



**TECHNION**  
Israel Institute  
of Technology

Kinematics, dynamics and control of robots

Homework project 3

Nir Manor 305229627  
Sheng Huang 808837678

## 1. Dynamics Equations

### a. Lagrange equations

In order to formulate dynamics equations

$$H(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + G(\mathbf{q}) = \mathbf{Q}$$

We first calculate Lagrangian L

$$L = T - U$$

where T is kinematic energy and U is potential energy, we first calculate T

$$T = \frac{1}{2} \sum_{i=1}^n (m_i |\mathbf{v}_{cm_i}|^2 + I_i \omega_i^2)$$

where  $m_i$  is mass for each link,  $\mathbf{v}_{cm_i}$  is velocity of center of mass,  $I_{m_i}$  is the tensor of inertia,  $\omega_i$  is the angular velocity. And then the potential U

$$U = \sum_{i=1}^n U_i = \sum_{i=1}^n (-m_i (\mathbf{r}_{cmi} \cdot (-g\hat{\mathbf{z}})))$$

For link 1:

$$\mathbf{r}_{cm_1} = \begin{bmatrix} 0 & 0 & \frac{l_1}{2} \end{bmatrix}^T \Rightarrow \mathbf{v}_{cm_1} = [0 \ 0 \ 0]^T \Rightarrow \frac{1}{2} m_1 |\mathbf{v}_{cm_1}|^2 = 0$$

$$\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \Rightarrow \frac{1}{2} \boldsymbol{\omega}_1^T \mathbf{I}_1 \boldsymbol{\omega}_1 = \frac{1}{2} [0 \ 0 \ \dot{\theta}_1] \begin{bmatrix} \frac{m_1 l_1^2}{12} & 0 & 0 \\ 0 & \frac{m_1 l_1^2}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = 0$$

$$U_1 = -m_1 (\mathbf{r}_{cm1} \cdot (-g\hat{\mathbf{z}})) = m_1 g \frac{l_1}{2}$$

For link 2:

$$\mathbf{r}_{cm_2} = \begin{bmatrix} \frac{l_2}{2} c_2 c_1 \\ \frac{l_2}{2} c_2 s_1 \\ l_1 + \frac{l_2}{2} s_2 \end{bmatrix} \Rightarrow \mathbf{v}_{cm_2} = \begin{bmatrix} \frac{l_2}{2} (-s_2 c_1 \dot{\theta}_2 - c_2 s_1 \dot{\theta}_1) \\ \frac{l_2}{2} (-s_2 s_1 \dot{\theta}_2 + c_2 c_1 \dot{\theta}_1) \\ \frac{l_2}{2} c_2 \dot{\theta}_2 \end{bmatrix} \Rightarrow \frac{1}{2} m_2 |\mathbf{v}_{cm_2}|^2 = \frac{m_2 l_2^2}{8} (\dot{\theta}_2^2 + c_2^2 \dot{\theta}_1^2)$$

$$\boldsymbol{\omega}_2 = \begin{bmatrix} 0 \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix} \Rightarrow \frac{1}{2} \boldsymbol{\omega}_2^T \mathbf{I}_2 \boldsymbol{\omega}_2 = \frac{1}{2} [0 \ -\dot{\theta}_2 \ \dot{\theta}_1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m_2 l_2^2}{12} & 0 \\ 0 & 0 & \frac{m_2 l_2^2}{12} \end{bmatrix} \begin{bmatrix} 0 \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix} = \frac{m_2 l_2^2}{24} (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$U_2 = -m_2 (\mathbf{r}_{cm2} \cdot (-g\hat{\mathbf{z}})) = m_2 g (l_1 + \frac{l_2}{2} s_2)$$

For link 3 & 4:

$$\mathbf{r}_{cm_3} = \left[ l_2 c_2 c_1 - (d_3 + l_4 - \frac{l}{2}) s_2 c_1; l_2 c_2 s_1 - (d_3 + l_4 - \frac{l}{2}) s_2 s_1; l_1 + l_2 s_2 + (d_3 + l_4 - \frac{l}{2}) c_2 \right]^T$$

$$\mathbf{v}_{cm_3} = \begin{bmatrix} (-l_2 s_2 - (d_3 + l_4 - \frac{l}{2}) c_2) c_1 \dot{\theta}_2 - (l_2 c_2 - (d_3 + l_4 - \frac{l}{2}) s_2) s_1 \dot{\theta}_1 - d_3 s_2 c_1 \\ (-l_2 s_2 - (d_3 + l_4 - \frac{l}{2}) c_2) s_1 \dot{\theta}_2 + (l_2 c_2 - (d_3 + l_4 - \frac{l}{2}) s_2) c_1 \dot{\theta}_1 - d_3 s_2 s_1 \\ l_2 c_2 \dot{\theta}_2 - (d_3 + l_4 - \frac{l}{2}) s_2 \dot{\theta}_2 + d_3 c_2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} m_3 |\mathbf{v}_{cm_3}|^2 = \frac{m_3}{8} \{ 4 \left( d_3 c_2 + l_2 \dot{\theta}_2 c_2 - \left( d_3 + l_4 - \frac{l}{2} \right) \dot{\theta}_2 s_2 \right)^2 + \\ \left( \dot{\theta}_1 s_1 (2l_2 c_2 + (-2d_3 - 2l_4 + l)s_2) + c_1 ((2d_3 + 2l_4 - l)\dot{\theta}_2 c_2 + 2(d_3 + l_2 \dot{\theta}_2)s_2) \right)^2 + \\ \left( \dot{\theta}_1 c_1 (2l_2 c_2 + (-2d_3 - 2l_4 + l)s_2) - s_1 ((2d_3 + 2l_4 - l)\dot{\theta}_2 c_2 + 2(d_3 + l_2 \dot{\theta}_2)s_2) \right)^2 \}$$

$$\boldsymbol{\omega}_3 = \begin{bmatrix} 0 \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix} \Rightarrow \frac{1}{2} \boldsymbol{\omega}_3^T \mathbf{I}_{3 \times 3} \boldsymbol{\omega}_3 = \frac{1}{2} [0 \ -\dot{\theta}_2 \ \dot{\theta}_1] \begin{bmatrix} \frac{m_3 l^2}{12} & 0 & 0 \\ 0 & \frac{m_3 l^2}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix} = \frac{m_3 l^2}{24} \dot{\theta}_2^2$$

$$U_3 = -(m_3 + m_4)(\mathbf{r}_{cm_3} \cdot (-g\hat{\mathbf{z}})) = (m_3 + m_4)g(l_1 + l_2 s_2 + (d_3 + l_4 - \frac{l}{2})c_2)$$

Note that we regard **m3** as the mass of link 3 & 4, and **I3** as the inertia value of link 3 & 4. This is according to the assumption that the center of mass of the link 3 is at a constant length **l/2** before the end effector, which take link 3 & 4 as one entity.

For M:

$$\mathbf{r}_{cM} = [l_2 c_2 c_1 - (d_3 + l_4) s_2 c_1; l_2 c_2 s_1 - (d_3 + l_4) s_2 s_1; l_1 + l_2 s_2 + (d_3 + l_4) c_2]^T \Rightarrow$$

$$\mathbf{v}_{cM} = \begin{bmatrix} (-l_2 s_2 - (d_3 + l_4) c_2) c_1 \dot{\theta}_2 - (l_2 c_2 - (d_3 + l_4) s_2) s_1 \dot{\theta}_1 - d_3 s_2 c_1 \\ (-l_2 s_2 - (d_3 + l_4) c_2) s_1 \dot{\theta}_2 + (l_2 c_2 - (d_3 + l_4) s_2) c_1 \dot{\theta}_1 - d_3 s_2 s_1 \\ l_2 c_2 \dot{\theta}_2 - (d_3 + l_4) s_2 \dot{\theta}_2 + d_3 c_2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} M |\mathbf{v}_{cM}|^2 = \frac{M}{2} \{ \left( (d_3 + l_2 \dot{\theta}_2) c_2 - (d_3 + l_4) \dot{\theta}_2 s_2 \right)^2 + \\ \left( \dot{\theta}_1 s_1 (l_2 c_2 - (d_3 + l_4) s_2) + c_1 ((d_3 + l_4) \dot{\theta}_2 c_2 + (d_3 + l_2 \dot{\theta}_2) s_2) \right)^2 + \\ \left( \dot{\theta}_1 c_1 (-l_2 c_2 + (d_3 + l_4) s_2) + s_1 ((d_3 + l_4) \dot{\theta}_2 c_2 + (d_3 + l_2 \dot{\theta}_2) s_2) \right)^2 \}$$

$$\boldsymbol{\omega}_M = \begin{bmatrix} 0 \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix} \Rightarrow \frac{1}{2} \boldsymbol{\omega}_M^T \mathbf{I}_M \boldsymbol{\omega}_M = \frac{1}{2} [0 \ -\dot{\theta}_2 \ \dot{\theta}_1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix} = 0$$

$$U_M = -M(\mathbf{r}_{cM} \cdot (-g\hat{\mathbf{z}})) = Mg(l_1 + l_2 s_2 + (d_3 + l_4) c_2)$$

Note that we treat the point mass **M** held by the end effector as an additional link with zero length.

Then we apply the Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q}$$

where  $\mathbf{q}$  is the vector of degree of freedom, and  $\mathbf{Q}$  is the generalized force

$$\mathbf{Q} = \sum_{j=1}^N \mathbf{F}_j \frac{\partial \mathbf{r}_j}{\partial \mathbf{q}} = \frac{\partial P_{nc}}{\partial \dot{\mathbf{q}}}$$

where  $\mathbf{F}_j$  is the force vector acting at the point  $\mathbf{r}_j$ , and  $P_{nc}$  is the power of non-conservative forces.

