(k,p)-Planarity Extensions

David Ariyibi, William Lenhart

Initial Questions:

1) Suppose input is $G = (V, E)$ along with partition $\chi = \{X_1, ..., X_L\}$ of V where each X_i has $|X_i| \leq 2$. Is there a $(2, 2)$-planar drawing of G with χ as the set of clusters?

   – Initial Thoughts: Similar proof to Tim's proof of hardness of $(k, 1)$-planarity hardness?

2) Think about the relationship between $(2, 2)$-planar drawings and vertex splitting.

   – Initial Thoughts: All clusters can be replaced with maximal clusters, so we only consider those. $(2, 2)$ clusters with non-alternating ports are reducible to an edge if the underlying vertices are adjacent or nothing if vertices are not, so we only consider clusters with alternating ports. Maximal $(2, 2)$ clusters with alternating ports are representations of a vertex split.

   – We also know that deciding $(2, 2)$-planarity is NP-complete [16]. Is there a way to re-prove this using vertex splitting?

     * Eligible Set Split Planar Graph – NP-Complete [15]

       · Given a graph $G = (V, E)$, a subset of vertices $S \subseteq V$, and a positive integer $K \leq |S|$, can G be transformed into a planar graph G_0 by K or less vertex splitting operations that involve only vertices in S? The vertices in S are called eligible vertices.

     * Split Planar Graph – NP-Complete [15]

       · Given a graph $G = (V, E)$ and a positive integer $K < |E|$, can G be transformed into a planar graph G_0 by K or less vertex splitting operations?

     * Maximum Planar Subgraph – NP-Complete [15]

       · Given a graph $G = (V, E)$ and a positive integer $K \leq |E|$, is there a subset $E_0 \subseteq E$ with $|E_0| \geq K$ such that the graph $G_0 = (V, E_0)$ is planar?

3) Imagine a version of $(2, 2)$-planarity where if a cluster $\{u, v\}$ is an edge of G then draw cluster as alternating; otherwise draw as non-alternating.

   – Initial Thoughts: In this model, there's no need to cluster non-adjacent vertices. This reduces to minimal $(2, 2)$-planar drawing.

Definitions:

- A (k, p) **cluster** is a cluster in a port drawing of a graph utilizing k vertices and at most p ports per vertex.
- In a drawing Γ, a (k, p) cluster is **irreducible** if it can't be replaced by a cluster of smaller k or p or a cluster with less ports in Γ. If it can be replaced, then the cluster is **reducible**. The action of replacing a reducible (k, p) cluster with a cluster of smaller k or p or a cluster with less ports is denoted as **reducing** the original cluster.
- A (k, p)-planar drawing of a graph is **minimal** if every cluster in the drawing is irreducible.
- A (k, p) cluster is **full** if it utilizes all $k \cdot p$ ports.
- A (k, p) cluster is **maximal** if it is irreducible and full.
- A graph is G **irreducibly** (k, p)**-planar** if there does not exist a drawing Γ of G such that Γ is $(k-1, p)$-planar or $(k, p-1)$-planar e.g. every (k, p)-planar drawing of G includes an irreducible (k, p) cluster.
- A graph G is **strongly** (k, p)**-planar** if there exists a (k, p)-planar drawing Γ of G such that the intracluster edges of Γ are planar. If such a drawing does not exist, G is **weakly** (k, p)**-planar**.

- An n-planar graph is IC n-planar if no two crossing "sections" share a vertex. A section is the crossing pairs included in the edge that crosses k other edges. ("section" is just a placeholder for better name)

Week 2:
Notes:

- Minimal (k, p)-planar graph drawings
  - We know that for a graph G to be (k, p)-planar, it must have a (k, p)-planar drawing Γ. We know that any (k, p)-planar drawing can be made minimal by reducing the clusters until they are irreducible or vertices. Thus finding a minimal (k, p)-planar drawing of a graph is of the same difficulty as finding a (k, p)-planar drawing.
- Minimal $(2, 2)$-planar graph drawings
  - All clusters have alternating ports. If one didn't, it could be reduced to an edge if the vertices are adjacent or nothing if the vertices aren't adjacent.
  - Cluster between adjacent vertices translates to 1 of the vertices being split
  - Every port in a cluster must be incident to an intercluster edge. If one wasn't, then it could be removed and the cluster reduced.
  - Since finding a $(2, 2)$-planar drawing of a graph is NP-hard, we know that is of same difficulty to find a minimal $(2, 2)$-planar drawing.
- How many vertices do I have to split until graph is planar? When vertices can only be split once?
  - vertices can only be split once because they can only be in one cluster
  - Splitting Number is NP-hard. [15, 11]
  - Splitting a vertex only once cannot work for graphs with too many edges. For example, K_{13} [10]
  - 1-splitting number is defined as the splitting number of a graph restricted such that one can only split vertices of the original graph (i.e. can only split a vertex once). Not all graphs have a 1-splitting number.
  - For the graphs with 1-splitting numbers, what graphs are they?

