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Module 1:- Probability

* Experiment:-

- An activity that gives us a result is called experiment.
- It is test, trials or procedure for the purpose of discovering something unknown.

* Random Experiment:-

In each trial of an experiment conducted under identical conditions, the outcomes is not unique but may be any one of the possible outcomes such experiment is called random experiment

* Sample Space:- (S)

It is a set of all possible outcomes of any a random experiment. is called sample space.

* Event:-

The subset of sample space is called event

- For example:-

Random experiment:- Tossing Two coins

Sample space:- $S = \{HH, HT, TH, TT\}$

Event A :- At least one head

$A = \{HH, HT, TH\}$

* **Probability :-**

let 'S' be the sample space
'A' be the event of a sample space S

Then
Probability of A :~~-P(A)~~ is denoted by P(A) and is defined as,

$$P(A) = \frac{\text{number of elements of } A}{\text{number of elements of } S}$$

$$\boxed{P(A) = \frac{n(A)}{n(S)}}$$

NOTE :-

① $0 \leq P(A) \leq 1$

② $P(S) = 1$

* **Addition Rule of Probability :-**

let A, B, C be events of a sample space S
Then,

1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

* **Mutually Exclusive Events :-**

The events A and B of a sample space S are said to be mutually exclusive if A and B are disjoined (completely diff.) i.e. $A \cap B = \phi$ (empty) or $P(A \cap B) = 0$

ex., Thrown of a dice

Sample space = $\{1, 2, 3, 4, 5, 6\}$
~~let A = getting number less than~~

let A = getting odd number = $\{1, 3, 5\}$
B = getting even number = $\{2, 4, 6\}$ } Mutually exclusive events

$A \cap B = \phi$
$P(A \cap B) = 0$

* **Mutually Exhaustive event :-**

The events A and B of a sample space S are said to be mutually exhaustive if $A \cup B = S$
i.e. ~~Poo~~ $P(A \cup B) = 1$

ex., Tossing Two coins

* Independent events :-

The events A and B of a sample space S are said to be independent if accurrence of event A doesn't affects the occurrence of event B and vice versa

ex., Tossing one coin and one dice

$S = \{(H,1), (T,1), (H,2), (T,2), (H,3), (T,3), (H,4), (T,4), (H,5), (T,5), (H,6), (T,6)\}$

$|S| = 12$

let A = getting heads

B = ~~getting~~ Number less than 4

$A = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)\}$

$|A| = 6$

$B = \{(H,1), (H,2), (H,3), (T,1), (T,2), (T,3)\}$

$|B| = 6$

NOTE :- The events A and are independent if
$P(A \cap B) = P(A) \cdot P(B)$

Now, from example,

$A \cap B = \{(H,1), (H,2), (H,3)\}$

$|A \cap B| = 3$

$P(A \cap B) = \frac{3}{12} = \frac{1}{4}$

$P(A) = \frac{6}{12} = \frac{1}{2}$ $\quad$ $\frac{P(B)}{1} = \frac{6}{12} = \frac{1}{2}$

$\therefore P(A \cap B) = P(A) \cdot P(B)$

$\therefore$ A & B are independent.

NOTE :-

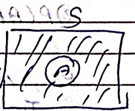
In general,

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \ldots \cdot P(A_n)$

if $A_1, A_2, A_3, \ldots A_n$ are independent

* Complement of event :-

The complement of event A of a sample space S is denoted by A^c, A' and is defined as $\boxed{A^c = S - A}$



$\boxed{P(A^c) = 1 - P(A)}$

* Conditional Probability :-

If probability of event A provided that event B has already occured is called the conditional probability and is defined as $P(A/B) = \frac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$

Similarly,

$P(B/A) = \frac{P(A \cap B)}{P(A)}$, $P(A) \neq 0$

NOTE:-

① If A, B are independent

Then, $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)\cdot P(B)}{P(B)} = \cancel{P(B)}\, P(A)$

② If A, B are mutually exclusive

Then, $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0}{P(B)} = 0$

$P(B|A) = 0$

③

* Multiplication Rule of probability.

1) $P(A\cap B) = P(B)\cdot \cancel{P(A)}\, P(A|B)$

2) $P(A\cap B) = P(A)\cdot P(B|A)$

ex., A box contain 4 bad and 6 good tubes Two are drawn out of box at a time one of them is tested and found to be good what is the probability that other one is also good

Sol^n:- let A = first drawn tube is good

B = second drawn tube is good

$\cancel{P(A\cap B)}\; P(A) = \frac{{}^6C_1}{{}^{10}C_1} = \frac{6}{10}$

$P(A\cap B) = \frac{{}^6C_2}{{}^{10}C_2}$ *

The probability of second drawn tube is good is

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{{}^6C_2}{{}^{10}C_2}}{\frac{6}{10}} = \frac{5}{9}$$

Q In a random experiment, $P(A) = \frac{1}{12}$, $P(B) = \frac{5}{12}$ and $P(B|A) = \frac{1}{15}$ find $P(A\cup B)$

$\Rightarrow$ $P(B|A) = \frac{P(A\cap B)}{P(A)}$

$\frac{1}{15} = \frac{P(A\cap B)}{1/12}$

$\frac{1}{15\times 12} = P(A\cap B)$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$= \frac{1}{12} + \frac{5}{12} - \frac{1}{15\times 12}$

$= \frac{1}{2} - \frac{1}{15\times 12}$

$= \frac{1}{2}\left[1 \quad \frac{}{90}\right]$

$P(A\cup B) = \frac{89}{180}$

* Total probability Theorem :-

Statement :- If $B_1, B_2, \dots, B_n$ are mutually exclusive and exhaustive events of a sample space S. and A be any event associated with the events $B_1, B_2, \dots, B_n$ ~~and~~

Then,

$$\boxed{P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}$$

* Baye's Theorem :-

Statement :-

If $A_1, A_2, \dots A_n$ be the mutually exclusive and exhaustive events of a sample space S. such that $P(A_i) \neq 0$ for every $i = 1, 2, \dots, n$. ~~and P(B~~ and B be any other events associated with the events $A_1, A_2, \dots, A_n$ such that $P(B) \neq 0$

Then,

$$P(A_i|B) = \frac{P(A_i) \cdot P(B/A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(B/A_i)}$$

Ex ① A box contain 7 red, 13 blue balls. 2 balls are selected randomly and are discarded without their colour being seen. If a third ball is drawn randomly and observed to be red. what is probabilit that both the discarded ball were blue

⇒ let A_1 = selected both balls are red
A_2 = selected balls are red and blue
A_3 = selected both balls are blue
B = third ball drawn is red

$$\text{now, } P(A_1) = \frac{{}^{7}C_2}{{}^{20}C_2} = \frac{21}{190}$$

$$P(A_2) = \frac{{}^{7}C_1 \times {}^{13}C_1}{{}^{20}C_2} = \frac{91}{190}$$

$$P(A_3) = \frac{{}^{13}C_2}{{}^{20}C_2} = \frac{78}{190}$$

$$P(B|A_1) = \frac{{}^{5}C_1}{{}^{18}C_1} = \frac{5}{18}$$

$$P(B|A_2) = \frac{{}^{6}C_1}{{}^{18}C_1} = \frac{6}{18}$$

$$P(B|A_3) = \frac{{}^{7}C_1}{{}^{18}C_1} = \frac{7}{18}$$

∴ By Baye's Theorem

$$P(A_3|B) = \frac{P(A_3)\cdot P(B|A_3)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + P(A_3)\cdot P(B|A_3)}$$

$$= \frac{78/190 \cdot 7/18}{\frac{21}{190}\cdot\frac{5}{18} + \frac{91}{190}\cdot\frac{6}{18} + \frac{78}{190}\cdot\frac{7}{18}}$$

$$= \frac{546}{1197}$$

$$= 0.456$$

Ex. ② There are in a bag 3 true coins and 1 false coin with head on both side. A coin is chosen at random and tossed 4 times. If head occurs all the times what is the probability that false coin was chosen and used

⇒

let A_1 = True coin is selected
A_2 = False coin is selected
B = Head occurs all the time

$$P(A_1) = \frac{{}^3C_1}{{}^4C_1} = \frac{3}{4}$$

$$P(A_2) = \frac{{}^1C_1}{{}^4C_1} = \frac{1}{4}$$

~~$P(B)$~~ =

$$P(B|A_1) = \cancel{{}^2C_1} \quad \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{16}$$

$$P(B|A_2) = 1$$

By ~~Baey'~~ Baye's Theorem

$$P(A_2|B) = \frac{P(A_2)\cdot P(B|A_2)}{P(A_1)P(B|A_1) + P(A_2)\cdot P(B|A_2)}$$

$$= \frac{1/4 \cdot 1}{\frac{3}{4}\cdot\frac{1}{16} + \frac{1}{4}\cdot 1}$$

$$= \frac{1/4}{1/4\left[\frac{3}{16}+1\right]}$$

$$P(A_2|B) = \frac{16}{19}$$

Ex ③ The chance that doctor A will diagnose a disease X correctly is 60%. The chance that a patient will die ~~is~~ by his treatment after correct diagnosis is 40% and chance of death by wrong diagnosis is 70%. A Patient of doc. A who has disease X, died. what is the chance that his disease was diagnosed correctly.

⇒

let A_1 =

Module 2

~~Module 2 :- Random Experiment~~ and

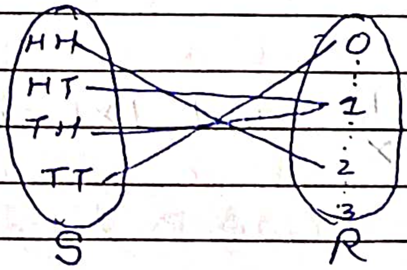
Module 2 :- Random variable and its Distribution.

- Random variable :-

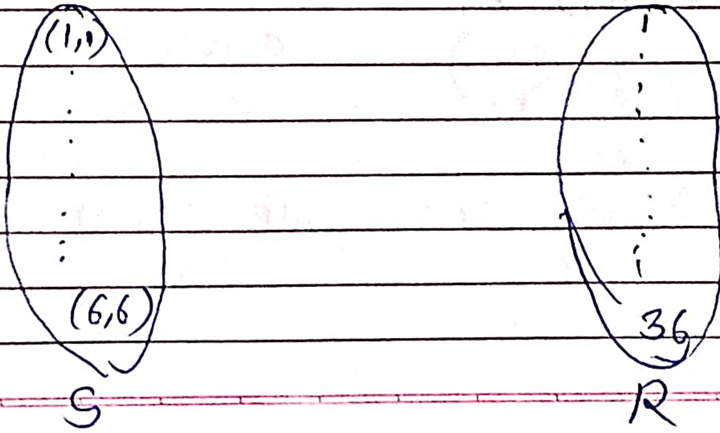
The random variable is a function that maps from the sample space of random experiment to the set of real number.
(i.e. $X : S \rightarrow R$)

ex.,

① R.E. :- Tossing 2 coins
$S = \{HH, HT, TH, TT\}$
X = Number of Heads



② R.E. :- 2 dices are thrown
$S = \{(1,1), \ldots, (6,6)\}$
X = product of nos. on dices



Random variables $\left\{ \begin{array}{l} \text{Descrete Random variable} \\ \text{Continous Random variable} \end{array} \right.$

* Descrete Random variable :- (D.R.V)

A variable X is said to be descrete r.v. if X takes finite or countably infinite values $x_0, x_1, x_2, \dots, x_n, \dots$

NOTE:-

1) If X_1 and X_2 are two R.V. associated with same sample space S
Then
for any constants C_1 and C_2, $C_1X_1 + C_2X_2$ is also R.V.