First we calculate the generalized forces

$$\mathbf{F}_e = [F_{ex} \ F_{ey} \ F_{ez}]^T$$

$$\mathbf{r}_{Fe} = [l_2 c_2 c_1 - (d_3 + l_4) s_2 c_1; l_2 c_2 s_1 - (d_3 + l_4) s_2 s_1; l_1 + l_2 s_2 + (d_3 + l_4) c_2]^T$$

$$P_{nc} = \tau \dot{\theta}^2$$

where  $\mathbf{F}_M$  is the a point mass  $M$  is held by the end effector,  $\mathbf{F}_e$  is the external force acting on the end effector,  $\tau$  is joint torque\force vector.

For  $q = \theta_1$

$$\boxed{\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = Q_1}$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) = & \frac{l_3^2 m_3 \ddot{\theta}_1}{2} + \frac{l^2 m_3 \ddot{\theta}_1}{6} + \frac{l_2^2 m_2 \ddot{\theta}_1}{6} + \frac{l_2^2 m_3 \ddot{\theta}_1}{2} + \frac{l_4^2 m_3 \ddot{\theta}_1}{2} + \frac{M d_3^2 \ddot{\theta}_1}{2} + \frac{M l_2^2 \ddot{\theta}_1}{2} + \frac{M l_4^2 \ddot{\theta}_1}{2} + d_3 \dot{d}_3 m_3 \dot{\theta}_1 - \frac{d_3 l m_3 \ddot{\theta}_1}{2} - \frac{\dot{d}_3 l m_3 \dot{\theta}_1}{2} \\ & + d_3 l_4 m_3 \ddot{\theta}_1 + \dot{d}_3 l_4 m_3 \dot{\theta}_1 - \frac{l l_4 m_3 \ddot{\theta}_1}{2} - \frac{M c_{22} d_3^2 \ddot{\theta}_1}{2} + \frac{M c_{22} l_2^2 \ddot{\theta}_1}{2} - \frac{M c_{22} l_4^2 \ddot{\theta}_1}{2} - \frac{c_{22} d_3^2 m_3 \ddot{\theta}_1}{2} - \frac{c_{22} l^2 m_3 \ddot{\theta}_1}{6} + \frac{c_{22} l_2^2 m_2 \ddot{\theta}_1}{6} \\ & + \frac{c_{22} l_2^2 m_3 \ddot{\theta}_1}{2} - \frac{c_{22} l_4^2 m_3 \ddot{\theta}_1}{2} + M d_3 \dot{d}_3 \dot{\theta}_1 + M d_3 l_4 \ddot{\theta}_1 + M \dot{d}_3 l_4 \dot{\theta}_1 - M c_{22} d_3 \dot{d}_3 \dot{\theta}_1 - M c_{22} d_3 l_4 \ddot{\theta}_1 - M c_{22} \dot{d}_3 l_4 \dot{\theta}_1 - M d_3 l_2 s_{22} \ddot{\theta}_1 \\ & - M d_3 l_2 s_{22} \dot{\theta}_1 - M l_2 l_4 s_{22} \ddot{\theta}_1 - c_{22} d_3 \dot{d}_3 \dot{\theta}_1 + \frac{c_{22} d_3 l m_3 \ddot{\theta}_1}{2} + \frac{c_{22} \dot{d}_3 l m_3 \dot{\theta}_1}{2} - c_{22} d_3 l_4 m_3 \dot{\theta}_1 - c_{22} \dot{d}_3 l_4 m_3 \dot{\theta}_1 + \frac{c_{22} l l_4 m_3 \ddot{\theta}_1}{2} \\ & - d_3 l_2 m_3 s_{22} \ddot{\theta}_1 - \dot{d}_3 l_2 m_3 s_{22} \dot{\theta}_1 + \frac{l l_2 m_3 s_{22} \ddot{\theta}_1}{2} - l_2 l_4 m_3 s_{22} \ddot{\theta}_1 + M d_3^2 s_{22} \dot{\theta}_1 \dot{\theta}_2 - M l_2^2 s_{22} \dot{\theta}_1 \dot{\theta}_2 + M l_4^2 s_{22} \dot{\theta}_1 \dot{\theta}_2 + d_3^2 m_3 s_{22} \dot{\theta}_1 \dot{\theta}_2 \\ & + \frac{l^2 m_3 s_{22} \dot{\theta}_1 \dot{\theta}_2}{3} - \frac{l_2^2 m_2 s_{22} \dot{\theta}_1 \dot{\theta}_2}{3} - l_2^2 m_3 s_{22} \dot{\theta}_1 \dot{\theta}_2 + l_4^2 m_3 s_{22} \dot{\theta}_1 \dot{\theta}_2 - l l_4 m_3 s_{22} \dot{\theta}_1 \dot{\theta}_2 - 2 M c_{22} d_3 l_2 \dot{\theta}_1 \dot{\theta}_2 - 2 M c_{22} l_2 l_4 \dot{\theta}_1 \dot{\theta}_2 \\ & + 2 M d_3 l_4 s_{22} \dot{\theta}_1 \dot{\theta}_2 - 2 c_{22} d_3 l_2 m_3 \dot{\theta}_1 \dot{\theta}_2 + c_{22} l l_2 m_3 \dot{\theta}_1 \dot{\theta}_2 - 2 c_{22} l_2 l_4 m_3 \dot{\theta}_1 \dot{\theta}_2 - d_3 l m_3 s_{22} \dot{\theta}_1 \dot{\theta}_2 + 2 d_3 l_4 m_3 s_{22} \dot{\theta}_1 \dot{\theta}_2 \end{aligned}$$

where  $s_{22} = \sin(2\theta_2)$ ,  $c_{22} = \cos(2\theta_2)$

$$\frac{\partial L}{\partial \theta_1} = 0, \quad Q_1 = F_{ex}(s_1 s_2(d_3 + l_4) - c_2 l_2 s_1) - F_{ey}(c_1 s_2(d_3 + l_4) - c_1 c_2 l_2)$$

For  $q = \theta_2$

$$\boxed{\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = Q_2}$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) &= \\ d_3^2 m_3 \ddot{\theta}_2 + \frac{l^2 m_3 \ddot{\theta}_2}{3} + \frac{l_2^2 m_2 \ddot{\theta}_2}{3} + l_2^2 m_3 \ddot{\theta}_2 + l_4^2 m_3 \ddot{\theta}_2 + M \ddot{d}_3 l_2 + \ddot{d}_3 l_2 m_3 + M d_3^2 \ddot{\theta}_2 + M l_2^2 \ddot{\theta}_2 \\ + M l_4^2 \ddot{\theta}_2 + 2 d_3 \dot{d}_3 m_3 \dot{\theta}_2 - d_3 l m_3 \ddot{\theta}_2 - \dot{d}_3 l m_3 \dot{\theta}_2 + 2 d_3 l_4 m_3 \ddot{\theta}_2 + 2 \dot{d}_3 l_4 m_3 \dot{\theta}_2 - l l_4 m_3 \ddot{\theta}_2 \\ + 2 M d_3 \dot{d}_3 \dot{\theta}_2 + 2 M d_3 l_4 \ddot{\theta}_2 + 2 M \dot{d}_3 l_4 \dot{\theta}_2 \\ \frac{\partial L}{\partial \theta_2} &= \\ d_3 g m_3 s_2 - c_2 g l_2 m_3 - \frac{c_2 g l_2 m_2}{2} - \frac{g l m_3 s_2}{2} + g l_4 m_3 s_2 + \frac{M d_3^2 s_{22} \dot{\theta}_1^2}{2} - \frac{M l_2^2 s_{22} \dot{\theta}_1^2}{2} + \\ \frac{M l_4^2 s_{22} \dot{\theta}_1^2}{2} + \frac{d_3^2 m_3 s_{22} \dot{\theta}_1^2}{2} + \frac{l^2 m_3 s_{22} \dot{\theta}_1^2}{6} - \frac{l_2^2 m_2 s_{22} \dot{\theta}_1^2}{6} - \frac{l_2^2 m_3 s_{22} \dot{\theta}_1^2}{2} + \frac{l_4^2 m_3 s_{22} \dot{\theta}_1^2}{2} \\ - M c_2 g l_2 + M d_3 g s_2 + M g l_4 s_2 - M c_{22} d_3 l_2 \dot{\theta}_1^2 - M c_{22} l_2 l_4 \dot{\theta}_1^2 + M d_3 l_4 s_{22} \dot{\theta}_1^2 \\ - c_{22} d_3 l_2 m_3 \dot{\theta}_1^2 + \frac{c_{22} l l_2 m_3 \dot{\theta}_1^2}{2} - c_{22} l_2 l_4 m_3 \dot{\theta}_1^2 - \frac{d_3 l m_3 s_{22} \dot{\theta}_1^2}{2} + d_3 l_4 m_3 s_{22} \dot{\theta}_1^2 - \frac{l l_4 m_3 s_{22} \dot{\theta}_1^2}{2} \end{aligned}$$

$$Q_2 = -F_{\text{ey}}(c_2 s_1(d_3 + l_4) + l_2 s_1 s_2) - F_{\text{ez}}(s_2(d_3 + l_4) - c_2 l_2) - F_{\text{ex}}(c_1 c_2(d_3 + l_4) + c_1 l_2 s_2)$$

