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Chapter 2 exercises

Exercise 2.1

Find an example of a set X & a monotone class $\mathcal{M}$ consisting of subsets of X s.t $\emptyset \in \mathcal{M}$, $X \in \mathcal{M}$, but $\mathcal{M}$ is not a σ-algebra

Consider $X = \{0, 1\}$ &
$\mathcal{M} = \{\emptyset, \{0\}, \{0,1\}\}$

First, I show $\mathcal{M}$ is a monotone class. First suppose $A_i \uparrow A$ where $\forall i \in \mathbb{N}$, $A_i \in \mathcal{M}$. Case 1: $A_i = \emptyset$ $\forall i \in \mathbb{N}$
Then $\bigcup_{i=0}^{\infty} A_i = \emptyset$. Since $\bigcup_{i=0}^{\infty} A_i = A$, we have $A = \emptyset \in \mathcal{M}$, as wanted.

Case 2: $A_k = \{0\}$ for some $k \in \mathbb{N}$, but $\forall i \in \mathbb{N}$, $A_i \neq \{0, 1\}$

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Ex 2.1 Since $\forall A \in \mathcal{M}$, $A \neq \{0,1\}$, $A \subseteq \{0\}$, we then have that $\bigcup_{i=0}^{\infty} A_i = \{0\}$. Since $\bigcup_{i=0}^{\infty} A_i = A$, that means $A = \{0\} \in \mathcal{M}$, as wanted.

Case 3: $A_i = \{0,1\}$ for some $i \in \mathbb{N}$

Then since $\forall A \in \mathcal{M}$, $A \subseteq \{0,1\}$, we have $\bigcup_{i=0}^{\infty} A_i = \{0,1\}$. Since $A = \bigcup_{i=0}^{\infty} A_i$

$\Rightarrow A = \{0,1\} \in \mathcal{M}$, as wanted.

So in all possible cases, $A_i \uparrow A$

$\Rightarrow A \in \mathcal{M}$, as wanted.

Now suppose $A_i \downarrow A$ & $\forall i \in \mathbb{N}$, $A_i \in \mathcal{M}$

Case 1: $A_i = \{0,1\}$, $\forall i \in \mathbb{N}$

Then $A = \bigcap_{i=0}^{\infty} A_i = \{0,1\} \in \mathcal{M}$,

i.e. $A \in \mathcal{M}$, as wanted.

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Chapter 2 exercises

Exercise 2.1

Case 2: $A_k = \{0\}$ for some $k \in \mathbb{N}$, & $\forall i \in \mathbb{N}$, $A_i \neq \emptyset$

Then since $\{0\} \subseteq A'$, $\forall A' \in \mathcal{M}$, $A' \neq \emptyset$, we have $A = \bigcap_{i=0}^{\infty} A_i = \{0\} \in \mathcal{M}$. ie $A \in \mathcal{M}$, as wanted.

Case 3: $A_k = \emptyset$ for some $k \in \mathbb{N}$.

Then since $\emptyset \subseteq A'$ $\forall A' \in \mathcal{M}$, we have $A = \bigcap_{i=0}^{\infty} A_i = \emptyset \in \mathcal{M}$, ie $A \in \mathcal{M}$.

So $A_i \downarrow A$ where $A_i \in \mathcal{M}$ $\forall i \in \mathbb{N}$ $\Rightarrow$ $A \in \mathcal{M}$. This concludes the verification that $\mathcal{M}$ is a monotone class. However, note $\mathcal{M}$ is not a σ-algebra, as $\{0\} \in \mathcal{M}$, but $X - \{0\} = \{1\} \notin \mathcal{M}$.

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chapter 2 exercises

Exercise 2.2

Find an example of a set X and two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, each consisting of subsets of X, s.t $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra.

By Example 2.5, if we let $X = \{1,2,3\}$ & let $\mathcal{A} = \{X, \emptyset, \{1\}, \{2,3\}\}$, then $\mathcal{A}$ is a σ-algebra on X. By symmetry, $\mathcal{A}' = \{X, \emptyset, \{3\}, \{1,2\}\}$ is also a σ-algebra on X. However, note $\mathcal{A} \cup \mathcal{A}' = \{X, \emptyset, \{1\}, \{3\}, \{1,2\}, \{2,3\}\}$ is not a σ-Algebra, as $\{1\} \in \mathcal{A} \cup \mathcal{A}'$, $\{3\} \in \mathcal{A}'$ but $\{1\} \cup \{3\} = \{1,3\} \notin \mathcal{A} \cup \mathcal{A}'$.

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Exercise 2.3

Suppose $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$ are σ-Algebras consisting of subsets of a set X.
Is $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ neccesarily a σ-Algebra?

If not, give a counterexample.

This is not true. I will give a counterexample. Consider $X = \mathbb{R}^{\mathbb{N}}$, the set of all sequences of Real Numbers.

Let

$$\mathcal{A}_n = \left\{ A \subset X \;\middle|\; \begin{array}{l} \text{All sequences in } A \text{ or } A^c \\ \text{have } \geq 1 \text{ and } \leq n \text{ of their} \\ \text{components in } \mathbb{N} \end{array} \right\}$$

($\mathcal{A}_n \subseteq \mathbb{R}^{\mathbb{N}}$ clearly)

First I will show each $\mathcal{A}_n$ is a σ-algebra.

Firstly, note that $X, \emptyset \in \mathcal{A}_n$ as $\emptyset$ vacously satisfies the condition to be in $\mathcal{A}_n$.

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chapter 2 exercises

Exercise 2.3

& $X^c = \emptyset$, & so X^c vacously satisfies the condition & so is in $\mathcal{A}_n$.

Next, note that if $A \in \mathcal{A}_n$, then by defn of $\mathcal{A}_n$, so is A^c, as
$(A^c)^c = A$.

Now I show closure under finite union/intersection. Consider $A_1, \dots, A_n \subset \mathcal{A}_n$

case 1: All sequences in each of $A_1, \dots, A_n$ have ≥ 1 & $\leq n$ of their components in $\mathbb{N}$

Then the same holds for $\bigcup_{i=1}^{n} A_i$,
& so $\bigcup_{i=1}^{n} A_i \in \mathcal{A}_n$.

