Tensor Calculus
J.L. Synge and A.Shild (Dover Publication)
Solutions to exercices

Bernard Carrette

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Remarks and warnings

Some notation conventions

$$\partial_r a_{mn} \equiv \frac{\partial a_{mn}}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \qquad \text{Christoffel symbol of the second kind}$$

Contents

Spaces and Tensors

1.1 p5-exercise

The parametric equations of a hypersurface in V_n are

$$
\begin{aligned}
x^1 &= a\cos\left(u^1\right)\\
x^2 &= a\sin\left(u^1\right)\cos\left(u^2\right)\\
x^3 &= a\sin\left(u^1\right)\sin\left(u^2\right)\cos\left(u^3\right)\\
&\vdots\\
x^{N-1} &= a\sin\left(u^1\right)\sin\left(u^2\right)\sin\left(u^3\right)\ldots\sin\left(u^{N-2}\right)\cos\left(u^{N-1}\right)\\
x^N &= a\sin\left(u^1\right)\sin\left(u^2\right)\sin\left(u^3\right)\ldots\sin\left(u^{N-2}\right)\sin\left(u^{N-1}\right)
\end{aligned}
$$

where a is a constant. Find the single equation of the hyperspace in the form 1.103.

We have:

$$
\begin{aligned}
(x^N)^2 + (x^{N-1})^2 &= a^2 \prod_{i=1}^{N-2} \sin^2(u^i)(\cos^2(u^{N-1}) + \sin^2(u^{N-1}))\\
&= a^2 \prod_{i=1}^{N-2} \sin^2(u^i)\\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i)\sin^2(u^{N-2})\\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i)(1-\cos^2(u^{N-2})\\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - a^2 \prod_{i=1}^{N-3} \sin^2(u^i)\cos^2(u^{N-2})\\
&= a^2 \prod_{i=1}^{N-3} \sin^2(u^i) - (x^{N-2})^2
\end{aligned}
$$

giving

$$(x^N)^2 + (x^{N-1})^2 + (x^{N-2})^2 = a^2 \prod_{i=1}^{N-3} \sin^2(u^i)$$

In general, by recursion

$$\sum_{i=0}^{k} (x^{N-i})^2 = a^2 \prod_{i=1}^{N-k-1} \sin^2(u^i) \quad (k \leq N-2)$$

be k = N - 2 (N - k - 1 = 1) and in the left term put j = N - i (j goes from 2 to N), we get

$$\begin{aligned}\sum_{j=2}^{N}(x^j)^2 &= a^2\prod_{i=1}^{1}\sin^2(u^i)\\ &= a^2(1-\cos^2(u^1))\\ &= a^2-(x^1)^2\end{aligned}$$

and thus the equation of the hyperspace is given by

$$\sum_{j=1}^{N}(x^j)^2 - a^2 = 0$$

> Determine whether the points $(\frac{1}{2}a, 0, 0, ...0)$, $(0, 0, ..., 0, 2a)$ lie on the same or opposite sides of the hyperspace.

For $(\frac{1}{2}a, 0, 0, ...0)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = -\frac{3a^2}{4} < 0$ and for $(0, 0, ..., 0, 2a)$ we have $\sum_{j=1}^{N}(x^j)^2 - a^2 = \frac{3a^2}{4} > 0$.
So the points lie on opposite sides of the hyperplane.

◆

1.2 p6-exercise

Let U_2 and W_2 be subspaces of V_N. Show that if N = 3 they will in general intersect in a curve; if N = 4 they will in general intersect in a finite number of points; and if $N > 4$ they will not in general intersect at all.

We have (see 1.102 page 5): $x^r = f^r(u^1, u^2, ..., u^M) \quad (r = 1, 2, ..., N)$
Case N=3:

For U_2 we have:

$$x^r = \phi^r(u^1, u^2) \quad (r = 1, 2, 3)$$

For W_2 we have:

$$x^r = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

The intersect of the two hyperplanes is given by the N equations:

$$\phi^r(u^1, u^2) = \psi^r(v^1, v^2) \quad (r = 1, 2, 3)$$

So we have 3 equations in 4 unknown u^1, u^2, v^1, v^2 and can choose (fix) one e.g. u^1 and solve the set of equations for u^2, v^1, v^2 giving

$$x^r = \theta^r(u^1) \quad (r = 1, 2, 3)$$

This is an equation of a curve in space (1 parameter equation)
Case N=4:
Using the same reasoning as with N=3, we get 4 equations for 4 unknown u^1, u^2, v^1, v^2.
Provided that the set of equation does not degenerate, these 4 equations will determine u^1, u^2, v^1, v^2 without any degree of freedom. So we get points as solutions. This solution does not to be unique, e.g. if the $\phi^r(u^1, u^2)$ are quadratic form, then the solutions

$$(u^1, u^2, v^1, v^2)$$

$$(-u^1, u^2, v^1, v^2)$$
$$(u^1, -u^2, v^1, v^2)$$
$$(-u^1, -u^2, v^1, v^2)$$

are possible.
Case N=5: There are more equations than variables. If the equations are not linear dependent, no solutions will be found.

