Summary: Second derivative test

It is easy to determine the shape and types of critical points for quadratic functions. Quadratic approximations allow us to make any function look like a quadratic function nearby, which then tells us about the behavior (i.e. local minimum, maximum, etc.) near that point. The second derivative test makes this procedure precise.

Mechanics

Second derivatives

Consider a function $f(x, y)$. The **second partial derivative with respect to x** is computed by taking the partial derivative with respect to x twice. The notation for this is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}. \tag{1}$$

We can also take higher order derivatives in different variables. For example, if we first take the partial of f with respect to y and then take the partial with respect to x, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}. \tag{2}$$

Notice the two different notations (Leibniz and subscript). Using Leibniz notation, we first take the derivative of f with respect to the variable written closest to f. So

$$\frac{\partial^2 f}{\partial x \partial y} \tag{3}$$

means we first take the derivative with respect to y to obtain a new function $\partial f/\partial y$. We then take the derivative of that function with respect to x. Using subscript notation, we still first take the derivative of f with respect to the variable written closest to f, but in this case, the order we write them looks reversed because the variables are on the other side of f. For example,

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \tag{4}$$

as above.

Important Fact: If a function has continuous second partial derivatives, then $f_{xy} = f_{yx}$. The functions we will explore in this course satisfy this criteria, so we will not need to worry about the order in which we take partial derivatives.

Second derivative test

Let (x_0, y_0) be a critical point of $f(x, y)$. Define

$$A = f_{xx}(x_0, y_0), \tag{5}$$
$$B = f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0), \text{ and} \tag{6}$$
$$C = f_{yy}(x_0, y_0). \tag{7}$$

Case 1: If $AC - B^2 < 0$, then (x_0, y_0) is a **saddle point** .
Case 2: If $AC - B^2 > 0$, then there are two subcases.

- If $AC - B^2 > 0$ and $A > 0$, then (x_0, y_0) is a **local minimum** .
- If $AC - B^2 > 0$ and $A < 0$, then (x_0, y_0) is a **local maximum** .

Case 3: If $AC - B^2 = 0$, then the test is inconclusive.

Ask yourself

Do I have to memorize everything in the second derivative test?
When do I have to use the second derivative test?

Additional resources

Disclaimer: The following OCW videos contain material that generally corresponds to material taught in this lecture. However, some of the material in the videos may not have been discussed here or may use different notation than what we have introduced.

OCW 18.02SC: Second Derivative Test