Tensor Calculus
J.L. Synge and A.Shild (Dover Publication)
Solutions to exercices

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Remarks and warnings

Some notation conventions

$$\partial_r a_{mn} \equiv \frac{\partial a_{mn}}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix}$$ Christoffel symbol of the second kind

Contents

Spaces and Tensors

1.1 p5-exercise

The parametric equations of a hypersurface in V_n are

$$
\begin{aligned}
x^1 &= a\cos\left(u^1\right)\\
x^2 &= a\sin\left(u^1\right)\cos\left(u^2\right)\\
x^3 &= a\sin\left(u^1\right)\sin\left(u^2\right)\cos\left(u^3\right)\\
&\vdots\\
x^{N-1} &= a\sin\left(u^1\right)\sin\left(u^2\right)\sin\left(u^3\right)\ldots\sin\left(u^{N-2}\right)\cos\left(u^{N-1}\right)\\
x^N &= a\sin\left(u^1\right)\sin\left(u^2\right)\sin\left(u^3\right)\ldots\sin\left(u^{N-2}\right)\sin\left(u^{N-1}\right)
\end{aligned}
$$

where a is a constant. Find the single equation of the hyperspace in the form 1.103.

We have:

$$
\begin{aligned}
(x^N)^2 + (x^{N-1})^2 &= a^2\prod_{i=1}^{N-2}\sin^2(u^i)(\cos^2(u^{N-1}) + \sin^2(u^{N-1}))\\
&= a^2\prod_{i=1}^{N-2}\sin^2(u^i)\\
&= a^2\prod_{i=1}^{N-3}\sin^2(u^i)\sin^2(u^{N-2})\\
&= a^2\prod_{i=1}^{N-3}\sin^2(u^i)(1-\cos^2(u^{N-2})\\
&= a^2\prod_{i=1}^{N-3}\sin^2(u^i) - a^2\prod_{i=1}^{N-3}\sin^2(u^i)\cos^2(u^{N-2})\\
&= a^2\prod_{i=1}^{N-3}\sin^2(u^i) - (x^{N-2})^2
\end{aligned}
$$

giving

$$
(x^N)^2 + (x^{N-1})^2 + (x^{N-2})^2 = a^2\prod_{i=1}^{N-3}\sin^2(u^i)
$$

In general, by recursion

$$
\sum_{i=0}^{k}(x^{N-i})^2 = a^2\prod_{i=1}^{N-k-1}\sin^2(u^i) \quad (k \le N-2)
$$

be k = N - 2 (N - k - 1 = 1) and in the left term put j = N - i (j goes from 2 to N), we get

$$\sum_{j=2}^{N}(x^j)^2 = a^2\prod_{i=1}^{1}\sin^2(u^i)$$
$$= a^2(1-\cos^2(u^1))$$
$$= a^2-(x^1)^2$$

and thus the equation of the hyperspace is given by

$$\sum_{j=1}^{N}(x^j)^2 - a^2 = 0$$

> Determine whether the points $(\frac{1}{2}a, 0, 0, ...0)$, $(0, 0, ..., 0, 2a)$ lie on the same or opposite sides of the hyperspace.

For $(\frac{1}{2}a, 0, 0, ...0)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = -\frac{3a^2}{4} < 0$ and for $(0, 0, ..., 0, 2a)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = \frac{3a^2}{4} > 0$.
So the points lie on opposite sides of the hyperplane.

◆

1.2 p6-exercise

Let U_2 and W_2 be subspaces of V_N. Show that if N = 3 they will in general intersect in a curve; if N = 4 they will in general intersect in a finite number of points; and if $N > 4$ they will not in general intersect at all.

We have (see 1.102 page 5): $x^r = f^r(u^1, u^2, ..., u^M) \quad (r = 1, 2, ..., N)$
Case N=3:

For U_2 we have:

$$x^r = \phi^r(u^1, u^2) \quad (r = 1, 2, 3)$$

For W_2 we have:

$$x^r = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

The intersect of the two hyperplanes is given by the N equations:

$$\phi^r(u^1, u^2) = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

So we have 3 equations in 4 unknown u^1, u^2, v^1, v^2 and can choose (fix) one e.g. u^1 and solve the set of equations for u^2, v^1, v^2 giving

$$x^r = \theta^r(u^1) \quad (r = 1, 2, 3)$$

This is an equation of a curve in space (1 parameter equation)
Case N=4:
Using the same reasoning as with N=3, we get 4 equations for 4 unknown u^1, u^2, v^1, v^2.
Provided that the set of equation does not degenerate, these 4 equations will determine u^1, u^2, v^1, v^2 without any degree of freedom. So we get points as solutions. This solution does not to be unique, e.g. if the $\phi^r(u^1, u^2)$ are quadratic form, then the solutions

$$(u^1, u^2, v^1, v^2)$$

$$(-u^1, u^2, v^1, v^2)$$

$$(u^1, -u^2, v^1, v^2)$$

$$(-u^1, -u^2, v^1, v^2)$$

are possible.
Case N=5: There are more equations than variables. If the equations are not linear dependent, no solutions will be found.

◆

1.3 p8-exercise

Show that $(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = 3a_{rst}x^r x^s x^t$

$(a_{rst}+a_{str}+a_{srt})x^r x^s x^t = a_{rst}x^r x^s x^t + a_{rts}x^r x^s x^t + a_{srt}x^r x^s x^t$ so by just renaming the dummy indices e.g. for the second term $r \mapsto s$, $s \mapsto t$ and $t \mapsto r$ we get the desired result.

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1.4 p8-exercise

If $\phi = a_{rs}x^r x^s$, show that

$$\frac{\partial \phi}{\partial x^r} = (a_{rs} + a_{sr})x^s \quad , \quad \frac{\partial^2 \phi}{\partial x^r \partial x^s} = a_{rs} + a_{sr}$$

Simplify these expressions in the case where $a^{rs} = a^{sr}$

We have

$$\frac{\partial \phi}{\partial x^t} = \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs} \frac{\partial x^r}{\partial x^t} x^s + a_{rs} x^r \frac{\partial x^s}{\partial x^t} \tag{1}$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs} \delta^r_t x^s + a_{rs} x^r \delta^s_t \tag{2}$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts} x^s + a_{rt} x^r \tag{3}$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts} x^s + a_{st} x^s \quad \text{(rename dummy variable in third term)} \tag{4}$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + (a_{ts} + a_{st}) x^s \tag{5}$$

Replace x^t by x^r, we get

$$\frac{\partial \phi}{\partial x^r} = \frac{\partial a_{rs}}{\partial x^r} x^r x^s + (a_{rs} + a_{sr}) x^s \tag{6}$$

So the asked expression is only true if a_{rs} is not a function of the x^s. Assuming that a_{rs} is not a function of the x^s, take the partial derivative of (6) with respect to x^t, we get

$$\frac{\partial^2 \phi}{\partial x^r \partial x^t} = (a_{rs} + a_{sr}) \frac{\partial x^s}{\partial x^t} \tag{7}$$

$$= (a_{rs} + a_{sr}) \delta^s_t \tag{8}$$

$$= (a_{rt} + a_{tr}) \tag{9}$$

Replace x^t by x^s, and we get the proposed expression.