◆

1.3 p8-exercise

Show that $(a_{rst} + a_{str} + a_{srt})x^r x^s x^t = 3a_{rst}x^r x^s x^t$

$(a_{rst}+a_{str}+a_{srt})x^r x^s x^t = a_{rst}x^r x^s x^t + a_{rts}x^r x^s x^t + a_{srt}x^r x^s x^t$ so by just renaming the dummy indices e.g. for the second term $r \mapsto s$, $s \mapsto t$ and $t \mapsto r$ we get the desired result.

◆

1.4 p8-exercise

If $\phi = a_{rs}x^r x^s$, show that

$$\frac{\partial \phi}{\partial x^r} = (a_{rs} + a_{sr})x^s \quad , \quad \frac{\partial^2 \phi}{\partial x^r \partial x^s} = a_{rs} + a_{sr}$$

Simplify these expressions in the case where $a^{rs} = a^{sr}$

We have

$$\frac{\partial \phi}{\partial x^t} = \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs}\frac{\partial x^r}{\partial x^t} x^s + a_{rs}x^r \frac{\partial x^s}{\partial x^t} \tag{1}$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{rs}\delta^r_t x^s + a_{rs}x^r \delta^s_t \tag{2}$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts}x^s + a_{rt}x^r \tag{3}$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + a_{ts}x^s + a_{st}x^s \quad \text{(rename dummy variable in third term)} \tag{4}$$

$$= \frac{\partial a_{rs}}{\partial x^t} x^r x^s + (a_{ts} + a_{st})x^s \tag{5}$$

Replace x^t by x^r, we get

$$\frac{\partial \phi}{\partial x^r} = \frac{\partial a_{rs}}{\partial x^r} x^r x^s + (a_{rs} + a_{sr})x^s \tag{6}$$

So the asked expression is only true if a_{rs} is not a function of the x^s. Assuming that a_{rs} is not a function of the x^s, take the partial derivative of (6) with respect to x^t, we get

$$\frac{\partial^2 \phi}{\partial x^r \partial x^t} = (a_{rs} + a_{sr})\frac{\partial x^s}{\partial x^t} \tag{7}$$

$$= (a_{rs} + a_{sr})\delta^s_t \tag{8}$$

$$= (a_{rt} + a_{tr}) \tag{9}$$

Replace x^t by x^s, and we get the proposed expression.

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1.5 p8-clarification on expression 1.210

$$\frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0$$

From 1.209:

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} + \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} = 0 \tag{1}$$

multiply (1) with $\frac{\partial x'^q}{\partial x^r}$

$$\frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} + \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{2}$$

$$\Longleftrightarrow \frac{\partial x^r}{\partial x'^n} \frac{\partial^2 x'^n}{\partial x^p \partial x^s} \frac{\partial x'^q}{\partial x^r} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{3}$$

in the first term we get $$\frac{\partial x'^q}{\partial x^r} \frac{\partial x^r}{\partial x'^n} = \frac{\partial x'^q}{\partial x'^n} = \delta^q_n \tag{4}$$

(3) becomes

$$\frac{\partial^2 x'^n}{\partial x^p \partial x^s} \delta^q_n + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{5}$$

$$\Longleftrightarrow \frac{\partial^2 x'^q}{\partial x^p \partial x^s} + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x'^m}{\partial x^p} \frac{\partial x'^n}{\partial x^s} \frac{\partial x'^q}{\partial x^r} = 0 \tag{6}$$

◆

1.6 p9-exercise

If A^r_s are the elements of a determinant A, and B^r_s the elements of a determinant B, show that the element of the product determinant is $A^r_n B^n_s$. Hence show that the product of the two jacobians

$$J = \left|\frac{\partial x^r}{\partial x^{,s}}\right|, \quad J^{'} = \left|\frac{\partial x^{,r}}{\partial x^s}\right|$$

is unity.

