Graph Theory

Because most of the major developments in graph theory have happened relatively recently and in a variety of different contexts, the terms used in the subject have not been standardized. For example, what a book calls a graph is sometimes called a multigraph, what a book calls a simple graph is sometimes called a graph, what a book calls a vertex is sometimes called a node, and what a book calls an edge is sometimes called an arc. Similarly, instead of the word trail, the word path is sometimes used; instead of the word path, the words simple path are sometimes used; and instead of the words simple circuit, the word cycle is sometimes used. Therefore, if you consult other sources, be sure to check their definitions.

What is a path?

If we think of the vertices in a graph as cities and the edges as roads, a path corresponds to a trip beginning at some city, passing through several cities, and terminating at some city.

If we start at a vertex v0, travel along an edge to vertex v1, travel along another edge to vertex v2, and so on, and eventually arrive at vertex vn, we call the complete tour a path from v0 to vn.

A path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along the edges of the graph.

A path is a circuit if it begins and ends at the same vertex.
A path or circuit is simple if it does not contain the same edge more than once.

In the simple graph shown in Figure 1, a, d, c, f, e is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, d, e, c, a is not a path, because $\{e, c\}$ is not an edge. Note that b, c, f, e, b is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice. ◀

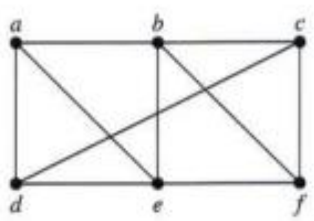
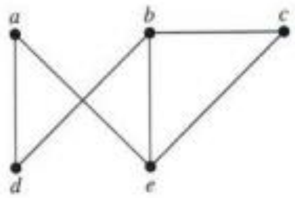


FIGURE 1 A Simple Graph.

Example 1:

Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, e, b, c, b **b)** a, e, a, d, b, c, a
c) e, b, a, d, b, e **d)** c, b, d, a, e, c



a) Path of length 4; not a circuit; not simple **b)** Not a path **c)** Not a path **d)** Simple circuit of length 5

Example 2:

How many paths of length four are there from a to d in the simple graph G in Figure 8?

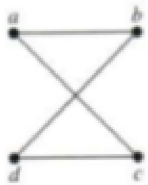


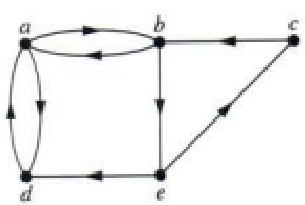
FIGURE 8
Graph G.

there are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths from a to d.

Example 3:

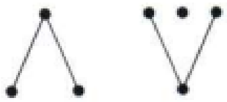
Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, b, e, c, b **b)** a, d, a, d, a
c) a, d, b, e, a **d)** a, b, e, c, b, d, a

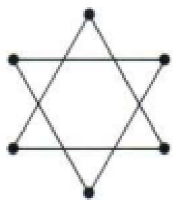


(a) Path of length 4; simple
(c) Not a path
(b) Path of length 4; not simple; circuit
(d) Not a path

Example 4:
Determine whether the given graphs are connected.

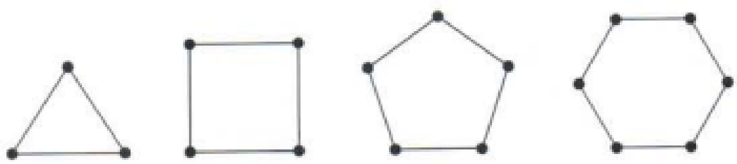


No



No

Cycles The **cycle** C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4.



C_3 C_4 C_5 C_6

Adjacency Matrices

Adjacency matrix of a simple graph

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The **adjacency matrix A** (or $\mathbf{A}_G$) of G, with respect to this listing of the vertices, is the $n \times n$ zero–one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is $\mathbf{A} = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Adjacency matrix of a simple graph is symmetric.

Example 1:

Use an adjacency matrix to represent the graph shown in Figure 3.

FIGURE 3
Simple Graph.