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1.5 p8-clarification on expression 1.210

$$\boxed{\frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0}$$

From 1.209:

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} + \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} = 0 \tag{1}$$

multiply (1) with $\frac{\partial x'^q}{\partial x^r}$

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} + \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{2}$$

$$\Longleftrightarrow \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} \frac{\partial x'^q}{\partial x^r} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{3}$$

in the first term we get $$\frac{\partial x'^q}{\partial x^r} \frac{\partial x^r}{\partial x'^n} = \frac{\partial x'^q}{\partial x'^n} = \delta^q_n \tag{4}$$

(3) becomes

$$\frac{\partial^2 x'^n}{\partial x^p \partial x^s} \delta^q_n + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{5}$$

$$\Longleftrightarrow \frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{6}$$

◆

1.6 p9-exercise

If A^r_s are the elements of a determinant A, and B^r_s the elements of a determinant B, show that the element of the product determinant is $A^r_n B^n_s$. Hence show that the product of the two jacobians

$$J = \left|\frac{\partial x^r}{\partial x'^s}\right|, \quad J' = \left|\frac{\partial x'^r}{\partial x^s}\right|$$

is unity.

Remark: Some nitpick about the formulation: A^r_s are not the elements of a determinant A, but elements of the matrix A which gives $\det\{A\}$ provided that A is square (which is not explicitly mentioned.). The same remark for B and $A^r_n B^n_s$.
Be A^i_k the elements of matrix A and B^k_j the elements of matrix B and C = A.B the resulting matrix of the multiplication of A and B, then

$$C^i_j = A^i_k B^k_j$$

are the elements of matrix C. Now, put $A^i_k = \frac{\partial x^i}{\partial x'^k}$ and $B^k_j = \frac{\partial x'^k}{\partial x^j}$ then,

$$C^i_j = A^i_k B^k_j \tag{1}$$

$$= \frac{\partial x^i}{\partial x'^k}\frac{\partial x'^k}{\partial x^j} \tag{2}$$

$$= \delta^i_k \tag{3}$$

So $C = JJ'$ becomes the unity matrix.

◆

1.7 p11-exercise

Show that a finite contravariant vector determines the ratios of the components of an infinitesimal displacement. (Consider the transformation of the equation $dx^r = \theta T^r$, where θ is an arbitrary factor which does not change under the transformation. Alternatively, show that the equations $T^r dx^s - T^s x^r = 0$ remain true when we transform the coordinates.)

Be T^q a contravariant vector.

$$T'^q = T^r \frac{\partial x'^q}{\partial x^r} \quad \text{(by definition)} \tag{1}$$

Be θ a small infinitesimal factor invariant for a coordinate transformation, define

$$dx^r = \theta T^r \tag{2}$$

$$\tag{3}$$

then

$$\frac{dx^r}{dx^s} = \frac{\theta T^r}{\theta T^s} \tag{4}$$

$$\Leftrightarrow T^s dx^r - T^r dx^s = 0 \tag{5}$$

Alternatively, multiply (5) with $\partial_{x^r} x'^q$, then

$$\frac{\partial x'^q}{\partial x^r} dx^r T^s - \frac{\partial x'^q}{\partial x^r} dx^s T^r = 0 \tag{6}$$

$$\Leftrightarrow \frac{\partial x'^q}{\partial x^r} dx^r T^s - dx^s T'^q = 0 \quad \text{(use (1) in the second term)} \tag{7}$$

$$\Leftrightarrow dx'^q T^s - dx^s T'^q = 0 \tag{8}$$

$$\tag{9}$$

Multiply (8) with $\partial_{x^s} x'^p$, then

$$dx'^q T^s \partial_{x^s} x'^p - dx^s T'^q \partial_{x^s} x'^p = 0 \tag{10}$$

$$\Leftrightarrow \quad T'^p dx^q - T'^q dx'^p = 0 \quad \text{(use (1) in the first term)} \tag{11}$$

and thus

$$\frac{dx'^q}{dx'^p} = \frac{T'^q}{T'^p}$$

◆

1.8 p12-exercise

Write down the equation of transformation, analogous to 1.305, of a contravariant tensor of the third order. Solve the equation so as to express the unprimed components in terms of the primed components.

Be

$$T^{,uvw} = T^{rst} \frac{\partial x^{,u}}{\partial x^r} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \quad \text{(by definition)} \tag{1}$$

a contravariant vector.
Multiply (1) by $\frac{\partial x^n}{\partial x^{,u}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{rst} \frac{\partial x^{,u}}{\partial x^r} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \tag{2}$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{rst} \delta^n_r \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \tag{3}$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} = T^{nst} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^{,w}}{\partial x^t} \tag{4}$$

Multiply (4) by $\frac{\partial x^m}{\partial x^{,v}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nst} \frac{\partial x^{,v}}{\partial x^s} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^{,w}}{\partial x^t} \tag{5}$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nst} \delta^m_s \frac{\partial x^{,w}}{\partial x^t} \tag{6}$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} = T^{nmt} \frac{\partial x^{,w}}{\partial x^t} \tag{7}$$

Multiply (7) by $\frac{\partial x^p}{\partial x^{,w}}$

$$T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmt} \frac{\partial x^{,w}}{\partial x^t} \frac{\partial x^p}{\partial x^{,w}} \tag{8}$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmt} \delta^p_t \tag{9}$$

$$\Leftrightarrow T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}} = T^{nmp} \tag{10}$$

Giving

$$T^{nmp} = T^{,uvw} \frac{\partial x^n}{\partial x^{,u}} \frac{\partial x^m}{\partial x^{,v}} \frac{\partial x^p}{\partial x^{,w}}$$

◆

1.9 p14-exercise

For a transformation from on set of rectangular Cartesian coordinates to another in Euclidean 3-space, show that the law of transformation of a contravariant vector is precisely the same as that of a covariant vector. Can this statements be extended to cover tensor of higher orders?