Remark: Some nitpick about the formulation: A^r_s are not the elements of a determinant A, but elements of the matrix A which gives $\det\{A\}$ provided that A is square (which is not explicitly mentioned.). The same remark for B and $A^r_n B^n_s$.
Be A^i_k the elements of matrix A and B^k_j the elements of matrix B and C = A.B the resulting matrix of the multiplication of A and B, then

$$C^i_j = A^i_k B^k_j$$

are the elements of matrix C. Now, put $A^i_k = \frac{\partial x^i}{\partial x^{,k}}$ and $B^k_j = \frac{\partial x^{,k}}{\partial x^j}$ then,

$$C^i_j = A^i_k B^k_j \tag{1}$$
$$= \frac{\partial x^i}{\partial x^{,k}} \frac{\partial x^{,k}}{\partial x^j} \tag{2}$$
$$= \delta^i_k \tag{3}$$

So $C = JJ^{,}$ becomes the unity matrix.

◆

1.7 p11-exercise

Show that a finite contravariant vector determines the ratios of the components of an infinitesimal displacement. (Consider the transformation of the equation $dx^r = \theta T^r$, where θ is an arbitrary factor which does not change under the transformation. Alternatively, show that the equations $T^r dx^s - T^s x^r = 0$ remain true when we transform the coordinates.)

Be T^q a contravariant vector.

$$T'^q = T^r \frac{\partial x'^q}{\partial x^r} \quad \text{(by definition)} \tag{1}$$

Be θ a small infinitesimal factor invariant for a coordinate transformation, define

$$dx^r = \theta T^r \tag{2}$$

$$\tag{3}$$

then

$$\frac{dx^r}{dx^s} = \frac{\theta T^r}{\theta T^s} \tag{4}$$

$$\Leftrightarrow T^s dx^r - T^r dx^s = 0 \tag{5}$$

Alternatively, multiply (5) with $\partial_{x^r} x'^q$, then

$$\frac{\partial x'^q}{\partial x^r} dx^r T^s - \frac{\partial x'^q}{\partial x^r} dx^s T^r = 0 \tag{6}$$

$$\Leftrightarrow \frac{\partial x'^q}{\partial x^r} dx^r T^s - dx^s T'^q = 0 \quad \text{(use (1) in the second term)} \tag{7}$$

$$\Leftrightarrow dx'^q T^s - dx^s T'^q = 0 \tag{8}$$

$$\tag{9}$$

Multiply (8) with $\partial_{x^s} x'^p$, then

$$dx'^q T^s \partial_{x^s} x'^p - dx^s T'^q \partial_{x^s} x'^p = 0 \tag{10}$$

$$\Leftrightarrow \quad T'^p dx^q - T'^q dx'^p = 0 \quad \text{(use (1) in the first term)} \tag{11}$$

and thus

$$\frac{dx'^q}{dx'^p} = \frac{T'^q}{T'^p}$$

◆

1.8 p12-exercise

Write down the equation of transformation, analogous to 1.305, of a contravariant tensor of the third order. Solve the equation so as to express the unprimed components in terms of the primed components.

Be

$$T'^{uvw} = T^{rst}\frac{\partial x'^u}{\partial x^r}\frac{\partial x'^v}{\partial x^s}\frac{\partial x'^w}{\partial x^t} \quad \text{(by definition)} \tag{1}$$

a contravariant vector.
Multiply (1) by $\frac{\partial x^n}{\partial x'^u}$

$$T'^{uvw}\frac{\partial x^n}{\partial x'^u} = T^{rst}\frac{\partial x'^u}{\partial x^r}\frac{\partial x^n}{\partial x'^u}\frac{\partial x'^v}{\partial x^s}\frac{\partial x'^w}{\partial x^t} \tag{2}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u} = T^{rst}\delta^n_r\frac{\partial x'^v}{\partial x^s}\frac{\partial x'^w}{\partial x^t} \tag{3}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u} = T^{nst}\frac{\partial x'^v}{\partial x^s}\frac{\partial x'^w}{\partial x^t} \tag{4}$$

Multiply (4) by $\frac{\partial x^m}{\partial x'^v}$

$$T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v} = T^{nst}\frac{\partial x'^v}{\partial x^s}\frac{\partial x^m}{\partial x'^v}\frac{\partial x'^w}{\partial x^t} \tag{5}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v} = T^{nst}\delta^m_s\frac{\partial x'^w}{\partial x^t} \tag{6}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v} = T^{nmt}\frac{\partial x'^w}{\partial x^t} \tag{7}$$