Solution: We order the vertices as a, b, c, d. The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2:

Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency matrix of an undirected graph

Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges. A loop at the vertex a_i is represented by a 1 at the (i, i)th position of the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero–one matrix, because the (i, j)th entry of this matrix equals the number of edges that are associated to $\{a_i, a_j\}$. All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

Example 1:

Use an adjacency matrix to represent the pseudograph shown in Figure 5.

FIGURE 5

Solution: The adjacency matrix using the ordering of vertices a, b, c, d is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

Adjacency matrix of a directed graph

The matrix for a directed graph $G = (V, E)$ has a 1 in its (i, j)th position if there is an edge from v_i to v_j, where $v_1, v_2, \ldots, v_n$ is an arbitrary listing of the vertices of the directed graph. In other words, if $\mathbf{A} = [a_{ij}]$ is the adjacency matrix for the directed graph with respect to this listing of the vertices, then

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from a_j to a_i when there is an edge from a_i to a_j.

Adjacency matrices can also be used to represent directed multigraphs. Again, such matrices are not zero–one matrices when there are multiple edges in the same direction connecting two vertices. In the adjacency matrix for a directed multigraph, a_{ij} equals the number of edges that are associated to (v_i, v_j).

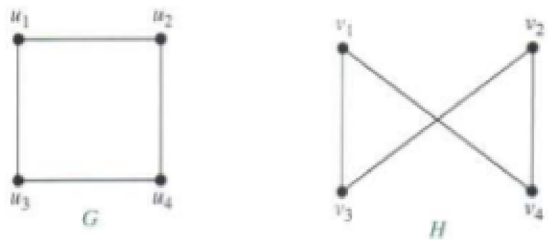
Isomorphism of Graphs

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism*.*

In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.

Example 1:

Show that the graphs G and H shown below are isomorphic.

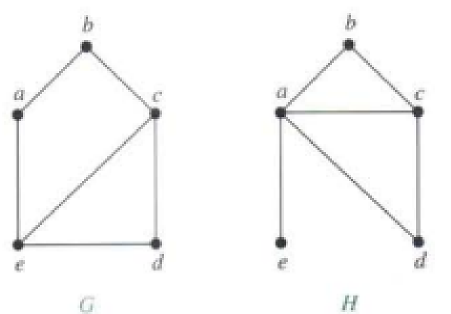


Solution: The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ are adjacent in H.

It is often difficult to determine whether two simple graphs are isomorphic. There are $n!$ possible one-to-one correspondences between the vertex sets of two simple graphs with n vertices. Testing each such correspondence to see whether it preserves adjacency and nonadjacency is impractical if n is at all large.

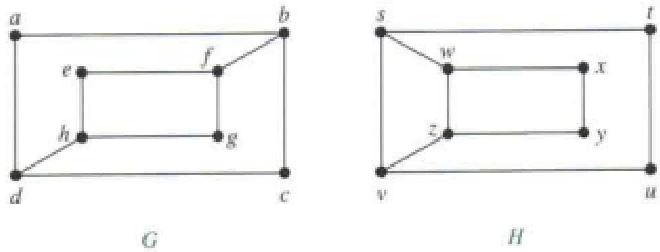
Example 2:

Show that the graphs G and H shown below are not isomorphic.



Solution: Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

Example 3:
Determine whether the graphs G and H shown below are isomorphic.

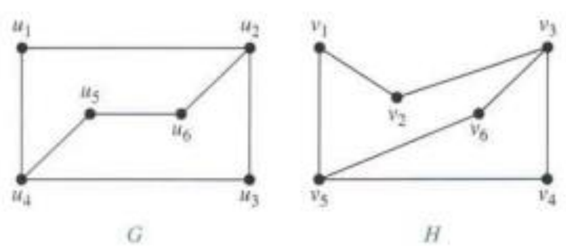


Solution: The graphs G and H both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three. Because these invariants all agree, it is still conceivable that these graphs are isomorphic.