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Chapter 2 exercises

Exercise 2-3

Case 2 There is a set in $A_1, \ldots, A_n$, call it A_{i_0} which contains a seqence with either <1 or $>n$ of its components in $\mathbb{N}$. Then since $A_{i_0} \in \mathcal{A}_n$, all seqvences in $(A_{i_0})^c$ have ≥ 1 & $\leq n$ of their components in $\mathbb{N}$.

Then

by De Morgan's law

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c \subseteq (A_{i_0})^c$$

Meaning all seqvences in $\left(\bigcup_{i=1}^{n} A_i\right)^c$ satisfy the condition, & so $\bigcup_{i=1}^{n} A_i \in \mathcal{A}_n$.

Closure under finite intersection follows from closure under compliments, closure under finite unions, & De Morgan's law.

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Chapter 2 exercises

Exercise 2.3

closure under countable unions/intersections follow from what is basically an analogous argument to that for closure under finite unions/intersections.

Thus, $\forall n \in \mathbb{N}$, $\mathcal{A}_n$ is a σ-algebra.

Clearly of their definitions,

$$\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots\cdots$$

Now consider $\bigcup_{i=1}^{\infty} \mathcal{A}_i$. <u>Note</u> that

$(\{a_n \in \mathbb{R}^{\mathbb{N}} \mid a_n \text{ has exactly } i \text{ components in } \mathbb{N}\})^c$

is in $\mathcal{A}_i$, for every i, & so each of them is thus in $\bigcup_{i=1}^{\infty} \mathcal{A}_i$.

Now consider

$$C = \bigcap_{i=1}^{\infty} \{a_n \in \mathbb{R}^{\mathbb{N}} \mid a_n \text{ has exactly } i \text{ components in } \mathbb{N}\}^c$$

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chapter 2 exercises

Exercise 2.3

Note $C = (\mathbb{R}-\mathbb{N})^{\mathbb{N}} \cup (\mathbb{N})^{\mathbb{N}}$
ie the set of all sequences with all entries in $\mathbb{R}-\mathbb{N}$ or all entries in $\mathbb{N}$.

This is not in A_n for any n, as all the sequences in it either have none of their entries in $\mathbb{N}$, or all of them will be in $\mathbb{N}$, &

~~Also note~~ $C^c = \left((\mathbb{R}-\mathbb{N})^{\mathbb{N}} \cup (\mathbb{N})^{\mathbb{N}}\right)^c$

$= \left((\mathbb{R}-\mathbb{N})^{\mathbb{N}}\right)^c \cap \left((\mathbb{N})^{\mathbb{N}}\right)^c$ By De Morgan's law

$= \{a_n \in \mathbb{R}^{\mathbb{N}} \mid a_n$ has at least one but finitely many components in $\mathbb{N}\}$

which is not in A_n for any n as the number of $\mathbb{N}$ entries in the sequences of this set are not bounded,

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Chapter 2 exercises

Ex 2.3

and there is at least one sequences with exactly i $\mathbb{N}$ entries in $\mathbb{R}^{\mathbb{N}}$, for each i.

So it is not in $\bigcup_{i=1}^{\infty} A_i$ & so $\bigcup_{i=1}^{\infty} A_i$ is not closed under ctble intersection & so is not a σ-algebra.

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chapter 2 exercises

Ex 2.4

Suppose $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ are monotone classes. Let $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$. Suppose $A_j \uparrow A$ & each $A_j \in \mathcal{M}$. Is $A \in \mathcal{M}$? If not give a counterexample.

No. consider $X = \mathbb{N}$, & $\mathcal{M}_n = \mathcal{P}(\{1, \dots, n\})$

Clearly, $\mathcal{M}_n$ is a monotone class, as if $A_i \uparrow A$ & $A_i \in \mathcal{M}_n$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{M}_n = \mathcal{P}(\{1, \dots, n\})$ as a union of

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Chapter 2 exercises

Ex 2.4

subsets of a given set is still a subset of that given set. $A_i \downarrow A$ & $A_i \in \mathcal{M}_n$ $\Rightarrow A \in \mathcal{M}_n$ follows similarly. Thus $\mathcal{M}_n$ is a monotone class.

Now consider $A_j = \{1, 2, \dots, j\}$.

Note that $A_j \in \mathcal{M}_j \subset \bigcup_{n=1}^{\infty} \mathcal{M}_n = \mathcal{M}$,

& so $A_j \in \mathcal{M}$ $\forall j \in \mathbb{N}$. Note $A_1 \subseteq A_2 \subseteq \cdots$ (by their definitions)

But note $\bigcup_{j=1}^{\infty} A_j = \mathbb{N} - \{0\}$

& $\mathbb{N} - \{0\} \notin \mathcal{M}_n$ for any n

(as $\mathbb{N} - \{0\} \not\subseteq \{1, \dots, n\}$ for any n)

so $A_j \uparrow \bigcup_{j=1}^{\infty} A_j$ but $\bigcup_{j=1}^{\infty} A_j \notin \mathcal{M}$, a counterexample.

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Chapter 2 exercises

Ex 2.5

Given: $(Y, \mathcal{A})$ is a measurable space

$f: X \to Y$ is a map

$\mathcal{B} = \{f^{-1}(A) : A \in \mathcal{A}\}$

WTP: $\mathcal{B}$ is a σ-algebra of subsets of X

proof: First note $\emptyset, X \in \mathcal{B}$

as $\emptyset \in \mathcal{A}$ (since $\mathcal{A}$ a σ-algebra)

& $f^{-1}(\emptyset) = \emptyset$ (by def of preimage)

& so $\emptyset = f^{-1}(\emptyset) \in \mathcal{B}$ by def of

$\mathcal{B}$. Also, $Y \in \mathcal{A}$ as $\mathcal{A}$ a σ-algebra

on Y. $\Rightarrow f^{-1}(Y) \in \mathcal{B}$

$\Rightarrow X \in \mathcal{B}$ (as $f^{-1}(Y) = X$, by def of preimage)

as wanted.

