Tensor Calculus

J.L. Synge and A.Schild (Dover Publication)

Solutions to exercises

Part II

Chapters V to VIII

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Remarks and warnings

You're welcome to use these notes, but they may contain errors, so proceed with caution. If you do find an error, however, I'd be happy to receive bug reports, suggestions, and the like through Github. An overview of the material covered in the book can be found in the separate document "Synge overview.pdf".

Some notation conventions

$$\partial_r \equiv \frac{\partial}{\partial x^r}$$

$$\Gamma^r_{mn} \equiv \begin{Bmatrix} r \\ mn \end{Bmatrix} \quad \text{Christoffel symbol of the second kind}$$

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Applications to Classical Mechanics

5.1 p153 - Exercise

If μ^α are the contravariant components of a unit vector in a surface S, show that $\mu^\alpha f_\alpha$ is the physical component of acceleration in the direction tangent to S defined by μ^α.

As we are in an Euclidean space we can interpret $a_{mn}\mu^\alpha f^\alpha$ as $|\mu|\,|f|\cos\theta$ with θ the angle between the two vectors. As $|\mu| = 1$ we have

$$a_{mn}\mu^\alpha f^\alpha = \mu^\alpha f_\alpha \tag{1}$$
$$= |f|\cos\theta \tag{2}$$

which is the projection of the vector f on the unit vector μ.

♦

5.2 p154 - Clarification to 5.226.

5.226. $$\mathbf{v\frac{dv}{ds} = 0}, \quad \overline{\kappa}\mathbf{v^2 = 0}$$

Assuming that the particle is not at rest $v \neq 0$, and therefore $\overline{\kappa} = 0$. ***Since this implies that the curve is a geodesic...***

The assertion in bold is a direct consequence

2.513. $$\frac{\delta \frac{dx^r}{ds}}{\delta s} = 0$$

As in **5.233** we have $\frac{\delta \lambda^\alpha}{\delta s} = \frac{\delta \frac{dx^\alpha}{ds}}{\delta s} = 0$, the considered curve follows the geodesic curve.

♦

5.3 p155 - Exercise

Show that in relativity the force 4-vector X^r lies along the first normal of the trajectory in space-time. Express the first curvature in terms of the proper mass m of the particle and the magnitude X of X^r.

Let us recall the first Frenet formula **2.705** without forgetting that the metric form is not positive-definite,

$$\frac{\delta\lambda^r}{\delta s} = \kappa\nu^r, \quad \epsilon_{(1)}\nu_n\nu^n = 1$$

As **5.299**

$$m\frac{\delta\lambda^r}{\delta s} = X^r$$

it is clear that $X^r = m\kappa\nu^r$ and is collinear with the first normal.

$$X^r = m\kappa\nu^r \tag{1}$$

$$\times \quad a_{mr}X^m \quad \Rightarrow \quad \underbrace{a_{mr}X^mX^r}_{=(X^1)^2+(X^2)^2+(X^3)^2-(X^4)^2} = m\kappa\underbrace{a_{mr}\nu^m\nu^r}_{=\epsilon_{(1)}} \tag{2}$$

$$\Rightarrow \qquad \kappa = \epsilon_{(1)}\frac{\left(X^1\right)^2+\left(X^2\right)^2+\left(X^3\right)^2-\left(X^4\right)^2}{m}$$

♦

5.4 p156 - Clarification

Interpretation of

5.231. $M_{rs} = \epsilon_{rsn} M_n = z_r F_s - z_s F_r$

What do the M_{rs} represent?

