Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

Part II

Chapters V to VIII

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github. An overview of the material covered in the book can be found in the separate document "Synge overview.pdf".

Some notation conventions

$$\partial_r \equiv \tfrac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \quad \text{Christoffel symbol of the second kind}$$

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Applications to Classical Mechanics

5.1 p153 - Exercise

If μ^α are the contra-variant components of a unit vector in a surface S, show that $\mu^\alpha f_\alpha$ is the physical component of acceleration in the direction tangent to S defined by μ^α.

As we are in an Euclidean space we can interpret $a_{mn}\mu^\alpha f^\alpha$ as $|\mu|\,|f|\cos\theta$ with θ the angle between the two vectors. As $|\mu| = 1$ we have

$$a_{mn}\mu^\alpha f^\alpha = \mu^\alpha f_\alpha \tag{1}$$

$$= |f|\cos\theta \tag{2}$$

which is the projection of the vector f on the unit vector μ.

♦

5.2 p154 - Clarification to 5.226.

5.226. $$\mathbf{v}\frac{\mathbf{dv}}{\mathbf{ds}} = \mathbf{0}, \quad \overline{\kappa}\mathbf{v^2} = \mathbf{0}$$

Assuming that the particle is not at rest $v \neq 0$, and therefore $\overline{\kappa} = 0$. ***Since this implies that the curve is a geodesic...***

The assertion in bold is a direct consequence

2.513. $$\frac{\delta \frac{dx^r}{ds}}{\delta s} = 0$$

As in **5.233** we have $\frac{\delta\lambda^\alpha}{\delta s} = \frac{\delta \frac{dx^\alpha}{ds}}{\delta s} = 0$, the considered curve follows the geodesic curve.

♦

5.3 p155 - Exercise

Show that in relativity the force 4-vector X^r lies along the first normal of the trajectory in space-time. Express the first curvature in terms of the proper mass m of the particle and the magnitude X of X^r.

Let us recall the first Frenet formula **2.705** without forgetting that the metric form is not positive-definite,

$$\frac{\delta\lambda^r}{\delta s} = \kappa\nu^r, \quad \epsilon_{(1)}\nu_n\nu^n = 1$$

As **5.299**

$$m\frac{\delta\lambda^r}{\delta s} = X^r$$

it is clear that $X^r = m\kappa\nu^r$ and is collinear with the first normal.

$$X^r = m\kappa\nu^r \tag{1}$$

$$\times \quad a_{mr}X^m \quad \Rightarrow \quad \underbrace{a_{mr}X^mX^r}_{=(X^1)^2+(X^2)^2+(X^3)^2-(X^4)^2} = m\kappa\,\underbrace{a_{mr}\nu^m\nu^r}_{=\epsilon_{(1)}} \tag{2}$$

$$\Rightarrow \qquad \kappa = \epsilon_{(1)}\frac{\left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2 - \left(X^4\right)^2}{m}$$

♦

5.4 p156 - Clarification to 5.231

Interpretation of

5.231. $M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r$

What do the M_{rs} represent?

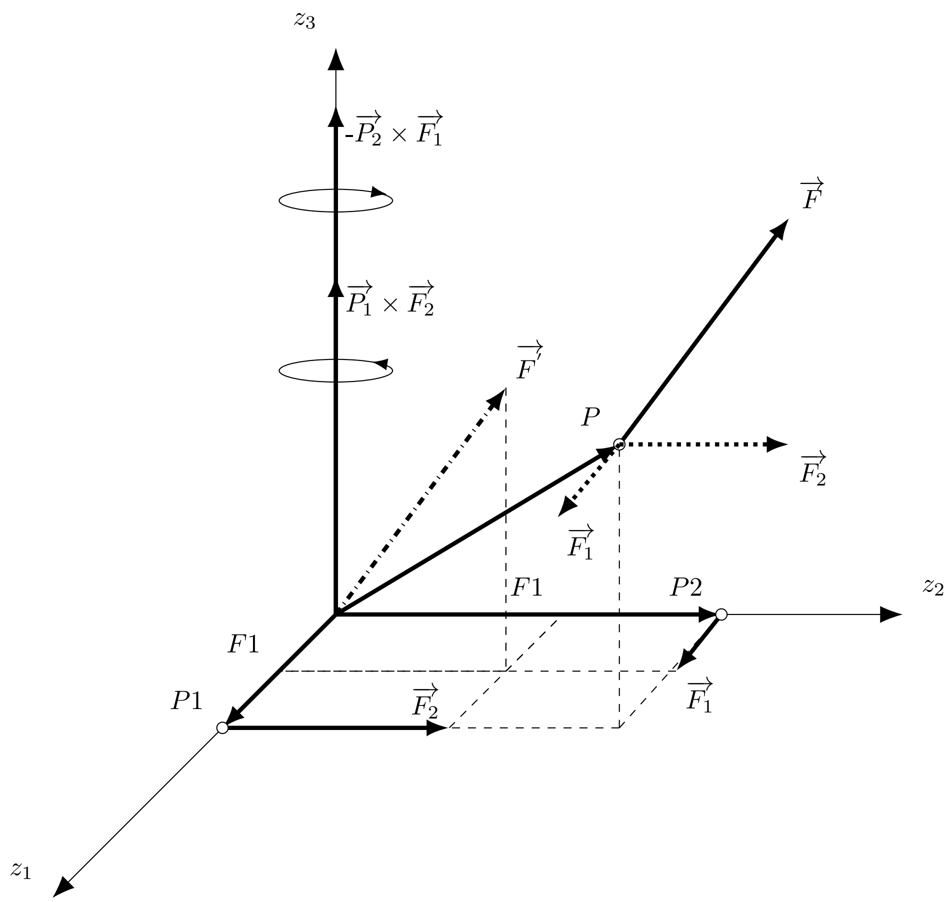


Figure 5.1: Interpretation of the tensor moment M_{12}

Let's consider a mass point P on which a force $\overrightarrow{F}$ is acting. The force has components (F_x, F_y, F_z) in the space V_3' (which is by the way not the space V_3 of the considered mass point).
Let's investigate the element M_{12} of the *tensor moment*.
$P_1 F_2 \overrightarrow{e_3}$ is the vector product $\overrightarrow{P_1} \times \overrightarrow{F_2}$ and is as such the torque of the component F_2 of $\overrightarrow{F}$ acting on the mass point situated at P_1. The origin being fixed, $\overrightarrow{F_2}$ tries to move P_1, clockwise along the z_3 axis. The same is true for the component $\overrightarrow{F_1}$ acting on the mass point situated at P_2, and is represented here by the vector $-\overrightarrow{P_2} \times \overrightarrow{F_1}$ ($\overrightarrow{F_1}$ tries to move P_2, counter clockwise along the z_3 axis). Hence, $P_1F_2 - P_2F_1$ is the net force trying to move the point P along the z_3 axis (i.e. in the plane $\|$ with the $z_3 = 0$ plane).

♦

5.5 p156 - Clarification to 5.234

5.234. $$\frac{dh_r}{dt} = M_r$$

$$h_r = m\epsilon_{rmn} z_m v_n \tag{1}$$

$$\Rightarrow \qquad \frac{dh_r}{dt} = m\epsilon_{rmn}\frac{dz_m}{dt}v_n + m\epsilon_{rmn}z_m\frac{dv_n}{dt} \tag{2}$$

$$= m\underbrace{\epsilon_{rmn}v_m v_n}_{=0} + \underbrace{\epsilon_{rmn}z_m F_n}_{=M_r} \tag{3}$$

$$= M_r \tag{4}$$

♦

5.6 p158-159 - Clarification to 5.313

5.313. $\omega_{rs} = -\omega_{sr}$

From 5.310 and the vector character of v_r and z_r (for transformations which do not change the origin), **it follows that ω_{rs} is a Cartesian tensor of second order**.

Be

$$v_r = -\omega_{rn} z_n \tag{1}$$

Considering orthogonal transformation in a flat space $z^{'}_m = A_{mr} z_r + B_m$ with $B_m = 0$ as we consider only transformations which do not change the origin. Differentiation with the parameter t gives

$$\begin{aligned} v^{'}_m &= A_{mr} v_r && (2) \\ &= -\omega_{rn} A_{mr} z_n && (3) \\ & && (4) \end{aligned}$$

But $z^{'}_q = A_{qr} z_r \quad \Rightarrow \quad A_{qn} z^{'}_q = A_{qn} A_{qr} z_r \quad \Rightarrow \quad A_{qn} z^{'}_q = z_n$ Hence

$$\begin{aligned} v^{'}_m &= -\omega_{rn} A_{mr} z_n && (5) \\ &= -\underbrace{\omega_{rn} A_{mr} A_{qn}}_{\stackrel{\text{def}}{=} \omega^{'}_{mq}} z^{'}_q && (6) \\ v^{'}_m &= -\omega^{'}_{mq} z^{'}_q && (7) \end{aligned}$$

♦

5.7 p159 - Exercise

Show that if a rigid body rotates about the point $z_r = b_r$ as fixed point, the velocity of a general point of the body is given by

$$v_r = -\omega_{rm}\left(z_m - b_m\right)$$

By **5.302**.:

$$\left(z_m^{(1)} - z_m^{(2)}\right)\left(dz_m^{(1)} - dz_m^{(2)}\right) = 0 \tag{1}$$

At the fixed point we have $z_m^{(2)} = b_m$ and $dz_m^{(2)} = 0$, hence

$$\left(z_m^{(1)} - b_m\right)\left(dz_m^{(1)}\right) = 0 \tag{2}$$

$$\Rightarrow \qquad z_m^{(1)} dz_m^{(1)} = b_m dz_m^{(1)} \tag{3}$$

As this is true for any point of the rigid mass, expanding (1) and using (3) we get when dividing by dt

$$\left(z_m^{(2)} - b_m\right) v_m^{(1)} + \left(z_m^{(1)} - b_m\right) v_m^{(2)} = 0 \tag{4}$$

Taking twice the partial derivative $\frac{\partial^2}{\partial z_p^{(1)} \partial z_q^{(1)}}$ we get

$$\left(z_m^{(2)} - b_m\right) \frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \tag{5}$$

As this is true for any arbitrary point in the rigid body we get

$$\frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \tag{6}$$

$$\Rightarrow \qquad v_m = K_{mr} z_r + B_m \tag{7}$$

At the fixed point we have

$$K_{mr} b_r + B_m = 0 \tag{8}$$

Plugging this in (7)

$$v_m = K_{mr}\left(z_r - b_m\right) \tag{9}$$

Putting $K_{mr} = -\omega_{mr}$ gives us indeed the asked expression.

♦

5.8 p161 - Clarification to 5.325 and 5.326

5.325. $$\Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p$$

and hence, since Ω_{np} is arbitrary,

5.326. $$\sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

To be complete the following step should be inserted

$$\Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p \qquad (1)$$

As Ω_{np} is skew-symmetric: $$-\Omega_{np} \sum (m f_p z_n) = -\Omega_{np} \sum F_p z_n \qquad (2)$$

(1)+(2) $$\Omega_{np} \sum m (f_n z_p - f_p z_n) = \Omega_{np} \sum (F_n z_p - F_p z_n) \qquad (3)$$

and hence, since Ω_{np} is arbitrary,

5.326. $$\sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

♦

5.9 p161 - Clarification to 5.329 and 5.330

5.329.
$$h_{np} = \sum m\left(\omega_{nq} z_q z_p - \omega_{pq} z_q z_n\right)$$
$$= J_{npqr}\omega_{rq}$$

where

5.330.
$$J_{npqr} = \sum m\left(\delta_{nr} z_q z_p - \delta_{pr} z_n z_q\right)$$

$$\begin{aligned}
h_{np} &= \sum m\left(\omega_{nq} z_q z_p - \omega_{pq} z_q z_n\right) && (1)\\
&= \sum m\left(\omega_{rq}\delta_{rn} z_q z_p - \omega_{rq}\delta_{rp} z_q z_n\right) && (2)\\
&= \omega_{rq}\sum m\left(\delta_{rn} z_q z_p - \delta_{rp} z_q z_n\right) && (3)\\
&= J_{npqr}\omega_{rq} && (4)
\end{aligned}$$

♦

5.10 p166 - Exercise

Deduce immediately from **5.420**. that the Coriolis force is perpendicular to the velocity.

$$
\begin{aligned}
& & G^{'}_s &= 2m\omega^{'}_{sm}(S^{'},S)v^{'}_m(S^{'}) && (1)\\
\times v^{'}_s(S^{'}) &\quad : & G^{'}_s v^{'}_s(S^{'}) &= m\left(\omega^{'}_{sm}(S^{'},S)v^{'}_m(S^{'})v^{'}_s(S^{'}) + \omega^{'}_{ms}(S^{'},S)v^{'}_m(S^{'})v^{'}_s(S^{'})\right) && (2)\\
& & &= 0 \quad \text{as } \omega^{'}_{ms} \text{ is skew-symmetric} && (3)
\end{aligned}
$$

♦

5.11 p166 - Exercise

Show that if $N = 3$ and $\dot{\omega}'_r(S', S) = 0$, then the centrifugal force may be written

$$\textbf{5.422.} \qquad C'_s = m\omega'_n(S', S)\omega'_n(S', S)z'_s - m\omega'_n(S', S)z'_n\omega'_s(S', S)$$

Deduce that C'_s is coplanar with the vectors $\omega'_s(S', S)$ and z'_n and perpendicular to the former.

By **5.420.** with $\dot{\omega}'_r(S', S) = 0$ and using **5.316.** $(\omega'_{rs} = \epsilon_{rsn}\omega'_n)$

$$\begin{aligned}
C'_s &= m\omega'_{sm}(S', S)\omega'_{nm}(S', S)z'_n && (1)\\
&= m\epsilon_{smk}\omega'_k(S', S)\epsilon_{nmp}\omega'_p(S', S)z'_n && (2)\\
&= m\epsilon_{msk}\epsilon_{mnp}\omega'_k(S', S)\omega'_p(S', S)z'_n && (3)\\
&= m\left(\delta_{sn}\delta_{kp} - \delta_{sp}\delta_{kn}\right)\omega'_k(S', S)\omega'_p(S', S)z'_n && (4)\\
&= m\delta_{sn}\delta_{kp}\omega'_k(S', S)\omega'_p(S', S)z'_n - m\delta_{sp}\delta_{kn}\omega'_k(S', S)\omega'_p(S', S)z'_n && (5)\\
&= m\omega'_p(S', S)\omega'_p(S', S)z'_s - m\omega'_n(S', S)\omega'_s(S', S)z'_n && (6)
\end{aligned}$$

To deduce that C'_s is coplanar with the vectors $\omega'_s(S', S)$ and z'_n we calculate the mixed triple product

$$\begin{aligned}
P &= \epsilon_{spr}C'_s\omega'_p(S', S)z'_r && (7)\\
&= m\underbrace{\epsilon_{spr}\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_p(S', S)z'_r}_{=0} - \underbrace{m\epsilon_{spr}\omega'_n(S', S)\omega'_s(S', S)z'_n\omega'_p(S', S)z'_r}_{=0} && (8)\\
&= 0 && (9)
\end{aligned}$$

Both terms vanish: the first by the presence of the terms $\epsilon_{spr}z'_sz'_r$ which cancel each other and for the second by the terms $\epsilon_{spr}\omega'_s(S', S)\omega'_p(S', S)$. As $P = 0$, the three vectors are coplanar.
We now calculate the inner product $C'_s\omega'_s(S', S)$

$$\begin{aligned}
P &= m\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_s(S', S) - \underbrace{m\omega'_n(S', S)\omega'_s(S', S)z'_n\omega'_s(S', S)}_{\Leftrightarrow\ m\omega'_n(S', S)\omega'_n(S', S)z'_s\omega'_s(S', S)} && (10)\\
&= 0 && (11)
\end{aligned}$$

♦

5.12 p168 - Exercise

Taking $N = 3$, show that **5.424** may be reduced to the usual Euler equations:

$$I_{11}\frac{d\omega_1^{'}(S^{'},S)}{dt} - \left(I_{22} - I_{33}^{'}\right)\omega_2(S^{'},S)\omega_3^{'}(S^{'},S) = M_1^{'}$$

and two similar equations.