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Ex 2.9 Now let $f^{-1}(A) \in \mathcal{B}$. WTS $X - f^{-1}(A) \in \mathcal{B}$.

since $x = f^{-1}(Y)$

Note $X - f^{-1}(A) = f^{-1}(Y) - f^{-1}(A)$

By preimage properties $= f^{-1}(Y \setminus A)$

since, by def of $\mathcal{B}$, $A \in \mathcal{A}$ & so $Y - A \in \mathcal{A}$ (as $\mathcal{A}$ a σ-algebra)

So $f^{-1}(Y \setminus A) \in \mathcal{B}$ by def of $\mathcal{B}$

$\Rightarrow X - f^{-1}(A) \in \mathcal{B}$, as wanted.

Now let $\{f^{-1}(A_i)\}_{i=1}^{n}$ be a finite collection of sets in $\mathcal{B}$.

WTS $\bigcup_{i=1}^{n} f^{-1}(A_i) \in \mathcal{B}$.

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chapter 2 exercises

Ex 2.5

well $\bigcup_{i=1}^{n} f^{-1}(A_i) = f^{-1}\left(\bigcup_{i=1}^{n} A_i\right)$ by preimage prop.

Since by defn of $\mathcal{B}$, $A_i \in \mathcal{A}$ $\forall 1 \leq i \leq n$, & $\mathcal{A}$ a σ-algebra, $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$.

$\Rightarrow$ $f^{-1}\left(\bigcup_{i=1}^{n} A_i\right) \in \mathcal{B}$ by defn of $\mathcal{B}$.

ie $\bigcup_{i=1}^{n} f^{-1}(A_i) \in \mathcal{B}$, as wanted.

closure under ctble union follows from an analogous argument.
Closure under finite / ctble intersection follows from closure under ctble union & finite union + closure under compliments & De Morgan's Law. This completes the proof that $\mathcal{B}$ is a σ-algebra of subsets of X. QED $\blacksquare$

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Chapter 2 exercises

Ex2.6

Given: X is non-empty. $\mathcal{A}$ is a σ-algebra w/ the property that whenever $A \in \mathcal{A}$ is non-empty, $\exists\, B, C \in \mathcal{A}$ w/ $B \cap C = \emptyset$, $B \cup C = A$ & neither B nor C is empty

WTP: $\mathcal{A}$ is uncountable

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Chapter 2 Exercises

Ex 2.6

<u>proof</u>: I define the following sequence of sets $\{A_i\}_{i=1}^{\infty}$ in $\mathcal{A}$: (recursively)

Since X is in $\mathcal{A}$ & $X \neq \emptyset$, $\exists\, B_1, C_1 \in \mathcal{A}$ s.t. $B_1 \cap C_1 = \emptyset$, $B_1 \cup C_1 = A$ & $B_1 \neq \emptyset$, $C_1 \neq \emptyset$.

Define $A_1 = B_1$.

Now, given A_n is defined, define A_{n+1} as follows:

<u>Recall</u> in the def<u>n</u> of A_n, we had some B_n & some $C_n \in \mathcal{A}$ s.t.

$B_n \cup C_n =$ $M \in \mathcal{M}$ (for some $M \in \mathcal{M}$) &

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Chapter 2 exercises

Ex 2.6

$B_n \cap C_n = \emptyset$ & $B_n, C_n \neq \emptyset$, & we defined $A_n = B_n$. Since $C_n \neq \emptyset$, $C_n \in \mathcal{A}$, then by property of $\mathcal{A}$, $\exists\ B_{n+1}, C_{n+1} \in \mathcal{A}$ s.t. $B_{n+1} \cap C_{n+1} = \emptyset$, $B_{n+1} \cup C_{n+1} = C_n$ & $B_{n+1}, C_{n+1} \neq \emptyset$. Define A_{n+1} as $A_{n+1} = B_{n+1}$.

<u>Note</u> by construction that $\{A_n\}_{n=1}^{\infty}$ is a non-empty collection of sets in $\mathcal{A}$. I will prove they are pairwise disjoint. Let $A_i, A_j \in \{A_n\}_{n=1}^{\infty}$ & wlog assume $i < j$.

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chapter 2 exercises

Ex 2.6

Note $\forall k \in \mathbb{N} - \{0\}$, $C_{k+1} \subseteq C_k$, by construction, ie $C_k \downarrow$ is a decreasing sequence wrt "$\subseteq$". Note also by construction that $A_j \subseteq C_{j-1}$ (Note since $j > i \geq 1$, we have $j > 1$ so C_{j-1} exists) If $j-1 = i$, then we are done, as $A_i \cap C_i = \emptyset$ by construction, & so $A_j \subseteq C_{j-1} = C_i \Rightarrow A_j \cap A_i = \emptyset$. If $j-1 \neq i$, then we must have $j-1 > i$. Since $A_j \subseteq C_{j-1}$ & $C_k \downarrow$, then we must have $A_j \subseteq C_{j-1} \subseteq C_i$ (as one can easily show by the finite induction principle) & so $A_j \subseteq C_i$. Since $A_i \cap C_i = \emptyset$ by construction, this implies $A_i \cap A_j = \emptyset$.

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Chapter 2 solns

Ex 2.6

as wanted. Thus $\{A_i\}_{i=1}^{\infty}$ is a disjoint collection of non-empty sets in $\mathcal{A}$.

Now I define the following function from $(0,1) \cap (\mathbb{R}-\mathbb{Q})$ to $\mathcal{A}$ as follows:

$$F: (0,1) \cap (\mathbb{R}-\mathbb{Q}) \to \mathcal{A}$$

$\forall x \in (0,1) \cap (\mathbb{R}-\mathbb{Q})$, recall from elementary real analysis, every $x \in (0,1) \cap (\mathbb{R}-\mathbb{Q})$ has a unique binary representation $\{a_n\}_{n=1}^{\infty}$ where $a_n : \mathbb{N}-\{0\} \to \mathbb{R}$ is a sequence st $a_n = 0$ or $a_n = 1$ $\forall n \in \mathbb{N}-\{0\}$.