For  $q = d_3$

$$\boxed{\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{d}_3} \right) - \frac{\partial L}{\partial d_3} = Q_3}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{d}_3} \right) = (\ddot{d}_3 + l_2 \ddot{\theta}_2) (M + m_3)$$

$$\frac{\partial L}{\partial d_3} =$$

$$\begin{aligned} m_3 \sigma_2 (\dot{\theta}_2 (c_1 c_2 \sigma_1 + c_1 l_2 s_2) - \dot{\theta}_1 (s_1 s_2 \sigma_1 - c_2 l_2 s_1) + c_1 \dot{d}_3 s_2) - M c_2 g - c_2 g m_3 \\ + m_3 \sigma_3 (\dot{\theta}_2 (c_2 s_1 \sigma_1 + l_2 s_1 s_2) + \dot{\theta}_1 (c_1 s_2 \sigma_1 - c_1 c_2 l_2) + \dot{d}_3 s_1 s_2) \\ + M \sigma_2 (\dot{\theta}_2 (c_1 c_2 (d_3 + l_4) + c_1 l_2 s_2) - \dot{\theta}_1 (s_1 s_2 (d_3 + l_4) - c_2 l_2 s_1) + c_1 \dot{d}_3 s_2) \\ + M \sigma_3 (\dot{\theta}_1 (c_1 s_2 (d_3 + l_4) - c_1 c_2 l_2) + \dot{\theta}_2 (c_2 s_1 (d_3 + l_4) + l_2 s_1 s_2) + \dot{d}_3 s_1 s_2) \\ - m_3 s_2 \dot{\theta}_2 (c_2 \dot{d}_3 + \dot{\theta}_2 (c_2 l_2 - s_2 \sigma_1)) - M s_2 \dot{\theta}_2 (c_2 \dot{d}_3 - \dot{\theta}_2 (s_2 (d_3 + l_4) - c_2 l_2)) \end{aligned}$$

$$\text{where } \sigma_1 = d_3 - \frac{l}{2} + l_4, \quad \sigma_2 = c_1 c_2 \dot{\theta}_2 - s_1 s_2 \dot{\theta}_1, \quad \sigma_3 = c_1 s_2 \dot{\theta}_1 + c_2 s_1 \dot{\theta}_2$$

$$Q_3 = F_{\text{ez}} c_2 - F_{\text{ex}} c_1 s_2 - F_{\text{ey}} s_1 s_2$$

In a matrix form:

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{B}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{Q}$$

$$\boxed{\mathbf{H}(\mathbf{q}) = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{22} & l_2(\mathbf{M} + \mathbf{m}_3) \\ \mathbf{0} & l_2(\mathbf{M} + \mathbf{m}_3) & \mathbf{M} + \mathbf{m}_3 \end{pmatrix}}$$

$$\mathbf{H}_{11} = \mathbf{M}(l_2 c_2 - (d_3 + l_4)s_2)^2 + \mathbf{m}_3 \left( \left( l_2 c_2 - \left( d_3 + l_4 - \frac{l}{2} \right) s_2 \right)^2 + \frac{l^2}{12} s_2^2 \right) + \mathbf{m}_2 \frac{l_2^2}{3} c_2^2$$

$$\mathbf{H}_{22} = \mathbf{M}(l_2^2 + (d_3 + l_4)^2) + \mathbf{m}_3 \left( l_2^2 + \left( d_3 + l_4 - \frac{l}{2} \right)^2 + \frac{l^2}{12} \right) + \mathbf{m}_2 \frac{l_2^2}{3}$$

$$\boxed{\mathbf{G}(\mathbf{q}) = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}(l_2 c_2 - (d_3 + l_4)s_2) + \mathbf{m}_3 \left( l_2 c_2 - \left( d_3 + l_4 - \frac{l}{2} \right) s_2 \right) + \mathbf{m}_2 \frac{l_2}{2} c_2 \\ (\mathbf{M} + \mathbf{m}_3)c_2 \end{bmatrix} g}$$

$$\mathbf{Q} = \boldsymbol{\tau} + [\mathbf{Q}_1 \ \mathbf{Q}_2 \ \mathbf{Q}_3]^T$$

### b. Partial Jacobian

In order to get the dynamics equations

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \boldsymbol{\tau} + J_L^{World^T} \mathbf{F}_e$$

We first calculate the partial Jacobian. From project 1 we know

$$J_L^{World} = \begin{bmatrix} -s_1(l_2 c_2 - (d_3 + l_4)s_2) & -c_1(l_2 s_2 + (d_3 + l_4)c_2) & -c_1 s_2 \\ c_1(l_2 c_2 - (d_3 + l_4)s_2) & -s_1(l_2 s_2 + (d_3 + l_4)c_2) & -s_1 s_2 \\ 0 & l_2 c_2 - (d_3 + l_4)s_2 & c_2 \end{bmatrix}$$

$$J_A^{World} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$${}^t R_0 = \begin{bmatrix} c_1 c_2 & -s_1 & -c_1 s_2 \\ s_1 c_2 & c_1 & -s_1 s_2 \\ s_2 & 0 & c_2 \end{bmatrix}$$

Therefore

$$J_L^{Tool} = {}^t R_0^T J_L^{World} = \begin{bmatrix} 0 & -(d_3 + l_4) & 0 \\ l_2 c_2 - (d_3 + l_4)s_2 & 0 & 0 \\ 0 & l_2 & 1 \end{bmatrix}$$

$$J_A^{Tool} = {}^t R_0^T J_A^{World} = \begin{bmatrix} s_2 & 0 & 0 \\ 0 & -1 & 0 \\ c_2 & 0 & 0 \end{bmatrix}$$

And then we could calculate the Jacobian for each link:

For link 1: ( $l_4 = 0, d_3 = 0, l_2 = 0, \theta_2 = 0$ )

$$J_{L1}^{World} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J_{A1}^{World} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_{L1}^{Tool} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J_{A1}^{Tool} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

For link 2: ( $l_4 = 0, d_3 = 0, l_2 = \frac{l_2}{2}$ )

$$J_{L2}^{World} = \begin{bmatrix} -\frac{l_2}{2}s_1c_2 & -\frac{l_2}{2}c_1s_2 & 0 \\ \frac{l_2}{2}c_1c_2 & -\frac{l_2}{2}s_1s_2 & 0 \\ 0 & \frac{l_2}{2}c_2 & 0 \end{bmatrix} \quad J_{A2}^{World} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_{L2}^{Tool} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{l_2}{2}c_2 & 0 & 0 \\ 0 & \frac{l_2}{2} & 0 \end{bmatrix} \quad J_{A2}^{Tool} = \begin{bmatrix} s_2 & 0 & 0 \\ 0 & -1 & 0 \\ c_2 & 0 & 0 \end{bmatrix}$$

For link 3 & 4: ( $l_4 = 0, d_3 = d_3 + l_4 - \frac{l}{2}$ )

$$J_{L3}^{World} = \begin{bmatrix} -s_1(l_2c_2 - (d_3 + l_4 - \frac{l}{2})s_2) & -c_1(l_2s_2 + (d_3 + l_4 - \frac{l}{2})c_2) & -c_1s_2 \\ c_1(l_2c_2 - (d_3 + l_4 - \frac{l}{2})s_2) & -s_1(l_2s_2 + (d_3 + l_4 - \frac{l}{2})c_2) & -s_1s_2 \\ 0 & l_2c_2 - (d_3 + l_4 - \frac{l}{2})s_2 & c_2 \end{bmatrix} \quad J_{A3}^{World} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_{L3}^{Tool} = \begin{bmatrix} 0 & -\left(d_3 + l_4 - \frac{l}{2}\right) & 0 \\ l_2c_2 - \left(d_3 + l_4 - \frac{l}{2}\right)s_2 & 0 & 0 \\ 0 & l_2 & 1 \end{bmatrix} \quad J_{A3}^{Tool} = \begin{bmatrix} s_2 & 0 & 0 \\ 0 & -1 & 0 \\ c_2 & 0 & 0 \end{bmatrix}$$

We assume each link as a thin rod, that the moment of inertia is negligible with respect to the longitudinal axis:

$$I_1 = \begin{bmatrix} \frac{m_1l_1^2}{12} & 0 & 0 \\ 0 & \frac{m_1l_1^2}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m_2l_2^2}{12} & 0 \\ 0 & 0 & \frac{m_2l_2^2}{12} \end{bmatrix} \quad I_3 = \begin{bmatrix} \frac{m_3l^2}{12} & 0 & 0 \\ 0 & \frac{m_3l^2}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the inertia matrix H is

$$\mathbf{H}(\mathbf{q}) = \sum_{i=1}^3 \left( \mathbf{m}_i \mathbf{J}_{Li}^{ToolT} \mathbf{J}_{Li}^{Tool} + \mathbf{J}_{Ai}^{ToolT} \mathbf{I}_i \mathbf{J}_{Ai}^{Tool} \right) + \mathbf{M} \mathbf{J}_L^{ToolT} \mathbf{J}_L^{Tool}$$

Note that we treat the point mass M held by the end effector as an additional link with zero length.

$$\boxed{\mathbf{H}(\mathbf{q}) = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{22} & \mathbf{l}_2(\mathbf{M} + \mathbf{m}_3) \\ \mathbf{0} & \mathbf{l}_2(\mathbf{M} + \mathbf{m}_3) & \mathbf{M} + \mathbf{m}_3 \end{pmatrix}}$$

$$H_{11} = M(l_2 c_2 - (d_3 + l_4)s_2)^2 + m_3 \left( \left( l_2 c_2 - \left( d_3 + l_4 - \frac{l}{2} \right) s_2 \right)^2 + \frac{l^2}{12} s_2^2 \right) + m_2 \frac{l_2^2}{3} c_2^2$$

$$H_{22} = M(l_2^2 + (d_3 + l_4)^2) + m_3 \left( l_2^2 + \left( d_3 + l_4 - \frac{l}{2} \right)^2 + \frac{l^2}{12} \right) + m_2 \frac{l_2^2}{3}$$