We first begin with an approach which leads to nothing. I probably made a reasoning error. I give here the whole calculation as this was interesting and also to, later, find my mistake. After this buggy solution, I will give a second version, which works. **5.424**:

$$M_{ab}^{'} = J_{abrq}^{'}\frac{d\omega_{rq}^{'}(S^{'},S)}{dt} + J_{cdrq}^{'}\left(\delta_{ac}\delta_{du}\delta_{bv} + \delta_{bd}\delta_{cu}\delta_{av}\right)\omega_{rq}^{'}(S^{'},S)\omega_{uv}^{'}(S^{'},S) = \tag{1}$$

$$\times\epsilon_{sab}: \quad 2M_s^{'} = \epsilon_{sab}J_{abrq}^{'}\frac{d\omega_{rq}^{'}(S^{'},S)}{dt} + \epsilon_{sab}J_{cdrq}^{'}\left(\delta_{ac}\delta_{du}\delta_{bv} + \delta_{bd}\delta_{cu}\delta_{av}\right)\omega_{rq}^{'}(S^{'},S)\omega_{uv}^{'}(S^{'},S) \tag{2}$$

Using $\omega_{rq}^{'}(S^{'},S) = \epsilon_{rqt}\omega_t^{'}(S^{'},S)$ and $I_{st} = \frac{1}{2}J_{abrq}^{'}\epsilon_{abs}\epsilon_{rqt}$

$$2M_s^{'} = 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} + \epsilon_{sab}\epsilon_{rqi}\epsilon_{uvj}J_{cdrq}^{'}\left(\delta_{ac}\delta_{du}\delta_{bv} + \delta_{bd}\delta_{cu}\delta_{av}\right)\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \tag{3}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ + \left(\epsilon_{sab}\epsilon_{rqi}\epsilon_{uvj}J_{cdrq}^{'}\delta_{ac}\delta_{du}\delta_{bv} + \epsilon_{sab}\epsilon_{rqi}\epsilon_{uvj}J_{cdrq}^{'}\delta_{bd}\delta_{cu}\delta_{av}\right)\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{4}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ + \left(\epsilon_{scb}\epsilon_{rqi}\epsilon_{dbj}J_{cdrq}^{'} + \epsilon_{sad}\epsilon_{rqi}\epsilon_{caj}J_{cdrq}^{'}\right)\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{5}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ + \left(\epsilon_{bcs}\epsilon_{bdj}\right)\epsilon_{rqi}J_{cdrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\ + \left(\epsilon_{asd}\epsilon_{acj}\right)\epsilon_{rqi}J_{cdrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{6}$$

$$= \begin{cases} 2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\ + \left(\delta_{cd}\delta_{sj} - \delta_{cj}\delta_{sd}\right)\epsilon_{rqi}J_{cdrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\ + \left(\delta_{sc}\delta_{dj} - \delta_{sj}\delta_{dc}\right)\epsilon_{rqi}J_{cdrq}^{'}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \end{cases} \tag{7}$$

$$
= \begin{cases}
2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\
+\epsilon_{rqi}J^{'}_{ccrq}\omega_i^{'}(S^{'},S)\omega_s^{'}(S^{'},S) \\
-\epsilon_{rqi}J^{'}_{jsrq}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\
+\epsilon_{rqi}J^{'}_{sjrq}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\
-\epsilon_{rqi}J^{'}_{ccrq}\omega_i^{'}(S^{'},S)\omega_s^{'}(S^{'},S)
\end{cases} \tag{8}
$$

giving

$$
2M_s^{'} = \begin{cases}
2I_{st}\frac{d\omega_t^{'}(S^{'},S)}{dt} \\
+\epsilon_{rqi}J^{'}_{sjrq}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S) \\
-\epsilon_{rqi}J^{'}_{jsrq}\omega_i^{'}(S^{'},S)\omega_j^{'}(S^{'},S)
\end{cases} \tag{9}
$$

For $s = 1$:

	$+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$	$-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$
ϵ_{123}	$+J_{1112}\omega_3\omega_1 + J_{1212}\omega_3\omega_2 + J_{1312}\omega_3\omega_3$	$-J_{1112}\omega_3\omega_1 - J_{2112}\omega_3\omega_2 - J_{3112}\omega_3\omega_3$
ϵ_{132}	$-J_{1113}\omega_2\omega_1 - J_{1213}\omega_2\omega_2 - J_{1313}\omega_2\omega_3$	$+J_{1113}\omega_2\omega_1 + J_{2113}\omega_2\omega_2 + J_{3113}\omega_2\omega_3$
ϵ_{213}	$-J_{1121}\omega_3\omega_1 - J_{1221}\omega_3\omega_2 - J_{1321}\omega_3\omega_3$	$+J_{1121}\omega_3\omega_1 + J_{2121}\omega_3\omega_2 + J_{3121}\omega_3\omega_3$
ϵ_{231}	$+J_{1123}\omega_1\omega_1 + J_{1223}\omega_1\omega_2 + J_{1323}\omega_1\omega_3$	$-J_{1123}\omega_1\omega_1 - J_{2123}\omega_1\omega_2 - J_{3123}\omega_1\omega_3$
ϵ_{321}	$-J_{1132}\omega_1\omega_1 - J_{1232}\omega_1\omega_2 - J_{1332}\omega_1\omega_3$	$+J_{1132}\omega_1\omega_1 + J_{2132}\omega_1\omega_2 + J_{3132}\omega_1\omega_3$
ϵ_{312}	$+J_{1131}\omega_2\omega_1 + J_{1231}\omega_2\omega_2 + J_{1331}\omega_2\omega_3$	$-J_{1131}\omega_2\omega_1 - J_{2131}\omega_2\omega_2 - J_{3131}\omega_2\omega_3$

Taking into account that $J_{abcd} = 0$ for $a \neq c \wedge b \neq d$

	$+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$	$-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$
ϵ_{123}	$+\cancel{J_{1112}\omega_3\omega_1} + J_{1212}\omega_3\omega_2 + J_{1312}\omega_3\omega_3$	$-\cancel{J_{1112}\omega_3\omega_1}$
ϵ_{132}	$-\cancel{J_{1113}\omega_2\omega_1} - J_{1213}\omega_2\omega_2 - J_{1313}\omega_2\omega_3$	$+\cancel{J_{1113}\omega_2\omega_1}$
ϵ_{213}	$-\cancel{J_{1121}\omega_3\omega_1}$	$+\cancel{J_{1121}\omega_3\omega_1} + J_{2121}\omega_3\omega_2 + J_{3121}\omega_3\omega_3$
ϵ_{231}	$+J_{1323}\omega_1\omega_3$	$-J_{2123}\omega_1\omega_2$
ϵ_{321}	$-J_{1232}\omega_1\omega_2$	$+J_{3132}\omega_1\omega_3$
ϵ_{312}	$+\cancel{J_{1131}\omega_2\omega_1}$	$-\cancel{J_{1131}\omega_2\omega_1} - J_{2131}\omega_2\omega_2 - J_{3131}\omega_2\omega_3$

Opposite sign terms vanish, giving

	$+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$	$-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$
ϵ_{123}	$+J_{1212}\omega_3\omega_2 + J_{1312}\omega_3\omega_3$	
ϵ_{132}	$-J_{1213}\omega_2\omega_2 - J_{1313}\omega_2\omega_3$	
ϵ_{213}		$+J_{2121}\omega_3\omega_2 + J_{3121}\omega_3\omega_3$
ϵ_{231}	$+J_{1323}\omega_1\omega_3$	$-J_{2123}\omega_1\omega_2$
ϵ_{321}	$-J_{1232}\omega_1\omega_2$	$+J_{3132}\omega_1\omega_3$
ϵ_{312}		$-J_{2131}\omega_2\omega_2 - J_{3131}\omega_2\omega_3$

Considering $J_{abcd} = -J_{badc}$

	$+\epsilon_{rqi}J_{1jrq}\omega_i\omega_j$	$-\epsilon_{rqi}J_{j1rq}\omega_i\omega_j$
ϵ_{123}	$+\cancel{J_{1212}}\omega_3\omega_2 + \cancel{J_{1312}}\omega_3\omega_3$	
ϵ_{132}	$-\cancel{J_{1213}}\omega_2\omega_2 - \cancel{J_{1313}}\omega_2\omega_3$	
ϵ_{213}		$+\cancel{J_{2121}}\omega_3\omega_2 + \cancel{J_{3121}}\omega_3\omega_3$
ϵ_{231}	$+\cancel{J_{1323}}\omega_1\omega_3$	$-\cancel{J_{2123}}\omega_1\omega_2$
ϵ_{321}	$-\cancel{J_{1232}}\omega_1\omega_2$	$+\cancel{J_{3132}}\omega_1\omega_3$
ϵ_{312}		$-\cancel{J_{2131}}\omega_2\omega_2 - \cancel{J_{3131}}\omega_2\omega_3$

?? We get

$$m_s^{'} = I_{st}\frac{d\omega_t^{'}(S^{'}, S)}{dt}$$

?????

◊

Let's try another approach. Start with **5.332**.: $\frac{d}{dt}\left(I_{st}\omega_t\right) = M_s$

$$\frac{d}{dt}\left(I_{st}(S^{'},S)\omega_t(S^{'},S)\right) = M_s(S^{'},S) \tag{10}$$

Cf. **5.408**.

$$\begin{aligned}
& & \omega^{'}_u(S^{'},S) &= A_{uq}\omega_q(S^{'},S) && (11)\\
\times A_{ut} &\quad \rightarrow & A_{ut}\omega^{'}_u(S^{'},S) &= A_{ut}A_{uq}\omega_q(S^{'},S) && (12)\\
& & &= \omega_t(S^{'},S) && (13)\\
& & \omega_t(S^{'},S) &= A_{ut}\omega^{'}_u(S^{'},S) && (14)\\
\textbf{(10)} &\quad \Rightarrow & M_s(S^{'},S) &= \frac{d}{dt}\left(I_{st}(S^{'},S)A_{ut}\omega^{'}_u(S^{'},S)\right) && (15)\\
\times A_{ps} &\quad \Rightarrow & M^{'}_p(S^{'},S) &= A_{ps}\frac{d}{dt}\left(I_{st}(S^{'},S)A_{ut}\omega^{'}_u(S^{'},S)\right) && (16)\\
& & I_{st}(S^{'},S) &= A_{as}A_{bt}I^{'}_{ab}(S^{'},S) && (17)\\
\textbf{(16)} &\quad \Rightarrow & M^{'}_p(S^{'},S) &= A_{ps}\frac{d}{dt}\left(A_{as}A_{bt}I^{'}_{ab}(S^{'},S)A_{ut}\omega^{'}_u(S^{'},S)\right) && (18)\\
& & &= A_{ps}\frac{d}{dt}\left(A_{as}I^{'}_{ak}(S^{'},S)\omega^{'}_k(S^{'},S)\right) && (19)
\end{aligned}$$

As we transformed $I_{st}(S^{'},S)$ to a coordinate system fixed to the body we have that the elements of $I^{'}_{ab}(S^{'},S)$ are constants.
Hence,

$$\begin{aligned}
& & M^{'}_p(S^{'},S) &= I^{'}_{ak}S^{'},S)A_{ps}\frac{d}{dt}\left(A_{as}\omega^{'}_k(S^{'},S)\right) && (20)\\
& & &= I^{'}_{ak}(S^{'},S)A_{ps}\left(\dot{A}_{as}\omega^{'}_k(S^{'},S) + A_{as}\dot{\omega}^{'}_k(S^{'},S)\right) && (21)\\
& & &= I^{'}_{ak}(S^{'},S)A_{ps}A_{as}\dot{\omega}^{'}_k(S^{'},S) + I^{'}_{ak}(S^{'},S)A_{ps}\dot{A}_{as}\omega^{'}_k(S^{'},S) && (22)\\
& & &= I^{'}_{pk}(S^{'},S)\dot{\omega}^{'}_k(S^{'},S) + I^{'}_{ak}(S^{'},S)A_{ps}\dot{A}_{as}\omega^{'}_k(S^{'},S) && (23)\\
\textbf{5.408.} &\quad \Rightarrow & A_{ps}\dot{A}_{as} &= \omega^{'}_{ap}(S^{'},S) && (24)\\
\textbf{(23)} &\quad \Rightarrow & M^{'}_p(S^{'},S) &= I^{'}_{pk}(S^{'},S)\dot{\omega}^{'}_k(S^{'},S) + I^{'}_{ak}(S^{'},S)\omega^{'}_{ap}(S^{'},S)\omega^{'}_k(S^{'},S) && (25)
\end{aligned}$$

Let's now calculate the last expression for $p = 1$

$$M^{'}_1(S^{'},S) = I^{'}_{1k}(S^{'},S)\dot{\omega}^{'}_k(S^{'},S) + I^{'}_{ak}(S^{'},S)\omega^{'}_{a1}(S^{'},S)\omega^{'}_k(S^{'},S) \tag{26}$$

As we want an arbitrary, fixed to the body of course, coordinate system, it is possible to chose one so that the $I^{'}_{kj}(S^{'},S) = 0$ for $k \neq j$ i.e. $I^{'}_{kj}(S^{'},S)$ is diagonal. This is possible because $I^{'}_{kj}(S^{'},S)$ is symmetric (the finite-dimensional spectral theorem says that any symmetric matrix whose entries are real can be diagonalized by an orthogonal matrix).
We get, noticing that $\omega^{'}_{ab}(S^{'},S)$ is skew-symmetric and hence $\omega^{'}_{11}(S^{'},S) = 0$:

$$M^{'}_1(S^{'},S) = I^{'}_{11}(S^{'},S)\dot{\omega}^{'}_1(S^{'},S) + I^{'}_{22}(S^{'},S)\omega^{'}_{21}(S^{'},S)\omega^{'}_2(S^{'},S) + I^{'}_{33}(S^{'},S)\omega^{'}_{31}(S^{'},S)\omega^{'}_3(S^{'},S) \tag{27}$$

Using **5.317**: $\omega^{'}_{21}(S^{'},S) = -\omega^{'}_{3}(S^{'},S)$ and $\omega^{'}_{31}(S^{'},S) = \omega^{'}_{2}(S^{'},S)$ we get the asked expression

$$M^{'}_{1}(S^{'},S) = I^{'}_{11}(S^{'},S)\dot{\omega}^{'}_{1}(S^{'},S) - \left(I^{'}_{22}(S^{'},S) - I^{'}_{33}(S^{'},S)\right)\omega^{'}_{2}(S^{'},S)\omega^{'}_{3}(S^{'},S) \tag{28}$$

♦

5.13 p169 - Exercise

Assign convenient generalized coordinates for the three systems (a), (b), and (c) mentioned at the beginning of this section, and calculate the kinematical metric form in each case

(a) **a particle on a surface** $(N = 2)$

No need here for fancy general coordinates: the V_2 coordinate system in the plane is the metric form of choice. Indeed $|v|^2 = a_{mn}v_m v_n$ and for a V_2

$$ds^2 = \left(a_{11}\left(v^1\right)^2 + 2a_{12}v^1v^2 + a_{22}\left(v^2\right)^2\right)dt^2$$

and if the space is Euclidean and the plane smooth, we can choose an orthogonal system where a_{12} will vanish.

(b) **a rigid body which can turn about a fixed point, as in the preceding section** $(N = 3)$

For a rigid body we can choose a coordinate system $S^{'}$ fixed to the body to describe the geometry of the rigid body. The kinetic energy referenced to a 'non-moving' (abuse of language) coordinate system S is

$$T = \frac{1}{2}\sum \rho v_n^{'}(S)v_n^{'}(S) \qquad \text{(summation over all masses in the rigid body)} \tag{1}$$

We know by **5.409**: $v_n^{'}(S) = v_n^{'}(S^{'}) + \omega_{mn}^{'}(S^{'}, S)z_m^{'}$. As the $v_n^{'}(S^{'})$ are fixed, we have $v_n^{'}(S^{'}) = 0$ giving

$$T = \frac{1}{2}\sum \rho z_m^{'}z_k^{'}\omega_{mn}^{'}(S^{'}, S)\omega_{kn}^{'}(S^{'}, S) \tag{2}$$

Note in (2) that we bring $\omega_{mn}^{'}(S^{'}, S)$ out of the summation as this expression is the same for all masses in the body.

$$\begin{aligned}
\omega_{mn}(S^{'}, S) &= \epsilon_{mnt}\omega_t^{'}(S^{'}, S) && (3)\\
\Rightarrow \quad T &= \frac{1}{2}\sum \rho \epsilon_{mnt}\epsilon_{kns}z_m^{'}z_k^{'}\omega_t^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S) && (4)\\
&= \frac{1}{2}\sum \rho\left(\delta_{mk}\delta_{ts} - \delta_{ms}\delta_{kt}\right)z_m^{'}z_k^{'}\omega_t^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S) && (5)\\
&= \frac{1}{2}\sum \rho\left(z_m^{'}z_m^{'}\omega_t^{'}(S^{'}, S)\omega_t^{'}(S^{'}, S) - z_s^{'}z_t^{'}\omega_t^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)\right) && (6)\\
&= \frac{1}{2}\sum \rho\left(\delta_{st}z_m^{'}z_m^{'}\omega_s^{'}(S^{'}, S)\omega_t^{'}(S^{'}, S) - z_s^{'}z_t^{'}\omega_t^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)\right) && (7)\\
&= \frac{1}{2}\sum \rho\left(\delta_{st}z_m^{'}z_m^{'} - z_s^{'}z_t^{'}\right)\omega_s^{'}(S^{'}, S)\omega_t^{'}(S^{'}, S) && (8)
\end{aligned}$$

By **5.335**. we have $I_{st} = \delta_{st}\sum \rho z_m z_m - \sum \rho z_s z_t$ and so (8) can be written as

$$T = \frac{1}{2}I_{st}\omega_s^{'}(S^{'}, S)\omega_t^{'}(S^{'}, S) \tag{9}$$

So we can choose the three angles $\Omega_s^{'}(S^{'},S)$ with $(\omega_s^{'}(S^{'},S) = \frac{d\Omega_s^{'}(S^{'},S)}{dt})$ as generalized coordinates and define

$$ds^2 = I_{st} d\Omega_s^{'}(S^{'},S) d\Omega_t^{'}(S^{'},S)$$

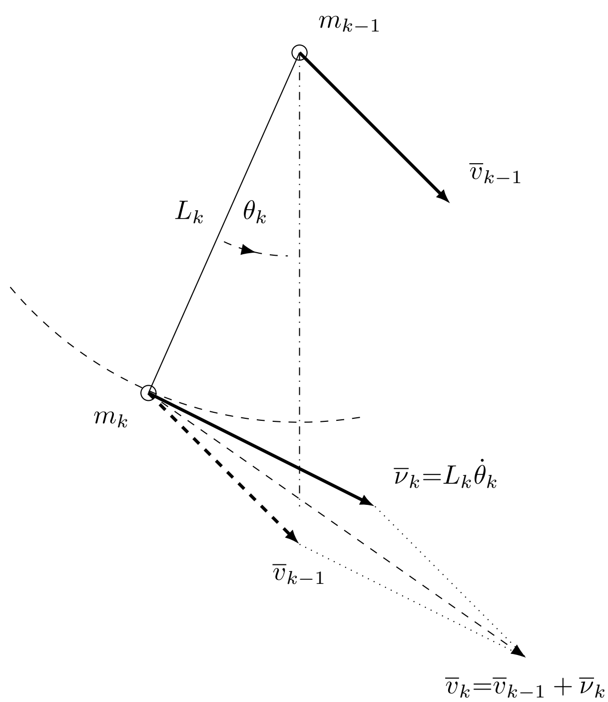
with

$$a_{mn} = I_{mn}$$

having constants as elements. Some check on consistency of the metric tensor defined by (14):
Positive definite ? : Yes, as T is positive by construction.
Symmetric ? : Yes, as $a_{mn} = I_{km}$ and I_{km} is symmetric.