Then define, $\forall x \in (0,1) \cap (\mathbb{R}-\mathbb{Q})$,

$$F(x) = \bigcup_{\{i : a_i = 1\}} A_i \quad \text{(where } a_i \text{ is } x\text{'s unique binary rep.)}$$

Note $\bigcup_{\{i : a_i = 1\}} A_i \in \mathcal{A}$ as $\mathcal{A}$ a σ-algebra & so

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Chapter 2 exercises

Ex 2.6

closed under ctble union. So indeed, F a fcn into $\mathcal{A}$. Now I show F is injective. Let $x, y \in (0,1) \cap (\mathbb{R} - \mathbb{Q})$, & suppose $x \neq y$. Let a_i & b_i be the unique binary representations of x & y resp. Then $\exists i \in \mathbb{N} - \{0\}$ s+ $a_i \neq b_i$. wlog suppose $a_i = 0$ & $b_i = 1$. Now consider $F(x) = \bigcup_{\{i : a_i = 1\}} A_i$ & $F(y) = \bigcup_{\{i : b_i = 1\}} A_i$. Note that

$A_i \subseteq F(y)$, by def of "$\bigcup$". Since $A_i \neq \emptyset$ (as recall $\{A_i\}_{i=1}^{\infty}$ is a non-empty collection of sets)

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Chapter 2 Solutions

Ex2.6

$\exists\, m \in A_i$. Note $m \in F(y)$ by def of union. Now suppose for a contradiction $m \in F(x)$. Then $m \in A_j$ for some j s.t. $a_j = 1$. Since $a_i = 0$, $A_j \neq A_i$. So $m \in A_j \cap A_i = \emptyset$ (as $\{A_i\}_{i=1}^{\infty}$ is a pairwise disjoint collection) a contradiction. $\times$ Thus our assumption was false & $m \notin F(x)$. So $\exists\, m \in F(y)$ s.t. $m \notin F(x)$, so $F(y) \neq F(x)$. Thus $x, y \in (0,1) \cap (\mathbb{R} - \mathbb{Q})$ & $x \neq y$

$\Rightarrow F(x) \neq F(y) \longrightarrow F$ is injective.

So thus, $|\mathbb{N}| < |\mathbb{R}| = |(0,1) \cap (\mathbb{R} - \mathbb{Q})| \leq |A|$

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Chapter 2 solutions

Ex 2.6

ie $|\mathcal{A}| > |\mathbb{N}|$, so $\mathcal{A}$ is uncountable.

QED ▨

Ex 2.7

Given: $\mathcal{F}$ is a collection of real-valued functions on X s.t. constant functions are in $\mathcal{F}$ & $f+g$, fg, & cf are in $\mathcal{F}$ whenever $f, g \in \mathcal{F}$ & $c \in \mathbb{R}$. $f \in \mathcal{F}$ whenever $f_n \to f$ ptwise & each $f_n \in \mathcal{F}$. For a set $A \subseteq X$ $\chi_A : X \to \mathbb{R}$ is defined as $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

WTP: $\mathcal{A} = \{A \subset X : \chi_A \in \mathcal{F}\}$ is a σ-algebra

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Chapter 2 solns

EX 2.7

<u>Proof</u>: First, note $X, \emptyset \in \mathcal{A}$.

As $\chi_\emptyset : X \to \mathbb{R}$ is defined as

$$\chi_\emptyset(x) = \begin{cases} 1 & \text{if } x \in \emptyset \\ 0 & \text{if } x \notin \emptyset \end{cases} \quad (\forall x \in X)$$

$$= 0 \quad (\text{as no } x \in X \text{ satisfies } x \in \emptyset)$$

ie $\chi_\emptyset(x)$ is the constant 0 fcn on X,

& so $\chi_\emptyset \in \mathcal{F}$ & so $\emptyset \in \mathcal{A}$.

Likewise, $\chi_X : X \to \mathbb{R}$ is defined as

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases} = 1 \quad (\forall x \in X)$$

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chapter 2 solns

Ex 2.7

So χ_X is the constant 1 fcn on X, & so $\chi_X \in \mathcal{F}$ & so $X \in \mathcal{A}$.

Now suppose $A \in \mathcal{A}$. WTS: $X - A \in \mathcal{A}$. Well by def of $\mathcal{A}$, $A \in \mathcal{A} \Rightarrow \chi_A \in \mathcal{F}$.

Note also the constant -1 fcn on X is in $\mathcal{F}$ (by property of $\mathcal{F}$) & so $\chi_A + (-1) \in \mathcal{F}$. Also, by property of $\mathcal{F}$, $(-1)\cdot(\chi_A + (-1)) \in \mathcal{F}$. But note:

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<u>Ex 2.7</u> $\forall x \in X$,

$$(-1)(\chi_A + (-1))(x)$$

$$= (-1)(\chi_A(x) + (-1))$$

$$= \begin{cases} (-1)\cdot(0) & \text{if } x \in A \\ (-1)\cdot(-1) & \text{if } x \notin A \end{cases}$$ By defn of χ_A

$$= \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in X - A \\ 0 & \text{if } x \notin X - A \end{cases}$$ as $x \in X$ & by defn of $X-A$

$$= \chi_{X-A}(x)$$

So thus, $\chi_{X-A} = (-1)(\chi_A + (-1)) \in \mathcal{F}$

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Ex 2.7

& so $X - A \in \mathcal{A}$ by def of $\mathcal{A}$, as wanted.

Now suppose $A_i \in \mathcal{A}$, for $1 \le i \le n$.

WTS $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$.

I will show this by induction.

Base case: $n=1$

Then since $A_1 \in \mathcal{A}$, & $\bigcup_{i=1}^{1} A_i = A_1$,

$\bigcup_{i=1}^{1} A_i = \mathcal{A}$, as wanted.

Inductive step: suppose per n,

$\bigcup_{i=1}^{n} A_i \in \mathcal{A}$

WTS for $n+1$, & $\{A_i\}_{i=1}^{n+1}$ a collection of sets in $\mathcal{A}$, $\bigcup_{i=1}^{n+1} A_i \in \mathcal{A}$.

