The coordinate Bethe ansatz for the six-vertex model

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Tiivistelmä

Tässä työssä tarkastellaan Bethe-ansatzia, eli Bethe yritettä, kuuden verteksin mallissa äärellisellä neliöhilalla. Muodostamme siirtymämatriisin V ja osoitamme, että vapaa energia hilan verteksiä kohti saadaan V:n suurimmasta ominaisarvosta. Tarkastelemme V:n alimatriisien antamia ominaisarvoyhtälöitä ja muotoilemme Bethe-ansatzin yleiseen muotoon.

Bethe-ansatz todistetaan mielivaltaisilla painoilla a, b, $c > 0$, jonka jälkeen johdetaan Bethe-yhtälöt. Esitämme lyhyen katsauksen Bethe-juuriin ja käymme läpi raja-arvon jatkumoon anisotropiaparametrin Δ eri arvoilla. Käsittelemme myös tapauksen, jossa yksi Bethen yhtälöiden ratkaisuista on nolla, ja todistamme yritteen tässä singulaarisessa tapauksessa.

Avainsanat Bethe yrite, kuuden verteksin malli, siirtymämatriisi, jään entropia, neliöjää, Bethe yhtälöt

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Abstract

In this work, we study the Bethe ansatz in the six-vertex model on a finite square lattice. Imposing the toroidal boundary conditions we construct the transfer matrix V and show that its maximal eigenvalue gives the per-site energy of the lattice. We consider the eigenvalue equations given by the sub-blocks of V, formulating the Bethe ansatz in a general case.

The proof of the Bethe ansatz is done for arbitrary weights a, b, $c > 0$ followed by the derivation of the Bethe equations. We provide a brief discussion of the Bethe roots and the continuum limit in different regimes of the anisotropy parameter Δ. We also cover the case where one of the solutions to Bethe's equations is zero and prove the ansatz in this singular case.

Keywords Bethe ansatz, the six-vertex model, transfer matrix, Bethe equations, ice-type model, the residual entropy of ice, square ice

Preface

I would like to thank my thesis supervisor, Professor Eveliina Peltola, for her unlimited support, invaluable insights, and encouragement. I also express my sincere gratitude to my thesis advisor, Dr. Augustin Lafay, for his guidance, enthusiasm, and limitless patience in myriad discussions. I am thankful to both for the introduction into the fascinating world of mathematical physics.

Otaniemi, 31.8.2023 Nikolai Argatoff

Contents

1 Introduction

The mathematical models provide fascinating theoretical insights into the complex physical phenomena they aim to imitate. When formulated from a simple enough rule set, certain models become exactly solvable and yield analytical solutions for physical quantities of interest. The six-vertex model not only falls within this class of models but also represents a captivating archetype of two-dimensional lattice models in statistical mechanics.

The origins of the six-vertex model trace back to an unidentified source of residual entropy in ice, an enigma that Pauling phenomenologically explained in 1935. The pursuit of the exact per-site entropy for a lattice of square ice became a precursor to the development of the six-vertex model and a challenging open question. Although the solution had to wait more than three decades, the technique to obtain it was already established by Bethe in 1935. To compute the eigenvectors of an isotropic spin-chain Hamiltonian, Bethe formulated his now-renowned *Bethe ansatz*. In 1967, already known as an effective method in quantum many-body physics, the *Bethe ansatz* further demonstrated its power when Lieb applied it to solve the ice problem via a transfer matrix method.

After 90 years of fluctuating interest, the spotlight on the Bethe ansatz and its applicable models shines brightly. The six-vertex model has been elevated in stature to a "rigorously solvable" model due to the increased interest and investigations of the mathematical physics community. While it remains specialized knowledge, vast amounts of research and literature have been written for beginners and experts alike.

Within this work, we aim to provide an accessible entry point to the Bethe ansatz within the context of the six-vertex model embedded in the toroidal lattice. The synthesis of the treatment in the classic book of Baxter [1] with the proof-oriented work of Duminil-Copin et al. [4] has been another major goal in this bachelor's thesis.

The second chapter gives a concise background on the basic statistical mechanical concepts along with historical developments. In the third chapter, we introduce the six-vertex model, leading us to the construction of the partition function through the powers of the transfer matrix. We gradually build up to the Bethe ansatz in the fourth chapter. Here, we apply the ansatz to specific sub-blocks of the transfer matrix, à la Baxter. The lengthy computations involved are detailed in Appendix B.

The majority of the fifth chapter is dedicated to the rigorous proof of the Bethe ansatz using the techniques of Duminil-Copin et al. In their work, two of the three six-vertex model's parameters are set to unity ($a = b = 1$). Following the authors' suggestion, we have extended their proof to arbitrary values of a, b, c. Although the nature of the task is similar, we benefit from the notation developed in the preceding chapters and provide detailed computations. The end of the fifth chapter derives and gives a brief overview of the *Bethe equations* and the free energy in the continuum limit. In the sixth chapter, again following the steps of Duminil-Copin et al., we prove the Bethe Ansatz in the singular case where Bethe equations permit a null solution. In Appendix A we prove that the transfer matrix satisfies the assumptions of the Perron-Frobenius theorem.

2 Physical foundations

2.1 Basics of statistical mechanics

Statistical mechanics is about providing a mathematical framework to describe how the macroscopic properties of a many-particle system emerge from the mesmerizing dance of given microscopic participants. Following the symphony of this chaotic waltz, dancers restlessly collide, intertwine, and interact with innumerable partners, only then to separate and embrace an ever-new ensemble. Due to the complexity, statistical mechanics does not concern itself with an individual dancer's movements, but instead aims to describe the collective choreography of the whole ballroom.

The macroscopic behavior of large mechanical systems is encoded into a weighted sum over all possible configurations, or *microstates*, of the system, known as *the partition function*

$$Z = \sum_{\varsigma} e^{-\beta E(\varsigma)}, \tag{2.1}$$

where $E(\varsigma)$ signifies the total energy of each microstate ς and $\beta = \frac{1}{k_{\mathrm{B}}T}$ is defined using Boltzmann's constant k_{B} and temperature T, all of which are real parameters.

Summed over quantity is known as the *Boltzmann's weight*. When normalized by Z, Boltzmann's weight gives the probability of the system being in a particular microstate ς denoted as $\mathcal{P}(\varsigma)$. Thus, we have

$$\mathcal{P}(\varsigma) = \frac{e^{-\beta E(\varsigma)}}{Z}. \tag{2.2}$$

From the partition function, we can obtain different important thermodynamic quantities of state. For example, the internal energy of the system U is given by the expected value of the energy

$$U := \frac{1}{Z}\sum_{\varsigma} E(\varsigma) e^{-\beta E(\varsigma)}. \tag{2.3}$$

Instead of having to evaluate another tedious sum, we can show that the internal energy can be easily calculated by manipulating the partition function and a standard application of the chain rule:

$$U = -\frac{\partial \ln Z}{\partial \beta}. \tag{2.4}$$

In a similar fashion, other classical thermodynamic quantities, such as Helmholtz free energy F and Gibbs' entropy S,

$$F := U - TS, \tag{2.5}$$

$$S := -k_{\mathrm{B}} \sum_{\varsigma} \mathcal{P}(\varsigma) \ln\Big(\mathcal{P}(\varsigma)\Big), \tag{2.6}$$

can be derived once the partition function is known:

$$F = -\frac{\ln Z}{\beta}, \tag{2.7}$$

$$S = -\frac{\partial F}{\partial T}. \tag{2.8}$$

Thus, the main problem of statistical mechanics is to track down the partition function by evaluating the sum over all allowed states. This is no easy task, and therefore one has to make some concessions, usually by using clever approximations.

However, an alternative route is to consider a model with a simpler set of rules, which would provide a way to pursue exact descriptions of this simplified version of our system. Such exactly solved models can provide useful information about the system, especially near special temperatures where estimates given by different approximation schemes can fail to deliver accurate results.

2.2 Pauling's ice model

In the late 1920s, physicists and chemists alike puzzled over the discrepancies between entropy changes measured in reactions involving water and their calculated counterparts. At the time, the disagreement between the experiment and the standard theory was attributed to the inaccuracies in contemporary data. However, even increasingly more precise experiments that followed couldn't eradicate the deviations. In 1933, Giauque and Ashley's work, later refined by Giauque and Stout, made it certain that the classical integration formula for entropy does not give the correct result. [8][10]

After graphically integrating over well-known measurements of the heat capacity of ice down to 10 K, they extrapolated the rest using Debye's model [3], and obtained a value that conflicted with the entropy computed from the band spectra of water vapor. Giauque and Ashley found that even at temperatures near absolute zero ice retains a significant amount of so-called residual entropy.

The unknown origin of the residual entropy was explained in 1935 when Linus Pauling proposed a theory for the molecular structure of ice [17]. At the time, the precise arrangement of oxygen atoms in the ice crystal was already known due to the x-ray investigations: each oxygen atom is surrounded tetrahedrally by four other oxygen atoms, each separated by a single hydrogen atom acting as a middleman.

As Pauling put it: "The question now arises as to whether a given hydrogen ion is midway between the two oxygen ions it connects or closer to one than to the other." Pauling proposed assumptions, which can now be summarized into *the ice rule*: each oxygen ion must have two "nearby" hydrogen ions and two "far out" hydrogen ions, as is illustrated in Fig. 1. As we shall see, this postulate is the origin of the six-vertex model.

When approaching 0 K, the ice crystal will not (in a reasonable time period) assume a unique ground state but instead freezes into some of the numerous configurations permitted by the ice rule. In his publication, Pauling calculated an estimate for the number of hydrogenic configurations Ω, thus obtaining the value of entropy $k_{\mathrm{B}} \ln \Omega$. Despite the crude approximations involved in Pauling's estimate, this theoretical value of residual entropy was in excellent agreement with the experimental value of Giauque and Ashley.

Soon after, the search for an exact solution began in hopes of improving the value of Ω, which would further back up the hypothesized structure of ice. The thirty-year chase ended when Lieb in 1967 solved the problem for two-dimensional, *square ice* by

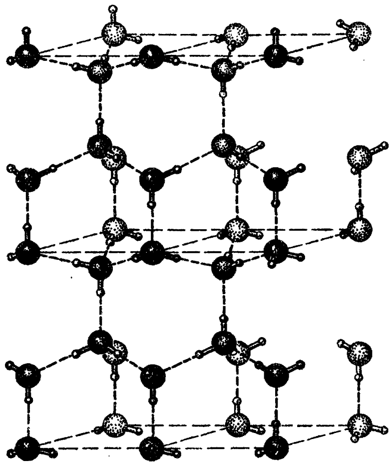
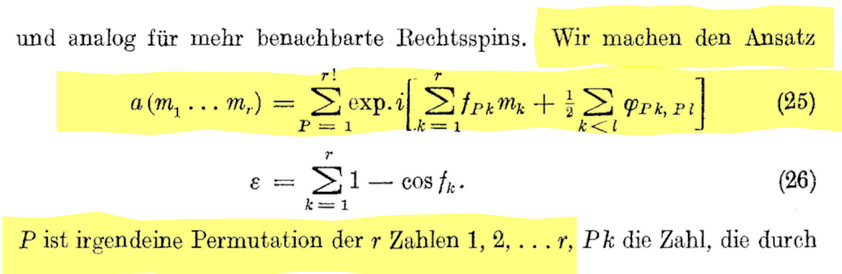


Figure 1: *The arrangement of molecules in the ice crystal.*[18]

evaluating the partition function of the lattice to the eigenvalues of *a transfer matrix* [14][15]. Techniques used by Lieb, such as solving the eigenvalue equation by means of Bethe ansatz, originate from *a priori* unrelated quantum mechanical model.

2.3 Origin of the Bethe ansatz

In 1928, Werner Heisenberg formulated a toy model to study magnetic phenomena through the spin interactions of neighboring atoms [11]. A particular case of the model, one-dimensional isotropic quantum spin chain (the XXX model), was solved by Hans Bethe in 1931 [2]. To obtain the wave function satisfying Schrödinger's equation, Bethe made an ingenious hypothesis for the eigenvector amplitudes of the Hamiltonian and demonstrated its validity. This method of proof by an educated guess is called an ansatz.



und analog für mehr benachbarte Rechtsspins. Wir machen den Ansatz

$$a(m_1 \dots m_r) = \sum_{P=1}^{r!} \exp . i \left[\sum_{k=1}^{r} f_{Pk} m_k + \tfrac{1}{2} \sum_{k<l} \varphi_{Pk,\, Pl} \right] \tag{25}$$

$$\varepsilon = \sum_{k=1}^{r} 1 - \cos f_k . \tag{26}$$

P ist irgendeine Permutation der r Zahlen $1, 2, \dots r$, Pk die Zahl, die durch

Figure 2: In his paper, Bethe writes: "We make the ansatz [...] P is any permutation of the r numbers"

Since then, the formidable-looking formula shown in Fig. 2 has not only found wide application as a powerful analytical tool in theoretical physics but has also become an object of study in its own right. The mathematical inquiry has revealed a rich underlying algebraic structure [7] so that now the original method covered in this work is called *coordinate* Bethe ansatz.

3 Mathematical framework

3.1 The six-vertex model

For referencing the definitions provided in the following sections, the reader is referred to Baxter's classic book [1]. Consider a finite, two-dimensional square lattice with M rows and N columns, where at each vertex we place an oxygen ion. At each of the four adjacent edges, a hydrogen ion is located in such a way that the *ice rule* is satisfied: each oxygen ion has two hydrogen ions nearby, and two hydrogen ions away. Equivalently, we can draw a dipole arrow in the direction of the vertex to which the hydrogen ion is closer.

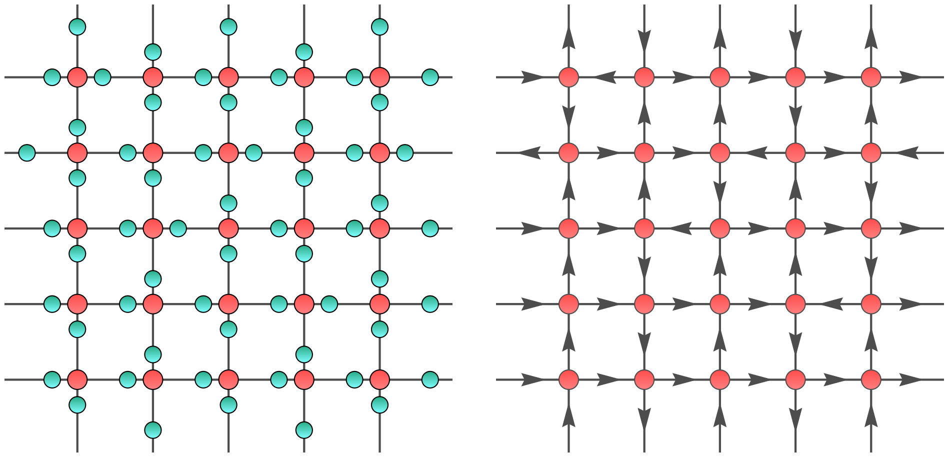


Figure 3: Particular hydrogenic arrangement satisfying the ice rule and their dipole arrows

The six arrow arrangements allowed by the ice rule, after which the model bears its name, are displayed in Fig. 4. Each such jth arrangement is assigned an energy ε_j.

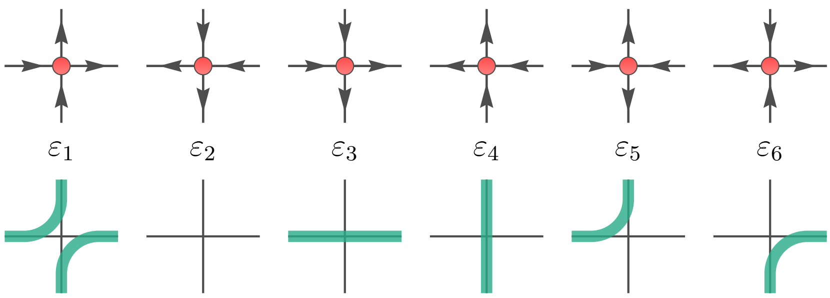


Figure 4: The six arrow configurations and their world line representation

The total energy of the lattice is defined as

$$E(\varsigma) = \sum_{j=1}^{6} \varepsilon_j n_j(\varsigma), \tag{3.1}$$

where ε_j is the energy of jth type of vertex and $n_j(\varsigma)$ is the multiplicity of this vertex in given configuration ς. Then the partition function of M by N lattice is given by

$$Z_{M,N} = \sum_{\varsigma} \prod_{j=1}^{6} e^{-\beta\varepsilon_j n_j(\varsigma)}, \tag{3.2}$$

where the sum is taken over all possible configurations of the ice lattice. We will continue with a model without an external electric field; therefore, upon reversing all dipole arrows, the situation is unchanged. This is equivalent to the restriction

$$\varepsilon_1 = \varepsilon_2, \quad \varepsilon_3 = \varepsilon_4, \quad \varepsilon_5 = \varepsilon_6, \tag{3.3}$$

Using the arrow reversal condition Eq. (3.3) we set for convenience

$$a = e^{-\beta\varepsilon_1} = e^{-\beta\varepsilon_2}, \quad b = e^{-\beta\varepsilon_3} = e^{-\beta\varepsilon_4}, \quad c = e^{-\beta\varepsilon_5} = e^{-\beta\varepsilon_6}. \tag{3.4}$$

Then, Eq. (3.2) becomes

$$\begin{aligned} Z_{M,N} &= \sum_{\varsigma} a^{n(\varsigma)_1 + n(\varsigma)_2} b^{n_3(\varsigma)+n_4(\varsigma)} c^{n_5(\varsigma)+n_6(\varsigma)} \\ &= \sum_{\varsigma} a^{n_a(\varsigma)} b^{n_b(\varsigma)} c^{n_c(\varsigma)}, \end{aligned} \tag{3.5}$$

where we set $n_a(\varsigma) = n_1(\varsigma) + n_2(\varsigma)$ and so on. One of the main goals in solving the six-vertex model is to compute the per-site free energy f in the thermodynamic limit,

$$f := \lim_{N,M\to\infty} -\frac{1}{NM}\frac{\ln Z_{M,N}}{\beta}. \tag{3.6}$$

3.2 The world lines

Fig. 4 demonstrates how the dipole arrangements can be represented more conveniently. We draw a *world line* on those edges where the arrow points up or to the right. The world lines flow through the lattice continuously with out intersecting each other, as can be seen in Fig. 5.

We impose periodic boundary conditions in both horizontal and vertical directions. Therefore, our lattice can be imagined to be embedded on a torus. More mathematically speaking, the set of vertices is formed as a Cartesian product of two cyclic groups, $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{Z}/M\mathbb{Z}$, where the integers are modulo N and modulo M, respectively. These *toroidal boundary conditions* will play a vital role in our construction, and we can note some of the consequences right away.

Lemma 3.1. *If toroidal (or cylindrical) boundary conditions are imposed, the number of arrangements* 5 *and* 6 *is equal.*

Proof. First, we point out that both arrows of the four first arrangements point in the same direction, either $\rightarrow\!\bullet\!\rightarrow$ or $\leftarrow\!\bullet\!\leftarrow$. They are "neutral" vertices, since flux of horizontal arrows through them is zero. Arrangement 5 is a "sink" $\rightarrow\!\bullet\!\leftarrow$, since

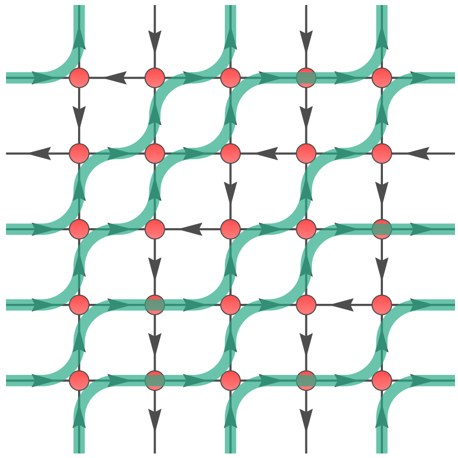


Figure 5: world lines drawn upon the configuration of Fig. 3 (with toroidal boundary conditions)

both horizontal arrows enter the vertex, and arrangement 6 is respectively a "source" .

There cannot be two consecutive (in the sense of being separated only by neutral vertices) sinks (resp. sources) since, in that case, neutral vertices should change direction at some point, thus creating a source (resp. sink) in between, which leads to a contradiction.

Consider a row that has only a single source (or sink). Due to the periodic boundaries, upon crossing, our source forms two consecutive sources with itself, which is not possible.

Therefore, sources and sinks must alternate (with possible neutral vertices in between) and comprise pairs all the way to the boundary. If we have a left-over source (or sink) at the end, we have again formed a consecutive source across the boundary. Because sources and sinks alternate and pair up, the claim is proven.

□

Remark 3.2. The restriction $\varepsilon_5 = \varepsilon_6$ is made redundant by Lemma 3.1. Since the number of sources and sinks is equal, their energies show up in the partition function only as a sum of $\varepsilon_5 + \varepsilon_6$. Therefore, they can be chosen equal without loss of generality.

Lemma 3.3. *Toroidal boundary conditions lead to line conservation: If there are n world lines on the first row of vertical edges, then there are n world lines on every row.*

Proof. Consider an arbitrary row of vertices separating the neighboring two rows of vertical edges E^j and E^{j+1}. Each vertex can be either neutral, a source or a sink with respect to the vertical arrows. Horizontally neutral arrangements 1 to 4 are also vertically neutral. The sink (arrangement 5) for horizontal arrows is a source for vertical arrows, and vice versa.

The world lines on vertical edges are due to the upward-pointing arrows. Since the two vertical arrows of neutral vertices point in the same direction, the number of up arrows corresponding to neutral vertices is the same on E^j and E^{j+1}.

Each vertical source contributes one up arrow to E^{j+1}, and none to E^j. Each vertical sink contributes one up arrow to E^j and none to E^{j+1}. But by Lemma 3.1 the number of horizontal sinks and sources is equal, therefore the number of vertical sinks and sources is equal. In conclusion, the number of up arrows on E^j and E^{j+1} are equal. The argument can be repeated for all rows of the lattice, proving the claim. $\square$

The conservation of world lines will aid us in covering every possible arrangement of our lattice with M rows and N columns. By a row, we now mean a row of N vertical edges. Associate each row with an index $j \in \{1, \ldots, M\}$ starting from the bottom.

Definition 3.4. Let $\Phi = \{\bullet,\ \circ\}^N$ be a Cartesian product of N instances of $\{\bullet,\ \circ\}$. Φ is thus a set containing every N-length sequence of $\bullet$ and $\circ$. Call an element of Φ a state. There are 2^N different states.

Consider a set of integers $I = \{1, 2, \ldots, 2^N\}$ and let $\varphi : I \to \Phi$ be any bijection from integers to N-tuples describing a particular order by which states are arranged. The world lines occupying each row of edges are thus described by some state $\varphi(i)$ where $i \in I$.

We mainly concern ourselves with which row is being described and not with which particular state $\varphi(i)$ the row is in. Thus, for now, we drop the input value and add a subscript, so that $\varphi(i) = \varphi_j$ describes some arrangement of world lines on the jth row.

Figure 6 depicts two particular states of $\varphi_j = (\bullet, \circ, \bullet, \bullet, \circ, \ldots, \bullet)$ and $\varphi_{j+1} = (\bullet, \bullet, \bullet, \circ, \circ, \ldots, \bullet)$.

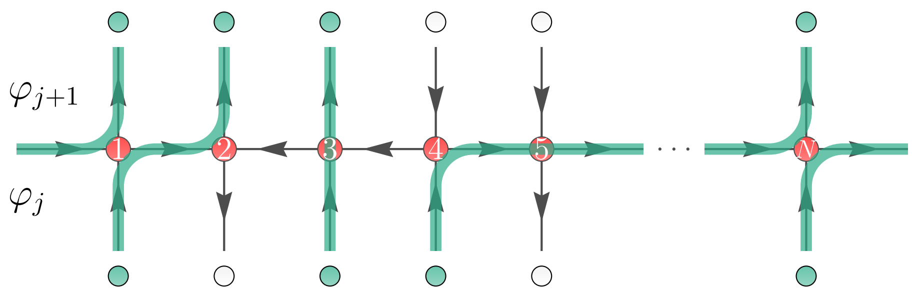


Figure 6: Two consecutive rows of vertical edges where $\vec{x} = (1, 3, 4, N)$ and $\vec{y} = (1, 2, 3, N)$

Given two states φ_j and φ_{j+1}, we denote in a vector $\vec{x}_{\varphi_j} = (x_1, \ldots, x_n)$ the coordinates of edges where the world lines flow in, and in a vector $\vec{y}_{\varphi_{j+1}} = (y_1, \ldots, y_n)$ the coordinates of edges where the world lines flow out. We drop the subscript when the context is clear. The integers are ordered, i.e., $1 \leq x_1 < \ldots < x_n \leq N$ where $0 \leq n \leq N$. Importantly, vectors $\vec{x}$ and $\vec{y}$ always *interlace*.

Lemma 3.5. *Two vectors $\vec{x}$ and $\vec{y}$ must always satisfy either*

$$x_1 \leq y_1 \leq x_2 \leq \ldots \leq x_n \leq y_n \quad or \quad y_1 \leq x_1 \leq y_2 \leq \ldots \leq y_n \leq x_n. \tag{3.7}$$

Proof. The line conservation guarantees that $\dim \vec{x} = \dim \vec{y} = n$. Since sinks and sources must alternate, world lines flow in and out in alternating fashion. The two ways of interlacement result from the choice of whether sink or source appears first.

For appropriate neutral vertices in between, either none of the world lines flow across (or), or they flow through directly (or). By the definition of state vectors, this proves the claim. □

The question arises as to whether, given the states of vertical arrows, the configuration of horizontal arrows is unique.

Lemma 3.6. *Let $\mathcal{H}(\varphi_j, \varphi_{j+1})$ be the set of all possible configurations of horizontal arrows when any two states φ_j and φ_{j+1} are specified.*

If $\varphi_j \neq \varphi_{j+1}$ and the corresponding $\vec{x} \neq \vec{y}$ are interlaced, there is only one allowed configuration. If $\varphi_j = \varphi_{j+1}$ there are exactly two possible horizontal arrow configurations. In summary

$$|\mathcal{H}(\varphi_j, \varphi_{j+1})| = \begin{cases} 1 & \text{if } \vec{x} \neq \vec{y} \text{ are interlaced,} \\ 2 & \text{if } \vec{x} = \vec{y}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Proof. All vertical arrows are determined by φ_j and φ_{j+1}.

When $\varphi_j = \varphi_{j+1}$ every world line flows through the vertices directly since $\vec{x} = \vec{y}$. All horizontal arrows can be locked either to the left or to the right .

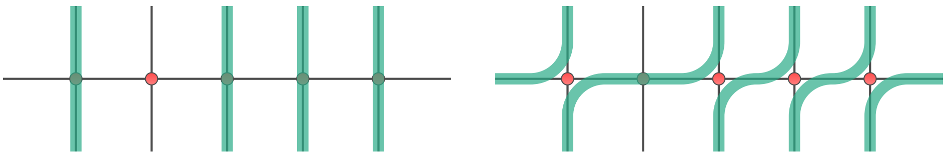


Figure 7: Two different horizontal arrow configurations when $\varphi_j = \varphi_{j+1}$

Otherwise, there exists at least one vertex where φ_j and φ_{j+1} differ in color, which is either a sink or a source (since they always come in pairs, there must be another site where the colors are reversed). By considering the sink at some kth vertex with known arrow configuration, the third arrow of vertex $k+1$ is also established. The ice rule gives the direction of the fourth arrow, which establishes the third arrow of $k+2$... Continuing the argument through the whole row, all vertical arrows are uniquely determined. □

3.3 Transfer matrix

The goal of this section is to restructure the partition function using a matrix V known as the *transfer matrix*.