We have to prove that, given that,

$$T'^{i} = T^{j}\frac{\partial x'^{i}}{\partial x^{j}} \quad T'_{i} = T_{j}\frac{\partial x^{j}}{\partial x'^{i}}$$

that also

$$T'^{i} = T^{j}\frac{\partial x^{j}}{\partial x'^{i}} \quad T'_{i} = T_{j}\frac{\partial x'^{i}}{\partial x^{j}} \tag{1}$$

$$\Leftrightarrow \frac{\partial x^{j}}{\partial x'^{i}} = \frac{\partial x'^{i}}{\partial x^{j}} \tag{2}$$

Be

$$\hat{e'^{i}} = g^{i}_{k}\hat{e^{k}} \quad \text{and } \hat{e^{i}} = h^{i}_{k}\hat{e'^{k}} \tag{3}$$

the transformation rules from one set of (rectangular Cartesian) basis vectors to another set of (rectangular Cartesian) basis vectors. Then,

$$\langle \hat{e'^{i}}, \hat{e'^{j}} \rangle = \langle g^{i}_{k}\hat{e^{k}}, g^{j}_{k}\hat{e^{k}} \rangle \text{ and } \langle \hat{e^{i}}, \hat{e^{j}} \rangle = \langle h^{i}_{k}\hat{e'^{k}}, h^{j}_{k}\hat{e'^{k}} \rangle \tag{4}$$

$$\Leftrightarrow \delta^{p}_{j} = g^{p}_{k}g^{j}_{k} \text{ and } \delta^{p}_{j} = h^{p}_{k}h^{j}_{k} \tag{5}$$

$$\tag{6}$$

Be $\vec{v}$ a random vector in the Euclidean space,

$$\vec{v} = x^{j}\hat{e^{j}} = x'^{j}\hat{e'^{j}} \tag{7}$$

then

$$(3) \;\Rightarrow x^{j}\hat{e^{j}} = x^{j}h^{j}_{k}\hat{e'^{k}} \text{ and } x'^{j}\hat{e'^{j}} = x'^{j}g^{j}_{k}\hat{e^{k}} \tag{8}$$

$$\Rightarrow x'^{j} = x^{m}h^{m}_{j} \text{ and } x^{m} = x'^{j}g^{j}_{m} \tag{9}$$

$$\Rightarrow x'^{j} = x'^{i}g^{i}_{m}h^{m}_{j} \text{ and } x^{m} = x^{k}h^{k}_{j}g^{j}_{m} \tag{10}$$

$$\Rightarrow \delta^{p}_{j} = g^{p}_{k}h^{k}_{j} \text{ and } \delta^{p}_{j} = g^{k}_{j}h^{p}_{k} \tag{11}$$

$$(5) \;\Rightarrow g^{p}_{k}g^{j}_{k} = g^{p}_{k}h^{k}_{j} \text{ and } h^{p}_{k}h^{j}_{k} = g^{k}_{j}h^{p}_{k} \tag{12}$$

$$\Rightarrow g^{j}_{k} = h^{k}_{j} \text{ and } h^{j}_{k} = g^{k}_{j} \tag{13}$$

From (9)

$$x^j = x^{,m} g_j^m \text{ and } x^{,k} = x^n h_k^n \tag{14}$$

$$\Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^n}{\partial x^j} h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \frac{\partial x^{,m}}{\partial x^{,k}} g_j^m \tag{15}$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = \delta_j^n h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \delta_k^m g_j^m \tag{16}$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = h_k^j \text{ and } \frac{\partial x^j}{\partial x^{,k}} = g_j^k \tag{17}$$

$$(13) \Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^j}{\partial x^{,k}} \tag{18}$$

So (13) matches (2), proving the assertion.

Can this statements be extended to cover tensor of higher orders? Consider

$$T^{,i,j,\dots,n} = T^{r,s,\dots w} \frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \cdots \frac{\partial x^{,n}}{\partial x^w} \text{ and } T^{r,s,\dots w} = T^{,i,j,\dots,n} \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \cdots \frac{\partial x^w}{\partial x^{,n}}$$

Using the same reasoning as in (1) to (2) we need

$$\frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \cdots \frac{\partial x^{,n}}{\partial x^w} = \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \cdots \frac{\partial x^w}{\partial x^{,n}}$$

As the conclusion (18) is independent of the order of the tensor, it is obvious that the above equality yields. Hence, the answer is YES.

◆

1.10 p16-exercise

In a space of 4 dimensions, the tensor A_{rst} is skew-symmetric in the last pair of suffixes. Show that only 24 of the 64 components may be chosen arbitrarily. If the further condition $A_{rst} + A_{str} + A_{trs} = 0$ is imposed, show that that only 20 components may be chosen arbitrarily.

We have, as A is skew-symmetric in the last pair of suffixes

$$A_{rst} = -A_{rts} \Rightarrow s = t\colon\ A_{rst} = 0$$

So, for each r (4 possible choices as N = 4) we have 4x4/2 - 4 = 6 degrees of freedom. [we have the term 4x4/2 as the tensor is (skew-)symmetric, e.g. once we choose element a_{12}, then a_{21} is also known. The term -4 takes into account the diagonal element which are 0 and thus cannot be chosen.]
So, we have 4x6 = 24 degrees of freedom.
What about the supplementary constraint $A_{rst} + A_{str} + A_{trs} = 0$:
Consider the two possible excluding cases:

i) $r = s \neq t$ ($\Leftrightarrow r = t \neq s$)
This case gives - without the additional constraint (1) - 4x(4x3/2-4) = 8 degrees of freedom. Does the constraint (1) reduces this degree of freedom?
We have,

$$A_{rst} + A_{str} + A_{trs} = 0 \tag{1}$$

$$\Rightarrow \underbrace{A_{rrt} + A_{rtr}}_{=0 \text{ (non-diagonal terms)}} + \underbrace{A_{trr}}_{=0 \text{ (diagonal terms)}} = 0 \tag{2}$$

So, no additional constraints are added by (1) to the restriction i) and the DOF remains 8.

ii) $t \neq r \neq s \neq t$
This case means that we have to choose a set of 3 elements out of 4 elements without repetition. This a *variation* of 3 elements out of 4.

$$V_k^n = \frac{n!}{(n-k)!} \text{ giving } V_3^4 = \frac{4!}{(4-3)!} = 24$$

The constraint (1) gives us 24 equations but as $A_{rst} = -A_{rts}$ only 12 equations have to be considered. So, with the additional constraints the DOF becomes 24-12 = 12.
As i) and ii) are independent and excluding events we can add the DOF of both events and we get 8+12 = 20 DOF.

◆

1.11 p16-exercise

If A^{rs} is skew-symmetric and B_{rs} is symmetric, prove that $A^{rs}B_{rs} = 0$. Hence show that the quadratic form $a_{ij}x^i x^j$ is unchanged if a_{ij} is replaced by its symmetric part.