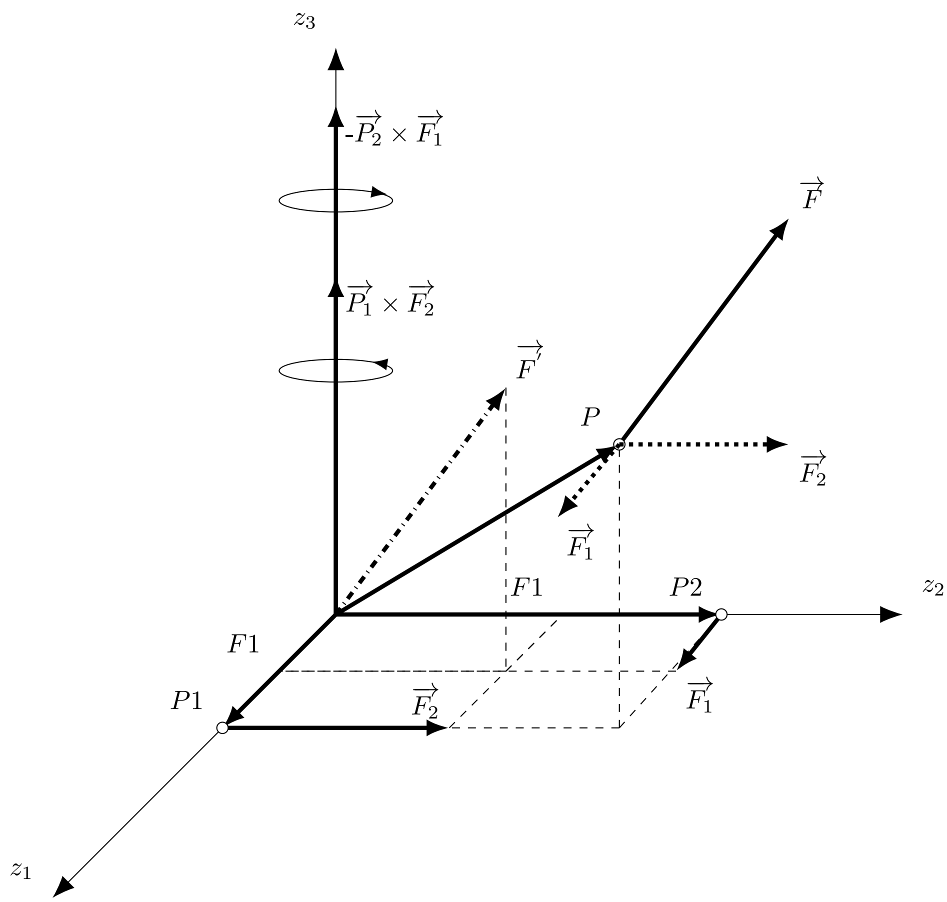


Figure 5.1: Interpretation of the tensor moment M_{12}

Let's consider a mass point P on which a force $\overrightarrow{F}$ is acting. The force has components (F_x, F_y, F_z) in the space V_3' (which is by the way not the space V_3 of the considered mass point).
Let's investigate the element M_{12} of the *tensor moment*.
$P_1 F_2 \overrightarrow{e_3}$ is the vector product $\overrightarrow{P_1} \times \overrightarrow{F_2}$ and is as such the torque of the component F_2 of $\overrightarrow{F}$ acting on the mass point situated at P_1. The origin being fixed, $\overrightarrow{F_2}$ tries to move P_1, clockwise along the z_3 axis. The same is true for the component $\overrightarrow{F_1}$ acting on the mass point situated at P_2, and is represented here by the vector $-\overrightarrow{P_2} \times \overrightarrow{F_1}$ ($\overrightarrow{F_1}$ tries to move P_2, counter clockwise along the z_3 axis).
Hence, $P_1 F_2 - P_2 F_1$ is the net force trying to move the point P along the z_3 axis (i.e. in the plane $\|$ with the $z_3 = 0$ plane).

♦

5.5 p156 - Clarification

5.234. $$\frac{dh_r}{dt} = M_r$$

$$
\begin{aligned}
& & h_r &= m\epsilon_{rmn} z_m v_n & (1)\\
&\Rightarrow & \frac{dh_r}{dt} &= m\epsilon_{rmn}\frac{dz_m}{dt} v_n + m\epsilon_{rmn} z_m \frac{dv_n}{dt} & (2)\\
& & &= m\underbrace{\epsilon_{rmn} v_m v_n}_{=0} + \underbrace{\epsilon_{rmn} z_m F_n}_{=M_r} & (3)\\
& & &= M_r & (4)
\end{aligned}
$$

♦

5.6 p158-159 - Clarification

5.313. $\omega_{rs} = -\omega_{sr}$

From 5.310 and the vector character of v_r and z_r (for transformations which do not change the origin), **it follows that ω_{rs} is a Cartesian tensor of second order**.

Be

$$v_r = -\omega_{rn} z_n \tag{1}$$

Considering orthogonal transformation in a flat space $z_m^{'} = A_{mr} z_r + B_m$ with $B_m = 0$ as we consider only transformations which do not change the origin. Differentiation with the parameter t gives

$$\begin{aligned} v_m^{'} &= A_{mr} v_r & (2)\\ &= -\omega_{rn} A_{mr} z_n & (3)\\ & & (4) \end{aligned}$$

But $z_q^{'} = A_{qr} z_r \quad \Rightarrow \quad A_{qn} z_q^{'} = A_{qn} A_{qr} z_r \quad \Rightarrow \quad A_{qn} z_q^{'} = z_n$ Hence

$$\begin{aligned} v_m^{'} &= -\omega_{rn} A_{mr} z_n & (5)\\ &= -\underbrace{\omega_{rn} A_{mr} A_{qn}}_{\stackrel{\text{def}}{=} \omega_{mq}^{'}} z_q^{'} & (6)\\ v_m^{'} &= -\omega_{mq}^{'} z_q^{'} & (7) \end{aligned}$$

♦

5.7 p159 - Exercise

Show that if a rigid body rotates about the point $z_r = b_r$ as fixed point, the velociy of a general point of the body is given by

$$v_r = -\omega_{rm}\left(z_m - b_m\right)$$

By **5.302**.:

$$\left(z_m^{(1)} - z_m^{(2)}\right)\left(dz_m^{(1)} - dz_m^{(2)}\right) = 0 \tag{1}$$

At the fixed point we have $z_m^{(2)} = b_m$ and $dz_m^{(2)} = 0$, hence

$$\left(z_m^{(1)} - b_m\right)\left(dz_m^{(1)}\right) = 0 \tag{2}$$

$$\Rightarrow \qquad z_m^{(1)} dz_m^{(1)} = b_m dz_m^{(1)} \tag{3}$$

As this is true for any point of the rigid mass, expanding (1) and using (3) we get when dividing by dt

$$\left(z_m^{(2)} - b_m\right) v_m^{(1)} + \left(z_m^{(1)} - b_m\right) v_m^{(2)} = 0 \tag{4}$$

Taking twice the partial derivative $\frac{\partial^2}{\partial z_p^{(1)} \partial z_q^{(1)}}$ we get

$$\left(z_m^{(2)} - b_m\right) \frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \tag{5}$$

As this is true for any arbitrary point in the rigid body we get

$$\frac{\partial^2 v_m}{\partial z_p^{(1)} \partial z_q^{(1)}} = 0 \tag{6}$$

$$\Rightarrow \qquad v_m = K_{mr} z_r + B_m \tag{7}$$

At the fixed point we have

$$K_{mr} b_r + B_m = 0 \tag{8}$$

Plugging this in (7)

$$v_m = K_{mr}\left(z_r - b_m\right) \tag{9}$$

Putting $K_{mr} = -\omega_{mr}$ gives us indeed the asked expression.

♦

5.8 p161 - Clarification

5.325. $$\Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p$$

and hence, since Ω_{np} is arbitrary,

5.326. $$\sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

To be complete the following step should be inserted

$$\Omega_{np} \sum (m f_n z_p) = \Omega_{np} \sum F_n z_p \tag{1}$$

As Ω_{np} is skew-symmetric: $$-\Omega_{np} \sum (m f_p z_n) = -\Omega_{np} \sum F_p z_n \tag{2}$$