Definition 3.7. Let Ω be a 2^N-dimensional real vector space spanned by a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{2^N}\}$. Associate each basis vector $\mathbf{e}_i$ to a unique state given by $\varphi(i) \in \Phi$ (recall the more explicit notation of states given in Definition 3.4).

Let $\varphi : I \to \Phi$ be a bijection where $\varphi(i)$ gives from now on ith state in ordering described as follows:

First, arrange the states in Φ in an increasing order by the number of world lines n they contain. The $N+1$ groups thus formed are of size $\binom{N}{n}$ and the total number of states is therefore

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N, \tag{3.9}$$

as it should be. Secondly, within formed groups, order the states so that world lines enter as early as possible. The position of an earlier world line takes precedence over the later ones.

As Fig. 8 showcases, this procedure fixes the subspaces

$$\Omega_n = \operatorname{span}\{\mathbf{e}_i : \dim \vec{x}_{\varphi(i)} = n, i \in I\}, \tag{3.10}$$

where $\vec{x}_{\varphi(i)}$ is the coordinate vector given by $\varphi(i)$.

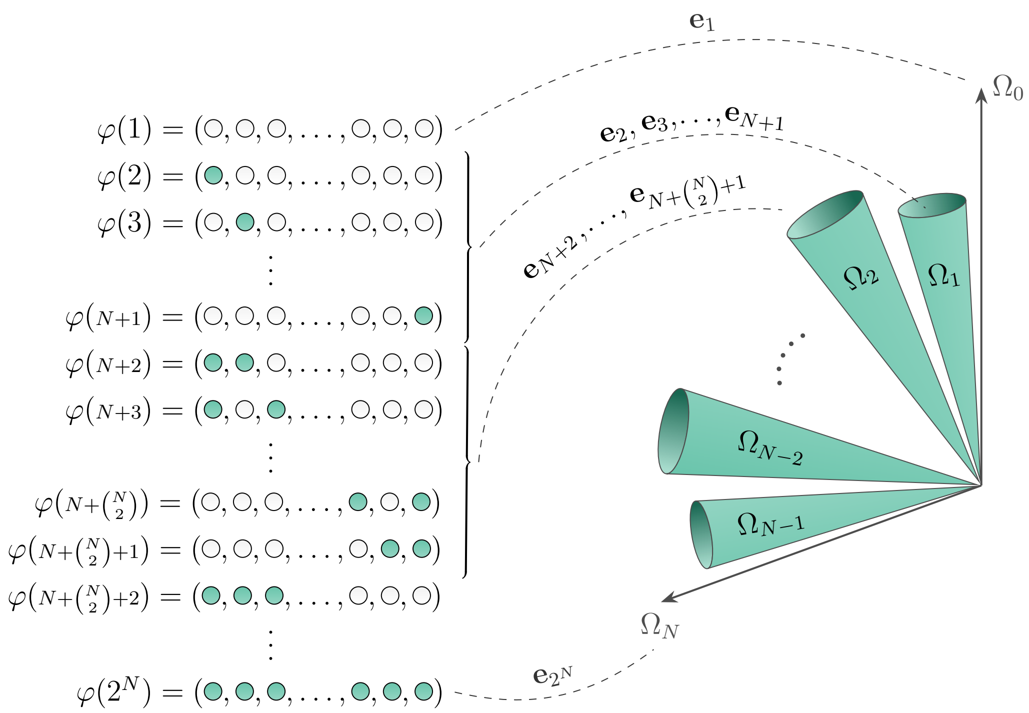
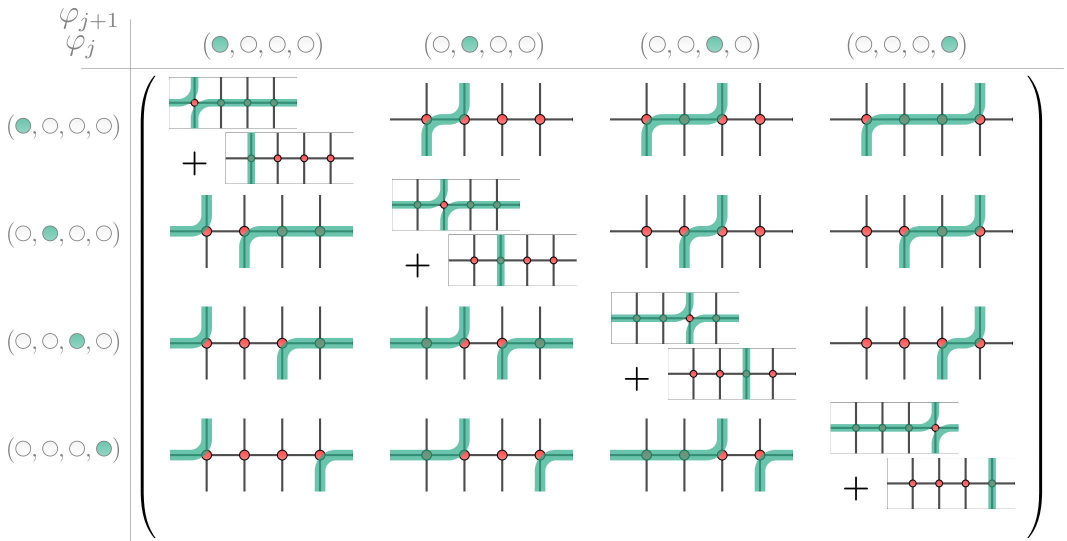


Figure 8: Construction of vector space Ω. The jet-code by Izaak Neutelings [16].

Definition 3.8. Let V be a real-valued 2^N by 2^N matrix given in the basis described above. Associate the kth row of V with a state $\varphi(k)$ that describes the entering sites $\vec{x}_{\varphi(k)}$ of world lines flowing in, and lth column with a state $\varphi(l)$ that describes exiting sites $\vec{y}_{\varphi(l)}$ of world lines flowing out. The matrix element $V(k, l)$ contains the total weight of this "transition" from one state to another.

From Lemma 3.6 it follows that off-diagonal elements of V are either zero (if transition is not possible or $\dim \vec{x}_{\varphi(k)} \neq \dim \vec{y}_{\varphi(l)}$) or give a single Boltzmann weight describing a unique configuration between two states. On the diagonal, we always have a sum of two Boltzmann weights. Consequently, V is a block-diagonal matrix where each nth block V_n describes the flow of n world lines.

As an example, a matrix block V_1 with a single world line is shown in Fig. 9, accompanied by a visual representation of the states.



$$
= \begin{pmatrix} ab^3 + a^3b & a^2c^2 & abc^2 & b^2c^2 \\ b^2c^2 & ab^3 + a^3b & a^2c^2 & abc^2 \\ abc^2 & b^2c^2 & ab^3 + a^3b & a^2c^2 \\ a^2c^2 & abc^2 & b^2c^2 & ab^3 + a^3b \end{pmatrix}
$$

Figure 9: The arrangements of V_1 block and their weights when $N = 4$

For the rest of this section, we show how the partition function can be constructed from the powers of V. Each multiplication of the transfer matrix "builds up" one row of edges on the lattice. Furthermore, from the leading eigenvalue of V, we will obtain the free energy per site of the six-vertex model. These results are stated in the following theorem, which is the main result of this section.

Theorem 3.9. *The partition function of the six-vertex model $Z_{M,N}$ can be written using a transfer matrix V as*

$$Z_{M,N} = \mathrm{Tr}\Big(V^M\Big). \tag{3.11}$$

Moreover, the free energy per vertex f in the limit $M \to \infty$ is given by

$$f = -\frac{1}{\beta N} \ln \Lambda, \tag{3.12}$$

where Λ is the leading eigenvalue of V.

Start by considering only the first two rows of vertical edges ($M = 2$). If we fix the sites of input φ_1, then all possible configurations are obtained by summing over all allowed sites of output φ_2. In this restricted case, the partition function becomes

$$\begin{aligned} Z_{M=2,N} = &\sum_{\substack{\varphi_2 \neq \varphi_1 \\ \vec{x} \neq \vec{y} \text{ are interlaced}}} a^{n_a(\varphi_1,\varphi_2)} b^{n_b(\varphi_1,\varphi_2)} c^{n_c(\varphi_1,\varphi_2)} \\ &+ \sum_{\substack{\mathcal{H}(\varphi_1,\varphi_1) \\ \vec{x}=\vec{y}}} a^{n_{\mathcal{H},a}(\varphi_1,\varphi_1)} b^{n_{\mathcal{H},b}(\varphi_1,\varphi_1)}, \end{aligned} \tag{3.13}$$

where the case of equal states $\varphi_1 = \varphi_2$ is considered separately since from Lemma 3.6 it follows that there are two possible horizontal configurations in $\mathcal{H}(\varphi_1, \varphi_1)$, while others are uniquely determined. It can be easily seen from Fig. 7 that

$$\sum_{\substack{\mathcal{H}(\varphi_1,\varphi_1) \\ \vec{x}=\vec{y}}} a^{n_{\mathcal{H},a}(\varphi_1,\varphi_1)} b^{n_{\mathcal{H},b}(\varphi_1,\varphi_1)} = a^n b^{N-n} + b^n a^{N-n}, \tag{3.14}$$

where $n = \dim \vec{x}_{\varphi_1}$. We now combine the sums by first summing over all vertical states φ_2 and then over their horizontal arrow configurations, if there are any.

$$\begin{aligned} Z_{M=2,N} &= \sum_{\varphi_2} \sum_{\mathcal{H}(\varphi_1,\varphi_2)}^{\mathbb{1}} a^{n_{\mathcal{H},a}(\varphi_1,\varphi_2)} b^{n_{\mathcal{H},b}(\varphi_1,\varphi_2)} c^{n_{\mathcal{H},c}(\varphi_1,\varphi_2)} \\ &= \sum_{\varphi_2} \mathcal{V}(\varphi_1, \varphi_2). \end{aligned} \tag{3.15}$$

Here, we use $\sum^{\mathbb{1}}_{\mathcal{H}(\varphi_1,\varphi_2)}$ as a shorthand to describe a "sum" which is just a single quantity if states are interlaced, a sum of two terms if states are equal, and zero otherwise. Consequently, we have defined a function $\mathcal{V} : \Phi \times \Phi \to \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} \mathcal{V}(\varphi_j, \varphi_{j+1}) &= \sum_{\mathcal{H}(\varphi_1,\varphi_2)}^{\mathbb{1}} a^{n_{\mathcal{H},a}(\varphi_1,\varphi_2)} b^{n_{\mathcal{H},b}(\varphi_1,\varphi_2)} c^{n_{c,\mathcal{H}}(\varphi_1,\varphi_2)} \\ &= \begin{cases} a^{n_a(\varphi_j,\varphi_{j+1})} b^{n_b(\varphi_j,\varphi_{j+1})} c^{n_c(\varphi_j,\varphi_{j+1})} & \text{if } \vec{x}_{\varphi_j} \neq \vec{y}_{\varphi_{j+1}} \text{ are interlaced,} \\ a^n b^{N-n} + b^n a^{N-n} & \text{if } \vec{x}_{\varphi_j} = \vec{y}_{\varphi_{j+1}}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.16}$$

where n_i gives the multiplicities of each vertex group $i \in \{a, b, c\}$ in a given configuration. The mapping $\mathcal{V}$ describes the weight of the transition from any of the 2^N possible states of φ_j to any 2^N possible states of φ_{j+1}.

Suppose we add another row of edges (and vertices) to our lattice. For every given state φ_1 and φ_2, we sum over possible sites of output φ_3 and its respective horizontal arrows. Thus, borrowing the previous argument gives

$$Z_{M=3,N} = \sum_{\varphi_2} \mathcal{V}(\varphi_1, \varphi_2) \sum_{\varphi_3} \mathcal{V}(\varphi_2, \varphi_3). \tag{3.17}$$

By repeating this reasoning, we build up the lattice with M rows. When the final row with toroidal boundaries is reached, the world lines of state φ_M flow into state φ_1. Summing over the output φ_1, we have considered every possible flow of world lines through the lattice, giving finally the partition function

$$\begin{aligned} Z_{M,N} &= \sum_{\varphi_2} \mathcal{V}(\varphi_1, \varphi_2) \sum_{\varphi_3} \mathcal{V}(\varphi_2, \varphi_3) \dots \sum_{\varphi_M} \mathcal{V}(\varphi_{M-1}, \varphi_M) \sum_{\varphi_1} \mathcal{V}(\varphi_M, \varphi_1) \\ &= \sum_{\varphi_1} \sum_{\varphi_2} \sum_{\varphi_3} \dots \sum_{\varphi_M} \mathcal{V}(\varphi_1, \varphi_2) \mathcal{V}(\varphi_2, \varphi_3) \dots \mathcal{V}(\varphi_{M-1}, \varphi_M) \mathcal{V}(\varphi_M, \varphi_1). \end{aligned} \tag{3.18}$$

We recognize that Eq. (3.18) can be regarded as M successive matrix multiplications and taking a trace at the end.

Lemma 3.10. *Consider the bijection $\varphi : I \to \Phi$ and the transfer matrix V defined by Definition 3.7 and Definition 3.8, respectively. Let each matrix element $V(k, l)$ be given by $\mathcal{V}\big(\varphi(k), \varphi(l)\big)$. The partition function in Eq. (3.18) can be rewritten using the transfer matrix V as*

$$Z_{M,N} = \mathrm{Tr}\Big(V^M\Big), \tag{3.19}$$

where Tr *is the trace of a matrix.*

Proof. From the definition of matrix multiplication, an element in position $(k,\, m)$ of a square of a matrix $A \in \mathbb{R}^{n \times n}$ is given by

$$A^2(k, m) = \sum_{l=1}^{n} A(k, l) A(l, m), \tag{3.20}$$

and of a cubed matrix A^3 by

$$\begin{aligned} A^3(k, m) &= \sum_{l=1}^{n} A(k, l) A^2(l, m) \\ &= \sum_{l=1}^{n} A(k, l) \sum_{l'=1}^{n} A(l, l') A(l', m). \end{aligned} \tag{3.21}$$

The trace of a matrix is defined as a sum of its diagonal elements

$$\mathrm{Tr}(A) = \sum_{l=1}^{n} A(l, l). \tag{3.22}$$

As the structure of repeated matrix multiplication is now clear, we rewrite Eq. (3.18) using matrix V as defined above. We first use looser notation where φ_j itself is a summation index for illustration purposes:

$$
\begin{aligned}
Z_{M,N} &= \sum_{\varphi_1} \dots \sum_{\varphi_{M-1}} V(\varphi_1, \varphi_2) \dots V(\varphi_{M-2}, \varphi_{M-1}) \sum_{\varphi_M} V(\varphi_{M-1}, \varphi_M) V(\varphi_M, \varphi_1) \\
&= \sum_{\varphi_1} \dots \sum_{\varphi_{M-1}} V(\varphi_1, \varphi_2) \dots V(\varphi_{M-2}, \varphi_{M-1}) V^2(\varphi_{M-1}, \varphi_1) \\
&= \sum_{\varphi_1} \dots \sum_{\varphi_{M-2}} V(\varphi_1, \varphi_2) \dots V(\varphi_{M-3}, \varphi_{M-2}) \sum_{\varphi_{M-1}} V(\varphi_{M-2}, \varphi_{M-1}) V^2(\varphi_{M-1}, \varphi_1) \\
&= \sum_{\varphi_1} \dots \sum_{\varphi_{M-1}} V(\varphi_1, \varphi_2) \dots V(\varphi_{M-3}, \varphi_{M-2}) V^3(\varphi_{M-2}, \varphi_1) \\
&\dots \\
&= \sum_{\varphi_1} \sum_{\varphi_2} V(\varphi_1, \varphi_2) V^{M-1}(\varphi_2, \varphi_1) = \sum_{\varphi_1} V^M(\varphi_1, \varphi_1) = \mathrm{Tr}\Big(V^M\Big).
\end{aligned}
\tag{3.23}
$$

To match the description in the beginning, summation indices should be written as integers k_j running from 1 to 2^N, and matrix elements should be $V(k_j, k_{j+1}) = \mathcal{V}\Big(\varphi(k_j), \varphi(k_{j+1})\Big)$. Written like this, the first row of Eq. (3.23) reads

$$
Z_{M,N} = \sum_{k_1=1}^{2^N} \dots \sum_{k_{M-1}=1}^{2^N} V(k_1, k_2) \dots V(k_{M-2}, k_{M-1}) \sum_{k_M=1}^{2^N} V(k_{M-1}, k_M) V(k_M, k_1).
\tag{3.24}
$$

Subsequently, the same proof follows in a more exact (but more cluttered) manner. □

From basic linear algebra, we have

$$
\mathrm{Tr}\Big(V^M\Big) = \sum_i n_i \lambda_i^M,
\tag{3.25}
$$

where λ_i is the ith eigenvalue of V and n_i is the size of the corresponding Jordan block. We have not proved the diagonalisability of V in our work. The existence of the leading eigenvalue Λ corresponding to a Jordan block of size one is ensured by the Perron-Frobenius theorem. The results of the Perron-Frobenius theorem are discussed in detail in Appendix A along with a proof that V satisfies its assumptions. It follows that we can separate Λ, and the free energy formula for M large becomes

$$
\begin{aligned}
f &= \lim_{M\to\infty} -\frac{1}{NM} \frac{\ln \mathrm{Tr}\Big(V^M\Big)}{\beta} \\
&= \lim_{M\to\infty} -\frac{1}{\beta NM} \ln\Big[\Lambda^M (1 + \sum_i n_i \Big(\frac{\lambda_i}{\Lambda}\Big)^M\Big] \\
&= \lim_{M\to\infty} -\frac{1}{\beta N} \ln \Lambda - \lim_{M\to\infty} \frac{1}{\beta NM} \ln\Big[\sum_i \Big(1 + n_i \Big(\frac{\lambda_i}{\Lambda}\Big)^M\Big)\Big] \\
&= -\frac{1}{\beta N} \ln \Lambda,
\end{aligned}
\tag{3.26}
$$

which concludes the proof of Theorem 3.9.

4 Formulating the ansatz

To calculate the free energy, one needs to solve the leading eigenvalue of the transfer matrix. We aim to first solve the eigenvalue Λ_n of a general block V_n. The candidate eigenvector Ψ_n is some linear combination of basis vectors living in some subspace Ω_n

$$\Psi_n := \sum_{\Omega_n} \psi(\vec{x}_{\varphi(i)})\mathbf{e}_i, \tag{4.1}$$

where the coefficient $\psi(\vec{x}_{\varphi(i)})$ is a function of the world line entering sites associated to $\mathbf{e}_i$. The Bethe ansatz is *a priori* an ingenious guess for the coefficient $\psi(\vec{x}_{\varphi(i)})$ that is substituted into the eigenvalue equation

$$\Lambda_n \psi(\vec{x}_{\varphi(i)}) = \sum_j V_n(i,j)\psi(\vec{y}_{\varphi(j)}), \tag{4.2}$$

and verified to satisfy it. The summation above is taken over all possible exit sites $\vec{y}_{\varphi(j)}$ while entering sites $\vec{x}_{\varphi(i)}$ are fixed. In this chapter, we provide intuition for the eigenvalue equation and justification for the ansatz in the general case V_n by considering the blocks V_0, V_1, V_2, and V_3. The general case V_n with the full Bethe ansatz itself is considered in the next chapter.

4.1 The V_0 block

There are no world lines flowing vertically through the two consecutive rows. Subspace Ω_0 is one dimensional, and V_0 is a one-by-one block containing an element

$$V(1,1) = a^N + b^N, \tag{4.3}$$

Thus, $\mathbf{e}_1$ is an eigenvector with an eigenvalue

$$\Lambda_0 = a^N + b^N. \tag{4.4}$$

The first term arises from a configuration of only empty vertices, and the second from a configuration of a world line flowing horizontally.

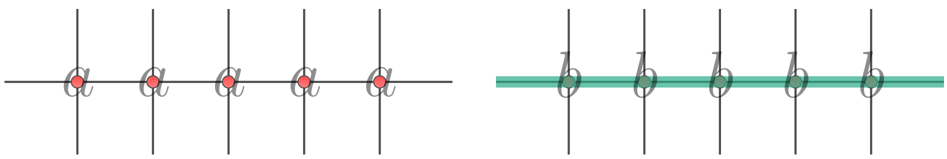


Figure 10: Arrow configurations when $\varphi_j = \varphi_{j+1} = \varphi(1)$

4.2 The V_1 block

There is a single world line entering at some vertex x_1 and leaving at some vertex y_1. The vector $\vec{x} = (x_1)$ contains a single coordinate, and we abbreviate $\psi(\vec{x}) = \psi(x)$. There are two general situations to consider:
In every case where $x < y$, all horizontal edges between them are filled by a world line, and all others are empty. In every case where $y < x$, then all horizontal edges between them are empty, and all others are filled.

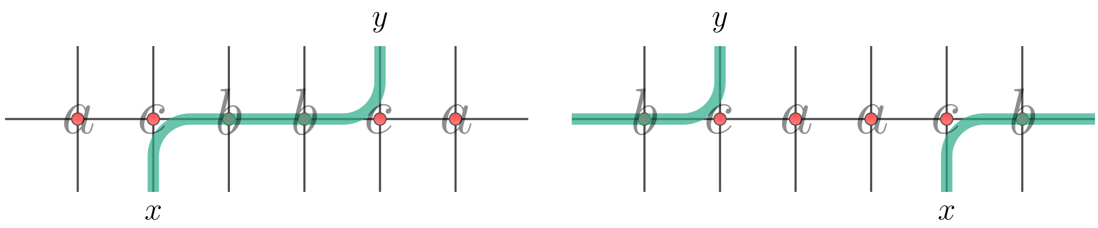


Figure 11: Each off-diagonal element of V_1 describes either situation $x < y$ or $y < x$

As always, when $x = y$ there are two horizontal arrow configurations. The eigenvalue equation takes the following form

$$\begin{aligned} \Lambda_1 \psi(x) = &\sum_{y=1}^{x-1} a^{x-y-1} b^{N+y-x-1} c^2 \psi(y) \\ &+ (a^{N-1}b + ab^{N-1})\psi(x) \\ &+ \sum_{y=x+1}^{N} a^{N+x-y-1} b^{y-x-1} c^2 \psi(y). \end{aligned} \tag{4.5}$$

It helps to visualize the matrix-vector multiplication: fix the entering site x, let the exit y run through the whole row of vertices, and see how the weights change. In Figure 9 fix some particular row and run through the columns. We can see how, in the entries to the left of the diagonal, y creeps up to x closing up the empty gap. At the diagonal, states become equal. To the right, y runs away from x extending the world line. These three scenarios are described by the three terms in Eq. (4.5). We try a natural substitution

$$\psi(x) = z^x, \quad z \in \mathbb{C}, \tag{4.6}$$

into Eq. (4.5) which results in geometric series if we restrict $z \neq \frac{a}{b}$. The rest is elementary (but tedious) algebra, for which we color code the terms.

$$\begin{aligned} \Lambda_1 z^x = &\; a^{x-1} b^{N-x-1} c^2 \sum_{y=1}^{x-1} (\frac{b}{a} z)^y + ab^{N-1} z^x \\ &+ a^{N-1} b z^x + a^{N+x-1} b^{-x-1} c^2 \sum_{y=x+1}^{N} (\frac{b}{a} z)^y . \end{aligned} \tag{4.7}$$

The orange term gives (we use a dashed underscore when each term is “finished“)

$$
\begin{aligned}
&= a^{x-1}b^{N-x-1}c^2\frac{\frac{b}{a}z-\left(\frac{b}{a}z\right)^x}{1-\frac{b}{a}z}+ab^{N-1}z^x\\
&= a^{x-1}b^{N-x-1}c^2\frac{bz-a\left(\frac{b}{a}z\right)^x}{a-bz}+ab^{N-1}z^x\\
&= a^{x-1}b^{N-x}c^2\frac{z}{a-bz}-b^{N-1}c^2\frac{z^x}{a-bz}+ab^{N-1}z^x\\
&= a^{x-1}b^{N-x}c^2\frac{z}{a-bz}-b^{N-1}\frac{c^2-a(a-bz)}{a-bz}z^x\\
&= a^{x-1}b^{N-x}c^2\frac{z}{a-bz}+b^N\frac{a^2-c^2-abz}{b(a-bz)}z^x.
\end{aligned}
\tag{4.8}
$$

The blue term gives respectively

$$
\begin{aligned}
&= a^{N-1}bz^x+a^{N+x-1}b^{-x-1}c^2\frac{\left(\frac{b}{a}z\right)^{x+1}-\left(\frac{b}{a}z\right)^{N+1}}{1-\frac{b}{a}z}\\
&= a^{N-1}bz^x+a^{N+x}b^{-x-1}c^2\frac{\left(\frac{b}{a}z\right)^{x+1}}{a-bz}-a^{N+x}b^{-x-1}c^2\frac{\left(\frac{b}{a}z\right)^{N+1}}{a-bz}\\
&= a^{N-1}bz^x+a^{N-1}c^2\frac{z^{x+1}}{a-bz}-a^{x-1}b^{N-x}c^2\frac{z}{a-bz}z^N\\
&= a^{N-1}\frac{b(a-bz)+c^2z}{a-bz}z^x-a^{x-1}b^{N-x}c^2\frac{z}{a-bz}z^N\\
&= a^N\frac{ab+(c^2-b^2)z}{a(a-bz)}z^x-a^{x-1}b^{N-x}c^2\frac{z}{a-bz}z^N.
\end{aligned}
\tag{4.9}
$$

We define functions

$$
\begin{aligned}
L(z) &:= \frac{ab+(c^2-b^2)z}{a(a-bz)},\\
M(z) &:= \frac{a^2-c^2-abz}{b(a-bz)}.
\end{aligned}
\tag{4.10}
$$

Combining the results we get

$$
\begin{aligned}
\Lambda_1 z^x = {} & b^N M(z)z^x + a^{x-1}b^{N-x}c^2\frac{z}{a-bz}\\
& + a^N L(z)z^x - a^{x-1}b^{N-x}c^2\frac{z}{a-bz}z^N.
\end{aligned}
\tag{4.11}
$$

The terms on the left containing a factor z^x) are what we look for, while the terms on the right are unwanted. These so-called “boundary terms“, arise from summation limits $y=1$ and $y=N$. We set

$$
z^N = 1, \tag{4.12}
$$

so that the unwanted terms cancel. The Eq. (4.11) simplifies to

$$\Lambda_1 z^x = \Big(a^N L(z) + b^N M(z)\Big) z^x, \tag{4.13}$$

giving the eigenvalue

$$\Lambda_1 = a^N L(z) + b^N M(z). \tag{4.14}$$

There are N solutions of Eq. (4.12), each being an Nth root of unity. Consequently, we have found all N eigenvectors Eq. (4.6) and their N corresponding eigenvalues Eq. (4.14).