Then calculate the velocity matrix C

$$q = [\theta_1 \quad \theta_2 \quad d_3]^T$$

$$\dot{q} = [\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{d}_3]^T$$

$$\ddot{q} = [\ddot{\theta}_1 \quad \ddot{\theta}_2 \quad \ddot{d}_3]^T$$

$$C_{11} = \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{11}}{\partial q_1} + \frac{\partial H_{11}}{\partial q_1} - \frac{\partial H_{11}}{\partial q_1} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{11}}{\partial q_2} + \frac{\partial H_{12}}{\partial q_1} - \frac{\partial H_{12}}{\partial q_1} \right] \dot{q}_2$$

$$+ \frac{1}{2} \left[ \frac{\partial H_{11}}{\partial q_3} + \frac{\partial H_{13}}{\partial q_1} - \frac{\partial H_{13}}{\partial q_1} \right] \dot{q}_3 = \frac{1}{2} \left( \frac{\partial H_{11}}{\partial q_2} \cdot \dot{\theta}_2 + \frac{\partial H_{11}}{\partial q_3} \cdot \dot{d}_3 \right)$$

$$C_{12} = \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{12}}{\partial q_1} + \frac{\partial H_{11}}{\partial q_2} - \frac{\partial H_{21}}{\partial q_1} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{12}}{\partial q_2} + \frac{\partial H_{12}}{\partial q_2} - \frac{\partial H_{22}}{\partial q_1} \right] \dot{q}_2$$

$$+ \frac{1}{2} \left[ \frac{\partial H_{12}}{\partial q_3} + \frac{\partial H_{13}}{\partial q_2} - \frac{\partial H_{23}}{\partial q_1} \right] \dot{q}_3 = \frac{1}{2} \cdot \frac{\partial H_{11}}{\partial q_2} \cdot \dot{\theta}_1$$

$$C_{13} = \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{13}}{\partial q_1} + \frac{\partial H_{11}}{\partial q_3} - \frac{\partial H_{31}}{\partial q_1} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{13}}{\partial q_2} + \frac{\partial H_{12}}{\partial q_3} - \frac{\partial H_{32}}{\partial q_1} \right] \dot{q}_2$$

$$+ \frac{1}{2} \left[ \frac{\partial H_{13}}{\partial q_3} + \frac{\partial H_{13}}{\partial q_3} - \frac{\partial H_{33}}{\partial q_1} \right] \dot{q}_3 = \frac{1}{2} \cdot \frac{\partial H_{11}}{\partial q_3} \cdot \dot{\theta}_1$$

$$C_{21} = \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{21}}{\partial q_1} + \frac{\partial H_{21}}{\partial q_1} - \frac{\partial H_{11}}{\partial q_2} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{21}}{\partial q_2} + \frac{\partial H_{22}}{\partial q_1} - \frac{\partial H_{12}}{\partial q_2} \right] \dot{q}_2$$

$$+ \frac{1}{2} \left[ \frac{\partial H_{21}}{\partial q_3} + \frac{\partial H_{23}}{\partial q_1} - \frac{\partial H_{13}}{\partial q_2} \right] \dot{q}_3 = -\frac{1}{2} \cdot \frac{\partial H_{11}}{\partial q_2} \cdot \dot{\theta}_1$$

$$C_{22} = \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{22}}{\partial q_1} + \frac{\partial H_{21}}{\partial q_2} - \frac{\partial H_{21}}{\partial q_2} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{22}}{\partial q_2} + \frac{\partial H_{22}}{\partial q_2} - \frac{\partial H_{22}}{\partial q_2} \right] \dot{q}_2$$

$$+ \frac{1}{2} \left[ \frac{\partial H_{22}}{\partial q_3} + \frac{\partial H_{23}}{\partial q_2} - \frac{\partial H_{23}}{\partial q_2} \right] \dot{q}_3 = \frac{1}{2} \cdot \frac{\partial H_{22}}{\partial q_3} \cdot \dot{d}_3$$

$$C_{23} = \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{23}}{\partial q_1} + \frac{\partial H_{21}}{\partial q_3} - \frac{\partial H_{31}}{\partial q_2} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{23}}{\partial q_2} + \frac{\partial H_{22}}{\partial q_3} - \frac{\partial H_{32}}{\partial q_2} \right] \dot{q}_2$$

$$+ \frac{1}{2} \left[ \frac{\partial H_{23}}{\partial q_3} + \frac{\partial H_{23}}{\partial q_3} - \frac{\partial H_{33}}{\partial q_2} \right] \dot{q}_3 = \frac{1}{2} \cdot \frac{\partial H_{22}}{\partial q_3} \cdot \dot{\theta}_2$$

$$\begin{aligned} C_{31} &= \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{31}}{\partial q_1} + \frac{\partial H_{31}}{\partial q_1} - \frac{\partial H_{11}}{\partial q_3} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{31}}{\partial q_2} + \frac{\partial H_{32}}{\partial q_1} - \frac{\partial H_{12}}{\partial q_3} \right] \dot{q}_2 \\ &\quad + \frac{1}{2} \left[ \frac{\partial H_{31}}{\partial q_3} + \frac{\partial H_{33}}{\partial q_1} - \frac{\partial H_{13}}{\partial q_3} \right] \dot{q}_3 = -\frac{1}{2} \cdot \frac{\partial H_{11}}{\partial q_3} \cdot \dot{\theta}_1 \end{aligned}$$

$$\begin{aligned} C_{32} &= \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{32}}{\partial q_1} + \frac{\partial H_{31}}{\partial q_2} - \frac{\partial H_{21}}{\partial q_3} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{32}}{\partial q_2} + \frac{\partial H_{32}}{\partial q_2} - \frac{\partial H_{22}}{\partial q_3} \right] \dot{q}_2 \\ &\quad + \frac{1}{2} \left[ \frac{\partial H_{32}}{\partial q_3} + \frac{\partial H_{33}}{\partial q_2} - \frac{\partial H_{23}}{\partial q_3} \right] \dot{q}_3 = -\frac{1}{2} \cdot \frac{\partial H_{22}}{\partial q_3} \cdot \dot{\theta}_2 \\ C_{33} &= \sum_{k=1}^3 c_{kji} \cdot \dot{q}_k = \frac{1}{2} \left[ \frac{\partial H_{33}}{\partial q_1} + \frac{\partial H_{31}}{\partial q_3} - \frac{\partial H_{31}}{\partial q_3} \right] \dot{q}_1 + \frac{1}{2} \left[ \frac{\partial H_{33}}{\partial q_2} + \frac{\partial H_{32}}{\partial q_3} - \frac{\partial H_{32}}{\partial q_3} \right] \dot{q}_2 \\ &\quad + \frac{1}{2} \left[ \frac{\partial H_{33}}{\partial q_3} + \frac{\partial H_{33}}{\partial q_3} - \frac{\partial H_{33}}{\partial q_3} \right] \dot{q}_3 = 0 \end{aligned}$$

$$C(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \begin{pmatrix} \frac{\partial H_{11}}{\partial q_2} \cdot \dot{\theta}_2 + \frac{\partial H_{11}}{\partial q_3} \cdot \dot{d}_3 & \frac{\partial H_{11}}{\partial q_2} \cdot \dot{\theta}_1 & \frac{\partial H_{11}}{\partial q_3} \cdot \dot{\theta}_1 \\ -\frac{\partial H_{11}}{\partial q_2} \cdot \dot{\theta}_1 & \frac{\partial H_{22}}{\partial q_3} \cdot \dot{d}_3 & \frac{\partial H_{22}}{\partial q_3} \cdot \dot{\theta}_2 \\ -\frac{\partial H_{11}}{\partial q_3} \cdot \dot{\theta}_1 & -\frac{\partial H_{22}}{\partial q_3} \cdot \dot{\theta}_2 & 0 \end{pmatrix}$$

where

$$\begin{aligned} \frac{\partial H_{11}}{\partial q_2} &= -2M(l_2 s_2 + (d_3 + l_4)c_2)(l_2 c_2 - (d_3 + l_4)s_2) \\ &\quad - 2m_3 \left( l_2 s_2 + \left( d_3 + l_4 - \frac{l}{2} \right) c_2 \right) \left( l_2 c_2 - \left( d_3 + l_4 - \frac{l}{2} \right) s_2 \right) + c_2 s_2 \frac{l^2 m_3}{6} \\ &\quad - 2c_2 s_2 \frac{l_2^2 m_2}{3} \end{aligned}$$

$$\frac{\partial H_{11}}{\partial q_3} = -2Ms_2(l_2 c_2 - (d_3 + l_4)s_2) - 2m_3 s_2 \left( l_2 c_2 - s_2 \left( d_3 + l_4 - \frac{l}{2} \right) \right)$$

$$\frac{\partial H_{22}}{\partial q_3} = 2M(d_3 + l_4) + 2m_3(d_3 + l_4 - \frac{l}{2})$$

And then calculate gravitational forces vector G

$$G(\mathbf{q}) = - \left( \sum_{i=1}^4 \mathbf{m}_i J_{L_i}^{World^T} + M J_L^{World^T} \right) \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -g \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} \\ \mathbf{M}(l_2 c_2 - (d_3 + l_4)s_2) + \mathbf{m}_3 \left( l_2 c_2 - \left( d_3 + l_4 - \frac{l}{2} \right) s_2 \right) + \mathbf{m}_2 \frac{l_2}{2} c_2 \\ (\mathbf{M} + \mathbf{m}_3) c_2 \end{bmatrix} g$$

Then the external force  $\mathbf{F}_e$  in world frame:

$$\boxed{\mathbf{F}_e = [F_{e_x} \ F_{e_y} \ F_{e_z}]^T}$$

Thus, the dynamic equation of motion is

$$\boxed{\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \boldsymbol{\tau} + J_L^{World^T} \mathbf{F}_e}$$

Note that we use Matlab to compare **the dynamic equation**,  $\mathbf{H}$ ,  $\mathbf{G}$  calculated by 1a and 1b, and **get the same result!** (Please check the details in section «**compare 1a 1b**» in the attached Matlab code) **But we can not get the same C**, because according to the course, the choice of C is not unique.

