

**Solution**

a) If there are more clubs than students, then no such set  $T$  exists because no individual person can be a treasurer for more than one group by the pigeonhole principle.

First make a bipartite graph  $G = (S, Y, E)$  where  $S = \{x_1, \dots, x_k\}$  is the set of students part of a club,  $Y$  is a collection of subsets of  $S$  denoting clubs and where elements of  $Y_i$  are students that are members of the club, and  $E = \{(x, Y_i) : x \in Y_i\}$ , meaning that an edge is made from a student in  $S$  to a club in  $Y$  if the student is part of the club. A successful set of treasurers corresponds to a set  $T = \{s_1, \dots, s_n\} \subseteq S$  such that each  $s_i \in Y_i$  and there exists only one treasurer per group.

To determine whether such a set  $T$  exists, transform  $G$  into a flow network  $G'$  by:

1. Directing all edges of  $G$  from  $S$  to  $Y$ .
2. Adding a vertex  $s$  with edges to all  $x \in S$ , and a vertex  $t$  with edges from all  $y \in Y$  to  $t$ .
3. Giving every edge in  $G'$  capacity 1.

Claim: The  $n$  clubs can each get a treasurer if and only if  $G'$  has a flow of value  $n$ .

*Proof.*  $\Rightarrow$  Assume the set of treasurers exists and let  $A$  be the set of edges representing the assignment of students to the club they are treasuring. Let  $f$  be a flow on  $G'$  with all edges having flow 0. Augment  $f$  as follows: For each edge  $\{x, y\} \in A$ , add 1 to the flow of the directed edges  $(s, x)$ ,  $(x, y)$ , and  $(y, t)$  in  $G'$ . Note that doing so preserves flow conservation, and increases the value of the flow  $f$  by 1. At the end of this process, no edge has flow exceeding capacity since no edges ever receive more than 1 unit of flow. Moreover, a student in  $S$  receiving a flow of 1 must not treasure another club because of flow conservation. Also, a student being treasurer means that they have an edge in  $A$ . The value of this flow is  $n$  since the set of treasurers exists so  $|A| = n$ , meaning that every club received a unit of flow. Thus, all edges from  $Y$  to  $t$  have 1 unit of flow.

$\Leftarrow$  Assume there is an integer flow  $f$  of value  $n$ . Then each edge  $(y, t)$  must have 1 unit of flow, and so each club  $y$  must have an incoming edge (from some  $x \in S$  with 1 unit of flow). Each  $x \in S$  can have at most 1 flow leaving it since edges  $(s, x)$  have a capacity of 1. Thus, the flow of  $n$  from clubs to the sink correspond to  $n$  students being treasurers of said club.  $\square$

Time Complexity:  $O(|S||Y|)$

b) The bipartite graph  $G$  we created might not always have a perfect matching. A certificate we can use to verify that there does not exist a perfect matching is one similar to Hall's Theorem, that is that for some  $X \subseteq S$ ,  $|\Gamma(X)| < |X|$ . Assume there is a sub-collection  $Y_{i_1}, \dots, Y_{i_k}$  such that  $|Y_{i_1}, \dots, Y_{i_k}| < k$ . This means that in this sub-collection of groups, there are less members than groups. In other words, some students in  $S$  do not have sufficient edges to  $Y$ . In fact, this means that a perfect matching cannot be made from  $S$  to  $Y$ . Conversely, if no such a set  $T \subseteq S$  exists, then some students do not have sufficient edges to a set of groups. Meaning that some groups are short of students.