MATH1081 notes

Nira (z5417727)

September 17th, 2022

Contents

1 Topic 1

1.1 Introduction

1. addition, multiplication, division and subtraction
2. Mainly dealing with finite sets

1.2 Sets and subsets

A set is a well defined collection of distinct objects

Example: $S = \{1, a, 3\}, A = \{\Pi, 1\}$.
1. $e \notin A$; it is not in A
2. For example, if A is a set of all integers; $\{$all even integers$\} = \{n \in \mathbb{R}|$n is even$\}$.
3. We can remove superfluos items (elements that occur more than one). $A = \{1, 2, 3, 3\}$ where 3 can be removed.

Example:
$A = \{1, 2, 3\}, B = \{2, 3, 1\}, C = \{1, 2, 3, 3\}, D = \{1, 3\}$.
Here, D is a proper subset of A, B, C; A, B, C are supersets of D.

$\subseteq$: Subset (proper subset),$\supseteq$: Superset.

1. To prove if a set is a proper subset; do the following:
For example, if $D \in A$, then check if $e \in D$
If $e \in D$, then $e \in A$. Thus, it would be a proper subset (here, e is just an element).

2. To prove that two sets are equal;
For example, if A = B, prove:
i) $A \subseteq B$; if an element is in A, then the element is in B.
ii) $B \subseteq A$; if an element is in B, then the element is in A.

1.3 Power Sets and Stability

Subsets of $A = \{1, 2, 3\}$:
1. Could throw everything out to get empty set Φ,
2. One element each: $\{1\}, \{2\}, \{3\}$,
3. Two elements: $\{1, 2\}, \{2, 3\}, \{1, 3\}$,
4. Set itself: A.

The set containing 1, 2, 3, 4 is called the powerset of A.

Given $A = \{1, 2, 3\}, B = \{1, 2, 3, 3\}, C = \{1, 3\}, D = \{1, 3\}$, where $A = B$, $C \subseteq A, B$ and $D \nsubseteq A, B, C$.
1. size of A = 3, B = 3, C = 2, D = 2.

[Exercise with A = 0, 1, 0, 1, B done in word].

1.4 Set Operations

Boolean Operators (”not” operation in programming):
1. Complement:
Let there be a set A in U (A: all of the people in the video, U: universal set of everyone in the world, A^c = complement of A).

$$\boxed{A^c = \{x \in U | x \notin A\}.}$$

2. Intersecting (”and” operation in programming):
If there is A, B, intersecting,

$$\boxed{A \cap B = \{x \in A | x \in B\}.}$$

3. Union (”or” operation in programming): If there is A, B, A or B is:

$$\boxed{A \cup B = \{x \in U | x \in A \text{ or } x \in B\}.}$$

4. Difference: If there is A, B, intersecting,

$$\boxed{A - B = \{x \in A | x \in B\}.}$$

[examples in word doc]

1.5 The Inclusion-Exclusion Principle

[example in Word]

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|.}$$

For three elements,

$$\boxed{|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.}$$

[example in word]

1.6 Sets Proofs

[proof question in word]
Hints for proofs:
1. To prove that $S \subseteq T$, we can assume that $x \in S$ and show that $x \in T$.
2. To prove that $S = T$, we can show that $S \subseteq T$ and $T \subseteq S$.

Scaffold:

Proof: Suppose that ...... (proof)we see that/ it follows ... (conclusion) (end with shaded box to indicat

Note that the "Suppose that" part of the proof is usually whatever the if statment mentions.

For example, if the question is "Prove that if $A \cap B = A$, then $A \cup B = B$, then the proof starts like this:

$\boxed{\underline{Proof}: \text{ Suppose that } A \cap B = A.}$

For questions like "is this statement true", there are two ways to approach the question:
1. If the statement is true (if you think it is true), then prove it.
2. If the statement is false, then give a counter-example that proves it false.
[examples in word]

1.7 Laws of Set Algebra

Laws of Set Algebra

1. $A \cap B = B \cap A$: Commutative Law.
2. $A \cap (B \cap C) = (A \cap B) \cap C$: Associative Law.
3. $A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$: Distributive Law.
4. $A \cap (A \cap B) = A$: Absorption Law.
5. $A \cap U = U \cap A = A$: Identity Law.
6. $A \cap A = A$: Idempotent Law.
7. $(A^c)^c = A$: Double Complement Law.
8. $A \cap \emptyset = \emptyset \cap A = \emptyset$: Domination Law.
9. $A \cap A^c = \emptyset$: Intersection with Complement Law.
10. $(A \cup B)^c = A^c \cap B^c$: De Moirve's Law.