(1)+(2) $$\Omega_{np} \sum m (f_n z_p - f_p z_n) = \Omega_{np} \sum (F_n z_p - F_p z_n) \tag{3}$$

and hence, since Ω_{np} is arbitrary,

5.326. $$\sum m (f_n z_p - f_p z_n) = \sum (F_n z_p - F_p z_n)$$

♦

5.9 p161 - Clarification

5.329.
$$\begin{aligned} h_{np} &= \sum m\left(\omega_{nq} z_q z_p - \omega_{pq} z_q z_n\right) \\ &= J_{npqr}\omega_{rq} \end{aligned}$$

where

5.330.
$$J_{npqr} = \sum m\left(\delta_{nr} z_q z_p - \delta_{pr} z_n z_q\right)$$

$$\begin{aligned} h_{np} &= \sum m\left(\omega_{nq} z_q z_p - \omega_{pq} z_q z_n\right) && (1) \\ &= \sum m\left(\omega_{rq}\delta_{rn} z_q z_p - \omega_{rq}\delta_{rp} z_q z_n\right) && (2) \\ &= \omega_{rq}\sum m\left(\delta_{rn} z_q z_p - \delta_{rp} z_q z_n\right) && (3) \\ &= J_{npqr}\omega_{rq} && (4) \end{aligned}$$

♦

5.10 p166 - Exercise

Deduce immediately from **5.420**. that the Coriolis force is perpendicular to the velocity.

$$
\begin{aligned}
& & G_s^{'} &= 2m\omega_{sm}^{'}(S^{'},S)v_m^{'}(S^{'}) && (1)\\
\times v_s^{'}(S^{'}) \quad & : & G_s^{'}v_s^{'}(S^{'}) &= m\left(\omega_{sm}^{'}(S^{'},S)v_m^{'}(S^{'})v_s^{'}(S^{'}) + \omega_{ms}^{'}(S^{'},S)v_m^{'}(S^{'})v_s^{'}(S^{'})\right) && (2)\\
& & &= 0 \quad \text{as } \omega_{ms}^{'} \text{ is skew-symmetric} && (3)
\end{aligned}
$$

♦

5.11 p166 - Exercise

Show that if $N = 3$ and $\dot{\omega}_r^{'}(S^{'}, S) = 0$, then the centrifugal force may be written

$$\textbf{5.422}. \qquad C_s^{'} = m\omega_n^{'}(S^{'}, S)\omega_n^{'}(S^{'}, S)z_s^{'} - m\omega_n^{'}(S^{'}, S)z_n^{'}\omega_s^{'}(S^{'}, S)$$

Deduce that $C_s^{'}$ is coplanar with the vectors $\omega_s^{'}(S^{'}, S)$ and $z_n^{'}$ and perpendicular to the former.

By **5.420**. with $\dot{\omega}_r^{'}(S^{'}, S) = 0$ and using **5.316**. $(\omega_{rs}^{'} = \epsilon_{rsn}\omega_n^{'})$

$$\begin{aligned}
C_s^{'} &= m\omega_{sm}^{'}(S^{'}, S)\omega_{nm}^{'}(S^{'}, S)z_n^{'} &(1)\\
&= m\epsilon_{smk}\omega_k^{'}(S^{'}, S)\epsilon_{nmp}\omega_p^{'}(S^{'}, S)z_n^{'} &(2)\\
&= m\epsilon_{msk}\epsilon_{mnp}\omega_k^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_n^{'} &(3)\\
&= m\left(\delta_{sn}\delta_{kp} - \delta_{sp}\delta_{kn}\right)\omega_k^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_n^{'} &(4)\\
&= m\delta_{sn}\delta_{kp}\omega_k^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_n^{'} - m\delta_{sp}\delta_{kn}\omega_k^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_n^{'} &(5)\\
&= m\omega_p^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)z_s^{'} - m\omega_n^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)z_n^{'} &(6)
\end{aligned}$$

To deduce that $C_s^{'}$ is coplanar with the vectors $\omega_s^{'}(S^{'}, S)$ and $z_n^{'}$ we calculate the mixed triple product

$$\begin{aligned}
P &= \epsilon_{spr}C_s^{'}\omega_p^{'}(S^{'}, S)z_r^{'} &(7)\\
&= m\underbrace{\epsilon_{spr}\omega_n^{'}(S^{'}, S)\omega_n^{'}(S^{'}, S)z_s^{'}\omega_p^{'}(S^{'}, S)z_r^{'}}_{=0} - \underbrace{m\epsilon_{spr}\omega_n^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)z_n^{'}\omega_p^{'}(S^{'}, S)z_r^{'}}_{=0} &(8)\\
&= 0 &(9)
\end{aligned}$$

Both terms vanish: the first by the presence of the terms $\epsilon_{spr}z_s^{'}z_r^{'}$ which cancel each other and for the second by the terms $\epsilon_{spr}\omega_s^{'}(S^{'}, S)\omega_p^{'}(S^{'}, S)$. As $P = 0$, the three vectors are coplanar.
We now calculate the inner product $C_s^{'}\omega_s^{'}(S^{'}, S)$

$$\begin{aligned}
P &= m\omega_n^{'}(S^{'}, S)\omega_n^{'}(S^{'}, S)z_s^{'}\omega_s^{'}(S^{'}, S) - \underbrace{m\omega_n^{'}(S^{'}, S)\omega_s^{'}(S^{'}, S)z_n^{'}\omega_s^{'}(S^{'}, S)}_{\Leftrightarrow\ m\omega_n^{'}(S^{'}, S)\omega_n^{'}(S^{'}, S)z_s^{'}\omega_s^{'}(S^{'}, S)} &(10)\\
&= 0 &(11)
\end{aligned}$$

♦

5.12 p169 - Exercise

Assign convenient generalized coordinates for the three systems (a), (b), and (c) mentioned at the beginning of this section, and calculate the kinematical metric form in each case

(a) **a particle on a surface** $(N = 2)$

No need here for fancy general coordinates: the V_2 coordinate system in the plane is the metric form of choice. Indeed $|v|^2 = a_{mn}v_m v_n$ and for a V_2

$$ds^2 = \left(a_{11}\left(v^1\right)^2 + 2a_{12}v^1v^2 + a_{22}\left(v^2\right)^2\right)dt^2$$

and if the space is Euclidean and the plane smooth, we can choose an orthogonal system where a_{12} will vanish.