4.3 The V_2 block

There are now two world lines and vectors $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$ contain two entering coordinates and two exiting coordinates, respectively. Using Lemma 3.5 we know that $\vec{x}$ and $\vec{y}$ must interlace each other, which satisfy either

$$x_1 \leq y_1 \leq x_2 \leq y_2 \quad \text{or} \quad y_1 \leq x_1 \leq y_2 \leq x_2. \tag{4.15}$$

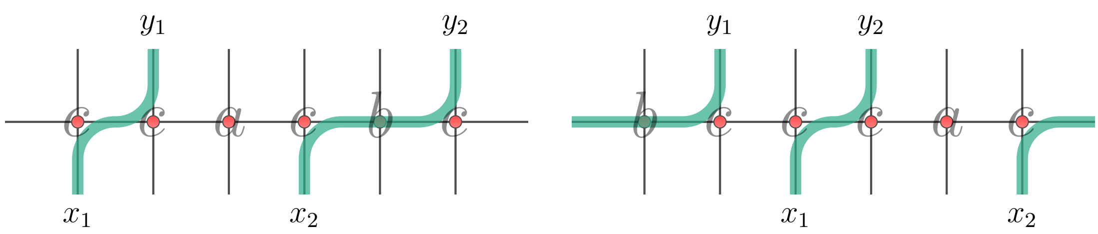


Figure 12: Vector $\vec{x}$ and $\vec{y}$ interlace in one of the two ways

Several new challenges emerge when considering multiple world lines. We describe an arrangement as "collapsed" if at least one of the entering sites x_i is the same as some exiting site y_j. In the case $n = 1$, there is only one collapsed state, $x = y$. However, in the case $n = 2$, the number of such states has substantially increased.

The overlapping of sites causes the weights to break away from the usual pattern. Looking at Fig. 12, if we squeeze some x_i and y_j together, the sink and source pair get suddenly replaced by neutral vertices. Therefore, collapsed states do not follow the same algebraic sequence as those where there is no overlapping.

In order to avoid the cumbersome task of considering each case separately, we introduce auxiliary functions D and E. They facilitate a direct summation over the exiting states y_1 and y_2 in Eq. (4.2) and yield the following expression:

$$\begin{aligned}\Lambda_2\psi(x_1, x_2) = &\sum_{y_1=x_1}^{x_2} {\sum_{y_2=x_2}^{N}}' a^{x_1-1} E(x_1, y_1) D(y_1, x_2) E(x_2, y_2) a^{N-y_2} c^4 \psi(y_1, y_2) \\ &+ \sum_{y_1=1}^{x_1} {\sum_{y_2=x_1}^{x_2}}' b^{y_1-1} D(y_1, x_1) E(x_1, y_2) D(y_2, x_2) b^{N-x_2} c^4 \psi(y_1, y_2).\end{aligned} \tag{4.16}$$

Let's now clarify the notation. First, note that in an uncollapsed arrangement, interlacement leads to alternating segments where horizontal edges are either all empty ($\mathfrak{D}$) or are occupied by a world line ($\mathcal{E}$), as Fig. 13 shows. These segments fall in between sources and sinks c. We exclude (for now) starting and ending segments from this classification.

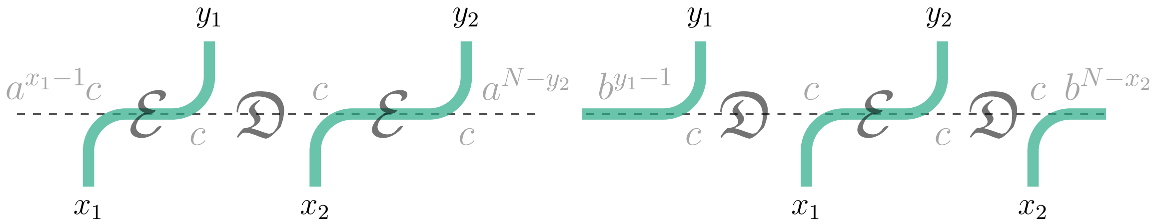


Figure 13: Two alternating domains $\mathcal{E}$ and $\mathfrak{D}$

We define a piecewise function D that gives the weight of an empty segment $\mathfrak{D}$ between one world line leaving at y and another entering at x:

$$D(y,x) := \begin{cases} a^{x-y-1}, & x > y, \\ \dfrac{a}{c^2}, & x = y. \end{cases} \tag{4.17}$$

The case $x = y$ represents the shrinking of segment $\mathfrak{D}$ to a collapsed state, where it vanishes as demonstrated by Fig. 14. A source and sink pair c^2 is then destroyed by a neutral vertex a.

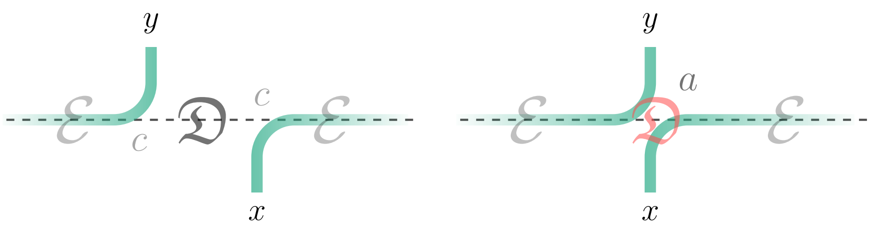


Figure 14: The shrinking of the segment $\mathfrak{D}$ to a collapsed state

We define another piecewise function E that gives the weight of an occupied segment $\mathcal{E}$ between a world line entering at x and leaving at y:

$$E(x,y) := \begin{cases} b^{y-x-1}, & y > x, \\ \dfrac{b}{c^2}, & x = y. \end{cases} \tag{4.18}$$

Collapse of segment $\mathcal{E}$ is shown in Fig. 15, where a source and sink pair c^2 is destroyed by a neutral vertex b.

We now turn our attention to the weights associated with the initial and final sequences, which were previously omitted from segments $\mathfrak{D}$ and $\mathcal{E}$. It is a pleasant surprise that these weights can also be expressed using the functions D and E:

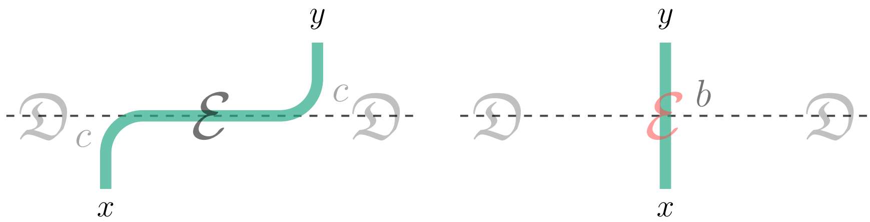


Figure 15: The shrinking of the segment $\mathcal{E}$ to a collapsed state

$$
\begin{aligned}
a^{x_1-1}a^{N-y_2} &= a^{x_1-(y_2-N)-1} = D(y_2-N, x_1),\\
b^{y_1-1}b^{N-x_2} &= b^{y_1-(x_2-N)-1} = E(x_2-N, y_1).
\end{aligned} \tag{4.19}
$$

The revelation is not too unexpected, as it can be understood as bridging the two ends across a periodic boundary into a single segment, as can be seen in Fig. 16. Coordinates $x_2 - N \leq 0$ and $y_2 - N \leq 0$ will play an important role, and we denote them as x_0 and y_0 respectively.

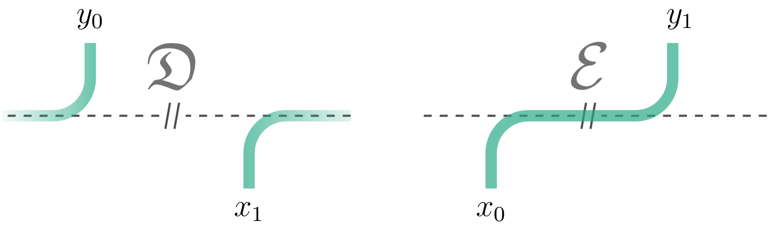


Figure 16: The construction of a segment across the boundary

Armed with a more powerful notation, we are ready to tackle Eq. (4.16). We make a simplified ansatz

$$
\psi(x_1,\, x_2) = z_1^{x_1} z_2^{x_2}, \quad z_1, z_2 \in \mathbb{C} \setminus \{\frac{a}{b}\}. \tag{4.20}
$$

Making this substitution into Eq. (4.2), utilizing the functions in Eqs. (4.17) and (4.18), and after quick rearrangement of the summations, we obtain the eigenvalue equation:

$$
\begin{aligned}
\Lambda_2 z_1^{x_1} z_2^{x_2} = &\sum_{y_1=x_1}^{x_2} E(x_1,\, y_1) D(y_1,\, x_2) z_1^{y_1} {\sum_{y_2=x_2}^{N}}' E(x_2,\, y_2) D(y_2-N, x_1) c^4 z_2^{y_2}\\
&+ \sum_{y_1=1}^{x_1} E(x_2-N, y_1) D(y_1,\, x_1) z_1^{y_1} {\sum_{y_2=x_1}^{x_2}}' E(x_1,\, y_2) D(y_2,\, x_2) c^4 z_2^{y_2},
\end{aligned} \tag{4.21}
$$

where the sums are color-coded for subsequent calculations. It is worth noting the following remarks:

Remark 4.1. All the sinks and sources are gathered together in the term c^4 at the end. In the uncollapsed case of n world lines, we would have n sink and source pairs gathered in the term c^{2n}. In the event of collapse, the functions D and E eliminate sinks and sources in pairs.

Remark 4.2. The primed sums indicate that terms with $y_1 = y_2$ are excluded from summation, as allowing them would result in two world lines merging into one, which is not permitted.

The most direct approach is to compute the sums without any constraints and subsequently subtract the weights of the forbidden cases. These *correction terms* will prove essential to the Bethe ansatz. To distinguish them, we will represent them using **bold** font.

Remark 4.3. The two weights of $\vec{y} = \vec{x}$ are separated into both sums: the orange sum contains the collapse of all $\mathcal{E}$ segments, and the blue sum contains the collapse of all $\mathfrak{D}$ segments.

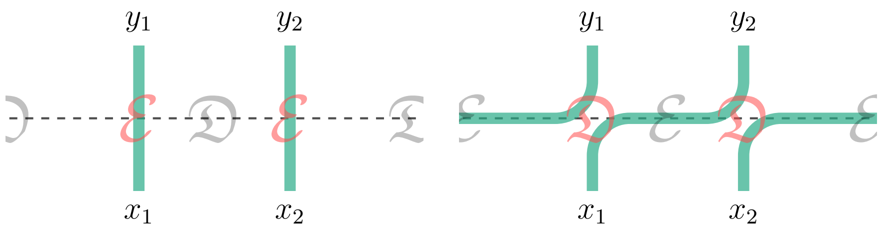


Figure 17: The shrinking of the segment $\mathcal{E}$ to a collapsed state

The direct computation of these sums is a straightforward yet tedious task, detailed in Appendix B. Consequently, we obtain a collection of terms as a result:

$$\begin{aligned}\Lambda_2 z_1^{x_1} z_2^{x_2} = &\left(a^N L_1 L_2 + b^N M_1 M_2 \right) z_1^{x_1} z_2^{x_2} \\ &+ \left(a^{x_1-(x_2-N)} b^{x_2-x_1} (z_1 z_2)^{x_2} + a^{x_2-x_1} b^{x_1-(x_2-N)} (z_1 z_2)^{x_1} \right) \Big[M_1 L_2 - \mathbf{1} \Big] \\ &+ a^{x_1} b^{N-x_2} \Big\{ \rho_1 \hat{R}_2(x_1,\, x_2) - \rho_2 \hat{R}_1(x_1, x_2) z_2^N \Big\}.\end{aligned} \tag{4.22}$$

In this process, we have introduced abbreviations $L_j = L(z_j)$, $M_j = M(z_j)$, and defined new functions $\rho_j = \rho(z_j)$ and $\hat{R}_j(x,\, y)$ in Equations (B3) and (B5), respectively. Per Remark 4.2, the bold font signifies that the specific term is a subtracted correction term.

The first row consists of the wanted terms, which, if all other terms cancel out, would yield the eigenvalue

$$\Lambda_2 = a^N L_1 L_2 + b^N M_1 M_2. \tag{4.23}$$

The second row consists of unwanted "internal terms". In the general case n, we will find that all internal terms will contain an expression

$$M(z_i)L(z_j)-\mathbf{1}, \tag{4.24}$$

as a function of different variables z_i and z_j. This quantity will be utilized to impose "internal relations" that provide many cancellations.
The third row consists of unwanted boundary terms for which we are interested in the expression in curly brackets

$$\rho_1 \hat{R}_2(x_1,\, x_2) - \rho_2 \hat{R}_1(x_1,\, x_2) z_2^N. \tag{4.25}$$

The function ρ will not be of interest in our treatment. We will later impose "boundary relations" which *a posteriori* allow us to combine orange and blue sums together and bypass these functions altogether.

As the emergence of unwanted terms suggests, the ansatz proposed in Eq. (4.20) proves insufficient. As a result, one might be inclined to introduce another pair of complex numbers, $(\tilde{z}_1,\, \tilde{z}_2)$, and attempt a linear combination:

$$A z_1^{x_1} z_2^{x_2} + \widetilde{A} \tilde{z}_1^{x_1} \tilde{z}_2^{x_2}, \tag{4.26}$$

with some appropriate coefficients. The most sensible option arises when we permute the original pair $(z_1,\, z_2)$, resulting in the following choice:

$$\tilde{z}_1 = z_2, \quad \tilde{z}_2 = z_1. \tag{4.27}$$

This gives the full ansatz

$$\psi(x_1,\, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2}, \tag{4.28}$$

where the coefficients $A_{12} = A(z_1,\, z_2)$ and $A_{21} = A(z_2,\, z_1)$ are expressed in terms of a function A that is yet to be determined. The transposition is justified by the conclusive results:

Lemma 4.4. *An eigenvector of the block V_2 is given by Eq. (4.28) with the corresponding eigenvalue*

$$\Lambda_2 = a^N L(z_1)L(z_2) + b^N M(z_1)M(z_2), \tag{4.29}$$

where assume that the coefficients A_{12} and A_{21} and distinct solutions $z_1,\, z_2 \in \mathbb{C}$ satisfy both the internal relation:

$$\frac{A_{12}}{A_{21}} = -\frac{M(z_1)L(z_2)-\mathbf{1}}{M(z_2)L(z_1)-\mathbf{1}}, \tag{4.30}$$

and the boundary relations:

$$z_1^N = \frac{A_{12}}{A_{21}}, \quad z_2^N = \frac{A_{21}}{A_{12}}. \tag{4.31}$$

Proof. By the linearity of matrix multiplication, our new ansatz produces a linear combination of exactly the same expressions as in Eq. (4.22) but with transposed suffixes of z:s (suffixes of x:s stay fixed).

Therefore, the eigenvalue given by Eq. (4.23) is unchanged, as the wanted terms become:

$$\begin{aligned}
&\Big(a^N L_1 L_2 + b^N M_1 M_2\Big) A_{12} z_1^{x_1} z_2^{x_2} + \Big(a^N L_2 L_1 + b^N M_2 M_1\Big) A_{21} z_2^{x_1} z_1^{x_2} \\
&= \Big(a^N L_1 L_2 + b^N M_1 M_2\Big)\Big(A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2}\Big) \\
&= \Big(a^N L_1 L_2 + b^N M_1 M_2\Big)\psi(x_1,\, x_2).
\end{aligned} \tag{4.32}$$

Within the unwanted internal terms, the expression in square brackets possesses a lengthy coefficient that is solely dependent on the product $z_1 z_2$ and therefore remains invariant under the transposition. Consequently, we obtain a linear combination of internal terms:

$$A_{12}\Big[M_1 L_2 - \mathbf{1}\Big] + A_{21}\Big[M_2 L_1 - \mathbf{1}\Big], \tag{4.33}$$

that vanishes when internal relation Eq. (4.30) is imposed.
The linear combination of the boundary terms may be regrouped as

$$\begin{aligned}
&A_{12}\Big(\rho_1 \hat{R}_2(x_1,\, x_2) - \rho_2 \hat{R}_1(x_1,\, x_2) z_2^N\Big) + A_{21}\Big(\rho_2 \hat{R}_1(x_1,\, x_2) - \rho_1 \hat{R}_2(x_1,\, x_2) z_1^N\Big) \\
&= \rho_1 \hat{R}_2(x_1,\, x_2)\Big(A_{12} - A_{21} z_1^N\Big) + \rho_2 \hat{R}_1(x_1,\, x_2)\Big(A_{21} - A_{12} z_2^N\Big),
\end{aligned} \tag{4.34}$$

which cancels out for all x_1 and x_2 if and only if

$$z_1^N = \frac{A_{12}}{A_{21}}, \quad z_2^N = \frac{A_{21}}{A_{12}}, \tag{4.35}$$

Combining the two relations Eqs. (4.30) and (4.31) gives two equations

$$z_1^N = -\frac{M(z_2)L(z_1) - \mathbf{1}}{M(z_1)L(z_2) - \mathbf{1}}, \quad z_2^N = -\frac{M(z_1)L(z_2) - \mathbf{1}}{M(z_2)L(z_1) - \mathbf{1}}, \tag{4.36}$$

which if solved for N solutions of z_1 and z_2 each, can be paired in $\binom{N}{2}$ ways giving a full set of eigenvectors of V_2. Remark that if $z_1 = z_2$ then we get the zero-vector. □

4.4 The V_3 block

The structure of the $n = 2$ case and the techniques developed for it are sufficient to generalize the argument for general n. However, it is beneficial to briefly cover the case $n = 3$, which offers a more comprehensive representation. We begin again with a rudimentary substitution

$$\psi(x_1, x_2, x_3) = z_1^{x_1} z_2^{x_2} z_3^{x_3}. \tag{4.37}$$

By calculating the eigenvalue equation, we obtain a tremendous number of terms that can be restructured as follows

$$\begin{aligned}
\Lambda_3 z_1^{x_1} z_2^{x_2} z_3^{x_3} = &\Big(a^N L_1 L_2 L_3 + b^N M_1 M_2 M_3 \Big) z_1^{x_1} z_2^{x_2} z_3^{x_3} \\
&+ \Big(a^{x_1-(x_2-N)} b^{x_2-x_1} L_3 z_3^{x_3} (z_1 z_2)^{x_2} + a^{x_2-x_1} b^{x_1-(x_2-N)} \hat{R}_3(x_2, x_3)(z_1 z_2)^{x_1} \Big) \Big[M_1 L_2 - \mathbf{1} \Big] \\
&+ \Big(a^{x_1-(x_3-N)} b^{x_3-x_2} \hat{R}_1(x_1, x_2)(z_2 z_3)^{x_3} + a^{x_3-x_2} b^{x_2-(x_3-N)} M_1 z_1^{x_1} (z_2 z_3)^{x_2} \Big) \Big[M_2 L_3 - \mathbf{1} \Big] \\
&+ a^{x_1} b^{N-x_3} \Big\{ \Big(\rho_1 \hat{R}_2(x_1, x_2) \hat{R}_3(x_2, x_3) - \rho_3 \hat{R}_1(x_1, x_2) \hat{R}_2(x_2, x_3) z_3^N \Big) \\
&+ a^{x_3-x_2} b^{x_2-x_1} \Big(\rho_3 z_3^N (z_1 z_2)^{x_2} - \rho_1 (z_2 z_3)^{x_2} \Big) \Big\}.
\end{aligned} \tag{4.38}$$

To extend the ansatz, we consider a linear combination of all possible *permutations* of the complex number tuple (z_1, z_2, z_3)

$$\psi(x_1, x_2, x_3) = \sum_{\sigma \in \mathfrak{S}_3} A_\sigma z_{\sigma(1)}^{x_1} z_{\sigma(2)}^{x_2} z_{\sigma(3)}^{x_3}, \tag{4.39}$$

where sum is taken over all permutations σ in the symmetric group of order 3 and $\sigma(i)$ is the image of $i \in \{1,\ 2,\ 3\}$ under σ. The coefficient A_σ is given by the yet-unknown function $A(z_{\sigma(1)},\ z_{\sigma(2)},\ z_{\sigma(3)})$ whose input is permuted by σ.

The eigenvalue equation becomes a linear combination of the expressions given by Eq. (4.38) but with suffixes of zs permuted accordingly. Once again, we aim to eliminate internal and boundary terms.

Observe that all internal terms contain again a square-bracketed factor $[M_j L_{j+1} - 1]$ and a coefficient that remains invariant under the transposition of z_j and z_{j+1}. Therefore, we pair up each permutation with two other permutations that have consecutive integers (either 1 and 2 or 2 and 3) transposed. Now, we cancel the internal terms by imposing for each permutation:

$$\begin{aligned}
A_\sigma \Big[M_{\sigma(1)} L_{\sigma(2)} - \mathbf{1} \Big] + A_{\sigma \circ (1\,2)} \Big[M_{\sigma(2)} L_{\sigma(1)} - \mathbf{1} \Big] &= 0 \\
A_\sigma \Big[M_{\sigma(2)} L_{\sigma(3)} - \mathbf{1} \Big] + A_{\sigma \circ (2\,3)} \Big[M_{\sigma(3)} L_{\sigma(2)} - \mathbf{1} \Big] &= 0,
\end{aligned} \tag{4.40}$$

where $(j,\ j+1)$ represents a transposition permuting j and $j+1$.

We take care of the boundary terms by pairing any permutation σ with another permutation:

$$\sigma \circ (1\,2\,3) = \Big(\sigma(3), \sigma(1), \sigma(2) \Big), \tag{4.41}$$

where $(1, 2, 3)$ bumps up 1 to 2 and 2 to 3, while sending 3 to 1.

The boundary terms in curly brackets vanish for all σ under the condition

$$z_{\sigma(1)}^N = \frac{A_\sigma}{A_{\sigma\circ(1\,2\,3)}}. \tag{4.42}$$

Upon combining the internal and boundary relations in Eqs. (4.40) and (4.42), we would arrive at equations that determine the complex numbers (z_1, z_2, z_3). The eigenvalue of V_3 is then given by

$$\Lambda_3 = a^N L(z_1)L(z_2)L(z_3) + b^N M(z_1)M(z_2)M(z_3), \tag{4.43}$$

provided that the solutions exist.

4.5 Some remarks

We are now ready to treat the general case with n world lines, by following the direction shown by cases $n = 2$ and $n = 3$. We can expect that the eigenvalue is

$$\Lambda_n = a^N \prod_{i=1}^n L(z_i) + b^N \prod_{i=1}^n M(z_i). \tag{4.44}$$

We can expect that the cancellation condition of internal terms is obtained by pairing up permutations with those that have some pair of consecutive indices transposed, obtaining a vanishing linear combination of factors

$$A_\sigma\Big[M(z_{\sigma(j)})L(z_{\sigma(j+1)}) - \mathbf{1}\Big]. \tag{4.45}$$

This "hypothesis" is particularly peculiar, considering that the contribution of $-\mathbf{1}$ is provided by the correction terms.

The boundary relations especially, seem to have been imposed *ad hoc* as if to force the calculations to work out. However, assuming them beforehand in the next section gives us a way to combine the sums in the eigenvalue equation into one unified quantity. The fact that two separate sums can be merged in hindsight explains why one has to endure such asymmetric algebraic manipulations.

There is an alternative path that leads to the boundary relations through the *translational invariance* of the lattice. The boundary conditions imply

$$\psi\Big(x_1, x_2, \ldots, x_{n-1}, x_n\Big) = \psi\Big(x_2, x_3 \ldots, x_n,\, x_1 + N\Big), \tag{4.46}$$

that is the eigenstate should remain unchanged when the world line is taken across the periodic boundary back to its original position. In case $n = 3$, we have

$$\begin{aligned}\psi\Big(x_1, x_2, x_3\Big) &= A_{(123)} z_1^{x_1} z_2^{x_2} z_3^{x_3} + A_{(132)} z_1^{x_1} z_3^{x_2} z_2^{x_3} + A_{(213)} z_2^{x_1} z_1^{x_2} z_3^{x_3} + \ldots \\ \psi\Big(x_2, x_3, x_1 + N\Big) &= A_{(231)} z_2^{x_2} z_3^{x_3} z_1^{x_1+N} + A_{(321)} z_3^{x_3} z_2^{x_2} z_1^{x_1+N} + A_{(132)} z_1^{x_2} z_3^{x_3} z_2^{x_1+N} + \ldots,\end{aligned} \tag{4.47}$$

For the above quantities to be equal for all x_j, we must have

$$A_{(231)} z_1^N = A_{(123)}, \quad A_{(321)} z_1^N = A_{(132)}, \quad A_{(132)} z_2^N = A_{(213)}, \ \ldots, \tag{4.48}$$

which are the boundary relations written explicitly for every permutation.

5 The Bethe ansatz

We aim to solve the eigenvalues of a general block, V_n. There are n world lines with entering and exiting sites given by coordinate vectors $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$, respectively.

Let σ be any permutation in $\mathfrak{S}_n$, the symmetric group of order n, and let $\mathbf{z}$ be an n-tuple of yet unknown complex numbers

$$\mathbf{z} := \Big(z_1,\, z_2, \dots, z_n\Big) \in \mathbb{C}^n. \tag{5.1}$$

For a given $\sigma \in \mathfrak{S}_n$ and a coordinate vector $\vec{x}$, set

$$Z_\sigma^{\vec{x}} := \prod_{j=1}^{n} z_{\sigma(j)}^{x_j}, \tag{5.2}$$

where elements of $\mathbf{z}$ have been permuted accordingly. Fix $1 \leq n \leq N/2$. We define a candidate eigenvector in subspace Ω_n by

$$\Psi_n := \sum_{\mathbf{e}_i \in \Omega_n} \psi(\vec{x}_i)\mathbf{e}_i, \tag{5.3}$$

where a coordinate vector $\vec{x}_i$ is corresponds to each basis vector $\mathbf{e}_i$. The Bethe Ansatz states that scalar projections $\psi(\vec{x}_i)$ are given by

$$\begin{aligned} \psi(\vec{x}_i) &:= \sum_{\sigma \in \mathfrak{S}_n} A_\sigma \prod_{j=1}^{n} z_{\sigma(j)}^{x_j} \\ &= \sum_{\sigma \in \mathfrak{S}_n} A_\sigma Z_\sigma^{\vec{x}_i}, \end{aligned} \tag{5.4}$$

where coefficients A_σ are expressed in terms of function $A(z_{\sigma(1)}, \dots,\, z_{\sigma(n)})$ that is yet to be determined. We impose two conditions:

Definition 5.1. (The Internal Relations)
For each $j \in \{1, \dots,\, n-1\}$ and any $\sigma \in \mathfrak{S}_n$, the following always holds

$$A_\sigma\Big[M(z_{\sigma(j)})L(z_{\sigma(j+1)}) - 1\Big] + A_{\sigma\circ(j;j+1)}\Big[M(z_{\sigma(j+1)}L(z_{\sigma(j)}) - 1\Big] = 0, \tag{5.5}$$

with $(j; j+1)$ being the transposition permuting j and $j+1$.