We can split the summation $A^{rs}B_{rs}$ in three subsummations:

$$A^{rs}B_{rs} = A^{rs}B_{rs}|_{r=s} \tag{1}$$
$$+A^{rs}B_{rs}|_{r>s} \tag{2}$$
$$+A^{rs}B_{rs}|_{r<s} \tag{3}$$

We have:
(1) = 0 as $A^{kk} = 0$ (skew-symmetric)
(2)+(3) = $A^{rs}B_{rs}|_{r>s} + A^{rs}B_{rs}|_{r<s}$
As $A^{rs} = -A^{sr}$ and $B^{rs} = B^{sr}$ we can write (2)+(3) as :

$$A^{rs}B_{rs}|_{r>s} + (-A^{sr})B_{sr}|_{r>s} = 0$$

So, $A^{rs}B_{rs} = 0$

Consider the quadratic form $\phi = a_{ij}x^i x^j$
Be $A_{ij} = (a_{ij})$ and $B_{ij} = (x^i x^j)$, then it is obvious that B_{ij} is symmetric and that $C_{ij} = -A_{ij}$ is the form where $-a_{ij}$ is replaced by its symmetric part (skew-symmetric). Hence $\phi = a_{ij}x^i x^j = a_{ij}b^{ij} = 0$ and so is $\phi = c_{ij}b^{ij} = 0$

◆

1.12 p18-exercise

What are the values (in a space of N dimensions) of the folllowing contractions formed from the Kronecker delta?

$$\delta^m_m, \delta^m_n \delta^n_m, \delta^m_n \delta^n_r \delta^r_m$$

We can split the summation $A^{rs}B_{rs}$ in three subsummations:

$$\delta^m_m = N \tag{1}$$

$$\delta^m_n \delta^n_m = \delta^m_m = N \tag{2}$$

$$\delta^m_n \delta^n_r \delta^r_m = \delta^m_n \delta^n_m = \delta^m_m = N \tag{3}$$

◆

1.13 p19-exercise

If X^r, Y^r are arbitrary contravariant vectors and $a_{rs}X^rY^s$ is an invariant, then a_{rs} are the components of a covariant tensor of the second order.

We have to prove that

$$a'_{rs} = a_{ij}\frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^s} \text{ or } a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{1}$$

$a_{rs}X^rY^s$ is an invariant, means

$$a'_{rs}X'^rY'^s = a_{rs}X^rY^s \tag{2}$$

As X^r, Y^r are arbitrary contravariant vectors, we have

$$X'^r = X^i\frac{\partial x'^r}{\partial x^i} \text{ and } Y'^s = Y^j\frac{\partial x'^s}{\partial x^j} \tag{3}$$

(3) in (2) gives

$$a'_{rs}X^i\frac{\partial x'^r}{\partial x^i}Y^j\frac{\partial x'^s}{\partial x^j} = a_{rs}X^rY^s \tag{4}$$

$$\Leftrightarrow a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j}X^iY^j = a_{ij}X^iY^j \tag{5}$$

$$\Leftrightarrow (a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij})X^iY^j = 0 \tag{6}$$

As X^r, Y^r are arbitrary contravariant vectors, we conclude that

$$a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij} = 0 \tag{7}$$

$$\Leftrightarrow a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{8}$$

(8) = (1): OK

◆

1.14 p19-exercise

If X_{rs} is an arbitrary covariant tensor of the second order, and $A_r^{mn}X_{mn}$ is a covariant vector, then A_r^{mn} has the mixed tensor character indicated by the positions of its suffixes

We have to prove that

$$A'^{vw}_r = A_k^{mn}\frac{\partial x^k}{\partial x'^r}\frac{\partial x'^v}{\partial x^m}\frac{\partial x'^w}{\partial x^n} \tag{1}$$

We have

$$P_r = A_r^{mn}X_{mn} \tag{2}$$

is a covariant vector

$$\Rightarrow P'_r = A_k^{mn}X_{mn}\frac{\partial x^k}{\partial x'^r} \tag{3}$$

but X_{mn} is a covariant tensor

$$\Rightarrow X_{mn} = X'_{ps}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n} \tag{4}$$

So (4) in (3) gives

$$P'_r = A_k^{mn}X'_{ps}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n}\frac{\partial x^k}{\partial x'^r} \tag{5}$$

$$\Leftrightarrow P'_r = \underbrace{A_k^{mn}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n}\frac{\partial x^k}{\partial x'^r}}_{(*)}X'_{ps} \tag{6}$$

Putting (*) as $A'^{ps}_r = A_k^{mn}\frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n}\frac{\partial x^k}{\partial x'^r}$ we see that (6) has the form (2) and that A'^{ps}_r obeys the rule of a mixed tensor (1).

◆

1.15 p21-exercise

If A_{rs} is a skew-symmetric covariant tensor, prove that B_{rst} defined as

$$B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$$

is a covariant tensor, and that it is skew-symmetric in all pairs of suffixes.

We have A_{rs} is a covariant tensor

$$A_{ij} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^i} \frac{\partial x^\beta}{\partial x^j} \tag{1}$$

$$\Rightarrow B_{rst} = \partial_r \left(A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}\right) + \partial_s \left(A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r}\right) + \partial_t \left(A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s}\right) \tag{2}$$

Note that

$$\partial_k \left(A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t}\right) = \partial_k (A_{\alpha\beta}) \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \partial_k \left(\frac{\partial x^\alpha}{\partial x^s}\right) \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \partial_k \left(\frac{\partial x^\beta}{\partial x^t}\right) \tag{3}$$

$$\tag{4}$$

so,

$$\begin{aligned} B_{rst} = {} & \partial_r A_{\alpha\beta} \tfrac{\partial x^\alpha}{\partial x^s} \tfrac{\partial x^\beta}{\partial x^t} + \underbrace{A_{\alpha\beta} \partial_r \tfrac{\partial x^\alpha}{\partial x^s} \tfrac{\partial x^\beta}{\partial x^t}}_{*} + \underbrace{A_{\alpha\beta} \tfrac{\partial x^\alpha}{\partial x^s} \partial_r \tfrac{\partial x^\beta}{\partial x^t}}_{**} \\ & + \partial_s A_{\alpha\beta} \tfrac{\partial x^\alpha}{\partial x^t} \tfrac{\partial x^\beta}{\partial x^r} + \underbrace{A_{\alpha\beta} \partial_s \tfrac{\partial x^\alpha}{\partial x^t} \tfrac{\partial x^\beta}{\partial x^r}}_{***} + \underbrace{A_{\alpha\beta} \tfrac{\partial x^\alpha}{\partial x^t} \partial_s \tfrac{\partial x^\beta}{\partial x^r}}_{*} \\ & + \partial_t A_{\alpha\beta} \tfrac{\partial x^\alpha}{\partial x^r} \tfrac{\partial x^\beta}{\partial x^s} + A_{\alpha\beta} \underbrace{\partial_t \tfrac{\partial x^\alpha}{\partial x^r} \tfrac{\partial x^\beta}{\partial x^s}}_{**} + \underbrace{A_{\alpha\beta} \tfrac{\partial x^\alpha}{\partial x^r} \partial_t \tfrac{\partial x^\beta}{\partial x^s}}_{***} \end{aligned} \tag{5}$$

In (5) consider the two terms with (*)

$$T = A_{\alpha\beta} \partial_r \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \partial_s \frac{\partial x^\beta}{\partial x^r} \tag{6}$$

$$= A_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x^s \partial x^r} \frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial^2 x^\beta}{\partial x^r \partial x^s} \tag{7}$$

$$= A_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x^s \partial x^r} \frac{\partial x^\beta}{\partial x^t} + A_{\beta\alpha} \frac{\partial x^\beta}{\partial x^t} \frac{\partial^2 x^\alpha}{\partial x^r \partial x^s} \text{ (by renaming dummy variables)} \tag{8}$$