Command Window	Command Window
<code>compare_H =</code>	<code>compare_eq1 =</code>
<code>[0, 0, 0]</code>	<code>0</code>
<code>[0, 0, 0]</code>	<code>compare_eq2 =</code>
<code>[0, 0, 0]</code>	<code>0</code>
 	<code>compare_eq3 =</code>
<code>compare_G =</code>	 
<code>0</code>	<code>fx 0</code>
<code>0</code>	
<code>fx 0</code>	

## 2. Inverse Dynamics

Firstly we calculate the mass for each link:

$$m_1 = \rho l_1 \pi \frac{d^2}{4} = 7800 \cdot 0.4\pi \frac{0.015^2}{4} = 0.5513[\text{kg}]$$

$$m_2 = \rho l_2 \pi \frac{d^2}{4} = 7800 \cdot 0.15\pi \frac{0.015^2}{4} = 0.2068[\text{kg}]$$

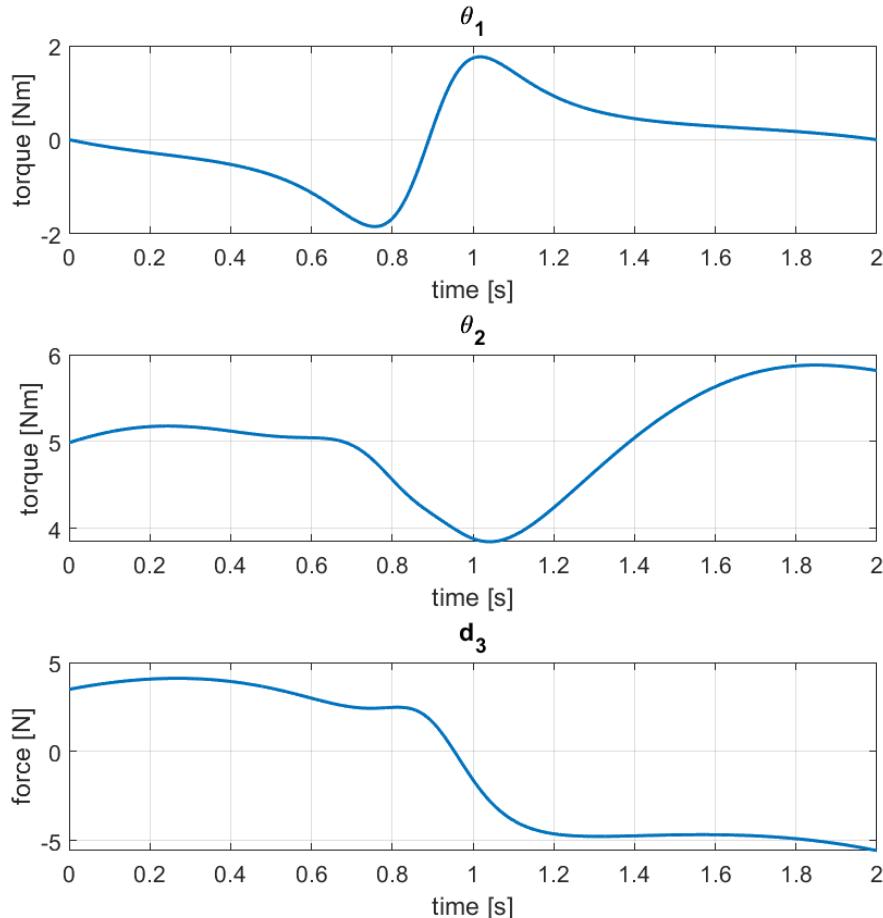
$$m_3 = \rho l \pi \frac{d^2}{4} = 7800 \cdot 0.6\pi \frac{0.015^2}{4} = 0.8270[\text{kg}]$$

$$M = 0.5[\text{kg}]$$

And then we get  $q$ ,  $q_{\text{dot}}$ ,  $q_{\text{dot2}}$  in the polynomial profile with discrete time value 0.001 (sec) from the MATLAB function `q_plan`, `q_dot_plan`, `q_dot2_plan`. Plug all values into MATLAB function `dynamics_mat`, we get the  $H$ ,  $C$ ,  $G$  for each time step. Finally, we could get the torque/force for each joint by the equation

$$\tau = H(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + G(\mathbf{q})$$

The figure below shows the control actuation forces/torques at the joints, as a function of time:



### 3. Reaction Force

Using Newton-Euler method to calculate the reaction force acting on joint 2 in the direction of the joint axis

In the following calculation,  $\omega_i$  is the angular velocity,  $\alpha_i$  is the angular acceleration,  $a_{c,i}$  is the acceleration of center of mass,  $a_{e,i}$  is the acceleration of end of link i.

Forward recursion:

$$\omega_0 = \alpha_0 = a_{c,0} = a_{e,0} = [0 \ 0 \ 0]^T$$

Joint 1 is a revolute, so  $\ddot{d}_1 = \dot{d}_1 = 0$  and we have:

$$r_{1,c1} = \left[ 0, 0, \frac{l_1}{2} \right]^T \quad r_{1,e1} = [0, 0, l_1]^T \quad {}^0R_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_2 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad u_1 = [0, 0, 1]^T$$

$$\begin{aligned} \omega_1 &= {}^0R_1^T \omega_0 + u_1 \dot{q}_1 = [0 \ 0 \ \dot{q}_1]^T \\ \alpha_1 &= {}^0R_1^T \alpha_0 + u_1 \ddot{q}_1 + \omega_1 \times u_1 \dot{q}_1 = [0 \ 0 \ \ddot{q}_1]^T \\ a_{c,1} &= {}^0R_1^T a_{e,0} + \alpha_1 \times r_{1,c1} + \omega_1 \times (\omega_1 \times r_{1,c1}) = [0 \ 0 \ 0]^T \\ a_{e,1} &= {}^0R_1^T a_{e,0} + \alpha_1 \times r_{1,e1} + \omega_1 \times (\omega_1 \times r_{1,e1}) = [0 \ 0 \ 0]^T \end{aligned}$$

Joint 2 is a revolute, so  $\ddot{d}_2 = \dot{d}_2 = 0$  and we have:

$$r_{2,c2} = \left[ \frac{l_2}{2}, 0, 0 \right]^T ; \quad r_{2,e2} = [l_2, 0, 0]^T \quad {}^1R_2 = \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix} \quad u_2 = [0, -1, 0]^T$$

$$\begin{aligned} \omega_2 &= {}^1R_2^T \omega_1 + u_2 \dot{q}_2 = [\dot{q}_1 s_2 \ -\dot{q}_2 \ \dot{q}_1 c_2]^T \\ \alpha_2 &= {}^1R_2^T \alpha_1 + u_2 \ddot{q}_2 + \omega_2 \times u_2 \dot{q}_2 = [\ddot{q}_1 s_2 + \dot{q}_1 \dot{q}_2 c_2 \ -\ddot{q}_2 \ \dot{q}_1 c_2 - \dot{q}_1 \dot{q}_2 s_2]^T \\ a_{c,2} &= \alpha_2 \times r_{2,c2} + \omega_2 \times (\omega_2 \times r_{2,c2}) = \frac{l_2}{2} \begin{bmatrix} -\dot{q}_1^2 c_2^2 - \dot{q}_2^2 \\ \dot{q}_1 c_2 - 2\dot{q}_1 \dot{q}_2 s_2 \\ \dot{q}_1^2 c_2 s_2 + \dot{q}_2 \end{bmatrix} \\ a_{e,2} &= \alpha_2 \times r_{2,e2} + \omega_2 \times (\omega_2 \times r_{2,e2}) = l_2 \begin{bmatrix} -\dot{q}_1^2 c_2^2 - \dot{q}_2^2 \\ \dot{q}_1 c_2 - 2\dot{q}_1 \dot{q}_2 s_2 \\ \dot{q}_1^2 c_2 s_2 + \dot{q}_2 \end{bmatrix} \end{aligned}$$

Joint 3 & 4 is prismatic, so  $\ddot{q}_3 = \dot{q}_3 = 0$

$$r_{3,c3} = \left[ 0, 0, d_3 + l_4 - \frac{l_1}{2} \right]^T \quad r_{3,e3} = [0, 0, d_3 + l_4]^T \quad {}^2R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad u_3 = [0, 0, 1]^T$$

$$\begin{aligned} \omega_3 &= {}^2R_3^T \omega_2 + u_3 \dot{q}_3 = [\dot{q}_1 s_2 \ -\dot{q}_2 \ \dot{q}_1 c_2]^T \\ \alpha_3 &= {}^2R_3^T \alpha_2 + u_3 \ddot{q}_3 + \omega_3 \times u_3 \dot{q}_3 = [\ddot{q}_1 s_2 + \dot{q}_1 \dot{q}_2 c_2 \ -\ddot{q}_2 \ \dot{q}_1 c_2 - \dot{q}_1 \dot{q}_2 s_2]^T \end{aligned}$$

$$\mathbf{a}_{c3} = {}^2R_3^T \mathbf{a}_{e2} + \boldsymbol{\alpha}_3 \times \mathbf{r}_{3,c3} + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{r}_{3,c3}) + \mathbf{u}_3 \ddot{\mathbf{d}}_3 + 2(\boldsymbol{\omega}_3 \times \mathbf{u}_3) \dot{\mathbf{d}}_3 =$$

$$\begin{bmatrix} (\dot{q}_1^2 s_2 c_2 - \ddot{q}_2) \left( d_3 + l_4 - \frac{l}{2} \right) - l_2 (\dot{q}_1^2 c_2^2 + \dot{q}_2^2) - 2 \dot{d}_3 \dot{q}_2 \\ l_2 (\ddot{q}_1 c_2 - 2 \dot{q}_1 \dot{q}_2 s_2) - (\dot{q}_1 s_2 + 2 \dot{q}_1 \dot{q}_2 c_2) \left( d_3 + l_4 - \frac{l}{2} \right) - 2 \dot{d}_3 \dot{q}_1 s_2 \\ \ddot{d}_3 - (\dot{q}_1^2 s_2^2 + \dot{q}_2^2) \left( d_3 + l_4 - \frac{l}{2} \right) + l_2 (s_2 c_2 \dot{q}_1^2 + \dot{q}_2) \end{bmatrix}$$

$$\mathbf{a}_{e3} = {}^2R_3^T \mathbf{a}_{e2} + \boldsymbol{\alpha}_3 \times \mathbf{r}_{3,e3} + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{r}_{3,e3}) + \mathbf{u}_3 \ddot{\mathbf{d}}_3 + 2(\boldsymbol{\omega}_3 \times \mathbf{u}_3) \dot{\mathbf{d}}_3 =$$

$$\begin{bmatrix} (\dot{q}_1^2 s_2 c_2 - \ddot{q}_2) (d_3 + l_4) - l_2 (\dot{q}_1^2 c_2^2 + \dot{q}_2^2) - 2 \dot{d}_3 \dot{q}_2 \\ l_2 (\ddot{q}_1 c_2 - 2 \dot{q}_1 \dot{q}_2 s_2) - (\dot{q}_1 s_2 + 2 \dot{q}_1 \dot{q}_2 c_2) (d_3 + l_4) - 2 \dot{d}_3 \dot{q}_1 s_2 \\ \ddot{d}_3 - (\dot{q}_1^2 s_2^2 + \dot{q}_2^2) (d_3 + l_4) + l_2 (s_2 c_2 \dot{q}_1^2 + \dot{q}_2) \end{bmatrix}$$

Note that we regard m3 as the mass of link 3 & 4, and I3 as the inertia value of link 3 & 4. This is according to the assumption that the center of mass of the link 3 is at a constant length l/2 before the end effector, which **take link 3 & 4 as one entity**.