Definition 5.2. (The Boundary Relations)
For any $\sigma \in \mathfrak{S}_n$, the following always holds

$$z_{\sigma(1)}^N = \frac{A_\sigma}{A_{\sigma\circ\tau}}, \tag{5.6}$$

where τ is the following permutation

$$\tau = (1\,2\,3\,\dots\,n-1;\, n), \tag{5.7}$$

that is $\tau(i) = i + 1$ for all $1 \leq i < n$ and $\tau(n) = 1$.

The trial eigenvector, as defined above, is the Bethe Ansatz for the eigenvalue equation of the transfer matrix V. The coefficients A_σ may already be solved directly from the internal relations. Both relations combined give n equations from which all elements of $\mathbf{z}$ can be solved in principle. These equations have been accordingly named *Bethe equations*. The end of the section is dedicated to these matters, allowing us now to focus on proving the eponymous ansatz:

Theorem 5.3. *(the Bethe ansatz) Consider Ψ_n as defined in Eq. (5.3) such that the A_σ and the $\mathbf{z} = (z_1, \ldots, z_n)$ satisfy internal and boundary relations above, and suppose all elements are distinct and $z_j \neq \frac{a}{b}$. Then, Ψ_n satisfies the eigenvalue equation $V\Psi_n = \Lambda_n \Psi_n$, where*

$$\Lambda_n = a^N \prod_{j=1}^{n} L(z_j) + b^N \prod_{j=1}^{n} M(z_j). \tag{5.8}$$

Remark 5.4. We have no guarantees that the eigenvector proposed by the Bethe Ansatz is non-trivial, that is $\Psi_n \neq 0$. We show later that if any two elements in $\mathbf{z}$ are equal, Ψ_n is identically zero. This leads to the condition that $z_1, \ldots, z_n$ be distinct. The second restriction is to avoid the singularity of functions L and M. The singular case, where some $z_j = \frac{a}{b}$, is covered in the next chapter.

When we locate the sub-block V_n such that Λ_n is, the maximal eigenvalue Λ of V, then the free energy can be evaluated from the limit

$$f = \lim_{N\to\infty} -\frac{1}{N\beta} \ln\Big[a^N \prod_{j=1}^{n} L(z_j) + b^N \prod_{j=1}^{n} M(z_j)\Big]. \tag{5.9}$$

5.1 The eigenvalue equation

We proceed to prove Theorem 5.3 in style of Duminil-Copin et al.[4]. The eigenvalue equation of $V\Psi_n$ along some basis vector $\mathbf{e}_i$ reads

$$\begin{aligned}\Lambda_n \psi(\vec{x}) = &\sum_{\sigma\in\mathfrak{S}_n} A_\sigma \sum_{y_1=x_1}^{x_2} \sum_{y_2=x_2}^{x_3}{}' \cdots \sum_{y_n=x_n}^{N}{}' E_{11} D_{12} E_{22} D_{23} \ldots E_{nn} D(y_n - N, x_1) c^{2n} Z_\sigma^{\vec{y}} \\ &+ \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \sum_{y_1=1}^{x_1} \sum_{y_2=x_1}^{x_2}{}' \cdots \sum_{y_n=x_{n-1}}^{x_n}{}' E(x_n - N, y_1) D_{11} E_{12} D_{22} E_{23} \ldots D_{nn} c^{2n} Z_\sigma^{\vec{y}},\end{aligned} \tag{5.10}$$

where we have abbreviated most of the functions as $D_{ij} = D(y_i, x_j)$ and $E_{ij} = E(x_i, y_j)$. The two sums arising from two different ways of interlacing are again color-coded. The primed sums indicate that no two coordinates of $\vec{y}$ can be equal.

We again reorganize the sums:

$$\Lambda_n\psi(\vec{x}) = \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Big[\sum_{y_j=x_j}^{x_{j+1}}{}' E_{jj}D_{j;j+1}z_{\sigma(j)}^{y_j}\Big]\sum_{y_n=x_n}^{N}{}' E_{nn}D(y_n-N,x_1)c^{2n}z_{\sigma(n)}^{y_n}$$
$$+\sum_{\sigma\in\mathfrak{S}_n} A_\sigma \sum_{y_1=1}^{x_1}{}' E(x_n-N,y_1)D_{11}z_{\sigma(1)}^{y_1}\prod_{j=1}^{n-1}\Big[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E_{j;j+1}D_{j+1;j+1}z_{\sigma(j+1)}^{y_{j+1}}\Big]c^{2n}. \tag{5.11}$$

The boundary relations allow us to combine the two sums together.

Lemma 5.5. *Assume that elements of* $\mathbf{z}$ *and coefficients* A_σ *satisfy the boundary relations in Eq. (5.6). We have*

$$\sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z_\sigma^{\vec{x}} = \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z_\sigma^{\tau^{-1}\vec{x}}, \tag{5.12}$$

where $\tau^{-1}\vec{x} = (x_0,\, x_1, \dots,\, x_{n-1})$ *with* $x_0 = x_n - N$*.*

Proof. From the boundary relations we obtain

$$A_{\sigma\circ\tau} = A_\sigma z_{\sigma(1)}^{-N}. \tag{5.13}$$

We make a change of variables $\sigma \to \sigma\circ\tau$. Recall that permutation τ bumps up $1 \le j < n$ to $j+1$ and sends n to 1.

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n} A_{\sigma\circ\tau} Z_{\sigma\circ\tau}^{\vec{x}} &= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma z_{\sigma(1)}^{-N}\prod_{j=1}^{n} z_{[\sigma\circ\tau](j)}^{x_j}\\ &= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma z_{\sigma(1)}^{-N} z_{\sigma(1)}^{x_n}\prod_{j=1}^{n-1} z_{\sigma(j+1)}^{x_j}\\ &= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=0}^{n-1} z_{\sigma(j+1)}^{x_j}\\ &= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z_\sigma^{\tau^{-1}\vec{x}},\end{aligned} \tag{5.14}$$

where in the third equality we denoted $x_n - N = x_0$. ☐

Corollary 5.6. *The two sums in Eq. (5.11) can be combined into one, giving*

$$V\psi(\vec{x}) = \sum_{\sigma\in\mathfrak{S}_n} A_\sigma R_\sigma, \tag{5.15}$$

where we defined for further consideration

$$R_\sigma := \prod_{j=1}^{n}\Big[\sum_{y_j=x_{j-1}}^{x_j}{}' E(x_{j-1},\, y_j)D(y_j,\, x_j)c^2 z_{\sigma(j)}^{y_j}\Big], \tag{5.16}$$

with the primed sums indicating the usual restriction on the y_j*:s.*

Proof. Apply the change of variables in Lemma 5.5 to the first sum. We highlight the changes between the rows

$$\begin{aligned}
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Bigg[\sum_{y_j=x_j}^{x_{j+1}}{}' E(x_j,\, y_j)D(y_j,\, x_{j+1})z_{\sigma(j)}^{y_j}\Bigg]\sum_{y_n=x_n}^{N}{}' E(x_n,\, y_n)D(y_n-N, x_1)c^{2n}z_{\sigma(n)}^{y_n} \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Bigg[\sum_{y_j=x_j}^{x_{j+1}}{}' E(x_j,\, y_j)D(y_j,\, x_{j+1})z_{\sigma(j+1)}{}^{y_j}\Bigg]\sum_{y_n=x_n}^{N}{}' E(x_n,\, y_n)D(y_n-N, x_1)c^{2n}z_{\sigma(1)}^{y_n-N}.
\end{aligned} \tag{5.17}$$

For $1 \le j < n$, y_j reindex y_j:s to y_{j+1} and reindex y_n to y_1. Then make the change of variables $y_1 - N \to y_1$. This procedure gives

$$\begin{aligned}
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Bigg[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E(x_j,\, y_{j+1})D(y_{j+1},\, x_{j+1})z_{\sigma(j+1)}^{y_{j+1}}\Bigg]\sum_{y_1=x_n}^{N}{}' E(x_n,\, y_1)D(y_1-N, x_1)c^{2n}z_{\sigma(1)}^{y_1-N} \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Bigg[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E(x_j,\, y_{j+1})D(y_{j+1},\, x_{j+1})z_{\sigma(j+1)}^{y_{j+1}}\Bigg]\sum_{y_1=x_n-N}^{0}{}' E(x_n,\, y_1+N)D(y_1, x_1)c^{2n}z_{\sigma(1)}^{y_1} \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Bigg[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E(x_j,\, y_{j+1})D(y_{j+1},\, x_{j+1})z_{\sigma(j+1)}^{y_{j+1}}\Bigg]\sum_{y_1=x_n-N}^{0}{}' E(x_n-N,\, y_1)D(y_1, x_1)c^{2n}z_{\sigma(1)}^{y_1},
\end{aligned} \tag{5.18}$$

where in the last equality we used the definition of E. The two sums in Eq. (5.11) are now easily combined, giving

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Bigg[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E_{j;j+1}D_{j+1;j+1}z_{\sigma(j+1)}^{y_{j+1}}\Bigg]\sum_{y_1=x_n-N}^{0}{}' E(x_n-N,\, y_1)D_{11}c^{2n}z_{\sigma(1)}^{y_1} \\
&+ \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \sum_{y_1=1}^{x_1}{}' E(x_n-N, y_1)D_{11}c^{2n}z_{\sigma(1)}^{y_1}\prod_{j=1}^{n-1}\Bigg[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E_{j;j+1}D_{j+1;j+1}z_{\sigma(j+1)}^{y_{j+1}}\Bigg] \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Bigg[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E_{j;j+1}D_{j+1;j+1}z_{\sigma(j+1)}^{y_{j+1}}\Bigg]c^{2n} \\
&\quad\times\Bigg(\sum_{y_1=x_n-N}^{0}{}' E(x_n-N,\, y_1)D_{11}z_{\sigma(1)}^{y_1} + \sum_{y_1=1}^{x_1}{}' E(x_n-N, y_1)D_{11}z_{\sigma(1)}^{y_1}\Bigg) \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^{n-1}\Bigg[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E_{j;j+1}D_{j+1;j+1}z_{\sigma(j+1)}^{y_{j+1}}\Bigg]c^{2n}\cdot\Bigg(\sum_{y_1=x_n-N}^{x_1}{}' E(x_n-N,\, y_1)D_{11}z_{\sigma(1)}^{y_1}\Bigg) \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=0}^{n-1}\Bigg[\sum_{y_{j+1}=x_j}^{x_{j+1}}{}' E(x_j,\, y_{j+1})D(y_{j+1},\, x_{j+1})z_{\sigma(j+1)}^{y_{j+1}}c^2\Bigg],
\end{aligned} \tag{5.19}$$

where in the last equality we set $x_0 = x_n - N$. ☐

The primary challenge now lies in evaluating the primed sums, which restrict our y_j:s from being equal. To tackle this obstacle, we initially compute the sums without any such restriction, dropping all the primes.

The "unprimed" sum has already been calculated in Eq. (B2) of the appendix to be

$$\sum_{y_j=x_{j-1}}^{x_j} E(x_{j-1},\, y_j)D(y_j,\, x_j)c^2 z_{\sigma(j)}^{y_j} = a^{x_j-x_{j-1}}L(z_{\sigma(j)})z_{\sigma(j)}^{x_{j-1}} + b^{x_j-x_{j-1}}M(z_{\sigma(j)})z_{\sigma(j)}^{x_j}. \tag{5.20}$$

Denote the counterpart to R_σ in Eq. (5.16) without restrictions on any of the sums as $R_\sigma(\varnothing)$. Using the above result, Eq. (5.20) $R_\sigma(\varnothing)$ becomes

$$R_\sigma(\varnothing) := \prod_{j=1}^{n}\Big[a^{x_j-x_{j-1}}L(z_{\sigma(j)})z_{\sigma(j)}^{x_{j-1}} + b^{x_j-x_{j-1}}M(z_{\sigma(j)})z_{\sigma(j)}^{x_j}\Big]. \tag{5.21}$$

We find ourselves in need of a means to express the product expansion while also identifying all the correction terms. Remarkably, these two needs are interconnected.

5.2 Words and correction terms

We expand the product using "words". Let $\mathscr{W} := \{L,\, M\}^n$, where $\mathscr{W}$ is a set containing every n-length sequence of L's and M's. The elements of $\mathscr{W}$ are called words and are denoted by $w = w_1w_2\ldots w_n$, where each w_i is either L or M. The product in Eq. (5.21) expands as:

$$\begin{aligned} R_\sigma(\varnothing) &= \sum_{w\in\mathscr{W}}\prod_{j:w_j=L} a^{x_j-x_{j-1}}L(z_{\sigma(k)})z_{\sigma(j)}^{x_{j-1}}\cdot\prod_{k:w_j=M} b^{x_j-x_{j-1}}M(z_{\sigma(j)})z_{\sigma(j)}^{x_j} \\ &= \sum_{w\in\mathscr{W}} v_\sigma(w,\varnothing), \end{aligned} \tag{5.22}$$

where we defined the summand as $v_\sigma(w,\varnothing)$. Words will also aid us in expressing all the correction terms, proving the following result in the rest of the section

Lemma 5.7. *For the product given by R_σ in Eq. (5.16), we find*

$$R_\sigma = \sum_{w\in\mathscr{W}} r_\sigma(w)Z_\sigma^{ab}(w), \tag{5.23}$$

where $r_\sigma(w)$ and $Z_\sigma^{ab}(w)$ are defined as

$$\begin{aligned} r_\sigma(w) &:= \Bigg[\prod_{\substack{j:\\ w_{j-1}w_j=LL}} L(z_{\sigma(j)})\prod_{\substack{j:\\ w_jw_{j+1}=MM}} M(z_{\sigma(j)})\prod_{\substack{j:\\ w_jw_{j+1}=ML}}\Big(M(z_{\sigma(j)})L(z_{\sigma(j+1)})-1\Big)\Bigg], \\ Z_\sigma^{ab}(w) &:= \prod_{j:\, w_j=L} a^{x_j-x_{j-1}}z_{\sigma(j)}^{x_{j-1}}\prod_{j:\, w_j=M} b^{x_j-x_{j-1}}z_{\sigma(j)}^{x_j}. \end{aligned} \tag{5.24}$$

We start by investigating what the unallowed cases *look* like. Recall the definitions of segments $\mathcal{E}$ and $\mathfrak{D}$ for which the functions E and D were implemented. In the product of Eq. (5.15) functions are grouped by pairs as $E(x_{j-1},\, y_j)D(y_j,\, x_j)$ with $1 \leq j \leq n$. Given such a factor, we can imagine a corresponding pair of neighboring

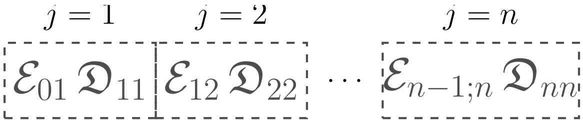


Figure 18: The pairing up of segments

segments, $\mathcal{E}_{j-1;j}$ and $\mathfrak{D}_{jj}$. We give an index j for each such pair, as can be seen in Fig. 18.

We need to subtract all terms where neighboring segments $\mathfrak{D}$ (belonging to some jth pair) and $\mathcal{E}$ (belonging to $j+1$th pair) are collapsed *both* to the same x_j. Such "double-collapse" is shown in Fig. 19

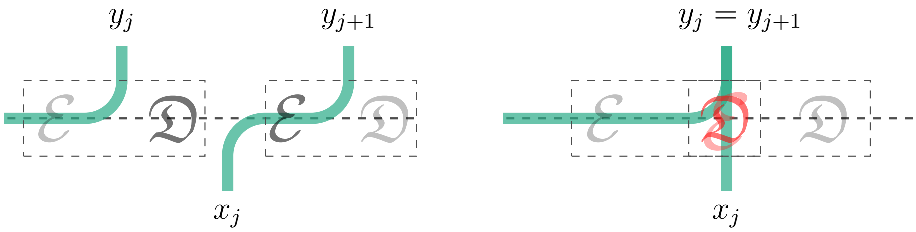


Figure 19: The double-collapse of segments $\mathfrak{D}$ and $\mathcal{E}$

Remark 5.8. Note that given pairs $j-1$ and k that are double-collapsed, it is not possible to simultaneously double-collapse j and $j+1$. Otherwise, that would mean $y_{j-1} = x_{j-1} = y_j$ and $y_j = x_{j+1} = y_{j+1}$ contradicting the fact that the coordinates of $\vec{x}$ are distinct.

We construct a subset $J \subset \{1, \ldots, n\}$, which dictates which neighboring segment pairs we choose to double-collapse.

Taking into account the Remark 5.8 above, J does not contain any successive integers. We again consider R_σ in Eq. (5.16) and drop all restrictions on the sums. However, for each $j \in J$ we fix $y_j = x_j = y_{j+1}$, hence double-collapsing the respective segments and degenerating the associated sums. We denote the quantity defined above as $R_\sigma(J)$.

Denote all correction terms as C. For each subset, J, quantity $R_\sigma(J)$ gives a sum over scenarios where some segments (restricted by J) are "fixed" to prohibited states while the rest (no restrictions) "experience" all states, the prohibited states included. Applying the inclusion-exclusion formula to all non-empty subsets J, we obtain all correction terms

$$C := \sum_{\varnothing \neq J \subset \{1,\ldots,n\}} (-1)^{|J|+1} R_\sigma(J). \tag{5.25}$$

Following the definition, $R_\sigma(J)$ is expressed as

$$\begin{aligned}
R_\sigma(J) :=& \prod_{j:\{j-1,k\}\cap J=\varnothing}\left[\sum_{y_j=x_{j-1}}^{x_k} E(x_{j-1},\, y_j)D(y_j,\, x_j)c^2 z_{\sigma(j)}^{y_j}\right] \\
&\times \prod_{j\in J}\left. E(x_{j-1},\, y_j)D(y_j,\, x_j)c^2 z_{\sigma(j)}^{y_j}\right|_{y_j=x_j}\left. E(x_j,\, y_{j+1})D(y_{j+1},\, x_{j+1})c^2 z_{\sigma(j+1)}^{y_{j+1}}\right|_{y_{j+1}=x_j} \\
=& \prod_{j:\{j-1,k\}\cap J=\varnothing}\left[a^{x_j-x_{j-1}}L(z_{\sigma(j)})z_{\sigma(j)}^{x_{j-1}} + b^{x_j-x_{j-1}}M(z_{\sigma(j)})z_{\sigma(j)}^{x_j}\right] \\
&\times \prod_{j\in J} a^{x_{j+1}-x_j} b^{x_j-x_{j-1}} z_{\sigma(j+1)}^{x_j} z_{\sigma(j)}^{x_j} \\
=& \prod_{j:\{j-1,k\}\cap J=\varnothing}\left[a^{x_j-x_{j-1}}L(z_{\sigma(j)})z_{\sigma(j)}^{x_{j-1}} + b^{x_j-x_{j-1}}M(z_{\sigma(j)})z_{\sigma(j)}^{x_j}\right] \\
&\times \prod_{j:\, j-1\in J} a^{x_j-x_{j-1}} z_{\sigma(j)}^{x_{j-1}} \cdot \prod_{j\in J} b^{x_j-x_{j-1}} z_{\sigma(j)}^{x_j},
\end{aligned} \tag{5.26}$$

where we repeated calculations in Eq. (5.21) for undegenerated sums. For these sums, we would like to expand the resulting product. The hope is to perform the expansion using the already established set of n-length words, $\mathscr{W}$.

We approach the task by reversing the direction: Instead of first fixing some subset J and expanding the product with words, we first fix some word and then find appropriate subsets J suited for it.

We are led to the right track by the following trivial fact: Given a word $w \in \mathscr{W}$ whose letters $w_{k-1}w_k = ML$, it is not possible that letters $w_k w_{k+1} = ML$ which is the same restriction as in Remark 5.8. Thus, the letter pairs ML can be associated with collapsed segments and, in the end, with all correction terms.

Definition 5.9. Let $\mathscr{W}_0 = \{L \ldots L,\, M \ldots M\}$ be the set of "constant words". Fix any word $w \in \mathscr{W} \setminus \mathscr{W}_0$. Let $S(w)$ be a set of indices k such that $w_k w_{k+1} = ML$ modulo n. $S(w)$ does not contain any consecutive indices and is not empty. Take any subset $S \subseteq S(w)$ and double-collapse the segment pairs k and $k+1$ for each $k \in S$. This procedure is illustrated in Fig. 20.

$$w = \overset{\cancel{w_1}}{\cancel{L}}\ \overset{w_2}{M}\,\overset{w_3}{M}\,\overset{w_4}{L}\,\overset{w_5}{L}\,\overset{w_6}{L}\,\overset{w_7}{L}\ \overset{\cancel{w_8}}{\cancel{M}}\,\overset{\cancel{w_9}}{\cancel{L}}\ \overset{w_{10}}{M}\,\overset{w_{11}}{M}\,\overset{w_{12}}{M}\,\overset{w_{13}}{L}\,\overset{w_{14}}{L}\,\overset{w_{15}}{L}\ \overset{\cancel{w_{16}}}{\cancel{M}}\,\overset{\cancel{w_{17}}}{\cancel{L}} \ldots \overset{w_{n-1}}{M}\ \overset{\cancel{w_n}}{\cancel{M}}$$

$$S(w) = \{3,\, 8,\, 12,\, 16,\, n\}$$
$$S = \{8,\, 16,\, n\}$$

Figure 20: Construction of a set $S(w)$ (in blue) and it's subset S (note how $w_n w_1 = ML$ is formed across the boundary)

The calculation in Eq. (5.26) can be repeated to obtain a quantity in which the sums indicated by $S \subseteq S(w)$) are degenerated:

$$
\begin{aligned}
R_\sigma(S) = &\prod_{j:\{j-1,j\}\cap S=\varnothing} \Big[a^{x_j-x_{j-1}} L(z_{\sigma(j)}) z_{\sigma(j)}^{x_{j-1}} + b^{x_j-x_{j-1}} M(z_{\sigma(j)}) z_{\sigma(j)}^{x_j}\Big] \\
&\times \prod_{\substack{w_{j-1}w_j=ML \\ j:\, j-1\in S}} a^{x_j-x_{j-1}} z_{\sigma(j)}^{x_{j-1}} \prod_{\substack{w_j w_{j+1}=ML \\ j\in S}} b^{x_j-x_{j-1}} z_{\sigma(j)}^{x_j},
\end{aligned} \tag{5.27}
$$

where we applied the definitions of sets S and $S(w)$.

We expand from the upper product just one term, which we denote as $v_\sigma(w,\, S)$. The expansion is dictated by the remaining letters of w: for those $j \notin S$, letters w_j designate which term in the product to choose.

$$
\begin{aligned}
v_\sigma(w,\, S) := &\prod_{\substack{j:\{j-1,j\}\cap S=\varnothing \\ w_j=L}} a^{x_j-x_{j-1}} L(z_{\sigma(j)}) z_{\sigma(j)}^{x_{j-1}} \prod_{\substack{j:\{j-1,j\}\cap S=\varnothing \\ w_j=M}} b^{x_j-x_{j-1}} M(z_{\sigma(j)}) z_{\sigma(j)}^{x_j} \\
&\times \prod_{\substack{w_{j-1}w_j=ML \\ j:\, j-1\in S}} a^{x_j-x_{j-1}} z_{\sigma(j)}^{x_{j-1}} \prod_{\substack{w_j w_{j+1}=ML \\ j\in S}} b^{x_j-x_{j-1}} z_{\sigma(j)}^{x_j}.
\end{aligned} \tag{5.28}
$$

Note that the letters $w_j w_{j+1} = ML$ for $j \in S(w) \setminus S$ are also included in the upper row. We collect all of the powers of a, b and z together, for which we define

$$
Z_\sigma^{ab}(w) := \prod_{j:\, w_j=L} a^{x_j-x_{j-1}} z_{\sigma(j)}^{x_{j-1}} \prod_{j:\, w_j=M} b^{x_j-x_{j-1}} z_{\sigma(j)}^{x_j}. \tag{5.29}
$$

This simplifies our term in Eq. (5.28) into

$$
\begin{aligned}
v_\sigma(w,\, S) &= \Big[\prod_{\substack{j:\{j-1,j\}\cap S=\varnothing \\ w_j=L}} L(z_{\sigma(j)}) \prod_{\substack{j:\{j-1,j\}\cap S=\varnothing \\ w_j=M}} M(z_{\sigma(j)})\Big] Z_\sigma^{ab}(w) \\
&= \Big[\prod_{\substack{j:\, j-1\notin S \\ w_j=L}} L(z_{\sigma(j)}) \prod_{\substack{j\notin S \\ w_j=M}} M(z_{\sigma(j)})\Big] Z_\sigma^{ab}(w),
\end{aligned} \tag{5.30}
$$

where in the last equality we dropped the redundant restrictions (if $j \in S$ then $w_j \neq L$ and if $j-1 \in S$ then $w_j \neq M$). Note how the degenerated sums have been all absorbed into $Z_\sigma^{ab}(w)$.

After separating the factors corresponding to letters $w_j w_{j+1} = ML$ that we didn't double-collapse, we have

$$
v_\sigma(w,\, S) = \Big[\prod_{\substack{j: \\ w_{j-1}w_j=LL}} L(z_{\sigma(j)}) \prod_{\substack{j: \\ w_j w_{j+1}=MM}} M(z_{\sigma(j)}) \prod_{\substack{j:\, w_j w_{j+1}=ML \\ j\in S(w)\setminus S}} M(z_{\sigma(j)}) L(z_{\sigma(j+1)})\Big] Z_\sigma^{ab}(w). \tag{5.31}
$$

Above, we have obtained just one term of many corresponding to the correction term associated with w. In it, we have double-collapsed only some of the pairs corresponding to letters $w_j w_{j+1} = ML$ with $j \in S$ whose weight is contained in

$Z^{ab}_\sigma(w)$. Those pairs corresponding also to the letters $w_j w_{j+1} = ML$ but with $j \in S(w) \setminus S$ contribute the full, not-degenerated, weight in terms of a product $M(z_{\sigma(j)})L(z_{\sigma(j+1)})$.

Denote a full correction term associated with w by $c(w)$. We obtain all terms of $c(w)$ by summing over every subset $S \subset S(w)$. Applying inclusion-exclusion gives

$$\begin{aligned} c(w) &:= \sum_{\varnothing \neq S \subset S(w)} (-1)^{|S|} v_\sigma(w, S) \\ &= \Bigg[\prod_{\substack{j: \\ w_{j-1}w_j = LL}} L(z_{\sigma(j)}) \prod_{\substack{j: \\ w_j w_{j+1} = MM}} M(z_{\sigma(j)}) \Bigg] Z^{ab}_\sigma(w) \sum_{\varnothing \neq S \subset S(w)} (-1)^{|S|+1} \prod_{\substack{j:\, w_j w_{j+1} = ML \\ j \in S(w) \setminus S}} M(z_{\sigma(j)}) L(z_{\sigma(j+1)}). \end{aligned} \tag{5.32}$$

By summing over all words $w \in \mathscr{W} \setminus \{\mathscr{W}_0\}$, we take into account all correction terms and thus succeeded in our task to express Eq. (5.25) through set $\mathscr{W}$.

$$C = \sum_{w \in \mathscr{W} \setminus \{\mathscr{W}_0\}} c(w). \tag{5.33}$$

However, we are interested in R_σ. Correction of $v(w, \varnothing)$ in the sum $R_\sigma(\varnothing)$ is $c(w)$. Subtracting $c(w)$ from $v(w, \varnothing)$ gives

$$\begin{aligned} & v_\sigma(w, \varnothing) - \sum_{\varnothing \neq S \subset S(w)} (-1)^{|S|+1} v_\sigma(w, S) \\ &= \sum_{S \subset S(w)} (-1)^{|S|} v_\sigma(w, S) \\ &= \Bigg[\prod_{\substack{j: \\ w_{j-1}w_j = LL}} L(z_{\sigma(j)}) \prod_{\substack{j: \\ w_j w_{j+1} = MM}} M(z_{\sigma(j)}) \Bigg] Z^{ab}_\sigma(w) \sum_{S \subset S(w)} (-1)^{|S|} \prod_{\substack{j:\, w_j w_{j+1} = ML \\ j \in S(w) \setminus S}} M(z_{\sigma(j)}) L(z_{\sigma(j+1)}) \\ &= \Bigg[\prod_{\substack{j: \\ w_{j-1}w_j = LL}} L(z_{\sigma(j)}) \prod_{\substack{j: \\ w_j w_{j+1} = MM}} M(z_{\sigma(j)}) \Bigg] Z^{ab}_\sigma(w) \prod_{\substack{j: \\ w_j w_{j+1} = ML}} \Big[M(z_{\sigma(j)}) L(z_{\sigma(j+1)}) - 1 \Big], \end{aligned} \tag{5.34}$$

where in the first equality we combined the two terms by allowing S to be an empty set and in the last equality we factorized the sum.