(c) **a chain of six rods smoothly hinged together, with one end fixed and all moving on a smooth plane** $(N = 6)$

To simplify the notation we will assume that the mass m_k of each rod (with length L_k) is concentrated at it's endpoint .
First we note that the velocity of a rod is composed of two vectors, one (labelled as $\overline{\nu}_k$) generated by its own rotation relative to the previous rod and the other (labelled as $\overline{v}_{k-1}$) generated by the velocity of the endpoint of the rod to which it is attached (see.fig. 5.2).



(a)

Figure 5.2: Composition of absolute and relative velocities of a chain of rods

If we take Cartesian coordinates it is easy to see that rod (1) will have components

$$\left(L_1\dot{\theta}_1\cos\theta, L_1\dot{\theta}_1\sin\theta_1\right)$$

rod (2)

$$\left(L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2, L_1\dot{\theta}_1\sin\theta_1 + L_2\dot{\theta}_2\sin\theta_2\right)$$

$$\vdots$$

rod (k)

$$\left(\sum_{i=1}^{k} L_i\dot{\theta}_i\cos\theta_i, \sum_{i=1}^{k} L_i\dot{\theta}_i\sin\theta_i\right)$$

and so

$$\begin{aligned}\left(v^{(k)}\right)^2 &= \left(\sum_{i=1}^{k} L_i\dot{\theta}_i\cos\theta_i\right)^2 + \left(\sum_{i=1}^{k} L_i\dot{\theta}_i\sin\theta_i\right)^2 &\quad (10)\\ &= \sum_{i=1}^{k}\left(L_i\dot{\theta}_i\right)^2 + 2\sum_{i=1}^{k}\sum_{j=1}^{k-i}\left(L_iL_{i+j}\dot{\theta}_i\dot{\theta}_{i+j}\cos\left(\theta_i - \theta_{i+j}\right)\right) &\quad (11)\end{aligned}$$

So the kinetic energy of one rod and the total kinetic energy of the system are

$$\begin{aligned}T^{(k)} &= \frac{1}{2}m_k\left[\sum_{i=1}^{k}\left(L_i\dot{\theta}_i\right)^2 + 2\sum_{i=1}^{k}\sum_{j=1}^{k-i}\left(L_iL_{i+j}\dot{\theta}_i\dot{\theta}_{i+j}\cos\left(\theta_i - \theta_{i+j}\right)\right)\right] &\quad (12)\\ T &= \sum_{k=1}^{N} T^{(k)} &\quad (13)\end{aligned}$$

For $N = 6$ we get

rod	$T^{(k)}$
1	$\frac{1}{2}m_1\left[\left(L_1\dot{\theta}_1\right)^2\right]$
2	$\frac{1}{2}m_2\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right)\right]$
3	$\frac{1}{2}m_3\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + 2L_1L_3\dot{\theta}_1\dot{\theta}_3\cos\left(\theta_1 - \theta_3\right) + \ldots\right]$
4	$\frac{1}{2}m_4\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + 2L_1L_3\dot{\theta}_1\dot{\theta}_3\cos\left(\theta_1 - \theta_3\right) + \ldots\right]$
5	$\frac{1}{2}m_5\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + \left(L_5\dot{\theta}_5\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + \ldots\right]$
6	$\frac{1}{2}m_6\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + \left(L_5\dot{\theta}_5\right)^2 + \left(L_6\dot{\theta}_6\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + \ldots\right]$

Giving for T

$$
2T = \begin{cases}
(m_1+m_2+m_3+m_4+m_5+m_6)\left(L_1\dot{\theta}_1\right)^2 \\
+(m_2+m_3+m_4+m_5+m_6)\left(L_2\dot{\theta}_2\right)^2 \\
+(m_3+m_4+m_5+m_6)\left(L_3\dot{\theta}_3\right)^2 \\
+(m_4+m_5+m_6)\left(L_4\dot{\theta}_4\right)^2 \\
+(m_5+m_6)\left(L_5\dot{\theta}_5\right)^2 \\
+(m_6)\left(L_6\dot{\theta}_6\right)^2 \\
+2(m_2+m_3+m_4+m_5+m_6)L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \\
+2(m_3+m_4+m_5+m_6)L_1L_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) \\
+2(m_3+m_4+m_5+m_6)L_2L_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3) \\
+2(m_4+m_5+m_6)L_1L_4\dot{\theta}_1\dot{\theta}_4\cos(\theta_1-\theta_4) \\
+2(m_4+m_5+m_6)L_2L_4\dot{\theta}_2\dot{\theta}_4\cos(\theta_2-\theta_4) \\
+2(m_4+m_5+m_6)L_3L_4\dot{\theta}_3\dot{\theta}_4\cos(\theta_3-\theta_4) \\
+2(m_5+m_6)L_1L_5\dot{\theta}_1\dot{\theta}_5\cos(\theta_1-\theta_5) \\
+2(m_5+m_6)L_2L_5\dot{\theta}_2\dot{\theta}_5\cos(\theta_2-\theta_5) \\
+2(m_5+m_6)L_3L_5\dot{\theta}_3\dot{\theta}_5\cos(\theta_3-\theta_5) \\
+2(m_5+m_6)L_4L_5\dot{\theta}_4\dot{\theta}_5\cos(\theta_4-\theta_5) \\
+2(m_6)L_1L_6\dot{\theta}_1\dot{\theta}_6\cos(\theta_1-\theta_6) \\
+2(m_6)L_2L_6\dot{\theta}_2\dot{\theta}_6\cos(\theta_2-\theta_6) \\
+2(m_6)L_3L_6\dot{\theta}_3\dot{\theta}_6\cos(\theta_3-\theta_6) \\
+2(m_6)L_4L_6\dot{\theta}_4\dot{\theta}_6\cos(\theta_4-\theta_6) \\
+2(m_6)L_5L_6\dot{\theta}_5\dot{\theta}_6\cos(\theta_5-\theta_6)
\end{cases} \tag{14}
$$

We define as general coordinates the angles θ^i and express ds^2 as

$$ds^2 = 2Tdt^2$$

and see that ds^2 is of the required form

$$ds^2 = a_{mn}d\theta^m d\theta^n$$

The metric tensor a_{mn} contains elements depending on the θ_k chosen as general coordinates of the system and is a good candidate as metric tensor. Some check on consistency of the metric tensor defined by (8):
Positive definite ? : Yes, as T is positive by definition
Symmetric ? : Yes, as the non-diagonal term a_{ij} contains $\cos(\theta_i - \theta_j) = \cos(\theta_j - \theta_i)$
Number of elements : the metric tensor a_{mn} for $N = 6$ should contain 6 diagonal elements and $\frac{6\times6-6}{2} = 15$ independent non-diagonal elements. Checking (8), one can find that the numbers yield.

♦

5.14 p174 - Exercise

Establish the general result

$$v\frac{dv}{ds} = X_r\lambda^r, \quad \kappa v^2 = X_r\nu^r$$

Deduce that, if no forces at on the system, the trajectory is a geodesic in configuration space and the magnitude of the velocity is constant.

In configuration space $f_r = X_r$. Hence by $\mathbf{5,515}$

$$X^r = v\frac{dv}{ds}\lambda^r + \kappa v^2\nu^r \tag{1}$$

$$\Rightarrow \qquad X^r\lambda_r = X_r\lambda^r = v\frac{dv}{ds} \quad \text{as } \lambda^r \perp \nu^r \tag{2}$$

$$\text{and} \qquad X^r\nu_r = X_r\nu^r = \kappa v^2 \quad \text{as } \lambda^r \perp \nu^r \tag{3}$$

$$\tag{4}$$

The trajectory is a geodesic if $\kappa = 0$ which is the case as $X_r = 0$ and

$$v\frac{dv}{ds} = 0 \Rightarrow \frac{dv}{ds} = 0 \Rightarrow v = C^t$$

♦

5.15 p174 - Clarification

> It is easy to see that the lines of force are the orthogonal trajectories of the equipotential surface $V = C^t$

Consider a curve given by $x^r = x^r(u)$.
Along that line we have $V = V(x^r(u))$. Take $u = s$ as parameter and let's impose that $V(s) = C^t$.
We have $\frac{dV}{ds} = \frac{\partial V}{\partial x^r}\frac{dx^r}{ds} = \frac{\partial V}{\partial x^r}\lambda^r = 0$ with $\lambda^r = \frac{dx^r}{ds}$ the tangent vector along that curve.
But $X_r = \frac{\partial V}{\partial x^r}$.
So, $X_r\lambda^r = 0$ and as X_r is collinear with dx^r (the infinitesimal line element of the line of force) we have $dx_n\lambda^n = a_{mn}dx^m\lambda^n = 0$ proving the perpendicularity of both curves.

♦

5.16 p176 - Exercise

For a spherical pendulum, show that the lines of force are geodesics on the sphere on which the particle is constrained to move. What does the theorem stated above tell us in this case?

For the spherical pendulum we have the following situation

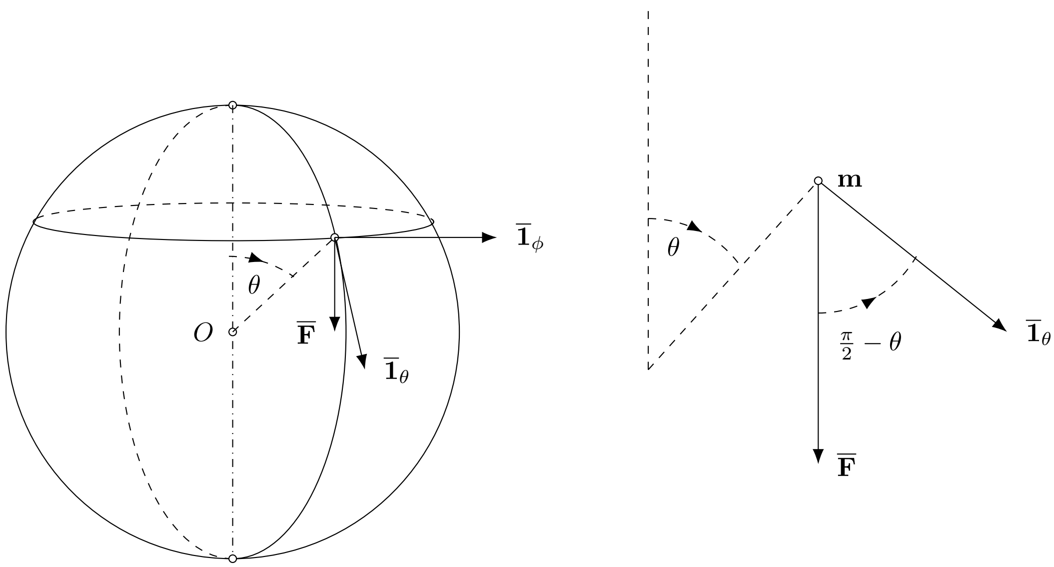


Figure 5.3: Physical components of the gravitational force tensor acting on a mass **m** on a sphere

From the figure it is clear that the only component of the gravitational force acting on the mass is restricted along the $\overline{1}_\theta$ vector which, with varying θ lays along a great circle of the sphere which is a geodesic. Hence the lines of force are great circle on the sphere.
For the theorem stated this means that as a mass is launched along a great circle, it will stay on that great circle.

♦

5.17 p176 - Exercise

A system starts from rest at a configuration O. Prove that the trajectory at O is tangent to the line of force through O, and that the first curvature of the trajectory is one-third of the first curvature of the line of force.

From **5.533** we have

$$v\frac{dv}{ds} = X_r\lambda^r, \quad \kappa v^2 = X_r\nu^r \tag{1}$$

From the second expression we have as $v = 0$ at O that $X_r\nu^r = 0$, meaning that X_r is perpendicular to ν^r. Also by **5.516**

$$f^r = \frac{dv}{dt}\lambda^r + \kappa v^2\nu^r \tag{2}$$

we know that the acceleration lies in the elementary two-space containing the tangent and the first normal to the trajectory implying by the previous result that X_r and λ^r are collinear. Note that from (1) we can not conclude (because $v = 0$) from the first expression that $X_r\lambda^r = 0$. Indeed, $v\frac{dv}{ds}$ is a derived expression form of $\frac{dv}{dt}$. As $\frac{dv}{dt}$ is not necessarily 0 (otherwise the system would for ever stay on the configuration at O meaning that $ds = 0$, making the expression $v\frac{dv}{ds}$ meaningless.)

Let's consider (2) with $f^r = X^r$:

$$\frac{dv}{dt}\lambda^r + \kappa v^2\nu^r = X^r \tag{3}$$

We know that at O, X^r is tangent to the trajectory and so $X^r = X\lambda^r$. At the same point we can also define $X^r = X\lambda^{'r}$, with $\lambda^{'r}$ the tangent vector to the line of force. Multiplying (3) with λ^r we see that $\frac{dv}{dt} = X$. So we get for (3)

$$X\lambda^r + \kappa v^2\nu^r = X\lambda^{'r} \tag{4}$$

$$\frac{\delta(4)}{\delta s} \quad \Rightarrow \quad \frac{dX}{ds}\lambda^r + X\underbrace{\frac{\delta\lambda^r}{\delta s}}_{\kappa\nu^r} + \frac{d\kappa}{ds}\underbrace{v^2}_{=0}\nu_r + 2\kappa\underbrace{v\frac{dv}{ds}}_{=\frac{dv}{dt}=X}\nu^r + \kappa\underbrace{v^2}_{=0}\frac{\delta\nu^r}{\delta s} = \frac{dX}{ds}\lambda^{'r} + X\underbrace{\frac{\delta\lambda^{'r}}{\delta s}}_{=\kappa^{'}\nu^{'r}} \Rightarrow 3\kappa \quad = \tag{5}$$

(we evaluate the expression at point O and define $\kappa^{'}\nu^{'r}$ as the first curvature tensor of the line of force evaluated at 0)

$$\frac{dX}{ds}\lambda^r + X\kappa\nu^r + 2\kappa X\nu^r = \frac{dX}{ds}\lambda^{'r} + X\nu^{'r} \tag{6}$$

$$\times \quad \nu^r \quad \Rightarrow \quad 3\kappa X = \frac{dX}{ds}\underbrace{\lambda^{'r}\nu^r}_{=0} + X\kappa^{'}\nu^{'r}\nu^r \tag{7}$$

$$\Rightarrow 3\kappa = \kappa^{'}\nu^{'r}\nu^r \tag{8}$$

Note that $\lambda^{'r}\nu^r = 0$ as $\lambda^{'r}$ coincides with λ^r. On the other hand we still have to prove that $\nu^{'r}$ coincides with ν^r at O.

$$(6) \times \nu^{'r} \quad \Rightarrow \quad X\kappa\nu^r\nu^{'r} + 2\kappa X\nu^r\nu^{'r} = X\kappa^{'} \tag{9}$$

$$3\kappa\nu^r\nu^{'r} = \kappa^{'} \tag{10}$$

From (8) and (10) we see that $\nu^r\nu^{'r} = 1$ and so

$$\mathbf{3}\kappa = \kappa^{'}$$

♦

5.18 p181-p182 - Clarification Figures 13., 14. and 15.

> There are several ways to perform a map of the configuration space of a rigid body with fixed point.

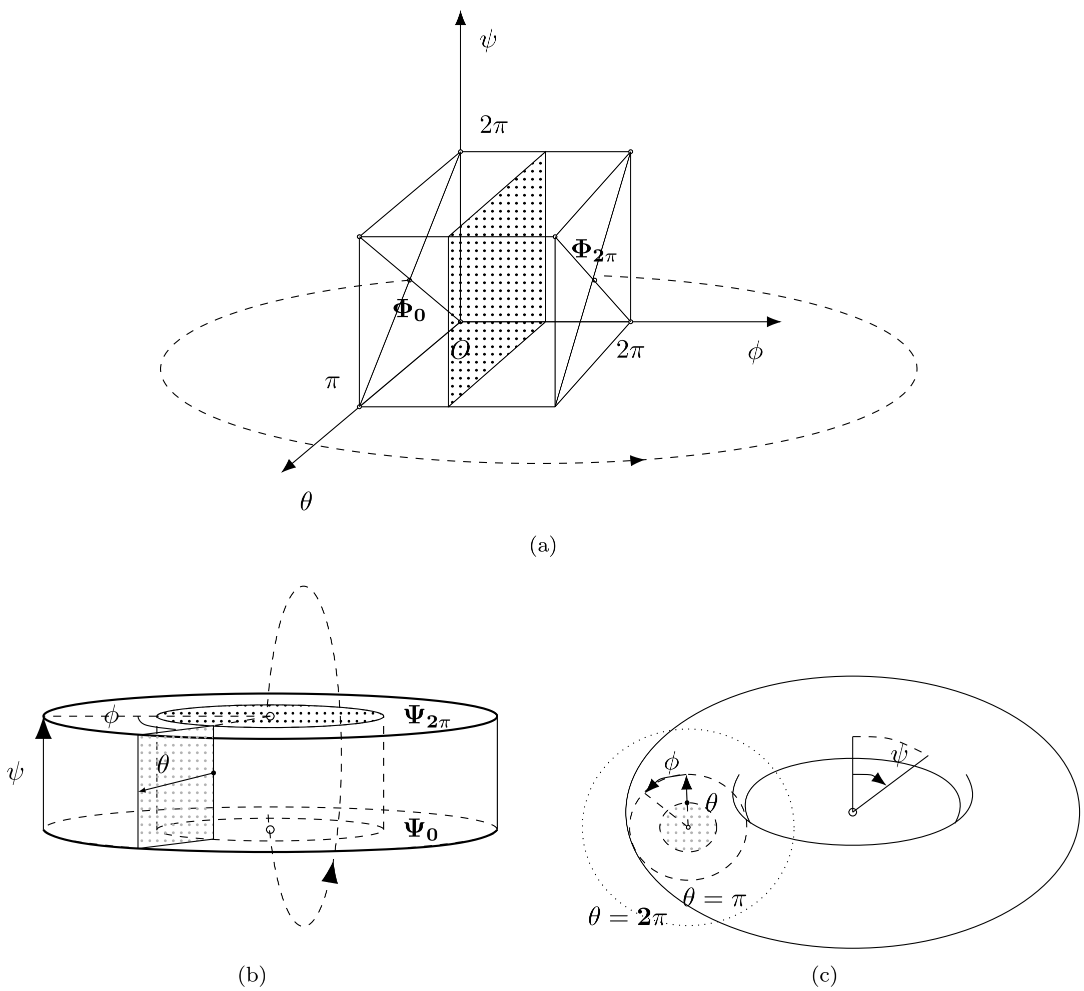


Figure 5.4: Map of the configuration space of a rigid body with fixed point.