However, G and H are not isomorphic. To see this, note that because $\deg(a) = 2$ in G, a must correspond to either t, u, x, or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

To show that a function f from the vertex set of a graph G to the vertex set of a graph H is an isomorphism, we need to show that f preserves the presence and absence of edges. One helpful way to do this is to use adjacency matrices. In particular, to show that f is an isomorphism, we can show that the adjacency matrix of G is the same as the adjacency matrix of H, when rows and columns are labeled to correspond to the images under f of the vertices in G that are the labels of these rows and columns in the adjacency matrix of G. We illustrate how this is done in Example

Example 4:
Determine whether the graphs G and H shown below are isomorphic.



Graphs *G* and *H*.

Solution: Both G and H have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three.

Because G and H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.

We now will define a function f and then determine whether it is an isomorphism. Because $\deg(u_1) = 2$ and because u_1 is not adjacent to any other vertex of degree two, the image of u_1 must be either v_4 or v_6, the only vertices of degree two in H not adjacent to a vertex of degree two. We arbitrarily set $f(u_1) = v_6$. [If we found that this choice did not lead to isomorphism, we would then try $f(u_1) = v_4$.] Because u_2 is adjacent to u_1, the possible images of u_2 are v_3 and v_5. We arbitrarily set $f(u_2) = v_3$. Continuing in this way, using adjacency of vertices and degrees as a guide, we set $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_1$, and $f(u_6) = v_2$. We now have a one-to-one correspondence between the vertex set of G and the vertex set of H, namely, $f(u_1) = v_6$, $f(u_2) = v_3$, $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_1$, $f(u_6) = v_2$. To see whether f preserves edges, we examine the adjacency matrix of G,

$$\mathbf{A}_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

and the adjacency matrix of H with the rows and columns labeled by the images of the corresponding vertices in G,

$$\mathbf{A}_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Because $\mathbf{A}_G = \mathbf{A}_H$, it follows that f preserves edges. We conclude that f is an isomorphism, so G and H are isomorphic. Note that if f turned out not to be an isomorphism, we would *not* have established that G and H are not isomorphic, because another correspondence of the vertices in G and H may be an isomorphism. ◀

Example 5:
Determine whether the graphs G and H shown below are isomorphic.

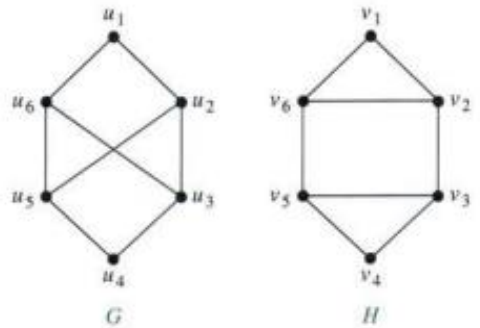


FIGURE 6 **The Graphs *G* and *H*.**

Solution: Both G and H have six vertices and eight edges. Each has four vertices of degree three, and two vertices of degree two. So, the three invariants—number of vertices, number of edges, and degrees of vertices—all agree for the two graphs. However, H has a simple circuit of length three, namely, v_1, v_2, v_6, v_1, whereas G has no simple circuit of length three, as can be determined by inspection (all simple circuits in G have length at least four). Because the existence of a simple circuit of length three is an isomorphic invariant, G and H are not isomorphic.

Example 6:
Determine whether the graphs G and H shown below are isomorphic.