Summing over all words in $\mathscr{W}$ gives the wanted expression R_σ, proving Lemma 5.7.

5.3 Proof of the Bethe ansatz

Combining Corollary 5.6 and Lemma 5.7, the eigenvalue equation right now reads

$$V\psi(\vec{x}) = \sum_{w \in \mathscr{W}} \sum_{\sigma \in \mathfrak{S}_n} A_\sigma r_\sigma(w) Z^{ab}_\sigma(w). \tag{5.35}$$

Thus far we have only used the boundary relations, which allowed us to merge the two sums given by the two ways of interlacing. By invoking the internal relations we find that all terms cancel in the sum above except two that are given by constant words in $\mathscr{W}_0 = \{L \dots L,\ M \dots M\}$.

Lemma 5.10. *Fix any $w \in \mathscr{W} \setminus \mathscr{W}_0$. Assume there exists an N-tuple $\mathbf{z}$ and coefficients A_σ that satisfy the internal and boundary relations. Then,*

$$\sum_{\sigma \in \mathfrak{S}_n} A_\sigma r_\sigma(w) Z_\sigma^{ab}(w) = 0. \tag{5.36}$$

Proof. Since $w \in \mathscr{W} \setminus \mathscr{W}_0$, there exists at least one index m such that $w_m w_{m+1} = ML$ since integers are modulo n. Fix any permutation σ and pair it with $\sigma \circ (m; m+1)$ where m and $m+1$ are transposed. Assume for now that $m \neq n$. Consider the combination

$$\begin{aligned}
&A_\sigma r_\sigma(w) Z_\sigma^{ab}(w) + A_{\sigma\circ(m;m+1)} v_{\sigma\circ(m;m+1)}(w) Z^{ab}_{\sigma\circ(m;m+1)}(w) \\
&= A_\sigma r_\sigma(w) Z_\sigma^{ab}(w)\Big[1 + \frac{A_{\sigma\circ(m;m+1)}}{A_\sigma} \cdot \frac{r_{\sigma\circ(m;m+1)}(w)}{v_\sigma(w)} \cdot \frac{Z^{ab}_{\sigma\circ(m;m+1)}(w)}{Z_\sigma^{ab}(w)}\Big] \\
&= A_\sigma v_\sigma(w) Z_\sigma^{ab}(w)\Big[1 + \frac{A_{\sigma\circ(m;m+1)}}{A_\sigma} \cdot \frac{M(z_{\sigma(m+1)})L(z_{\sigma(m)}) - 1}{M(z_{\sigma(m)})L(z_{\sigma(m+1)}) - 1} \cdot \frac{z_{\sigma(m+1)}^{x_m} z_{\sigma(m)}^{x_{(m+1)-1}}}{z_{\sigma(m)}^{x_m} z_{\sigma(m+1)}^{x_{(m+1)-1}}}\Big] \\
&= A_\sigma v_\sigma(w) Z_\sigma^{ab}(w)\Big[1 + \frac{A_{\sigma\circ(m;m+1)}}{A_\sigma} \cdot \frac{M(z_{\sigma(m+1)})L(z_{\sigma(m)}) - 1}{M(z_{\sigma(m)})L(z_{\sigma(m+1)}) - 1} \cdot 1\Big].
\end{aligned} \tag{5.37}$$

From the internal relations in Definition 5.1 we obtain for every $m \in \{1, \dots, n-1\}$ and any $\sigma \in \mathfrak{S}_n$

$$\frac{A_{\sigma\circ(m;m+1)}}{A_\sigma} = -\frac{M(z_{\sigma(m)})L(z_{\sigma(m+1)}) - 1}{M(z_{\sigma(m+1)})L(z_{\sigma(m)}) - 1}. \tag{5.38}$$

Therefore for every $m \in \{1, \dots, n-1\}$ we obtain

$$A_\sigma r_\sigma(w) Z_\sigma^{ab}(w) + A_{\sigma\circ(m;m+1)} r_{\sigma\circ(m;m+1)}(w) Z^{ab}_{\sigma\circ(m;m+1)}(w) = 0. \tag{5.39}$$

Next, we consider the less direct case $m = n$, where $n+1$ is regarded periodically as 1. Considering the ratios separately, we first obtain

$$\begin{aligned}
\frac{Z^{ab}_{\sigma\circ(n;n+1)}(w)}{Z_\sigma^{ab}(w)} &= \frac{z_{\sigma(n+1)}^{x_n} z_{\sigma(n)}^{x_{(n+1)-1}}}{z_{\sigma(n)}^{x_n} z_{\sigma(n+1)}^{x_{(n+1)-1}}} \\
&= \frac{z_{\sigma(1)}^{x_n} z_{\sigma(n)}^{x_0}}{z_{\sigma(n)}^{x_n} z_{\sigma(1)}^{x_0}} \\
&= \frac{z_{\sigma(1)}^{x_n} z_{\sigma(n)}^{x_n - N}}{z_{\sigma(n)}^{x_n} z_{\sigma(1)}^{x_n - N}} \\
&= \left(\frac{z_{\sigma(1)}}{z_{\sigma(n)}}\right)^N.
\end{aligned} \tag{5.40}$$

The ratio of r:s is straightforward

$$\frac{r_{\sigma\circ(n;n+1)}}{r_\sigma} = \frac{M(r_{\sigma(1)})L(z_{\sigma(n)}) - 1}{M(z_{\sigma(n)})L(z_{\sigma(1)}) - 1}, \tag{5.41}$$

but the ratio of coefficients A_σ and $A_{\sigma\circ(n\,1)}$ is not. Since the transposition is done on neighboring indices "across the boundary", we recall the boundary relations in Definition 5.2. For any permutation $\sigma \in \mathfrak{S}_n$ we have

$$z_{\sigma(1)}^N = \frac{A_\sigma}{A_{\sigma\circ\tau}}, \tag{5.42}$$

where $\tau = (1\,2\,3\,\ldots\,n-1;\,n)$. As a direct consequence, we find

$$z_{\sigma(n)}^N = z_{\sigma\circ\tau^{-1}(1)}^N = \frac{A_{\sigma\circ\tau^{-1}}}{A_{\sigma\circ\tau^{-1}\circ\tau}} = \frac{A_{\sigma\circ\tau^{-1}}}{A_\sigma}. \tag{5.43}$$

We decompose transposition $(n\,1)$ using τ as follows

$$\begin{aligned}(n\,1) &= (n; n-1\ldots 3\,2\,1)\circ(1\,2)\circ(1\,2\,3\ldots\,n-1;n)\\ &= \tau^{-1}\circ(1\,2)\circ\tau.\end{aligned} \tag{5.44}$$

Using this decomposition we obtain

$$\begin{aligned}\frac{A_{\sigma\circ(n\,1)}}{A_\sigma} &= \frac{A_{\sigma\circ\tau^{-1}\circ(1\,2)\circ\tau}}{A_\sigma}\\ &= \frac{A_{\sigma\circ\tau^{-1}}}{A_\sigma}\cdot\frac{A_{\sigma\circ\tau^{-1}\circ(1\,2)}}{A_{\sigma\circ\tau^{-1}}}\cdot\frac{A_{\sigma\circ\tau^{-1}\circ(1\,2)\circ\tau}}{A_{\sigma\circ\tau^{-1}\circ(1\,2)}}\\ &= z_{\sigma(n)}^N\cdot\Big[-\frac{M(z_{\sigma\circ\tau^{-1}(1)})L(z_{\sigma\circ\tau^{-1}(2)})-1}{M(z_{\sigma\circ\tau^{-1}(2)})L(z_{\sigma\circ\tau^{-1}(1)})-1}\Big]\cdot z_{\sigma(1)}^{-N}\\ &= -\Big(\frac{z_{\sigma(n)}}{z_{\sigma(1)}})\Big)^N\cdot\frac{M(z_{\sigma(n)})L(z_{\sigma(1)})-1}{M(z_{\sigma(1)})L(z_{\sigma(n)})-1},\end{aligned} \tag{5.45}$$

where on the third line we used internal and boundary relations in Eq. (5.5), Eq. (5.6) and Eq. (5.43). Combining the obtained results in case $m = n$ the investigated combination gives

$$\begin{aligned}&1+\frac{A_{\sigma\circ(n;1)}}{A_\sigma}\cdot\frac{r_{\sigma\circ(n;1)}(w)}{r_\sigma(w)}\cdot\frac{Z^{ab}_{\sigma\circ(n;1)}(w)}{Z^{ab}_\sigma(w)}\\ &= 1-\Big(\frac{z_{\sigma(n)}}{z_{\sigma(1)}})\Big)^N\frac{M(z_{\sigma(n)})L(z_{\sigma(1)})-1}{M(z_{\sigma(1)})L(z_{\sigma(n)})-1}\cdot\frac{M(z_{\sigma(1)})L(z_{\sigma(n)})-1}{M(z_{\sigma(n)})L(z_{\sigma(1)})-1}\Big(\frac{z_{\sigma(1)}}{z_{\sigma(n)}}\Big)^N\\ &= 0.\end{aligned} \tag{5.46}$$

We finish the proof by rewriting the sum of the lemma as

$$\begin{aligned}&\sum_{\sigma\in\mathfrak{S}_n} A_\sigma r_\sigma(w)Z^{ab}_\sigma(w)\\ &= \frac{1}{2}\sum_{\sigma\in\mathfrak{S}_n}\Big[A_\sigma r_\sigma(w)Z^{ab}_\sigma(w) + A_{\sigma\circ(m;m+1)}r_{\sigma\circ(m;m+1)}(w)Z^{ab}_{\sigma\circ(m;m+1)}(w)\Big]\\ &= 0.\end{aligned} \tag{5.47}$$

□

Remark 5.11. Combining the usual internal relations in Definition 5.1 with Eq. (5.45), we obtain for all $m \in \{1, \ldots, n\}$ the formula

$$\frac{A_{\sigma\circ(m;m+1)}}{A_\sigma} = -\Big(\frac{z_{\sigma(m)}}{z_{\sigma(m+1)}}\Big)\Big)^{\mathbb{1}_{m=n}N} \cdot \frac{M(z_{\sigma(m)})L(z_{\sigma(m+1)})-1}{M(z_{\sigma(m+1)})L(z_{\sigma(m)})-1}, \tag{5.48}$$

when $n+1$ is considered periodically as 1.

We compute the terms associated with the constant words. The contribution of the word $M \ldots M$ gives

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_n} A_\sigma r_\sigma(M\ldots M) Z^{ab}_\sigma(M\ldots M) \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^n M(z_{\sigma(j)}) \prod_{j=1}^n b^{x_j - x_{j-1}} z_{\sigma(j)}^{x_j} \\
&= b^{x_n - x_0} \prod_{j=1}^n M(z_j) \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^n z_{\sigma(j)}^{x_j} \\
&= b^N \prod_{j=1}^n M(z_j) \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z^{\vec{x}}_\sigma,
\end{aligned} \tag{5.49}$$

where the last equality follows from the definition of $Z^{\vec{x}}_\sigma$ in Eq. (5.2). Similarly the contribution of the word $L \ldots L$ gives

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_n} A_\sigma r_\sigma(L\ldots L) Z^{ab}_\sigma(L\ldots L) \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^n L(z_{\sigma(j)}) \prod_{j=1}^n a^{x_j - x_{j-1}} z_{\sigma(j)}^{x_{j-1}} \\
&= a^{x_n - x_0} \prod_{j=1}^n L(z_j) \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{j=1}^n z_{\sigma(j)}^{x_{j-1}} \\
&= a^N \prod_{j=1}^n L(z_j) \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z^{\tau^{-1}\vec{x}}_\sigma \\
&= a^N \prod_{j=1}^n L(z_j) \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z^{\vec{x}}_\sigma,
\end{aligned} \tag{5.50}$$

where in the last equality we used the change of variables formula of Lemma 5.5.

Combining Lemma 5.10 with the computations above we simplify the eigenvalue equation into a sum of only two terms,

$$\begin{aligned}
V\psi(\vec{x}) &= \sum_{w\in\mathscr{W}_0}\sum_{\sigma\in\mathfrak{S}_n} A_\sigma r_\sigma(w) Z_\sigma^{ab}(w) + \overbrace{\sum_{w\in\mathscr{W}\setminus\mathscr{W}_0}\sum_{\sigma\in\mathfrak{S}_n} A_\sigma r_\sigma(w) Z_\sigma^{ab}(w)}^{=0} \\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma r_\sigma(M\ldots M) Z_\sigma^{ab}(M\ldots M) + \sum_{\sigma\in\mathfrak{S}_n} A_\sigma r_\sigma(L\ldots L) Z_\sigma^{ab}(L\ldots L) \\
&= \left[a^N \prod_{j=1}^n L(z_j) + b^N \prod_{j=1}^n M(z_j)\right] \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z_\sigma^{\vec{x}} \\
&= \left[a^N \prod_{j=1}^n L(z_j) + b^N \prod_{j=1}^n M(z_j)\right] \psi(\vec{x}),
\end{aligned} \tag{5.51}$$

which concludes the proof of Theorem 5.3.

The Bethe Ansatz has yielded an eigenvalue of general block V_n. In the formulation of Theorem 5.3 we have put aside the coefficients A_σ and unknown tuple of complex numbers $\mathbf{z}$ by assuming their existence. This gave us the convenience to focus on the computational part of the ansatz. To finish the treatment, we turn our attention to the elephant in the room.

5.4 Bethe equations

The coefficients A_σ and complex numbers $\mathbf{z}$ are subject to internal relations in Definition 5.1 and boundary relations in Definition 5.2. We begin by solving the coefficients. Through a direct computation detailed in Eq. (B12) of the appendix, we find the following factorization

$$M(z_k)L(z_l) - 1 = -c^2 \frac{S(z_k,\, z_l)}{(a-bz_k)(a-bz_l)}, \tag{5.52}$$

with the function S defined as

$$S(z_k, z_l) := 1 - 2\Delta\, z_l + z_k z_l, \tag{5.53}$$

and where the parameter Δ is given by

$$\Delta := \frac{a^2+b^2-c^2}{2ab}. \tag{5.54}$$

There is both mathematical significance and physical interpretations connected to S and Δ, as will be seen later on. Inserting Eq. (5.52) into internal relations, we obtain a simplified internal condition

$$A_\sigma S(z_{\sigma(j)}, z_{\sigma(j+1)}) + A_{\sigma\circ(j;j+1)} S(z_{\sigma(j+1)}, z_{\sigma(j)}) = 0, \tag{5.55}$$

from which coefficients A_σ can now be solved.

Lemma 5.12. *The internal condition Eq. (5.55) has a solution (up to a constant of normalization)*

$$A_\sigma = \varepsilon(\sigma) \prod_{1\le k<l\le n} S(z_{\sigma(l)}, z_{\sigma(k)}), \tag{5.56}$$

where $\varepsilon(\sigma)$ is a signature of the permutation:

$$\varepsilon(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases} \tag{5.57}$$

Proof. In the product, group up the factors in a way that gives special consideration to certain indices j and $j+1$:

$$\begin{aligned}
&\prod_{1\le k\le l\le n} S(z_{\sigma(l)}, z_{\sigma(k)}) \\
&= \overbrace{\left[\prod_{1\le k\le j-1} S(z_{\sigma(j)}, z_{\sigma(k)})\right]}^{\text{The terms where: } l=j} \overbrace{\left[\prod_{1\le k\le j-1} S(z_{\sigma(j+1)}, z_{\sigma(k)})\right]}^{l=j+1} \overbrace{\left[\prod_{\substack{1\le k<l\le n\\ k,l\ne j,j+1}} S(z_{\sigma(l)}, z_{\sigma(k)})\right]}^{k,l\ne j,j+1} \\
&\quad\times \overbrace{\left[\prod_{j+2\le l\le n} S(z_{\sigma(l)}, z_{\sigma(j)})\right]}^{k=j} \overbrace{\left[\prod_{j+2\le l\le n} S(z_{\sigma(l)}, z_{\sigma(j+1)})\right]}^{k=j+1} \cdot \overbrace{S(z_{\sigma(j+1)}, z_{\sigma(j)})}^{k=j,\ l=j+1} \\
&= \left[\prod_{1\le k\le j-1} S(z_{\sigma(j)}, z_{\sigma(k)}) S(z_{\sigma(j+1)}, z_{\sigma(k)})\right] \left[\prod_{\substack{1\le k<l\le n\\ k,l\ne j,j+1}} S(z_{\sigma(l)}, z_{\sigma(k)})\right] \\
&\quad\times \left[\prod_{j+2\le l\le n} S(z_{\sigma(l)}, z_{\sigma(j)}) S(z_{\sigma(l)}, z_{\sigma(j+1)})\right] \cdot S(z_{\sigma(j+1)}, z_{\sigma(j)}).
\end{aligned} \tag{5.58}$$

In the last equality, all square bracketed expressions are invariant under permutation $(j; j+1)$. Therefore, the substitution of the proposed solution Eq. (5.56) into the ratio

$$\begin{aligned}
\frac{A_\sigma}{A_{\sigma\circ(j;j+1)}} &= \frac{\varepsilon(\sigma)}{\varepsilon(\sigma\circ(j;j+1))} \frac{\prod_{1\le k<l\le n} S(z_{\sigma(l)}, z_{\sigma(k)})}{\prod_{1\le k<l\le n} S(z_{[\sigma\circ(j;j+1)](l)}, z_{[\sigma\circ(j;j+1)](k)})} \\
&= \frac{\varepsilon(\sigma)}{-1\cdot\varepsilon(\sigma)} \frac{S(z_{\sigma(j+1)}, z_{\sigma(j)})}{S(z_{[\sigma\circ(j;j+1)](j+1)}, z_{[\sigma\circ(j;j+1)](j)})} \\
&= -\frac{S(z_{\sigma(j+1)}, z_{\sigma(j)})}{S(z_{\sigma(j)}, z_{\sigma(j+1)})},
\end{aligned} \tag{5.59}$$

proves the claim. □

Corollary 5.13. *If at least two elements in $\mathbf{z} = (z_1, \ldots, z_n)$ are equal, then Ψ_n given by Eq. (5.3) is identically equal to zero.*

Proof. Fix $\sigma \in \mathfrak{S}_n$ and $\vec{x} = (x_1, \dots, x_n)$. Let m and s be the distinct indices that are mapped by σ to the two equal elements in $\mathbf{z}$. Consider the contribution of σ in the coefficient $\psi(\vec{x})$, which is equal to

$$A_\sigma \prod_{j=1}^{n} z_{\sigma(j)}^{x_j} = \varepsilon(\sigma) \prod_{1 \le k < l \le n} S\big(z_{\sigma(l)}, z_{\sigma(k)}\big) \prod_{j=1}^{n} z_{\sigma(j)}^{x_j}. \tag{5.60}$$

Since $z_{\sigma(m)} = z_{\sigma(s)}$, the LHS above is invariant under substitution $\sigma \to \sigma \circ (m\, s)$ except for the signature $\varepsilon\big(\sigma \circ (m\, s)\big) = -\varepsilon(\sigma)$. Therefore, by pairing the contribution of each σ with that of $\sigma \circ (m\, s)$, the coefficient vanishes. $\square$

Substituting the obtained solution into the boundary relations gives

$$\begin{aligned} z_{\sigma(1)}^{N} &= \frac{A_\sigma}{A_{\sigma\circ\tau}} \\ &= \frac{\varepsilon(\sigma) \prod_{1 \le k < l \le n} S\big(z_{\sigma(l)}, z_{\sigma(k)}\big)}{\varepsilon(\sigma \circ \tau) \prod_{1 \le k < l \le n} S\big(z_{\sigma\circ\tau(l)}, z_{\sigma\circ\tau(k)}\big)} \\ &= (-1)^{n-1} \prod_{j=2}^{n} \frac{S\big(z_{\sigma(j)}, z_{\sigma(1)}\big)}{S\big(z_{\sigma(1)}, z_{\sigma(j)}\big)}, \end{aligned} \tag{5.61}$$

where the last equality follows from cancellations between the two products and the fact that if n is even (resp. odd), then $\tau = (1\,2 \dots n)$ is an odd (resp. even) permutation. The above equation can also be obtained straight from the internal conditions. Using the decomposition $\tau = (1\,2) \circ (2\,3) \circ \dots \circ (n-2; n-1) \circ (n-1; n)$, we find

$$\begin{aligned} z_{\sigma(1)}^{N} &= \frac{A_\sigma}{A_{\sigma\circ(1\,2)\circ(2\,3)\circ\dots\circ(n-2;n-1)\circ(n-1;n)}} \\ &= \frac{A_\sigma}{A_{\sigma\circ(1\,2)}} \frac{A_{\sigma\circ(1\,2)}}{A_{\sigma\circ(1\,2)\circ(2\,3)}} \cdots \frac{A_{\sigma\circ(1\,2)\circ(2\,3)\circ\dots\circ(n-2;n-1)}}{A_{\sigma\circ(1\,2)\circ(2\,3)\circ\dots\circ(n-2;n-1)\circ(n-1;n)}} \\ &= \left[-\frac{S\big(z_{\sigma(2)}, z_{\sigma(1)}\big)}{S\big(z_{\sigma(1)}, z_{\sigma(2)}\big)}\right] \cdot \left[-\frac{S\big(z_{[\sigma\circ(1\,2)](3)}, z_{[\sigma\circ(1\,2)](2)}\big)}{S\big(z_{[\sigma\circ(1\,2)](2)}, z_{[\sigma\circ(1\,2)](3)}\big)}\right] \cdots \\ &\quad \cdot \left[-\frac{S\big(z_{[\sigma\circ\circ(1\,2)\circ(2\,3)\circ\dots\circ(n-2;n-1)](n)}, z_{[\sigma\circ\circ(1\,2)\circ(2\,3)\circ\dots\circ(n-2;n-1)](n-1)}\big)}{S\big(z_{[\sigma\circ\circ(1\,2)\circ(2\,3)\circ\dots\circ(n-2;n-1)](n-1)}, z_{[\sigma\circ\circ(1\,2)\circ(2\,3)\circ\dots\circ(n-2;n-1)](n)}\big)}\right] \\ &= \left[-\frac{S\big(z_{\sigma(2)}, z_{\sigma(1)}\big)}{S\big(z_{\sigma(1)}, z_{\sigma(2)}\big)}\right] \cdot \left[-\frac{S\big(z_{\sigma(3)}, z_{\sigma(1)}\big)}{S\big(z_{\sigma(1)}, z_{\sigma(3)}\big)}\right] \cdot \dots \cdot \left[-\frac{S\big(z_{\sigma(n)}, z_{\sigma(1)}\big)}{S\big(z_{\sigma(1)}, z_{\sigma(n)}\big)}\right] \\ &= (-1)^{n-1} \prod_{j=2}^{n} \frac{S\big(z_{\sigma(j)}, z_{\sigma(1)}\big)}{S\big(z_{\sigma(1)}, z_{\sigma(j)}\big)} \\ z_{\sigma(1)}^{N} &= (-1)^{n-1} \prod_{j=2}^{n} \frac{S\big(z_j, z_{\sigma(1)}\big)}{S\big(z_{\sigma(1)}, z_j\big)}. \end{aligned} \tag{5.62}$$

In the last equality, we replaced $\sigma(j)$ by j since the equation above is symmetric under any permutation of elements $\sigma(2), \dots, \sigma(n)$. That is for any two permutations σ, $\pi \in \mathfrak{S}_n$ for which $\sigma(1) = \pi(1)$, the RHS of the above equation is invariant.

Since the obtained equation is valid for all permutations $\sigma \in \mathfrak{S}_n$, we cover all elements in the set $\{1, \ldots, n\}$ to which 1 can be mapped to by the permutations. Therefore, we obtain n independent equations in the form of

$$z_j^N = (-1)^{n-1} \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{S(z_k, z_j)}{S(z_j, z_k)}, \qquad \forall j \in \{1, 2, \ldots, n\}. \tag{5.63}$$

The above coupled, non-linear equations have not been solved exactly for finite values of both n and N. In the thermodynamic limit, where N approaches infinity, these equations can be employed to analytically determine the free energy of the six-vertex model. However, a rigorous analysis is essential to establish the existence of solutions that genuinely yield the maximum eigenvalue. We outline the essential outcomes and reference some of the numerous associated works.

In Eq. (5.63), the parameters a, b, c of the six-vertex model are expressed in Δ. Any choice of a', b', c' gives the same eigenvectors as long as $\Delta(a, b, c) = \Delta(a', b', c')$. Hence, the behavior of the model is governed by Δ only. In particular, the solutions $z_1, \ldots, z_n$, the maximum eigenvalue, and the analytical expression of the free energy are all contingent on whether $\Delta > 1$, $-1 < \Delta < 1$, or $\Delta < -1$. These four regimes are illustrated in Fig. 21.

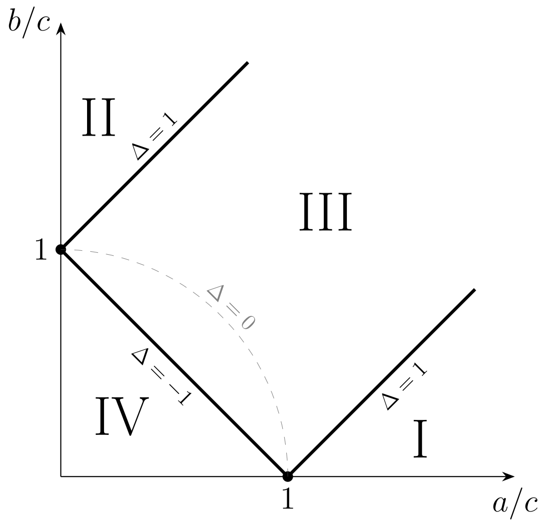


Figure 21: The phase diagram of the six-vertex model and it's four regions: I and II are called ferroelectric ($\Delta > 1$), III is disordered ($-1 < \Delta < 1$) and IV is antiferroelectric ($\Delta < -1$).