The intersection can be swapped with the union to form another law (like, $A \cup B = B \cup A$ swapped as $A \cap B = B \cap A$.). Similarily, U should be swapped with $\emptyset$ and vice versa.
[examples in word]

1.8 Generalised Set Operations

Unions and Intersections; A saga:

1. $\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cap A_n$,
2. $\cap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$.

Example:

$$A_k = k, k+1;$$
$$= \cup_{i=1}^{3} A_k = A_1(\{1,2\}) \cup A_2(\{2,3\}) \cup A_3(\{3,4\}),$$
$$= \{1,2,3,4\}.$$

[example in word]

1.9 Russel's Paradox

A set may contain another set as one of its elements.
This raises the possibility that a set may contain itself as an element.

Problem: Try to let S be the set of all sets that are not elements of themselves, i.e., $\mathbf{S} = \{\mathbf{A} | \mathbf{A}$ is a set and $\mathbf{A} \notin \mathbf{A}\}$.
Is S an element of itself?
i) If $S \in S$, then the definition of S implies that $S \notin S$, a contradiction.
ii) If $S \notin S$, then the definition of S implies that $S \in S$, also a contradiction.
Hence neither $S \in S$ nor $S \notin S$. This is Russell's paradox.

1.10 Cartesian Product

[example in word]
The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs from A to B:

$$\boxed{A \times B = \{(a, b) | a \in A \text{ and } b \in B\}}$$

If $|A| = m$ and $|B| = n$, then we have $|A \times B| = mn$.

Sets with more than 2 elements:

Example: $A = \{a, b\}, B = \{1, 2, 3\}$.

$$\text{Cartesian Product } (A \times B) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

(all of the ordered pairs – combinations)

[example in word]

When X and Y are small finite sets, we can use an arrow diagram to represent a subset S of $X \times Y$: we list the elements of X and the elements of Y , and then we draw an arrow from x to y for each pair $(x, y) \in S$.

1.11 Functions

Example: Take 2 sets X and Y, for which we have to find a function.

$$X = \{\text{all MATH 1081 students}\}, Y = \{0, 1, \dots, 84, 85, \dots, 100\}.$$

X: number of students; Y: marks from $0 - 100$.
Take function $f : X \to Y$; where X is the domain and Y is the co domain.
Ie, $f(x) =$ X's mark (Y).
Function $f : X \to Y$ satisfies $\{(x, f(x))|x \in X\} \subseteq X \times Y$ so that, for each $x \in X$;
1. $f(x)$ exists
2. $f(x)$ is unique

[example in word]

Note: be vary of the one-to-one function property lol

Floor function and ceiling functions:

1. Floor function (rounds down; smallest integer):

$\boxed{\lfloor x \rfloor = \max \{z \in Z | z \leq x\}.}$

2. Ceiling function (rounds up; largest integer):

$\boxed{\lceil x \rceil = \min \{z \in Z | z \geq x\}.}$

[example in word] Domain/codomain: $\lfloor x \rfloor / \lceil x \rceil : \mathbb{R} \to \mathbb{Z}$.

Range$(\lceil x \rceil) = \mathbb{Z}$.

[example in word]

1.12 Image and Inverse Image

- The image of a set $A \subseteq X$ under a function $f : X \to Y$ is $f(A) = \{y \in Y — y = f(x)$ for some $x \in A\} = \{f(x)|x \in A\}$.

- The inverse image of a set $B \subseteq Y$ under a function $f : X \to Y is f^{-1}(B) = \{x \in X|f(x) \in B\}$.

(image is just function values in the doman and inverse image is function values in range).

$\boxed{\text{note: this is just function and inverse functions.}}$

[example in word]

1.13 Injective, Surjective, Bijective

Formal Definitions:

Recall that if f is a function from X to Y , then for every $x \in X$, there is exactly one $y \in Y$ such that $f(x) = y$.

1. We say that a function $f : X \to Y$ is injective or one-to-one if, for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.
Example: for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

2. We say that a function $f : X \to Y$ is surjective or onto if, for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$. the range of f is the same as the codomain of f (range(f) = Y).

3. We say that a function $f : X \to Y$ is bijective if f is both injective and surjective (one-to-one and onto).

for every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.
[example in word]

1.14 Composition of Functions

For functions $f : X \to Y$ and $g : Y \to Z$, the composite of f and g is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

The composite function $g \circ f$ exists whenever the range of f is a subset of the domain of g.

In general, $g \circ f$ and $f \circ g$ are not the same composite functions. Associativity of composition (assuming they exist): $h \circ (g \circ f) = (h \circ g) \circ f$.

Example: Take sets $X = \{$ **all MATH1081 students** $\}, Y = \{0, 1, \dots, 100\}, Z = \{F, P, CR, D, HD\}$.

Maps: $f = X \to Y; g = Y \to Z$.