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Chap 2 solns

Ex 2.7

Note $\bigcup_{i=1}^{n+1} A_i = \bigcup_{i=1}^{n} A_i \cup A_{n+1}$

Denote $\bigcup_{i=1}^{n} A_i = C$. By IH, $C \in \mathcal{A}$.

By hypothesis, $A_{n+1} \in \mathcal{A}$. By defn of $\mathcal{A}$, that means χ_C & $\chi_{A_{n+1}} \in \mathcal{F}$.

By property of $\mathcal{F}$, $\chi_C \cdot \chi_{A_{n+1}} \in \mathcal{F}$.

But note $\chi_{C \cap A_{n+1}} = \chi_C \cdot \chi_{A_{n+1}}$, as $\forall x \in X$

$$\chi_C \cdot \chi_{A_{n+1}}(x) = \begin{cases} 1 \cdot 1 & \text{if } x \in C \text{ \& } x \in A_{n+1} \\ 1 \cdot 0 & x \in C \text{ \& } x \notin A_{n+1} \\ 0 \cdot 1 & x \notin C \text{ \& } x \in A_{n+1} \\ 0 \cdot 0 & x \notin C \text{ \& } x \notin A_{n+1} \end{cases}$$

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$$= \begin{cases} 1 & \text{if } x \in C \cap A_{n+1} \\ 0 & \text{o/w} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in C \cap A_{n+1} \\ 0 & \text{if } x \notin C \cap A_{n+1} \end{cases}$$

$= \chi_{C \cap A_{n+1}}(x)$ as wanted.

so $\chi_{C \cap A_{n+1}} = \chi_C \cdot \chi_{A_{n+1}} \in \mathcal{F}$.

Since $\mathcal{F}$ is also closed under addition of fcns, $\Rightarrow$ $\chi_C + \chi_{A_{n+1}} - \chi_{C \cap A_{n+1}}$ $\in \mathcal{F}$ (as $(-1)\chi_{C \cap A_{n+1}} = -\chi_{C \cap A_{n+1}} \in \mathcal{F}$)

But note $\chi_{C \cup A_{n+1}} = \chi_C + \chi_{A_{n+1}} - \chi_{C \cap A_{n+1}}$

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chapter 2 exercises

Ex 2.7 as $\forall x \in X$,

~~as~~ $\chi_C + \chi_{A_{n+1}} - \chi_{C \cap A_{n+1}}(x)$

$$= \begin{cases} 1+1-1 & \text{if } x \in C \text{ \& } x \in A_{n+1} \text{ \& } x \in C \cap A_{n+1} \\ 1 & \text{if } x \in C \text{ \& } x \notin A_{n+1} \\ 1 & \text{if } x \in A_{n+1} \text{ \& } x \notin C \\ 0 & \text{if } x \notin A_{n+1}, x \notin C \end{cases}$$

$$\begin{pmatrix}\text{By defn} \\ \text{of } C \cup A_{n+1}\end{pmatrix} = \begin{cases} 1 & \text{if } x \in C \cup A_{n+1} \\ 0 & \text{if } x \notin C \cup A_{n+1} \end{cases}$$

$= \chi_{C \cup A_{n+1}}$ as wanted.

so $\chi_{C \cup A_{n+1}} \in \mathcal{F}$, & so $C \cup A_{n+1}$

$= \bigcup_{i=1}^{n+1} A_i \in \mathcal{A}$, as I wanted to show. This

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Chapter 2 exercises

Ex 2.7

completes the proof ~~..~~ by induction & so shows $\mathcal{A}$ is closed under finite union.

Now let $\{A_i\}_{i=1}^{\infty}$ be an infinite collection of sets in $\mathcal{A}$. WTS $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. By the previously result, $\forall n \in \mathbb{N}$, $\chi_{\bigcup_{i=1}^{n} A_i} \in \mathcal{F}$ (as $\mathcal{A}$ closed under finite union).

Claim: $\chi_{\bigcup_{i=1}^{n} A_i} \to \chi_{\bigcup_{i=1}^{\infty} A_i}$ Ptwise

Let $x \in X$. Let $\varepsilon > 0$ be arbitrary.

Case 1 $x \in \bigcup_{i=1}^{\infty} A_i$. Then $\exists$ $k \in \mathbb{N} - \{0\}$ s.t $x \in A_k$ (defn of "$\cup$")

RAGS

Chapter 2 exercises

Ex 2.7

Then consider $K \in \mathbb{N}$ & suppose $n \geq K$.

Then $\left|\chi_{\bigcup_{i=1}^{n} A_i}(x) - \chi_{\bigcup_{i=1}^{\infty} A_i}(x)\right|$

$= |1 - 1|$ (as $x \in A_K \subseteq \bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{\infty} A_i$)

$= |0| < \varepsilon$

as wanted.

Case 2: $x \notin \bigcup_{i=1}^{\infty} A_i$

Then $x \notin A_i \quad \forall i \in \mathbb{N} - \{0\}$ Then consider $1 \in \mathbb{N}$ & suppose $n \geq 1$

Then $\left|\chi_{\bigcup_{i=1}^{n} A_i}(x) - \chi_{\bigcup_{i=1}^{\infty} A_i}(x)\right|$

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Chapter 2 Solutions

Ex 2.7

$= |0-0|$ (as $x \notin \bigcup_{i=1}^{n} A_i$ & $x \notin \bigcup_{i=1}^{\infty} A_i$ as $x \notin A_i$ $\forall i \in \mathbb{N} - \{0\}$)

$= 0 < \varepsilon$ as wanted.

This shows $\chi_{\bigcup_{i=1}^{n} A_i} \to \chi_{\bigcup_{i=1}^{\infty} A_i}$ ptwise.

Since $\chi_{\bigcup_{i=1}^{n} A_i} \in \mathcal{F}$ $\forall n \in \mathbb{N}$, by property of $\mathcal{F}$, $\chi_{\bigcup_{i=1}^{\infty} A_i} \in \mathcal{F}$.