Conjecture 5.14. *If $\Delta > 1$, the distinct solutions Eq. (5.63) $\mathbf{z} = (z_1, \ldots, z_n)$ are positive real numbers. In the thermodynamic limit, the maximum eigenvalue is*

$$\Lambda = \Lambda_0 = a^N + b^N. \tag{5.64}$$

Corollary 5.15. *If $\Delta > 1$ the free energy per site in the thermodynamic limit is*

$$f = -\beta \max\{\ln a,\, \ln b\} = \beta \min\{\varepsilon_1,\, \varepsilon_2\}. \tag{5.65}$$

Proof. Using $\Delta > 1$ we obtain an inequality

$$a^2 + b^2 - c^2 > 2ab \iff |a - b| > c. \tag{5.66}$$

Let $a > b$, then using above lemma L_0 is the maximum eigenvalue. For N large, we have

$$f = -\lim_{N\to\infty} \frac{1}{\beta N} \ln\Big[a^N(1 + \big(\frac{b}{a}\big)^N)\Big] = -\frac{1}{\beta}\ln a = -\frac{1}{\beta}\ln e^{-\beta\varepsilon_1} = \varepsilon_1. \tag{5.67}$$

Letting $b > a$ and repeating the argument, concludes the proof. $\square$

It turns out that for $\Delta < 1$ the solutions of Eq. (5.63) that provide the maximal eigenvalue lie on the unit circle, are symmetrically distributed about unity and are packed as closely as possible.

Conjecture 5.16. *Let* $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$. *If* $\Delta < 1$, *the solutions of Eq. (5.63)* $z_1, \ldots, z_n$ *that give the maximal eigenvalue of* V *are such that*

$$z_j = e^{ip_j}, \quad \forall j \in \{1,\, 2, \ldots,\, n\}, \tag{5.68}$$

where we require that numbers in $\mathbf{p}$ *are symmetric, distinct and ordered*

$$\begin{aligned} p_j &= -p_{n+1-j}, \\ p_j &< p_{j+1}, \end{aligned} \tag{5.69}$$

for all $j \in \{1,\, 2, \ldots,\, n\}$.

Conjecture 5.17. *If* $-1 < \Delta < 1$, *parameterise* $\Delta = -\cos\mu$ *with a unique* $\mu \in [0,\, \pi]$. *If* $\Delta < 1$, *set* $\mu = 0$. *Define a set* $\mathscr{D}_\Delta = (-\pi + \mu,\, \pi - \mu)$. *The real numbers* $p_1, \ldots, p_n$ *all lie in* $\mathscr{D}_\Delta$.

We define the function $\Theta : \mathscr{D}_\Delta \to \mathbb{R}$ by

$$\exp\Big(i\Theta(p_k,\, p_l)\Big) := \frac{S(e^{ip_k},\, e^{ip_l})}{S(e^{ip_l},\, e^{ip_k})} = \frac{1 - 2\,\Delta e^{ip_l} + e^{ip_k + ip_l}}{1 - 2\,\Delta e^{ip_k} + e^{ip_k + ip_l}}. \tag{5.70}$$

It is straightforward to deduce that

$$\Theta(p_k,\, p_l) = 2\tan^{-1}\Big[\frac{\Delta \sin\frac{1}{2}(p_k - p_l)}{\cos\frac{1}{2}(p_k + p_l) - \Delta\cos\frac{1}{2}(p_k - p_l)}\Big], \tag{5.71}$$

so $\Theta(p_k,\, p_l)$ is a real function. Equation (5.63) can now be written in unimodular form

$$
\begin{aligned}
e^{iNp_j} &= (-1)^{n-1} \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{S(e^{ip_k}, e^{ip_j})}{S(e^{ip_j}, e^{ip_k})} \\
&= (-1)^{n-1} \prod_{\substack{k=1 \\ k \neq l}}^{n} \exp\Big(i\Theta(p_k,\, p_j)\Big) \qquad (5.72) \\
&= (-1)^{n-1} \exp\Big(-i \sum_{k=1}^{n} \Theta(p_j,\, p_k)\Big), \qquad \forall j \in \{1,\, 2, \ldots,\, n\}.
\end{aligned}
$$

where used the antisymmetry of $\Theta(p_k,\, p_l)$ and the property $\Theta(p, p) = 0$. We are suggested to take logarithms of both sides. By the hypothesis of Conjecture 5.16 all $p_1, \ldots, p_n$ are real, so we find

$$
Np_j = 2\pi I_j - \sum_{k=1}^{n} \Theta(p_j,\, p_k), \quad \forall j \in \{1,\, 2, \ldots,\, n\}, \qquad \text{(BE)}
$$

where I_j is an integer (resp. half-integer) if n is odd (resp. even). The equations above (or their exponentiated forms in Eq. (5.72)) are often called *Bethe's equations*.

When solving the spectra of isotropic Heisenberg spin chain Hamiltonian in 1931 Bethe formulated exactly the same relations, with some differences in the definitions. In 1967, Lieb leveraged the extensively studied results of "Bethe's hypothesis" in his solution to the square ice problem. Consequently, the terminology and concepts of quantum spin chains have left their imprint on the nomenclature of the six-vertex model. Numbers $\mathbf{p}$ are referred to as "wave-numbers" or "quasi-momenta" and the ansatz itself Eq. (5.4) as a "plane-wave". Functions S and Θ have been named as "scattering phases" and "scattering phase shift" respectively and Δ is called an "anisotropy parameter".

The solutions of Eq. (BE), sometimes referred as *Bethe roots*, would directly provide the solution of the six-vertex model but obtaining them is easier said than done. As Lieb himself put it: "We are indeed very fortunate that [these equations] have appeared previously in the literature in another context and have been discussed extensively".

The conjectures Conjecture 5.14, Conjecture 5.16 and Conjecture 5.17 were discussed by Yang and Yang in their celebrated three-part study of the anisotropic Heisenberg spin chain (the XXZ model) in 1966. For $\Delta < -1$, a complete rigorous proof was recently given by Duminil-Copin et al [5].

5.5 Continuum limit

In their publication, Yang and Yang selected $I_j = j - \frac{(n+1)}{2}$ and demonstrated that the values $p_1, \ldots, p_n$ are real as hypothesized, unique, and fall within the range of $\mathscr{D}_\Delta$. With this choice Eq. (BE) becomes

$$
Np_j = 2\pi j - \pi(n+1)I_j - \sum_{k=1}^{n} \Theta(p_j,\, p_k), \quad \forall j \in \{1,\, 2, \ldots,\, n\}. \qquad (5.73)
$$

The thermodynamic limit is taken as $n,\, N \to \infty$ with the ratio n/N fixed to some limit in $[0,\, 1/2]$. The distribution of solutions $p_1, \ldots, p_n$ becomes continuous as they pack closer and closer in $\mathscr{D}_\Delta$. From this *condensation* of Bethe roots, we define a density function $\rho(q)$ such that $N\rho(q)\mathrm{d}q$ is the number of Bethe roots lying between q and $q + \mathrm{d}q$. Given that the total count of Bethe roots is n, we obtain a normalization condition

$$\int_{-Q_\Delta(n/N)}^{Q_\Delta(n/N)} \rho(q)\mathrm{d}q = \frac{n}{N}, \tag{5.74}$$

where the interval $\big(-Q_\Delta(n/N),\, Q_\Delta(n/N)\big)$ is determined from the given ratio n/N. This ratio gives the probability that the vertical arrows on the lattice point upward. In Eq. (5.73), the sums turn into integrals, producing

$$Nq = 2\pi N \int_{-Q_\Delta(n/N)}^{q} \rho(q')\mathrm{d}q' - N \int_{-Q_\Delta(n/N)}^{Q_\Delta(n/N)} \Theta(q,\, q')\rho(q')\mathrm{d}q' - \pi(n+1). \tag{5.75}$$

Dividing by N and differentiating with respect to q, results in a linear integral equation

$$2\pi\rho(k) = 1 + \int_{-Q_\Delta(n/N)}^{Q_\Delta(n/N)} \frac{\partial\Theta(q,q')}{\partial q}\rho(q')\mathrm{d}q'. \tag{5.76}$$

The above *continuum Bethe equation* can be resolved using a suitable Möbius transformation, leading to a more manageable integral equation from which the density function is obtained using basic Fourier methods. The ratio n/N yielding the maximum eigenvalue Λ turns out to be $1/2$ that is Λ is given by the sub-block $V_{N/2}$. Then, the free energy formula Eq. (5.9) turns in the thermodynamic limit into

$$f = -\frac{1}{\beta}\max\Big\{\log a + \int_{-Q_\Delta(1/2)}^{Q_\Delta(1/2)} \Big[\ln L(e^{iq})\Big]\rho(q)\mathrm{d}q,\, \log b + \int_{-Q_\Delta(1/2)}^{Q_\Delta(1/2)} \Big[\ln M(e^{iq})\Big]\rho(q)\mathrm{d}q\Big\}. \tag{5.77}$$

The computation of the free energy for various ranges of Δ is carried out in Baxter's work[1]. The condensation hypothesis of Bethe roots has been proved rigorously on different occasions. In 2018, Kozlowski proved condensation for $\Delta < 1$ and any value of $n/N \in [0,\, 1/2]$ [12] and more recently Duminil-Copin, Kozlowski, and others gave alternative proofs of the existence and condensation of Bethe roots [6]. We summarize the results for the free energy via their theorem.

Theorem 5.18. *[6] For every $a \geq b > 0$ and $c \geq 0$ such that $\Delta < 1$, the per-site free energy is given by*

$$f(a,\,b,\,c) = -\frac{1}{\beta}\begin{cases} \ln b + \int\limits_{-\infty}^{\infty} \dfrac{1}{2t}\dfrac{\sinh[\frac{2(\pi-\theta)}{\pi}\mu t]}{\cosh[\mu t]}\dfrac{\sinh[(\pi-\theta)t]}{\sinh[\pi t]}\mathrm{d}t & \text{if } -1 < \Delta < 1, \\ \ln b + \int\limits_{-\infty}^{\infty} \dfrac{\sinh[\frac{2(\pi-\theta)}{\pi}t]}{\cosh[t]}\dfrac{e^{-|t|}}{2t}\mathrm{d}t & \text{if } \Delta = -1, \\ \ln a + \frac{\mu\theta}{\pi} + \sum\limits_{n=1}^{\infty}\dfrac{e^{-n\mu}}{n}\dfrac{\sinh[2n\mu\theta/\pi]}{\cosh(n\mu)} & \text{if } \Delta < -1, \end{cases} \tag{5.78}$$

where θ and μ are parametrized as follows: For

- $-1 < \Delta < 1$, *set* $\Delta = -\cos\mu$ *with* $\mu \in (0,\,\pi)$,

$$a\sin\frac{\mu}{2} := r\sin\left(1-\frac{\theta}{\pi}\right)\mu, \quad b\sin\frac{\mu}{2} := r\sin\frac{\theta\mu}{\pi}, \quad c := 2r\cos\frac{\mu}{2} \tag{5.79}$$

- $\Delta = -1$, *set*

$$a := 2r\frac{\pi-\theta}{\pi}, \quad b := 2r\frac{\theta}{\pi}, \quad c := 2r, \tag{5.80}$$

- $\Delta < -1$, *set* $\Delta = -\cosh\mu$ *with* $\mu \in \mathbb{R}_+$,

$$a\sinh\frac{\mu}{2} := r\sinh\left(1-\frac{\theta}{\pi}\right)\mu, \quad b\sinh\frac{\mu}{2} := r\sinh\frac{\theta\mu}{\pi}, \quad c := 2r\cosh\frac{\mu}{2}, \tag{5.81}$$

with $\theta \in (0,\,\pi)$ and $r \in \mathbb{R}_+$ representing a scaling parameter.

In particular, for the problem of square ice the energies of all vertex types are equal and can be rescaled to zero: $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_6 = 0$. From Theorem 5.18, it follows that the per-site free energy of the ice model is

$$f(1,\,1,\,1) = -\frac{1}{\beta}\ln\Big(\frac{4}{3}\Big)^{3/2}. \tag{5.82}$$

Taking a derivative with respect to T, we obtain the residual entropy of square ice

$$S = -\frac{\partial f}{\partial T} = k_{\mathrm{B}}\ln\Big(\frac{4}{3}\Big)^{3/2}, \tag{5.83}$$

which is the famous result obtained by Lieb and hence the number $(\frac{4}{3})^{3/2}$ is called Lieb's constant.

6 The singular eigenvalue

In the previous chapter, we made an assumption $z_j \neq \frac{a}{b}$ for all $z_j \in \mathbf{z}$ allowing us to compute the geometric series. This procedure left behind a singularity in functions L and M. When considering solutions on the unit circle the singularity occurs only when one of the Bethe roots is null and parameters a and b are equal. For the rest of the chapter, we set $b = a$ which results in the following identities

$$L(z) = 1 + \frac{c^2 z}{a^2(1-z)}, \quad M(z) = 1 - \frac{c^2}{a^2(1-z)}, \quad \Delta = 1 - \frac{c^2}{2a^2}. \tag{6.1}$$

This chapter is dedicated to proving the Bethe ansatz with a degenerate Bethe root via a perturbative approach. This proof was first given by Duminil-Copin et al. [4], and the treatment below follows closely their work.

Theorem 6.1. *[4] Let elements of n-tuple $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be the distinct solutions of Eq. (BE) in the regime $\Delta < 1$. For some unique l, let $p_l = 0$. Consider Ψ_n as defined in Eq. (5.3) with $z_j = \exp(ip_j)$ for all $j \in \{1, 2, \ldots, n\}$. Then, Ψ_n satisfies the eigenvalue equation $V\Psi_n = \Lambda_n \Psi_n$ where*

$$\Lambda_n = a^N \prod_{\substack{j=1 \\ j \neq l}}^{n} M(e^{ip_j}) \cdot \left[2 + \frac{c^2}{a^2} \left(N - 1 + \sum_{j=2}^{n} \partial_1 \Theta(0, p_j) \right) \right]. \tag{6.2}$$

Using the symmetry under the permutation group, we assume without loss of generality that $p_1 = 0$. We reuse the results from the previous chapter: From Corollary 5.6, which does not rely on the assumption that p_j:s are nonzero, we have

$$V\psi(\vec{x}) = \sum_{\sigma \in \mathfrak{S}_n} A_\sigma R_\sigma. \tag{6.3}$$

The computation of R_σ in Lemma 5.7 requires that the p_j:s are non-zero. Therefore, we use a small parameter $\varepsilon \notin \{p_1 = 0,\, p_2, \ldots,\, p_n\}$ and set $z_\varepsilon = \exp(i\varepsilon)$ allowing us to repeat this step with an aim of taking the limit $\varepsilon \to 0$ later on.

Let R^ε_σ, $r^\varepsilon_\sigma(w)$ and $Z^{ab,\varepsilon}_\sigma(w)$ be the expressions as defined previously but with z_ε instead of $z_1 = \exp(ip_1) = 1$. Since Lemma 5.7 does not rely on Bethe equations, we obtain

$$R^\varepsilon_\sigma = \sum_{w \in \mathscr{W}} r^\varepsilon_\sigma(w) Z^{ab,\varepsilon}_\sigma(w). \tag{6.4}$$

As coefficients A_σ need beforehand to satisfy both internal relations (giving their expression) and boundary relations (to combine the sums in Corollary 5.6) they are dependent $(p_1 = 0, \ldots, p_n)$ and not ε.

R^ϵ_σ is a polynomial in z_ε and equal to R_σ for $z_\varepsilon = 0$, thus

$$V\psi(\vec{x}) = \lim_{\varepsilon \to 0} \sum_{\sigma \in \mathfrak{S}_n} A_\sigma R^\varepsilon_\sigma. \tag{6.5}$$

In the singular case there are also many cancellations but the number of non-vanishing terms is increased. We recall the set of constant words $\mathscr{W}_0$ and introduce the set

$\mathscr{W}_1$ of those words with a unique index m such that $w_m w_{m+1} = ML$. Integers are considered periodically with $n+1$ being associated with 1. Thus, words in $\mathscr{W}_1$ are formed from one non-empty sequence of letters M and one non-empty sequence of letters L in either order. We abbreviate

$$\sum_{\sigma\in\mathfrak{S}_n} A_\sigma R_\sigma^\varepsilon = \sum_{(w,\sigma)\in\mathscr{W}\times\mathfrak{S}_n} A_\sigma r_\sigma^\varepsilon(w) Z_\sigma^{ab,\varepsilon}(w) = \sum_{(w,\sigma)\in\mathscr{W}\times\mathfrak{S}_n} g_\sigma^\varepsilon(w). \tag{6.6}$$

Fix any $w \in \mathscr{W}$ such that there exists an index m with $w_m w_{m+1} = ML$ and σ such that $\sigma(m)$ and $\sigma(m+1)$ are not 1. For every such pair (w,σ) we may apply the computation of Lemma 5.10 to cancel it out with $\big(w, \sigma\circ(m;m+1)\big)$. Therefore, the only remaining terms are given by pairs (w,σ) such that

i. $w\in\mathscr{W}_0$ and any $\sigma\in\mathfrak{S}_n$

ii. $w\in\mathscr{W}_1$ and $\sigma\in\mathfrak{S}_n$ such that $w_m w_{m+1} = ML$ and $\sigma(m)=1$ or $\sigma(m+1)=1$.

This gives two terms to evaluate:

$$\sum_{\sigma\in\mathfrak{S}_n} A_\sigma R_\sigma^\varepsilon = \underbrace{\sum_{(w,\sigma)\in\mathscr{W}_0\times\mathfrak{S}_n} g_\sigma^\varepsilon(w)}_{T_0(\varepsilon)} + \underbrace{\sum_{\substack{(w,\sigma)\in\mathscr{W}_1\times\mathfrak{S}_n\\ w_m w_{m+1}=ML\\ \sigma(m)=1 \text{ or } \sigma(m+1)=1}} g_\sigma^\varepsilon(w)}_{T_1(\varepsilon)}. \tag{6.7}$$

Before computing the limits of $T_0(\varepsilon)$ and $T_1(\varepsilon)$, we note some consequences of a vanishing root $p_1 = 0$ combined with restriction $a=b$.

Lemma 6.2. *For one vanishing Bethe root $p_1 = 0$, we have*

$$\Pi := \prod_{k=2}^{n} M(z_k) = \prod_{k=2}^{n} L(z_k). \tag{6.8}$$

Proof. For $k\geq 2$, we find

$$\begin{aligned}\frac{S(z_k,1)}{S(1,z_k)} &= \frac{1-2\Delta\cdot 1 + z_k\cdot 1}{1-2\Delta\, z_k + 1\cdot z_k}\\ &= \frac{1-2+\dfrac{c^2}{a^2}+z_k}{1-2z_k+\dfrac{c^2}{a^2}z_k+z_k}\\ &= \frac{z_k-1+\dfrac{c^2}{a^2}}{1-z_k+\dfrac{c^2 z_k}{a^2}}\\ &= -\frac{M(z_k)}{L(z_k)}.\end{aligned} \tag{6.9}$$

When above is combined with precursors to the Bethe equations (Eq. (5.63)) for $z_1 = 1$, we get

$$\prod_{k=2}^{n} \frac{M(z_k)}{L(z_k)} = (-1)^{n-1} \prod_{k=2}^{n} \frac{S(z_k,1)}{S(1,z_k)} = z_1^N = 1, \tag{6.10}$$

proving the claim.

□

6.1 Limit of $T_0(\varepsilon)$

The term $T_0(\varepsilon)$, as introduced in Eq. (6.7), can now be given by the following lemma.

Lemma 6.3. *We have*

$$\lim_{\varepsilon\to 0} T_0(\varepsilon) = a^N\Big(2 - \frac{c^2}{a^2}\Big)\Pi\psi(\vec{x}) + a^N \frac{c^2}{a^2} N\Pi \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(n)=1}} A_\sigma Z_\sigma^{\vec{x}}, \tag{6.11}$$

where Π *is defined in Eq. (6.8).*

Proof. For any $\sigma \in \mathfrak{S}_n$ we have

$$Z_\sigma^{ab,\varepsilon}(M\dots M) = a^N \prod_{j=1}^{n} \Big(z_{\sigma(j)}^{x_j} z_\varepsilon^{x_j \mathbb{1}_{\sigma(j)=1}} \Big) = a^N Z_\sigma^{\vec{x},\varepsilon}, \tag{6.12}$$

where $Z_\sigma^{\vec{x},\varepsilon}$ is given by Eq. (5.2), but with z_ε instead of $z_1 = 1$. Note that for k such that $\sigma(k) = 1$ we get $z_{\sigma(k)}^{x_k} z_\varepsilon^{x_k} = z_\varepsilon^{x_k}$ since $z_{\sigma(k)} = z_1 = 1$. Similarly

$$Z_\sigma^{ab,\varepsilon}(L\dots L) = a^N \prod_{j=1}^{n} \Big(z_{\sigma(j)}^{x_{j-1}} z_\varepsilon^{x_{j-1} \mathbb{1}_{\sigma(j)=1}} \Big) = a^N Z_\sigma^{\tau^{-1}\vec{x},\varepsilon}, \tag{6.13}$$

where $Z_\sigma^{\tau^{-1}\vec{x},\varepsilon}$ is likewise a counterpart to $Z_\sigma^{\tau^{-1}\vec{x}}$ from Lemma 5.5. The contribution of the word $M\dots M$ gives

$$\begin{aligned} &\sum_{\sigma\in\mathfrak{S}_n} g_\sigma^\varepsilon(M\dots M) \\ &= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma r_\sigma^\varepsilon(M\dots M) Z_\sigma^{ab,\epsilon}(M\dots M) \\ &= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \Big[M(z_\varepsilon) \prod_{j=2}^{n} \big[M(z_{\sigma(j)}) \big] \Big] a^N Z_\sigma^{\vec{x},\varepsilon} \\ &= a^N M(z_\varepsilon) \Pi \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z_\sigma^{\vec{x},\varepsilon}, \end{aligned} \tag{6.14}$$

The contribution of the word $L\dots L$ gives

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_n} g^{\varepsilon}_{\sigma}(L\dots L)\\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma v^{\epsilon}_{\sigma}(L\dots L) Z^{ab,\epsilon}_{\sigma}(L\dots L)\\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \Big[L(z_\varepsilon)\prod_{j=2}^{n}\big[L(z_{\sigma(j)})\big]\Big] a^N Z^{\tau^{-1}\vec{x},\varepsilon}_{\sigma}\\
&= a^N L(z_\varepsilon)\Pi \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z^{\tau^{-1}\vec{x},\varepsilon}_{\sigma},
\end{aligned} \tag{6.15}$$

where in the last equality we used Lemma 6.2. The change of variables formula of Lemma 5.5 does not apply directly. Recall that $\tau^{-1}(j) = j-1$ for all $2 \le j < n$ and $\tau^{-1}(1) = n$. Also from Eq. (5.43) we have $A_{\sigma\circ\tau^{-1}} = A_\sigma z^N_{\sigma(n)}$. With this, we make the change of variables

$$\begin{aligned}
\sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z^{\tau^{-1}\vec{x},\varepsilon}_{\sigma} &= \sum_{\sigma\in\mathfrak{S}_n} A_{\sigma\circ\tau^{-1}} Z^{\tau^{-1}\vec{x},\varepsilon}_{\sigma\circ\tau^{-1}}\\
&= \sum_{\sigma\in\mathfrak{S}_n} \left(A_\sigma z^N_{\sigma(n)}\right)\left[\prod_{j=1}^{n} z^{x_{j-1}}_{\sigma\circ\tau^{-1}(j)} z_\varepsilon^{x_{j-1}\mathbb{1}_{\sigma\circ\tau^{-1}(j)=1}}\right]\\
&= \sum_{\sigma\in\mathfrak{S}_n} \left(A_\sigma z^N_{\sigma(n)}\right)\left[z^{x_0}_{\sigma(n)} z_\varepsilon^{x_0\mathbb{1}_{\sigma(n)=1}} \prod_{j=2}^{n} z^{x_{j-1}}_{\sigma(j-1)} z_\varepsilon^{x_{j-1}\mathbb{1}_{\sigma(j-1)=1}}\right]\\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma z^{x_n}_{\sigma(n)} z_\varepsilon^{(x_n-N)\mathbb{1}_{\sigma(n)=1}}\left[\prod_{j=1}^{n-1} z^{x_j}_{\sigma(j)} z_\varepsilon^{x_j\mathbb{1}_{\sigma(j)=1}}\right]\\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma z_\varepsilon^{-N\mathbb{1}_{\sigma(n)=1}}\left[\prod_{j=1}^{n} z^{x_j}_{\sigma(j)} z_\varepsilon^{x_j\mathbb{1}_{\sigma(j)=1}}\right]\\
&= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma z_\varepsilon^{-N\mathbb{1}_{\sigma(n)=1}} Z^{\vec{x},\varepsilon}_{\sigma}.
\end{aligned} \tag{6.16}$$

Combining the results, we find

$$\begin{aligned}
T_0(\varepsilon) &= \sum_{\sigma\in\mathfrak{S}_n} g^{\varepsilon}_{\sigma}(M\dots M) + \sum_{\sigma\in\mathfrak{S}_n} g^{\varepsilon}_{\sigma}(L\dots L)\\
&= a^N\Pi\Big(M(z_\varepsilon)\sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z^{\vec{x},\varepsilon}_{\sigma} + L(z_\varepsilon)\sum_{\sigma\in\mathfrak{S}_n} A_\sigma z_\varepsilon^{-N\mathbb{1}_{\sigma(n)=1}} Z^{\vec{x},\varepsilon}_{\sigma}\Big)\\
&= a^N\Pi\Bigg(\Big[M(z_\varepsilon)+L(z_\varepsilon)\Big]\sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z^{\vec{x},\varepsilon}_{\sigma} + \sum_{\sigma\in\mathfrak{S}_n} A_\sigma L(z_\varepsilon)\Big[z_\varepsilon^{-N\mathbb{1}_{\sigma(n)=1}}-1\Big] Z^{\vec{x},\varepsilon}_{\sigma}\Bigg).
\end{aligned} \tag{6.17}$$

By taking the limit $z_\varepsilon \to 1$ we obtain

$$\lim_{z_\epsilon\to 1}\Big(M(z_\varepsilon)+L(z_\varepsilon)\Big)=2-\frac{c^2}{a^2},$$
$$\lim_{z_\epsilon\to 1} Z_\sigma^{\vec{x},\varepsilon}=Z_\sigma^{\vec{x}}, \tag{6.18}$$
$$\lim_{z_\epsilon\to 1} L(z_\varepsilon)\Big[z_\varepsilon^{-N\mathbb{1}_{\sigma(n)=1}}-1\Big]=\frac{c^2}{a^2}N\mathbb{1}_{\sigma(n)=1},$$

concluding the proof. □

6.2 The zipper

Evaluating the limit of $T_1(\varepsilon)$, as introduced in Eq. (6.7), is more laborious. Before computing it, we need to show a consequence of combining the pairs (w,σ) of the case ii. with the vanishing Bethe root. We are able to "change" the letters L into the letters M one by one in a quantity $\hat{g}_\sigma(w)$ defined below that emerges in the limit.