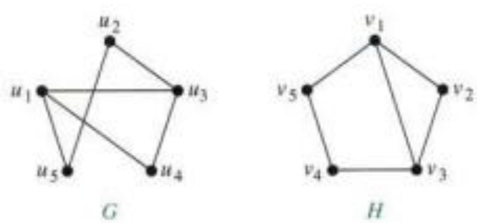


FIGURE 7 **The Graphs *G* and *H*.**

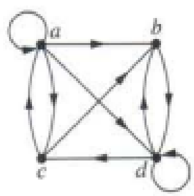
Solution: Both G and H have five vertices and six edges, both have two vertices of degree three and three vertices of degree two, and both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five. Because all these isomorphic invariants agree, G and H may be isomorphic. To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree. For example, the paths u_1, u_4, u_3, u_2, u_5 in G and v_3, v_2, v_1, v_5, v_4 in H both go through every vertex in the graph; start at a vertex of degree three; go through vertices of degrees two, three, and two, respectively; and end at a vertex of degree two. By following these paths through the graphs, we define the mapping f with $f(u_1) = v_3$, $f(u_4) = v_2$, $f(u_3) = v_1$, $f(u_2) = v_5$, and $f(u_5) = v_4$. The reader can show that f is an isomorphism, so G and H are isomorphic, either by showing that f preserves edges or by showing that with the appropriate orderings of vertices the adjacency matrices of G and H are the same.

Exercises

1. Represent the following graph with an adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

2. Represent the following graph with an adjacency matrix.



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

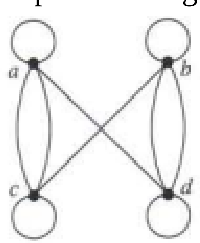
3. Draw a graph for the given adjacency matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

4. Represent the given graph using an adjacency matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

5. Represent the given graph using an adjacency matrix.



$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

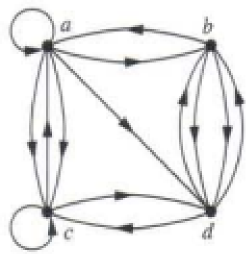
6. Draw an undirected graph represented by the given adjacency matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

7. Find the adjacency matrix of the given directed multigraph.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

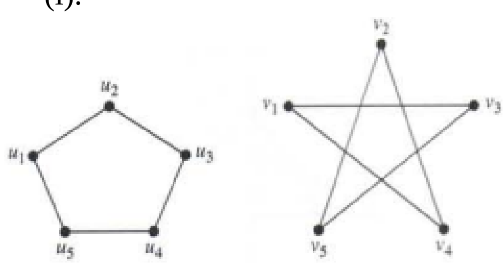
8. Find the adjacency matrix of the given directed multigraph.



$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

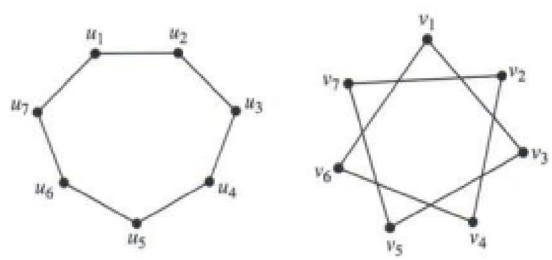
9. Determine whether the given pair of graphs is Isomorphic.

(i).



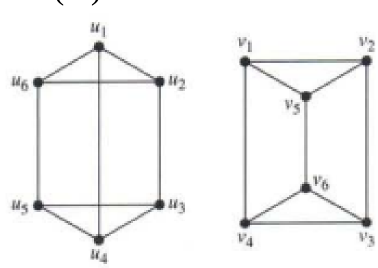
Isomorphic

(ii).



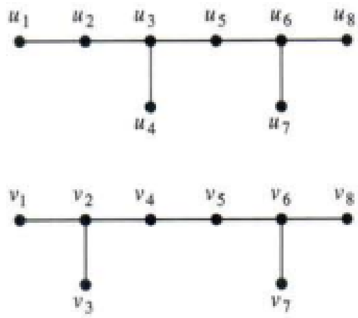
Isomorphic

(iii).



Isomorphic

(iv).



Not Isomorphic

Euler and Hamilton Paths

Introduction

Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly once? Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly once? Although these questions seem to be similar, the first question, which asks whether a graph has an *Euler circuit*, can be easily answered simply by examining the degrees of the vertices of the graph, while the second question, which asks whether a graph has a *Hamilton circuit*, is quite difficult to solve for most graphs.

Euler Paths and Circuits

An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

Examples 1 and 2 illustrate the concept of Euler circuits and paths.