Multiply (7) by $\frac{\partial x^p}{\partial x'^w}$

$$T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v}\frac{\partial x^p}{\partial x'^w} = T^{nmt}\frac{\partial x'^w}{\partial x^t}\frac{\partial x^p}{\partial x'^w} \tag{8}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v}\frac{\partial x^p}{\partial x'^w} = T^{nmt}\delta^p_t \tag{9}$$

$$\Leftrightarrow T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v}\frac{\partial x^p}{\partial x'^w} = T^{nmp} \tag{10}$$

Giving

$$T^{nmp} = T'^{uvw}\frac{\partial x^n}{\partial x'^u}\frac{\partial x^m}{\partial x'^v}\frac{\partial x^p}{\partial x'^w}$$

◆

1.9 p14-exercise

For a transformation from on set of rectangular Cartesian coordinates to another in Euclidean 3-space, show that the law of transformation of a contravariant vector is precisely the same as that of a covariant vector. Can this statements be extended to cover tensor of higher orders?

We have to prove that, given that,

$$T^{,i} = T^j \frac{\partial x^{,i}}{\partial x^j} \quad T^{,}_i = T_j \frac{\partial x^j}{\partial x^{,i}}$$

that also

$$T^{,i} = T^j \frac{\partial x^j}{\partial x^{,i}} \quad T^{,}_i = T_j \frac{\partial x^{,i}}{\partial x^j} \tag{1}$$

$$\Leftrightarrow \frac{\partial x^j}{\partial x^{,i}} = \frac{\partial x^{,i}}{\partial x^j} \tag{2}$$

Be

$$\hat{e^{,i}} = g^i_k \hat{e^k} \quad \text{and } \hat{e^i} = h^i_k \hat{e^{,k}} \tag{3}$$

the transformation rules from one set of (rectangular Cartesian) basis vectors to another set of (rectangular Cartesian) basis vectors. Then,

$$\langle \hat{e^{,i}}, \hat{e^{,j}} \rangle = \langle g^i_k \hat{e^k}, g^j_k \hat{e^k} \rangle \text{ and } \langle \hat{e^i}, \hat{e^j} \rangle = \langle h^i_k \hat{e^{,k}}, h^j_k \hat{e^{,k}} \rangle \tag{4}$$

$$\Leftrightarrow \delta^p_j = g^p_k g^j_k \text{ and } \delta^p_j = h^p_k h^j_k \tag{5}$$

$$\tag{6}$$

Be $\vec{v}$ a random vector in the Euclidean space,

$$\vec{v} = x^j \hat{e^j} = x^{,j} \hat{e^{,j}} \tag{7}$$

then

$$(3) \;\Rightarrow x^j \hat{e^j} = x^j h^j_k \hat{e^{,k}} \text{ and } x^{,j} \hat{e^{,j}} = x^{,j} g^j_k \hat{e^k} \tag{8}$$

$$\Rightarrow x^{,j} = x^m h^m_j \text{ and } x^m = x^{,j} g^j_m \tag{9}$$

$$\Rightarrow x^{,j} = x^{,i} g^i_m h^m_j \text{ and } x^m = x^k h^k_j g^j_m \tag{10}$$

$$\Rightarrow \delta^p_j = g^p_k h^k_j \text{ and } \delta^p_j = g^k_j h^p_k \tag{11}$$

$$(5) \;\Rightarrow g^p_k g^j_k = g^p_k h^k_j \text{ and } h^p_k h^j_k = g^k_j h^p_k \tag{12}$$

$$\Rightarrow g^j_k = h^k_j \text{ and } h^j_k = g^k_j \tag{13}$$

From (9)

$$x^j = x^{,m} g_j^m \text{ and } x^{,k} = x^n h_k^n \tag{14}$$

$$\Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^n}{\partial x^j} h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \frac{\partial x^{,m}}{\partial x^{,k}} g_j^m \tag{15}$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = \delta_j^n h_k^n \text{ and } \frac{\partial x^j}{\partial x^{,k}} = \delta_k^m g_j^m \tag{16}$$

$$\Leftrightarrow \frac{\partial x^{,k}}{\partial x^j} = h_k^j \text{ and } \frac{\partial x^j}{\partial x^{,k}} = g_j^k \tag{17}$$

$$(13) \;\Rightarrow \frac{\partial x^{,k}}{\partial x^j} = \frac{\partial x^j}{\partial x^{,k}} \tag{18}$$

So (13) matches (2), proving the assertion.