Lemma 6.4. *Fix any $w\in\mathscr{W}_1$ and let $1\le m\le n$ be the unique index such that $w_{m-1}w_m=ML$. Consider any $\sigma\in\mathfrak{S}_n$ that satisfies $\sigma(m)=1$. Define a quantity $\hat{g}_\sigma(w)$ by*

$$\hat{g}_\sigma(w):=A_\sigma\prod_{\substack{j:\,w_j=L\\ \sigma(j)\neq 1}}L(z_{\sigma(j)})\prod_{\substack{j:\,w_j=M\\ \sigma(j)\neq 1}}M(z_{\sigma(j)})Z_\sigma^{ab}(w), \tag{6.19}$$

and let w' be the word obtained from w by changing $w_m=L$ to $w'_m=M$. For any pair (w,σ) as defined above, we have

$$\hat{g}_\sigma(w)=\hat{g}_{\sigma\circ(m;m+1)}(w'). \tag{6.20}$$

Proof. Quantity $\hat{g}_\sigma(w)$ is defined such that we remove the contribution of the letter $w_{\sigma^{-1}(1)}$. Imagine a zipper positioned at the first letter L after a series of M in w as shown in Fig. 22. The index of the letter is by construction $\sigma^{-1}(1)=m$ and therefore it does not contribute to $\hat{g}_\sigma(w)$. When moving the zipper one step to the right change the first letter L to M and compose σ with transposition $(m;m+1)$. The zipper is now at position $m+2$ which is again the pre-image of 1 but this time by $\sigma\circ(m;m+1)$. Therefore the contribution of a letter w'_{m+1} to which the zipper is moved to is left out a swell.

We show that the "zipping" operation does not affect the quantity $\hat{g}_\sigma(w)$. From the definitions above, we have

$$\frac{\hat{g}_{\sigma\circ(m;m+1)}(w')}{\hat{g}_\sigma(w)}=\frac{A_{\sigma\circ(m;m+1)}}{A_\sigma}\cdot\frac{M(z_{\sigma(m+1)})}{L(z_{\sigma(m+1)})}\cdot\frac{Z^{ab}_{\sigma\circ(m;m+1)}(w')}{Z_\sigma^{ab}(w)}. \tag{6.21}$$

Applying the generalized relations given by Eq. (5.48), we obtain

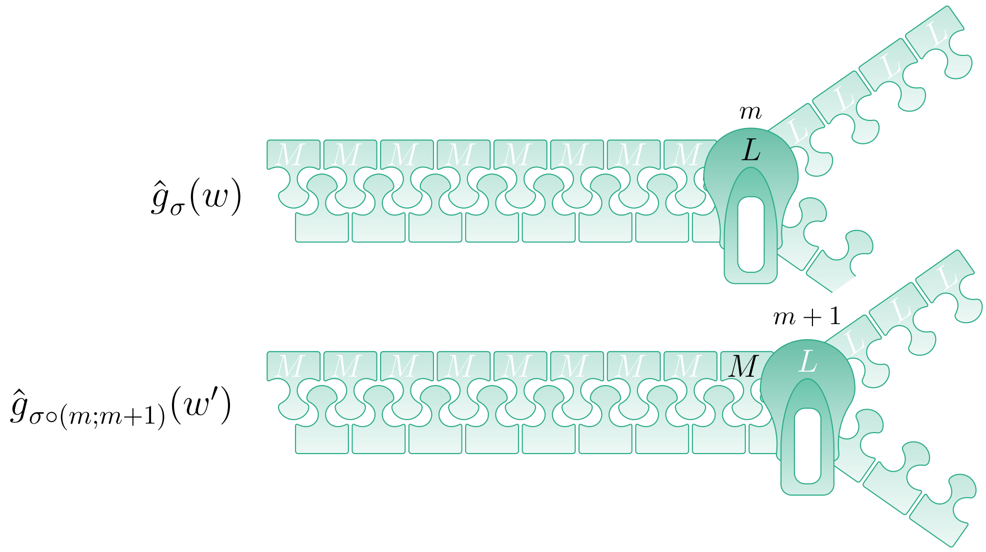


Figure 22: One move of the zipper to the right

$$\begin{aligned}\frac{A_{\sigma\circ(m;m+1)}}{A_\sigma} &= -\left(\frac{z_{\sigma(m)}}{z_{\sigma(m+1)}}\right)^{N\mathbb{1}_{m=n}} \cdot \frac{S(z_{\sigma(m)}, z_{\sigma(m+1)})}{S(z_{\sigma(m+1)}, z_{\sigma(m)})} \\ &= -\left(\frac{z_1}{z_{\sigma(1)}}\right)^{N\mathbb{1}_{m=n}} \cdot \frac{S(z_1, z_{\sigma(m+1)})}{S(z_{\sigma(m+1)}, z_1)} \\ &= -\left(z_{\sigma(1)}\right)^{-N\mathbb{1}_{m=n}} \cdot \frac{S(1, z_{\sigma(m+1)})}{S(z_{\sigma(m+1)}, 1)} \\ &= -\left(z_{\sigma(1)}\right)^{-N\mathbb{1}_{m=n}} \cdot \frac{L(z_{\sigma(m+1)})}{M(z_{\sigma(m+1)})},\end{aligned} \tag{6.22}$$

where in the last equality we used Eq. (6.9).

The ratio of Z functions gives

$$\begin{aligned}\frac{Z^{ab}_{\sigma\circ(m;m+1)}(w')}{Z^{ab}_\sigma(w)} &= \frac{z^{x_m}_{[\sigma\circ(m;m+1)](m)} z^{x_{(m+1)-1}}_{[\sigma\circ(m;m+1)](m+1)}}{z^{x_{m-1}}_{\sigma(m)} z^{x_{(m+1)-1}}_{\sigma(m+1)}} \\ &= \frac{z^{x_m}_{\sigma(m+1)} z^{x_{(m+1)-1}}_{\sigma(m)}}{z^{x_{m-1}}_{\sigma(m)} z^{x_{(m+1)-1}}_{\sigma(m+1)}} \\ &= \frac{z^{x_m}_{\sigma(m+1)} \cdot 1^{x_{(m+1)-1}}}{1^{x_{m-1}} \cdot z^{x_{(m+1)-1}}_{\sigma(m+1)}} \\ &= z^{x_m - x_{(m+1)-1}}_{\sigma(m+1)} \\ &= z^{(x_n - x_0)\mathbb{1}_{m=n}}_{\sigma(n+1)} \\ &= z^{N\mathbb{1}_{m=n}}_{\sigma(1)},\end{aligned} \tag{6.23}$$

where we used the periodicity of integers. Combining Eqs. (6.21) to (6.23) proves the claim.

□

We can now zip off all the letters L and obtain a quantity depending on the constant word $M \dots M$.

Corollary 6.5. *Fix any $w \in \mathscr{W}_1$ and let l, m be the unique indices such that $w_m w_{m+1} = ML$ and $w_l w_{l+1} = LM$. Consider any $\sigma \in \mathfrak{S}_n$ that satisfies $\sigma(m+1) = 1$. Let $[l,\, m]$ be a following permutation*

$$[l,\, m] = (l; l-1; \dots; m+2; m+1), \tag{6.24}$$

that is

$$[l,\, m](j) = \begin{cases} l & \text{if } j = m+1, \\ j-1 & \text{if } j \in \{m+2; m+3; \dots; l-1; l\}, \\ j & \text{otherwise}, \end{cases} \tag{6.25}$$

where all the indices are considered periodically. For any pair (w, σ) as defined above, we have

$$\hat{g}_\sigma(w) = \hat{g}_{\sigma \circ [l,\, m]}(M \dots M), \tag{6.26}$$

where $\hat{g}_\sigma(w)$ is given by Eq. (6.19).

Proof. Applying Lemma 6.4 repeatedly, we obtain

$$\hat{g}_\sigma(w) = \hat{g}_{\sigma \circ (m;m+1) \circ (m+1;m+2) \circ \dots \circ (l-1;l)}(M \dots M). \tag{6.27}$$

The decomposition $(m; m+1) \circ (m+1; m+2) \circ \dots \circ (l-1; l)$ is exactly $[l,\, m]$. □

6.3 Limit of $T_1(\varepsilon)$

We have now the tools to compute the limit of $T_1(\varepsilon)$. While in the previous chapter contributions of all words, $w \notin \mathscr{W}_0$ are canceled by pairing up permutations σ with $\sigma \circ (m, m+1)$, in the singular case the pairs (w, σ) contributing to T_1 do not. Here, such pairing cancels the singular terms resulting in a finite, non-zero contribution.

Lemma 6.6. *We have that*

$$\lim_{\varepsilon \to 0} T_1(\varepsilon) = a^N \frac{c^2}{a^2} \Pi \Big[\sum_{j=2}^{n} \partial_1 \Theta(0, p_{(j)}) \psi(\vec{x}) + N \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n) \neq 1}} A_\sigma Z_\sigma^{\vec{x}} \Big]. \tag{6.28}$$

Proof. We begin with pairing the permutation σ with $\sigma \circ (m; m+1)$ and write

$$g_\sigma^\varepsilon(w) + g_{\sigma \circ (m;m+1)}^\varepsilon(w) = g_\sigma^\varepsilon(w) \Big(1 + \frac{g_{\sigma \circ (m;m+1)}^\varepsilon(w)}{g_\sigma^\varepsilon(w)} \Big). \tag{6.29}$$

The first part of the proof is spent on showing the following result.

Claim 6.7. *Fix any $w \in \mathscr{W}_1$ and let $1 \le m \le n$ be the unique index such that $w_{m1}w_{m+1} = ML$. Consider any $\sigma \in \mathfrak{S}_n$ that satisfies $\sigma(m+1) = 1$.*

$$\lim_{\varepsilon \to 0} g^{\varepsilon}_{\sigma}(w) + g^{\varepsilon}_{\sigma \circ (m;m+1)}(w) = D_\sigma \hat{g}_\sigma(w), \tag{6.30}$$

where $\hat{g}_\sigma$ is given in Eq. (6.19) and D_σ is defined by

$$D_\sigma := \partial_1 \Theta(0, p_{\sigma(m)}) + N\mathbb{1}_{m=n}. \tag{6.31}$$

Proof. We investigate the asymptotics of the two terms in Eq. (6.29) separately using "little o"-notation. By $f(\varepsilon) = o\big(g(\varepsilon)\big)$ we imply

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0. \tag{6.32}$$

Note that we can extract $\hat{g}_\sigma(w)$ from the first term and obtain

$$\begin{aligned} g^{\varepsilon}_{\sigma}(w) &= \hat{g}_\sigma(w) \cdot \frac{M(z_{\sigma(m)})L(z_\varepsilon) - 1}{M(z_{\sigma(m)})} \\ &= \hat{g}_\sigma(w) \cdot \frac{M(z_{\sigma(m)})\Big(1 + \dfrac{c^2 z_\varepsilon}{a^2(1 - z_\varepsilon)}\Big) - 1}{M(z_{\sigma(m)})} \\ &= \frac{c^2}{a^2}\hat{g}_\sigma(w) \cdot \Big[\frac{z_\varepsilon}{1 - z_\varepsilon} + o(\frac{1}{\varepsilon})\Big] \\ &= \frac{c^2}{a^2}\hat{g}_\sigma(w) \cdot \Big[\frac{1}{e^{-i\varepsilon} - 1} + o(\frac{1}{\varepsilon})\Big] \\ &= \frac{c^2}{a^2}\hat{g}_\sigma(w) \cdot \Big(\frac{1}{-i\varepsilon(1 + \frac{(-i\varepsilon)}{2} + \dots)} + o(\frac{1}{\varepsilon})\Big] \\ &= \frac{c^2}{a^2}\hat{g}_\sigma(w) \cdot \Big[-\frac{1}{i\varepsilon} + o(\frac{1}{\varepsilon})\Big]. \end{aligned} \tag{6.33}$$

The ratio of functions $g^{\varepsilon}(w)_\sigma = A_\sigma r_\sigma(w) Z^{ab,\varepsilon}_\sigma(w)$ in the second term is obtained by applying the computations in Lemma 5.10 to the definitions of r^{ε}_σ and $Z^{ab,\varepsilon}_\sigma(w)$. We find that

$$\frac{r^{\varepsilon}_{\sigma \circ (m;m+1)}(w)}{r^{\varepsilon}_\sigma(w)} = \frac{M(z_\varepsilon)L(z_{\sigma(m)}) - 1}{M(z_{\sigma(m)})L(z_\varepsilon) - 1} = \frac{S(z_\varepsilon, z_{\sigma(m)})}{S(z_{\sigma(m)}, z_\varepsilon)} = \exp\Big(i\Theta(\varepsilon, p_{\sigma(m)})\Big), \tag{6.34}$$

where we utilized the function Θ defined by Eq. (5.70). Also, from Eqs. (5.37) and (5.40), we have

$$\frac{Z^{ab,\varepsilon}_{\sigma \circ (m;m+1)}(w)}{Z^{ab,\varepsilon}_\sigma(w)} = \, = \Big(\frac{z_\varepsilon}{z_{\sigma(n)}}\Big)^{N\mathbb{1}_{m=n}}. \tag{6.35}$$

Lastly, by the use of Θ with Eq. (6.22), we obtain

$$\begin{aligned}\frac{A_{\sigma\circ(m;m+1)}}{A_\sigma} &= -\Big(z_{\sigma(n)}\Big)^{N\mathbb{1}_{m=n}} \exp\Big(i\Theta(p_{\sigma(m)},p_{\sigma(m+1)}\Big)\\ &= -\Big(z_{\sigma(n)}\Big)^{N\mathbb{1}_{m=n}} \exp\Big(-i\Theta(0,p_{\sigma(m)}\Big),\end{aligned} \tag{6.36}$$

where in the last equality we used $p_{\sigma(m+1)} = 0$ and the antisymmetry of Θ. Using Eqs. (6.34) to (6.36) we get

$$\begin{aligned}1+\frac{g^\varepsilon_{\sigma\circ(m;m+1)}(w)}{g^\varepsilon_\sigma(w)} &= 1-\Big(z_\varepsilon\Big)^{N\mathbb{1}_{m=n}}\exp\Big(i\Theta(\varepsilon,p_{\sigma(m)})-i\Theta(0,p_{\sigma(m)})\Big)\\ &= 1-\exp\Big(i\Theta(\varepsilon,p_{\sigma(m)})-i\Theta(0,p_{\sigma(m)})+N\varepsilon\mathbb{1}_{m=n}\Big)\\ &= 1-\Big[1+i\varepsilon(\partial_1\Theta(0,p_{\sigma(m)})+N\mathbb{1}_{m=n}+o(\varepsilon)\Big]\\ &= -i\varepsilon\Big(\partial_1\Theta(0,p_{\sigma(m)})+N\mathbb{1}_{m=n}\Big)+o(\varepsilon).\end{aligned} \tag{6.37}$$

where we performed a Taylor expansion. Multiplying out Eqs. (6.33) and (6.37), we find

$$\begin{aligned}&g^\varepsilon_\sigma(w)+g^\varepsilon_{\sigma\circ(m;m+1)}(w)\\ &=\frac{c^2}{a^2}\hat{g}_\sigma(w)\cdot\Big[-\frac{1}{i\varepsilon}+o(\frac{1}{\varepsilon})\Big]\cdot\Big[-i\varepsilon\Big(\partial_1\Theta(0,p_{\sigma(m)})+N\mathbb{1}_{m=n}\Big)+o(\varepsilon)\Big]\\ &=\frac{c^2}{a^2}\hat{g}_\sigma(w)\cdot(\partial_1\Theta(0,p_{\sigma(m)})+N\mathbb{1}_{m=n}\Big)+\frac{o(\varepsilon)}{i\varepsilon}+o(\varepsilon)\cdot o(\frac{1}{\varepsilon}).\end{aligned} \tag{6.38}$$

From the definition, if f and g are functions such that $f(\varepsilon)=o(\varepsilon)$ and $g(\varepsilon)=o(\frac{1}{\varepsilon})$, it follows that

$$\lim_{\varepsilon\to 0} o(\varepsilon)\cdot o(\frac{1}{\varepsilon}) = \lim_{\varepsilon\to 0} f(\varepsilon)g(\varepsilon) = \lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\varepsilon}\frac{g(\varepsilon)}{\frac{1}{\varepsilon}} = 0. \tag{6.39}$$

Consequently, letting ϵ approach zero in Eq. (6.38) proves the claim. □

The above result gives

$$\lim_{\varepsilon\to 0} T_1(\varepsilon) = \sum_{\substack{(w,\sigma)\in\mathscr{W}_1\times\mathfrak{S}_n\\ w_m w_{m+1}=ML\\ \sigma(m)=1 \text{ or } \sigma(m+1)=1}} \frac{c^2}{a^2} D_\sigma \hat{g}_\sigma(w), \tag{6.40}$$

which we can rewrite into

$$\lim_{\varepsilon\to 0} T_1(\varepsilon) = \sum_m \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(m+1)=1}} \sum_{\substack{w\in\mathscr{W}_1\\ w_{m-1}w_m=ML}} \frac{c^2}{a^2} D_\sigma \hat{g}_\sigma(w). \tag{6.41}$$

Now, the work with the zipper pays off. Let l be the unique index for each word $w \in \mathscr{W}_1$ such that $w_l w_{l+1} = LM$. Applying Corollary 6.5 we obtain

$$\lim_{\varepsilon\to 0} T_1(\varepsilon) = \sum_m \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(m+1)=1}} \sum_{l:\, l\neq m} \frac{c^2}{a^2} D_\sigma \hat{g}_{\sigma\circ[l,\, m]}(M\dots M), \tag{6.42}$$

where we used the fact that as words in $\mathscr{W}_1$ with $w_{m-1}w_m$ are summed over, l ranges over all values $\{1,\dots,n\}\setminus m$. Performing the change of variables $\sigma \to \sigma\circ[l,\, m]$ and exchanging sums, we find

$$\lim_{\varepsilon\to 0} T_1(\varepsilon) = \sum_l \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(l)=1}} \frac{c^2}{a^2} \hat{g}_\sigma(M\dots M)\Big(\sum_{m:\, m\neq l} D_{\sigma\circ[l,\, m]^{-1}}\Big). \tag{6.43}$$

Inspecting the bracketed expression, we find

$$\begin{aligned} \sum_{m:\, m\neq l} D_{\sigma\circ[l,\, m]^{-1}} &= \sum_{m:\, m\neq l} \Big(N\mathbb{1}_{m=n} + \partial_1\Theta(0, p_{\sigma\circ[l,\, m]^{-1}(m)})\Big) \\ &= N\mathbb{1}_{l\neq n} + \sum_{j=2}^{n} \partial_1\Theta(0, p_{(j)}), \end{aligned} \tag{6.44}$$

where in the last equality we used the fact that for every σ given by the second sum $\sigma\circ[l,\, m]^{-1}(m) \neq \sigma(l) = 1$. This gives

$$\begin{aligned} \lim_{\varepsilon\to 0} T_1(\varepsilon) &= \frac{c^2}{a^2} \sum_{\sigma\in\mathfrak{S}_n} \hat{g}_\sigma(M\dots M)\Big(N\mathbb{1}_{\sigma(n)\neq 1} + \sum_{j=2}^{n} \partial_1\Theta(0, p_{(j)})\Big) \\ &= \frac{c^2}{a^2} \sum_{\sigma\in\mathfrak{S}_n} A_\sigma a^N \Pi Z_\sigma^{\vec{x}}\Big(N\mathbb{1}_{\sigma(n)\neq 1} + \sum_{j=2}^{n} \partial_1\Theta(0, p_{(j)})\Big) \\ &= a^N \frac{c^2}{a^2} \Pi\Big[\sum_{j=2}^{n} \partial_1\Theta(0, p_{(j)}) \sum_{\sigma\in\mathfrak{S}_n} A_\sigma Z_\sigma^{\vec{x}} + N \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(n)\neq 1}} A_\sigma Z_\sigma^{\vec{x}}\Big], \end{aligned} \tag{6.45}$$

which yields the desired expression. □

Combing the results of this chapter which culminate in Lemmas 6.3 and 6.6, we get

$$\begin{aligned} &\lim_{\varepsilon\to 0} T_0(\varepsilon) + T_1(\varepsilon) \\ &= a^N \Pi\psi(\vec{x})\Big[\Big(2 - \frac{c^2}{a^2}\Big) + \frac{c^2}{a^2}\sum_{j=2}^{n} \partial_1\Theta(0, p_{(j)})\Big] + a^N \frac{c^2}{a^2} N\Pi\Big(\sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(n)=1}} A_\sigma Z_\sigma^{\vec{x}} + \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(n)\neq 1}} A_\sigma Z_\sigma^{\vec{x}}\Big) \\ &= a^N \Pi\Big[2 + \frac{c^2}{a^2}\Big(N - 1 + \sum_{j=2}^{n} \partial_1\Theta(0, p_{(j)})\Big)\Big]\psi(\vec{x}), \end{aligned} \tag{6.46}$$

concluding the proof of Theorem 6.1.

7 Conclusions

The pursuit of an exact solution for the six-vertex model using rigorous methods has turned out to be anything but a straightforward task. By imposing periodic boundary conditions, the partition function can be derived through the transfer matrix approach. The challenge of calculating the spectrum of the transfer matrix necessitates the application of the Bethe ansatz. The formal proof of the ansatz within our work is lengthy and from a mathematical perspective is itself a pleasing result. However, this satisfaction is contingent on the further study of Bethe equations, the existence of Bethe roots, and the condensation hypothesis in the continuum limit.

The six-vertex model and Bethe ansatz have shown their rich mathematical structure. Nonetheless, in the realm of mathematical *physics*, it pays off to also emphasize the latter subject. The experimental measurements that lead up to Pauling's hypothesis are

- Calorimetric data: 44.23 e.u. ([8], 1933) and 44.28 e.u. ([10], 1936)
- Spectroscopic data: 45.10 e.u. ([9], 1934)

Substracting the two values we obtain the discrepancy in entropy of ice to be

$$S_{\text{Data}} = 45.10\,\text{e.u.} - 44.28\,\text{e.u.} = 0.82\,\text{e.u} = 0.412R \tag{7.1}$$

where we replaced the "entropy units" to an ideal gas constant R. Lieb's transfer matrix solution for square ice is

$$S_{\text{Lieb}} = \frac{3}{2}\ln\Big(\frac{4}{3}\Big)R \approx 0.432R\ldots \tag{7.2}$$

So the fruits provided by decades of analysis of the square ice deviate just by 5 % from the empirical value. We conclude our discussion, by approximating the residual entropy using Pauling's method of 1935.

Consider a lattice with N vertices and $2N$ edges for N large. Without constraints, each edge may choose between two possible states, yielding a total of 2^{2N} potential lattice configurations. The ice rule asserts that only 6/16 configurations are admissible for each vertex. Assuming the independence of individual vertices (a *huge* approximation), the number of configurations becomes

$$\Omega = 2^{2N} \cdot \Big(\frac{6}{16}\Big)^N = \Big(\frac{3}{2}\Big)^N. \tag{7.3}$$

Hence, for the residual entropy per site, estimated through Pauling's rough approximation, we obtain

$$S_{\text{Pauling}} = \frac{1}{N}R\ln\Omega^N = \ln\Big(\frac{3}{2}\Big)R \approx 0.405\,R\ldots, \tag{7.4}$$

which is only 1, 7 % off from the value provided by nature. Mathematics can sometimes be a cruel game.

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A The Perron-Frobenius condition

A square matrix with all entries being positive real is called *positive*. The Perron-Frobenius theorem states that every positive matrix has a unique real eigenvalue that is largest in absolute value.

Theorem A.1. *[13] Every positive matrix P has a dominant eigenvalue, denoted by $\lambda(P)$, which has the following properties:*

i) $\lambda(P)$ is positive and the associated eigenvector $\mathbf{h}$ has positive entries:

$$P\mathbf{h} = \lambda(P)\mathbf{h}. \tag{A1}$$

ii) $\lambda(P)$ is a simple eigenvalue.

iii) Every other eigenvalue κ of P is less than $\lambda(P)$ in absolute value:

$$|\kappa| < \lambda(P). \tag{A2}$$

iv) P has no other eigenvector with non-negative entries.

We are set to prove that a particular power of the sub-block V_n satisfies the assumptions of Theorem A.1.

Theorem A.2. *Fix $1 \leq n \leq N/2$ and let V_n be the nth sub-block of the transfer matrix. Then, V_n^n is a positive matrix.*

If an entry $V_n(k, l) > 0$ by the definition there exists at least one configuration of n world lines entering a row of vertices at $\vec{x}_{\varphi(k)}$ and leaving at $\vec{y}_{\varphi(l)}$. Each multiplication by the transfer matrix "adds" a row of the lattice. Thus, $V_n^n(k, l)$ is the partition function of the six-vertex model on a cylinder with n rows where the entering and exiting sites of n world lines are fixed by $\vec{x}_{\varphi(k)}$ and $\vec{y}_{\varphi(l)}$.

Lemma A.3. *Given $\vec{x}$ and $\vec{y}$ such that $x_i \leq y_i$ for all i, there exists a configuration of world lines permitted by the ice rule.*

Proof. One configuration can be drawn explicitly as follows: extend the jth wordline entering at site x_j vertically to the $n - j + 1$th row of vertices. There extend it horizontally until it is below the site y_j where again extend it vertically. With this procedure, all world lines can be drawn without any intersections as shown in Fig. A1 □

We show that all entries of V_n^n are positive by proving that on a cylinder with n rows there is always at least one configuration given any pairing of $\vec{x}$ and $\vec{y}$. That configuration is obtained by a rotation of the lattice such that after the rotation we have $x_i \leq y_i$ for all i. To prove that such a rotation exists we introduce use words made of parentheses.

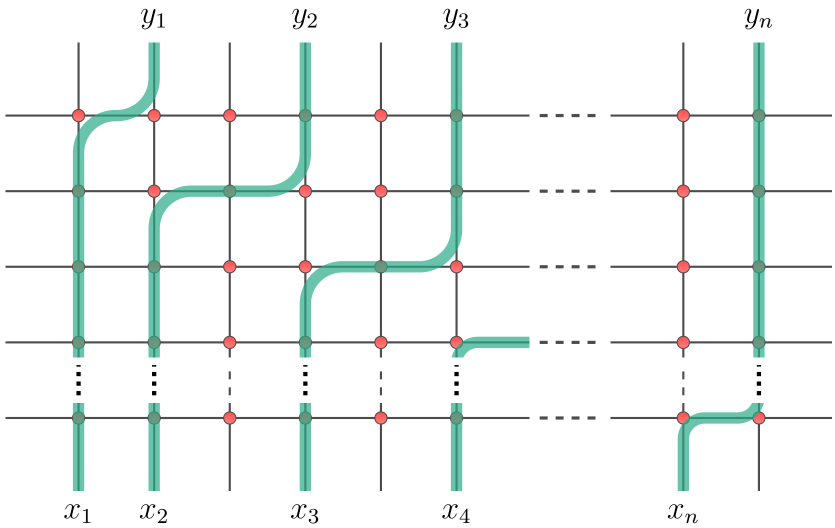

Figure A1: A configuration of world lines when $x_i \leq y_i$ for all i.

Definition A.4. Let w be a word of size $2n$ in the alphabet $\{\langle, \rangle\}$ and let w_i be ith letter in w, so $w = w_1 w_2 \dots w_{2n}$. A *paranthesizing* is a word w comprised of exactly n symbols $\langle$ and n symbols $\rangle$.