(b) **a rigid body which can turn about a fixed point, as in the preceding section** $(N = 3)$

For a rigid body we can choose a coordinate system $S^{'}$ fixed to the body to describe the geometry of the rigid body. The kinetic energy referenced to a 'non-moving' (abuse of language) coordinate system S is

$$T = \frac{1}{2}\sum \rho v^{'}_n(S)v^{'}_n(S) \qquad \text{(summation over all masses in the rigid body)} \tag{1}$$

We know by **5.409**: $v^{'}_n(S) = v^{'}_n(S^{'}) + \omega^{'}_{mn}(S^{'}, S)z^{'}_m$. As the $v^{'}_n(S^{'})$ are fixed, we have $v^{'}_n(S^{'}) = 0$ giving

$$T = \frac{1}{2}\sum \rho z^{'}_m z^{'}_k \omega^{'}_{mn}(S^{'}, S)\omega^{'}_{kn}(S^{'}, S) \tag{2}$$

Note in (2) that we bring $\omega^{'}_{mn}(S^{'}, S)$ out of the summation as this expression is the same for all masses in the body.

$$\begin{aligned}
\omega_{mn}(S^{'}, S) &= \epsilon_{mnt}\omega^{'}_t(S^{'}, S) && (3)\\
\Rightarrow \quad T &= \frac{1}{2}\sum \rho \epsilon_{mnt}\epsilon_{kns} z^{'}_m z^{'}_k \omega^{'}_t(S^{'}, S)\omega^{'}_s(S^{'}, S) && (4)\\
&= \frac{1}{2}\sum \rho \left(\delta_{mk}\delta_{ts} - \delta_{ms}\delta_{kt}\right) z^{'}_m z^{'}_k \omega^{'}_t(S^{'}, S)\omega^{'}_s(S^{'}, S) && (5)\\
&= \frac{1}{2}\sum \rho \left(z^{'}_m z^{'}_m \omega^{'}_t(S^{'}, S)\omega^{'}_t(S^{'}, S) - z^{'}_s z^{'}_t \omega^{'}_t(S^{'}, S)\omega^{'}_s(S^{'}, S)\right) && (6)\\
&= \frac{1}{2}\sum \rho \left(\delta_{st} z^{'}_m z^{'}_m \omega^{'}_s(S^{'}, S)\omega^{'}_t(S^{'}, S) - z^{'}_s z^{'}_t \omega^{'}_t(S^{'}, S)\omega^{'}_s(S^{'}, S)\right) && (7)\\
&= \frac{1}{2}\sum \rho \left(\delta_{st} z^{'}_m z^{'}_m - z^{'}_s z^{'}_t\right)\omega^{'}_s(S^{'}, S)\omega^{'}_t(S^{'}, S) && (8)
\end{aligned}$$

By **5.335**. we have $I_{st} = \delta_{st}\sum \rho z_m z_m - \sum \rho z_s z_t$ and so (8) can be written as

$$T = \frac{1}{2}I_{st}\omega^{'}_s(S^{'}, S)\omega^{'}_t(S^{'}, S) \tag{9}$$

So we can choose the three angles $\Omega_s^{'}(S^{'},S)$ with $(\omega_s^{'}(S^{'},S) = \frac{d\Omega_s^{'}(S^{'},S)}{dt})$ as generalized coordinates and define

$$ds^2 = I_{st} d\Omega_s^{'}(S^{'},S) d\Omega_t^{'}(S^{'},S)$$

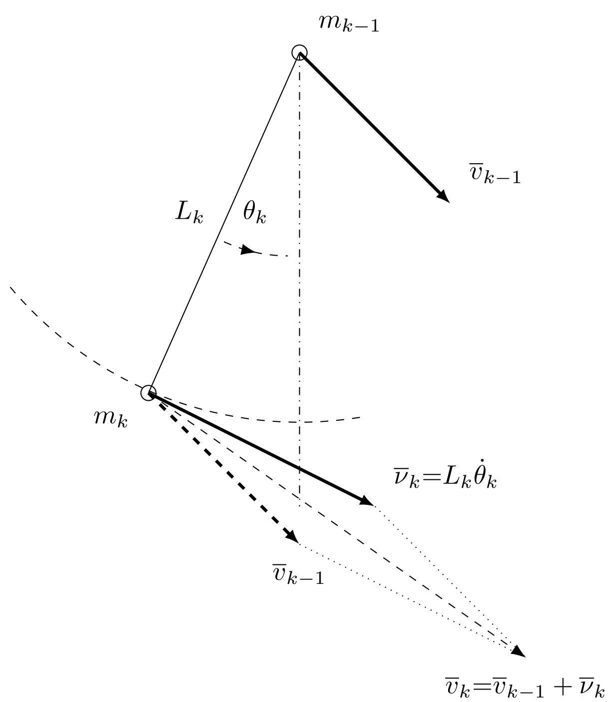
with

$$a_{mn} = I_{mn}$$

having constants as elements. Some check on consistency of the metric tensor defined by (14):
Positive definite ? : Yes, as T is positive by construction.
Symmetric ? : Yes, as $a_{mn} = I_{km}$ and I_{km} is symmetric.

(c) **a chain of six rods smoothly hinged together, with one end fixed and all moving on a smooth plane** $(N = 6)$

To simplify the notation we will assume that the mass m_k of each rod (with length L_k) is concentrated at it's endpoint .
First we note that the velocity of a rod is composed of two vectors, one (labelled as $\overline{\nu}_k$) generated by its own rotation relative to the previous rod and the other (labelled as $\overline{v}_{k-1}$) generated by the velocity of the endpoint of the rod to which it is attached (see.fig. 5.2).



(a)

Figure 5.2: Composition of absolute and relative velocities of a chain of rods

If we take Cartesian coordinates it is easy to see that rod (1) will have components

$$\left(L_1\dot{\theta}_1\cos\theta, L_1\dot{\theta}_1\sin\theta_1\right)$$

rod (2)

$$\left(L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2, L_1\dot{\theta}_1\sin\theta_1 + L_2\dot{\theta}_2\sin\theta_2\right)$$

$$\vdots$$

rod (k)

$$\left(\sum_{i=1}^{k} L_i\dot{\theta}_i\cos\theta_i, \sum_{i=1}^{k} L_i\dot{\theta}_i\sin\theta_i\right)$$

and so

$$\left(v^{(k)}\right)^2 = \left(\sum_{i=1}^{k} L_i\dot{\theta}_i\cos\theta_i\right)^2 + \left(\sum_{i=1}^{k} L_i\dot{\theta}_i\sin\theta_i\right)^2 \tag{10}$$

$$= \sum_{i=1}^{k}\left(L_i\dot{\theta}_i\right)^2 + 2\sum_{i=1}^{k}\sum_{j=1}^{k-i}\left(L_iL_{i+j}\dot{\theta}_i\dot{\theta}_{i+j}\cos\left(\theta_i - \theta_{i+j}\right)\right) \tag{11}$$