A) $g \circ f : X \to Z$.
$(f \circ g)(y) = f(g(y))$.
[examples in word]

1.15 Identity and Inverse Functions

Identity Function:

$\boxed{i_x : x \to x; i_x(x) = x.}$

For any function $f : X \to Y$, we have $f \circ i_x = f = i_y \circ f$. A function $g : Y \to X$ is an inverse of $f : X \to Y$ if $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$,
or equivalently, $g \circ f = i_x$ and $f \circ g = i_y$.

$\boxed{\text{1. A function can have at most one inverse.}}$

If $f : X \to Y$ has an inverse, then we say that f is invertible, and we denote the inverse off by f^{-1}. Thus,$f^{-1} \circ f = i_x$ and $f \circ f^{-1} = i_y$.

If g is the inverse of f, then f is the inverse of g. Thus, $(f^{-1})^{-1} = f$.

[example in word]

1.16 Inverse Function Proofs

Theorem and Proof:

$\boxed{\text{1. A function } f : X \to Y \text{ has at most 1 inverse}}$

Proof:

$$
\begin{aligned}
&\text{Let } g_1, g_2 : Y \to X \text{ be inverse of } f. \\
&\text{Then } : g_1 = g_i \circ i_y \\
&= g_i \circ (f \circ g_2) \\
&= (g_i \circ f) \circ g_2 \\
&= i_x \circ g_2 \\
&= g_2 \text{ End of proof .}
\end{aligned}
$$

[example in word]

2 Number Theory and Relations

2.1 Numbers and Divisibility

[topic 2 done in word (SteelsSlides1): maybe put in definitions here ?? that depends]

Number Set Notation:

1. The positive integers: $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$,
2. The natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$,
3. The integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
4. The rational numbers: $\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{Z}^+\}$,
5. The real numbers, $\mathbb{R}$ and the complex numbers $\mathbb{C}$.

Tests (Divisibility):

1. 2 — N if and only if the decimal expansion of N ends in an even integer
2. 5 — N if and only if the last decimal digit of N is 5 or 0.
3. 3 — N if and only if the sum of the decimal digits of N is divisible by 3.
3': 9 — N if and only if the sum of the decimal digits of N is divisible by 9.
4. 11 — Nif the alternating sum of the decimal digits of N is divisible by 11.
(example: $1232 = 1 - 2 + 3 - 2 = 0$)

[proof in word]

2.2 Primes

[in word]

<u>Primes Definition: Formal:</u> Another way of saying this is if p is prime:

$$x — p \text{ implies } x \in \{-1, 1, -p, p\}$$

.

<u>Theorems:</u>

1. If p is prime and $p|ab$, then $p|a$ or $p|b$,
2. If n is composite, then it has a prime factor less than or equal to $\sqrt[2]{n}$,
3. If no prime less than or equal to $\sqrt[2]{n}$ divides n then n is a prime,
4. Every integer $n \geq 2$ can be written uniquely as a product of a finite number of primes in increasing order i.e. $n = p_1^{m_1} * p_2^{m_2} \ldots p_k^{m_k}$ for primes $p_1 < p_2 < \cdots < p_k$ and exponents $m_1, m_2, \ldots, m_k \in \mathbb{Z}^+$.

Open Results about Primes:

1. A prime of the form $2^n + 1$ is called a Fermat prime.
2. A prime of the form $2^n - 1$ is called a Mersenne prime.
3. Two primes that differ by 2, are called twin primes. For example, 3 and 5 are twin primes; so are 29 and 31.
4. The Goldbach Conjecture is that they are: it has been proved true for all numbers with fewer than about 17 digits.

2.3 Common Divisors and Multiples

[mostly on word]
All $a, b \in \mathbb{Z}$ have (at least) one common divisor, namely 1, and so we can define the following:
For $a, b \in \mathbb{Z}$, not both zero, the positive integer d such that

$$1.d — aandd — b,$$

$$2.If c — aandc — bthenc \leq d.$$

is called the greatest common divisor of a and b. We write $d = gcd(a, b)$.

$\boxed{\text{Begin by writing } a \text{ and } b \text{ as a product of primes.}}$

Properties of GCD:

1. gcd (a, b) is not affected by the signs of a or b
2. Condition (2) in the definition of gcd can be replaced by $(2')$ if $c — a$ and $c — b$ then $c — d$.
3. For $a \in \mathbb{Z}^+$, gcd $(a, 0) = a$.

Least Common Multiple

All $a, b \in \mathbb{Z}$ have (at least) one common multiple, namely ab, and so we can define the following: For $a, b \in \mathbb{Z}$, not both zero, the positive integer l such that

$$1)a — l \text{ and } b — l$$

2)If $a — c$ and $b — c$ then $l \leq c$ is called the least common multiple of a and b.