### Backwards recursion:

The force  $\mathbf{f}_M$  produced by the gripper M:

$${}^0R_3 = \begin{bmatrix} c_1 c_2 & -s_1 & -c_1 s_2 \\ s_1 c_2 & c_1 & -s_1 s_2 \\ s_2 & 0 & c_2 \end{bmatrix}$$

$$\mathbf{f}_M = M \mathbf{a}_{e3} - M {}^0R_3^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} =$$

$$\begin{bmatrix} M g s_2 - M \left( \ddot{q}_2 (d_3 + l_4) + 2 \dot{d}_3 \dot{q}_2 + l_2 (c_2^2 \dot{q}_1^2 + \dot{q}_2^2) - c_2 s_2 \dot{q}_1^2 (d_3 + l_4) \right) \\ -M \left( (d_3 + l_4) \left( s_2 \ddot{q}_1 + c_2 \dot{q}_1 \dot{q}_2 \right) - l_2 \left( c_2 \ddot{\theta}_1 - 2 s_2 \dot{q}_1 \dot{q}_2 \right) + 2 \dot{d}_3 s_2 \dot{q}_1 + c_2 \dot{q}_1 \dot{q}_2 (d_3 + l_4) \right) \\ M \left( d_3 - \dot{q}_2^2 (d_3 + l_4) + l_2 \left( s_2 c_2 \dot{q}_1^2 + \dot{q}_2 \right) - s_2^2 \dot{q}_1^2 (d_3 + l_4) \right) + M c_2 g \end{bmatrix}$$

The force  $\mathbf{f}_3$  on link 3:

$${}^3R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{f}_3 = m_3 \mathbf{a}_{e3} + {}^3R_t \mathbf{f}_M - m_3 {}^0R_3^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

$$\begin{aligned} f_3(1) = & -M \left( 2 \dot{d}_3 \dot{q}_2 + (q_2 - s_2 c_2 \dot{q}_1^2) (d_3 + l_4) l_2 (c_2^2 \dot{q}_1^2 + \dot{q}_2^2) - g s_2 \right) \\ & - m_3 \left( 2 \dot{d}_3 \dot{q}_2 + (q_2 - s_2 c_2 \dot{q}_1^2) \left( d_3 - \frac{l}{2} + l_4 \right) + l_2 (c_2^2 \dot{q}_1^2 + \dot{q}_2^2) - g s_2 \right) \end{aligned}$$

$$\begin{aligned} f_3(2) = & -M \left( \left( s_2 \ddot{q}_1 + 2 c_2 \dot{q}_1 \dot{q}_2 \right) (d_3 + l_4) - l_2 \left( c_2 \ddot{q}_1 - 2 s_2 \dot{q}_1 \dot{q}_2 \right) + 2 \dot{d}_3 s_2 \dot{q}_1 \right) \\ & - m_3 \left( \left( s_2 \ddot{q}_1 + 2 c_2 \dot{q}_1 \dot{q}_2 \right) \left( d_3 - \frac{l}{2} + l_4 \right) - l_2 \left( c_2 \ddot{q}_1 - 2 s_2 \dot{q}_1 \dot{q}_2 \right) + 2 \dot{d}_3 s_2 \dot{q}_1 \right) \end{aligned}$$

$$\begin{aligned} f_3(3) = & M \left( q_3 - (\dot{q}_2^2 + s_2^2 \dot{q}_1^2) (d_3 + l_4) + l_2 \left( s_2 c_2 \dot{q}_1^2 + q_2 \right) + c_2 g \right) \\ & + m_3 \left( q_3 - (\dot{q}_2^2 + s_2^2 \dot{q}_1^2) \left( d_3 - \frac{l}{2} + l_4 \right) + l_2 \left( s_2 c_2 \dot{q}_1^2 + q_2 \right) + c_2 g \right) \end{aligned}$$

The force  $f_2$  on link 2:

$${}^2R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f_2 = m_2 \mathbf{a}_{e2} + {}^2R_3 f_3 - m_2 {}^0R_2^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

$$f_2(1) = -M \left( q_2 (d_3 + l_4) + 2 \dot{d}_3 \dot{q}_2 + l_2 (c_2^2 \dot{q}_1^2 + \dot{q}_2^2) - \frac{s_{22} \dot{q}_1^2 (d_3 + l_4)}{2} - g s_2 \right)$$

$$- m_3 \left( 2 \dot{d}_3 \dot{q}_2 + (\theta_2 - s_2 c_2 \dot{q}_1^2) (d_3 - \frac{l}{2} + l_4) + l_2 (c_2^2 \dot{q}_1^2 + \dot{q}_2^2) - g s_2 \right)$$

$$- \frac{l_2 m_2 (c_2^2 \dot{q}_1^2 + \dot{q}_2^2)}{2} + g m_2 s_2$$

$$f_2(2) = -M \left( (d_3 + l_4) \left( s_2 \ddot{q}_1 + c_2 \dot{q}_1 \dot{q}_2 \right) - l_2 (c_2 \ddot{q}_1 - 2 s_2 \dot{q}_1 \dot{q}_2) + 2 \dot{d}_3 s_2 \dot{q}_1 + c_2 \dot{q}_1 \dot{q}_2 (d_3 + l_4) \right)$$

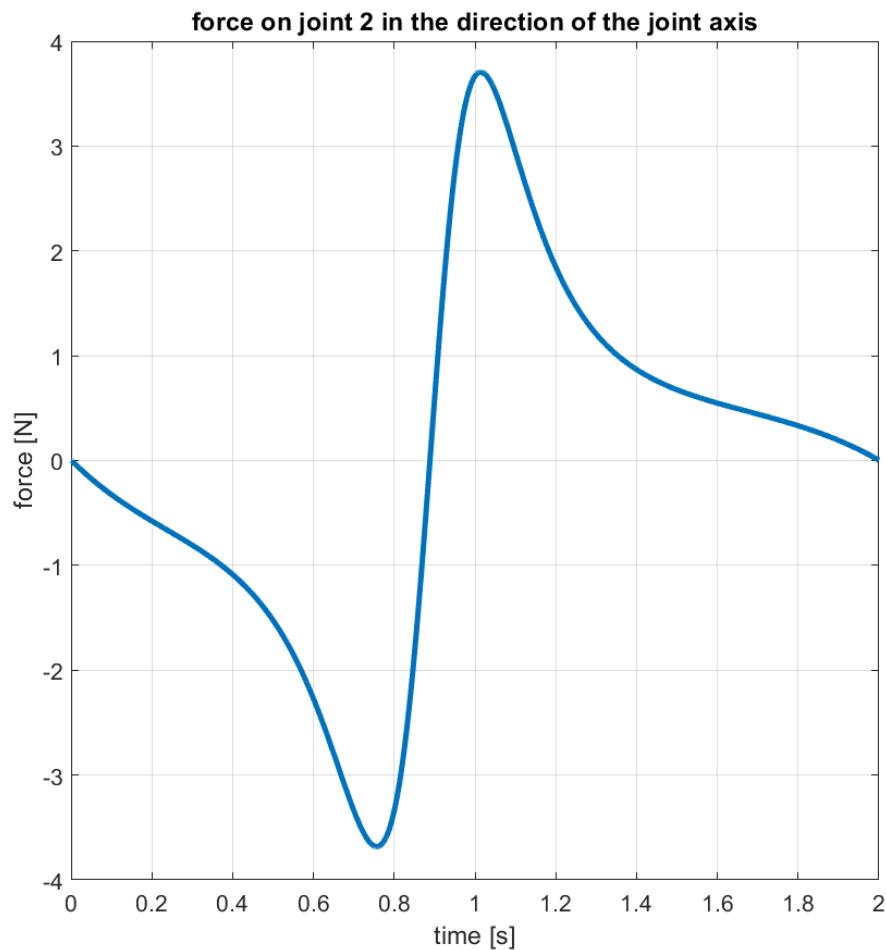
$$- m_3 \left( \left( s_2 \ddot{q}_1 + c_2 \dot{q}_1 \dot{q}_2 + c_2 \dot{q}_1 \dot{q}_2 \right) \left( d_3 - \frac{l}{2} + l_4 \right) - l_2 (c_2 \ddot{q}_1 - 2 s_2 \dot{q}_1 \dot{q}_2) + 2 \dot{q}_3 s_2 \dot{q}_1 \right)$$

$$+ \frac{l_2 m_2 (c_2 \ddot{q}_1 - 2 s_2 \dot{q}_1 \dot{q}_2)}{2}$$

$$\begin{aligned}
f_2(3) = & M \left( q_3 - \dot{q}_2^2 (d_3 + l_4) + (s_2 c_2 \dot{q}_1^2 + q_2) l_2 - s_2^2 \dot{q}_1^2 (d_3 + l_4) + c_2 g \right) \\
& + m_3 \left( q_3 - (\dot{q}_2^2 + s_2^2 \dot{q}_1^2) (d_3 + l_4 - \frac{l}{2}) + (s_2 c_2 \dot{q}_1^2 + q_2) l_2 + c_2 g \right) \\
& + \frac{m_2 l_2 (s_2 c_2 \dot{q}_1^2 + q_2)}{2} + c_2 g m_2
\end{aligned}$$

Here  $f_2(2)$  is the explicit expression of the reaction force acting on joint 2 in the direction of the joint axis.

