Summary: Finding vectors normal to level curves

Big Picture

Many questions and equations involving vectors can be rewritten in terms of “hidden" dot products. These hidden dot products can give us insights into the geometry of what is happening.

The main example in this lecture is the line defined by

$$ax + by = c \tag{1}$$

for some constants a, b, and c. This equation can be rewritten in terms of a dot product

$$\langle a, b\rangle \cdot \langle x, y\rangle = c. \tag{2}$$

Because we know that the line $ax+by = c$ is parallel to the line $ax+by = 0$, which is defined by the dot product equation

$$\langle a, b\rangle \cdot \langle x, y\rangle = 0. \tag{3}$$

We know the vector $\langle a, b\rangle$ is perpendicular to any line of the form $ax + by = c$.

Mechanics

1. If two vectors $\vec{v}$ and $\vec{w}$ are **perpendicular**, then

   $$\vec{v} \cdot \vec{w} = 0. \tag{4}$$

   Similarly, if $\vec{v} \cdot \vec{w} = 0$, then $\vec{v}$ and $\vec{w}$ are perpendicular.

2. If two vectors $\vec{v}$ and $\vec{w}$ are **parallel**, then there exists some constant $\lambda \neq 0$ such that

   $$\vec{v} = \lambda\vec{w}. \tag{5}$$

3. Given any vectors $\vec{v}$ and $\vec{a}$, we can **decompose** the vector $\vec{v}$ into a sum of components, one tangent to $\vec{a}$ and one perpendicular to $\vec{a}$. Let $\vec{b}$ be perpendicular to $\vec{a}$, then

   $$\vec{v} = \left(\frac{\vec{v} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\right)\vec{a} + \left(\frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\right)\vec{b}.$$

4. Given a vector $\vec{v}$ and a vector $\vec{a}$, the **projection** of $\vec{v}$ onto the vector $\vec{a}$ is the component of $\vec{v}$ that points into the same (or opposite) direction as $\vec{a}$. It is given by

   $$\left(\frac{\vec{v} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\right)\vec{a}$$

Ask Yourself

Can the dot product of two vectors be a negative number?