Consider figure 5.2(a). We can stretch like an accordion the cuboid along the ϕ axis and bent it so that the planes $\phi = 0$ and $\phi = 2\pi$ join. We get (b), a torus with square sections. The dimension ϕ is dealt with as a point $P(\theta, \phi, \psi)$ in the configuration space returns to the same point when varying ϕ to $\phi + 2k\pi$.

We can apply the same procedure of stretching and bending for the ψ dimension so that the planes $\Psi = 0$ and $\Psi = 2\pi$ join. We get (c), a torus-like object.

The only dimension left is θ which our multi-dimensional crippled mind can't find a way to reshape this pseudo-torus so that when varying θ we can come back to the same point as started.

♦

5.19 p183 - Clarification for 5.561

The kinetic energy is

$$\textbf{5.561}. \qquad T = \frac{1}{2} I \left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta \right)$$

We first determine the general form of the kinetic energy for a rigid body rotating around a fixed point. From **5, 310** we have

$$v_r = -\omega_{rm} z_m = -\epsilon_{rst}\omega_s z_t \tag{1}$$

$$T = \frac{1}{2}\sum m v_r v_r \tag{2}$$

$$\Rightarrow \qquad T = \frac{1}{2}\sum m \epsilon_{rst}\omega_s z_t \epsilon_{ruv}\omega_u z_v \tag{3}$$

$$= \frac{1}{2}\sum m \left(\delta_{su}\delta_{tv}\omega_s\omega_u z_t z_v - \delta_{sv}\delta_{tu}\omega_s\omega_u z_t z_v\right) \tag{4}$$

For the case $N = 3$ we get from (4):

$$T = \frac{1}{2}\sum m \left[\omega_1^2\left(z_2^2 + z_3^2\right) + \omega_2^2\left(z_1^2 + z_3^2\right) + \omega_3^2\left(z_1^2 + z_2^2\right) - 2\omega_1\omega_2 z_1 z_2 - 2\omega_1\omega_3 z_1 z_3 - 2\omega_2\omega_3 z_2 z_3\right] \tag{5}$$

Using the result from **5.336** this can be written as

$$T = \frac{1}{2}\left[I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2 + 2I_{12}\omega_1\omega_2 + 2I_{13}\omega_1\omega_3 + 2I_{23}\omega_2\omega_3\right] \tag{6}$$

Considering that the matrix I_{ij} is symmetric, one can always find an appropriate basis so that the matrix becomes diagonal. Hence (6) can be simplified to

$$T = \frac{1}{2}\left[I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2\right] \tag{7}$$

Of course the ω_i in (7) are not the Euler angles and we have to express the ω_i as functions of the Euler angles.

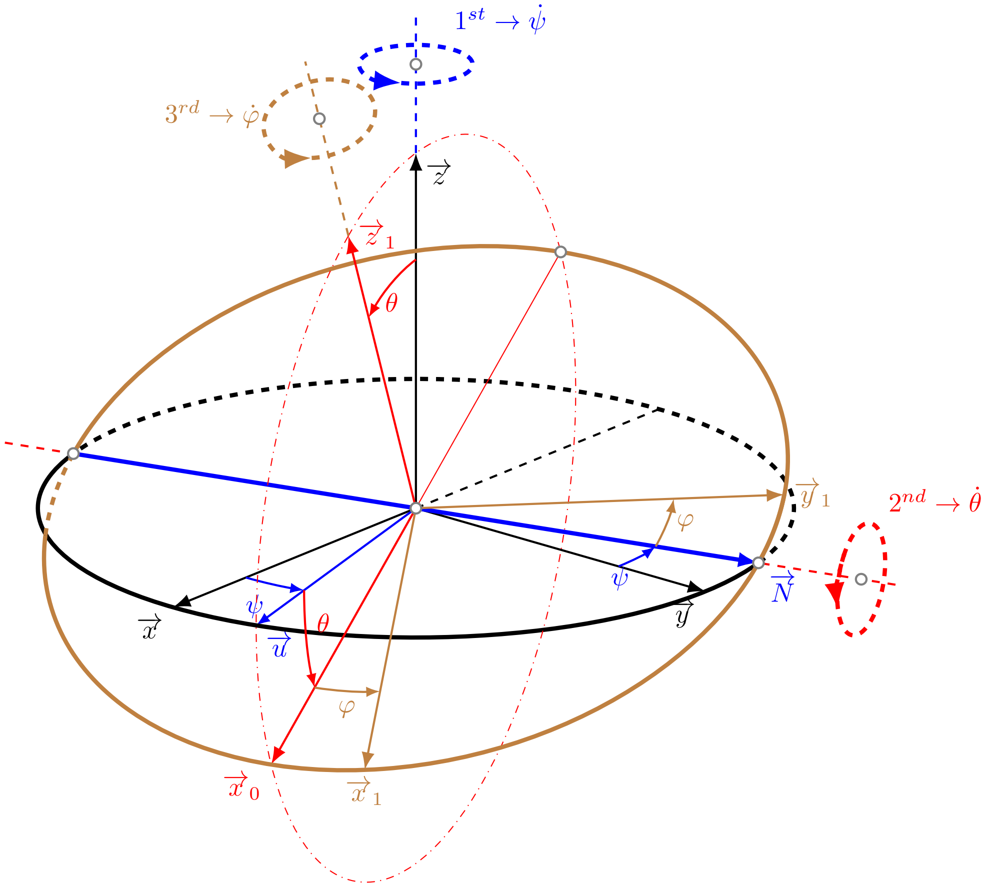


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Figure 5.5: Euler angles

Consider the Euler angles as in figure 5.5. The resulting angular velocity of the rigid body can be expressed as

$$\overline{\omega} = \dot{\psi}\overline{z} + \dot{\theta}\overline{N} + \dot{\phi}\overline{z}_1 \tag{8}$$

The projection of $\overline{\omega}$ on the basis $\overline{x}_1, \overline{y}_1, \overline{z}_1$ (which we choose fixed to the rigid body) will then coincide with the ω_i.
We determine the components of $\overline{z}, \overline{N}, \overline{z}_1$ with $\overline{x}_1, \overline{y}_1, \overline{z}_1$ as basis.
We have

$$\begin{cases} \overline{N} = \cos\phi\ \overline{y}_1 + \sin\phi\ \overline{x}_1 \\ \overline{z} = \cos\theta\ \overline{z}_1 - \sin\theta\ \overline{x}_0 \\ \overline{x}_0 = \cos\phi\ \overline{x}_1 - \sin\phi\ \overline{y}_1 \end{cases} \tag{9}$$

$$\Rightarrow \qquad \begin{cases} \overline{N} = \cos\phi\ \overline{y}_1 + \sin\phi\ \overline{x}_1 \\ \overline{z} = \cos\theta\ \overline{z}_1 - \sin\theta\ \cos\phi\ \overline{x}_1 + \sin\theta\ \sin\phi\ \overline{y}_1 \end{cases} \tag{10}$$

Hence,

$$\overline{\omega} = \dot{\psi}\cos\theta\ \overline{z}_1 - \dot{\psi}\sin\theta\ \cos\phi\ \overline{x}_1 + \dot{\psi}\sin\theta\ \sin\phi\ \overline{y}_1 + \dot{\theta}\cos\phi\ \overline{y}_1 + \dot{\theta}\sin\phi\ \overline{x}_1 + \dot{\phi}\overline{z}_1 \tag{11}$$

giving

$$\begin{cases} \omega_1 = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\ \cos\phi \\ \omega_2 = \dot{\psi}\sin\theta\ \sin\phi + \dot{\theta}\cos\phi \\ \omega_3 = \dot{\psi}\cos\theta + \dot{\phi} \end{cases} \tag{12}$$

In the case considered $I_{11} = I_{22} = I_{33} = I$. Plugging (12) in (7) gives indeed

$$T = \frac{1}{2}I\left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta\right)$$

♦

5.20 p186 - Exercise 1

If a vector at the point with coordinates $(1,1,1)$ in Euclidean 3-space has components $(3,-1,2)$, find the contra-variant, covariant and physical components in spherical polar co-ordinates.

The tensor T_n to consider is $(3,-1,2)-(1,1,1)=(2,-2,1)$.
The Jacobian matrix for the transformation $z^n \to x^k$, evaluated at the point $(1,1,1)$ is

$$J_{(1,1,1)} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{2}$$

$$\Rightarrow \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'^n} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \tag{4}$$

We have the metric tensor evaluated at $(1,1,1)$

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{5}$$

$$\Rightarrow \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \tag{7}$$

And the physical components

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \tag{9}$$

Another way to find the physical components is to project orthogonally the tensor on the unit vectors of a local Cartesian coordinate system, oriented along the unit vectors $\overline{e}_r, \overline{e}_\theta, \overline{e}_\phi$ corresponding to the vector $P\,(1,1,1)$ with modulus $|P| = \sqrt{3}$. We have for the tensor $T_n(2,-2,1)$ with modulus $|T_n| = 3$ as component along $\overline{e}_r$:

$$|T_n|\cos\alpha = |T_n|\,\frac{\langle T_n, P\rangle}{|T_n|\,|P|} \tag{10}$$

$$= |T_n|\,\frac{2-2+1}{|T_n|\,|P|} \tag{11}$$

$$= \frac{1}{\sqrt{3}} \tag{12}$$

For the component along $\overline{e}_\theta$ we first have to determine the vector $\overline{e}_\theta$. As first equation we have the orthogonality condition with $\overline{e}_r$ and putting $\overline{e}_\theta = (a, b, c)$, get $\langle \overline{e}_r, \overline{e}_\theta \rangle = a + b + c = 0$. As $\overline{e}_\theta$ lies in the plane $(1, 1, 0) - (0, 0, 0) - (0, 0, 1)$ we can put $a = b$ and get $\overline{e}_\theta = \frac{1}{\sqrt{6}}(1, 1, -2)$ and get for the tensor $T_n(2, -2, 1)$ as component along $\overline{e}_\theta$:

$$|T_n| \cos \beta = |T_n| \frac{\langle T_n, \overline{e}_\theta \rangle}{|T_n|} \tag{13}$$

$$= |T_n| \frac{2 - 2 - 2}{|T_n| \sqrt{6}} \tag{14}$$

$$= -\frac{\sqrt{2}}{\sqrt{3}} \tag{15}$$

For the component along $\overline{e}_\phi$ we first have to determine the vector $\overline{e}_\phi$. As first equation we have the orthogonality condition with the pair $\overline{e}_r, \overline{e}_\theta$ and get $\overline{e}_\phi = \overline{e}_r \times \overline{e}_\theta = \frac{1}{\sqrt{3}\sqrt{6}}(-3, 3, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$. For the tensor $T_n(2, -2, 1)$ as component along $\overline{e}_\phi$:

$$|T_n| \cos \gamma = |T_n| \frac{\langle T_n, \overline{e}_\phi \rangle}{|T_n|} \tag{16}$$

$$= |T_n| \frac{-2 - 2}{|T_n| \sqrt{2}} \tag{17}$$

$$= -\frac{4}{\sqrt{2}} \tag{18}$$

$$= -2\sqrt{2} \tag{19}$$

giving

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \tag{20}$$

as in (9).

♦

5.21 p186 - Exercise 2

In cylindrical coordinates (r, ϕ, z) in Euclidean 3-space, a vector field is such that the vector at each point points along the parametric line of ϕ, in the sense of ϕ increasing, and its magnitude is kr, where k is a constant. Find the contra-variant, covariant and physical components of this vector field.

We can work backwards, with the physical components as starting point. Indeed, at a point $P\,(r, \phi, z)$ the tensor of this vector field will have $(0, kr, 0)$ as physical components in the cylindrical coordinates (r, ϕ, z) system.
We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Giving

$$\begin{cases} X_1 = & h_1 X_1^{phys.} & = 0 \\ X_2 = & h_2 X_2^{phys.} & = kr^2 \\ X_3 = & h_3 X_3^{phys.} & = 0 \end{cases} \tag{2}$$

and

$$\begin{cases} X^1 = & \frac{X_1^{phys.}}{h_1} & = 0 \\ X^2 = & \frac{X_2^{phys.}}{h_2} & = k \\ X^3 = & \frac{X_3^{phys.}}{h_3} & = 0 \end{cases} \tag{3}$$

♦

5.22 p186 - Exercise 3

Find the physical components of velocity and acceleration along the parametric lines of cylindrical coordinates in terms of the and their derivatives with respect to time.

We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

and the contravariant velocities

$$\begin{cases} v^1 = & \frac{dr}{dt} \\ v^2 = & \frac{d\phi}{dt} \\ v^3 = & \frac{dz}{dt} \end{cases} \tag{2}$$

giving by $v_K^{phys.} = h_K v^K$

$$\begin{cases} v_r = & \frac{dr}{dt} \\ v_\phi = & r\frac{d\phi}{dt} \\ v_z = & \frac{dz}{dt} \end{cases} \tag{3}$$

For the acceleration using $f^r = \frac{\delta v^r}{\delta t}$ and the Christoffel symbols being

$$\begin{cases} \Gamma^m_{nk} = 0 \quad \forall \quad (nk) \neq (r,\theta), (\theta,\theta) \\ \Gamma^\theta_{r\theta} = \frac{1}{r} \quad \text{and} \quad \Gamma^r_{\theta\theta} = -r \end{cases} \tag{4}$$

we have

$$\begin{cases} f^1 = & \frac{dv^1}{dt} - r\,\underbrace{v^2 \dfrac{dx^2}{dt}}_{=(v^2)^2} \\ \\ f^2 = & \frac{dv^2}{dt} + \underbrace{\dfrac{1}{r}v^1 \dfrac{dx^2}{dt} + \dfrac{1}{r}v^2 \dfrac{dx^2 1}{dt}}_{=\frac{2}{r}v^1 v^2} \\ \\ f^3 = & \frac{dv^3}{dt} \end{cases} \tag{5}$$

giving by $f_K^{phys.} = h_K f^K$

$$\begin{cases} f_r = & \frac{dv^1}{dt} - r\left(v^2\right)^2 \\ f_{phi} = & r\frac{dv^2}{dt} + r\frac{2}{r}v^1 v^2 \\ f_z = & \frac{dv^3}{dt} \end{cases} \tag{6}$$

$$\Rightarrow \quad \begin{cases} f_r = & \frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \\ f_{phi} = & r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} \\ f_z = & \frac{d^2 z}{dt^2} \end{cases} \tag{7}$$

♦

5.23 p186 - Exercise 4

A particle moves on a sphere under the action of gravity. Find the contra-variant an co-variant components of the force, using colatitude and azimuth, and write down the equation of motion.

We determine first the physical components of the force.

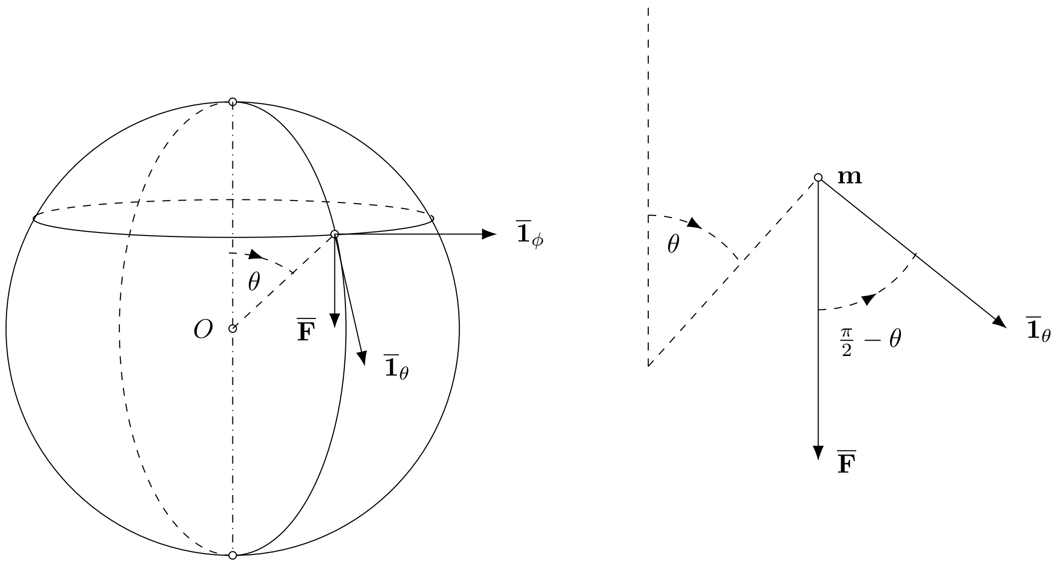


Figure 5.6: Physical components of the gravitational force tensor acting on a mass **m** on a sphere

We note first that the unit vector $\overline{1}_\phi$ is perpendicular to the place formed by the vectors $\overline{1}_\theta, \overline{F}$ and s the force has no components projected on this vector. The vector $\overline{F}$ is parallel with the axis of reference of the sphere with radius R and so the physical components become

$$\begin{cases} F_\phi^{phys} = 0 \\ \\ F_\theta^{phys} = mg\sin\theta \\ \end{cases} \tag{1}$$

$$\Rightarrow \qquad \begin{cases} F_\phi = 0 & F_\phi = 0 \\ \\ F^\theta = \frac{1}{R}mg\sin\theta & F_\theta = Rmg\sin\theta \\ \end{cases} \tag{2}$$

We use equation **5.212**.