For a paranthesizing, denote by $w_{\leq i}$ the sub-word made of the $1 \leq i \leq 2n$ first letters

$$w_{\leq i} := w_1 w_2 \dots w_i. \tag{A3}$$

Similarly denote by $w_{\geq i}$ the sub-word made of the $2n - i + 1$ last letters

$$w_{\geq i} := w_i w_{i+1} \dots w_{2n}. \tag{A4}$$

A *good* parenthesizing is a parenthesizing such that

$$n_{\langle}(w_{\leq i}) \geq n_{\rangle}(w_{\leq i}), \quad \forall i \in \{1, \dots, 2n\}, \tag{A5}$$

where $n_{\langle}(w)$ (resp. $n_{\rangle}(w)$) is the number of symbols $\langle$ (resp. $\rangle$) in w. In general, we say a word, not necessarily a parenthesizing, with the above property is an *increasing* word.

Lemma A.5. *A parenthesizing is good if and only if*

$$n_{\langle}(w_{\geq i}) \leq n_{\rangle}(w_{\geq i}), \quad \forall i \in \{1, \dots, 2n\}. \tag{A6}$$

Proof. From the definition of parenthesizing, we have

$$\begin{aligned} & n_{\langle}(w) = n_{\rangle}(w) \\ \iff & n_{\langle}(w_{\leq i}) + n_{\langle}(w_{\geq i}) = n_{\rangle}(w_{\leq i}) + n_{\rangle}(w_{\geq i}) \\ \iff & n_{\langle}(w_{\leq i}) - n_{\rangle}(w_{\leq i}) = n_{\rangle}(w_{\geq i}) - n_{\langle}(w_{\geq i}), \quad \forall i \in \{1, \dots, 2n\}. \end{aligned} \tag{A7}$$

If w is a good parenthesizing the LHS above is non-negative and so is RHS. If for some i we have $n_{\rangle}(w_{\geq i}) - n_{\langle}(w_{\geq i}) \leq 0$, then w is not a good parenthesizing. This proves the claim. $\square$

In general, a word, not necessarily a parenthesizing which satisfies the above property is said to be a *decreasing* word.

Lemma A.6. *The concatenation of two good parenthesizing is a good parenthesizing.*

Proof. Let v and w be good parenthesizing, and denote their concatenation by u. Then,

$$u = v_1 v_2 \ldots v_{2n} w_1 w_2 \ldots w_{2n}. \tag{A8}$$

Fix any $1 \leq i \leq 2n$, then the sub word $u_{\leq i} = v_{\leq i}$ and Eq. (A5) is valid for a given range. Next, fix any $2n + 1 \leq i \leq 4n$, then

$$\begin{aligned} n_{\langle}(u_{\leq i}) &= n_{\langle}(v_{\leq 2n}) + n_{\langle}(w_{\leq i-2n}) \\ &= n + n_{\langle}(w_{\leq i-2n}) \\ &\geq n + n_{\rangle}(w_{\leq i-2n}) \\ &= n_{\rangle}(u_{\leq i}). \end{aligned} \tag{A9}$$

Thus, Eq. (A5) holds for $1 \leq i \leq 4n$ and u is a good parenthesizing. $\square$

Lemma A.7. *Let w be a parenthesizing. If there is i_0 such that $w_{\geq i_0}$ is a decreasing word, then there exists $i_1 \leq i_0$ such that $w_{\geq i_1}$ is a good parenthesizing.*

Proof. Let $v = w_{\geq i_0}$. From the definition of a decreasing word, we have

$$n_{\langle}(v_{\geq i}) \leq n_{\rangle}(v_{\geq i}), \quad \forall i \in \{i_0, \ldots, 2n\}. \tag{A10}$$

The claim follows using the finite induction. Case $i_0 = 1$ is trivial: $w_{\geq 1} = w$ is a parenthesizing and a decreasing word. Thus, using Lemma A.5 w is a good parenthesizing and we set $i_1 = i_0 = 1$.

Induction hypothesis: If there is $i_0 \in \{1, 2, \ldots, k\}$ such that $w_{\geq i_0}$ is a decreasing word, then there exists $i_1 \leq i_0$ such that $w_{\geq i_1}$ is a good parenthesizing.
Consider the case $i_0 = k + 1$ such that $w_{\geq i_0}$ is a decreasing word. If $w_{\geq i_0}$ is a parenthesizing, we set $i_1 = i_0$. Thus, assume $v = w_{\geq i_0}$ is not a parenthesizing. Then, we obtain a strict inequality

$$n_{\langle}(v_{\geq i}) < n_{\rangle}(v_{\geq i}), \quad \forall i \in \{k + 1, \ldots, 2n\}. \tag{A11}$$

Therefore $w_{\geq k}$ is still a decreasing word. By induction hypothesis, there exists $i_1 \leq k$ such that $w_{\geq i_1}$ is a good parenthesizing. This concludes the proof. $\square$

Corollary A.8. *If there is i_0 such that $w_{\leq i_0}$ is an increasing word, then there exists $i_1 \geq i_0$ such that $w_{\leq i_0}$ is a good parenthesizing.*

Corollary A.9. *Let w be a parenthesizing. Let $i_{max}(w)$ be the maximum of $\Big\{i \in \{0, 2, 4, \ldots\} \mid w_{\leq i}$ is a good parenthesizing$\Big\}$ if this set is non-empty and $i_{max} = 0$ otherwise. If $i_{max}(w) \neq 2n$, then $w_{i_{max}(w)+1} = \rangle$ and the word $w_{\leq i_{max}(w)+1}$ is a decreasing word.*

Proof. Consider $i_{\max}(w) \neq 2n$. Assume $w_{i_{\max}(w)+1}$ is $\langle$. Since $w_{\leq i_{\max}(w)}$ is a good parenthesizing, we have

$$n_{\langle}(w_{\geq i_{\max}+1}) \leq n_{\rangle}(w_{\geq i_{\max}+1}), \tag{A12}$$

and therefore $w_{\leq i_{\max}(w)+1}$ is an increasing word. Using the above lemma, there exists $i_1 \geq i_{\max}(w) + 1$ such that $w_{\leq i_1}$ is a good parenthesizing. We constructed a contradiction with the definition of $i_{\max}(w)$ which proves both claims. □

Lemma A.10. *Let w be a parenthesizing. If $i_{max}(w) \neq 2n$, define*

$$f(w) = w_{i_{max}(w)+2} w_{i_{max}(w)+3} \ldots w_{2n} w_1 w_2 \ldots w_{i_{max}(w)+1}, \tag{A13}$$

an set $f(w) = w$ if $i_{max} = 2n$, i.e w is a good parenthesizing. We have,

$$f(w) = uv, \tag{A14}$$

where v is non-empty good parenthesizing.

Proof. By the definition $f(w)_{\geq 2n - i_{\max}} = w_1 w_2 \ldots w_{i_{\max}+1}$ is a decreasing word. Lemma A.7 gives that there exists $i_1 \leq 2n - i_{\max}$ such that $f(w)_{\geq i_1}$ is a good parenthesizing. Denoting $v = f(w)_{\geq i_1}$ gives the desired result. □

Lemma A.11. *Let $i_{min(w)}$ be the minimum of $\Big\{i \in \{1, 3, 5, \ldots, 2n+1\} \mid w_{\geq i}$ is a good parenthesizing$\Big\}$ if this set is non-empty, $i_{min(w)} = 2n+1$ otherwise. Show that $i_{min(f(w))} \leq i_{min(w)}$ with equality if and only if w is a good parenthesizing.*

Proof. Let w be a good parenthesizing. Then by the definition $f(w) = w$ and $i_{\min(f(w))} = i_{\min(w)} = 1$.

Let w be a parenthesizing that is not good. We show that $i_{\max(w)} < i_{\min(w)}$. Assume on the contrary that $i_{\min(w)} < i_{\max(w)}$. Then,

$$w = \underbrace{\overbrace{w_1 \ldots w_{i_{\min(w)}-1} \, w_{i_{\min(w)}} \ldots w_{i_{\max(w)}}}^{\text{good par.}}}_{w'} \ldots w_n. \tag{A15}$$

The sub-word w' is trivially a good parenthesizing, thus by applying Lemma A.6 we arrive at contradiction. Thus, we write

$$w = w_{\leq i_{\max(w)}+1} w' w_{\geq i_{\min(w)}}, \tag{A16}$$

where w' may be an empty word. Applying function f, we have

$$f(w) = w' \underbrace{w_{\geq i_{\min(w)}}}_{\text{good par.}} \underbrace{w_{\leq i_{\max(w)}+1}}_{\text{decreasing}} . \tag{A17}$$

The concatenation of a good parenthesizing and a decreasing word is easily shown to be a decreasing word. Thus $w_{\geq i_{\min(w)}} w_{\leq i_{\max(w)}+1}$ is a decreasing sub-word of $f(w)$. Denote it's the index of the sub-word's first letter as i_0. Using Lemma A.7 there exists an index $i_1 \leq i_0$ such that $f(w)_{\geq i_1}$ is a good parenthesizing. Split $w' = w''u$ where u may be an empty word that starts with the letter w_{i_1}. Then,

$$f(w) = w'' \underbrace{u w_{\geq i_{\min(w)}} w_{\leq i_{\max(w)}+1}}_{\text{good par.}} = w'' \hat{w}. \tag{A18}$$

Denote $l(v)$ as the length of a word v. Since $w_{\leq i_{\max(w)}+1}$ is not empty, we have

$$l(\hat{w}) > l(w_{\geq i_{\min(w)}}) = n - i_{\min}(w) + 1. \tag{A19}$$

The $i_{\min(f(w))}$ is at most the index corresponding to the first letter of $\hat{w}$ (previously set as i_1). Then, we have

$$l(\hat{w}) = n - i_1 + 1 \leq n - i_{\min(f(w))} + 1. \tag{A20}$$

We conclude the proof by combining the two inequalities above which gives

$$n - i_{\min}(w) + 1 < l(\hat{w}) \leq n - i_{\min(f(w))} + 1. \tag{A21}$$

□

Corollary A.12. *Let w be a parenthesizing. There exists a cyclic permutation σ such that $w_{\sigma(1)} w_{\sigma(2)} \ldots w_{\sigma(n-1)} w_{\sigma(n)}$ is a good parenthesizing.*

Proof. If w is a good parenthesizing we are done. Thus, assume w is not a good parenthesizing. Composing f with itself, we obtain using the above lemma a strict inequality

$$i_{\min\{(f \circ f)(w)\}} < i_{\min\{f(w)\}} < i_{\min(w)}. \tag{A22}$$

Iterating n times we obtain a decreasing sequence that at some iteration $k < n$ reaches a fixed point

$$i_{\min\{f^n(w)\}} = 1 = i_{\min\{f^k(w)\}} < \ldots < i_{\min\{(f \circ f)(w)\}}. \tag{A23}$$

Thus, $f^n(w)$ is a good parenthesizing. The proof concludes by noting that f is a cyclic permutation and composition of cyclic permutations is a cyclic permutation. □

Lemma A.13. *Given $\vec{x}$ and $\vec{y}$, let $a_i \in \{x_j \mid j = 1, \ldots, k\} \cup \{y_j + \frac{1}{2} \mid j = 1, \ldots, k\}$, with $i \in \{1, \ldots, 2k\}$ such that $a_1 < a_2 < \ldots < a_{2k}$.*

Consider the parenthesizing $w = w_1 w_2 \ldots w_{2k}$ such that $w_i = \langle$ if $a_i \in \{x_j, j = 1, \ldots, k\}$ and $w_i = \rangle$ if $a_i \in \{y_j + \frac{1}{2} \mid j = 1, \ldots, k\}$. w is a good parenthesizing if and only if $x_i \leq y_i$ for all i.

Proof. It follows directly from the definitions that if w is a good parenthesizing then $x_i \leq y_i$ for all i.

Let k be the first such index such that $y_k + \frac{1}{2} < x_k$ and assume that w is a good parenthesizing. Then sub-word $w_{\leq 2k-1} = w_1 \dots w_{2k-1}$ is a good parenthesizing. By assumption $w_{2k} = \rangle$ and thus $w_{\leq 2k}$ is a decreasing sub-word of w which contradicts the definition of good parenthesizing. $\square$

We associate given $\vec{x}$ and $\vec{y}$ to a parenthesizing w as formulated above. From Corollary A.12 there exists cycling permutation such that $w_{\sigma(1)}w_{\sigma(2)} \dots w_{\sigma(2n)}$ is a good parenthesizing. This cyclic permutation corresponds to the rotation of the lattice after which $x_i \leq y_i$ for all i. For this arrangement, we have shown that there exists a configuration of world lines. This concludes the proof of Theorem A.2.

B The eigenvalue equation of V_2

We calculate the right-hand side of Eq. (4.21) displayed again below:

$$\sum_{y_1=x_1}^{x_2} E(x_1,\, y_1)D(y_1,\, x_2)z_1^{y_1} \sum_{y_2=x_2}^{N}{}' E(x_2,\, y_2)D(y_2-N, x_1)c^4 z_2^{y_2} + \sum_{y_1=1}^{x_1} E(x_2-N, y_1)D(y_1,\, x_1)z_1^{y_1} \sum_{y_2=x_1}^{x_2}{}' E(x_1,\, y_2)D(y_2,\, x_2)c^4 z_2^{y_2}\,, \tag{B1}$$

where the primed sums again indicate the constraint $y_1 \neq y_2$.

The most straightforward approach is to compute the sums without any constraints and then subtract the weights of the forbidden cases afterward. We evaluate the unconstrained sums in parts (we highlight some terms in gray for readability). The left term of the orange sum gives

$$\begin{aligned}
&= \sum_{y_1=x_1}^{x_2} E(x_1,\, y_1)D(y_1,\, x_2)z_1^{y_1} \\
&= c^2\Big(E(x_1,\, x_1)D(x_1,\, x_2)z_1^{x_1} + \sum_{y_1=x_1+1}^{x_2-1} E(x_1,\, y_1)D(y_1,\, x_2)z_1^{y_1} + E(x_1,\, x_2)D(x_2,\, x_2)z_1^{x_2}\Big) \\
&= a^{x_2-x_1-1}bz_1^{x_1} + \frac{a^{x_2-1}}{b^{x_1+1}}c^2 \sum_{y_1=x_1+1}^{x_2-1} (\frac{b}{a}z_1)^{y_1} + ab^{x_2-x_1-1}z_1^{x_2} \\
&= a^{x_2-x_1-1}bz_1^{x_1} + \frac{a^{x_2-1}}{b^{x_1+1}}c^2 \frac{(\frac{b}{a}z_1)^{x_1+1} - (\frac{b}{a}z_1)^{x_2}}{1-\frac{b}{a}z_1} + ab^{x_2-x_1-1}z_1^{x_2} \\
&= a^{x_2-x_1-1}bz_1^{x_1} + a^{x_2-x_1}\frac{c^2 z_1}{a(a-bz_1)}z_1^{x_1} - b^{x_2-x_1}\frac{c^2}{b(a-bz_1)}z_1^{x_2} + ab^{x_2-x_1-1}z_1^{x_2} \\
&= a^{x_2-x_1}\frac{ab+(c^2-b^2)z_1}{a(a-bz_1)}z_1^{x_1} + b^{x_2-x_1}\frac{a^2-c^2-abz_1}{b(a-bz_1)}z_1^{x_2} \\
&= a^{x_2-x_1}L(z_1)z_1^{x_1} + b^{x_2-x_1}M(z_1)z_1^{x_2} \\
&= \hat{R}_1(x_1,\, x_2)
\end{aligned} \tag{B2}$$

where we have defined a function

$$\hat{R}_j(x, y) := a^{y-x}L(z_j)z_j^x + b^{y-x}M(z_j)z_j^y. \tag{B3}$$

$$\begin{aligned}
&= \Big(E(x_2,\, x_2) D(x_2 - N, x_1) z_2^{x_2} + \sum_{y_2 = x_2+1}^{N} E(x_2,\, y_2) D(y_2 - N, x_1) z_2^{y_2} \Big) \\
&= \Big(a^{x_1 - (x_2 - N) - 1} b z_2^{x_2} + \frac{a^{x_1 - (1-N)}}{b^{x_2+1}} c^2 \sum_{y_2 = x_2+1}^{N} (\frac{b}{a} z_2)^{y_2} \Big) \\
&= \Big(a^{x_1 - (x_2 - N) - 1} b z_2^{x_2} + \frac{a^{x_1 - (1-N)}}{b^{x_2+1}} c^2 \frac{(\frac{b}{a} z_2)^{x_2+1} - (\frac{b}{a} z_2)^{N+1}}{1 - \frac{b}{a} z_2} \Big) \\
&= \Big(a^{x_1 - (x_2 - N) - 1} b z_2^{x_2} + a^{x_1 - (x_2 - N)} \frac{c^2 z_2}{a(a - b z_2)}) z^{x_2} - a^{x_1} b^{N - x_2} \frac{c^2 z_2}{a(a - b z_2)} z_2^N \Big) \\
&= \Big(a^{x_1 - (x_2 - N)} \frac{ab + (c^2 - b^2) z_2}{a(a - b z_2)}) z_2^{x_2} - a^{x_1} b^{N - x_2} \frac{c^2 z_2}{a(a - b z_2)} z_2^N \Big) \\
&= \cdot \Big(a^{x_1 - (x_2 - N)} L(z_2) z_2^{x_2} - a^{x_1} b^{N - x_2} \rho(z_2) z_2^N \Big),
\end{aligned} \tag{B4}$$

where we have introduced another new function

$$\rho(z) = \frac{c^2 z}{a(a - bz)}. \tag{B5}$$

We abbreviate $L_j = L(z_j)$, $M_j = M(z_j)$ and $\rho_j = \rho(z_j)$. Computing the product, the orange sum gives in total

$$\begin{aligned}
\cdot &= a^N L_1 L_2 z_1^{x_1} z_2^{x_2} + a^{x_1 - (x_2 - N)} b^{x_2 - x_1} M_1 L_2 z_1^{x_2} z_2^{x_2} \\
&\quad - a^{x_1} b^{N - x_2} \rho_2 \Big(a^{x_2 - x_1} L_1 z_1^{x_2} + b^{x_2 - x_1} M_1 z_1^{x_1} \Big) z_2^N \\
&= a^N L_1 L_2 z_1^{x_1} z_2^{x_2} + a^{x_1 - (x_2 - N)} b^{x_2 - x_1} M_1 L_2 (z_1 z_2)^{x_2} \\
&\quad - a^{x_1} b^{N - x_2} \rho_2 \hat{R}_1(x_1, x_2) z_2^N.
\end{aligned} \tag{B6}$$

Similarly, we compute the unconstrained blue sum. Notice that the right term of the blue sum is exactly the same as the left term of the orange sum with the substitutions $z_2 \to z_1$ and $y_2 \to y_1$. Thus the right blue term becomes

$$= a^{x_2 - x_1} L_2 z_2^{x_1} + b^{x_2 - x_1} M_2 z_2^{x_2} \quad = \hat{R}_2(x_1,\, x_2). \tag{B7}$$

Thus, we need to evaluate only the first sum

$$
\begin{aligned}
&= c^2 \sum_{y_1=1}^{x_1} E(x_2-N, y_1) D(y_1,\, x_1) z_1^{y_1} \\
&= c^2 \Big(\sum_{y_1=1}^{x_1-1} E(x_2-N, y_1) D(y_1,\, x_1) z_1^{y_1} + E(x_2-N, x_1) D(x_1,\, x_1) z_1^{x_1} \Big) \\
&= c^2 \Big(\sum_{y_1=1}^{x_1-1} b^{y_1-(x_2-N)-1} a^{x_1-y_1-1} z_1^{y_1} + b^{x_1-(x_2-N)-1} \frac{a}{c^2} z_1^{x_1} \Big) \\
&= \frac{a^{x_1-1}}{b^{x_2-N+1}} c^2 \sum_{y_1=1}^{x_1-1} (\frac{b}{a} z_1)^{y_1} + a b^{x_1-(x_2-N)-1} z_1^{x_1} \\
&= \frac{a^{x_1-1}}{b^{x_2-N+1}} c^2 \frac{\frac{b}{a} z_1 - (\frac{b}{a} z_1)^{x_1}}{1 - \frac{b}{a} z_1} + a b^{x_1-(x_2-N)-1} z_1^{x_1} \\
&= \frac{a^{x_1}}{b^{x_2-N}} \frac{c^2 z_1}{a(a-bz_1)} - b^{x_1-(x_2-N)} \frac{c^2}{b(a-bz_1)} z_1^{x_1} + a b^{x_1-(x_2-N)-1} z_1^{x_1} \\
&= a^{x_1} b^{N-x_2} \rho(z_1) + b^{x_1-(x_2-N)} \frac{a^2 - c^2 - abz_1}{b(a-bz_1)} z_1^{x_1} \\
&= a^{x_1} b^{N-x_2} \rho_1 + b^{x_1-(x_2-N)} M_1 z_1^{x_1}.
\end{aligned}
\tag{B8}
$$

Observe that in both the orange and blue cases, ρ arises from the boundary limits $y_2 = N$ or $y_1 = 1$, respectively.

Expanding the derived expressions, we obtain the unconstrained sum of the blue term.

$$
\begin{aligned}
\cdot &= \Big(a^{x_1} b^{N-x_2} \rho_1 + b^{x_1-(x_2-N)} M_1 z_1^{x_1} \Big) \Big(a^{x_2-x_1} L_2 z_2^{x_1} + b^{x_2-x_1} M_2 z_2^{x_2} \Big) \\
&= b^N M_1 M_2 z_1^{x_1} z_2^{x_2} + a^{x_2-x_1} b^{x_1-(x_2-N)} M_1 L_2 z_1^{x_1} z_2^{x_1} \\
&\quad + a^{x_1} b^{N-x_2} \rho_1 \Big(a^{x_2-x_1} L_2 z_2^{x_1} + b^{x_2-x_1} M_2 z_2^{x_2} \Big) \\
&= b^N M_1 M_2 z_1^{x_1} z_2^{x_2} + a^{x_2-x_1} b^{x_1-(x_2-N)} M_1 L_2 (z_1 z_2)^{x_1} \\
&\quad + a^{x_1} b^{N-x_2} \rho_1 \hat{R}_2(x_1,\, x_2).
\end{aligned}
\tag{B9}
$$

Finally, we evaluate the correction terms C that must be subtracted. These terms arise from the cases where $y_1 = x_2 = y_2$ and $y_1 = x_1 = y_2$ in the orange and blue double sum, respectively.

$$
\begin{aligned}
C(\quad) + C(\quad) &= E(x_1,\, x_2) D(x_2,\, x_2) z_1^{x_2} E(x_2,\, x_2) D(x_2-N, x_1) c^4 z_2^{x_2} \\
&\quad + E(x_2-N, x_1) D(x_1,\, x_1) z_1^{x_1} E(x_1,\, x_1) D(x_1,\, x_2) c^4 z_2^{x_1} \\
&= b^{x_2-x_1-1} \frac{a}{c^2} z_1^{x_2} \frac{b}{c^2} a^{x_1-(x_2-N)-1} c^4 z_2^{x_2} \\
&\quad + b^{x_1-(x_2-N)-1} \frac{a}{c^2} z_1^{x_1} \frac{b}{c^2} a^{x_2-x_1-1} c^4 z_2^{x_1} \\
&= a^{x_1-(x_2-N)} b^{x_2-x_1} (z_1 z_2)^{x_2} + a^{x_2-x_1} b^{x_1-(x_2-N)} (z_1 z_2)^{x_1}.
\end{aligned}
\tag{B10}
$$

The desired expression in Eq. (B1) is obtained by summing the terms in Eq. (B6) and Eq. (B9), and subtracting the correction given by Eq. (B10):

$$
\begin{aligned}
& a^N L_1 L_2 z_1^{x_1} z_2^{x_2} + a^{x_1-(x_2-N)} b^{x_2-x_1} M_1 L_2 (z_1 z_2)^{x_2} - a^{x_1} b^{N-x_2} \rho_2 \hat{R}_1(x_1, x_2) z_2^N \\
&\quad + b^N M_1 M_2 z_1^{x_1} z_2^{x_2} + a^{x_2-x_1} b^{x_1-(x_2-N)} M_1 L_2 (z_1 z_2)^{x_1} + a^{x_1} b^{N-x_2} \rho_1 \hat{R}_2(x_1,\, x_2) \\
&\quad - a^{x_1-(x_2-N)} b^{x_2-x_1} (z_1 z_2)^{x_2} - a^{x_2-x_1} b^{x_1-(x_2-N)} (z_1 z_2)^{x_1} \\
&= \Big(a^N L_1 L_2 + b^N M_1 M_2 \Big) z_1^{x_1} z_2^{x_2} \\
&\quad + \Big(a^{x_1-(x_2-N)} b^{x_2-x_1} (z_1 z_2)^{x_2} + a^{x_2-x_1} b^{x_1-(x_2-N)} (z_1 z_2)^{x_1} \Big) \Big[M_1 L_2 - 1 \Big] \\
&\quad + a^{x_1} b^{N-x_2} \Big[\rho_1 \hat{R}_2(x_1,\, x_2) - \rho_2 \hat{R}_1(x_1, x_2) z_2^N \Big],
\end{aligned}
\tag{B11}
$$

where we have organized the terms in three rows. This concludes the computation of the RHS of Eq. (4.21) for Section 4.3.

The expression within the square brackets plays a significant role in the proof of Bethe ansatz and Bethe equations and can be factorized as follows

$$
\begin{aligned}
M(z_1)L(z_2) - 1 &= \frac{a^2 - c^2 - abz_1}{b(a - bz_1)} \cdot \frac{ab + (c^2 - b^2) z_2}{a(a - bz_2)} - 1 \\
&= \frac{ab(a^2 - c^2 - abz_1) + (a^2 - c^2)(c^2 - b^2) z_2 - ab(c^2 - b^2) z_1 z_2 - ab(a - bz_1)(a - bz_2)}{ab(a - bz_1)(a - bz_2)} \\
&= \frac{\cancel{a^2} - c^2 - \cancel{abz_1} + \frac{(a^2-c^2)(c^2-b^2)}{ab} z_2 + (\cancel{b^2} - c^2) z_1 z_2 - \Big(\cancel{a^2} - ab(\cancel{z_1} + z_2) + \cancel{b^2 z_1 z_2} \Big)}{(a - bz_1)(a - bz_2)} \\
&= \frac{-c^2 + \dfrac{a^2 c^2 + b^2 c^2 - c^4 - \cancel{a^2 b^2}}{ab} z_2 - c^2 z_1 z_2 + \cancel{abz_2}}{(a - bz_1)(a - bz_2)} \\
&= -c^2 \frac{1 - \dfrac{a^2 + b^2 - c^2}{ab} z_2 + z_1 z_2}{(a - bz_1)(a - bz_2)} \\
&= -c^2 \frac{1 - 2\Delta z_2 + z_1 z_2}{(a - bz_1)(a - bz_2)} = -c^2 \frac{S(z_1,\, z_2)}{(a - bz_1)(a - bz_2)},
\end{aligned}
\tag{B12}
$$

where we introduced an important parameter for the six-vertex model

$$
\Delta = \frac{a^2 + b^2 - c^2}{2ab}, \tag{B13}
$$

and a function

$$
S(x, y) = 1 - 2\Delta y + xy. \tag{B14}
$$

These results are used in Section 5.4 for deriving Bethe equations.