So the kinetic energy of one rod and the total kinetic energy of the system are

$$T^{(k)} = \frac{1}{2}m_k\left[\sum_{i=1}^{k}\left(L_i\dot{\theta}_i\right)^2 + 2\sum_{i=1}^{k}\sum_{j=1}^{k-i}\left(L_iL_{i+j}\dot{\theta}_i\dot{\theta}_{i+j}\cos\left(\theta_i - \theta_{i+j}\right)\right)\right] \tag{12}$$

$$T = \sum_{k=1}^{N} T^{(k)} \tag{13}$$

For $N = 6$ we get

rod	$T^{(k)}$
1	$\frac{1}{2}m_1\left[\left(L_1\dot{\theta}_1\right)^2\right]$
2	$\frac{1}{2}m_2\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right)\right]$
3	$\frac{1}{2}m_3\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + 2L_1L_3\dot{\theta}_1\dot{\theta}_3\cos\left(\theta_1 - \theta_3\right) + \ldots\right]$
4	$\frac{1}{2}m_4\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + 2L_1L_3\dot{\theta}_1\dot{\theta}_3\cos\left(\theta_1 - \theta_3\right) + \ldots\right]$
5	$\frac{1}{2}m_5\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + \left(L_5\dot{\theta}_5\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + \ldots\right]$
6	$\frac{1}{2}m_6\left[\left(L_1\dot{\theta}_1\right)^2 + \left(L_2\dot{\theta}_2\right)^2 + \left(L_3\dot{\theta}_3\right)^2 + \left(L_4\dot{\theta}_4\right)^2 + \left(L_5\dot{\theta}_5\right)^2 + \left(L_6\dot{\theta}_6\right)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + \ldots\right]$

Giving for T

$$2T = \begin{cases} (m_1+m_2+m_3+m_4+m_5+m_6)\left(L_1\dot{\theta}_1\right)^2 \\ +(m_2+m_3+m_4+m_5+m_6)\left(L_2\dot{\theta}_2\right)^2 \\ +(m_3+m_4+m_5+m_6)\left(L_3\dot{\theta}_3\right)^2 \\ +(m_4+m_5+m_6)\left(L_4\dot{\theta}_4\right)^2 \\ +(m_5+m_6)\left(L_5\dot{\theta}_5\right)^2 \\ +(m_6)\left(L_6\dot{\theta}_6\right)^2 \\ +2(m_2+m_3+m_4+m_5+m_6)L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \\ +2(m_3+m_4+m_5+m_6)L_1L_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) \\ +2(m_3+m_4+m_5+m_6)L_2L_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3) \\ +2(m_4+m_5+m_6)L_1L_4\dot{\theta}_1\dot{\theta}_4\cos(\theta_1-\theta_4) \\ +2(m_4+m_5+m_6)L_2L_4\dot{\theta}_2\dot{\theta}_4\cos(\theta_2-\theta_4) \\ +2(m_4+m_5+m_6)L_3L_4\dot{\theta}_3\dot{\theta}_4\cos(\theta_3-\theta_4) \\ +2(m_5+m_6)L_1L_5\dot{\theta}_1\dot{\theta}_5\cos(\theta_1-\theta_5) \\ +2(m_5+m_6)L_2L_5\dot{\theta}_2\dot{\theta}_5\cos(\theta_2-\theta_5) \\ +2(m_5+m_6)L_3L_5\dot{\theta}_3\dot{\theta}_5\cos(\theta_3-\theta_5) \\ +2(m_5+m_6)L_4L_5\dot{\theta}_4\dot{\theta}_5\cos(\theta_4-\theta_5) \\ +2(m_6)L_1L_6\dot{\theta}_1\dot{\theta}_6\cos(\theta_1-\theta_6) \\ +2(m_6)L_2L_6\dot{\theta}_2\dot{\theta}_6\cos(\theta_2-\theta_6) \\ +2(m_6)L_3L_6\dot{\theta}_3\dot{\theta}_6\cos(\theta_3-\theta_6) \\ +2(m_6)L_4L_6\dot{\theta}_4\dot{\theta}_6\cos(\theta_4-\theta_6) \\ +2(m_6)L_5L_6\dot{\theta}_5\dot{\theta}_6\cos(\theta_5-\theta_6) \end{cases} \tag{14}$$

We define as general coordinates the angles θ^i and express ds^2 as

$$ds^2 = 2Tdt^2$$

and see that ds^2 is of the required form

$$ds^2 = a_{mn}d\theta^m d\theta^n$$

The metric tensor a_{mn} contains elements depending on the θ_k chosen as general coordinates of the system and is a good candidate as metric tensor. Some check on consistency of the metric tensor defined by (8):

Positive definite ? : Yes, as T is positive by definition

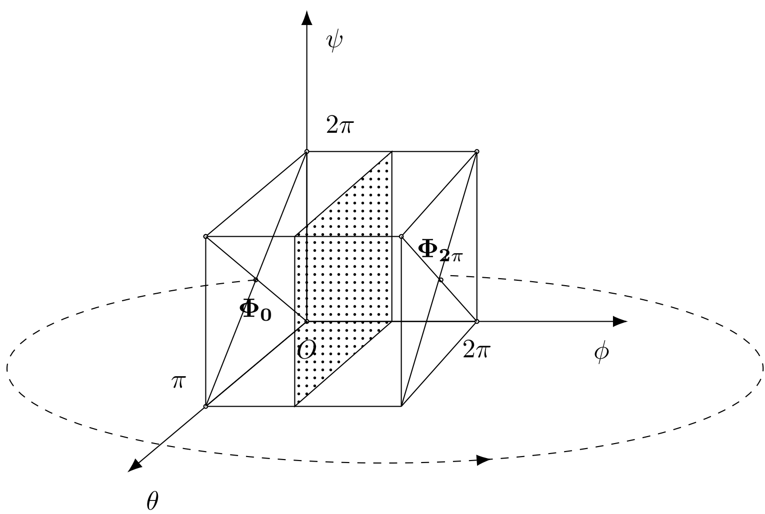
Symmetric ? : Yes, as the non-diagonal term a_{ij} contains $\cos(\theta_i - \theta_j) = \cos(\theta_j - \theta_i)$

Number of elements : the metric tensor a_{mn} for $N = 6$ should contain 6 diagonal elements and $\frac{6\times 6-6}{2} = 15$ independent non-diagonal elements. Checking (8), one can find that the numbers yield.

