Introduction to Modern Communications

361-1-3211

Computer Assignment 2

Spring 2022

General Instructions

- Electronic submission via the course website by **30.06.22, 23:59**.
  - All figures must be labeled and appropriately referenced in the text.
  - All figures must have a meaningful caption.
  - All figures must be labeled correctly on both the x and y axes.
  - In case a figure includes more than one plot, a legend should be added. Use markers to facilitate readability in the case of black and white printing.
  - The assignments has three questions: Question 1 has an analysis part and a simulation part. The analysis part is to be done without the aid of a computer. The simulation part is intended to be carried out using MATLAB. Questions 2 and 3 should be done analytically.
  - Analytic expressions should be simplified and explicit as much as possible.
- Submission should be done in pairs.
- All solutions and graphs should be submitted as a PDF file. MATLAB code should be attached to the end of the file.
- **Only one person should upload the assignment to the moodle, and the pdf file should contain the names and IDs of both students.**

Question 1: Error probability analysis of BPSK constellation with Gaussian and Laplace noise (MATLAB assignment) (50%)

In this section we analyze and simulate transmission over a discrete memoryless additive white noise (AWN) channel using a BPSK constellation. The channel is described by the following input-output relationship

$$R_n = S_n + W_n \ ; \ S_n \perp\!\!\!\perp W_n \tag{1}$$

where W_n denotes the white noise.

The symbols of the BPSK constellation are $S_n \in \left\{-\sqrt{E_s}, \sqrt{E_s}\right\}$, and the a-priori probabilities are equal (i.e., $\mathrm{P}\left(S_n = \sqrt{E_s}\right) = \mathrm{P}\left(S_n = -\sqrt{E_s}\right) = 0.5$).

1.1. Analysis (20%)

For the next two items assume that $W_n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

1.1.1. Derive the MAP decision rule for the given constellation. Draw the optimal decision regions.

1.1.2. Derive an analytic expression for the average error probability (also denoted as SER) at the output of the MAP decoder as a function of the signal to noise ratio (SNR) per symbol defined as $SNR = E_s / N_0$. Simplify the expression as much as you can.

The zero mean Laplace distribution with a variance equal to σ^2 is characterized by the following probability density function: $f_X(x) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{\frac{2}{\sigma^2}}|x|}$

For the next two items assume that W_n is a Laplace noise with zero mean and variance $\frac{N_0}{2}$.

1.1.3. Derive the MAP decision rule for the given constellation. Draw the optimal decision regions.

1.1.4. Derive an analytic expression for the average error probability (also denoted as SER) at the output of the MAP decoder as a function of the signal to noise ratio (SNR) per symbol defined as $SNR = E_s/N_0$. Simplify the expression as much as you can.

1.1.5. For a fixed SNR, under which noise distribution is the probability of error expected to be lower? Under Gaussian or Laplace distribution? Try to explain by drawing the Gaussian and Laplace distributions on the same figure.

1.2. Simulations (30%)

For the following simulations, $E_s = 1$.

1.2.1. Generate 10^5 symbols from the given constellation with equal a priori probability. You may use the MATLAB function “randsrc”.

1.2.2. Simulate transmission over the discrete-time baseband additive white **Gaussian** noise channel given in (1) and generate the received noisy vector r. The channel simulation should use SNR as a parameter. You may use the MATLAB function “randn” in order to generate the Gaussian noise.

1.2.3. Decode the received noisy signal into symbols for $SNR = -6, -5, ..., 5, 6[dB]$ using the MAP rule.

1.2.4. Compute the probability of error for the decoded symbol stream.

1.2.5. Plot the probability of error of the decoded symbols stream as a function of SNR.

1.2.6. Repeat items 1.2.1-1.2.5 for the **Laplace** noise. You can use the laprnd.m MATLAB function found at the course website to generate i.i.d Laplace random variables.

1.2.7. Plot in a single graph the probability of error of the Gaussian and Laplace channel as a function of SNR, and confirm your analytic analysis from item 1.1.5.

Question 2 (30%)

A source transmits one of two symbols s_0, s_1 with equal a-priori probabilities.

- When the symbol s_0 is transmitted, the received signal R has a *uniform distribution*:

$$R \left| s_0 \sim U\left(-\frac{A}{2}, \frac{A}{2}\right) \right., \text{ i.e., } f_{R|S}\left(r \middle| S = s_0\right) = \begin{cases} \frac{1}{A}, & r \in \left[-\frac{A}{2}, \frac{A}{2}\right] \\ 0, & r \notin \left[-\frac{A}{2}, \frac{A}{2}\right] \end{cases}$$

- When the symbol s_1 is transmitted, the received signal R has a *Gaussian distribution*:

$$R|s_1 \sim \mathcal{N}(0, \sigma^2), \text{ i.e., } f_{R|S}\left(r \middle| S = s_1\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r^2}{2\sigma^2}}$$

2.1. Write the MAP decision rule and the decisions regions. All possible cases should be listed as a function of A, σ^2.

2.2. for each of the cases you listed above, derive the probability of error of the MAP decision rule.

Question 3 (20%)

Consider the 4-QAM constellation in Figure 2.1:

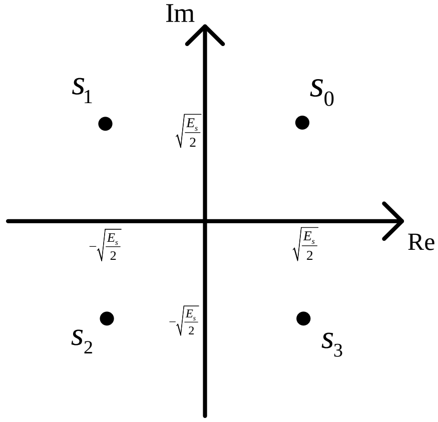

Figure 2.1

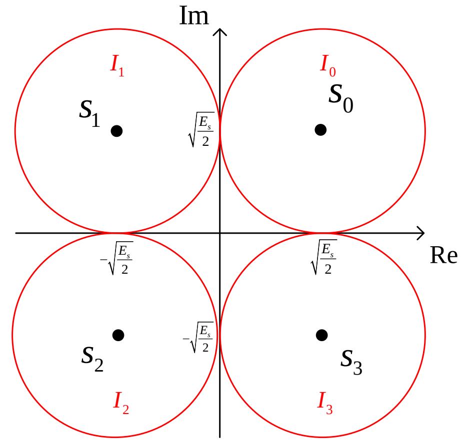

Figure 2.2

The symbols are transmitted with equal a-priori probabilities over an AWGN channel:

$$\underline{r} = \underline{s} + \underline{w} \; ; \; \underline{s} \perp\!\!\!\perp \underline{w} \; ,$$

where $\underline{W}$ is a zero mean Gaussian noise, with a covariance matrix:

$$\Delta = \begin{pmatrix} \frac{N_0}{2} & 0 \\ 0 & \frac{N_0}{2} \end{pmatrix}$$

Instead of using the optimal decision regions, it has been suggested to use the decision regions described in Figure 2.2. The decision regions for symbol s_i is the circle with radius $\sqrt{E_s/2}$ centered around the symbol s_i (denoted by I_i).

Find the probability that the received noisy vector is in the decision region I_i given that the transmitted symbol is s_i, for $i = 0,1,2,3$ (e.g., the probability that $\underline{r}$ is in I_0 given that s_0 is transmitted), as a function of $\sqrt{E_s}, N_0$.