Example 1:

Which of the undirected graphs in Figure 3 have an Euler circuit? Of those that do not, which have an Euler path?

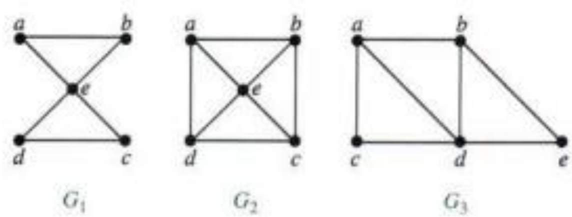


FIGURE 3 **The Undirected Graphs G_1, G_2, and G_3.**

Solution: The graph G_1 has an Euler circuit, for example, a, e, c, d, e, b, a. Neither of the graphs G_2 or G_3 has an Euler circuit (the reader should verify this). However, G_3 has an Euler path, namely, a, c, d, e, b, d, a, b. G_2 does not have an Euler path (as the reader should verify). ◀

Example 2:

Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?

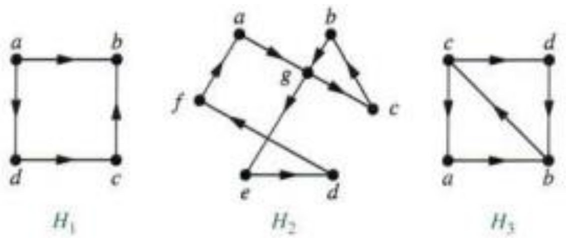


FIGURE 4 The Directed Graphs H_1, H_2, and H_3.

Solution: The graph H_2 has an Euler circuit, for example, $a, g, c, b, g, e, d, f, a$. Neither H_1 nor H_3 has an Euler circuit (as the reader should verify). H_3 has an Euler path, namely, c, a, b, c, d, b, but H_1 does not (as the reader should verify). ◀

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Example:

Which graphs shown in Figure 7 have an Euler path?

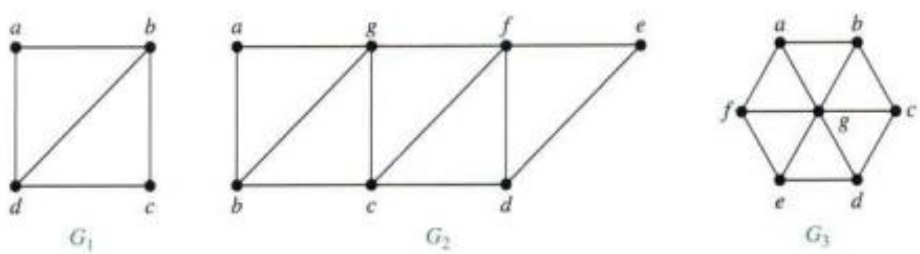


FIGURE 7 Three Undirected Graphs.

Solution: G_1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints. One such Euler path is d, a, b, c, d, b. Similarly,

G_2 has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints. One such Euler path is $b, a, g, f, e, d, c, g, b, c, f, d$. G_3 has no Euler path because it has six vertices of odd degree. ◀

Hamilton Paths and Circuits

A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit.* That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

Example 1:

Which of the simple graphs in Figure 10 have a Hamilton circuit or, if not, a Hamilton path?

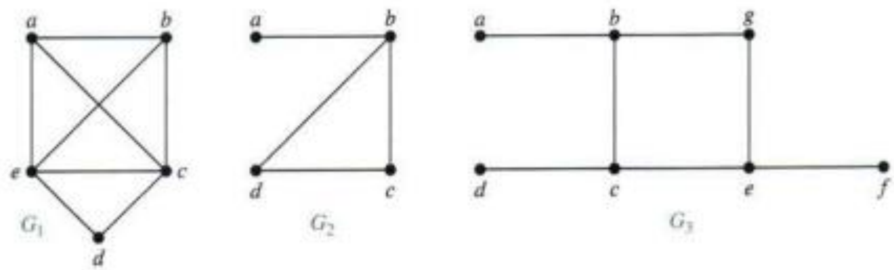


FIGURE 10 Three Simple Graphs.