We write $l =$ lcm (a, b).

Theorem:

$\boxed{\text{For all positive integers } a \text{ and } b\text{; gcd}(a, b)x \text{ lcm } (a, b) = ab.}$

Quotient and Remainder

[mostly in word]

The Quotient-Remainder Theorem (aka The Division Algorithm)

If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, then there exist unique $q, r \in Z$ such that (q: quotient; r: remainder):

$$a = bq + r \text{ and } 0 \leq r < b.$$

Note: q can be found using floor function; $q = \lfloor a/b \rfloor$.; then $r = a - qb$.

2.4 The Euclidean Algorithm

[mostly in word]

$\boxed{\text{If } a = bq + r \text{ then } \gcd(a, b) = \gcd(b, r).}$

The Euclidian Algorithm: General Case [steps]

1) Let a and b be integers with $a > b \geq 0$.
2) If $b = 0$, then $\gcd(a, b) = a$.
3) If $b > 0$, use the Quotient-Remainder theorem to write $a = bq + r$ where $0 \leq r < b$. Then by our prevous result, $\gcd(a, b) = \gcd(b, r)$.
4) Repeat steps 2 and 3 to find $\gcd(b, r)$.

Example: Find $\gcd(708, 540)$

$$708 = 540 \cdot 1 + 168,$$

$$540 = 168 \cdot 3 + 36,$$

$$168 = 36 \cdot 4 + 24,$$

$$36 = 24 \cdot 1 + 12,$$

$$24 = 12 \cdot 2 + 0.$$

So,

$$\gcd\ (708, 540) = 12.$$

$\boxed{\text{Note: gcd is the last non-zero remainder.}}$

Bezout's Identity

For $a, b \in Z$ not both zero, there exist integers x and y (not unique) such that:

$\boxed{\gcd\ (a, b) = ax + by.}$

$\boxed{\text{Theorem: Integers } a \text{ and } b \text{ are relatively prime if and only if there exists } x, y \in Z \text{ such that } ax + by = 1.}$

Extended Euclidean Theorem: The Extended Euclidean Algorithm is a more efficient way of finding the numbers in Bézout's Identity: In looking for $\gcd(a, b)$, assume $a > b > 0$.
1. We make up a table with five columns labelled i, q_i, r_i, x_i, y_i, where i labels the rows.
2. We set row 1 to be $1, 0, a, 1, 0$ and row 2 to be $2, 0, b, 0, 1$. Thus $q_1 = q_2 = 0; r_1 = a, r_2 = b; x_1 = y_2 = 1; x_2 = y_1 = 0$.
3. Then for i from 3 onwards, q_i is the quotient on dividing r_{i-2} by r_{i-1} (a divided by b in the first case).

4. Then subtract q_i times the rest of row $i-1$ from row $i-2$.
5. Repeat until we get $r_{n+1}=0$ for some n, then stop. Then the gcd is r_n and $r_n = ax_n + by_n$, that is the last row before r_i was zero gives the gcd, the x and the y.
In fact a similar identity holds at each step: $r_i = ax_i + by_i$.

2.5 Modular Arithmetic

[mostly in word]

Let $m \geq 2$ be an integer. We say that a and b are congruent modulo m if $m|(a-b)$.
We write this as:

$$a \cong b(mod m).$$

The reason we have taken our modulus m to be greater than 2 is that
1) As $m|(a-b)$ iff $-m|(a-b)$, there is nothing to be gained from using negative moduli.
2) All numbers are congruent modulo 1, so that is not interesting.
3) divisibility by 0 is not defined.

Theorem
For integers $a, b$and $m, a \cong b(mod m)$ if and only if there is an integer k such that $a = b + km$.

Arithmetic with Congrences
Suppose $a \cong b(mod m)$ and $c \cong d(mod m)$.
Then

$$(1a)(a + c) \cong (b + d)(mod m).$$

$$(1b)(a - c) \cong (b - d)(mod m).$$

$$(2)ac \cong bd(mod m).$$

$$(3)an \cong bn(mod m) for all n \in N.$$

$$(4)If k — m then a \cong b(mod k).$$

$\boxed{note : never divide congruences}$

Applications of Congruence Arithmetic:
1. Pseudo-random Numbers
2. Equations with no solutions

3 Logic and Proofs

3.1 Introduction to Logic and Proofs

Mathematical proof: consists of logical deduction on the basis of agreed premises. Apart from human error the results are certain.

Techniques for Proof:

- Always explain what you are doing, and your reasons for drawing your conclusions.
- Simplify!
- Keep the aim in mind.
- Plan a solution.
- Work on one side of an equation or inequation to relate it to the other.

To study proofs: you always have to practice!! No matter what. There are techniques, but most of the time you will have to practice proofs.