Can this statements be extended to cover tensor of higher orders? Consider

$$T^{,i,j,\dots,n} = T^{r,s,\dots w} \frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \cdots \frac{\partial x^{,n}}{\partial x^w} \text{ and } T^{r,s,\dots w} = T^{,i,j,\dots,n} \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \cdots \frac{\partial x^w}{\partial x^{,n}}$$

Using the same reasoning as in (1) to (2) we need

$$\frac{\partial x^{,i}}{\partial x^r} \frac{\partial x^{,j}}{\partial x^s} \cdots \frac{\partial x^{,n}}{\partial x^w} = \frac{\partial x^r}{\partial x^{,i}} \frac{\partial x^s}{\partial x^{,j}} \cdots \frac{\partial x^w}{\partial x^{,n}}$$

As the conclusion (18) is independent of the order of the tensor, it is obvious that the above equality yields. Hence, the answer is YES.

◆

1.10 p16-exercise

In a space of 4 dimensions, the tensor A_{rst} is skew-symmetric in the last pair of suffixes. Show that only 24 of the 64 components may be chosen arbitrarily. If the further condition $A_{rst} + A_{str} + A_{trs} = 0$ is imposed, show that that only 20 components may be chosen arbitrarily.

We have, as A is skew-symmetric in the last pair of suffixes

$$A_{rst} = -A_{rts} \Rightarrow s = t\colon\ A_{rst} = 0$$

So, for each r (4 possible choices as N = 4) we have 4x4/2 - 4 = 6 degrees of freedom. [we have the term 4x4/2 as the tensor is (skew-)symmetric, e.g. once we choose element a_{12}, then a_{21} is also known. The term -4 takes into account the diagonal element which are 0 and thus cannot be chosen.]
So, we have 4x6 = 24 degrees of freedom.
What about the supplementary constraint $A_{rst} + A_{str} + A_{trs} = 0$:
Consider the two possible excluding cases:

i) $r = s \neq t$ ($\Leftrightarrow r = t \neq s$)
This case gives - without the additional constraint (1) - 4x(4x3/2-4) = 8 degrees of freedom. Does the constraint (1) reduces this degree of freedom?
We have,

$$A_{rst} + A_{str} + A_{trs} = 0 \tag{1}$$

$$\Rightarrow \underbrace{A_{rrt} + A_{rtr}}_{= 0 \text{ (non-diagonal terms)}} + \underbrace{A_{trr}}_{= 0 \text{ (diagonal terms)}} = 0 \tag{2}$$

So, no additional constraints are added by (1) to the restriction i) and the DOF remains 8.

ii) $t \neq r \neq s \neq t$
This case means that we have to choose a set of 3 elements out of 4 elements without repetition. This a *variation* of 3 elements out of 4.

$$V_k^n = \frac{n!}{(n-k)!} \text{ giving } V_3^4 = \frac{4!}{(4-3)!} = 24$$

The constraint (1) gives us 24 equations but as $A_{rst} = -A_{rts}$ only 12 equations have to be considered. So, with the additional constraints the DOF becomes 24-12 = 12.
As i) and ii) are independent and excluding events we can add the DOF of both events and we get 8+12 = 20 DOF.

◆

1.11 p16-exercise

If A^{rs} is skew-symmetric and B_{rs} is symmetric, prove that $A^{rs}B_{rs} = 0$. Hence show that the quadratic form $a_{ij}x^i x^j$ is unchanged if a_{ij} is replaced by its symmetric part.

We can split the summation $A^{rs}B_{rs}$ in three subsummations:

$$\begin{aligned} A^{rs}B_{rs} &= A^{rs}B_{rs}|_{r=s} && (1)\\ &+ A^{rs}B_{rs}|_{r>s} && (2)\\ &+ A^{rs}B_{rs}|_{r<s} && (3)\end{aligned}$$

We have:
(1) = 0 as $A^{kk} = 0$ (skew-symmetric)
(2)+(3) = $A^{rs}B_{rs}|_{r>s} + A^{rs}B_{rs}|_{r<s}$
As $A^{rs} = -A^{sr}$ and $B^{rs} = B^{sr}$ we can write (2)+(3) as :

$$A^{rs}B_{rs}|_{r>s} + (-A^{sr})B_{sr}|_{r>s} = 0$$

So, $A^{rs}B_{rs} = 0$

Consider the quadratic form $\phi = a_{ij}x^i x^j$
Be $A_{ij} = (a_{ij})$ and $B_{ij} = (x^i x^j)$, then it is obvious that B_{ij} is symmetric and that $C_{ij} = -A_{ij}$ is the form where $-a_{ij}$ is replaced by its symmetric part (skew-symmetric). Hence $\phi = a_{ij}x^i x^j = a_{ij}b^{ij} = 0$ and so is $\phi = c_{ij}b^{ij} = 0$