So $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, as wanted. Closure under finite/ctble intersection follows from closure under finite/ctble union & De Morgan's law. Thus, $\mathcal{A}$ is a σ-algebra. QED ▨

RAGS

Chapter 2 exercises

Ex 2.8

WTP: There does not exist a σ-algebra which has ctbly many elements, but not finitely many

proof: Suppose for a contradiction that for a set X, $\exists$ a σ-algebra $\mathcal{A}$ on X s.t $\mathcal{A}$ is ctbly infinite.

Let $x \in X$, & consider $A_x = \bigcap_{\substack{A' \in \mathcal{A} \\ x \in A'}} A'$

(Note $A_x \neq \emptyset$ as $X \in \mathcal{A}$ & $x \in X$ as $\mathcal{A}$ a σ-algebra)

since $\mathcal{A}$ is ctble, we have this is a ctble intersection & so $A_x \in \mathcal{A}$ as $\mathcal{A}$ a σ-algebra.

claim: Given $x, y \in X$, $A_x = A_y$ or $A_x \cap A_y = \emptyset$

RAGS

Chapter 2 solutions

<u>Ex 2.8</u> Suppose $A_x \cap A_y \neq \emptyset$. Then

Let $m \in A_x \cap A_y \subseteq X$. <u>Note</u>

then by defn of A_m (since $m \in X$),

we have $A_m \subseteq A_x$ (since $A_x \in \mathcal{A}$

& $m \in A_x \cap A_y \Rightarrow m \in A_x$, so

A_x is in the intersection defining A_m

by defn of A_m)

we have then 2 cases:

<u>Case 1</u>: $x \in A_m$

Then by defn of A_x, we also

have $A_x \subseteq A_m$, by analogous

argument to above, & so $A_m = A_x$.

RAGS

Chap 2 sols

Ex 2.8

Case 2: $x \notin A_m$

Note since $A_x, A_m \in \mathcal{A}$, then $A_m^c \in \mathcal{A}$ & so $A_x \cap A_m^c \in \mathcal{A}$ as $\mathcal{A}$ a σ-algebra.

Note $x \in A_x \cap A_m^c$ (as $x \notin A_m$, $x \in A_x$ by def obv.)

But $m \notin A_x \cap A_m^c$ (as $m \in A_m$ by def of A_m, obv)

But this contradicts $m \in A_x$, as $m \in A_x \Rightarrow m \in B \ \forall B \in \mathcal{A}$ s.t. $x \in B$, & $A_x \cap A_m^c \in \mathcal{A}$, $x \in A_x \cap A_m^c$. So this case is not possible.

RAGS

chap 2 Solns

Ex 2.8

Thus, in all possible cases, $A_m = A_x$. A completely analogous argument shows in All possible cases, $A_m = A_y$. So $A_x = A_y$, in the case that $A_x \cap A_y \neq \emptyset$, as wanted. So either $A_x \cap A_y = \emptyset$ or $A_x \cap A_y \neq \emptyset$ & $A_x = A_y$. So $\forall x, y \in X$, $A_x = A_y$ or $A_x \cap A_y = \emptyset$.

Now note, if $C \in \mathcal{A}$ & $x \in C$, then $A_x \subset C$ by defn of A_x. Note then that this implies $\bigcup_{x \in C} A_x \subset C$. Also, $C \subset \bigcup_{x \in C} A_x$ as $\forall x \in C$, $x \in A_x$, & so $C = \bigcup_{x \in C} A_x$. So $\forall C \in \mathcal{A}$,

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Chapter 2 solns

Ex 2.8

ie $\forall C \in \mathcal{A}$, C is a union of sets in $\{A_x\}_{x\in X}$. **Note** this means $\{A_x\}_{x\in X}$ cannot be finite, because $\mathcal{A}$ is infinite & there are at most finitely many distinct unions of a finite collection of sets. So $\{A_x\}_{x\in X}$ is infinite. Since $\{A_x\}_{x\in X} \subseteq \mathcal{A}$
$\Rightarrow |\{A_x\}_{x\in X}| \leq |\mathcal{A}| = |\mathbb{N}|$, & so $\{A_x\}_{x\in X}$ is at most ctble & so ctble as it is infinite. So $\{A_x\}_{x\in X} = \{A_i\}_{i=1}^{\infty}$. So we have a ctble collection of pairwise disjoint sets in $\mathcal{A}$. By an analogous argument as given in exercise 2.6, $\exists$ an injection $F: (0,1) \cap (\mathbb{R} - \mathbb{Q}) \rightarrow \mathcal{A}$,

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chapter 2 solutions

Ex 2.8

& so $|\mathbb{R}| = |(0,1) \cap (\mathbb{R}-\mathbb{Q})| \leq |\mathcal{A}|$.

This contradicts $\mathcal{A}$ being ctble. $\times$
Thus our assumption was false &
there can exist no such σ-algebra
$\mathcal{A}$ s.t $\mathcal{A}$ is ctble but not finite. QED

Ex 2.9 (1)

Given: $\{A_i\}_{i=0}^{\infty}$ is a sequence of sets

$\liminf_i (A_i) := \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i$ $\quad \limsup_i (A_i) = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} A_i$

WTP: $\liminf_i (A_i) = \{x : x \in A_i$ for all but finitely many $i\}$

$\limsup_i (A_i) = \{x : x \in A_i$ for infinitely many $i\}$

proof: First, let $y \in \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i$. Then
by defn of union, $y \in \bigcap_{i=j}^{\infty} A_i$ for some $j \in \mathbb{N}$,

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Chap 2 exercises

Ex 2.9(i)

$j \geq 1$. By defn of "$\cap$", $y \in A_i$ $\forall i \geq j$.

Since there are only finitely many $i < j$, this means that $y \in A_i$ for all but finitely many i, ie

$y \in \{x : x \in A_i$ for all but finitely many $i\}$ by defn of this set. Now suppose $y \in \{x : x \in A_i$ for all but finitely many $i\}$.

