Summary: Gradients

When we are considering a function of 2 (or more) variables, it is natural to wonder what the slope of the graph of this function is. However, the function has a slope that may be different depending on the direction you move away from this point. This geometric notion of the slope of the graph along a particular direction is the **directional derivative** , and can be computed with a dot product with the gradient of the function.

Mechanics

Directional derivatives definition

Definition

The **directional derivative** of a function $f(x, y)$ in the direction of the unit vector $\hat{u}$ at the point (x, y) is given by

$$D_{\hat{u}} f(x, y) = \nabla f \cdot \hat{u}.$$

For the directional derivative along any non-zero vector $\vec{v}$, we use $D_{\vec{v}} = D_{\vec{v}/|\vec{v}|}$.

Extension to higher dimension: Directional derivatives

In n dimensions, the definition of the directional derivative is the same. We would have an n-dimensional unit vector $\hat{u} = \langle u_1, u_2, \ldots, u_n \rangle$. Then

$$\begin{aligned} D_{\hat{u}} f(x_1, x_2, \ldots, x_n) &= \nabla f(x_1, x_2, \ldots, x_n) \cdot \hat{u} \\ &= \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle \cdot \langle u_1, u_2, \ldots, u_n \rangle \\ &= f_{x_1}(x_1, x_2, \ldots, x_n) u_1 + f_{x_2}(x_1, x_2, \ldots, x_n) u_2 + \cdots + f_{x_n}(x_1, x_2, \ldots, x_n) u_n. \end{aligned}$$

Directional derivatives given an angle

Let θ be an angle measured from the positive x-axis.The rate of change of $f(x, y)$ in the direction of the angle θ is given by

$$D_{\hat{u}} f(x, y) = f_x \cos\theta + f_y \sin\theta$$

This is the same as the directional derivative of f in the direction of $\hat{u} = \langle \cos\theta, \sin\theta \rangle$.

Directional derivatives with non-unit vectors

Given a vector $\vec{v}$ whose magnitude is not 1, we can obtain a unit vector in the direction of $\vec{v}$ by computing

$$\hat{u} = \frac{1}{|\vec{v}|} \vec{v}.$$

Then the directional derivative of f in the direction of $\vec{v}$ is

$$D_{\vec{u}} f(x, y) = \frac{\nabla f \cdot \vec{v}}{|\vec{v}|}.$$

Direction of maximal change in f

We can write the directional derivative as the dot product

$$D_{\hat{u}} f(x,y) = \nabla f(x,y) \cdot \hat{u} = |\nabla f| \cos\theta$$

where θ is the angle between ∇f and $\hat{u}$. This quantity is maximized when $\theta = 0$, which implies that **the gradient is the direction of the maximum rate of change of** f.

Ask Yourself

Is the directional derivative a vector or a scalar?
Why is the directional derivative useful?
What does it mean if the directional derivative in a certain direction is zero?