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ \\ T = \frac{1}{2}ma_{pq}\dot{x}^p\dot{x}^q,\ \dot{x}^s = \frac{dx^s}{dt} \\ \end{cases} \tag{3}$$

with for our case

$$T = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \sin^2\theta \; \dot{\phi}^2 \right) \tag{4}$$

and get the set of equation of motion (the second column gives the dimensional analysis as a check for consistency)

$$\begin{cases} \frac{\ddot{\phi}}{\dot{\phi}} = -2 \cot\theta \; \dot{\theta} & : \quad \frac{[T]^{-2}}{[T]^{-1}} \cong [T]^{-1} \\ \\ \ddot{\theta} - \left(\dot{\phi}\right)^2 \sin\theta \cos\theta = \frac{g}{R} \sin\theta & : \quad [T]^{-2} + \left([T]^{-1}\right)^2 \cong \frac{[L][T]^{-2}}{[L]} \\ \end{cases} \tag{5}$$

Let's check the special case when $\dot{\phi} = 0$.
The first equation can be rewritten and gives of course $\phi = C$ while the second equation becomes

$$\ddot{\theta} = \frac{g}{R} \sin\theta$$

which is similar to the equation of the simple gravity pendulum.

♦

5.24 p186 - Exercise 5

Consider the motion of a particle on a smooth torus under no forces except normal reaction. The geometrical line element may be written

$$ds^2 = (a - b\cos\theta)^2 \, d\phi^2 + b^2 d\theta^2$$

where ϕ is an azimuthal angle and θ an angular displacement from the equatorial plane. Show that the path of a particle satisfies the following two differential equations in which h is a constant

$$(a) \qquad (a - b\cos\theta)^2 \frac{d\phi}{ds} = h$$

$$(b) \qquad b^2 \left(\frac{d\theta}{d\phi}\right)^2 = \frac{(a - b\cos\theta)^4}{h^2} - (a - b\cos\theta)^2$$

We use equation **5.212**. and **5.212**.

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ \\ T = \frac{1}{2} m a_{pq} \dot{x}^p \dot{x}^q, \ \dot{x}^s = \frac{dx^s}{dt} \end{cases} \tag{1}$$

with for our case

$$T = \frac{1}{2} m \left(b^2 \dot{\theta}^2 + (a - b\cos\theta)^2 \ \dot{\phi}^2\right) \tag{2}$$

$$\begin{cases} \frac{\partial T}{\partial \dot{\phi}} = m (a - b\cos\theta)^2 \dot{\phi} & \frac{\partial T}{\partial \phi} = 0 \\ \\ \frac{\partial T}{\partial \dot{\theta}} = m b^2 \dot{\theta} & \frac{\partial T}{\partial \theta} = m b (a - b\cos\theta) \dot{\phi}^2 \sin\theta \end{cases} \tag{3}$$

giving

$$\begin{cases} (a - b\cos\theta)^2 \ddot{\phi} + 2b (a - b\cos\theta) \dot{\theta}\dot{\phi}\sin\theta = 0 \\ \\ b^2 \ddot{\theta} - b (a - b\cos\theta) \dot{\phi}^2 \sin\theta = 0 \end{cases} \tag{4}$$

$$\Rightarrow \qquad \begin{cases} (a - b\cos\theta) \ddot{\phi} = -2b\dot{\theta}\dot{\phi}\sin\theta \\ \\ b^2 \ddot{\theta} - b (a - b\cos\theta) \dot{\phi}^2 \sin\theta = 0 \end{cases} \tag{5}$$

In the first equation, put $y \equiv \dot{\phi}$ giving for the first equation:

$$\frac{dy}{y} = -2b\frac{\sin\theta d\theta}{(a - b\cos\theta)} \tag{6}$$

$$\Leftrightarrow \qquad \frac{dy}{y} = -2\frac{d\,(a - b\cos\theta)}{(a - b\cos\theta)} \tag{7}$$

$$\Rightarrow \qquad \log y = -2\log\,(a - b\cos\theta) + \log C \tag{8}$$

$$\Rightarrow \qquad \dot{\phi} = C\,(a - b\cos\theta)^{-2} \tag{9}$$

Note that $\dot{\phi}$ is a time derivative. But as we are on a geodesic, **5.226**. stands and so v is constant as $\frac{dv}{ds} = 0$. Using $v = \frac{ds}{dt}$, (9) can be written as

$$(a - b\cos\theta)^2\,\frac{d\phi}{dt} = C \tag{10}$$

$$\Leftrightarrow \qquad (a - b\cos\theta)^2\,\frac{d\phi}{ds}\underbrace{\frac{ds}{dt}}_{=v} = C \tag{11}$$

$$\Leftrightarrow \qquad (a - b\cos\theta)^2\,\frac{d\phi}{ds} = h \quad \text{with } h = \frac{C}{v} \tag{12}$$

Next, we don't use the second equation in (5) but the line element equation instead

$$ds^2 = (a - b\cos\theta)^2\,d\phi^2 + b^2 d\theta^2 \tag{13}$$

$$\Rightarrow \qquad \left(\frac{ds}{d\phi}\right)^2 = (a - b\cos\theta)^2 + b^2\left(\frac{d\theta}{d\phi}\right)^2 \tag{14}$$

$$\Rightarrow \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = \left(\frac{d\phi}{ds}\right)^{-2} - (a - b\cos\theta)^2 \tag{15}$$

$$(12) \quad : \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = \frac{(a - b\cos\theta)^4}{h^2} - (a - b\cos\theta)^2 \tag{16}$$

♦

5.25 p186 - Exercise 6

Consider the motion of a particle under gravity on the smooth torus o the previous problem, the equatorial plane of the torus being horizontal. Taking the mass of the particle to unity, so that $V = bg\sin\theta$, show that the path of the particle satisfies the following two differential equations.

$$(a) \qquad (E-V)\,(a-b\cos\theta)^2\,\frac{d\phi}{ds} = h$$

$$(b) \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = (E-V)\,\frac{(a-b\cos\theta)^4}{h^2} - (a-b\cos\theta)^2$$

where E is the total energy, h is a constant and $d\sigma$ is the action line element.

The line of reasoning is quite the same as problem (5). We use equation **5.212**. and **5.212**.

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ \\ T = \frac{1}{2}ma_{pq}\dot{x}^p\dot{x}^q,\ \dot{x}^s = \frac{dx^s}{dt} \end{cases} \tag{1}$$

with for our case

$$T = \frac{1}{2}m\left(b^2\dot{\theta}^2 + (a-b\cos\theta)^2\ \dot{\phi}^2\right) \tag{2}$$

$$\begin{cases} \frac{\partial T}{\partial \dot{\phi}} = m\,(a-b\cos\theta)^2\,\dot{\phi} & \frac{\partial T}{\partial \phi} = 0 \\ \\ \frac{\partial T}{\partial \dot{\theta}} = mb^2\dot{\theta} & \frac{\partial T}{\partial \theta} = mb\,(a-b\cos\theta)\,\dot{\phi}^2\sin\theta \end{cases} \tag{3}$$

giving (as $F_\phi = -\partial_\phi V = 0$ and $F_\theta = -\partial_\theta V = -bg\cos\theta$)

$$\begin{cases} (a-b\cos\theta)^2\,\ddot{\phi} + 2b\,(a-b\cos\theta)\,\dot{\theta}\dot{\phi}\sin\theta = 0 \\ \\ b^2\ddot{\theta} - b\,(a-b\cos\theta)\,\dot{\phi}^2\sin\theta = -bg\cos\theta \end{cases} \tag{4}$$

$$\Rightarrow \qquad \begin{cases} (a-b\cos\theta)\,\ddot{\phi} = -2b\dot{\theta}\dot{\phi}\sin\theta \\ \\ b^2\ddot{\theta} - b\,(a-b\cos\theta)\,\dot{\phi}^2\sin\theta = -bg\cos\theta \end{cases} \tag{5}$$

In the first equation, put $y \equiv \dot{\phi}$ giving for the first equation:

$$\frac{dy}{y} = -2b\frac{\sin\theta d\theta}{(a - b\cos\theta)} \tag{6}$$

$$\Leftrightarrow \qquad \frac{dy}{y} = -2\frac{d(a - b\cos\theta)}{(a - b\cos\theta)} \tag{7}$$

$$\Rightarrow \qquad \log y = -2\log(a - b\cos\theta) + \log C \tag{8}$$

$$\Rightarrow \qquad \dot{\phi} = C(a - b\cos\theta)^{-2} \tag{9}$$

Note that $\dot{\phi}$ is a time derivative. Using $\frac{ds}{dt} = v = \sqrt{2T} = \sqrt{2}\sqrt{E - V}$, (9) can be written as

$$(a - b\cos\theta)^2 \frac{d\phi}{dt} = C \tag{10}$$

$$\Leftrightarrow \qquad (a - b\cos\theta)^2 \frac{d\phi}{d\sigma} \underbrace{\frac{d\sigma}{ds}}_{=\sqrt{E-V}} \underbrace{\frac{ds}{dt}}_{=\sqrt{2}\sqrt{E-V}} = C \tag{11}$$

$$\Leftrightarrow \qquad (E - V)(a - b\cos\theta)^2 \frac{d\phi}{d\sigma} = h \tag{12}$$

with $h = \frac{C}{\sqrt{2}}$.
Next, we don't use the second equation in (5) but the line element equation instead

$$ds^2 = (a - b\cos\theta)^2 d\phi^2 + b^2 d\theta^2 \tag{13}$$

$$\Rightarrow \qquad \left(\frac{ds}{d\phi}\right)^2 = (a - b\cos\theta)^2 + b^2\left(\frac{d\theta}{d\phi}\right)^2 \tag{14}$$

$$\Rightarrow \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = \left(\frac{d\phi}{ds}\right)^{-2} - (a - b\cos\theta)^2 \tag{15}$$

$$\Leftrightarrow \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = \left(\frac{d\phi}{d\sigma}\right)^{-2}\left(\frac{d\sigma}{ds}\right)^{-2} - (a - b\cos\theta)^2 \tag{16}$$

$$(12) \quad : \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = (E - V)^2 \frac{1}{(E - V)} \frac{(a - b\cos\theta)^4}{h^2} - (a - b\cos\theta)^2 \tag{17}$$

$$\Rightarrow \qquad b^2\left(\frac{d\theta}{d\phi}\right)^2 = (E - V)\frac{(a - b\cos\theta)^4}{h^2} - (a - b\cos\theta)^2 \tag{18}$$

♦

5.26 p187 - Exercise 7

A dynamical system consists of a thin straight smooth tube which can rotate in a horizontal plan about one end O, together with a bead$B$ inside the tube connected to O by a spring. Taking as coordinates $r = OB$ and $\theta =$ angle of rotation of the tube about O, the potential energy V is a function of r only. Show that in configuration space , all the lines of force are geodesics for the kinematical line element.

Well understanding the question is of course paramount:

- The tube mentioned plays only a functional role to hold the spring "stiff" along the line OB as its mass can be neglected.It will play no further role in the dynamics of the system.
- Nothing is said that the system contains any force that keeps the angular velocity at a constant speed ω.

That being clarified, one can expect that the system will behave as a harmonic oscillator along the line OB and that ,given an initial rotational momentum, the angular momentum will be a constant during the trajectory of the bead. This means that the bead will oscillate along OB but as the angular momentum is a constant and given $m\omega r^2 = C(m =$ mass of the bead), the instant radial speed will vary.
The only conservative force acting on the bead will be that of the spring and will be $V = \frac{1}{2}k\left(r - r_0\right)^2$, r_0 being the point along OB where the spring is not stretched. The generalized forces are $F_r = -k\left(r - r_0\right)$ and $F_\theta = 0$ meaning the lines of force are straight lines pointing to the origin O.
About the geodesics. Clearly the instantaneous velocity of the bead is $\overrightarrow{v} = \dot{r}\overrightarrow{1}_r + \dot{\theta}r\overrightarrow{1}_\theta$ giving as kinetic energy $T = \frac{1}{2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right)$ giving as kinematic line element

$$ds^2 = 2Tdt = dr^2 + r^2 d\theta^2$$

Referring to **3.101**, the configuration space is flat and the geodesics are straight lines. As the line forces are straight lines towards the origin O, these line of force are also geodesics in the configuration space equipped with the kinematical line element.

♦

5.27 p187 - Exercise 8

Show that if a line of force is a geodesic for the kinematical line element, it is also a geodesic for the action line element.

From **5.516** and **5.529** we have

$$X^r = v\frac{dv}{ds}\lambda^r + \kappa v^2\nu^r \tag{1}$$

As the line of force is a geodesic, we can start with a velocity tangent to the line of force, ensuring that the trajectory of the dynamical system will lie on the geodesic line of force (see page 175) and thus $\kappa = 0$ for the trajectory. Hence,

$$X^r = v\frac{dv}{ds}\lambda^r \tag{2}$$

expressing now the function of the action line element $d\sigma = \sqrt{E-V}ds$ we have

$$\begin{align} X^r &= v\frac{dv}{ds}\lambda^r \tag{3}\\ &= v\frac{dv}{ds}\frac{dx^r}{d\sigma}\frac{d\sigma}{ds} \tag{4}\\ &= v\frac{dv}{ds}\frac{dx^r}{d\sigma}\sqrt{E-V} \tag{5}\\ &= \sqrt{E-V}v\frac{dv}{ds}\lambda'^r \tag{6}\end{align}$$

As stated page 177, this dynamical system will describe in configuration space a geodesic for the action metric, meaning that λ'^r is tangent to this geodesic and that X^r, being collinear with λ'^r (with the factor $\sqrt{E-V}v\frac{dv}{ds}$), is also tangent to this geodesic. Hence, this line of force is also a geodesic for the action line element.

♦

5.28 p187 - Exercise 9

Using the methods of Chapter II and **5.532**, show that the trajectories of a dynamical system with kinetic energy T and potential energy V satisfy the variational equation

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

Let's start with a function L defined by

$$dL = (T - V)du \tag{1}$$
$$\tag{2}$$

As in figure 2 page 38 we will make L a function of two parameters, u and v , the latter defining a family of curves between the begin point u_1 and the endpoint u_2.

$$L = L(u, v) \tag{3}$$

with

$$(T - V)(u_1, v) = (T - V)_1 \quad (T - V)(u_2, v) = (T - V)_2 \quad \forall v \tag{4}$$

We will try to minimize (with respect to v) the following functional

$$L == \int_{u_1}^{u_2} (T - V)(u, v)du \tag{5}$$

It's derivative with respect to v

$$\frac{dL}{dv} = \int_{u_1}^{u_2} \frac{\partial (T - V)(u, v)}{\partial v} du \tag{6}$$

We express $(T - V)(u, v)$ as a function of the generalized coordinates x^r and their derivatives. Then,

$$\frac{\partial (T - V)(u, v)}{\partial v} = \frac{\partial (T - V)(u, v)}{\partial \dot{x^r}} \frac{\partial \dot{x^r}}{\partial v} + \frac{\partial (T - V)(u, v)}{\partial x^r} \frac{\partial x^r}{\partial v} \tag{7}$$

where $\dot{x^r} = \frac{\partial x^r}{\partial u}$.
We have

$$\frac{\partial \dot{x^r}}{\partial v} = \frac{\partial}{\partial v} \frac{\partial x^r}{\partial u} = \frac{\partial}{\partial u} \frac{\partial x^r}{\partial v} \tag{8}$$

So,

$$\frac{\partial (T - V)(u, v)}{\partial v} = \frac{\partial (T - V)(u, v)}{\partial \dot{x^r}} \frac{\partial}{\partial u} \frac{\partial x^r}{\partial v} + \frac{\partial (T - V)(u, v)}{\partial x^r} \frac{\partial x^r}{\partial v} \tag{9}$$

Consider the expression

$$\int_{u_1}^{u_2} d(AB) = \int_{u_1}^{u_2} Ad(B) + \int_{u_1}^{u_2} Bd(A) \tag{10}$$

$$\Rightarrow \qquad \int_{u_1}^{u_2} Ad(B) = \int_{u_1}^{u_2} d(AB) - \int_{u_1}^{u_2} Bd(A) \tag{11}$$

Put $B = \frac{\partial x^r}{\partial v}$ and $A = \frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}$ and putting this inside (9) and (6):

$$\frac{dL}{dv} = \int_{u_1}^{u_2} d\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\frac{\partial x^r}{\partial v}\right) - \int_{u_1}^{u_2} \frac{\partial x^r}{\partial v} d\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) + \int_{u_1}^{u_2} \frac{\partial (T-V)(u,v)}{\partial x^r}\frac{\partial x^r}{\partial v} du \tag{12}$$

$$= \left.\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\frac{\partial x^r}{\partial v}\right|_{u_1}^{u_2} - \left[\int_{u_1}^{u_2} \left(\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u,v)}{\partial x^r}\right)\frac{\partial x^r}{\partial v} du\right] \tag{13}$$

We express now the results in term of infinitesimals. A change in "length" δL when we pas from a curve v to a curve $v + dv$ is

$$\delta L = \frac{dL}{dv}\delta v \tag{14}$$

$$= \left.\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\frac{\partial x^r}{\partial v}\delta v\right|_{u_1}^{u_2} - \int_{u_1}^{u_2} \left(\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u,v)}{\partial x^r}\right)\frac{\partial x^r}{\partial v}\delta v du \tag{15}$$

$$= \left.\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\delta x^r\right|_{u_1}^{u_2} - \int_{u_1}^{u_2} \left(\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u,v)}{\partial x^r}\right)\delta x^r du \tag{16}$$

The first term vanish as at the endpoints the δx^r are zero and hence we get

$$\delta L = -\int_{u_1}^{u_2} \left(\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u,v)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u,v)}{\partial x^r}\right)\delta x^r du \tag{17}$$

As the δx^r are arbitrary, we must have for $\delta L = 0$

$$\frac{\partial}{\partial u}\left(\frac{\partial (T-V)(u)}{\partial \dot{x^r}}\right) - \frac{\partial (T-V)(u)}{\partial x^r} = 0 \tag{18}$$

This is the same equation as **5.532** which describe the motion of a system with a conservative force.

♦

5.29 p188 - Exercise 10

Using the definition **5.5335** for I_{rs}, prove that if X_r is any non-zero vector, then $I_{rs}X_rX_s \geq 0$, and that the equality occurs only if all particles of the system are distributed on a single line.