As $A_{ij} = -A_{ji}$ (skew-symmetric tensor), we get $T = 0$. The same yields for the (**) and (***) terms. So, B_{rst} reduces to

$$B_{rst} = \partial_r A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \partial_s A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \partial_t A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \tag{9}$$

$$\Leftrightarrow B_{rst} = \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^r} \frac{\partial x^\alpha}{\partial x^s} \frac{\partial x^\beta}{\partial x^t} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^s} \frac{\partial x^\alpha}{\partial x^t} \frac{\partial x^\beta}{\partial x^r} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^t} \frac{\partial x^\alpha}{\partial x^r} \frac{\partial x^\beta}{\partial x^s} \tag{10}$$

By adequate renaming of the dummy variable in the 3 terms:

$$\begin{bmatrix} 1^{st}term \\ 2^{nd}term \\ 3^{rd}term \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma \to \alpha & \alpha \to \beta & \beta \to \gamma \\ \beta \to \alpha & \gamma \to \beta & \alpha \to \gamma \\ \alpha \to \alpha & \beta \to \beta & \gamma \to \gamma \end{bmatrix}$$

we get

$$B_{rst} = (\frac{\partial A_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial A_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma})\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}\frac{\partial x^\gamma}{\partial x^t} \tag{11}$$

$$\Leftrightarrow B_{rst} = (\underbrace{\partial_\alpha A_{\beta\gamma} + \partial_\beta A_{\gamma\alpha} + \partial_\gamma A_{\alpha\beta}}_{(****)})\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}\frac{\partial x^\gamma}{\partial x^t} \tag{12}$$

The expression (****) has exactly the required form $B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$ and is transformed (12) according the rules of a covariant tensor.
Let's prove now that it is skew-symmetric in all pairs of suffixes. We have to consider the following permutations

$$\begin{bmatrix} rst \\ rts \\ srt \\ str \\ trs \\ tsr \end{bmatrix}$$

E.g. *srt*

$$B_{rts} = \partial_r A_{ts} + \partial_t A_{sr} + \partial_s A_{rt} \tag{13}$$

$$= -\partial_r A_{st} - \partial_t A_{rs} - \partial_s A_{tr} \tag{14}$$

$$= -B_{rst} \tag{15}$$

The same calculations can be done for the other permutations.

◆

1.16 p23-exercise 1.

In a V_4 there are two 2-spaces with equations

$$x^r = f^r(u^1, u^2),\ x^r = g^r(u^3, u^4)$$

Prove that if these 2-spaces have a curve of intersection, then the determinal equation

$$\left|\frac{\partial x^r}{\partial u^s}\right| = 0$$

is satisfied along the curve.

Having a curve means that one of the parameters u^i can be freely chosen while the other 3 are determined by the chosen parameter.
We have,

$$\left|\frac{\partial x^r}{\partial u^s}\right| = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} & \frac{\partial x^1}{\partial u^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^4}{\partial u^1} & \frac{\partial x^4}{\partial u^2} & \frac{\partial x^4}{\partial u^3} & \frac{\partial x^4}{\partial u^4} \end{vmatrix} \tag{1}$$

Suppose we choose u^4 as parameter. This means $u^i = \phi^i(u^4)$ for i=1,2,3 and thus we can write

$$\frac{\partial x^i}{\partial u^4} = \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} + \frac{\partial x^i}{\partial u^4} \qquad \text{with j=1,2,3} \qquad \text{i = 1,2,3,4} \tag{2}$$

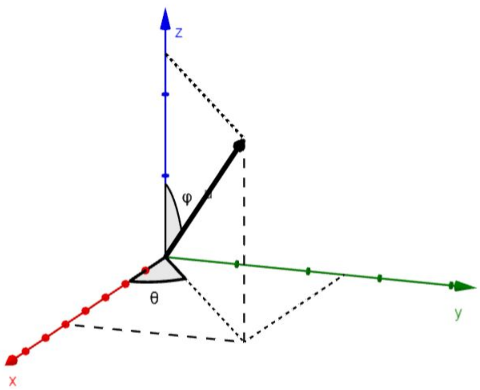
$$\Rightarrow \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} = 0 \tag{3}$$

This means that in (1) the three first columns a not linearly independent and thus have $\left|\frac{\partial x^r}{\partial u^s}\right| = 0$

◆

1.17 p23-exercise 2.

In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates x, y, z and spherical polar coordinates r,θ,ϕ.
Find the Jacobian of the transformation. Where is it zero or infinite?



We use the latitude ψ instead of the co-latitude ϕ.

$$\left\{ \begin{array}{c} x = r\cos(\psi)\cos(\theta) \\ y = r\cos(\psi)\sin(\theta) \\ z = r\sin(\psi) \end{array} \right\}$$

Partial differentiating of (x,y,z) with respect to (r,ψ,θ) gives the Jacobian

$$J = \begin{vmatrix} \cos(\psi)\cos(\theta) & -r\sin(\psi)\cos(\theta) & -r\cos(\psi)\sin(\theta) \\ \cos(\psi)\sin(\theta) & -r\sin(\psi)\sin(\theta) & r\cos(\psi)\cos(\theta) \\ \sin(\psi) & r\cos(\psi) & 0 \end{vmatrix} \tag{1}$$

$$\begin{aligned} J &= \cos(\psi)\cos(\theta)(-r^2)\cos^2(\psi)\cos(\theta)) && (2) \\ &+ r\sin(\psi)\cos(\theta)(-r\cos(\psi)\cos(\theta)\sin(\psi)) && (3) \\ &- r\cos(\psi)\sin(\theta)(r\cos^2(\psi)\sin(\theta) + r\sin^2(\psi)\sin(\theta)) && (4) \\ &= -r^2\cos^3(\psi)\cos^2(\theta) - r^2\sin^2(\psi)\cos^2(\theta)\cos(\psi) - r^2\cos(\psi)\sin^2(\theta) && (5) \end{aligned}$$

Noting that the 2^{nd} term in (5) can be written as $-r^2\cos^2(\theta)\cos(\psi) + r^2\cos^2(\theta)\cos^3(\psi)$, we get

$$\begin{aligned} J &= -r^2(\cos^3(\psi)\cos^2(\theta) + \cos^2(\theta)\cos(\psi) - \cos^3(\psi)\cos^2(\theta) + \cos(\psi)\sin^2(\theta)) && (6) \\ &= -r^2\cos(\psi) && (7) \end{aligned}$$

J=0: for $r = 0$ or $\psi = \frac{\pi}{2}|_{r\epsilon(-\infty,+\infty)}$ and $J \to \pm\infty$ or $\mp\infty$ for $r \to \pm\infty|_{\psi\neq 0}$. But what about the case $r \to \pm\infty|_{\psi\to 0}$? This case is not determined as long as no path is chosen in the (r,ψ) configuration space.

◆

1.18 p23-exercise 3.

If X, Y, Z are the components of a contravariant vector for rectangular Cartesian coordinates in Euclidean 3-space, find it's components for spherical polar coordinates.

