} - \partial_k a_{mr}\right) a^{sk} \overline{S}^{rm} \end{cases} \tag{13}$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}}\, \partial_r S^{rs} + S^{rs} \partial_r\, (\epsilon(a)a)^{-\frac{1}{2}} \\ \\ + (\epsilon(a)a)^{-\frac{1}{2}}\, (\epsilon(a)a)^{-\frac{1}{2}}\, S^{ms} \partial_m\, (\epsilon(a)a)^{\frac{1}{2}} \\ \\ + \frac{1}{2}\, (\epsilon(a)a)^{-\frac{1}{2}} \left(S^{rm} \partial_m a_{rk} + S^{rm} \partial_r a_{mk} - S^{rm} \partial_k a_{mr}\right) a^{sk} \end{cases} \tag{14}$$

(15)

Using the skew-symmetry of S_{rm} we get

$$\left[\overline{S}^{rs}\right]_{|r} = \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}}\,\partial_r S^{rs} \\ \underbrace{S^{rs}\partial_r\left(\epsilon(a)a\right)^{-\frac{1}{2}} + \left(\epsilon(a)a\right)^{-1} S^{ms}\partial_m\left(\epsilon(a)a\right)^{\frac{1}{2}}}_{=0} \\ +\frac{1}{2}\left(\epsilon(a)a\right)^{-\frac{1}{2}}\left(\underbrace{S^{rm}\partial_m a_{rk} + S^{rm}\partial_r a_{mk}}_{=0} - S^{rm}\partial_k a_{mr}\right) a^{sk} \end{cases} \tag{16}$$

$$= \begin{cases} (\epsilon(a)a)^{-\frac{1}{2}}\,\partial_r S^{rs} \\ -\frac{1}{2}\left(\epsilon(a)a\right)^{-\frac{1}{2}}\left(a^{sk}\partial_k a_{mr} S^{rm}\right) \end{cases} \tag{17}$$

$$\tag{18}$$

In the last term, put $b_{smr} \doteq a^{sk}\partial_k a_{mr}$ which is symmetric in the last two indexes. Due to the skew-symmetry of S^{rm} we have then $b_{srm}S^{rm} = 0$ and get

$$\left[\overline{S}^{rs}\right]_{|r} = \left(\epsilon(a)a\right)^{-\frac{1}{2}}\,\partial_r S^{rs} \tag{19}$$

$$\text{(2) and (19):} \qquad S^{rs}_{\cdot\cdot|r} = \partial_r S^{rs} \tag{20}$$

♦