By **5.335**

$$I_{rs} = \delta_{rs} \sum m z_q z_q - \sum m z_r z_s \tag{1}$$

Multiplying by $X_r X_s$:

$$I_{rs}X_rX_s = \underbrace{X_rX_s\delta_{rs}}_{=X_rX_r} \sum m z_q z_q - \sum m \underbrace{z_rX_r}_{=|z|_{(m)}|X|\cos\theta_m} \underbrace{z_sX_s}_{=|z|_{(m)}|X|\cos\theta_m} \tag{2}$$

with θ_m the angle between the vector X_r and the position vector z_m of a particle.

$$I_{rs}X_rX_s = |X|^2 \sum m\, |z|^2_{(m)} - |X|^2 \sum m\, |z|^2_{(m)} \cos^2\theta_m \tag{3}$$

$$= |X|^2 \sum m\, |z|^2_{(m)} \left(1 - \cos^2\theta_m\right) \tag{4}$$

As we have $\left(1 - \cos^2\theta_m\right) \in [0, 1]$ it is clear that $I_{rs}X_rX_s \geq 0$ and that it only will be zero when $\theta_m = 0 \quad \forall m$ which means that all position vectors are collinear wit X_r and are on a line.

♦

5.30 p188 - Exercise 11

Let $Oz_1z_2z_3$ and $O^{'}z_1^{'}z_2^{'}z_3^{'}$ be two sets of Cartesian axes parallel to one another. Consider a mass distribution and let $I_{rs}, I_{rs}^{'}$be its moment of inertia tensors calculated for these two axes in accordance with **5.335**. Writing $I_{rs}^{'} = I_{rs} + K_{rs}$, evaluate K_{rs}.

By **5.335**

$$I_{rs} = \delta_{rs} \sum m z_q z_q - \sum m z_r z_s \tag{1}$$

As the axes of both coordinate systems are parallel, we can write

$$z_q^{'} = z_q + b_q \tag{2}$$

which gives for (1):

$$I_{rs}^{'} = \delta_{rs} \sum m \left(z_q + b_q\right) \left(z_q + b_q\right) - \sum m \left(z_r + b_r\right) \left(z_s + b_s\right) \tag{3}$$

$$= \begin{cases} \delta_{rs} \sum m z_q z_q - \sum m z_r z_s \\ +\delta_{rs} \sum m b_q z_q - \sum m b_r z_s \\ +\delta_{rs} \sum m b_q z_q - \sum m b_s z_r \\ +\delta_{rs} \sum m b_q b_q - \sum m b_r b_s \end{cases} \tag{4}$$

$$= \begin{cases} I_{rs} \\ +\delta_{rs} \sum m b_q z_q - \sum m b_r z_s \\ +\delta_{rs} \sum m b_q z_q - \sum m b_s z_r \\ +\delta_{rs} \sum m b_q b_q - \sum m b_r b_s \end{cases} \tag{5}$$

$$\tag{6}$$

The last term $\delta_{rs} \sum m b_q b_q - \sum m b_r b_s$ can be interpreted as a moment of inertia tensor for a single virtual mass $M = \sum m$ situated at the point b_q seen from the axes $Oz_1z_2z_3$. Let's denote it with $\tilde{I}_{rs} = \sum m \left(\delta_{rs} b_q b_q - b_r b_s\right)$.
The other two terms can also be seen as a rigid body of particles distributed in a plane perpendicular to one of the axis i.e. all particles are transported perpendicularity to a plane. We note that $\delta_{rs} \sum m b_q z_q - \sum m b_r z_s = \delta_{rs} \sum m b_q z_q - \sum m b_s z_r$. This follows immediately from the symmetric character of $I_{rs}^{'}, I_{rs}, \tilde{I}_{rs}$.
Denoting $\overline{I}_{rs} = \delta_{rs} \sum m b_q z_q - \sum m b_r z_s + \delta_{rs} \sum m b_q z_q - \sum m b_s z_r$ giving

$$K_{rs} = I_{rs} + \overline{I}_{rs} + \tilde{I}_{rs}$$

♦

5.31 p188 - Exercise 12

A rigid body is turning about a fixed point. Referred to right-handed axes $Oz_1z_2z_3$, its angular velocity tensor has components

$$\omega_{23} = 1, \quad \omega_{31} = 2, \quad \omega_{12} = 3$$

If we refer the same motion to the axis $O^{'}z_1^{'}z_2^{'}z_3^{'}$, such that the axis $O^{'}z_1^{'}$ is $Oz_1^{'}$ reversed, while z_2z_3 coincide with $O^{'}z_2^{'}z_3^{'}$, what are the $\omega_{rs}^{'}$ and $\omega_{rs}^{'}$?

We use the following identities

$$\begin{cases} \mathbf{5.312} & \omega_{rm} = -\omega_{mr} \\ \mathbf{5.316} & \omega_{rs} = \epsilon_{rsn}\omega_n \\ \mathbf{5.317} & \omega_1 = \omega_{23} \quad \omega_2 = \omega_{31} \quad \omega_3 = \omega_{12} \end{cases} \tag{1}$$

The angular velocity tensor is

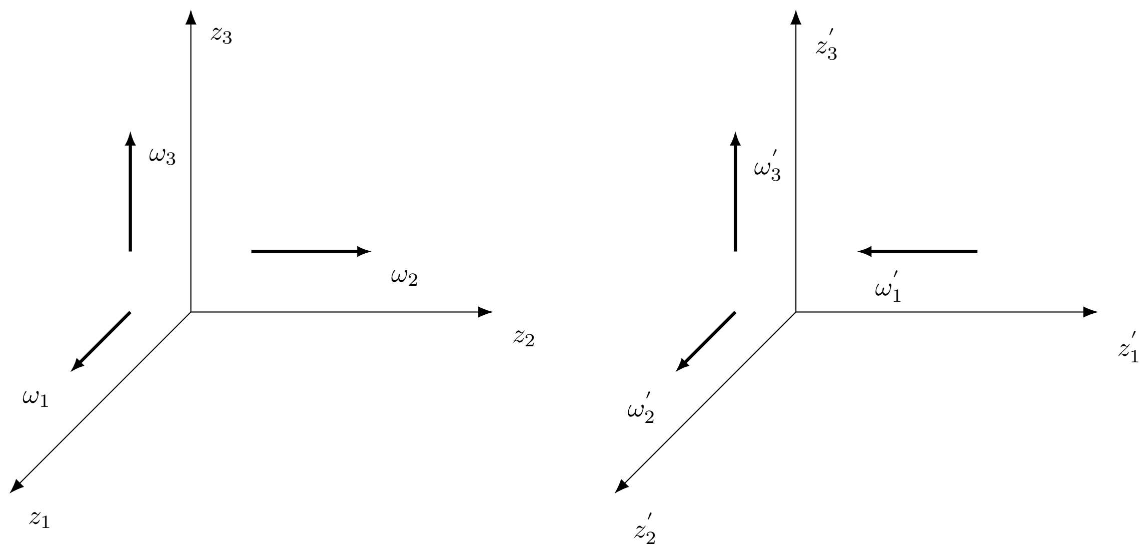
$$\Omega = \begin{pmatrix} 0 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \tag{2}$$

giving by **5.317**

$$\omega_1 = \omega_{23} \quad \omega_2 = \omega_{31} \quad \omega_3 = \omega_{12} \tag{3}$$

From pure geometrical consideration we can conclude that

$$\omega_1^{'} = -\omega_1 \quad \omega_2^{'} = \omega_2 \quad \omega_3^{'} = \omega_3 \tag{4}$$



(a)

Figure 5.7: Angular velocity vectors in mirrored axis

Indeed, the ω_i can be considered as vectors, objects independent from the chosen coordinate system. Reversing the direction of the first axis, will for the observer looking along the positive direction, look as if the ω_1 is reversed. We now use $\omega_{rs} = \epsilon_{rsn}\omega_n$ but here we have to be careful with ϵ_{rsn} when using the equation in the transformed coordinate system.

Looking at **4.312** $\epsilon_{stu}^{'} = \epsilon_{mnr}\frac{\partial z_m}{\partial z_s^{'}}\frac{\partial z_n}{\partial z_t^{'}}\frac{\partial z_r}{\partial z_u^{'}}$ and noting that $\frac{\partial z_1}{\partial z_1^{'}} = -1$ and 1 or 0 for the others, we have $\epsilon_{stu}^{'} = -\epsilon_{mnr}$. Now with **5.316** we get

$$\omega_{rs}^{'} = -\epsilon_{rsn}\omega_n^{'} \tag{5}$$

giving

$$\omega_{12}^{'} = -\omega_{12} \quad \omega_{13}^{'} = -\omega_{13} \quad \omega_{23}^{'} = \omega_{23} \tag{6}$$

Giving

$$\Omega^{'} = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \tag{7}$$

♦

5.32 p188 - Exercise 13

Consider three rigid bodies, $S, S^{'}, S^{"}$, turning about a common point. If all angular velocities are referred to common axes, show that the angular velocity tensors of $S^{"}$ relative to S is the sum of the angular velocity tensors of $S^{'}$ relative to S and of $S^{"}$ relative to $S^{'}$.

Consider the following three transformation from one axes system to another

$$\begin{cases} z_r^{'} = A_{rm} z_m & z_r = A_{mr} z_m^{'} & A_{mp} A_{mq} = \delta_{pq} & A_{pm} A_{qm} = \delta_{pq} \\ z_r^{"} = B_{rm} z_m^{'} & z_r^{'} = B_{mr} z_m^{"} & B_{mp} B_{mq} = \delta_{pq} & B_{pm} B_{qm} = \delta_{pq} \\ z_r^{"} = C_{rm} z_m & z_r = C_{mr} z_m^{"} & C_{mp} C_{mq} = \delta_{pq} & C_{pm} C_{qm} = \delta_{pq} \end{cases} \tag{1}$$

We then have,

$$\begin{cases} \omega_{pq}^{'} \left(S^{'}, S \right) = -A_{pm} \dot{A_{qm}} \\ \omega_{pq}^{"} \left(S^{"}, S^{'} \right) = -B_{pm} \dot{B_{qm}} \\ \omega_{pq}^{"} \left(S^{"}, S \right) = -C_{pm} \dot{C_{qm}} \end{cases} \tag{2}$$

From (1) we see that

$$C_{rq} = B_{rm} A_{mq} \tag{3}$$

And thus

$$\omega_{pq}^{"} \left(S^{"}, S \right) = -B_{pk} A_{km} (B_{qn} \dot{A}_{nm}) \tag{4}$$

$$\Rightarrow \qquad = \underbrace{-A_{km} \dot{A_{nm}}}_{=\omega_{kn}^{'}(S^{'},S)} B_{pk} B_{qn} - \underbrace{A_{km} A_{nm}}_{=\delta_{kn}} B_{pk} \dot{B_{qn}} \tag{5}$$

$$\Rightarrow \qquad = \omega_{kn}^{'} \left(S^{'}, S \right) B_{pk} B_{qn} - \underbrace{B_{pn} \dot{B_{qn}}}_{=-\omega_{pq}^{"}(S^{"},S^{'})} \tag{6}$$

The fist term of the right side expression is a bilinear map of the tensor $\omega_{kn}^{'} \left(S^{'}, S \right)$ from the reference axis $S^{'}$ to $S^{"}$. Hence we get

$$\omega_{pq}^{"} \left(S^{"}, S \right) = \omega_{pq}^{"} \left(S^{"}, S^{'} \right) + \omega_{pq}^{"} \left(S^{'}, S \right) \tag{7}$$

♦

5.33 p188 - Exercise 14

A freely moving particle is observed from a platform which rotates with angular velocity $\omega_r = n\delta_{r3}$, where n is constant, relative to a Newtonian frame S in which z_r are rectangular Cartesians. Use **5.421** to find the equations of motion relative to $S^{'}$ in terms of coordinates $z_r^{'}$ in $S^{'}$, such that the axis of $z_3^{'}$ coincides permanently with the axis of z_3.

5.421 gives (where the equation is expressed in term of the $z_r^{'}$

$$\begin{cases} mf_s = F_s^{'} + C_s^{'} + G_s^{'} \\ C_s^{'} = m\left[\dot{\omega}_{sn}^{'}\left(S^{'},S\right) + \omega_{sm}^{'}\left(S^{'},S\right)\omega_{nm}^{'}\left(S^{'},S\right)\right]z_n^{'} \\ C_s^{'} = 2m\omega_{sm}^{'}v_m^{'}\left(S^{'}\right) \end{cases} \tag{1}$$

We note the particle is free, so $F_s^{'} = 0$ and the angular velocity is a constant, so $\dot{\omega}_{sn}^{'}\left(S^{'},S\right) = 0$, and the equation simplify to

$$\begin{cases} f_s^{'} = K_s^{'} + J_s^{'} \\ K_s^{'} = \left[\omega_{sm}^{'}\left(S^{'},S\right)\omega_{nm}^{'}\left(S^{'},S\right)\right]z_n^{'} \\ J_s^{'} = 2\omega_{sm}^{'}v_m^{'}\left(S^{'}\right) \end{cases} \tag{2}$$

As $\omega_s = n\delta_{s3}$ and by the requirement that the axis of $z_3^{'}$ coincides permanently with the axis of z_3, it is not hard to see that

$$\begin{cases} \omega_{12}\left(S^{'},S\right) = n \\ \omega_{12}^{'}\left(S^{'},S\right) = n \\ \omega_{12}\left(S,S^{'}\right) = -n \\ \omega_{12}^{'}\left(S,S^{'}\right) = -n \end{cases} \tag{3}$$

while all other elements vanish.
We get

$$\begin{cases} K_1^{'} = \omega_{12}^{'}\left(S^{'},S\right)\omega_{12}^{'}\left(S^{'},S\right)z_1^{'} = n^2 z_1^{'} \\ K_1^{'} = \omega_{21}^{'}\left(S^{'},S\right)\omega_{21}^{'}\left(S^{'},S\right)z_1^{'} = n^2 z_1^{'} \\ K_3^{'} = 0 \end{cases} \tag{4}$$

$$\begin{cases} J_1^{'} = 2\omega_{12}^{'}\left(S^{'},S\right)v_2^{'}\left(S^{'}\right) = 2nv_2^{'}\left(S^{'}\right) \\ J_2^{'} = 2\omega_{21}^{'}\left(S^{'},S\right)v_1^{'}\left(S^{'}\right) = -2nv_1^{'}\left(S^{'}\right) \\ J_3^{'} = 0 \end{cases} \tag{5}$$

and get as equations of motion

$$\begin{cases} f_1' = n^2 z_1' + 2nv_2' \left(S'\right) \\ f_2' = n^2 z_1' - 2nv_1' \left(S'\right) \\ f_3' = 0 \end{cases} \tag{6}$$

♦

5.34 p188 - Exercise 15

If the tensor I_{st} is defined by **5.335** for N dimensions, and J_{nprq} is defined by **5.330**, establish the following relations:

$$J_{nprq} = (N-1)^{-1} I_{ss} (\delta_{nr}\delta_{pq} - \delta_{nq}\delta_{pr}) - \delta_{nr} I_{pq} + \delta_{pr} I_{nq}$$

$$J_{nppq} = I_{ss}$$

$$I_{nq} = (N-1)^{-1} (J_{nprq} - \delta_{nq} J_{nprq})$$

5.421 and **5.421**:

$$\begin{cases} I_{st} = \delta_{st} \sum m z_q z_q - \sum m z_s z_t \\ J_{nprq} = \sum m (\delta_{nr} z_p z_q - \delta_{pr} z_n z_q) \end{cases} \tag{1}$$

The first equation can be expressed as $\sum m z_p z_q = \delta_{pq} \sum m z_k z_k - I_{pq}$ and $\sum m z_n z_q = \delta_{st} \sum m z_k z_k - I_{nq}$

giving

$$J_{nprq} = \delta_{nr}\delta_{pq} \sum m z_k z_k - \delta_{nr} I_{pq} - \delta_{pr}\delta_{st} \sum m z_k z_k + \delta_{pr} I_{nq} \tag{2}$$

$$= \sum m z_k z_k (\delta_{nr}\delta_{pq} - \delta_{nr} I_{pq}) - \delta_{nr} I_{pq} + \delta_{pr} I_{nq} \tag{3}$$

Now, consider the expressions

$$\begin{cases} I_{11} = \sum m z_q z_q - \sum m z_1 z_1 \\ I_{11} = \sum m z_q z_q - \sum m z_1 z_1 \\ \vdots \\ I_{NN} = \sum m z_q z_q - \sum m z_N z_N \end{cases} \tag{4}$$

Summin up these N expressions we have

$$I_{ss} = N \left(\sum m z_q z_q \right) - \sum m z_q z_q \tag{5}$$

$$= (N-1) \sum m z_q z_q \tag{6}$$

$$\Rightarrow \qquad \sum m z_q z_q = I_{ss} (N-1)^{-1} \tag{7}$$

Plugging this in (3) we get

$$J_{nprq} = I_{ss} (N-1)^{-1} (\delta_{nr}\delta_{pq} - \delta_{nr} I_{pq}) - \delta_{nr} I_{pq} + \delta_{pr} I_{nq} \tag{8}$$

♦

5.35 p188 - Exercise 16

> The motion of a dynamical system is represented by a curve in configuration-space. Using the kinematical line element, express the curvature as a function of its total energy E, and deduce that as E tends to infinity, the trajectory tends to become a geodesic . Illustrate by considering a particle moving under gravity on a smooth sphere.

We have **5.512** and **5.533**:

$$\begin{cases} v^2 = a_{mn} v^m v^n = 2T \\ \kappa v^2 = X_r \nu^r \end{cases} \quad \Rightarrow \quad \kappa = \frac{X_r \nu^r}{2T} \tag{1}$$

First, we have to note that nothing is said about the nature of the generalized forces (conservative or not) and therefore we use **5.517**

$$dW = X_r dx^r \tag{2}$$

From this we can express T as

$$\begin{aligned} T(s) &= T_0 + \int_0^s dW && (3) \\ &= T_0 + \int_0^s X_r dx^r && (4) \\ & && (5) \end{aligned}$$

where T_0 is the kinetic energy at the initial configuration $s = 0$.
Suppose now that the for $s \to +\infty$, $\int_0^s dW \to +\infty$. In that case, the kinetical energy will represent the total energy of the system, $T \to E$ and $\kappa \to 0$ for $E \to +\infty$ provided that $X_r \nu^r \not\to \pm\infty$. Consider **5.535** $dW = X ds \cos\phi$ where X is the magnitude of the generalized force and ϕ the angle between the line of force and the trajectory.