2) If X is R.V. then $\frac{1}{X}$ and $|X|$ is also R.V.

* Expectation ⇒ ~~(Ex)~~ (E(X))
let X be descrete R.V.
Then E(X) is defined as

$$E(X) = \sum_{i} P(x_i)\, x_i$$

* Mean of $\overline{X}$ = E(X)

* Variance of X = $V(X) = E(X^2) - [E(X)]^2$

- Probability mass function (pmf) :-

let X is D.R.V. and $x_0, x_1, x_2, \dots, x_n, \dots$ are values of X and $P(x_0), P(x_1), P(x_2), \dots P(x_n), \dots$ are corresponding probabilities.

Then, The function P is called ~~pm~~ pmf ~~if~~
if

(i) $P(x_i) \geq 0 \quad \forall\, i = 1, 2, \dots, n, \dots$

(ii) $\sum_i P(x_i) = 1$

- Probability Distribution function (pdf)

let X be D.R.V. Then the probability distributio
function of X is defined as

$$F(X) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i)$$

Ex. ① If x is DRV

X	1	2	3	4	5	6	7
$P(X=x)$	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

find i) k. ii) $P(X<5)$, iii) $P(x<5 \mid 2<x\leq 6)$

⊙ i) Note that $\sum_i P(x_i) = 1$

$\Rightarrow$ i] $k+2k+3k+k^2+k^2+k+2k^2+4k^2 = 1$

$7k+8k^2 = 1$

$8k^2+7k-1=0$

$8k^2+8k-k-1=0$

$8k(k+1)-k(k+1)=0$

-8
7
+8 -1

$k=-1 \qquad k = 1/8$

$k \neq -1 \qquad (\because p(x_i) \geq 0)$

Take $k = 1/8$

ii] $P(X<5) = P(X=1,2,3,4)$

$= P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$= k+2k+3k+k^2$

$= 6k+k^2$

$= \frac{1}{8}\left(6+\frac{1}{8}\right)$

$= \frac{1}{8}\cdot\frac{49}{8}$

$= \frac{49}{64}$

iii) $P(x<5 \mid 2<x\leq 6) =$

$P(A|B) = \frac{P(A\cap B)}{P(B)}$

~~$P(x<5 \cap$~~

$P(2<x\leq 6) = P(X=3)+P(X=4)+P(X=5)+P(X=6)$

$= 3k+k^2+k^2+k+2k^2$

$= 4k^2+4k$

$= 4\times\frac{1}{8}\left(\frac{1}{8}+1\right)$

$= \frac{4}{8}\left(\frac{9}{8}\right)$

$= \frac{8}{16}\ \frac{36}{64}$

$P[(x<5)\cap(2<x\leq 6)] = P(X=3)+P(X=4)$

$= 3k+k^2$

$= \frac{1}{8}\left[3+\frac{1}{8}\right]$

$= \frac{25}{64}$

$P(x<5 \mid 2<x\leq 6) = \frac{P(x<5 \cap 2<x\leq 6)}{P(x=2<x\leq 6)}$

$= \frac{25/64}{36/64}$

$= \frac{25}{36}$ //

Q4 Probability Dist. of R.V. X is

X	0	1	2	3
$P(X=x)$	0.1	0.3	0.5	0.1

If $Y = x^2 + 2x$
find $P(Y \leq 5)$
and variance of Y

~~Y — 0~~

$P(Y \leq 5) = P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4) + P(Y=5)$

=

Y	0	3	8	15
$P(Y=y)$	0.1	0.3	0.5	0.1

$P(Y \leq 5) = P(0) + P(3)$
$= 0.1 + 0.3$
$= 0.4$

$E(Y) = \sum Y_i P(Y_i)$

$= 0(0.1) + 3(0.3) + 8(0.5) + 15(0.1)$

$= 0 + 0.9 + 4 + 1.5$
$= \cancel{2.8}\ 6.4$

$E(Y^2) = \sum_i Y_i^2 \cdot P(Y_i)$

$= 0 + 9(0.3) + 64(0.5) + 225(0.1)$

$= 2.7 + 32 + 22.5$
$= 57.2$

$\therefore Var(Y) = \cancel{E(X^2)}\ E(Y^2) - [E(Y)]^2$

$= 57.2 - (6.4)^2$
$= 57.2 - 40.96$
$= 16.24$

Q) If R.V. X Takes 1, 2, 3, 4 such that

$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K$

find P.D. and mean.

$\Rightarrow$ $P(X=1) = \frac{K}{2}$ $P(X=2) = \frac{K}{3}$ $P(X=3) = \frac{K}{1}$

$P(X=4) = \frac{K}{5}$

$\sum_i P(X=x_i) = 1$

$\frac{K}{2} + \frac{K}{3} + \frac{K}{1} + \frac{K}{5} = 1$

$K\left[\frac{5}{6} + \frac{6}{5}\right) = 1$

$K \times \left(\frac{25+36}{30}\right) = 1$

$\boxed{K = \frac{30}{61}}$

Probability Distribution Table :-

X	1	2	3	4
$P(X=x)$	15/61	10/61	30/61	6/61

mean

$$E(X) = \sum_i x_i P(x_i)$$

$$= 1\left(\frac{15}{61}\right) + 2\cdot\left(\frac{10}{61}\right) + 3\left(\frac{30}{61}\right) + 4\left(\frac{6}{61}\right)$$

$$E(X) = \frac{149}{61}$$

$$\text{mean} = \frac{149}{61}$$

* Example on Total Probab- Theorem

coin tossing

① In an experiment if coin shows head, one die is thrown and a no. is recorded If coin shows tail, two dice are thrown and sum of numbers is recorded what is probability that recorded no. will be 2

$$\Rightarrow P(A) = \sum P(B_i)\, P(A|B_i)$$

let

B_1 = Getting head

B_2 = Getting tail

A = Recorded no. is 2

now, $P(B_1) = \frac{1}{2}$

$P(B_2) = \frac{1}{2}$

$P(A|B_1) = \frac{1}{6}$

$P(A|B_2) = \frac{1}{36}$

By Total Probability theorem

$$P(A) = \sum P(B_i) P(A|B_i)$$

$$= P(B_1)P(A|B_1) + P(B_2)\cdot P(A|B_2)$$

$$= \frac{1}{2}\cdot\frac{1}{6} + \frac{1}{2}\cdot\frac{1}{36}$$

$$= \frac{1}{12}\left[1 + \frac{1}{6}\right]$$

$$= \frac{7}{72}$$

② Continous Random variable (C.R.V)

Random variable X is said to be continous if X ~~make~~ takes uncountably Infinite values in the given interval

* Probability Density Function (PDF)

let X be the C.R.V, and f(x) be the continous function defined on [a,b] then probability density function of X is defined as ~~$P(a \le X \le b)$~~

$$\boxed{P(\alpha \le X \le \beta) = \int_{\alpha}^{\beta} f(x) \cdot dx}$$, where ~~αx~~

& $a < \alpha < \beta < b$

* Expectation

If X is CRV then $\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$

* $\boxed{\text{mean} = E(X)}$

* Variance ~~Ex~~ $\boxed{V(x) = E(X^2) - [E(X)]^2}$

③ Cumulative Distribution function (C.D.F)

let f(x) is a continous function on interval [a,b] and X be the continous random variable.
Then, Cumulative distribution function of X with p.d.f for f(x) is defined as.

$$F(X) = P(X < x) = \int_{-\infty}^{x} f(x) \cdot dx$$

Q) A Random variable X has p.d.f. ~~f(x) =~~

$$f(x) = \begin{cases} 2x & , 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$$

find i) $P(X < 1/2)$ ii) $P(1/4 < X < 1/2)$ iii) $P(X > 1/3 \,/\, X > 1/2)$

iv) CDF of X

⇒ Note that $P(\alpha < x < \beta) = \int_{\alpha}^{\beta} f(x)\, dx$

i) $P(X < 1/2) = \int_{-\infty}^{1/2} f(x)\, dx$

$$= \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{1/2} f(x)\, dx + \cancel{\int_{1/2}^{\infty} f(x)\, dx}$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{1/2} 2x \cdot dx + \cancel{\int_{1}^{\infty} 0 \cdot dx}$$

$$= 0 + 2 \cdot \left[\frac{x^2}{2}\right]_0^{1/2} \cancel{+0}$$

$$= \frac{1}{4}$$

ii] $P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx$

$$= \int_{1/4}^{1/2} 2x\,dx$$

$$= \left[x^2\right]_{1/4}^{1/2}$$

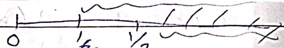
$$= \frac{1}{4} - \frac{1}{16}$$

$$= \frac{1}{4}\left[\frac{3}{4}\right]$$

$$= \frac{3}{16}$$

iii] Note that $P(A|B) = \frac{P(A\cap B)}{P(B)}$

for $P(X>1/3 \,/\, X>1/2)$



$$P\left(X>\tfrac{1}{3} \cap X>\tfrac{1}{2}\right) = P\left(X>\frac{1}{2}\right)$$

$$= 1 - P\left(X<\frac{1}{2}\right)$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

$$\therefore P\left(X>\tfrac{1}{3} \,/\, X>\tfrac{1}{2}\right) = \frac{P\left(X>1/3 \,/\, X>1/2\right)}{P(X>1/2)} = \frac{3/4}{3/4} = 1$$

iv] C.D.F. of X

~~$F(X) = \int$~~

if $x<0$, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0\cdot dx = 0$

if ~~x~~ $0<x<1$, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \cancel{\int_{-\infty}^{x} 2x\,dx}$

$$\cancel{= \left[x^2\right]_{-\infty}^{x}}$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\cdot dx + \int_{0}^{x} \cancel{f(x)\ x^2}\ 2x\,dx$$

$$= 0 + \left[x^2\right]_0^x$$

$$= x^2$$

if $x\geq 1$, $F(X) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\cancel{\infty}\,x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 2x\,dx + \int_{1}^{x} 0\cdot dx$$

$$= 0 + \left[x^2\right]_0^1 + 0 = 1$$

∴ CDF of x is

$$F(x) = \begin{cases} 0 & ; x \leq 0 \\ x^2 & ; 0 < x < 1 \\ 1 & ; 0 \geq 1 \end{cases}$$

Ex. 2 The pdf of R.V. X is $f(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1 \leq x < 2 \\ 0 & x > 2 \end{cases}$

Find CDF of X.