Be x^α the components of a contravariant vector in spherical polar coordinates and x^i it's components in rectangular Cartesian coordinates. As we have

$$\begin{aligned} x^\rho &= \sqrt{x^j x^j} \\ x^\theta &= \operatorname{atan} \tfrac{x^2}{x^1} \\ x^\phi &= \operatorname{asin} \tfrac{x^3}{\sqrt{x^j x^j}} \end{aligned} \qquad \text{and} \qquad A^\alpha = A^i \frac{\partial x^\alpha}{\partial x^i} \tag{1}$$

$$\Rightarrow \left[A^\alpha\right] = \left[A^i \frac{\partial x^\alpha}{\partial x^i}\right] = \begin{bmatrix} \frac{x^1}{\sqrt{x^j x^j}} & \frac{x^2}{\sqrt{x^j x^j}} & \frac{x^3}{\sqrt{x^j x^j}} \\ -\frac{x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} & 0 \\ -\frac{x^3 x^1}{(x^j x^j)\sqrt{(x^1)^2+(x^2)^2}} & -\frac{x^3 x^2}{(x^j x^j)\sqrt{(x^1)^2+(x^2)^2}} & \frac{\sqrt{(x^1)^2+(x^2)^2}}{(x^j x^j)} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} \tag{2}$$

◆

1.19 p23-exercise 4.

In a space of three dimensions, how many different expressions are represented by the product $A^m_{np}B^{pq}_{rs}C^s_{tu}$? How many terms occur in each such expression, when written out explicitly?

As we have V_3 and considering that in $A^m_{np}B^{pq}_{rs}C^s_{tu}$ the six indices m, n, q, r, t, u are not dummy indices, we get 3^6 different expressions (first choose m: you have three choices, then n: also three choices giving 3x3 possibilities, etc for q, r, t, u).
For the second question, as in $A^m_{np}B^{pq}_{rs}$ there is only summation on over index (p) we get three terms for this part. As the summation with $A^m_{np}B^{pq}_{rs}$ and C^s_{tu} occurs only on one index also (s) we get 3x3 terms in the expression.

◆

1.20 p23-exercise 5.

If A is an invariant in V_n, are the second derivatives $\frac{\partial^2 A}{\partial x^r \partial x^s}$ the components of a tensor?

As A is invariant (note: different alphabets in the indices indicates different coordinate systems):

$$A(x^\rho) = A(x^i) \tag{1}$$

$$\Rightarrow \frac{\partial A(x^\rho)}{\partial x^i} = \frac{\partial A(x^j)}{\partial x^i} \tag{2}$$

To simplify the notation, we put $A(x^\rho) = A$' and $A(x^j) = A$' then (2) can be written as

$$\frac{\partial A'}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^i} = \frac{\partial A}{\partial x^i} \tag{3}$$

Conclusion: $\frac{\partial A}{\partial x^i}$ is a covariant tensor.

Consider now $\frac{\partial A}{\partial x^i} = \frac{\partial A'}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^i}$. Then,

$$\frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A'}{\partial x^\rho \partial x^j}\frac{\partial x^\rho}{\partial x^i} + \frac{\partial A'}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \tag{4}$$

$$\Leftrightarrow \frac{\partial^2 A}{\partial x^i \partial x^j} = \frac{\partial^2 A'}{\partial x^\rho \partial x^\gamma}\frac{\partial x^\gamma}{\partial x^j}\frac{\partial x^\rho}{\partial x^i} + \frac{\partial A'}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} \tag{5}$$

The first term on the right side, behaves as covariant tensor but the presence of the second term makes that generally, $\frac{\partial^2 A}{\partial x^i \partial x^j}$ has not a tensor character. This is only when $\frac{\partial A'}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x^i \partial x^j} = 0$, which means that x^ρ, x^i are a linear map of each other.

◆

1.21 p23-exercise 6.

Suppose that in V_2 the components of a contravariant tensor field T^{mn} in a coordinate system x^r are

$$T^{11} = 1 \quad T^{12} = 0$$

$$T^{21} = 1 \quad T^{22} = 0$$

Find the components $T^{,mn}$ in a coordinate system $x^{,r}$, where

$$x^{,1} = (x^1)^2 \quad x^{,2} = (x^2)^2$$

Write down the values of these components in particular at the point $x^1 = 1, x^2 0 =$.

As we have a contravariant tensor field :

$$T^{,mn} = T^{ij} \frac{\partial x^{,m}}{\partial x^i} \frac{\partial x^{,n}}{\partial x^j} \tag{1}$$

$$\begin{aligned} x^{,1} = (x^1)^2 \Rightarrow \quad \tfrac{\partial x^{,1}}{\partial x^1} = 2x^1 \quad \tfrac{\partial x^{,1}}{\partial x^2} = 0 \\ x^{,2} = (x^2)^2 \Rightarrow \quad \tfrac{\partial x^{,2}}{\partial x^1} = 0 \quad \tfrac{\partial x^{,2}}{\partial x^2} = 2x^2 \end{aligned} \tag{2}$$

$$\tag{3}$$

$$\Rightarrow T^{,11} = 4(x^1)^2 + 4(x^2)^2 \tag{4}$$

$$\Rightarrow T^{,12} = T^{,21} = 0 \tag{5}$$

$$\Rightarrow T^{,22} = 4(x^1)^2 + 4(x^2)^2 \tag{6}$$

The components in at the point $x^1 = 1, x^2 = 0$ are

$$T^{,}(1,0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

◆

1.22 p24-exercise 7.

Given that if T_{mnrs} is a covariant tensor, and

$$T_{mnrs} + T_{mnsr} = 0$$

in a coordinate system x^p, establish directly that

$$T'_{mnrs} + T'_{mnsr} = 0$$

in any other coordinate system x, q.

Note: in the following, different alphabets in the indices indicates different coordinate systems. As we T_{mnrs} is a covariant tensor :

$$T_{\alpha\beta\gamma\delta} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} \tag{1}$$

$$\Rightarrow T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} = T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} + T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\delta}\frac{\partial x^s}{\partial x^\gamma} \tag{2}$$

Now, swap the dummy indices r and s in the second term on the right and as $T_{mnrs} = -T_{mnsr}$:

$$\begin{aligned}
T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} &= T_{mnrs}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^r}{\partial x^\gamma}\frac{\partial x^s}{\partial x^\delta} + T_{mnsr}\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^s}{\partial x^\delta}\frac{\partial x^r}{\partial x^\gamma} && (3)\\
&= (T_{mnrs} + T_{mnsr})\frac{\partial x^m}{\partial x^\alpha}\frac{\partial x^n}{\partial x^\beta}\frac{\partial x^s}{\partial x^\delta}\frac{\partial x^r}{\partial x^\gamma} && (4)\\
&= 0 && (5)
\end{aligned}$$

◆

1.23 p24-exercise 8.

Prove that if A_r is a covariant vector, then $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ is a skew-symmetric covariant tensor of the second order (use the notation of 1.7).