♦

5.36 p189 - Exercise 17

A particle moves on a smooth sphere under action of gravity. Using he action line element, calculate the Gaussian curvature of configuration-space as a function of total energy E and height z above the centre of the sphere. Show that if the total energy is not sufficient to raise the particle to the top of the sphere, but only to a level $z = h$, then the Gaussian curvature tends to infinity as z approaches h from below.

Using polar spherical coordinates, the line element on the sphere is

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{1}$$

For the potential energy, we use the lowest point (along the axis of the gravitational filed) as reference. Hence the potential is given by

$$\begin{aligned} V &= mgR + mgz &(2)\\ &= R\left(1 + mg\cos\theta\right) &(3) \end{aligned}$$

Giving for the action line element (with a total energy of the system E_0)

$$d\sigma^2 = \left(E_0 - mgR - mgR\cos\theta\right) ds^2 \tag{4}$$

Be $E = E_0 - mgR$

$$(a_{mn}) = \begin{pmatrix} R^2\left(E - mgR\cos\theta\right) & 0 \\ 0 & R^2\left(E - mgR\cos\theta\right)\sin^2\theta \end{pmatrix} \tag{5}$$

3.114 and the exercise (Riemann curvature of a 2-space) on page 112 gives:

$$\begin{cases} G = \frac{R_{1212}}{a_{11}a_{22}} \\ \\ R_{1212} = -\frac{1}{2}\partial^2_{11}a_{22} - \frac{1}{4}a^{11}\partial_1 a_{11}\partial_1 a_{22} + \frac{1}{4}a^{22}\partial_1 a_{22}\partial_1 a_{22} \end{cases} \tag{6}$$

With

$$\begin{cases} \partial_1 a_{11} = mgR^3 \sin\theta \\ \\ \partial_1 a_{22} = 2ER^2 \sin\theta\cos\theta + R^3 mg\left(\sin^3\theta - 2\sin\theta\cos^2\theta\right) \\ \\ \partial_{11}^2 a_{22} = 2ER^2\left(\cos^2\theta - \sin^2\theta\right) + R^3 mg\left(3\cos\theta\sin^2\theta - 2\left(\cos^3\theta - 2\sin^2\theta\cos\theta\right)\right) \\ \quad = 2ER^2\left(\cos^2\theta - \sin^2\theta\right) + R^3 mg\left(7\cos\theta\sin^2\theta - 2\cos^3\theta\right) \\ \\ a^{11} = \frac{1}{R^2(E - mgR\cos\theta)} \\ \\ a^{22} = \frac{1}{R^2(E - mgR\cos\theta)\sin^2\theta} \end{cases} \tag{7}$$

We first try now to replace the expressions in θ with expressions in $R\cos\theta = z$ and $R\sin\theta = \sqrt{R^2 - z^2}$

$$\begin{cases} \partial_1 a_{11} = mgR^2\sqrt{R^2 - z^2} \\ \\ \partial_1 a_{22} = 2Ez\sqrt{R^2 - z^2} + mg\left(\left(R^2 - z^2\right)\sqrt{R^2 - z^2} - 2z^2\sqrt{R^2 - z^2}\right) \\ \\ \partial_{11}^2 a_{22} = 2E\left(2z^2 - R^2\right) + mg\left(7z\left(R^2 - z^2\right) - 2z^3\right) \\ \\ a^{11} = \frac{1}{R^2(E - mgz)} \\ \\ a^{22} = \frac{1}{(E - mgz)(R^2 - z^2)} \end{cases} \tag{8}$$

$$\Rightarrow \begin{cases} \partial_1 a_{11} = mgR^2\sqrt{R^2 - z^2} & ML^4T^{-2} \\ \\ \partial_1 a_{22} = \left(mgR^2 + 2Ez - 3mgz^2\right)\sqrt{R^2 - z^2} & ML^4T^{-2} \\ \\ \partial_{11}^2 a_{22} = 2E\left(2z^2 - R^2\right) + mgz\left(7R^2 - 9z^2\right) & ML^4T^{-2} \\ \\ a^{11} = \frac{1}{R^2(E - mgz)} & M^{-1}L^{-4}T^2 \\ \\ a^{22} = \frac{1}{(E - mgz)(R^2 - z^2)} & M^{-1}L^{-4}T^2 \\ \\ a_{11} = R^2\left(E - mgz\right) & ML^4T^{-2} \\ \\ a_{22} = \left(E - mgz\right)\left(R^2 - z^2\right) & ML^4T^{-2} \end{cases} \tag{9}$$

giving

$$R_{1212} = -\frac{1}{2}\partial_{11}^2 a_{22} - \frac{1}{4}a^{11}\partial_1 a_{11}\partial_1 a_{22} + \frac{1}{4}a^{22}\partial_1 a_{22}\partial_1 a_{22} \tag{10}$$

$$= \begin{cases} -E\left(2z^2 - R^2\right) - \frac{1}{2}mgz\left(7R^2 - 9z^2\right) \\ \\ -\frac{1}{4}\frac{1}{(E-mgz)}mg\left(R^2 - z^2\right)\left(mgR^2 + 2Ez - 3mgz^2\right) \\ \\ +\frac{1}{4}\frac{1}{(E-mgz)}\left(mgR^2 + 2Ez - 3mgz^2\right)^2 \end{cases} \tag{11}$$

$$= \begin{cases} \frac{1}{4}\frac{1}{(E-mgz)}\left[(E-mgz)\left(-E\left(2z^2 - R^2\right) - \frac{1}{2}mgz\left(7R^2 - 9z^2\right)\right)\right. \\ \\ -mg\left(R^2 - z^2\right)\left(mgR^2 + 2Ez - 3mgz^2\right) \\ \\ \left. + \left(mgR^2 + 2Ez - 3mgz^2\right)^2\right] \end{cases} \tag{12}$$

$$= \frac{R^2 - z^2}{(E - mgz)}\left[3m^2g^2z^2 - 4Emgz + E^2\right] \tag{13}$$

$$= \frac{R^2 - z^2}{(E - mgz)}(E - mgz)\left[E - 3mgz\right] \tag{14}$$

$$= \left(R^2 - z^2\right)\left[E - 3mgz\right] \tag{15}$$

For the Gauss curvature we get then

$$G = \frac{R_{1212}}{R^2\left(E - mgz\right)^2\left(R^2 - z^2\right)} \tag{16}$$

and so

$$G = \frac{E - 3mgz}{R^2\left(E - mgz\right)^2} \tag{17}$$

Be $h = \frac{E}{mg}$. From (17) we see that as long $z < \frac{h}{3}$, G is defined and positive. It becomes 0 for $z = \frac{h}{3}$ and negative for $z > \frac{h}{3}$ to become $-\infty$ for $z \to h$.
Remember that $E = E_0 - mgR$ with E_0 the total energy of the system and that the maximum potential energy is $V_{max} = 2mgR$. In order to reach the top, a particle starting from the bottom of the sphere ($V = 0$), should have at least a total energy $E_0 = 2mgR$.
Suppose now, that we configure the system so that the particle starts from the bottom and gets zero velocity at a point $z = h$. Then $E_0 = mg\left(R + h\right) < 2mgR$ and so $E = E_0 - mgR = mgh$.
(17) becomes

$$G = \frac{h - 3z}{mgR^2\left(h - z\right)^2} \tag{18}$$

$$\Rightarrow \qquad \lim_{z \to h} G = -\infty \tag{19}$$

♦

5.37 p189 - Exercise 18

Show that the equations of motion of a rigid body with a fixed point may be written in either of the forms

(a) $\qquad \dot{h}_r^{'} + \omega_{mr}^{'}\left(S^{'}, S\right) h_m^{'} = M_{rs}^{'},$

(b) $\qquad \dot{h}_r^{'} - K_{rmn}^{'} h_m^{'} h_n^{'} = M_{rs}^{'},$

where $h_r^{'}$ are the components on $z^{'}$-axes (moving with the body) of angular moment as given in **5.338** and $K_{rmn}^{'}$ is a certain moment of inertia tensor. Evaluate the components $K_{rmn}^{'}$ in terms of the moments and products of inertia.

We use **5.329** ,**5.231**,**5.233** and **5.424**:

$$M_{rs}^{'} = \epsilon_{rsn} M_n^{'} \tag{1}$$

$$h_{rs}^{'} = \epsilon_{rsn} h_n^{'} \tag{2}$$

$$h_{np}^{'} = J_{nprq}^{'} \omega_{rq}^{'} \tag{3}$$

$$M_{ab}^{'} = J_{abrq}^{'} \dot{\omega}_{rq}^{'}\left(S^{'}, S\right) + J_{cdrq}^{'}\left(\delta_{ac}\delta_{du}\delta_{bv} + \delta_{bd}\delta_{cu}\delta_{av}\right)\omega_{uv}^{'}\left(S^{'}, S\right)\omega_{rq}^{'}\left(S^{'}, S\right) \tag{4}$$

Then, using $(1), (2), (3)$ in (4) and contracting the terms in δ_{ij}

$$M'_{ab} = \dot{h_{ab}}^{'} + h_{au}^{'}\omega_{ub}^{'}\left(S^{'}, S\right) + h_{ub}^{'}\omega_{ua}^{'}\left(S^{'}, S\right) \tag{5}$$

$$\Leftrightarrow \qquad \epsilon_{abn} M'_n = \epsilon_{abn}\dot{h_n}^{'} + \epsilon_{aun} h_n^{'}\omega_{ub}^{'}\left(S^{'}, S\right) + \epsilon_{ubn} h_n^{'}\omega_{ua}^{'}\left(S^{'}, S\right) \tag{6}$$

$$\times\epsilon_{abt} \quad \Rightarrow \quad \epsilon_{abt}\epsilon_{abn} M'_n = \epsilon_{abt}\epsilon_{abn}\dot{h_n}^{'} + \epsilon_{abt}\epsilon_{aun} h_n^{'}\omega_{ub}^{'}\left(S^{'}, S\right) + \epsilon_{bat}\epsilon_{bun} h_n^{'}\omega_{ua}^{'}\left(S^{'}, S\right) \tag{7}$$

$$\Rightarrow \quad 2M'_t = 2\dot{h_t}^{'} + \left(\delta_{bu}\delta_{tn} - \delta_{bn}\delta_{tu}\right) h_n^{'}\omega_{ub}^{'}\left(S^{'}, S\right) + \left(\delta_{au}\delta_{tn} - \delta_{an}\delta_{tu}\right) h_n^{'}\omega_{ua}^{'}\left(S^{'}, S\right) \tag{8}$$

$$= 2\dot{h_t}^{'} + h_t^{'}\omega_{bb}^{'}\left(S^{'}, S\right) - h_b^{'}\omega_{tb}^{'}\left(S^{'}, S\right) + h_t^{'}\omega_{uu}^{'}\left(S^{'}, S\right) - h_a^{'}\omega_{ta}^{'}\left(S^{'}, S\right) \tag{9}$$

And so,

$$M'_r = \dot{h_r}^{'} + \omega_{mr}^{'}\left(S^{'}, S\right) h_m^{'} \tag{10}$$

$\Diamond$

Let's try to express equation (4) but with the inertia tensor I_{ij} as parameter. We have **5.332**:

$$\frac{d\left(I_{st}\omega_t\left(S^{'}, S\right)\right)}{dt} = M_s \tag{11}$$

Let's express this in the coordinate system $S^{'}$ so that $I_{sr}^{'}$ will not depend of the time. Be A_{ij} the map from $S^{'}$ to S. Then:

$$\frac{d\left(A_{ks}I^{'}_{st}\omega^{'}_{t}\left(S^{'},S\right)\right)}{dt} = M_s \tag{12}$$

$$\times \quad A_{ps} \qquad A_{ps}\frac{d\left(A_{ks}I^{'}_{st}\omega^{'}_{t}\left(S^{'},S\right)\right)}{dt} = M^{'}_{p} \tag{13}$$

$$A_{ps}\dot{A_{ks}}I^{'}_{kt}\omega^{'}_{t}\left(S^{'},S\right) + A_{ps}A_{ks}I^{'}_{kt}\dot{\omega}^{'}_{t}\left(S^{'},S\right) = M^{'}_{p} \tag{14}$$

We have

$$\begin{cases} \textbf{5.408} \quad \omega^{'}_{ts}\left(S^{'},S\right) = A_{tm}\dot{A}_{sm} \\ \textbf{5.401} \quad A_{mp}A_{mq} = \delta_{pq} \qquad\qquad A_{pm}A_{qm} = \delta_{pq} \end{cases} \tag{15}$$

So, (14) becomes

$$\omega^{'}_{pk}\left(S^{'},S\right)I^{'}_{kt}\omega^{'}_{t}\left(S^{'},S\right) + I^{'}_{pt}\dot{\omega}^{'}_{t}\left(S^{'},S\right) = M^{'}_{p} \tag{16}$$

We use

$$\omega^{'}_{pk}\left(S^{'},S\right) = \epsilon_{pkm}\omega^{'}_{m}\left(S^{'},S\right) \tag{17}$$

and get for (16):

$$I^{'}_{pt}\dot{\omega}^{'}_{t}\left(S^{'},S\right) + \epsilon_{pkm}\omega^{'}_{m}\left(S^{'},S\right)I^{'}_{kt}\omega^{'}_{t}\left(S^{'},S\right) = M^{'}_{p} \tag{18}$$

Note also that $h^{'}_{s} = I_{sr}\omega^{'}_{r}\left(S^{'},S\right)$. Indeed,

$$h^{'}_{np} = J^{'}_{npqr}\omega^{'}_{rq}\left(S^{'},S\right) \tag{19}$$

$$= J^{'}_{npqr}\epsilon_{rqm}\omega^{'}_{m}\left(S^{'},S\right) \tag{20}$$

$$\times \quad \frac{1}{2}\epsilon_{snp} \qquad h^{'}_{s} = \underbrace{\frac{1}{2}J^{'}_{npqr}\epsilon_{snp}\epsilon_{rqm}}_{=I^{'}_{sm}}\omega^{'}_{m}\left(S^{'},S\right) \tag{21}$$

$$\Rightarrow \qquad h^{'}_{s} = I^{'}_{sm}\omega^{'}_{m}\left(S^{'},S\right) \tag{22}$$

and (18) becomes

$$\dot{h}^{'}_{p}\left(S^{'},S\right) + \epsilon_{pkm}I^{'}_{kt}\omega^{'}_{m}\left(S^{'},S\right)\omega^{'}_{t}\left(S^{'},S\right) = M^{'}_{p} \tag{23}$$

Let's examine the term $\epsilon_{pkm}I^{'}_{kt}\omega^{'}_{m}\left(S^{'},S\right)\omega^{'}_{t}\left(S^{'},S\right)$ and let's write tentatively

$$K'_{pqn}h^{'}_{q}h^{'}_{n} = \epsilon_{pkm}I^{'}_{kt}\omega^{'}_{m}\left(S^{'},S\right)\omega^{'}_{t}\left(S^{'},S\right) \tag{24}$$

using $h_r^{'} = I_{rv}^{'}\omega_v^{'}\left(S^{'}, S\right)$ in (24) we get

$$K_{pkn}^{'}I_{kt}^{'}I_{nm}^{'}\omega_m^{'}\left(S^{'}, S\right)\omega_t^{'}\left(S^{'}, S\right) = \epsilon_{pkm}I_{kt}^{'}\omega_m^{'}\left(S^{'}, S\right)\omega_t^{'}\left(S^{'}, S\right) \tag{25}$$

$$\Rightarrow \qquad K_{pkn}^{'}I_{kt}^{'}I_{nm}^{'} = \epsilon_{pkm}I_{kt}^{'} \tag{26}$$

$$\Rightarrow \qquad K_{pkn}^{'}I_{nm}^{'} = \epsilon_{pkm} \tag{27}$$

$$\times \quad \epsilon_{pkt} \Rightarrow \qquad \epsilon_{pkt}K_{pkn}^{'}I_{nm}^{'} = \delta_{mt} \tag{28}$$

Let's write

$$I_{tn}^{'-1} = \epsilon_{pkt}K_{pkn}^{'}$$

We can truly consider $I_{tn}^{'-1}$ as the inverse of $I_{tn}^{'}$ due to (28) and the fact that $I_{tm}^{'}$ is represented as a symmetric square matrix with real numbers as elements and hence has a non-zero determinant and has indeed an inverse. Multiplying (27) by $I_{mt}^{'-1}$ gives us finally

$$\mathbf{K_{pkt}^{'}} = \epsilon_{\mathbf{pkm}}\mathbf{I_{mt}^{'-1}}$$

Q: Why the minus sign in the question?