Soln:-

Note That CDF of X is

$$F(X) = \int_{-\infty}^{x} f(x)\,dx$$

if $x < 0$, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0 \cdot dx = 0$

$0 < x < 1$, $F(x) = \int_{\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} x\,dx \; \cancel{\frac{x^2}{2}}$

$$= \int_{-\infty}^{0} \overset{f(x)}{\cancel{x}}\,dx + \int_{0}^{x} x\,dx$$

$$= \int_{-\infty}^{0} 0 \cdot dx + \left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{x^2}{2}$$

if $1 \leq x < 2$, $F(x) = \cancel{\int_{-\infty}^{x} 0 \cdot dx} \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2}$$

$$= 2x - \frac{x^2}{2} \cancel{+1} - 1$$

if $x \geq 2$, $F(x) = \int_{-\infty}^{2} f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx + \int_{2}^{x} 0\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \frac{1}{2} + \cancel{4} - \frac{\cancel{4}}{2} - \cancel{2} + \frac{1}{2}$$

$$= 1$$

∴ The CDF of X is

$$F(X) = \begin{cases} 0 & , x \leq 0 \\ x^2/2 & , 0 \leq \cancel{x} < 1 \\ 2x - x^2/2 - 1 & , 1 \leq x < 2 \\ 1 & , x > 2 \end{cases}$$

E(x). 3 A R.V. X has a pdf $f(x) = kx(1-x), \ 0 \le x \le 1$
Find ⓘ k ⓘⓘ determine the no. 'b' such that
$P(X \ge b) = P(X \le b)$

Sol^n:- Note that $P(-\infty < X < \infty) = 1$

i]
$$P(\alpha \le X < \beta) = \int_{\alpha}^{\beta} f(x)\,dx$$

$$P(-\infty \le X < \infty) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} kx(1-x)\,dx + \int_{1}^{\infty} 0 \cdot dx = 1$$

$$= k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$k\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

$$\Rightarrow k = 6$$

~~ii] Given $P(X \ge b) = P(X \le b)$~~

~~$P(b \le X < \infty) = P(-\infty < X \le b)$~~

~~$\int_{b}^{\infty} \cancel{f(x)\,dx} = \int_{-\infty}^{b} f(x)\,dx$~~

~~$\int_{b}^{1} kx(1-x)\,dx + \int_{1}^{\infty} 0\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{b} 6x(1-x)\,dx$~~

ii] Given $P(X \ge b) = P(X \le b)$

$$P(b \le b < 1) = P(0 < X \le b)$$

$$\int_{b}^{1} f(x)\,dx = \int_{0}^{b} f(x)\,dx$$

$$\int_{b}^{1} 6x(1-x)\,dx = \int_{0}^{b} 6x(1-x)\,dx$$

$$6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_b^1 = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b$$

$$\frac{1}{2} - \frac{1}{3} - \frac{b^2}{2} + \frac{b^3}{3} = \frac{b^2}{2} - \frac{b^3}{3}$$

$$\frac{1}{6} - b^2 + \frac{2b^3}{3} = 0$$

$$\frac{2b^3}{3} + b^2 - \frac{1}{6} = 0$$

$$-2b^3 + 3b^2 - \frac{1}{2} = 0$$

$$-4b^3 + 6b^2 - 1 = 0$$

~~$a^3 + 3ab^2$~~

$$\boxed{b = \frac{1}{2}}, \ \cancel{\frac{1+\sqrt{3}}{2}}$$

* Two Dimensional Random variable

let X and Y be the random variables defined on some sample space, then the function (X, Y) that maps S to the R^2 (R×R) is called two-dimensional Random variable

NOTE:- $\{x \le a, Y \le b\}$ denotes the event of all elements ~~s~~ s ∈ S such that $X(s) \le a$ and $Y(s) \le b$

① Discrete Random Variable :-

* Joined Probability Distribution of (X, Y) :-

let, (X, Y) be the two dimensional descrete R.V. then the element $X(S) = (x_1, x_2, \cdots x_n)$ and $Y(S) = (y_1, y_2, \cdots, y_m)$ are image sets.

Hence, the function P on sets X(S) x Y(S) is defined by.

$$P(x_i, y_j) = P(X = x_i, Y = y_j) = P_{ij}$$

and is represented by using the following way.

X \ Y	y_1	y_2,	y_3,	- - - -	, y_m	Total
x_1	P_{11}	P_{12}	P_{13}	. . . . .	P_{1m}	P_{1*}
x_2	P_{21}	P_{22}	P_{23}		⋮	P_{2*}
⋮						⋮
x_n	P_{n1}	P_{n2}	P_{n3}	- - -	P_{nm}	P_{n*}
Total	P_{*1}	P_{*2}	P_{*3}	. . . .	P_{*m}	1

* Two-dimensional probability mass function

let (X, Y) be two dimensional D.R.V. then the function P is said to be 2-dim pmf if

① $P_{ij} \ge 0 \ (\forall i, j$

② $\sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij} = 1$

* Cumulative Distribution function : (Two-D. distribu. P.D.F.)

let (X, Y) be 2-dim D.R.V.

CDF of (X, Y) is denoted by F(x, y) & is given by

$$F(x, y) = P(X \le x, Y \le y) = \sum_{x_i < x} \sum_{y_j < y} P(x_i, y_j)$$

* Marginal Probability distribution :-

The M.P.D of X is

1] $P(X = x_i) = \sum_{j=1}^{m} P_{ij} = P_{i*} = P_{11} + P_{12} + \cdots P_{im}$

2] M.P.D of Y is

$$P(Y = y_j) = \sum_{i=1}^{n} P_{ij} = P_{*j} = P_{1j} + P_{2j} + \cdots + P_{nj}$$

* Conditional Probability Distribution

① The C.P. of X given that $Y = y_j$ is

$$P(X = x_i / Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}$$

② The C.P. of Y given that $X = x_i$ is

$$P(Y = y_j / X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)}$$

* Independent Random Variable The variable X and Y are said to be independent

if $P_{ij} = P_{i*} \cdot P_{*j}$, $\forall i, j$

i.e. $P(X = x_i, Y = y_j) = P(X = x_i) \cdot P(Y = y_j)$

ex. ① The (X, Y)
$P(x, y) = k(2x + 3y)$, $x = 0, 1, 2$ and $y = 1, 2, 3$
find marginal and Conditional Probability
① $P(X = 2, Y \leq 2)$ ② $P(X \leq 2, Y = 3)$ ③ $P(X = 2)$

④ $P(Y = 2)$ ⑤ $P(X \leq 1 / Y \leq 2)$ ⑥ $P(Y \leq 2 / X \leq 2)$

Sol^n The probability distribution for (X, Y) is

X \ Y	~~y_1~~ 1	~~y_2~~ 2	3	Total
0	3k	6k	9k	18k
1	5k	8k	11k	24k
2	7k	10k	13k	30k
	15k	24k	33k	1

Note that $\sum_{i=1}^{3} \sum_{i=1}^{3} P_{ij} = 72k = 1$

$$\boxed{k = \frac{1}{72}}$$

∴ The Prob. dist. table is

X \ Y	1	2	3	Total	
0	3/72	6/72	9/72	18/72	P(X=0)
1	5/72	8/72	11/72	24/72	P(X=1)
2	7/72	10/72	13/72	30/72	P(X=2 ~~P(X~~
Total	15/72	24/72	33/72	1	
	P(Y=1)	P(Y=2)	P(Y=3)		

① $P(X=2, Y\le 2) = P(X=2, Y=1) + P(X=2, Y=2)$

$= \frac{7}{72} + \frac{10}{72} = \frac{17}{72}$

② $P(X\le 2, Y=3) = P(X=0, Y=3) + P(X=1, Y=3) + P(X=2, Y=3)$

$= \frac{9}{72} + \frac{11}{72} + \frac{13}{72}$

$= \frac{33}{72}$

③ $P(X=2) = P(X=2, Y=1) + P(X=2, Y=2) + P(X=3, Y=3)$

$= \frac{7}{72} + \frac{10}{72} + \frac{13}{72}$

$= \frac{30}{72}$

④ $P(Y=2) = P(X=0, Y=2) + P(X=1, Y=2) + P(X=2, Y=2)$

$= \frac{6}{72} + \frac{8}{72} + \frac{10}{72}$

$= \frac{24}{72}$

⑤ $P(X\le 1 / Y\le 2) = \frac{P(X\le 1 \cap (Y\le 2))}{P(Y\le 2)}$

$$= \frac{P(X=0, Y=1) + P(X=0', Y=2) + P(X=\cancel{1}, Y=0) + P(X=1, \cancel{Y}=2) + P(X=0, Y=1) + P(X=0, \cancel{Y}=2)}{P(X=0, Y=1) + P(X=0, Y=1) + P(X=2, Y=1) + P(X=2, Y=2) + P(X=1, Y=1) + P(X=1), Y=2)}$$

=

$$= \frac{3/72 + 6/72}{\frac{3}{72} + 5/72 + 7/72 + \frac{6}{72} + \frac{8}{72} + \frac{10}{72}}$$

$$= \frac{9/72}{39/72}$$

$$= \frac{9}{39}$$

~~③~~ ⑤ $P(X\le 1 / Y\le 2) = \frac{P(X\le 1, Y\le 2)}{P(Y\le 2}$

$$= \frac{P(X=0, Y\le 2) + P(X=1, Y\le 2)}{P(Y=1) + P(Y=2)}$$

$$= \frac{P(X=0, Y=1) + P(X=0, Y=2) + P(X=1, Y=1) + P(X=0, Y=2)}{P(Y=1 + P(Y=2)}$$

$$= \frac{3/72 + 6/72 + 5/72 + 8/72}{15/72 + 24/72} = \frac{2}{3}$$

⑥ $P(Y \leq 2 \mid X \leq 2) = \dfrac{P(Y=1, X \leq 2) + P(Y=2, X \leq 2)}{P(X \leq 2)}$

$$= \frac{P(Y=1, X=0)+P(Y=1, X=1)+P(Y=1, X=2) + P(Y=2, X=0)+P(Y=2, X=1)+P(Y=2, X=2)}{P(X=0)+P(X=1)+P(X=2)}$$

$$= \frac{15/72 + 24/72}{1}$$

$$= \frac{39}{72}$$

---

②

Q. 2 Given that joint probability distribution

Y \ X	-1	0	1	Total
0	1/15	2/15	1/15	4/15
1	3/15	2/15	1/15	6/15
2	2/15	1/15	2/15	5/15
Total	6/15	5/15	4/15	1

ⓐ Find marginal probability distribution of X and Y

ⓑ Find conditional probability distribution of Y given that X = 0

ⓒ Check whether X, Y are independent.

⇒ ⓐ ~~$P(X=0) = \frac{1}{15} + \frac{2}{15} + \frac{1}{15} = \frac{4}{15}$~~

~~$P(X=1) =$~~

⇒ ⓐ marginal Probability of Y is

$$P(X = x_i) = \sum_{j=1}^{m} P_{ij}$$

$P(Y=0) = P(Y=0, X=-1) + P(Y=0, X=0) + P(Y=0, X=1)$

$= \frac{1}{15} + \frac{2}{15} + \frac{1}{15} = \frac{4}{15}$

$P(Y=1) = \frac{3}{15} + \frac{2}{15} + \frac{1}{15} = \frac{6}{15}$

$P(Y=2) = \frac{2}{15} + \frac{1}{15} + \frac{2}{15} = \frac{5}{15}$

Marginal probability of X

$$P(X=x_i) = \sum_{j=1}^{m} P_{ij}$$

① $P(X=-1) = P(X=-1, Y=0) + P(X=-1, Y=1) + P(X=-1, Y=2)$

$= \frac{1}{15} + \frac{3}{15} + \frac{2}{15}$

$= \frac{6}{15}$

② $P(X=0) = \frac{5}{15}$

③ $P(X=1) = \frac{4}{15}$

ⓑ $P(Y \leq 2 / X=0) = \frac{P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=0)}{P(X=0)}$

$= \frac{2/15 + 2/15 + 1/15}{5/15}$

$= 1$

ⓒ Note that X, Y are independent

if $P_{ij} = P(x_i, y_j) = P(X=x_i) \cdot P(Y=y_j)$

i.e.