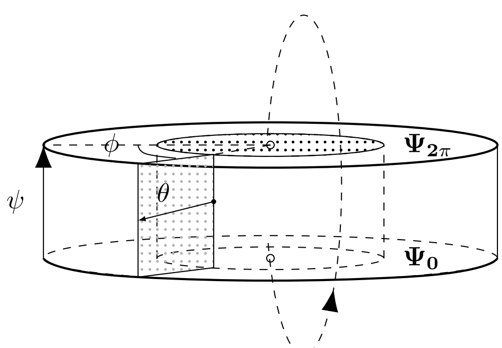
♦

5.13 p181 and p182 - Clarification Figures 13., 14. and 15.

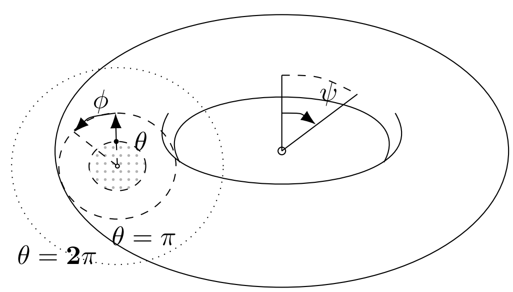
There are several ways to perform a map of the configuration space of a rigid body with fixed point.



(a)



(b)



(c)

Figure 5.3: Map of the configuration space of a rigid body with fixed point.

Consider figure 5.2(a). We can stretch like an accordion the cuboid along the ϕ axis and bent it so that the planes $\phi = 0$ and $\phi = 2\pi$ join. We get (b), a torus with square sections. The dimension ϕ is dealt with as a point $P(\theta, \phi, \psi)$ in the configuration space returns to the same point when varying ϕ to $\phi + 2k\pi$.

We can apply the same procedure of stretching and bending for the ψ dimension so that the planes $\Psi = 0$ and $\Psi = 2\pi$ join. We get (c), a torus-like object.

The only dimension left is θ which our multi-dimensional crippled mind can't find a way to reshape this pseudo-torus so that when varying θ we can come back to the same point as started.

♦

5.14 p186 - Exercise 1

If a vector at the point with coordinates $(1, 1, 1)$ in Euclidean 3-space has components $(3, -1, 2)$, find the contravariant, covariant and physical components in spherical polar coordinates.

The tensor T_n to consider is $(3, -1, 2) - (1, 1, 1) = (2, -2, 1)$.
The Jacobian matrix for the transformation $z^n \to x^k$, evaluated at the point $(1, 1, 1)$ is

$$J_{(1,1,1)} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{2}$$

$$\Rightarrow \quad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'^n} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \tag{4}$$

We have the metric tensor evaluated at $(1,1,1)$

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{5}$$

$$\Rightarrow \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T_n'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{3} \\ -2 \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \tag{7}$$

And the physical components

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T_{ph.}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{2} \\ -4 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \tag{9}$$

Another way to find the physical components is to project orthogonally the tensor on the unit vectors of a local Cartesian coordinate system, oriented along the unit vectors $\overline{e}_r, \overline{e}_\theta, \overline{e}_\phi$ corresponding to the vector $P\,(1,1,1)$ with modulus $|P| = \sqrt{3}$. We have for the tensor $T_n(2,-2,1)$ with modulus $|T_n| = 3$ as component along $\overline{e}_r$:

$$|T_n|\cos\alpha = |T_n|\,\frac{\langle T_n, P\rangle}{|T_n|\,|P|} \tag{10}$$

$$= |T_n|\,\frac{2-2+1}{|T_n|\,|P|} \tag{11}$$

$$= \frac{1}{\sqrt{3}} \tag{12}$$

For the component along $\overline{e}_\theta$ we first have to determine the vector $\overline{e}_\theta$. As first equation we have the orthogonality condition with $\overline{e}_r$ and putting $\overline{e}_\theta = (a, b, c)$, get $\langle \overline{e}_r, \overline{e}_\theta \rangle = a + b + c = 0$. As $\overline{e}_\theta$ lies in the plane $(1, 1, 0) - (0, 0, 0) - (0, 0, 1)$ we can put $a = b$ and get $\overline{e}_\theta = \frac{1}{\sqrt{6}}(1, 1, -2)$ and get for the tensor $T_n(2, -2, 1)$ as component along $\overline{e}_\theta$:

$$
\begin{aligned}
|T_n| \cos\beta &= |T_n| \frac{\langle T_n, \overline{e}_\theta \rangle}{|T_n|} && (13)\\
&= |T_n| \frac{2 - 2 - 2}{|T_n| \sqrt{6}} && (14)\\
&= -\frac{\sqrt{2}}{\sqrt{3}} && (15)
\end{aligned}
$$

For the component along $\overline{e}_\phi$ we first have to determine the vector $\overline{e}_\phi$. As first equation we have the orthogonality condition with the pair $\overline{e}_r, \overline{e}_\theta$ and get $\overline{e}_\phi = \overline{e}_r \times \overline{e}_\theta = \frac{1}{\sqrt{3}\sqrt{6}}(-3, 3, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$. For the tensor $T_n(2, -2, 1)$ as component along $\overline{e}_\phi$:

$$
\begin{aligned}
|T_n| \cos\gamma &= |T_n| \frac{\langle T_n, \overline{e}_\phi \rangle}{|T_n|} && (16)\\
&= |T_n| \frac{-2 - 2}{|T_n| \sqrt{2}} && (17)\\
&= -\frac{4}{\sqrt{2}} && (18)\\
&= -2\sqrt{2} && (19)
\end{aligned}
$$

giving

$$
\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}_{T'_{ph.}} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \\ -2\sqrt{2} \end{pmatrix} \tag{20}
$$

as in (9).