Solution: G_1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G_2 (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice), but G_2 does have a Hamilton path, namely, a, b, c, d. G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

Example 2:

Show that neither graph displayed in Figure 11 has a Hamilton circuit.

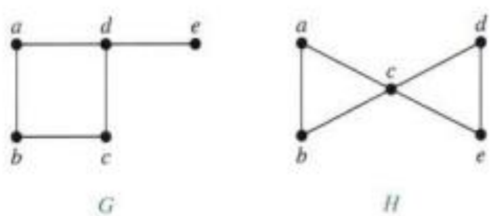


FIGURE 11 Two Graphs That Do Not Have a Hamilton Circuit.

Solution: There is no Hamilton circuit in G because G has a vertex of degree one, namely, e. Now consider H. Because the degrees of the vertices a, b, d, and e are all two, every edge incident with these vertices must be part of any Hamilton circuit. It is now easy to see that no Hamilton circuit can exist in H, for any Hamilton circuit would have to contain four edges incident with c, which is impossible.

Exercises

1.

determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

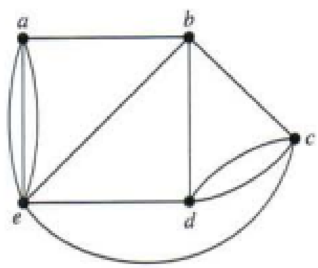
(i).

Neither

(ii).

No Euler circuit; $a, e, c, e, b, e, d, b, a, c, d$

(iii).



$a, b, c, d, c, e, d, b, e, a, e, a$

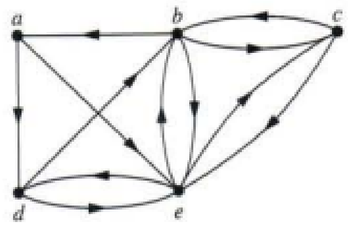
2.

determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path. Construct an Euler path if one exists.

(i).

Neither

(ii).

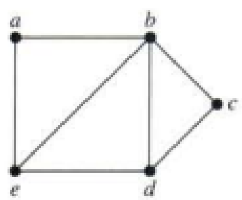


No Euler circuit, $a, d, e, d, b, a, e, c, e, b, c, b, e$

3.

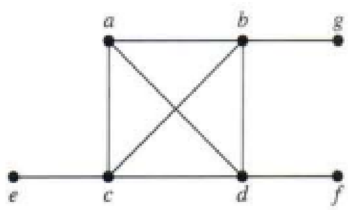
determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

(i).



a, b, c, d, e, a is a Hamilton circuit.

(ii).



No Hamilton circuit exists, because once a purported circuit has reached e it would have nowhere to go.

(iii).

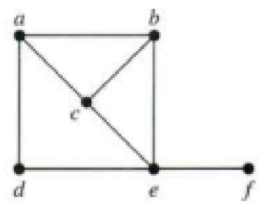
No Hamilton circuit exists, because every edge in the graph is incident to a vertex of degree 2 and therefore must be in the circuit.

4. Determine whether the given graph has a Hamilton path.

(i).

a,b,c,f,d,e is a Hamilton path

(ii).



f,e,d,a,b,c is a Hamilton path

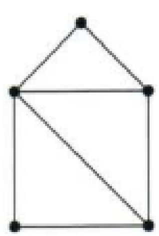
5.

For each of these graphs, determine (*i*) whether Dirac's Theorem can be used to show that the graph has a Hamilton circuit, (*ii*) whether Ore's Theorem can be used to show that the graph has a Hamilton circuit, and (*iii*) whether the graph has a Hamilton circuit.

a)

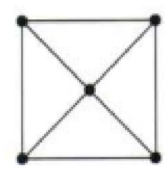
(i).No (ii). No (iii). Yes

b)



(i).No (ii). No (iii). Yes

c)



(i).Yes (ii). Yes (iii). Yes

d)

(i).Yes (ii). Yes (iii). Yes