$P(X=-1, Y=0) = \frac{1}{15}$

~~$P(X=x_i)$~~ $P(X=-1) \cdot P(Y=0) = \frac{\cancel{6}^{2}}{\cancel{15}_{5}} \cdot \frac{4}{15} = \frac{\cancel{24}}{\cancel{225}}$

$= \frac{8}{75}$

$P(X=-1, Y=0) \neq P(X=-1) \cdot P(Y=0)$

$\therefore X, Y$ are dependent.

Q.3 The joint P.D. of (X, Y) is given below check whether X, Y are independent.

$X \backslash Y$	1	3	5	Marginal P.D. of X	
2	0.1	0.2	0.1	0.4	$P(X=2)$
4	0.15	0.30	0.15	0.6	$P(X=4)$
Marginal P.D. of Y	0.25	0.5	0.25	1	

$P(Y=1)$ $P(Y=3)$ $P(Y=5)$

if $P_{ij} = P(\cancel{x_i}, y_j) \; P_{i*} \cdot P_{*j}$

① $P(X=2, Y=1) = 0.1$

~~P~~ $P(X=2) \cdot P(Y=1) = 0.4 \times 0.25 \Rightarrow 0.100$

$\therefore P(X=2, Y=1) = P(X=2) \cdot P(Y=1)$

② $P(X=2, Y=3) = 0.2$, $P(X=2) \cdot P(Y=3) = 0.4 \cdot 0.5 = 0.$

$P(X=2, Y=3) = P(X=2) \cdot P(Y=3)$

③ $P(X=2, Y=5) = 0.1$, $P(X=2) \cdot Y(5) = 0.4 \times 0.25 = 0.$

④ $P(X=4, Y=1) = 0.15$, $P(X=4) \times P(Y=1) = 0.6 \times 0.25 = 0.15$

⑤ $P(X=4, Y=3) = 0.3$, $P(X=4) \times P(Y=3) = 0.6 \times 0.5 = 0.3$

⑥ $P(X=4, Y=5) = 0.15$, $P(X=4) \times P(Y=5) = 0.6 \times 0.25 = 0.15$

$\therefore X, Y$ are independent

C.D.F.

$$F(x,y) = P(X \leq x, Y \leq y)$$
$$= \sum_{x_i \leq x} \sum_{y_j \leq y} P(X = x_i, Y = y_j)$$

Ex. ④ Find P.D.F. for following probability distribution of (X, Y)

X \ Y	1	3	5
2	0.1	0.2	0.1
4	0.15	0.3	0.15

Sol

Solⁿ Note that the probability distribution function of (X, Y) is $F(X,Y) = P(X \leq x, Y \leq y)$

$$= \sum_{x_i \leq x} \sum_{y_j \leq y} P(X = x_i, Y = y_j)$$

if $x = 2, y = 1$, $F(x,y) = P(X \leq 2, Y \leq 1)$
$= P(X = 2, Y = 1)$
$= 0.1$

if $x = 2, y = 3$, $F(x,y) = P(X \leq 2, Y \leq 3)$
$= P(X=2, Y=1) + P(X=2, Y=3)$
$= 0.1 + 0.2$
$= 0.3$

if $x = 2, y = 5$, $F(x,y) = P(X \leq 2, Y \leq 5)$
$= P(X=2, Y=1) + P(X=2, Y=3) + P(X=2, Y=5)$
$= 0.1 + 0.2 + 0.1$
$= 0.4$

if $x = 4, y = 1$, $F(X,Y) = P(X \leq 4, Y \leq 1)$
$= P(X=2, Y=1) + P(X=4, Y=1)$
$= 0.1 + 0.15$
$= 0.25$

if $x = 4, y = 3$, $F(X,Y) = P(X \leq 4, Y \leq 3)$
$= P(X=2, Y=1) + P(X=2, Y=3) +$
$P(X=4, Y=1) + P(X=4, Y=3)$
$= 0.1 + 0.2 + 0.15 + 0.3$
$= 0.75$

if $x = 4, y = 5$, $F(X,Y) = P(X \leq 4, Y \leq 5)$
$= P(X=2, Y=1) + P(X=2, Y=3) + P(X=2, Y=$
$+ P(X=4, Y=1) + P(X=4, Y=3) + P(X=4, Y=5$
$= 0.1 + 0.2 + 0.1 + 0.15 + 0.3 + 0.1$
$=$ 1.00

∴ ~~The probability~~

∴ The pdf of (X, Y)

$$F(x,y) = \begin{cases} 0.1 & \text{if } x=2, y=1 \\ 0.3 & \text{if } x=2, y=3 \\ 0.4 & \text{if } x=2, y=5 \\ 0.25 & \text{if } x=4, y=1 \\ 0.75 & \text{if } x=4, y=3 \\ 1 & \text{if } x=4, y=5 \end{cases}$$

* Two-D Continous Random variable

* 2-D joint probability density function

let, (X,Y) be the 2-D ~~con~~ C.R.V. and $f(x,y)$ be the bivariate Continous function such that ~~$P(x-\frac{dx}{2})$~~

$$P\left(x-\frac{dx}{2} < X < x+\frac{dx}{2},\ y-\frac{dy}{2} < Y < y+\frac{dy}{2}\right)$$

$$= df(x,y)$$

Then $f(x,y)$ is said to be joint probability density function if

① $f(x,y) \geq 0$, $\forall (x,y)$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Note :- $P(a < X < b,\ c < Y < d) = \int_{c}^{d}\int_{a}^{b} f(x,y)\,dx\,dy$

* Marginal P.D.F.

① Marginal probability density function of X is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

~~* 2-D joined pro~~

② Marginal p.d.f. of Y is defined as :-

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

* Conditional p.d.f.

① The conditional p.d.f. of X given that Y is defined as ~~$f(x|y)$~~ $f(x|y) = \frac{f(x,y)}{f_Y(y)}$

② The C.P.D.F. of Y given that X is defined as

$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

* probability distribution function :- (cumulative dist-fun

let (X,Y) be the 2-D continous random variable and $f(x,y)$ be the joint prob. density function. then C.D.F is denoted by

$$F(x,y) = P(X < x,\ Y < y)$$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$

* Independent random variables

The R.V. X and Y are said to be independent if $f(x,y) = f_X(x) \cdot f_Y(y)$

NOTE :-

① If X, Y are Independent then $f(x|y)$

$f(x|y) = f_Y(y)$, $f(y|x) = \cancel{f_Y(y)}\ f_X(x)$

Ex. ① find the value of k if the joined probability density function is $f(x,y) = \begin{cases} k(1-x)(1-y) & 0<x<4,\ 1<y<5 \\ 0, & \text{otherwise} \end{cases}$

Sol^n :-

If joint probability function $f(x,y)$

That $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_1^5\int_0^4 k(1-x)(1-y)\,dx\,dy = 1$$

$$\int_1^5 k(1-y)\cdot\left[x-\frac{x^2}{2}\right]_0^4 dy = 1$$

$$\int_1^5 k(1-y)\left[4-8\right]dy = 1$$

$$\int_1^5 k(1-y)\cdot(-4)\,dy = 1$$

$$-4k\left[y-\frac{y^2}{2}\right]_1^5 = 1$$

$$-4k\left[5-\frac{25}{2}-1+\frac{1}{2}\right] = 1$$

$$-4k\left[4-12\right] = 1$$

$$32k = 1$$

$$\boxed{k = \frac{1}{32}}$$

Ex. ② Find value of k if joint p.d.f. is

$f(x,y) = kye^{-x}$, $0<x<\infty$, $0<y<1$

$= 0$, otherwise

$\Rightarrow$ $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^1\int_0^\infty \cancel{f(x)}\ kye^{-x}\,dx\,dy = 1$$

$$-\int_0^1 ky\left[e^{-x}\right]_0^\infty = 1$$

$$-\int_0^1 ky\left[0-1\right] = 1$$

$$+\ k\left[\frac{y^2}{2}\right]_0^1 = 1$$

$$k\left[\frac{1}{2}\right] = 1$$

$$\boxed{k = 2}$$

* ~~Marginal p.d.f.~~

Ex. ③ Given the join p.d.f.

$$f(x,y) = \begin{cases} x e^{-x(y+1)} & 0<x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$$

ⓐ Find marginal p.d.f.
ⓑ Find conditional p.d.f.
ⓒ Check whether X and Y are independent

⇒ ① Marginal p.d.f of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{1} 0 \cdot dy + \int_{1}^{5} x e^{-x(y+1)}\,dy$$

$$= x\left[\frac{e^{-x(y+1)}}{-x}\right]_1^5 + \int_5^{\infty} 0\,dy$$

$$= \left[-e^{-6x} + e^{-2x}\right] = e^{-2x} - e^{-6x}$$

Now, Marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^4 x e^{-x(y+1)}\,dx$$

$$= \left[x\left(\frac{e^{-x(y+1)}}{-(y+1)}\right) - (1)\left(\frac{e^{-x(y+1)}}{(y+1)^2}\right)\right]_0^4$$

$$= \left[\frac{-4e^{-4(y+1)}}{(y+1)} - \frac{e^{-4(y+1)}}{(y+1)^2} - \left[0 - \frac{e^0}{(y+1)^2}\right]\right]$$

$$= \frac{1 - e^{-4(y+1)}}{(y+1)^2} - \frac{4e^{-4(y+1)}}{y+1}$$

ⓑ The conditional p.d.f of X given that Y is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x e^{-x(y+1)}}{\frac{1-e^{-4(y+1)}}{(y+1)^2} - \frac{4e^{-4(y+1)}}{y+1}}$$

$$= \frac{x(y+1)^2 e^{-x(y+1)}}{1 - e^{-4(y+1)} - 4(y+1)e^{-4(y+1)}}$$

and conditional p.d.f of Y given that X

$$\text{if } f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x e^{-x(y+1)}}{e^{-2x} - e^{-6x}}$$

③ Note that X, Y are independent if
$f(x,y) = f_X(x) \cdot f_Y(y)$
here $f_X(x) \cdot f_Y(y) = (e^{-2x} - e^{-6x})\left[\frac{1-e^{-4(y+1)}}{(y+1)^2} - \frac{4e^{-4(y+1)}}{(y+1)}\right]$

$$\neq f(x,y)$$

i.e. $f(x,y) \neq f_X(x) \cdot f_Y(y)$

X, Y are not independent

④ Joint p.d.f of (X, Y) is given by

$f(x,y) = \begin{cases} kxy, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$

i] find k ii] find $P(X+Y<1)$

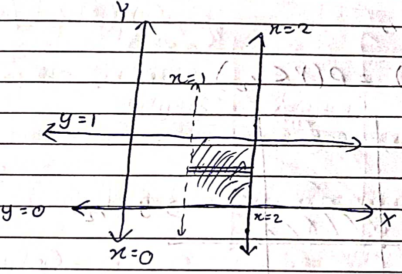
iii] Are x, Y independent.