♦

5.15 p186 - Exercise 2

In cylindrical coordinates (r, ϕ, z) in Euclidean 3-space, a vector field is such that the vector at each point points along the parametric line of ϕ, in the sense of ϕ increasing, and its magnitude is kr, where k is a constant. Find the contravariant, covariant and physical components of this vector field.

We can work backwards, with the physical components as starting point. Indeed, at a point $P\,(r, \phi, z)$ the tensor of this vector field will have $(0, kr, 0)$ as physical components in the cylindrical coordinates (r, ϕ, z) system.
We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Giving

$$\begin{cases} X_1 = & h_1 X_1^{phys.} & = 0 \\ X_2 = & h_2 X_2^{phys.} & = kr^2 \\ X_3 = & h_3 X_3^{phys.} & = 0 \end{cases} \tag{2}$$

and

$$\begin{cases} X^1 = & \frac{X_1^{phys.}}{h_1} & = 0 \\ X^2 = & \frac{X_2^{phys.}}{h_2} & = k \\ X^3 = & \frac{X_3^{phys.}}{h_3} & = 0 \end{cases} \tag{3}$$

♦

5.16 p186 - Exercise 3

Find the physical components of velocity and acceleration along the parametric lines of cylindrical coordinates in terms of the and their derivatives with respect to time.

We have the metric tensor

$$a_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

and the contravariant velocities

$$\begin{cases} v^1 = & \frac{dr}{dt} \\ v^2 = & \frac{d\phi}{dt} \\ v^3 = & \frac{dz}{dt} \end{cases} \tag{2}$$

giving by $v_K^{phys.} = h_K v^K$

$$\begin{cases} v_r = & \frac{dr}{dt} \\ v_\phi = & r\frac{d\phi}{dt} \\ v_z = & \frac{dz}{dt} \end{cases} \tag{3}$$

For the acceleration using $f^r = \frac{\delta v^r}{\delta t}$ and the Christoffel symbols being

$$\begin{cases} \Gamma^m_{nk} = 0 \quad \forall \quad (nk) \neq (r,\theta),(\theta,\theta) \\ \Gamma^\theta_{r\theta} = \frac{1}{r} \quad \text{and} \quad \Gamma^r_{\theta\theta} = -r \end{cases} \tag{4}$$

we have

$$\left\{\begin{array}{ll} f^1 = & \frac{dv^1}{dt} - r\underbrace{v^2\frac{dx^2}{dt}}_{=(v^2)^2} \\ f^2 = & \frac{dv^2}{dt} + \underbrace{\frac{1}{r}v^1\frac{dx^2}{dt} + \frac{1}{r}v^2\frac{dx^2 1}{dt}}_{=\frac{2}{r}v^1v^2} \\ f^3 = & \frac{dv^3}{dt} \end{array}\right. \tag{5}$$

giving by $f_K^{phys.} = h_K f^K$

$$\left\{\begin{array}{ll} f_r = & \frac{dv^1}{dt} - r\left(v^2\right)^2 \\ f_{phi} = & r\frac{dv^2}{dt} + r\frac{2}{r}v^1v^2 \\ f_z = & \frac{dv^3}{dt} \end{array}\right. \tag{6}$$

$$\Rightarrow \left\{\begin{array}{ll} f_r = & \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \\ f_{phi} = & r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} \\ f_z = & \frac{d^2z}{dt^2} \end{array}\right. \tag{7}$$

♦

5.17 p186 - Exercise 4

A particle moves on a sphere under the action of gravity. Find the contravariant an covaraiant components of the force, using colatitude and azimuth, and write down the equation of motion.

We determine first the physical components of the force.

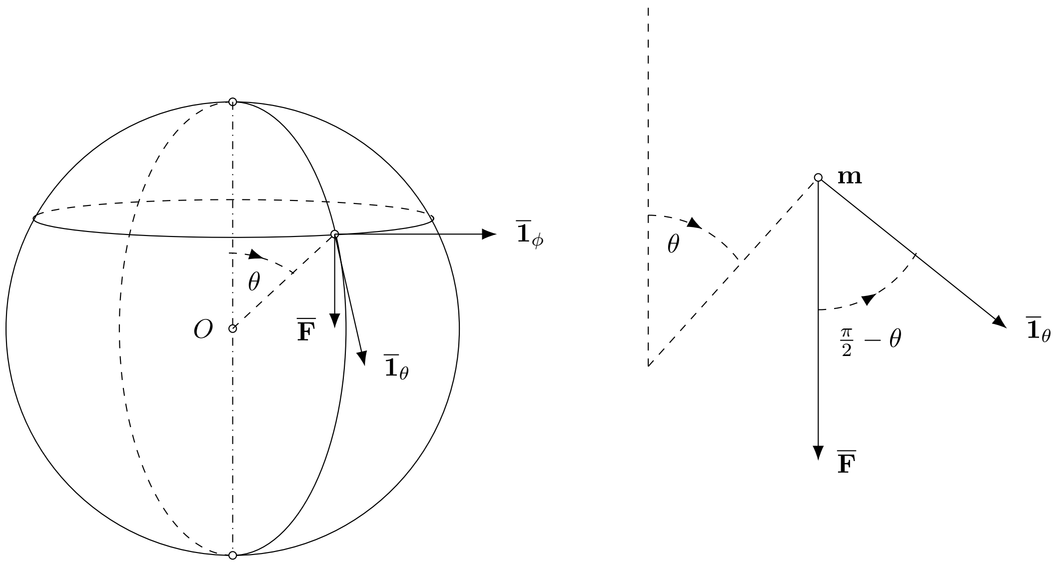


Figure 5.4: Physical components of the gravitational force tensor acting on a mass **m** on a sphere

We note first that the unit vector $\overline{1}_\phi$ is perpendicular to the place formed by the vectors $\overline{1}_\theta, \overline{F}$ and s the force has no components projected on this vector. The vector $\overline{F}$ is parallel with the axis of reference of the sphere with radius R and so the physical components become

$$\begin{cases} F_\phi^{phys} = 0 \\ F_\theta^{phys} = mg\sin\theta \end{cases} \tag{1}$$

$$\Rightarrow \quad \begin{cases} F_\phi = 0 & F_\phi = 0 \\ F^\theta = \frac{1}{R} mg\sin\theta & F_\theta = Rmg\sin\theta \end{cases} \tag{2}$$

We use equation **5.212**.