Then $\exists j \in \mathbb{N}$ s.t $\forall i \geq j$, $y \in A_i$ (since $y \in A_i$ for all but finitely many i, then there must exist a largest k s.t $y \notin A_k$)

$\Rightarrow y \in \bigcap_{i=j}^{\infty} A_i$, by defn of "$\cap$"

RAGS

Chap 2 exercises

Ex. 2.9(1)

$\Longrightarrow y \in \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i$ By defn of union,

so $y \in \liminf_i (A_i)$ by defn, as wanted.

Thus $\liminf_i (A_i) = \{x : x \in A_i$ for all but finitely many $i\}$

Now let $y \in \limsup_i (A_i)$

$$= \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} A_i$$

Then $y \in \bigcup_{i=j}^{\infty} A_i \quad \forall j \in \mathbb{N} - \{0\}$

$\Longrightarrow \forall j \in \mathbb{N} - \{0\}, \exists i$ s.t $i \geq j$ & $y \in A_i$, by defn of union

$\Longrightarrow \exists$ infinitely many A_i s.t $y \in A_i$

(since there is no upper bound j s.t $y \notin A_i \; \forall i > j$)

RAGS

Chapter 2 exercises

Ex 2.9(1)

$\Rightarrow y \in \{x : x \in A_i$ for infinitely many $i\}$
as wanted.

Now suppose $y \in \{x : x \in A_i$ for infinitely many $i\}$

Then $y \in A_i$ for infinitely many i

$\Rightarrow \forall j \in \mathbb{N} - \{0\}, \exists i \geq j$ s.t

$y \in A_i$ (since o/w j would be an upper bound on the i s.t $y \in A_i$, & so y would be in A_i for only finitely many i)

$\Rightarrow y \in \bigcup_{i=j}^{\infty} A_i \quad \forall j \in \mathbb{N} - \{0\}$ (By defn of union)

$\Rightarrow y \in \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} A_i$ by def of intersect

$\Rightarrow y \in \limsup (A_i)$, as wanted

RAGS

Chapter 2 exercizes

Ex 2.9(1)

so $\limsup(A_i) = \{x : x \in A_i \text{ for infinitely many } i\}$

& $\liminf(A_i) = \{x : x \in A_i \text{ for all but finitely many } i\}$

as wanted. QED ■

Ex 2.9(2)

Give an example where $\liminf_i (A_i) \neq \limsup_i (A_i)$

Define the sequence of sets $\{A_i\}_{i=1}^{\infty}$ as follows:

If i is even: $A_i = \left\{k \in \mathbb{N} \,\middle|\, \begin{matrix} k \text{ is even}, \\ k \leq i \end{matrix}\right\}$

& if i is odd: $A_i = \left\{k \in \mathbb{N} \,\middle|\, \begin{matrix} k \text{ is odd}, \\ k \leq i \end{matrix}\right\}$

consider $2 \in \mathbb{N}$. Note: $2 \in \limsup(A_i)$

$= \{x : x \in A_i \text{ for infinitely many } i\}$

RAGS

Chapter 2 exercises

Ex 2.9(2)

Since 2 is even, so $\forall i \in \mathbb{N}$, i even, $i \geq 2$, we have $2 \in A_i = \{k \in \mathbb{N} \mid k \text{ even}, k \leq i\}$ by defn of A_i for even i. Since there are infinitely many even i, $i \geq 2$, we have $2 \in A_i$ for infinitely many i, ie $2 \in \limsup_i (A_i)$. But note $2 \notin \liminf_i (A_i)$, since $\forall i \in \mathbb{N}$, i odd, $2 \notin A_i = \{k \in \mathbb{N} \mid k \text{ odd}, k \leq i\}$ as 2 is not odd & by defn of A_i for i odd, & since there are infinitely many $i \in \mathbb{N}$ that are odd, there are infinitely many A_i s.t. $2 \notin A_i$ $\Rightarrow$ it is not the case that $2 \in A_i$ for all but finitely many i

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Chap 2 exercises

Ex 2.9(2)

$\Rightarrow 2 \notin \liminf_i (A_i) = \{x : x \in A_i$ for all but finitely many $i\}$

So $2 \in \limsup_i (A_i)$ & $2 \notin \liminf_i (A_i)$, so

$\limsup_i (A_i) \neq \liminf_i (A_i)$.

Ex 2.9(3)

Given: given a set D, define χ_D by

$\chi_D(x) = 1$ if $x \in D$ & $\chi_D(x) = 0$ if $x \notin D$.

WTP: For each $x_0 \in \mathbb{R}$,

$$\chi_{(\liminf A_n)}(x) = \liminf_n \chi_{A_n}(x)$$

& $$\chi_{(\limsup A_n)}(x) = \limsup_n \chi_{A_n}(x)$$

proof: Let $x \in \mathbb{R}$.

Case 1: $x \in \liminf_n (A_n)$.

RAGS

chapter 2 exercises

Ex 2.9(3)

Then by 2.9(1), $x \in A_{n_j}$ for all but finitely many n_j & $\chi_{(\liminf A_n)}(x) = 1$

Now consider

$$\liminf_n \chi_{A_n}(x) = \sup_n \inf_{m \geq n} \{\chi_{A_m}(x)\}$$

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Chapter 2 exercises

Ex 2.9(3) Let $n \in \mathbb{N}$ be arbitrary.

Consider first $\{\chi_{A_m}(x) \mid m \geq n\}$.

Since $x \in A_i$ for all but finitely many i, we must have that $\exists\ m \geq n$ s.t. $x \in A_m$, ie s.t. $\chi_{A_m}(x) = 1$. Now, either we have

$\forall m \geq n$, $x \in A_m$, & so $\{\chi_{A_m}(x) \mid m \geq n\} = \{1\}$, or $\exists\ m \geq n$ s.t. $x \notin A_m$, & so $\{\chi_{A_m}(x) \mid m \geq n\} = \{0, 1\}$.

So either $\inf\{\chi_{A_m}(x) \mid m \geq n\} = 0$

or $\inf\{\chi_{A_m}(x) \mid m \geq n\} = 1$

Now, <u>note</u> that b/c $x \in A_i$ for all but finitely many i, $\exists\ n' \in \mathbb{N}$ s.t.