Ex. ⑤ The joint probability density function of (X, Y) is given by $f(x,y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le$

(i) find $P(X>1)$ ② $P(2Y<1)$ ③ $P(X>1 / Y<\frac{1}{2})$

④ $P(X<Y)$ ⑤ $P(X>Y \cap X<Y)$

Solⁿ :- (I) To find $P(X>1)$



limits :- $1 \le x \le 2$, $0 \le y \le 1$

* $P(X>1) = P(1 \le x \le 2,\ 0 \le Y \le 1)$

$$= \int_0^1\int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_1^2 dy$$

$$= \int_0^1 \left[\frac{y^2(4)}{2} + \frac{8}{24} - \frac{y^2}{2} - \frac{1}{24}\right] dy$$

$$= \int_0^1 \cancel{y}^2\left[\frac{3}{2}y^2 + \frac{7}{24}\right] dy$$

=

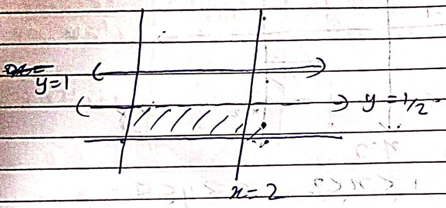
$= \left[\frac{1}{2}y^3 + \frac{7}{24}y\right]_0^1$

$= \frac{1}{2} + \frac{7}{24}$

$= \frac{12}{24} + \frac{7}{24}$

$= \frac{19}{24}$

$P(2Y<1) = P(Y<1/2)$



limit ⇒ ~~x=0~~ $0 \le x \le 2$, $0 \le y \le 1/2$

$\int_0^{1/2}\int_0^2 (\cancel{x^2}\ xy^2 + \frac{x^2}{8})\,dx\,dy$

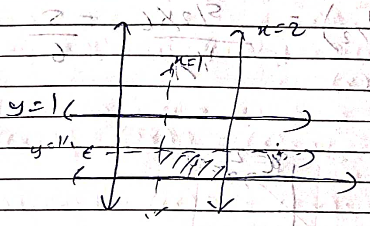
$\int_0^{1/2}\left[y^2\frac{x^2}{2} + \frac{x^3}{24}\right]_0^2 dy$

$\int_0^{1/2} y^2\frac{(4)}{2} + \frac{8}{24}\,dy$

$2\left[\frac{y^3}{3}\right]_0^{1/2} + \cancel{8}\frac{1}{3}[y]_0^{1/2}$ $\qquad \frac{1}{\cancel{3}} \times \frac{\cancel{6}4}{\cancel{8}4}$

$\frac{\cancel{2}}{3}\left[\frac{1}{\cancel{8}4}\right] + \frac{1}{3}\left[\frac{1}{2}\right] = \frac{1}{3}\left[\frac{1}{4}+\frac{1}{2}\right] = \boxed{\frac{1}{4}}$

③ $P(X>1 / Y<1/2) = \frac{P(X>1,\ Y<1/2)}{P(Y<1/2)} = \frac{P(X>1 \cap Y<1/2)}{P(Y<1/2)}$



~~x~~⇒ $1 \cancel{0} < x < 2$ $\qquad 0 < y < 1/2$

$\int_0^{1/2}\int_1^2 xy^2 + \frac{\cancel{x}^2}{8}]\,dx\,dy$

$\int_0^{1/2}\left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_1^2 dy$

$\int_0^{1/2}\left[2y^2 + \frac{8}{24} - \frac{y^2}{2} - \frac{1}{24}\right]$

$\int_0^{1/2}\left[\frac{3y^2}{2} + \frac{7}{24}\right] y$ $\qquad \cancel{P(Y<}1/2) = \int$

$\left[\frac{y^3}{2} + \frac{7}{24}y\right]_0^{1/2}$

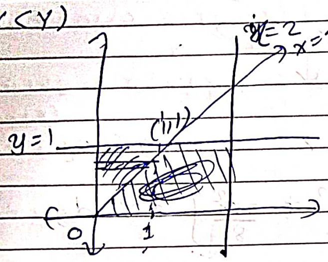
$\frac{1}{16} + \frac{7}{\cancel{8}48}$

$\frac{10}{48} = \frac{\cancel{8}\,5}{24}$

$P(Y<1/2) = 1/4$

$\therefore P(X>1 / Y<1/2) = \frac{5/2\cancel{4}6}{1/\cancel{4}} = \frac{5}{6}$

$P(X<Y)$



$0<X<y \qquad 0<Y<1$

$$\int_0^1\int_0^y \cancel{x^2+}\left[y^2x + \frac{x^2}{8}\right] dx\,dy$$

$$\int_0^1 \left[\frac{y^2x^2}{2} + \frac{x^3}{24}\right]_0^y dy$$

$$\int_0^1 \left[\frac{y^4}{2} + \frac{y^3}{24}\right] dy$$

$$\left[\frac{y^5}{10} + \frac{y^4}{24\times4}\right]_0^1$$

$$\frac{1}{10} + \frac{1}{96} \Rightarrow \cancel{96+}\ \frac{106}{960} = \frac{53}{480}$$

---

⑤ $P(X>Y \cap X \leq Y) = \cancel{0}\, P(\phi) = 0$

* Expectation of 2-D R.V.

① For discrete random variables :-

* $E(X) = \sum_i P(x_i)\cdot x_i = \sum_{x_i} P(X=x_i)\cdot x_i$

~~$= \sum_{x_i} P(X=x_i)$~~

* $E(Y) = \sum_j P(y_j)\cdot y_j = \sum P((Y=y_j))\cdot y_j$

* $E(XY) = \sum_j\sum_i P(X=x_i, Y=y_j)\,x_i\,y_j$

* $E(X/Y) = \sum_i P(X=x_i, Y=y_j)\,x_i$

* $E(Y/X) = \sum_j (P(X=x_i, Y=y_j)\,y_j$

* $E(X \pm Y) = E(X) \pm E(Y)$

② For Continous R.V.

* $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

* $E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\cdot dy$

* $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy$

$* E(x|Y) = \int_{-\infty}^{\infty} x f(x/y)\, dx$

$* E(Y|X) = \int_{-\infty}^{\infty} y f(y/x)\, dy$

⊙ Mean of $X = E(X)$

⊙ Variance of $X = Var(X) = E(X^2) - [E(X)]^2$

⊙ Co-variance $= Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

$Cov(X,Y) = E(XY) -$

⊙ Properties ~~of Covari~~ :-

① If X, Y are Independent
Then $Cov(X,Y) = 0$

② If $Cov(X,Y) = 0$, then X, Y are uncorrelated

③ $E(aX+bY) = a E(X) + b E(Y)$, $a, b \in R$.

④ $Var(aX+b) = a^2 Var(X) + 0$

⑤ $Var(k) = 0$, $k \in R$

⑥ $Cov(aX+bY, ~~cx~~ (X+dY)) = ac\, Var(X) + bd\, Var(Y) + (ad+bc) \cdot Cov(X,Y)$

⑦ ~~Var(ax+by~~)

6

⑦ $Var(aX \pm bY) = a^2 Var(X) + b^2 Var(Y) \pm 2ab\, Cov(X,Y)$

Ex. ① Given the following bivariate p.d.

Y \ X	-1	0	1		
-1	0	0.1	0.1	0.2	$P(Y=-1)$
0	0.2	0.2	0.2	0.6	$P(Y=0)$
1	0	0.1	0.1	0.2	$P(Y=1)$
	0.2 $P(X=-1)$	0.4 $P(X=0)$	0.4 $P(X=1)$		

Find $E(X)$, $E(Y)$, $E(XY)$, $Var(X)$ and Prove that X, Y are uncorrelated.

$\Rightarrow$ ① $E(X) = \sum_i P(X=x_i)\, x_i$

$= (-1) \cdot P(X=-1) + P(X=0) \cdot 0 + P(X=1) \cdot (1)$

$= (0.2)(-1) + 0 + (0.4)(1)$

$= 0.4 - 0.2$

$= 0.2$

② $E(Y) = \sum_j P(Y=y_j) \cdot y_j$

$= (-1)(P(Y=-1)) + 0 \cdot P(Y=0) + (1)(P(Y=1))$

$= (-1)(0.2) + 0 + (1)(0.2)$

$= 0$

③ $E(XY) = \sum_j \sum_i P(X=x_i, Y=y_j)\, x_i \cdot y_j$

$= (-1)(-1)\, P(X=-1, Y=-1) + (-1)(0) \cdot P(X=-1, Y=0) + (-1)(1)\, P(X=-1, Y=1)$
$+ (0)(-1)\, P(X=0, Y=-1) + (0)(0)\, P(X=0, Y=0) + (0)(1)\, P(X=0, Y=1)$
$+ (1)(-1)\, P(X=1, Y=-1) + (1)(0)\, P(X=1, Y=0) + (1)(1)\, P(X=1, Y=1)$

$= 0 + 0 + (-0.1) + 0 + 0 + 0.1$

$= 0.1 - 0.1$
$= 0$

④ $Var(X)$
$= E(X^2) - [E(X)]^2$

i] $E(X^2) = \sum_i P(X=x_i)\, x_i^2$

$= (-1)^2 \cdot P(X=-1) + 0 + (1)^2 \cdot P(X=1)$

$= 0.2 + 0.4$
$= 0.6$

$Var(X) = E(X^2) - [E(X)]^2$
$= 0.6 - (0.2)^2$
$= 0.6 - 0.04$
$= 0.56$

⑤ $Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$
$= 0 - (0.2) \cdot (0)$
$= 0 - 0$
$Cov(X,Y) = 0$

$\therefore$ X, Y are unrelated

Ex. ② If $Y = -2X+3$ find $Cov(X,Y)$

$\Rightarrow$ ① $E(XY) = E(X \cdot -2X+3)$

$= E(-2X^2 + 3X)$

$=$ ~~$4E(X^2)$~~ $-2E(X^2) + 3E(X)$

② ~~$E(X)$~~ $=$

② $E(Y) = E(-2X+3)$
$= -2E(X) + 3E(1)$
$= -2E(X) + 3$

~~Cov (X, -2X+3) = E(X·(-~~

$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

$= -2E(X^2) + 3\cancel{E(X)} + 2[E(X)]^2 \mp 3\cancel{E(X)}$

$=$ ~~$-2[E(X^2) + [E(X)]^2]$~~

$=$ ~~$-2 Var(X)$~~

$=$ ~~$-2E(X^2) + 6E(X) - 2[E(X)]^2$~~
$-2 Var(X)$ ~~$+ 6E(X)$~~

Ex. ③ The join pdf of (X,Y) is

$f(x,y) = x+y \,, \quad 0 \le x \,, y \le 1$

find $E(X)$, $E(Y)$, $E(XY)$

→ ①

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^1 (x+y)\, dy$$

$$= \left[xy + \frac{y^2}{2}\right]_0^1$$

$$= x + \frac{1}{2}$$

$$E(X) = \int_{\cancel{-\infty}\,0}^{\cancel{\infty}\,1} x \cdot \left(x + \frac{1}{2}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

