Problem

There's another well-known algorithm to compute the diameter of a tree that may surprise you. Here it is

- Pick any vertex v in T
- Find a vertex u that maximizes $dist(v, u)$, using a traversal, for example
- Find a vertex w that maximizes $dist(u, w)$
- Return $dist(u, w)$ as the diameter of T.

Prove that this algorithm is correct. Suggestion: Draw some pictures. Consider the sub-tree T' that consists of the (unique!) paths from v to u and u to w. Imagine some longer path exists—how does it connect to T'?

Solution

Proof. The algorithm finds a vertex u that is farthest away from v, leaves are the only vertices that can be farthest away from another vertice so u and w are leaves. Consequently, any other leaf is not further away from v or from u. Now take a subtree T' that only contains paths from v to u and u to w. If there was a longer path containing u, then our algorithm would have nade that other leaf w. Also, if there was a longer path from v, then our algorithm would have nade that other leaf u. Suppose there was a longer path from u to w. This would mean that there are two paths from u to w, which means that T has a cycle. T is a tree so it cannot have cycles. We have reached a contradiction, so there is no longer path from u to w. □