◆

1.12 p18-exercise

What are the values (in a space of N dimensions) of the folllowing contractions formed from the Kronecker delta?

$$\delta^m_m, \delta^m_n \delta^n_m, \delta^m_n \delta^n_r \delta^r_m$$

We can split the summation $A^{rs}B_{rs}$ in three subsummations:

$$\delta^m_m = N \tag{1}$$

$$\delta^m_n \delta^n_m = \delta^m_m = N \tag{2}$$

$$\delta^m_n \delta^n_r \delta^r_m = \delta^m_n \delta^n_m = \delta^m_m = N \tag{3}$$

◆

1.13 p19-exercise

If X^r, Y^r are arbitrary contravariant vectors and $a_{rs}X^rY^s$ is an invariant, then a_{rs} are the components of a covariant tensor of the second order.

We have to prove that

$$a'_{rs} = a_{ij}\frac{\partial x^i}{\partial x'^r}\frac{\partial x^j}{\partial x'^s} \text{ or } a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{1}$$

$a_{rs}X^rY^s$ is an invariant, means

$$a'_{rs}X'^rY'^s = a_{rs}X^rY^s \tag{2}$$

As X^r, Y^r are arbitrary contravariant vectors, we have

$$X'^r = X^i\frac{\partial x'^r}{\partial x^i} \text{ and } Y'^s = Y^j\frac{\partial x'^s}{\partial x^j} \tag{3}$$

(3) in (2) gives

$$a'_{rs}X^i\frac{\partial x'^r}{\partial x^i}Y^j\frac{\partial x'^s}{\partial x^j} = a_{rs}X^rY^s \tag{4}$$

$$\Leftrightarrow a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j}X^iY^j = a_{ij}X^iY^j \tag{5}$$

$$\Leftrightarrow (a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij})X^iY^j = 0 \tag{6}$$

As X^r, Y^r are arbitrary contravariant vectors, we conclude that

$$a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} - a_{ij} = 0 \tag{7}$$

$$\Leftrightarrow a_{ij} = a'_{rs}\frac{\partial x'^r}{\partial x^i}\frac{\partial x'^s}{\partial x^j} \tag{8}$$

(8) = (1): OK

◆

1.14 p19-exercise

If X_{rs} is an arbitrary covariant tensor of the second order, and $A_r^{mn} X_{mn}$ is a covariant vector, then A_r^{mn} has the mixed tensor character indicated by the positions of its suffixes

We have to prove that

$$A_r^{,vw} = A_k^{mn} \frac{\partial x^k}{\partial x^{,r}} \frac{\partial x^{,v}}{\partial x^m} \frac{\partial x^{,w}}{\partial x^n} \tag{1}$$

We have

$$P_r = A_r^{mn} X_{mn} \tag{2}$$

is a covariant vector

$$\Rightarrow P_r^{,} = A_k^{mn} X_{mn} \frac{\partial x^k}{\partial x^{,r}} \tag{3}$$

but X_{mn} is a covariant tensor

$$\Rightarrow X_{mn} = X_{ps}^{,} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \tag{4}$$

So (4) in (3) gives

$$P_r^{,} = A_k^{mn} X_{ps}^{,} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}} \tag{5}$$

$$\Leftrightarrow P_r^{,} = \underbrace{A_k^{mn} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}}_{(*)} X_{ps}^{,} \tag{6}$$

Putting (*) as $A_r^{,ps} = A_k^{mn} \frac{\partial x^{,p}}{\partial x^m} \frac{\partial x^{,s}}{\partial x^n} \frac{\partial x^k}{\partial x^{,r}}$ we see that (6) has the form (2) and that $A_r^{,ps}$ obeys the rule of a mixed tensor (1).

◆

1.15 p21-exercise

If A_{rs} is a skew-symmetric covariant tensor, prove that B_{rst} defined as

$$B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$$

is a covariant tensor, and that it is skew-symmetric in all pairs of suffixes.

