Summary: Introduction to vectors and dot products

Vectors are objects with magnitude and direction that have an algebra that allows us to perform mathematical operations such as addition, scaling, and more.

Mechanics

Definition of vectors

A **vector** is a quantity that has both magnitude and direction.
A vector in two dimensions has two components and can be written as

$$\vec{v} = \langle v_1, v_2 \rangle \tag{1}$$

or

$$\vec{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} \tag{2}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors in the x and y directions, respectively. In other words,

$$\begin{aligned}\hat{\mathbf{i}} &= \langle 1, 0 \rangle \\ \hat{\mathbf{j}} &= \langle 0, 1 \rangle.\end{aligned}$$

A vector in three dimensions has three components and can be written as

$$\vec{v} = \langle v_1, v_2, v_3 \rangle \tag{3}$$

or

$$\vec{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}} \tag{4}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors in the x, y, and z directions, respectively. In other words,

$$\begin{aligned}\hat{\mathbf{i}} = &\quad \langle 1, 0, 0 \rangle \\ \hat{\mathbf{j}} = &\quad \langle 0, 1, 0 \rangle \\ \hat{\mathbf{k}} = &\quad \langle 0, 0, 1 \rangle.\end{aligned}$$

Magnitude

The **magnitude** of a vector $\vec{v}$ is equal to its length and is denoted by $|\vec{v}|$.
By the Pythagorean theorem, the magnitude of a 2-dimensional vector $\vec{v} = \langle v_1, v_2 \rangle$ is given by

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2}. \tag{5}$$

Extension to higher dimension: Magnitude

Consider a vector with n components given by $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$. The magnitude of $\vec{v}$ is given by

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}. \tag{6}$$

Scalar multiplication

We can multiply vectors by real numbers (called **scalars**). Multiplication of a vector $\vec{v}$ by a scalar c will scale v by c. If $c > 0$, the vector $c\vec{v}$ will be in the same direction as $\vec{v}$ and have length $c|\vec{v}|$. If $c < 0$, the vector $c\vec{v}$ will be in the opposite direction of $\vec{v}$ and have length $|c||\vec{v}|$.
To multiply a vector $\vec{v} = \langle v_1, v_2\rangle$ by a scalar c, we multiply each component by c as follows:

$$c\vec{v} = c\langle v_1, v_2\rangle = \langle cv_1, cv_2\rangle. \tag{7}$$

Extension to higher dimension: Scalar Multiplication
Consider a scalar c and a vector with n components given by $\vec{v} = \langle v_1, v_2, \ldots, v_n\rangle$. Then $c\vec{v}$ is given by

$$c\vec{v} = c\langle v_1, v_2, \cdots, v_n\rangle = \langle cv_1, cv_2, \cdots, cv_n\rangle. \tag{8}$$

Vector addition

We can add two vectors that have the same number of components by adding each of their components. For example, the vectors $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$ add up to

$$\vec{v} + \vec{w} = \langle v_1, v_2\rangle + \langle w_1, w_2\rangle = \langle v_1 + w_1, v_2 + w_2\rangle. \tag{9}$$

Extension to higher dimension: Vector addition
Consider two vectors with n components each given by $\vec{v} = \langle v_1, v_2, \ldots, v_n\rangle$ and $\vec{w} = \langle w_1, w_2, \ldots, w_n\rangle$. Then the sum $\vec{v} + \vec{w}$ is given by

$$\vec{v} + \vec{w} = \langle v_1, v_2, \cdots, v_n\rangle + \langle w_1, w_2, \cdots, w_n\rangle = \langle v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n\rangle. \tag{10}$$

Dot products

The **dot product** between vectors $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$ is defined as

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2\rangle \cdot \langle w_1, w_2\rangle = v_1w_1 + v_2w_2. \tag{11}$$

Extension to higher dimension: Dot product
Consider two vectors of length n given by $\vec{v} = \langle v_1, v_2, \ldots, v_n\rangle$ and $\vec{w} = \langle w_1, w_2, \ldots, w_n\rangle$. The dot product of these vectors is given by

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2, \ldots, v_n\rangle \cdot \langle w_1, w_2, \ldots, w_n\rangle = v_1w_1 + v_2w_2 + \cdots + v_nw_n. \tag{12}$$

The dot product between two vectors $\vec{v}$ and $\vec{w}$ can also be computed as

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\theta) \tag{13}$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

Ask Yourself

Given a vector, how do you find a unit vector that points in the same direction? Is the dot product useful?