② $$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^1 (x+y)\, dx$$

$$= \left[\frac{x^2}{2} + xy\right]_0^1$$

$$= \left[\frac{1}{2} + y\right]$$

$$\therefore E(Y) = \int_0^1 y \left(\frac{1}{2} + y\right) dy$$

$$= \left[\frac{y^2}{4} + \frac{y^3}{3}\right]_0^1$$

$$= \frac{7}{12}$$

③ $$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 x\, y\, (x+y)\, dx\, dy$$

$$= \int_0^1\int_0^1 y\, (x^2 + xy)\, dx\, dy$$

$$= \int_0^1 y \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^1 dy$$

$$= \int_0^1 y \left[\frac{1}{3} + \frac{y}{2}\right] dy$$

$$= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

④ Join p.d.f of (X,Y) is $f(x,y) = \begin{cases} 24xy, & x>0, y>0, x+y \le 1 \\ 0, & \text{otherwise} \end{cases}$

find conditional mean and variance of Y given X

=)

$f(x,y) = \begin{cases} 24xy, & x>0, y>0, x+y \le 1 \\ 0, & \text{otherwise} \end{cases}$

$E(Y/X)$, $Var(Y/X)$

=)

$$E(Y/X) = \int_{-\infty}^{\infty} y \cdot f(y/x)\,dy$$

$$= \int_{-\infty}^{\infty} y \cdot \frac{f(x,y)}{f_X(x)}\,dy$$

1] $f(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{24xy}{\int_{-\infty}^{\infty} f(x,y)\,dy} = \frac{24xy}{\int_0^{1-x} 24xy\,dy} = \frac{\cancel{24x}y}{\cancel{24x}\left[\frac{y^2}{2}\right]_0^{1-x}} = \frac{2y}{(1-x)^2}$

$$E(Y/X) = \int_{-\infty}^{\infty} y \cdot \frac{2y}{(1-x)^2}\,dy$$

$$= \frac{2}{(1-x)^2}\int_0^{1-x} y^2\,dy$$

$$= \left[\frac{y^3}{3}\right]_0^{1-x} \cdot \frac{2}{(1-x)^2}$$

$$= \frac{(1-x)^3}{3} \cdot \frac{2}{(1-x)^2}$$

$$= \frac{2}{3}(1-x)$$

2] $Var(Y/X) = E(Y^2/X) - [E(Y/X)]^2$

=)

$$E(Y^2/X) = \int_{-\infty}^{\infty} y^2 \cdot f(y/x)\,dy$$

$$= \int_0^{1-x} y^2 \cdot \frac{2y}{(1-x)^2}\,dy$$

$$= \frac{2}{(1-x)^2}\left[\frac{y^4}{4}\right]_0^{1-x}$$

$$= \frac{(1-x)^2}{2}$$

$$Var(Y/X) = \frac{(1-x)^2}{2} - (1-x)^2 \cdot \frac{4}{9}$$

$$\boxed{\hat{Var}(Y/X) = \frac{(1-x)^2}{18}}$$

x.④ The joint p.d.f of x,y is

$$f(x,y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}, \quad 0<x<\infty,\ 0\le y\le\infty$$

Find marginal distribution of X and Y.
The conditional dist. of Y given that X = x
and expected value of conditional distribution.

⇒ ① marginal p.f.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^\infty \frac{9(1+x+y)}{2(1+x)(1+y)^4}\, dy$$

y→0 t→1
y→
1+y = t
dy = dt

$$= \frac{9}{2(1+x)} \int_0^\infty \frac{1+x+y}{(1+y)^4}\, dy$$

$$= \frac{9}{2(1+x)} \int \left[(1+y)^{-4} + x(1+y)^{-4} + y(1+y)^{-4}\right] dy$$

$$= \frac{9}{2(1+x)} \left[\int_0^\infty (1+x)(1+y)^{-4} + \int_{\cancel{0}\,1}^\infty (t-1)\cdot t^{-4}\, dt\right.$$

$$= \frac{9}{2(\cancel{1+x})} \left[\cancel{(1+x)} \left[\frac{(1+y)^{-4+1}}{-4+1}\right]_0^\infty + \int_1^\infty t^{-3} - t^{-4}\, dt\right.$$

$$= \frac{9}{2(1+x)} \left[\frac{(1+x)}{-3\cdot(1+y)^3}\right]_0^\infty + \left[\frac{t^{-3+1}}{-3+1} - \frac{t^{-4+1}}{-4+1}\right]_1^\infty$$

$$= \frac{9}{2(1+x)} \left[-\frac{(1+x)}{3}\left(\frac{1}{\infty} - \frac{1}{\cancel{0}\,1}\right) + \left(-\frac{1}{2t^2} + \frac{1}{3t^3}\right)_1^\infty\right]$$

$$\frac{9}{2(1+x)} \left[\frac{(1+x)}{3} + \left(-\frac{1}{\infty} + \frac{1}{\infty} - \left(\frac{-1}{2} + \frac{1}{3}\right)\right)\right]$$

$$\frac{9}{2(1+x)} \left[\frac{(1+x)}{3} + \frac{1}{2} - \frac{1}{3}\right]$$

$$\frac{9}{2(1+x)} \cdot \left[\frac{x+\cancel{0}}{3} + \frac{1}{2}\right]$$

$$\frac{\cancel{9}\,3}{2(1+x)} \quad \frac{(2x+3)}{\cancel{6}\,2}$$

$6x+9$

$$\left(\frac{2x+3}{\cancel{6}\,2}\right) \cdot \frac{\cancel{9}\,3}{2(1+x)^4}$$

$$\frac{3}{4}\, \frac{(2x+3)}{(1+x)^4}$$

$$\int_0^\infty \frac{1+y}{(1+y)^4} + \int_0^\infty \frac{x}{(1+y)^4}$$

$$\frac{x}{3} + \frac{1}{2}$$

$$\left(\frac{(1+y)^{-3+1}}{-3+1}\right)_0^\infty + \frac{x(1+y)^{-4+1}}{-4+1}$$

$$\frac{2x+3}{6}$$

$$\left(\frac{0 \quad 1}{-2(1+y)^2}\right)_0^\infty + \left(\frac{x}{-3[1+y]^{-3}}\right)_0^\infty$$

$$\left[\frac{1}{\cancel{-\infty}} - \frac{1}{-2}\right] + \frac{x}{\cancel{-\infty}} + \frac{x}{3}$$

⑥ ∴ -

② Marginal p.d.f of Y is

$$f_Y(y) = \frac{3(2y+3)}{4(1+y)^4}$$

③ Conditional pdf of Y given that X = x

$$f(y/x=x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{\dfrac{\cancel{9}^{3}(1+x+y)}{\cancel{2}(1+\cancel{x})^4(1+\cancel{y})^3}}{\dfrac{\cancel{3}(2\cancel{y}+3)}{2\cancel{4}(1+\cancel{y})^4}}$$

$$= \frac{6(1+x+y)}{(2x+3)(1+y)^4}$$

$$f(x,y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^3}$$

$$f_X(x) \cdot f_Y(y) = \frac{3(2x+3)}{4(1+x)^4} \cdot \frac{3(2y+3)}{4(1+y)^4}$$

$$= \frac{9(\cancel{2}4xy+6x+6y+9)}{16(1+x)^4(1+y)^3}$$

∴ X, Y are not independent

④ $E(Y/X) = \int_{-\infty}^{\infty} y \cdot f(y/x)\,dy$

$$= \int_{\cancel{-\infty}\,0}^{\infty} y \cdot \frac{6(1+x+y)}{(2y+3)(y+1)^4}\,dy$$

$$= 6\int_0^{\infty} \frac{y(1+x+y)}{(2\cancel{y}x+3)(1+y)^4}\,dy$$

$1+y = t$

$dy = dt$

$$= 6\int_1^{\infty} \frac{(t-1)(\cancel{1}+x+t\cancel{-1})}{(2(t-1)+3)(t^4)}\,dy$$

$$= 6\int_1^{\infty} \frac{(t-1)(x+t)}{(2t+1)(t^4)}\,dy$$

$$= 6\int_1^{\infty} \frac{tx+t^2-x-t}{2t^5+t^4}$$

$$= 6\int_1^{\infty} \frac{t^2+(x-1)t-x}{2t^5+t^4}$$

$$= \frac{\cancel{6}}{\cancel{2x+3}}\ \frac{x+3}{2x+3}$$

* Raw a ments :-

let x b

① The r^{th} about origin is
$\mu'_r = E(x^r)$,

② The r^{th} moment of x about mean $(\bar{x})$

$\mu_r = E[(x-\bar{x})^r]$ $\qquad \bar{x} = E(x)$

[Central moment]

③ The r^{th} moment of x about point A
(Row moment)

$\mu'_r = E[(x-A)^r]$

ⓒ Relation between moments about origin and central moments :-

consider, $\mu_r = E[(x-\bar{x})^r]$

$\cancel{= E[x^r}$

$= E[$

$\cancel{= E[\Sigma\, {}^rC_i\, x}$

$= E[{}^rC_0\, x^r - {}^rC_1\, x^{r-1}\bar{x} + {}^rC_2\, x^{r-2}\bar{x}^2$

$+ (-1)^r\, {}^rC_r\, \bar{x}^r]$

$= E(x^r) - rE(x^{r-1})\bar{x} + \frac{r(r-1)}{2!} E($

$+ (-1)^r (\bar{x}^r]$

$\mu_r = \mu'_r - r\mu'_{r-1}\mu'_1 + \frac{r(r-1)}{2!}\mu'_{r-2}\mu_1'^2$

$+ (-1)^r \mu$

if $r=1 \Rightarrow \mu_1 = \mu'_1 - 1(\mu'_0)(\mu_1) = 0$

$r=2 \Rightarrow \mu_2 = \mu'_2 - 2(\mu'_1)\mu_1 + \frac{2(1)}{2}\mu'_0\mu_1'^2$

$= \mu'_2 - 2\mu_1'^2 + \mu_1'^2 = \mu'_2 - \mu_1'^2$

if $r=3 \Rightarrow \mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$

if $r=4 \Rightarrow \mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_1\mu'_2 - 3\mu_1'^4$

* Relation between Raw moment and central moment

here , $\mu_r' = E[(X-A)^r]$

$\left[\bar{x} = \mu_1' + A\right)$

$\mu_1' = \cancel{\mu_1 + A}\ \bar{x} - A$

$\mu_2' = \mu_2 + \mu_1'^2$

$\mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3$

$\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1'^2 + 4\mu_1'^4$

& soon...