Be $B_{rs} = \frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$.
i) B_{rs} is skew-symmetric: It is obvious that:

$$-B_{rs} = -\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r} = \frac{\partial A_s}{\partial x^r} - \frac{\partial A_r}{\partial x^s} \equiv B_{sr}$$

ii) B_{rs} is covariant:
Note: in the following, different alphabets in the indices indicates different coordinate systems.
Let

$$C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta. \tag{1}$$

We know that $A_i = A_\gamma X^\gamma_i$ as A_i is covariant. Hence,

$$\begin{aligned} \partial_j A_i &= \partial_j A_\gamma X^\gamma_i + A_\gamma \partial_j X^\gamma_i &(2)\\ &= \partial_\alpha A_\gamma X^\alpha_j X^\gamma_i + A_\gamma \partial_j X^\gamma_i &(3) \end{aligned}$$

Using (3), we compute the first term in (1)

$$\begin{aligned} \partial_s A_r X^r_\alpha X^s_\beta &= \partial_\rho A_\gamma X^\rho_s X^\gamma_r X^r_\alpha X^s_\beta + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta &(4)\\ &= \partial_\rho A_\gamma X^\rho_\beta X^\gamma_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta &(5)\\ &= \partial_\rho A_\gamma \delta^\rho_\beta \delta^\gamma_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta &(6)\\ &= \partial_\beta A_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta &(7) \end{aligned}$$

In the same way, we get for the second term in (1)

$$\partial_r A_s X^s_\alpha X^r_\beta = \partial_\alpha A_\beta + A_\gamma \partial_r X^\gamma_s X^r_\alpha X^s_\beta \tag{8}$$

And thus,

$$\begin{aligned} C_{\alpha\beta} = (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta &= \partial_\beta A_\alpha + A_\gamma \partial_s X^\gamma_r X^r_\alpha X^s_\beta - \partial_\alpha A_\beta - A_\gamma \partial_r X^\gamma_s X^r_\alpha X^s_\beta &(9)\\ \Rightarrow \partial_\beta A_\alpha - \partial_\alpha A_\beta &= (\partial_s A_r - \partial_r A_s) X^r_\alpha X^s_\beta &(10) \end{aligned}$$

So, i) and (10) proves that $\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}$ is skew-symmetric tensor of the second order.

◆

1.24 p24-exercise 9.

Let $x^r, \overline{x}^r, y^r, \overline{y}^r$ be four systems of coordinates. Examine the tensor character of $\frac{\partial x^r}{\partial y^s}$ with respect to the following transformations:
i) A transformation $x^r = f^r(\overline{x}^1, \ldots, \overline{x}^N)$, with y^r unchanged;
ii) A transformation $y^r = g^r(\overline{y}^1, \ldots, \overline{y}^N)$, with x^r unchanged;

Note: in the following, different alphabets in the indices indicates different coordinate systems.

i) Let's compute the expression $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta}$. Obviously, the right side is an expression of a (possible) mixed tensor of the second order $(\frac{\partial x^r}{\partial y^s})$ under transformation from the (r) coordinate system to the (α) coordinate system. Then,

$$A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta} \tag{1}$$
$$= \frac{\partial x^\alpha}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} \tag{2}$$
$$= \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} \tag{3}$$

If we consider the $\overline{y}^r$ coordinate system as the y^ρ coordinate system and as $\overline{y}^r = y^r$ then $\frac{\partial y^\rho}{\partial y^s} = \delta^\rho_s$ and we get from (3)

$$A(\alpha, \beta) = \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial x^s}{\partial x^\beta} \tag{4}$$
$$= \frac{\partial x^\alpha}{\partial y^\rho}\delta^\rho_s\frac{\partial x^s}{\partial x^\beta} \tag{5}$$
$$= \frac{\partial x^\alpha}{\partial y^\rho}\frac{\partial x^\rho}{\partial x^\beta} \tag{6}$$
$$= \frac{\partial x^\alpha}{\partial y^\rho}\delta^\rho_\beta \tag{7}$$
$$= \frac{\partial x^\alpha}{\partial y^\beta} \tag{8}$$
$$\text{(1) and (8)} \Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^s}{\partial x^\beta} \tag{9}$$

So $A(r, s) = \frac{\partial x^r}{\partial y^s}$ is a mixed tensor of type A^r_s

ii) Let's compute the expression $A(\alpha, \beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta}$. Obviously, the right side is an expression of a (possible) mixed tensor of the second order $(\frac{\partial x^r}{\partial y^s})$ under transformation from the (r) coordinate system to the (α) coordinate system. Then,

$$A(\alpha,\beta) = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{10}$$

$$= \frac{\partial x^r}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{11}$$

$$= \frac{\partial x^r}{\partial y^\rho}\frac{\partial y^\rho}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{12}$$

$$= \frac{\partial x^r}{\partial y^\rho}\delta^\rho_\beta\frac{\partial y^\alpha}{\partial y^r} \tag{13}$$

$$= \frac{\partial x^r}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{14}$$

$$= \frac{\partial x^r}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{15}$$

If we consider the $\overline{x}^r$ coordinate system as the x^σ coordinate system and as $\overline{x}^r = x^r$ then $\frac{\partial x^\sigma}{\partial x^r} = \delta^\sigma_r$ and we get from (15)

$$A(\alpha,\beta) = \frac{\partial x^r}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{16}$$

$$= \delta^r_\sigma\frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^r} \tag{17}$$

$$= \frac{\partial x^\sigma}{\partial y^\beta}\frac{\partial y^\alpha}{\partial y^\sigma} \tag{18}$$

$$= \frac{\partial x^\sigma}{\partial y^\beta}\delta^\alpha_\sigma \tag{19}$$

$$= \frac{\partial x^\alpha}{\partial y^\beta} \tag{20}$$

$$\text{(10) and (19)} \Rightarrow \frac{\partial x^\alpha}{\partial y^\beta} = \frac{\partial x^r}{\partial y^s}\frac{\partial y^\alpha}{\partial y^r}\frac{\partial y^s}{\partial y^\beta} \tag{21}$$

So $A(r,s) = \frac{\partial x^r}{\partial y^s}$ is a mixed tensor of type A^r_s

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1.25 p23-exercise 10.

If x^r, y^r, z^r are three systems of coordinates, prove the follwoing rule for the multiplication of Jacobians.

$$\left|\frac{\partial x^m}{\partial y^n}\right|\left|\frac{\partial y^r}{\partial z^s}\right| = \left|\frac{\partial x^t}{\partial z^u}\right|$$

As we have

$$\frac{\partial x^t}{\partial z^u} = \frac{\partial x^t}{\partial y^k}\frac{\partial y^k}{\partial z^u} \tag{1}$$

$$\begin{bmatrix} \frac{\partial x^1}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial z^N} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial y^k}\frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^1}{\partial y^k}\frac{\partial y^k}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^k}\frac{\partial y^k}{\partial z^1} & \cdots & \frac{\partial x^N}{\partial y^k}\frac{\partial y^k}{\partial z^N} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^N}{\partial y^1} & \cdots & \frac{\partial x^N}{\partial y^k} \end{bmatrix}\begin{bmatrix} \frac{\partial y^1}{\partial z^1} & \cdots & \frac{\partial y^1}{\partial z^N} \\ \vdots & \vdots & \vdots \\ \frac{\partial y^N}{\partial z^1} & \cdots & \frac{\partial y^N}{\partial z^N} \end{bmatrix} \tag{3}$$

$$\Rightarrow \left|\frac{\partial x^m}{\partial y^n}\right|\left|\frac{\partial y^r}{\partial z^s}\right| = \left|\frac{\partial x^t}{\partial z^u}\right| \tag{4}$$

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1.26 p23-exercise 11.