Ideas:

- Try and map how many times a vertex must be split to the type of cluster it could be a part of, thus the (k, p)-planarity of the graph. Example, vertex v_i needs to be split twice in order for graph to be planar, so the minimal (k, p)-planar drawing need to have a cluster where

Week 3:
Notes:

- Minimal $(2, 2)$-planar drawings and vertex splitting
  - Can split up to half of the vertices. In particular, at most $\left\lfloor \frac{|V|}{2} \right\rfloor$
  - A $(2, 2)$ cluster is analogous to splitting a vertex u split "over" a vertex v. The restriction is that u is now split into two vertices u_1 and u_2, where u_1 and u_2 are in adjacent faces, the faces being connected by v.
  - How does this relate to regular vertex splitting? Is this even a restriction at all?
  - For general vertex splitting, if a vertex is split and the two remaining vertices are in the same face, they can be brought back together. Therefore, any non-trivial vertex split will be split into two different faces in the final planar graph. However, can a vertex be split into
  - for a 1-planar graph, if a vertex u is split into two adjacent faces, connected by a vertex v, then v can't be split because....

- If given partition of graph G with parts at most size 2, is $(2, 2)$-planarity still NP-hard?
  - Don't consider parts of size less than 2
- Crossing number reduction via vertex splitting [15]
  - Splitting a vertex does not always reduce crossing number
  - If graph $G = (V, E)$ is not crossing-critical, then there exists a proper subgraph of G, $G' = (V', E')$, with the same crossing number as G. If $V' \subsetneq V$, then split a vertex in $V \setminus V'$. The crossing number of G is unchanged.
  - Is it the case that there exists non-crossing-critical graphs such that $V' \subsetneq V$?
- Relating planar split thickness and (k, p)-planar graphs
  - (k, p)-planar $\subset$ p-splittable
  - K_{12} is 2-splittable, but not $(2, 2)$-planar since it needs all vertices to be 2-split [10]
  - Determining 2-splittability is NP-complete (our 1-splitting number) [10]
- Minimal (k, p)-planar drawings
  - For $(2, 2)$-planar drawing, at most $\left\lfloor \frac{|V|}{2} \right\rfloor$ vertices can be 2-split
  - For $(2, 3)$-planar drawing, at most $\left\lfloor \frac{|V|}{2} \right\rfloor$ vertices can be 3-split and at most all vertices can be 2-split
  - For $(2, 4)$-planar drawing, at most $\left\lfloor \frac{|V|}{2} \right\rfloor$ vertices can be 4-split, at most $\left\lfloor \frac{|V|}{2} \right\rfloor$ vertices can be 3-split
  - If a graph is $(2, p)$-planar but not $(2, p-1)$-planar, it must include a "fully alternating" cluster, which is analagous to at most a p-split and least a $\left(\left\lfloor \frac{p}{2} \right\rfloor + 1\right)$-split. All other clusters can be reduced to those of smaller p
  - For $(k, 2)$-planar drawing, at most $\left\lfloor \frac{(k-1)|V|}{k} \right\rfloor$ vertices can be 2-split
- Graphs that are $(2, 2)$-planar
  - IC-planar, AcNIC-planar, TrNIC-planar, however not NIC-planar
  - If 1-planar, a crossing can be connected to another by at most 2 vertices.
  - An NIC-planar graph G is planar if there exists an NIC-planar drawing Γ such that each crossing pair has at least one vertex connected to no other vertices than those in the crossing pair. (Has this already been discovered?)
  - cp-vertex graph is the same as a cp-cut graph with the other vertices of each crossing pair added. There are two types of vertices. cp-vertices and
  - ... if you can form a triangle for each cp-vertex, made up of one cp-vertex and two non cp-vertices such that no two triangles share exactly one vertex. (no two new edges created to form triangles are adjacent)
  - A 1-planar graph G is $(2, 2)$-planar if it admits a 1-planar drawing such that the cp-vertex graph is (above)
  - We can prove that this requirement encompasses TrNIC-planar graphs as well

Ideas:

- Directed (k, p)-planar graphs with restriction that in a cluster some vertices are strictly for incoming edges and some are strictly for outgoing edges.

