Summary: Limits and asymptotics

Curve Sketching

General Strategy

1. Plot
   - discontinuities (especially infinite ones)
   - end points (or $x \to \pm\infty$)
   - easy points ($x = 0$, or $y = 0$) (This is optional.)
2. Plot critical points and values. (Solve $f'(x) = 0$ or undefined.)
3. Decide whether $f' < 0$ or $f' > 0$ on each interval between endpoints, critical points, and discontinuities. (Must be consistent with steps 1 and 2.)
4. Identify where $f'' < 0$ and $f'' > 0$ (concave down and concave up). Identify inflection points.
5. Combine into graph.

l'Hôpital's Rule

l'Hôpital's Rule Version 1: Indeterminate form $\frac{0}{0}$

If

$$\begin{array}{l} f(x) \to 0 \\ g(x) \to 0 \end{array} \qquad \text{as } x \to a,$$

and the functions f and g are differentiable near the point $x = a$, then limit

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{1}$$

provided that the right hand limit exists or is $\pm\infty$.

l'Hôpital's Rule Version 2: Indeterminate form $\dfrac{\infty}{\infty}$

If

$$\begin{array}{l} f(x) \to \pm\infty \\ g(x) \to \pm\infty \end{array} \qquad \text{as } x \to a,$$

and the functions f and g are differentiable near the point $x = a$, then limit

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{2}$$

provided that the right hand limit exists or is $\pm\infty$.

Note that

- We can replace a with a^+ or a^- and the results (versions 1 and 2) still hold.
- We can replace a with $\pm\infty$, and the results (versions 1 and 2) still hold.

Other indeterminate forms

Other indeterminate forms $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0 should be rearranged to be of the form $0/0$ or ∞/∞ in order to apply l'Hôpital's rule.