And then we get  $q$ ,  $q_{\text{dot}}$ ,  $q_{\text{dot2}}$  in the polynomial profile with discrete time value 0.001 (sec) from the MATLAB function `q_plan`, `q_dot_plan`, `q_dot2_plan`. Plug all values into  $f_2(2)$ , and plot a graph of this force as a function of time:



## 4. Forward Dynamics

To solve the forward dynamics, we first set the dynamic equation as

$$\ddot{q} = H^{-1}(q)[\tau - C(q, \dot{q}) - G(q)]$$

Then in order to use **ode45** to perform the dynamic simulation, we have to rewrite the dynamic equation as

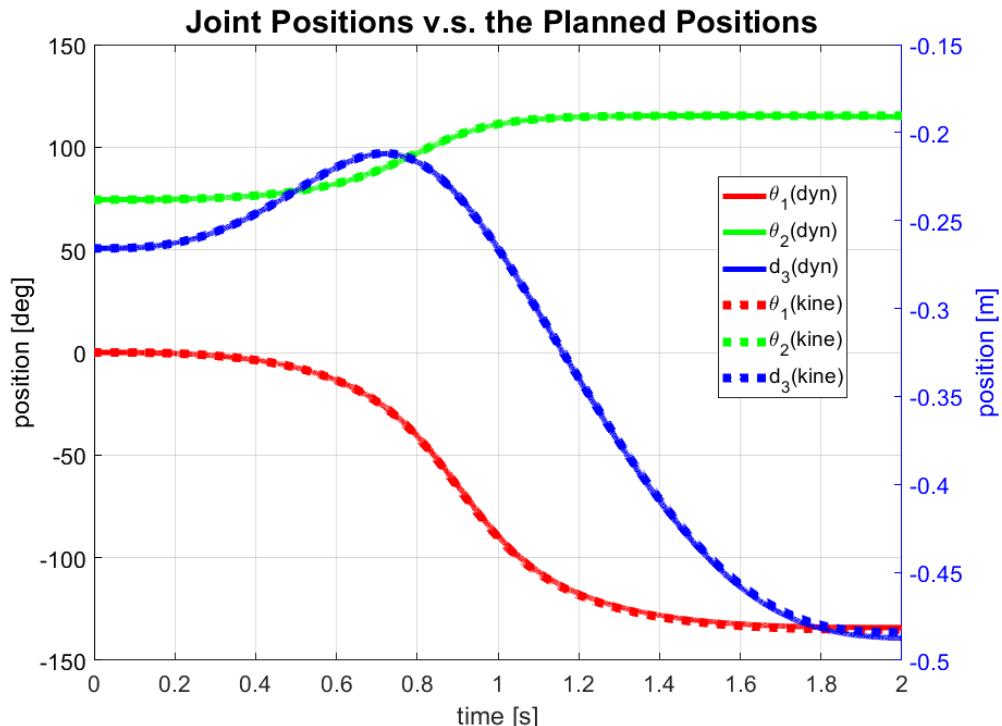
$$\dot{X} = \begin{bmatrix} z_2 \\ H(z_l)^{-1}(\tau - C(z_l, z_2) \cdot z_2 - G(z_l)) \end{bmatrix}$$

$$\text{where } X = [z_l, z_2]^T \quad z_l = [q_1, q_2, q_3]^T$$

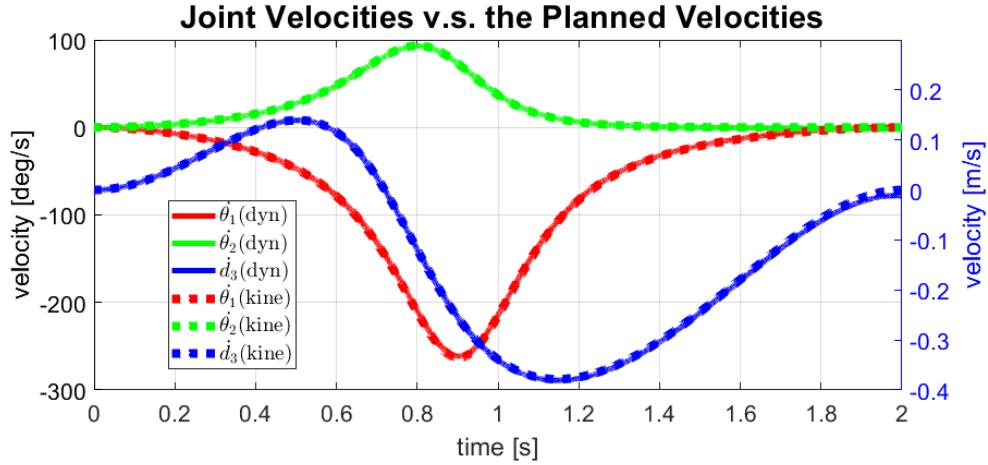
We write the function **state\_eq** from the above equation, note that we use function **interp1** inside the function **state\_eq** in order to modify **tau** fitting to **ode45**. Then we choose the integration step as 0.0001 sec, which is 10 times smaller than  $\Delta T$ .

As for the initial condition, we use **inv\_kin** to calculate the corresponding joint positions to initial position A. Specifically, we take the first row of **q** calculated by **q\_plan** as the initial joint value.

Lastly, we set the maximal time step as 1e-8 and relative error as 1e-8. Put all into **ode45**, we get the dynamics simulation of the robot motion. We compare this motion to the planned values of joints, which is plotted in dots.



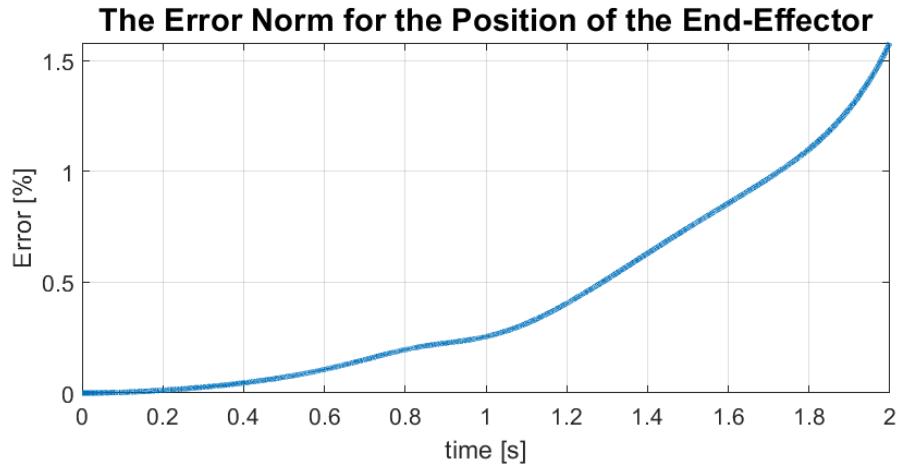
The figure above is the joint positions as a function of time compared to the planned positions, and the figure below is the joint velocities as a function of time compared to the planned velocities.



We also calculate the error norm for the position of the end-effector

$$\text{error} = \frac{\sqrt{(x - x_d)^2 + (y - y_d)^2 + (z - z_d)^2}}{\| \text{full path length} \|}$$

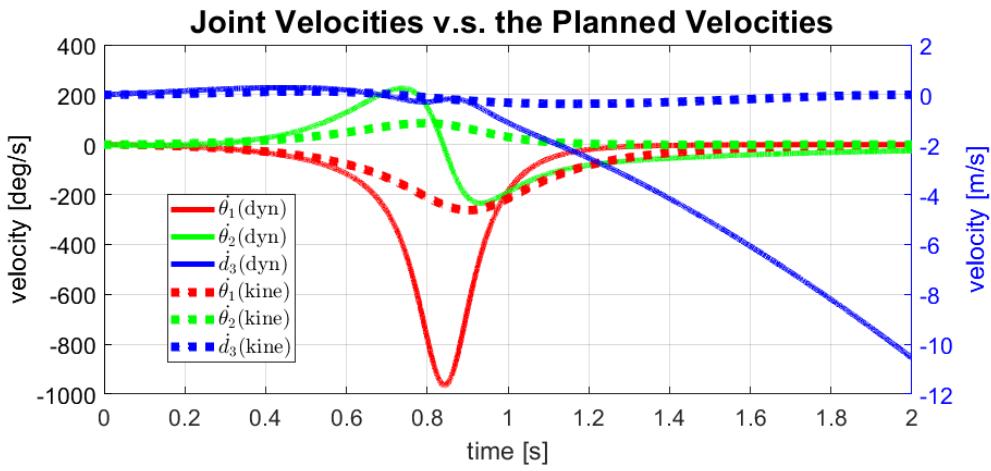
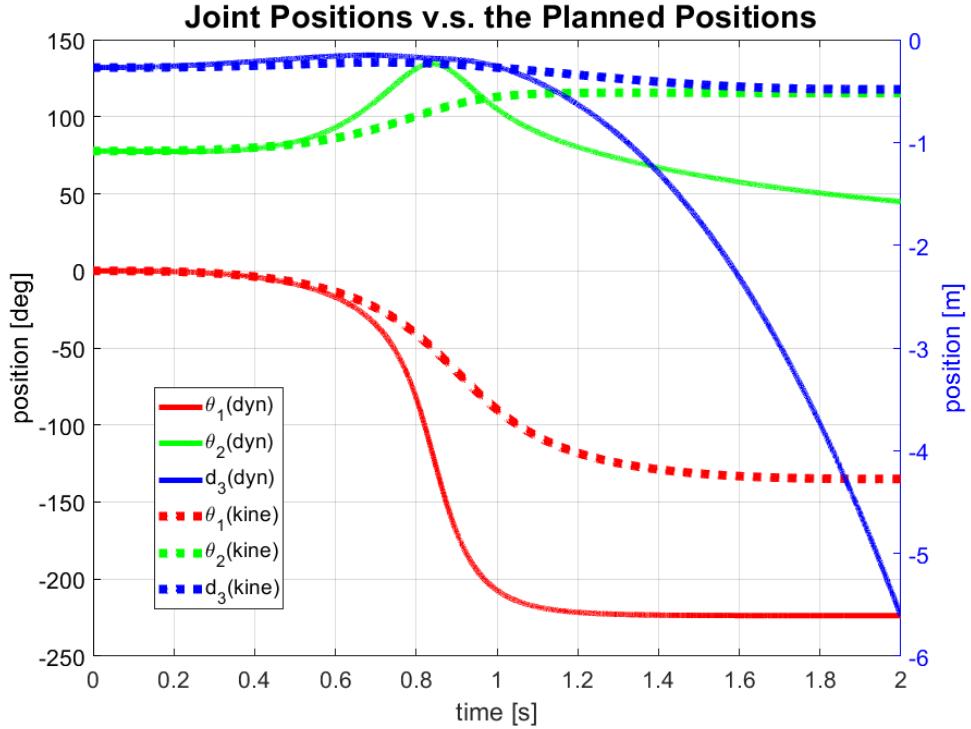
where full path length is  $\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$