Let's now calculate $I_{tn}^{'-1}$

$$I_{tn}^{'} = \begin{pmatrix} I_{11}^{'} & I_{12}^{'} & I_{13}^{'} \\ I_{12}^{'} & I_{22}^{'} & I_{23}^{'} \\ I_{13}^{'} & I_{23}^{'} & I_{33}^{'} \end{pmatrix} \tag{29}$$

The determinant

$$\Delta = I_{12}^{'}I_{22}^{'}I_{33}^{'} + 2I_{12}^{'}I_{13}^{'}I_{23}^{'} - I_{11}^{'}I_{23}^{'2} - I_{22}^{'}I_{13}^{'2} - I_{33}^{'}I_{12}^{'2} \tag{30}$$

giving

$$I_{tn}^{'-1} = \frac{1}{\Delta}\begin{pmatrix} I_{22}^{'}I_{33}^{'} - I_{23}^{'2} & -I_{12}^{'}I_{33}^{'} + I_{13}^{'}I_{23}^{'} & I_{12}^{'}I_{23}^{'} - I_{22}^{'}I_{13}^{'} \\ -I_{12}^{'}I_{33}^{'} + I_{13}^{'}I_{23}^{'} & I_{11}^{'}I_{33}^{'} - I_{13}^{'2} & -I_{11}^{'}I_{23}^{'} + I_{12}^{'}I_{13}^{'} \\ I_{12}^{'}I_{23}^{'} - I_{22}^{'}I_{13}^{'} & -I_{11}^{'}I_{23}^{'} + I_{12}^{'}I_{13}^{'} & I_{11}^{'}I_{22}^{'} - I_{12}^{'2} \end{pmatrix} \tag{31}$$

giving for $\mathbf{K}^{'}_{\mathbf{pkt}} = \epsilon_{\mathbf{pkm}} \mathbf{I}^{'-\mathbf{1}}_{\mathbf{mt}}$

$$\begin{cases} K^{'}_{121} = \frac{1}{\Delta}\left(I^{'}_{12}I^{'}_{23} - I^{'}_{22}I^{'}_{13}\right) \\ K^{'}_{122} = \frac{1}{\Delta}\left(-I^{'}_{11}I^{'}_{23} + I^{'}_{12}I^{'}_{13}\right) \\ K^{'}_{123} = \frac{1}{\Delta}\left(I^{'}_{11}I^{'}_{22} - I^{'2}_{12}\right) \\ K^{'}_{131} = -\frac{1}{\Delta}\left(-I^{'}_{12}I^{'}_{33} + I^{'}_{13}I^{'}_{23}\right) \\ K^{'}_{132} = \frac{1}{\Delta}\left(I^{'}_{11}I^{'}_{33} - I^{'2}_{13}\right) \\ K^{'}_{133} = -\frac{1}{\Delta}\left(-I^{'}_{11}I^{'}_{23} + I^{'}_{12}I^{'}_{13}\right) \\ K^{'}_{231} = -\frac{1}{\Delta}\left(I^{'}_{22}I^{'}_{33} - I^{'2}_{23}\right) \\ K^{'}_{232} = \frac{1}{\Delta}\left(-I^{'}_{11}I^{'}_{23} + I^{'}_{12}I^{'}_{13}\right) \\ K^{'}_{233} = \frac{1}{\Delta}\left(I^{'}_{12}I^{'}_{23} - I^{'}_{22}I^{'}_{13}\right) \end{cases} \tag{32}$$

all others can be found by symmetry considerations.

♦

5.38 p189 - Exercise 19

A rigid body turns about a fixed point Oin a flat space of N dimensions. prove that if N is odd, there exists at any instant a line OP of particles instantaneously at rest, but that, if N is even, no point other than O is, in general, instantaneously at rest. Show that if $N = 4$, there are are points other than O instantaneously at rest if, and only if,

$$\omega_{23}\omega_{14} + \omega_{31}\omega_{24} + \omega_{12}\omega_{34} = 0$$

Consider **5.310**

$$v_p = -\omega_{pr} z_r \tag{1}$$

What we seek, is a vector z_r so that

$$v_p = -\theta\omega_{pr} z_r = 0 \quad \theta \in \mathbb{R} \tag{2}$$

which means that we have to solve the homogeneous system of linear equations

$$\Omega\mathbf{z} = 0 \tag{3}$$

with Ω the skew-symmetric matrix containing the elements of the tensor ω_{pr}. From algebra, we know that when the dimension of a skew-symmetric matrix is odd, then it's determinant is zero, and hence the homogeneous system will have an infinity of solutions that can be of the form $z_r = a_r Z_N \quad r = \{1, 2, \ldots, N-1\}$, (we take the last coordinate as free parameter). This represents a line along which, all velocities are zero.
On the contrary if N is even, the determinant might be non-zero and the system will not have any solution except the trivial solution $z_r = 0$.

◊

Let's investigate this for $N = 4$. We have for (3):

$$\Omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix} \tag{4}$$

giving

$$\det\{\Omega\} = -\omega_{12}\begin{vmatrix} -\omega_{12} & \omega_{23} & \omega_{24} \\ -\omega_{13} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{34} & 0 \end{vmatrix} + \omega_{13}\begin{vmatrix} -\omega_{12} & 0 & \omega_{24} \\ -\omega_{13} & -\omega_{23} & \omega_{34} \\ -\omega_{14} & -\omega_{24} & 0 \end{vmatrix} - \omega_{14}\begin{vmatrix} -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \\ -\omega_{14} & -\omega_{24} & -\omega_{34} \end{vmatrix} \tag{5}$$

$$= \begin{cases} -\omega_{12}\left(-\omega_{12}\omega_{34}\omega_{34} - \omega_{23}\omega_{14}\omega_{34} + \omega_{24}\omega_{13}\omega_{34}\right) \\ +\omega_{13}\left(-\omega_{12}\omega_{14}\omega_{34} + \omega_{24}\left(\omega_{13}\omega_{24} - \omega_{23}\omega_{14}\right)\right) \\ -\omega_{14}\left(-\omega_{12}\omega_{23}\omega_{34} + \omega_{23}\left(\omega_{13}\omega_{24} - \omega_{23}\omega_{14}\right)\right) \end{cases} \tag{6}$$

$$= \begin{cases} +\omega_{12}\omega_{12}\omega_{34}\omega_{34} + \omega_{12}\omega_{14}\omega_{23}\omega_{34} - \omega_{12}\omega_{13}\omega_{24}\omega_{34} \\ -\omega_{12}\omega_{13}\omega_{14}\omega_{34} + \omega_{13}\omega_{13}\omega_{24}\omega_{24} - \omega_{13}\omega_{14}\omega_{23}\omega_{24} \\ +\omega_{12}\omega_{14}\omega_{23}\omega_{34} - \omega_{13}\omega_{14}\omega_{23}\omega_{24} + \omega_{14}\omega_{14}\omega_{23}\omega_{23} \end{cases} \tag{7}$$

Define

$$\begin{cases} A = \omega_{12}\omega_{34} \\ B = \omega_{13}\omega_{24} \\ C = \omega_{14}\omega_{23} \end{cases} \tag{8}$$

then we can write (7) as

$$\det\{\Omega\} = A^2 + B^2 + C^2 - 2AB - 2BC + 2AB \tag{9}$$

$$= (A - B + C)^2 \tag{10}$$

So, in the space of even dimensions,the system of homogeneous linear equations will have non-trivial solutions, only if

$$\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0$$

$\lozenge$

Let's find now some possible instantaneous lines of rotation.
Suppose N=4.
Let's define a line with

$$(z_r) = \theta\begin{pmatrix} \omega_{23} - \omega_{24} + \omega_{34} \\ -\omega_{13} + \omega_{14} + \omega_{34} \\ \omega_{12} - \omega_{14} - \omega_{24} \\ -\omega_{12} + \omega_{13} + \omega_{23} \end{pmatrix} \tag{11}$$

Then calculating the velocities with (1) we get

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = -\theta \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix} \begin{pmatrix} \omega_{23} - \omega_{24} + \omega_{34} \\ -\omega_{13} + \omega_{14} + \omega_{34} \\ \omega_{12} - \omega_{14} - \omega_{24} \\ -\omega_{12} + \omega_{13} + \omega_{23} \end{pmatrix} \tag{12}$$

$$= -\theta \begin{pmatrix} \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} \\ -\omega_{12}\omega_{34} + \omega_{13}\omega_{24} - \omega_{14}\omega_{23} \\ -\omega_{12}\omega_{34} + \omega_{13}\omega_{24} - \omega_{14}\omega_{23} \\ -\omega_{12}\omega_{34} + \omega_{13}\omega_{24} - \omega_{14}\omega_{23} \end{pmatrix} \tag{13}$$

So the velocities will vanish when

$$\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0$$

$$\lozenge$$

Suppose N is odd. Let's define the following vector

$$\omega_{i_1} = \frac{1}{2^{\frac{N-1}{2}} \frac{N-1}{2}!} \epsilon_{i_1 i_2 \ldots i_N} \prod_{k=1}^{\frac{N-1}{2}} \omega_{i_{2k} i_{2k+1}} \tag{14}$$

and a line

$$z_{i_1} = \theta \omega_{i_1} \quad (\theta \in \mathbb{R}) \tag{15}$$

First we note that ω_{i_1} (and hence z_{i_1}) is not a null-vector:
Let's consider in (14) the terms consisting of the permutation of the sequence of pairs

$$\{(i_2, i_3), (i_4, i_5), (i_6, i_7), \ldots, (i_{N-1}, i_N)\}$$

This sequence contains $\frac{N-1}{2}$ pairs and so can be arranged in $\frac{N-1}{2}!$ ways. As for each pair we have two valid possibilities e.g. (i_2, i_3) and (i_3, i_2) and as a sequence contains $\frac{N-1}{2}$ pairs, we will have for a given order of pairs $2^{\frac{N-1}{2}}$ possibilities. So in (1) there will be $2^{\frac{N-1}{2}} \frac{N-1}{2}!$ terms consisting of the permutation of the sequence of pairs $\{(i_2, i_3), (i_4, i_5), (i_6, i_7), \ldots, (i_{N-1}, i_N)\}$.
Without loss of generality, suppose that $\epsilon_{i_1 i_2 \ldots i_N}$ is positive and also all $\omega_{i_{2k} i_{2k+1}}$ are positive. Let's first consider a permutation of two pairs in the sequence $\{(i_2, i_3), (i_4, i_5), (i_6, i_7), \ldots, (i_{N-1}, i_N)\}$. Obviously, this does not change the product of the $\omega_{i_{2k} i_{2k+1}}$. Also $\epsilon_{i_1 i_2 \ldots i_N}$ will hold it's initial sign as the considered permutation needs two permutation of indices.
Next consider a permutation in one of the pairs of the sequence. Obviously $\epsilon_{i_1 i_2 \ldots i_N}$ will change sign but also the picked $\omega_{i_{2k} i_{2k+1}}$(skew-symmetric).
Conclusion, all $2^{\frac{N-1}{2}} \frac{N-1}{2}!$ terms can be reduced to the sum of $2^{\frac{N-1}{2}} \frac{N-1}{2}!$ of a same quantity and the ω_{i_1} will not trivially be zero.

Let's consider now **5.310**

$$v_p = -\omega_{pi_1} z_{i_1} \tag{16}$$

$$(14): \qquad v_p = -\frac{1}{2^{\frac{N-1}{2}} \frac{N-1}{2}!} \theta \epsilon_{i_1 i_2 \ldots i_N} \omega_{pi_1} \prod_{k=1}^{\frac{N-1}{2}} \omega_{i_{2k} i_{2k+1}} \tag{17}$$

On the right side, for having a non-zero term, we need that $p \neq i_1$ (ω_{st} skew-symmetric). This leaves us with only $N-1$ possible choices in the indices but as $\epsilon_{i_1 i_2 \ldots i_N}$ needs N mutual different indices it is obvious that each term in (17) will have a $\epsilon_{i_1 i_2 \ldots i_N} = 0$
Conclusion, all v_p are zero and hence the defined line in (15) is an instantaneous line of rotation.

♦

5.39 p189 - Exercise 20

The equations **5.329** do not determine J_{nprq} uniquely. Why? As an alternative to **5.330**, we can require J_{npqr} to be skew-symmetric in the last two suffixes. Show that this defines J_{nprq} uniquely as follows:

$$J_{nprq} = \frac{1}{2}\sum m\left(\delta_{nr}z_p z_q + \delta_{pq}z_n z_r - \delta_{nq}z_p z_r - \delta_{pr}z_n z_q\right)$$

Prove thatJ_{nprq}, as defined here, has the same symmetries as the covariant curvature tensor (see **3.115**, **3.116**) and that, for $N = 3$, we have

$$I_{st} = \frac{1}{2}\epsilon_{snp}\epsilon_{trq}J_{nprq}, \quad J_{nprq} = \frac{1}{2}\epsilon_{snp}\epsilon_{trq}I_{st}$$

The equations **5.329** , $h_{np} = J_{nprq}\omega_{rq}$ do not determine J_{nprq} uniquely because ω_{rq} is skew symmetric, so all elements at the positions $J_{np(rr)}$ can be chosen arbitrarily and still comply with the equation.
Consider now the expression

$$\begin{aligned}
J'_{nprq} &= \frac{1}{2}\left(J_{nprq} - J'_{npqr}\right) && (1)\\
\Rightarrow \qquad h'_{np} &= J'_{nprq}\omega_{rq} && (2)\\
&= \frac{1}{2}\left(J_{nprq}\omega_{rq} - J_{npqr}\omega_{rq}\right) && (3)\\
&= \frac{1}{2}\left(J_{nprq}\omega_{rq} + J_{npqr}\omega_{qr}\right) && (4)\\
&= \frac{1}{2}\left(J_{nprq}\omega_{rq} + J_{nprq}\omega_{rq}\right) && (5)\\
&= h_{np} && (6)
\end{aligned}$$

So this expression $J_{nprq} = \frac{1}{2}\sum m\left(\delta_{nr}z_p z_q + \delta_{pq}z_n z_r - \delta_{nq}z_p z_r - \delta_{pr}z_n z_q\right)$ still describes the dynamical system and we note that this expression is skew-symmetric in the two last suffixes:

$$\begin{aligned}
J_{np(rr)} &= \frac{1}{2}\sum m\left(\delta_{nr}z_p z_r + \delta_{pr}z_n z_r - \delta_{nr}z_p z_r - \delta_{pr}z_n z_r\right) = 0 && (7)\\
J_{npqr} &= \frac{1}{2}\sum m\left(\delta_{nq}z_p z_r + \delta_{pr}z_n z_q - \delta_{nr}z_p z_q - \delta_{pq}z_n z_r\right) && (8)\\
&= -\frac{1}{2}\sum m\left(-\delta_{nq}z_p z_r - \delta_{pr}z_n z_q + \delta_{nr}z_p z_q + \delta_{pq}z_n z_r\right) && (9)\\
&= -J_{nprq} && (10)
\end{aligned}$$

Symmetries to prove:

$$\begin{cases} J_{nprq} = -J_{pnrq}, \quad J_{nprq} = -J_{npqr}, \quad J_{nprq} = J_{rqnp} \\ J_{nprq} + J_{nrqp} + J_{nqpr} = 0 \end{cases} \qquad (11)$$

The second identity of (11) is already proven as J_{nprq} is skew-symmetric in the last two suffixes.

For the rest:

$$J_{nprq} = \frac{1}{2}\sum m\left(\delta_{nr}z_p z_q + \delta_{pq}z_n z_r - \delta_{nq}z_p z_r - \delta_{pr}z_n z_q\right) \tag{12}$$
$$= -\frac{1}{2}\sum m\left(-\delta_{nr}z_p z_q - \delta_{pq}z_n z_r + \delta_{nq}z_p z_r + \delta_{pr}z_n z_q\right) \tag{13}$$
$$= J_{pnrq} \tag{14}$$

and

$$J_{nprq} = \frac{1}{2}\sum m\left(\delta_{nr}z_p z_q + \delta_{pq}z_n z_r - \delta_{nq}z_p z_r - \delta_{pr}z_n z_q\right) \tag{15}$$
$$J_{rqnp} = \frac{1}{2}\sum m\left(\delta_{rn}z_q z_p + \delta_{qp}z_r z_n - \delta_{rp}z_q z_n - \delta_{qn}z_r z_p\right) \tag{16}$$
$$= J_{pnrq} \tag{17}$$

and

$$J_{nprq} + J_{nrqp} + J_{nqpr} = \begin{cases} +\frac{1}{2}\sum m\left(\delta_{nr}z_p z_q + \delta_{pq}z_n z_r - \delta_{nq}z_p z_r - \delta_{pr}z_n z_q\right) \\ +\frac{1}{2}\sum m\left(\delta_{nq}z_p z_r + \delta_{rp}z_n z_q - \delta_{np}z_q z_r - \delta_{rq}z_n z_p\right) \\ +\frac{1}{2}\sum m\left(\delta_{np}z_r z_q + \delta_{qr}z_n z_p - \delta_{nr}z_q z_p - \delta_{qp}z_n z_r\right) \end{cases} = 0 \tag{18}$$

For the last part: From **5.33**, (J'_{nprq} being a not necessarily skew-symmetric tensor)

$$I_{st} = \frac{1}{2}J'_{nprq}\epsilon_{rqt}\epsilon_{snp} \tag{19}$$

or

$$I_{st} = \frac{1}{2}J'_{npqr}\epsilon_{qrt}\epsilon_{snp} \tag{20}$$
$$= -\frac{1}{2}J'{}_{npqr}\epsilon_{rqt}\epsilon_{snp} \tag{21}$$

Adding (19) and (21) gives

$$2I_{st} = \underbrace{\frac{1}{2}\left(J'_{npqr} - J'{}_{npqr}\right)}_{=J_{npqr}}\epsilon_{rqt}\epsilon_{snp} \tag{22}$$
$$\Rightarrow \qquad I_{st} = \frac{1}{2}J_{npqr}\epsilon_{rqt}\epsilon_{snp} \tag{23}$$

And

$$I_{st}\epsilon_{trq}\epsilon_{snp} = \frac{1}{2}J'_{kjuv}\underbrace{\epsilon_{tuv}\epsilon_{trq}}_{=\delta_{ur}\delta_{vq}-\delta_{uq}\delta_{vr}}\underbrace{\epsilon_{skj}\epsilon_{snp}}_{=\delta_{kn}\delta_{jp}-\delta_{kp}\delta_{jn}} \tag{24}$$
$$= -\frac{1}{2}J'{}_{npqr}\epsilon_{rqt}\epsilon_{snp} \tag{25}$$

expanding the right product we get

$$I_{st}\epsilon_{trq}\epsilon_{snp} = \frac{1}{2}\left(J_{nprq} + J_{pnqr} - J_{npqr} - J_{pnrq}\right) \tag{26}$$

And considering the symmetries described previously we get

$$\begin{aligned} I_{st}\epsilon_{trq}\epsilon_{snp} &= \frac{1}{2}4J_{nprq} &(27)\\ &= 2J_{nprq} &(28)\\ \Rightarrow \qquad J_{nprq} &= \frac{1}{2}I_{st}\epsilon_{trq}\epsilon_{snp} &(29)\end{aligned}$$

♦