* Moment generating function (MGF)

The MGF of R.V. X about origin is denoted by $\boxed{M_x(t) = E(e^{tx})}$

Note:- ① If x is discrete R.V. then ~~$M_x(t)$~~ =

$$M_x(t) = E(e^{tx}) = \sum_x e^{tx} \cdot p(x)$$

→ ② If x is continous R.V. then

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

→ ③ moments using power series:-

$$M_x(t) = E(e^{tx}) = E\left[1 + tx + (tx)^2 + t^3x^3 + \dots + t^r x^r + \dots\right]$$

$$M_x(t) = 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \dots + \frac{t^r}{r!}E(x^r) + \dots$$

$$M_x(t = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

$$\boxed{\mu_r' = \text{Coefficient of } \frac{t^r}{r!}}$$

→ ④ moments using derivative

$$\mu_1' = \left[\frac{d\,[M_x(t)]}{dt}\right]_{t=0}$$

$$\mu_2' = \left[\frac{d^2\,[M_x(t)]}{dt^2}\right]_{t=0}$$

In general,

$$\mu_r' = \left[\frac{d^r\,[M_x(t)]}{dt^r}\right]_{t=0}$$

→ ⑤ MGF ~~for~~ will not always exist

Ex ① ~~Find~~ The first four moment of the distribution about $X=4$ are 1, 4, 10 and 45 respectively show that mean is 5, variance is 3 and $\mu_3 = 0$, $\mu_4 = 26$

⇒ $M_x(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3}\mu_3' + \dots$

let
$\mu_1' = 1$, $\mu_2' = 4$, $\mu_3' = 10$, $\mu_4' = 45$
and $A = 4$

* mean $= \mu_1' + A = 1+4 = 5$

* variance $= E(x^2) - [E(x)]^2$

$= \mu_2' - (\mu_1')^2$

$= 4 - 1^2$

variance $= 3$

$\mu_4 = 45 - 40 + 24 - 3$
$= 5 + 21$
$\boxed{\mu_4 = 26}$

* $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= 10 - 3(4)(1) + 2(1)^3$

$= 10 - 12 + 2$

$\boxed{\mu_3 = 0}$

* $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 45 - 4(10)(1) + 6(4)(1)^2$

1
1 ① 1
1 ② 1
1 3 ③ 1
1 4 6 ④ 1
1 5 10 10 ⑤ 1

$\mu_1 = 0$
$\mu_2 = \mu_2' - (\mu_1')^2$
$\mu_3 = \mu_3' - 3\mu_2'\mu_1'^2 + 2\mu_1'^3$
$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

Ex ② If $f(x) = \begin{cases} 3e^{-3x} & ; x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$ is p.d.f.

of R.V. x using moment generating function find first four moments about origin and first four central moments

⇒ $M_x(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \dots$

∴ $M_x(t) = E(e^{tx})$

$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$= \int_0^{\infty} \cancel{3e^{-3x}}\; e^{tx} \cdot 3e^{-3x}\,dx$

$= 3\int_0^{\infty} e^{tx-3x}\,dx$

$= 3\left[\frac{e^{(t-3)x}}{t-3}\right]_0^{\infty}$

$= \frac{3}{(t-3)}\left[e^{(t-3)x}\right]_0^{\infty}$

let $(t-3) < 0$
$t < 3$

$= \frac{3}{t-3}\left[e^{-\infty} - e^{0}\right] = \frac{3}{3-t}$

$M_x(t) = \frac{3}{3(1-\frac{t}{3})} = \frac{1}{1-t/3}$, $t<3$

$\cancel{\left(\frac{1}{1-1/x}\right)}$ $\boxed{\frac{1}{1-x} = 1+x+x^2+x^3+\ldots \quad |x|<1}$

$\therefore M_x(t) = 1 + \frac{t}{3} + \frac{t^2}{3^2} + \frac{t^3}{3^3} + \ldots + \frac{t^r}{3^r}$

we know,

$\mu_r' =$ coefficient of $\frac{t^r}{r!}$

④ $\mu_1' =$ coefficient of $\frac{t}{1}$

$\boxed{\mu_1' = \frac{1}{3}}$

② $\mu_2' =$ coeff of $\frac{t^2}{2!}$

$\left(\frac{\cancel{t^2}}{3^2} = \mu_2' \cdot \frac{\cancel{t^2}}{2!}\right.$

$\mu_2' = \frac{2}{9}$

$\boxed{\mu_2' = \frac{2}{9}}$

③ $\mu_3' =$ coeff of $\frac{t^3}{3!}$

$\left(\frac{\cancel{t^3}}{3^{\cancel{3}}} = \mu_3' \cdot \frac{\cancel{t^3}}{\cancel{3}\times 2}\right.$

$\mu_3' = \frac{2}{9}$

$\boxed{\mu_3' = \frac{2}{9}}$

④ $\mu_4' =$ coeff of $\frac{t^4}{4!}$

$\left(\frac{\cancel{t^4}}{3^{\cancel{4}}} = \mu_4' \cdot \frac{\cancel{t^4}}{4\times\cancel{3}\times 2}\right.$

$\boxed{\mu_4' = \frac{8}{27}}$

$\mu_4' = \frac{8}{27}$

① $\mu_1 = 0$

② $\mu_2 = \mu_2' - \mu_1'$

$= \frac{2}{9} - \frac{1}{3} \Rightarrow -\frac{1}{9}$

③ $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= \frac{2}{9} - 3\left(\frac{2}{9}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^3$

$= \frac{2}{9} - \frac{2}{9} + \frac{2}{3^3}$

$= \frac{0}{9} + \frac{2}{27}$

$= \cancel{\frac{3-2}{27}} = \frac{2}{27}$

$\cancel{= \frac{1}{27}}$

④ $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= \frac{8}{27} - 4\left(\frac{2}{9}\right)\left(\frac{1}{3}\right) + 6\left(\frac{2}{9}\right)\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right)^4$

$= \cancel{\frac{8}{27}} - \cancel{\frac{8}{27}} + \frac{12}{9\times 9} - \frac{3}{9\times 9}$

$\frac{81}{\;} = \frac{1}{9}$

: Module 3:-
Special Probability Distribution

* Descrete probability Distribution

① Binomial Distribution

let n be the no. of trials and ~~p~~ p be the probability of success in each trial and q be the probability of failure. Then r.v. X is set to follow the Binomial Distribution if it has probability $P(X=x)$

$$\boxed{P(X=x) = {}^nC_x\, p^x q^{n-x}} \qquad x = 0,1,2,\dots,n$$

Note :-

① n must be finite

② mean $= E(X) = \sum_x P(x)\,x$

$$= \sum_{x=0}^{n} \left({}^nC_x\, p^x q^{n-x}\right) x$$

$$= \sum_{x=0}^{n} \frac{n!}{(n-x)!\,x!}\, p^x q^{n-x} \cdot x$$

$$= \sum_{x=1}^{n} \frac{n!}{(n-x)!\,(x-1)!} \cdot p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(n-x)!\,(x-1)!}\, p \cdot p^{x-1} \cdot q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!\,(x-1)!}\, p^{x-1} q^{n-x}$$

$$= np\,(p+q)^{n-1}$$

$$= np\,(1)^{n-1}$$

$$= np$$

③ Variance $= npq$

④ mean $\geq$ variance

⑤ MGF $= M_X(t) = E(e^{tx}) = e^{-npt}(q+pe^t)^n$

⑥ $M_{(x_1+x_2)}(t) = M_{x_1}(t) + M_{x_2}(t)$

Ex. ① If the mean and variance of Binomial distributio are 16 and 8, find $P(X \geq 3)$

⇒ mean $= 16$

variance $= 8$

$\therefore\ np = 16$ —①

$npq = 8$ —②

$n \times \frac{1}{2} = 16$

$\boxed{n = 32}$

~~① ÷ ②~~ ② ÷ ①

$\boxed{q = 0.5}$

$\boxed{p = 0.5}$

$P(X \geq 3) = 1 - \left[P(X=0) + P(X=1) + P(X=2) + P(X=3)\right]$

$= {}^{32}C_0\, p^0 q^{32} + {}^{32}C_1\, p^1 q^{31} + {}^{32}C_2\, p^2 q^{30} + {}^{32}C_3\, p^3 q^{29}$

$$\left[1 - \left({}^{32}C_0 + {}^{32}C_1 + {}^{32}C_2 + {}^{32}C_3\right)(0.5)^{32}\right]$$

$$=1-\left\{\left(1+32+\frac{\overset{16}{\cancel{32}}\times 31}{\cancel{2}}+\frac{\overset{16}{\cancel{32}}\times 31\times\overset{10}{\cancel{30}}}{\cancel{3}\times\cancel{2}}\right)\times\left(\frac{1}{2}\right)^{32}\right.$$

$$=1-\left\{(33+496+4960)\times\frac{1}{2^{32}}\right.$$

$$=1-\frac{\overset{529}{\cancel{5489}}}{2^{32}}$$

= ~~1-0.000~~

Ex ② In 256 sets of 12 tosses of fair coin in how many cases may one expects 8 head and 4 tails?

$\Rightarrow$ ~~n = 256~~ $n = 12$

$p = \text{head} = 0.5$

$q = \text{tails} = 0.5$

~~P(x=8~~

$$-P(X=8) = {}^{12}C_8\, p^8 q^4$$

$$= {}^{12}C_4 \left(\frac{1}{2}\right)^{12}$$

$$= \frac{\cancel{12}\times 11\times\overset{5}{\cancel{10}}\times 9}{\cancel{4}\times\cancel{3}\times\cancel{2}}$$

$$= \frac{55\times 9}{2^{12}}$$

$$= \frac{495}{2^{12}}$$

No. of ~~cha~~ cases $= 256 \times \frac{495}{2^{12}}$

$$= 2^8 \times \frac{495}{2^{12}}$$

$$= \frac{495}{2^4}$$

$$= 31$$

② Poisson distribution :-

The discrete random variable x is said to follow the poisson distribution if it has the probability

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad \lambda > 0$$

NOTE :-

① here n is infinetly large

② Mean :- $E(x) = \sum x_i \, P(X=x_i)$

$$= \sum x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

~~$= \sum x \, e^{\lambda}$~~

$$= e^{-\lambda} \sum_{x=1} \frac{\lambda^{x}}{(x-1)!}$$

$$= e^{-\lambda} \sum \lambda \cdot \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$= \lambda e^{-\lambda} \cdot [e^{\lambda}]$$

$$\boxed{E(x) = \lambda}$$

③ Variance $= \lambda$

④ mean = variance

⑤ MGF $= m_x(t) = e^{\lambda(e^t - 1)}$

⑥ $\boxed{\lambda = np}$

Ex. ① If X and Y are independent poisson variates such that $P(X=1) = P(X=2)$, $P(Y=1) = P(Y=3)$ Find variance of $(X-2Y)$

$\Rightarrow \quad P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

① $P(X=1) = \dfrac{e^{-\lambda}\lambda}{1!}$

② $P(X=2) = \dfrac{e^{-\lambda}\lambda^2}{2!}$

$P(X=1) = P(X=2)$

$\therefore \dfrac{\cancel{e^{-\lambda}}\lambda}{1!} = \dfrac{\cancel{e^{-\lambda}}\lambda^2}{2!}$

$\boxed{\lambda = 2}$

③ $P(Y=3) = \dfrac{e^{-\lambda}\lambda^3}{3!}$

④ $P(Y=1) = \dfrac{e^{-\lambda}\lambda}{1!}$

~~$P(Y=3) = P(Y=1)$~~

~~$\dfrac{e^{-\lambda}\lambda^3}{6} = \dfrac{e^{-\lambda}\lambda}{1}$~~

$\boxed{\lambda'^2 = 6}$

$\boxed{\lambda' = \sqrt{6}}$

$\therefore \boxed{\lambda \neq -ve}$

$$Var(aX \pm bY) = a^2 Var(X) + b^2 Var(Y) \pm 2ab\, Cov(X,Y)$$

$Var(X-2Y) = Var(X) + 4\,Var(Y) -$ ~~$2(2)\,Cov(X,Y)$~~

$= 2 + 4(\sqrt{6})$

$= 2 + 4\sqrt{6}$

$Cov(\cancel{X,Y}) = E(XY) - E(X) \cdot E(Y)$

$= 0 \quad - \cancel{\lambda \lambda'}$

Ex ② In a component manufacturing Industry there is a small chance of $\frac{1}{500}$ for any component to be defective.
The component are supplied in pockets of 10. Use poisson distribution to calculate the app. no. of pockets containing
- i) no defective
- ii) One defective
- iii) Two defective components respectively in ~~in~~ a consignment of 10,000 packets