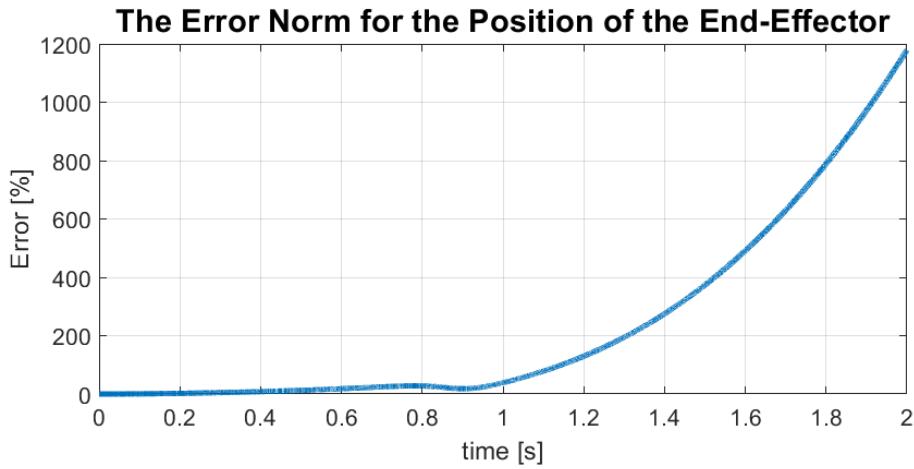
We could see that the error for the position of the end-effector is less than 2% in the whole trajectory.

## 5. Forward Dynamics (new initial position)

We repeat section 4 with the initial position of the end-effector moved 1cm above point A. The new  $Z_A = 0.51$  m. And we get the following figures.



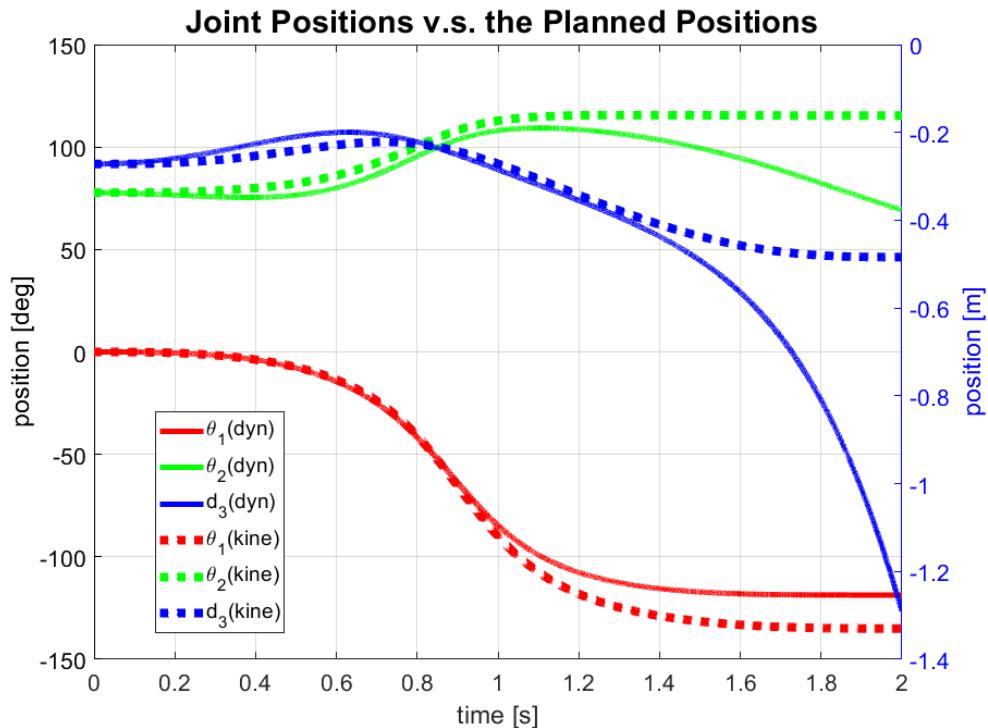
We could observe that just a slight change in the initial position causes a great difference to the planned joint values. This is expressed in the next figure, that is, the error of the position of the end effector.



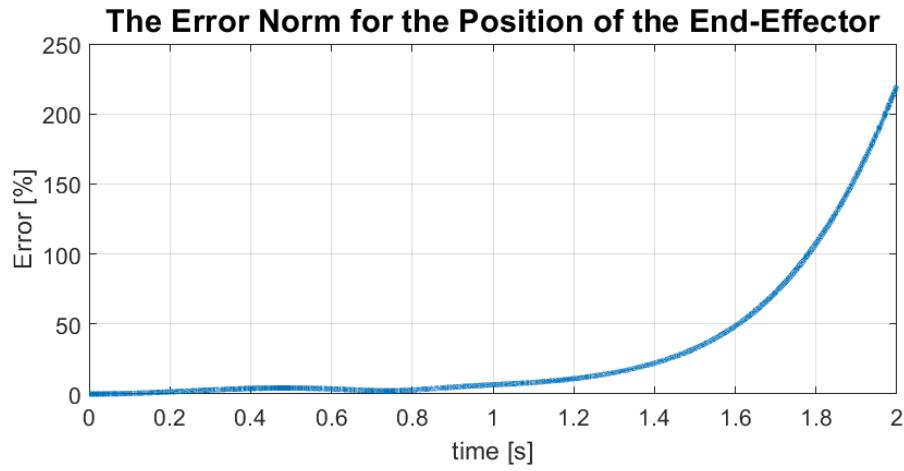
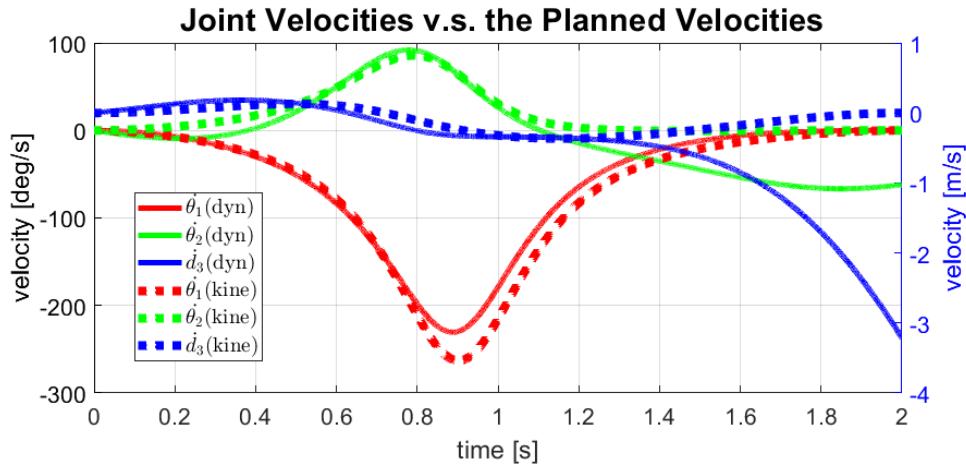
In this case, the position of the end effector is totally away from the planned position, the error even reach 1200%!

## 6. Forward Dynamics (new load mass)

Lastly, we repeat section 4 with initial conditions at point A but the load mass is changed to  $M = 0.6 \text{ kg}$ . We also get the following figures.



This time is better than section 5, yet not good as section 4. This performance also expressed in the following figures.



We could see this time the performance is better than section 5: The trajectories of the joint position and velocity are in the same trend but not coincide, and the largest error of the position of the end effector is 200%.

## 7. Summary

In this project, our primary focus was on formulating dynamic equations for the serial robot system. In Section 1, we explored two different approaches to obtain the same equation: differentiating the energies and utilizing the partial Jacobian. Initially, these two methods yielded different results, but through careful comparison, we gradually arrived at the correct answer. This process not only led us to the desired outcome but also enhanced our understanding of techniques for formulating dynamic systems in serial robots.

Moving on to Section 2, we utilized the equation obtained in the previous section to calculate the forces and torques acting on the joints as functions of time. We accomplished this task with the aid of Matlab. Simultaneously, in Section 3, we determined the reaction force acting on joint 2 in the direction of the joint axis. This explicit expression was derived using the Newton-Euler method. This "inverse dynamics" aspect allowed us to experience the utilization of dynamic equations to obtain interesting forces or torques acting on the joints.

Finally, Sections 4 to 6 introduced us to the challenge of "forward dynamics." It was truly gratifying when we observed the dynamic simulation of robot motion aligning with the planned trajectory based on the robot's kinematics. While it required considerable effort to identify and fix bugs, the satisfaction of obtaining the correct results surpassed all obstacles. We also noticed that even a slight change in initial conditions could result in a significant disparity between the dynamic simulation and the kinematic path planned for robot motion. This realization emphasized the importance of control, which will be covered in subsequent classes. By mastering robot control, we can prevent situations similar to those encountered in Sections 5 and 6.