We have A_{rs} is a covariant tensor

$$A_{ij} = A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^i}\frac{\partial x^\beta}{\partial x^j} \tag{1}$$

$$\Rightarrow B_{rst} = \partial_r(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}) + \partial_s(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r}) + \partial_t(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s}) \tag{2}$$

Note that

$$\partial_k(A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t}) = \partial_k(A_{\alpha\beta})\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \ A_{\alpha\beta}\partial_k(\frac{\partial x^\alpha}{\partial x^s})\frac{\partial x^\beta}{\partial x^t} + \ A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\partial_k(\frac{\partial x^\beta}{\partial x^t}) \tag{3}$$

$$\tag{4}$$

so,

$$\begin{aligned} B_{rst} &= \partial_r A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^s}\tfrac{\partial x^\beta}{\partial x^t} + \underbrace{A_{\alpha\beta}\partial_r\tfrac{\partial x^\alpha}{\partial x^s}\tfrac{\partial x^\beta}{\partial x^t}}_{*} + \underbrace{A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^s}\partial_r\tfrac{\partial x^\beta}{\partial x^t}}_{**} \\ &+\partial_s A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^t}\tfrac{\partial x^\beta}{\partial x^r} + \underbrace{A_{\alpha\beta}\partial_s\tfrac{\partial x^\alpha}{\partial x^t}\tfrac{\partial x^\beta}{\partial x^r}}_{***} + \underbrace{A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^t}\partial_s\tfrac{\partial x^\beta}{\partial x^r}}_{*} \\ &+\partial_t A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^r}\tfrac{\partial x^\beta}{\partial x^s} + A_{\alpha\beta}\underbrace{\partial_t\tfrac{\partial x^\alpha}{\partial x^r}\tfrac{\partial x^\beta}{\partial x^s}}_{**} + \underbrace{A_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial x^r}\partial_t\tfrac{\partial x^\beta}{\partial x^s}}_{***} \end{aligned} \tag{5}$$

In (5) consider the two terms with (*)

$$\begin{aligned} T &= A_{\alpha\beta}\partial_r\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\partial_s\frac{\partial x^\beta}{\partial x^r} && (6)\\ &= A_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x^s\partial x^r}\frac{\partial x^\beta}{\partial x^t} + A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial^2 x^\beta}{\partial x^r\partial x^s} && (7)\\ &= A_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x^s\partial x^r}\frac{\partial x^\beta}{\partial x^t} + A_{\beta\alpha}\frac{\partial x^\beta}{\partial x^t}\frac{\partial^2 x^\alpha}{\partial x^r\partial x^s} \text{ (by renaming dummy variables)} && (8) \end{aligned}$$

As $A_{ij} = -A_{ji}$ (skew-symmetric tensor), we get $T = 0$. The same yields for the (**) and (***) terms. So, B_{rst} reduces to

$$B_{rst} = \partial_r A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \partial_s A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r} + \partial_t A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s} \tag{9}$$

$$\Leftrightarrow\ B_{rst} = \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^r}\frac{\partial x^\alpha}{\partial x^s}\frac{\partial x^\beta}{\partial x^t} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^s}\frac{\partial x^\alpha}{\partial x^t}\frac{\partial x^\beta}{\partial x^r} + \frac{\partial A_{\alpha\beta}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^t}\frac{\partial x^\alpha}{\partial x^r}\frac{\partial x^\beta}{\partial x^s} \tag{10}$$

By adequate renaming of the dummy variable in the 3 terms:

$$\begin{bmatrix} 1^{st} term \\ 2^{nd} term \\ 3^{rd} term \end{bmatrix} \longrightarrow \begin{bmatrix} \gamma \to \alpha & \alpha \to \beta & \beta \to \gamma \\ \beta \to \alpha & \gamma \to \beta & \alpha \to \gamma \\ \alpha \to \alpha & \beta \to \beta & \gamma \to \gamma \end{bmatrix}$$

we get

$$B_{rst} = \left(\frac{\partial A_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial A_{\gamma\alpha}}{\partial x^{\beta}} + \frac{\partial A_{\alpha\beta}}{\partial x^{\gamma}}\right)\frac{\partial x^{\alpha}}{\partial x^{r}}\frac{\partial x^{\beta}}{\partial x^{s}}\frac{\partial x^{\gamma}}{\partial x^{t}} \tag{11}$$

$$\Leftrightarrow B_{rst} = (\underbrace{\partial_{\alpha} A_{\beta\gamma} + \partial_{\beta} A_{\gamma\alpha} + \partial_{\gamma} A_{\alpha\beta}}_{(****)})\frac{\partial x^{\alpha}}{\partial x^{r}}\frac{\partial x^{\beta}}{\partial x^{s}}\frac{\partial x^{\gamma}}{\partial x^{t}} \tag{12}$$

The expression (****) has exactly the required form $B_{rst} = \partial_r A_{st} + \partial_s A_{tr} + \partial_t A_{rs}$ and is transformed (12) according the rules of a covariant tensor.
Let's prove now that it is skew-symmetric in all pairs of suffixes. We have to consider the following permutations

$$\begin{bmatrix} rst \\ rts \\ srt \\ str \\ trs \\ tsr \end{bmatrix}$$

E.g. *srt*

$$B_{rts} = \partial_r A_{ts} + \partial_t A_{sr} + \partial_s A_{rt} \tag{13}$$

$$= -\partial_r A_{st} - \partial_t A_{rs} - \partial_s A_{tr} \tag{14}$$

$$= -B_{rst} \tag{15}$$

The same calculations can be done for the other permutations.