Week 4:
Notes:

- How big is the class of pseudoforestal 1-planar graphs? (PF 1-planar)
  - Pf1-planar $\subsetneq$ 1-planar. By definition, PF 1-planar $\subset$ 1-planar, but there exists 1-planar graphs that do not permit a pseudoforestal 1-plane drawing, so PF 1-planar $\neq$ 1-planar.
  - TrNIC-planar $\subsetneq$ PF 1-planar. Given a TrNIC-planar drawing Γ of a graph G, we know that by definition, the cp-cut graph of Γ is acyclic. Thus, no partition of the crossing edges in Γ form a cycle. It follows that the ce-graph of Γ is acyclic and is thus pseudoforestal.
  - NIC-planar $\not\subset$ PF 1-planar. PF 1-planar $\subset$ $(2,2)$-planar, and NIC-planar $\not\subset$ $(2,2)$-planar.
  - PF 1-planar $\not\subset$ NIC-planar. PF 1-planar graphs include graphs where two crossing pairs can be connected to each other by 2 vertices, thereby including graphs not NIC-planar by definition.
- IC k-planar graphs
  - We define IC k-planar graphs as subfamilies of their k-planar family where no two crossing "sections" share a vertex. A section is the crossing pairs included in the edge that crosses k other edges. ("section" is just a placeholder for better name)
  - For example, IC 1-planar graphs are IC-planar graphs.

Week 5:
Notes:

- Subsets of PF 1-planarity
  - If cp-cut graph of a graph G is pseudoforestal, then ce graph of G is pseudoforestal. Thus G is PF 1-planar.
  - If all cycles in the cp-cut graph of a graph G have length divisible by 4, then ce graph of G is pseudoforestal. Thus G is PF 1-planar.
- Finding IC k-planar graphs
  - We define a relation on vertices. For a graph $G = (V, E)$, such that $u, v, w \in V$, $u \sim v$ if u and v share an edge or an edge incident to u crosses an edge incident to v.
  - $u \sim u$; $u \sim v$ implies $v \sim u$; $u \sim v$ and $v \sim w$ implies $u \sim w$.
  - "Natural" clusters form based on these relations, with each class forming a cluster $(k, 1)$ cluster such the k is the number of vertices in a class.
  - Intercluster edges are planar edges since all edges that cross another edge are in a cluster.
  - The largest class size , s, provides the $(s, 1)$-planarity. Of all possible graph drawings, the smallest s gives the smallest $(s, 1)$-planarity.
  - Notice that the graph is IC r-planar for some r, since the clusters
  - $r = \lfloor \frac{s-2}{2} \rfloor \cdot \lceil \frac{s-2}{2} \rceil$
- Types of (k, p) clusters
  - A (k, p) cluster is irreducible if it can't be reduced to a cluster of smaller k or p.
  - A (k, p) cluster is maximal if irreducible and it utilizes all $k \cdot p$ ports (all ports have an intercluster edge).
  - A (k, p) cluster is the maximum if maximal and "perfectly" alternating.
  - A graph is considered irreducibly (k, p)-planar if every (k, p)-planar drawing includes an irreducible (k, p) cluster.
- Strong vs Weak (k, p)-planarity

- A graph G is strongly (k, p)-planar if there exists a (k, p)-planar drawing Γ of G such that the intracluster edges can be drawn to be planar. If they can't, the drawing is weakly (k, p)-planar.
- This changes theorems for (k, p)-planarity and p-splittability, because we need for the intracluster edges to be planar for us to say that the vertices of the graph can always be p-split
- For which k and p are strong and weak (k, p)-planarity equivalent?
  $(2, 2), (3, 2), (4, 2), (5, 2), (6, 2)$

Week 6:
Notes:

- Classes of strong (k, p)-planarity
  - Using k-gon triangulation argument, for a (k, p)-planar drawing of a graph, the drawing can only be strongly (k, p)-planar if
$$\binom{k}{2} - k \leq kp - 3$$
$$k \leq \frac{2p + 3 \pm \sqrt{(2p+3)^2 - 24}}{2} \approx 2p + 3$$
  - If looking at edges connecting ports neighboring along perimeter of cluster, there are at least k distinct edges since there are k distinct vertices. Thus equation can be generalized for specific cluster patterns to $k + i$:
$$\binom{k}{2} - (k + i) \leq kp - 3$$
$$k \leq \frac{2p + 3 \pm \sqrt{(2p+3)^2 + 8i + 24}}{2}$$
  - every strongly $(k, 1)$-planar graph is planar
  - for $1 \leq k \leq 3$ and $p \geq 1$, every (k, p)-planar graph is strongly (k, p)-planar
  - for $4 \leq k \leq 6$ and $p \geq 2$, every (k, p)-planar graph is strongly (k, p)-planar
  - every (k, p)-planar graph is strongly $(k, p + k - 1)$-planar
  - In order to show that a (k, p)-planarity is strong, you must show that it is possible for every maximal (k, p) cluster to be strong. This can be done by showing it is possible for every irreducible (k, p) cluster of the smallest size, $p + k - 1$, since they can all be replaced with maximal (k, p) clusters with the same corresponding intracluster edges.
- Restrictions of strong (k, p)-planarity
  - 3 consecutive vertices that appear once can't be made planar (1st needs to be connected to the 3rd, cutting off the 2nd)
  - all vertices that appear once have to be on the same side of all diagonal edges not incident to them
  - any two vertices hat appear only once have to be connected by a diagonal if not consecutive
  - there can only by 3 vertices that can appear only once
- (k, p)-planarizing complete graphs of size n (for bounds on all graphs)
  - $(n-1, 1)$, $(n-2, 2)$, $(n-3, 3)$, ... wlog, any complete graph is $(\frac{n}{2}, \frac{n}{2})$-planar
  - Let $1 \leq a < n$, then $(\max\{a, n-a\}, a)$-planar. thus $(\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil)$-planar
  - Let $a + b < n$ and $a, b > 0$, then $(\max\{a, b, n-a-b\}, \max\{a, b, n-a-b\} + 1)$-planar. thus $(\lceil \frac{n}{3} \rceil, \lceil \frac{n}{3} \rceil + 1)$-planar