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ T = \frac{1}{2} m a_{pq} \dot{x}^p \dot{x}^q, \ \dot{x}^s = \frac{dx^s}{dt} \end{cases} \tag{3}$$

with for our case

$$T = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \sin^2\theta \; \dot{\phi}^2 \right) \tag{4}$$

and get the set of equation of motion (the second column gives the dimensional analysis as a check for consistency)

$$\begin{cases} \frac{\ddot{\phi}}{\dot{\phi}} = -2\cot\theta \; \dot{\theta} & : \quad \frac{[T]^{-2}}{[T]^{-1}} \cong [T]^{-1} \\ \\ \ddot{\theta} - \left(\dot{\phi}\right)^2 \sin\theta\cos\theta = \frac{g}{R}\sin\theta & : \quad [T]^{-2} + \left([T]^{-1}\right)^2 \cong \frac{[L][T]^{-2}}{[L]} \\ \end{cases} \tag{5}$$

Let's check the special case when $\dot{\phi} = 0$.
The first equation can be rewritten and gives of course $\phi = C$ while the second equation becomes

$$\ddot{\theta} = \frac{g}{R}\sin\theta$$

which is similar to the equation of the simple gravity pendulum.

♦

5.18 p186 - Exercise 5

Consider the motion of a particle on a smooth torus under no forces except normal reaction. The geometrical line element may be written

$$ds^2 = (a - b\cos\theta)^2 \, d\phi^2 + b^2 d\theta^2$$

where ϕ is an azimuthal angle and θ an angular displacement from the equatorial plane. Show that the path of a particle satisfies the following two differential equations in which h is a constant

$$(a) \qquad (a - b\cos\theta)^2 \frac{d\phi}{ds} = h$$

$$(b) \qquad b^2 \left(\frac{d\theta}{d\phi}\right)^2 = \frac{(a - b\cos\theta)^4}{h^2} - (a - b\cos\theta)^2$$

We use equation **5.212**.

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^s} - \frac{\partial T}{\partial x^s} = F_s \\ \\ T = \frac{1}{2} m a_{pq} \dot{x}^p \dot{x}^q, \ \dot{x}^s = \frac{dx^s}{dt} \end{cases} \tag{1}$$

with for our case

$$T = \frac{1}{2} m \left(b^2 \dot{\theta}^2 + (a - b\cos\theta)^2 \ \dot{\phi}^2 \right) \tag{2}$$

giving

$$\begin{cases} \frac{\partial T}{\partial \dot{\phi}} = m (a - b\cos\theta)^2 \dot{\phi} & \frac{\partial T}{\partial \phi} = 0 \\ \\ \frac{\partial T}{\partial \dot{\theta}} = m b^2 \dot{\theta} & \frac{\partial T}{\partial \theta} = m b (a - b\cos\theta) \dot{\phi}^2 \sin\theta \end{cases} \tag{3}$$

$$\Rightarrow \qquad \begin{cases} (a - b\cos\theta)^2 \ddot{\phi} + 2b (a - b\cos\theta) \dot{\theta}\dot{\phi} \sin\theta = 0 \\ \\ b^2 \ddot{\theta} - b (a - b\cos\theta) \dot{\phi}^2 \sin\theta = 0 \end{cases} \tag{4}$$

$$\Rightarrow \qquad \begin{cases} (a - b\cos\theta) \ddot{\phi} = -2b\dot{\theta}\dot{\phi} \sin\theta \\ \\ b^2 \ddot{\theta} - b (a - b\cos\theta) \dot{\phi}^2 \sin\theta = 0 \end{cases} \tag{5}$$

In the first equation, put $y \equiv \dot{\phi}$ giving for the first equation:

$$\begin{aligned}
& & \frac{dy}{y} = -2b\frac{\sin\theta d\theta}{(a - b\cos\theta)} & \quad (6) \\
&\Leftrightarrow & \frac{dy}{y} = -2\frac{d\,(a - b\cos\theta)}{(a - b\cos\theta)} & \quad (7) \\
&\Rightarrow & \log y = -2\log\left(a - b\cos\theta\right) + \log C & \quad (8) \\
&\Rightarrow & \dot{\phi} = C\left(a - b\cos\theta\right)^{-2} & \quad (9)
\end{aligned}$$

Note that $\dot{\phi}$ is a time derivative. But as we are on a geodesic, **5.226**. stands and so v is constant as $\frac{dv}{ds} = 0$. Using $v = \frac{ds}{dt}$, (9) can be written as

$$\begin{aligned}
& & \left(a - b\cos\theta\right)^2 \frac{d\phi}{dt} = C & \quad (10) \\
&\Leftrightarrow & \left(a - b\cos\theta\right)^2 \frac{d\phi}{ds}\underbrace{\frac{ds}{dt}}_{=v} = C & \quad (11) \\
&\Leftrightarrow & \left(a - b\cos\theta\right)^2 \frac{d\phi}{ds} = h \quad \text{with } h = \frac{C}{v} & \quad (12)
\end{aligned}$$

We don't use the second equation in (5) but the line element equation instead

$$\begin{aligned}
& & ds^2 = \left(a - b\cos\theta\right)^2 d\phi^2 + b^2 d\theta^2 & \quad (13) \\
&\Rightarrow & \left(\frac{ds}{d\phi}\right)^2 = \left(a - b\cos\theta\right)^2 + b^2\left(\frac{d\theta}{d\phi}\right)^2 & \quad (14) \\
&\Rightarrow & b^2\left(\frac{d\theta}{d\phi}\right)^2 = \left(\frac{d\phi}{ds}\right)^{-2} - \left(a - b\cos\theta\right)^2 & \quad (15) \\
(12) \quad &: & b^2\left(\frac{d\theta}{d\phi}\right)^2 = \frac{\left(a - b\cos\theta\right)^4}{h^2} - \left(a - b\cos\theta\right)^2 & \quad (16)
\end{aligned}$$

♦