Prove that with respect to transformations

$$x'^r = C_{rs}x^s$$

where the coefficients are constants satisfying

$$C_{mr}C_{ms} = \delta^r_s$$

contravariant and covariant vectors have the same formula of transformation

$$A'^r = C_{rs}A^s, \; A_{,r} = C_{rs}A_s$$

i) $A'^r = C_{rs}A^s$

Be $A'^r = A^s\frac{\partial x'^r}{\partial x^s}$ and as $x'^r = C_{rs}x^s$ we have $\frac{\partial x'^r}{\partial x^s} = C_{rs}$. Hence,

$$A'^r = C_{rs}A^s$$

.

i) $A_{,r} = C_{rs}A_s$
Be $A_{,r} = A_s\frac{\partial x^s}{\partial x'^r}$ and as $x'^r = C_{rs}x^s$ we have

$$\frac{\partial x'^r}{\partial x'^t} = C_{rs}\frac{\partial x^s}{\partial x'^t} \tag{1}$$

$$\Rightarrow \delta^r_t = C_{rs}\frac{\partial x^s}{\partial x'^t} \tag{2}$$

Now, multiply (2) by C_{rq}. We get,

$$\delta^r_t C_{rq} = C_{rq}C_{rs}\frac{\partial x^s}{\partial x'^t} \tag{3}$$

$$C_{tq} = C_{rq}C_{rs}\frac{\partial x^s}{\partial x'^t} \tag{4}$$

$$\text{as } C_{mr}C_{ms} = \delta^r_s \Rightarrow \qquad C_{tq} = \delta^q_s\frac{\partial x^s}{\partial x'^t} \tag{5}$$

$$\Rightarrow C_{tq} = \frac{\partial x^q}{\partial x'^t} \text{ or } C_{rs} = \frac{\partial x^s}{\partial x'^r} \tag{6}$$

$$\text{as } A_{,r} = A_s\frac{\partial x^s}{\partial x'^r} \Rightarrow A_{,r} = C_{rs}\frac{\partial x^s}{\partial x'^r} \tag{7}$$

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1.27 p23-exercise 12.

Prove that

$$\frac{\partial ln\left|\frac{\partial x^m}{\partial y^n}\right|}{\partial x^r} = \frac{\partial^2 y^m}{\partial x^r \partial x^n}\frac{\partial x^n}{\partial y^m}$$

.

Lemma:
Be A a square matrix NxN; Be f a C^1 function $f : \mathbb{R}^{NxN} \to \mathbb{R}$.
Define A' as $(A'_{ij}) = \frac{df}{dA_{ij}}$. Then,

$$(ln\,|A|)' = (A^{-1})^T \text{ wih } f = |A|$$

Proof:
By definition of the determinant, we have

$$|A| = A_{iK}C^i_K \quad \text{(no summation on K!)} \tag{1}$$

with $(C^i_K) = (-1)^{i+K}M^i_K$ being the cofactor of element A_{iK} and M^i_K the minor (N-1)x(N-1) matrix associated with the cofactor A_{K^i}. Be $C = (C_{ij})$ the NxN matrix formed with all posssible cofactor elements C^i_j $(i, j = 1\ldots, N)$.
We have

$$A^{-1} = \frac{C^T}{|A|} \tag{2}$$

$$\Rightarrow (A^{-1})^T = \frac{C}{|A|} \tag{3}$$

$$\text{differentiating (1)} \;\Rightarrow \frac{\partial\,|A|}{\partial A_{mn}} = \frac{\partial A_{iK}}{\partial A_{mn}}C^i_K + A_{iK}\frac{\partial C^i_K}{\partial A_{mn}} \tag{4}$$

$$\text{we have for i = m} \quad \begin{matrix} \frac{\partial A_{iK}}{\partial A_{mn}} = 1 & K = n \\ \frac{\partial A_{iK}}{\partial A_{mn}} = 0 & K \neq n \end{matrix} \tag{5}$$

Also, $\forall K : \frac{\partial C^i_K}{\partial A_{in}} = 0$ as by definition of the cofactor matrix, A_{ij} is not contained in C_{ij}.Hence, (4) becomes

$$\frac{\partial |A|}{\partial A_{ij}} = C^i_j \tag{6}$$

$$\text{But,} \quad \frac{\partial ln |A|}{\partial A_{ij}} = \frac{\frac{\partial |A|}{\partial A_{ij}}}{|A|} \tag{7}$$

$$\text{(6) and (7) gives} \quad \frac{\partial ln |A|}{\partial A_{ij}} = \frac{C^i_j}{|A|} \tag{8}$$

$$\text{(3) and (8) gives} \quad \frac{\partial ln |A|}{\partial A_{ij}} = \frac{(A^{-1}_{ij})^T |A|}{|A|} = (A^{-1}_{ij})^T \tag{9}$$

$$\Rightarrow (ln |A|)' = (A^{-1})^T \tag{10}$$

Now the main proof:
Let,

$$A \equiv [a_{mn}] = \left[\frac{\partial y^m}{\partial x^n} \right] \tag{11}$$

$$\Rightarrow \frac{\partial ln |A|}{\partial x^r} = \sum_{m,n=1}^{N,N} \frac{\partial ln |A|}{\partial a_{mn}} \frac{\partial a_{mn}}{\partial x^r} \tag{12}$$

$$\text{from (10) we get} \quad \frac{\partial ln |A|}{\partial a_{mn}} = (A^{-1})^T_{mn} \tag{13}$$

$$\text{But A is a Jacobian, so } (A^{-1})_{mn} = \frac{\partial x^m}{\partial y^n} \tag{14}$$

$$\text{and thus } (A^{-1})^T_{mn} = \frac{\partial x^n}{\partial y^m} \tag{15}$$

$$\text{(13) can be written as } \frac{\partial ln |A|}{\partial x^r} = \sum_{m,n=1}^{N,N} \frac{\partial x^n}{\partial y^m} \frac{\partial a_{mn}}{\partial x^r} \tag{16}$$

$$\Rightarrow \frac{\partial ln |A|}{\partial x^r} = \sum_{m,n=1}^{N,N} \frac{\partial x^n}{\partial y^m} \frac{\partial^2 y^m}{\partial x^r \partial x^n} \tag{17}$$

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