- Let $a+b+c<n$ and $a,b,c>0$, then $\left(\left\lceil \frac{n}{4} \right\rceil, \max\left\{\left\lceil \frac{n}{2} \right\rceil + 1, 3\right\}\right)$-planar

Theorems:

- If a graph G is strongly (k,p)-planar, then G is p-splittable.
  - Let G be a strongly (k,p)-planar graph and Γ be a strongly (k,p)-planar drawing of G. Note that for every cluster in Γ, a vertex is split into at most p different vertices, denoted as ports. Since Γ is a strongly (k,p)-planar drawing, both the intra- and intercluster edges are planar. Removing the boundaries of the clusters, the remaining drawing is planar with each vertex in the original graph being at most p-split. Therefore G is p-splittable.
- If a graph G is irreducibly $(2,p)$-planar, then G is k-splittable for some $k \leq \left\lfloor \frac{p}{2} \right\rfloor + 1$.
  - Let G be an irreducible $(2,p)$-planar graph and Γ be an irreducible (k,p)-planar drawing of G.
  - Prove. (split into lemmas about what we know about the underlying vertex splits, and their possible splitting arrangements)
- If a graph G is $(3,p)$-planar and does not utilize the maximum $(3,p)$ cluster, then G is $(p-1)$-splittable.
  - Prove.
- If a graph H is a subgraph of graph G, and G is (k,p)-planar, then H is (k,p)-planar.
  - Since G is (k,p)-planar, let Γ be a (k,p)-drawing of G. We then remove the corresponding edge set $\{e \in E(G) \mid e \notin E(H)\}$ and corresponding vertex set $\{v \in V(G) \mid v \notin V(H)\}$ from Γ. Note Γ is now a (k,p)-planar drawing of H. Thus H is (k,p)-planar.
- (k,p)-planarity is not minor-closed.
  - We prove by contradiction. We first assume that $(k.p)$-planarity is minor-closed on IC-planar graph who's minor is not IC-planar. Therefore, minor is not $(4,1)$-planar, while original graph is.
- (k,p)-planarity is minor-closed for complete graphs.
  - Proof.
- (k,p)-planarity is immersion minor-closed.
  - Proof.
- If a graph G is optimal 1-planar, then G is $(2,2)$-planar.
  - A 1-planar graph G is optimal if and only if G can be obtained from a 3-connected planar quadrangulation then inserting crossing edges inside each face. [14]
  - The optimal 1-planar graph on 8 vertices H is a is a minor of every optimal 1-planar graph.
  - H is not $(2,2)$-planar, therefore no optimal 1-planar graph is $(2,2)$-planar.
- A graph G is IC n-planar if and only if G is $(2n+2,1)$-planar.
  - Prove. (Use Tim's thesis and Properties from [6] as guides)
- If a graph G is IC n-planar, but not IC $(n-1)$-planar, then G is $(k,1)$-planar for some $k \geq n+3$.
  - Prove.
- If a graph G is $(k,1)$-planar, then G is at most r-planar for some $r \leq \left\lfloor \frac{k-2}{2} \right\rfloor \cdot \left\lceil \frac{k-2}{2} \right\rceil$.
  - Prove.
- If a graph G is strongly (k,p)-planar, then $\binom{k}{2} - (k+i) \leq kp - 3$ where $0 \leq i \leq \binom{k}{2} - k$.

[14, 15, 11, 10, 5, 4, 1, 12, 13, 3, 7, 8, 9, 6, 2, 16]

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