$\inf\{\chi_{A_m}(x) \mid m \geq n'\} = 1$.

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Chapter 2 solutions

Ex 2-9(3)

So since $\forall n \in \mathbb{N}$, $\inf_{m \geq n} \{\chi_{A_m}(x)\} = 0$

or 1, & there exists

$n' \in \mathbb{N}$ s.t. $\inf_{m \geq n'} \{\chi_{A_m}(x)\} = 1$, we must

have that $\sup_n \inf_{m \geq n} \{\chi_{A_m}(x)\} = 1$, ie

$$\liminf_n \chi_{A_n}(x) = 1$$

$$= \chi_{\liminf(A_n)}(x)$$

as wanted.

RAGS

Chapter 2 solns

Ex 2.9(3)

~~Since~~ Case 2: $x \notin \liminf(A_i)$

Then since $\liminf(A_i) = \{x \in A_i \mid \text{for all but finitely many } i\}$,
we must have that $\forall n \in \mathbb{N}$, $\exists\, m \geq n$ s.t $x \notin A_m$. I.e $\exists\, m \geq n$ s.t $\chi_{A_m}(x) = 0$.

So $\forall n \in \mathbb{N}$, either $x \notin A_m \ \forall m \geq n$ & so $\{\chi_{A_m}(x) \mid m \geq n\} = \{0\}$ or $\exists$ an $m' \geq n$ s.t $x \in A_m$ & so $\{\chi_{A_m}(x) \mid m \geq n\} = \{0, 1\}$.

In both cases, $\inf\{\chi_{A_m}(x) \mid m \geq n\} = 0$

RAGS

Chapter 2 exercises

Ex 2.9(3)

& so thus $\liminf_n \chi_{A_m}(x) = \sup_n \inf_{m \geq n} \{\chi_{A_m}(x)\}$

$= \sup_n \{0\}$

$= 0$

$= \chi_{\liminf(A_n)}(x)$

as wanted

So in all possible cases, $\chi_{\liminf(A_n)}(x)$

$= \liminf_n \chi_{A_n}(x)$

as wanted. Now I show $\forall x \in \mathbb{R}$,

$$\chi_{\limsup(A_n)}(x) = \limsup_n \chi_{A_n}(x).$$

Case 1: $x \in \limsup(A_n)$

Then recall $x \in \{x \mid x \in A_n \text{ for infinitely many } n\}$

RAGS

Chapter 2 exercises

Ex 2.9(3)

Note also then $\chi_{\limsup(A_n)}(x) = 1$.

Now consider $\limsup_n \chi_{A_n}(x)$

$$= \inf_n \sup_{m \geq n} \{\chi_{A_m}(x)\}$$

Let $n \in \mathbb{N}$ be arbitrary.

Note that there must exist $m' \geq n$ s.t $x \in A_m$, or in other words, $\chi_{A_m}(x) = 1$. So either $\forall m' \geq n$, $\chi_{A_m}(x) = 1$, & so $\{\chi_{A_m}(x) \mid m \geq n\} = \{1\}$, or $\exists \bar{m}' \geq n$ s.t $\chi_{A_m}(x) = 0$, & so $\{\chi_{A_m}(x) \mid m \geq n\} = \{0, 1\}$. In either case, $\sup\{\chi_{A_m}(x) \mid m \geq n\} = 1$.

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Chapter 2 exercises

Ex 2.9(3)

Thus $\limsup_n \chi_{A_n}(x) = \inf_n \sup_{m\geq n} \{\chi_{A_m}(x)\}$

$= \inf_n \{1\}$

$= 1$

$= \chi_{\limsup(A_n)}(x)$

as wanted.

Case 2: $x \notin \limsup(A_n)$.

Then this means x is not in infinitely many of the A_n, or in other words, x is in only finitely many of the A_n.

Note also that $\chi_{\limsup(A_n)}(x) = 0$.

now consider $\limsup_n \chi_{A_n}(x)$

$= \inf_n \sup_{m\geq n} \{\chi_{A_m}(x)\}$

KAGS

chapter 2 exercises

Ex 2.9(3)

Let $n \in \mathbb{N}$ be arbitrary. Since x is in only finitely many of the A_n, there must exist $m' \geq n$ s.t $x \notin A_{m'}$, ie s.t $\chi_{A_{m'}}(x) = 0$.

So either $\forall m \geq n$, $\chi_{A_m}(x) = 0$, or $\exists \bar{m}' \geq n$ s.t $\chi_{A_{\bar{m}}}(x) = 1$. Thus

$\{\chi_{A_m}(x) \mid m \geq n\} = \{0\}$ <u>or</u>

$\{\chi_{A_m}(x) \mid m \geq n\} = \{0, 1\}$.

ie $\sup\{\chi_{A_m}(x) \mid m \geq n\} = 0$

<u>or</u> $\sup\{\chi_{A_m}(x) \mid m \geq n\} = 1$.

<u>Note</u> Since $x \in A_n$ for only finitely many n, there must exist $n' \in \mathbb{N}$ s.t $\forall m \geq n'$; $\chi_{A_m}(x) = 0$, or in other words, s.t $\sup\{\chi_{A_m}(x) \mid m \geq n\} = 0$.

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chap 2 exercises

Ex 2.9(3)

Thus $\limsup_n \chi_{A_n}(x) = \inf_n \sup_{m \geq n} \{\chi_{A_m}(x)\}$

$= 0$

$= \chi_{\limsup(A_n)}(x)$

as wanted.

(since $\sup_{m \geq n} \{\chi_{A_m}(x)\} = 0$ or 1

as we have shown, & we have shown there is some $n' \in \mathbb{N}$ s.t. $\sup_{m \geq n'} \{\chi_{A_m}(x)\} = 0\}$

Thus, in all possible cases of $x \in \mathbb{R}$, $\limsup_n \chi_{A_n}(x) = \chi_{\limsup(A_n)}(x)$ as wanted QED ■