Summary: Critical points

Big Picture

When we are interested in finding the maximum or minimum value of a function, we look to see mathematically what properties of that function are necessary to have a maximum or minimum.

Local maxima and minima of a function $f(x, y)$ occur at points where the gradient is zero (or undefined). We call points where the gradient is zero **critical points.** A critical point can be a local maximum, a local minimum, or neither, which is called a saddle point.

The gradient or level curves of a function give us graphical information about the behavior of a function that allows us to determine the type of critical point we have.

Mechanics

Mechanics

Critical points

Definition. Let $f(x, y)$ be a function of two variables. A **critical point** of $f(x, y)$ is a point (x_0, y_0) at which $\nabla f(x_0, y_0) = \vec{0}$. In other words, when $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ simultaneously.

Extension to higher dimension: Critical points

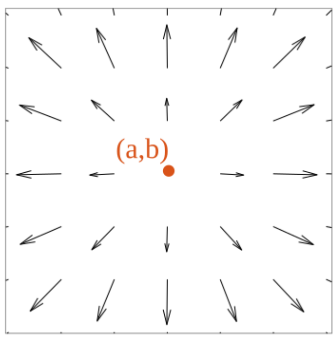
Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables. A **critical point** of $f(x_1, x_2, \ldots, x_n)$ is a point $(x_1^*, x_2^*, \ldots, x_n^*)$ at which $\nabla f(x_1^*, x_2^*, \ldots, x_n^*) = \vec{0}$. In other words, when $f_{x_1}(x_1^*, x_2^*, \ldots, x_n^*) = 0$, $f_{x_2}(x_1^*, x_2^*, \ldots, x_n^*) = 0$, $\ldots$, and $f_{x_n}(x_1^*, x_2^*, \ldots, x_n^*) = 0$ simultaneously.

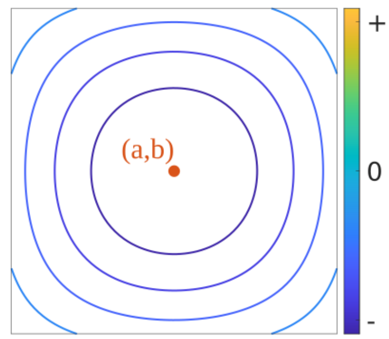
Graphical methods

Suppose $(x, y) = (a, b)$ is a critical point of $f(x, y)$ (meaning $\nabla f(a, b) = \langle 0, 0\rangle$).

Case 1: If the vectors representing $\nabla f(x, y)$ surrounding (a, b) are pointing away from (a, b), then $f(x, y)$ is decreasing as we approach (a, b) from every direction. This means (a, b) is a local minimum of $f(x, y)$.

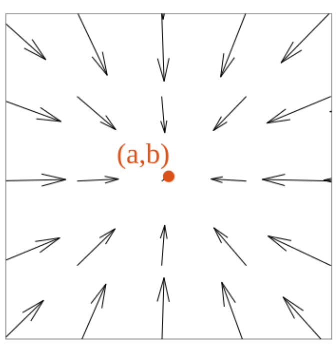
The figure below on the left shows the gradient field near a local minimum. The figure below on the right shows the corresponding level curves.

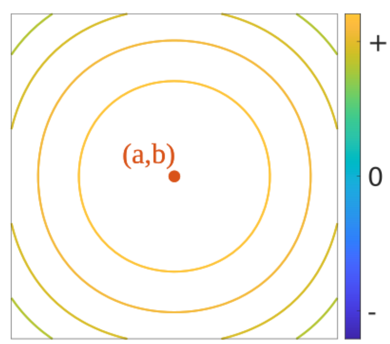




Case 2: If the vectors representing $\nabla f(x, y)$ surrounding (a, b) are pointing towards (a, b), then $f(x, y)$ is increasing as we approach (a, b) from every direction. This means (a, b) is a local maximum of $f(x, y)$.

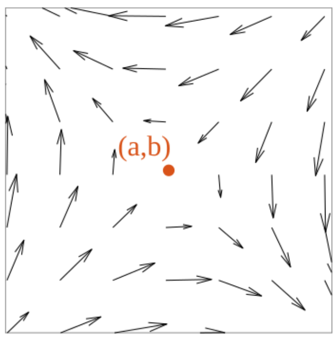
The figure below on the left shows the gradient field near a local maximum. The figure below on the right shows the corresponding level curves.

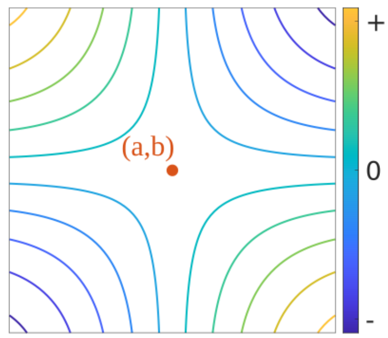




Case 3: If some vectors representing $\nabla f(x, y)$ near (a, b) point towards (a, b) and some point away from (a, b), then $f(x, y)$ is increasing as we approach (a, b) from some directions and decreasing as we approach (a, b) from other directions. This means (a, b) is a saddle point of $f(x, y)$.

The figure below on the left shows the gradient field near a saddle point. The figure below on the right shows the corresponding level curves.





Ask yourself

Ask Yourself

What does "finding critical points" have to do with "maximizing a function of two variables"?

What does it mean to "maximize a function of two variables"?