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1.16 p23-exercise 1.

In a V_4 there are two 2-spaces with equations

$$x^r = f^r(u^1, u^2),\ x^r = g^r(u^3, u^4)$$

Prove that if these 2-spaces have a curve of intersection, then the determinal equation

$$\left|\frac{\partial x^r}{\partial u^s}\right| = 0$$

is satisfied along the curve.

Having a curve means that one of the parameters u^i can be freely chosen while the other 3 are determined by the chosen parameter.
We have,

$$\left|\frac{\partial x^r}{\partial u^s}\right| = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} & \frac{\partial x^1}{\partial u^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^4}{\partial u^1} & \frac{\partial x^4}{\partial u^2} & \frac{\partial x^4}{\partial u^3} & \frac{\partial x^4}{\partial u^4} \end{vmatrix} \tag{1}$$

Suppose we choose u^4 as parameter. This means $u^i = \phi^i(u^4)$ for i=1,2,3 and thus we can write

$$\frac{\partial x^i}{\partial u^4} = \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} + \frac{\partial x^i}{\partial u^4} \quad \text{with j=1,2,3} \quad \text{i = 1,2,3,4} \tag{2}$$

$$\Rightarrow \frac{\partial x^i}{\partial u^j}\frac{d\phi^j}{du^4} = 0 \tag{3}$$

This means that in (1) the three first columns a not linearly independent and thus have $\left|\frac{\partial x^r}{\partial u^s}\right| = 0$

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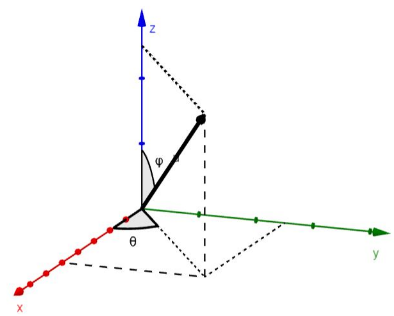
1.17 p23-exercise 2.

In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates x, y, z and spherical polar coordinates r,θ,ϕ.
Find the Jacobian of the transformation. Where is it zero or infinite?



$$\left\{\begin{array}{c} x = r\cos(\phi)\cos(\theta) \\ y = r\cos(\phi)\sin(\theta) \\ z = r\sin(\phi) \end{array}\right\}$$

Partial differentiating of (x,y,z) with respect to (r,ϕ,θ) gives the Jacobian

$$J = \begin{vmatrix} \cos(\phi)\cos(\theta) & -r\sin(\phi)\cos(\theta) & -r\cos(\phi)\sin(\theta) \\ \cos(\phi)\sin(\theta) & -r\sin(\phi)\sin(\theta) & r\cos(\phi)\cos(\theta) \\ \sin(\phi) & r\cos(\phi) & 0 \end{vmatrix} \tag{1}$$

$$\begin{aligned} J &= \cos(\phi)\cos(\theta)(-r^2)\cos^2(\phi)\cos(\theta)) & (2) \\ &+ r\sin(\phi)\cos(\theta)(-r\cos(\phi)\cos(\theta)\sin(\phi)) & (3) \\ &- r\cos(\phi)\sin(\theta)(r\cos^2(\phi)\sin(\theta) + r\sin^2(\phi)\sin(\theta)) & (4) \\ &= -r^2\cos^3(\phi)\cos^2(\theta) - r^2\sin^2(\phi)\cos^2(\theta)\cos(\phi) - r^2\cos(\phi)\sin^2(\theta) & (5) \end{aligned}$$

Noting that the 2^{nd} term in (5) can be written as $-r^2\cos^2(\theta)\cos(\phi) + r^2\cos^2(\theta)\cos^3(\phi)$, we get

$$\begin{aligned} J &= -r^2(\cos^3(\phi)\cos^2(\theta) + \cos^2(\theta)\cos(\phi) - \cos^3(\phi)\cos^2(\theta) + \cos(\phi)\sin^2(\theta)) & (6) \\ &= -r^2\cos(\phi) & (7) \end{aligned}$$

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