⇒

$p \Rightarrow$ defective

$\lambda = np$

$= \cancel{10 \times} \left(1 - \frac{1}{500}\right) \quad 10 \times \frac{1}{500}$

$= \cancel{\frac{499}{50}} \quad \frac{1}{50}$

$=$

$P(X=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-1/50} = \frac{1}{e^{1/50}}$

$P(X=1) = \frac{e^{-\lambda}\lambda^1}{1!} = \frac{1}{50} e^{-1/50}$

$P(X=2) = \frac{e^{-\lambda}\lambda^2}{2!} = \frac{1}{(50)^2} \cdot \frac{e^{-1/50}}{2}$

① packet containing zero defective

$= 10000 \times e^{-1/50}$

$= 10000 \times 0.98$

$\approxeq 9800$ packet

② Packet contaning 1 defective

$= \overset{200}{\cancel{10000}} \times \frac{1}{\cancel{50}} \times e^{-1/50}$

$= 200 \times 0.98$

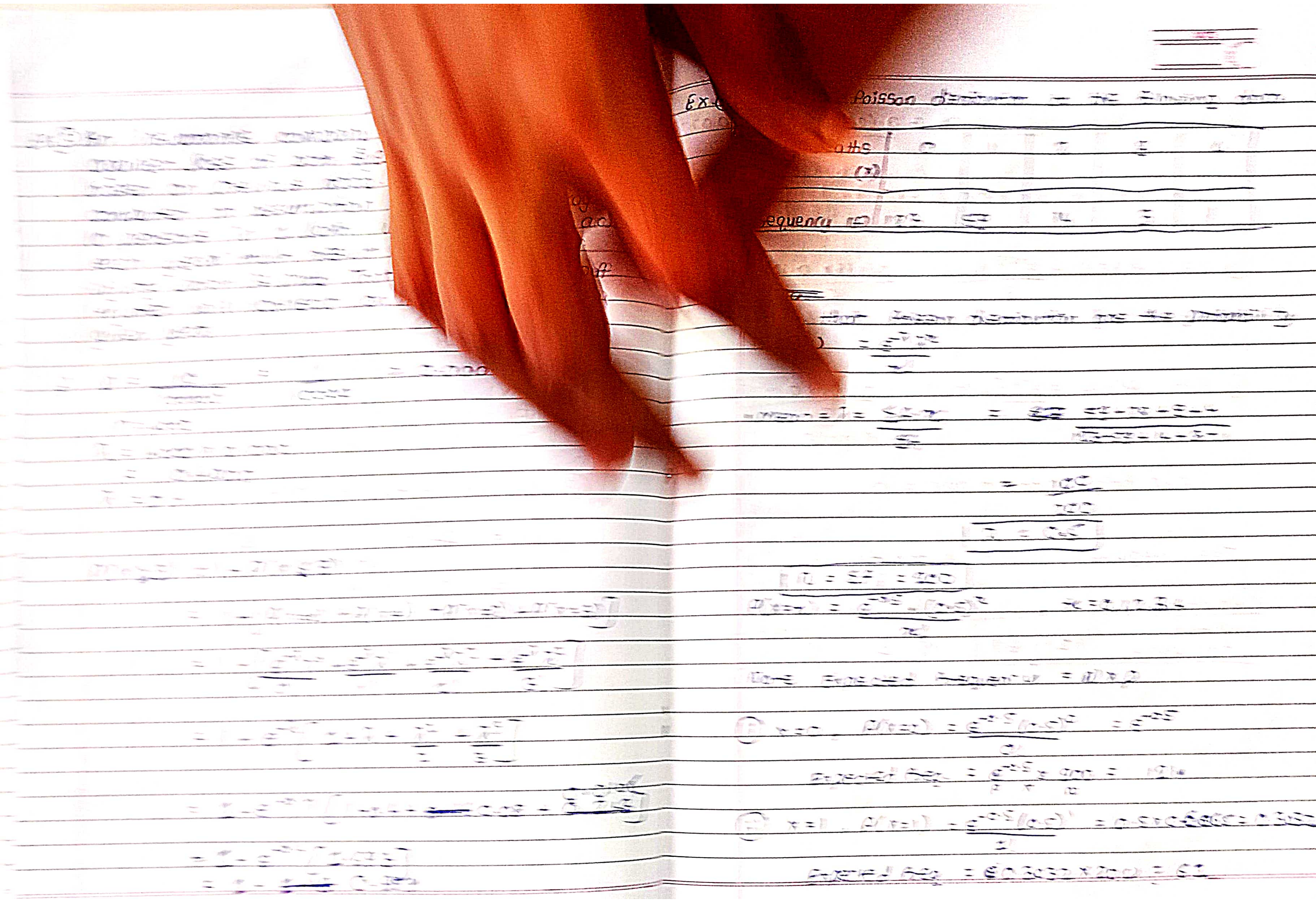
$= 196$ packet

③ Packet containing 2 defective

$= \overset{2}{\cancel{\overset{4}{\cancel{20}}}}\cancel{10000} \times \frac{1}{\cancel{50} \times \cancel{50} \times \cancel{2}} \quad e^{-1/50}$

$= 2 \times 0.98$

≈ 2 packet



Ex. [illegible] Poisson [illegible]

[illegible] deaths [illegible]

[illegible]equency [illegible]

Ex ③ An insuarance company ensure 4000 people against loss of both eyes in a car accident. based on previous data. The rates where computed or assumption that on average of 10 persons in 1 lakh will have car accident each year that result in this type of injury. what is prob. That more than 3 of injured will collect on their policy in a given year.

$\Rightarrow$ $p = \frac{10}{100000} = \frac{1}{10000} = 0.0001$

$n = 4000$

$\lambda = 4000 \times 0.0001$

$= 0.4000$

$\lambda = 0.4$

$P(X \geq 3) = 1 - P(X \leq 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - \left[\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda}\lambda^3}{3!}\right]$

$= 1 - e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3}\right]$

$= 1 - e^{-0.4}[1 + 0.4 + \cancel{0.16}\, 0.08 + \overset{0.0106}{\cancel{0.213}}]$

$= 1 - e^{-0.4}[1.693]$

$= 1 - \cancel{1.13}\ 0.999$

$= 0.0008$

Ex. ④ Fit a Poisson distribution to the following data.

No. of deaths (X)	0	1	2	3	4
frequency (f)	123	59	14	3	1

Solⁿ:

~~Mean~~

Note that Poisson distribution has the probability

$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

- Mean $= \lambda = \frac{\Sigma f_i x_i}{\Sigma f_i} = \cancel{\Sigma f}\ \frac{59 + 28 + 9 + 4}{123 + 59 + 14 + 3 + 1}$

$= \frac{100}{200}$

$\boxed{\lambda = 0.5}$

$\boxed{N = \Sigma f_i = 200}$

$P(X=x) = \frac{e^{-0.5} \cdot (0.5)^x}{x!}$, $x = 0, 1, 2, 3, 4$

Note Expected frequency $= N \times p$

ⓘ $x = 0$, $P(X=0) = \frac{e^{-0.5}(0.5)^0}{0!} = e^{-0.5}$

Expected freq. $= \underset{p}{e^{-0.5}} \underset{\times}{\times} \underset{N}{200} = 121$

(ii) $x = 1$, $P(X=1) = \frac{e^{-0.5}(0.5)^1}{1!} = 0.5 \times 0.6065 = 0.3032$

Expected freq $= \cancel{e}\ 0.3032 \times 200 = 61$

(iii) $x = 2$, $P(X=2) = \frac{e^{-0.5}(0.5)^2}{2!} = \frac{0.6065 \times 0.25}{2} = 0.0758$

Exp. freq. $= 200 \times 0.0758 = 15$

(iv) $x = 3$, $P(X=3) = \frac{e^{-0.5} \times (0.5)^3}{3!} = \frac{0.6065 \times 0.125}{6} = 0.0126$

Exp. freq $= 200 \times 0.0126 =$ ~~2~~ 3

(v) $X = 4$, $P(X=4) = \frac{e^{-0.5} \times (0.5)^4}{4!} = \frac{0.6065 \times 0.0625}{24}$

$= 0.0015$

Exp. freq $= 200 \times 0.0015 = 0$

The required poison distribution is

No of death	0	1	2	3	4
expeted frequencies	121	6·1	15	3	0

* Continous Distribution

① Normal Distribution

The random variable X is said to follow the Normal Distribution if X has the probabiliry density function.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

Note :-

① mean of $X = \bar{X} = m$

② Variance $= Var(X) = \sigma^2$

③ mean = median = mode

④ MGF $= M_X(t) = e^{\left(mt + \frac{t^2\sigma^2}{2}\right)}$

* Standard Normal variable :-

A R.V. X with the parameter m and σ,

$\boxed{Z = \frac{X-m}{\sigma}}$ is said to be standard normal variable.

* Area Property :-

If $X = m \Rightarrow Z = 0$

If $X = x_1 \Rightarrow Z = \frac{x_1 - m}{\sigma} = z_1$ (say)

~~The area will between~~

The area under the normal variates X between $x = m$ and $x = x_1$ is equals to the area under the standard normal curve Z.

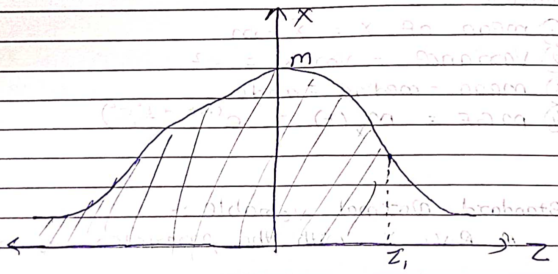
between $Z = 0$ to $Z = z_1$

i.e.,

$$P(m \le X \le x_1) = P(0 \le Z \le z_1) = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{1}{2}x^2} dx$$

Note :-

$$P(-z_1 \le Z \le z_2) = P(0 \le Z \le z_1) + P(0 \le Z \le z_2)$$



Ex. ① If x is normal variates with mean 10 and S.D. 4 find

① $P(5 \le x \le 18)$ ii) $P(x \le 12)$ iii) $P(|x-14| < 1)$

⇒ $m = 10$
$\sigma = 4$

Note that the standard normal variable is

$$Z = \frac{x - m}{\sigma}$$

① $x = 5$

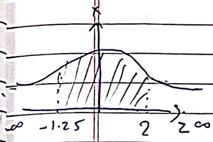
$$z_1 = \frac{5-10}{4} = \frac{-5}{4} = -1.25$$

$x = 18$ ⇒

$$z_2 = \frac{18-10}{4} = 2$$

$\therefore P(5 \le x \le 18) = P(-1.25 \le Z \le 2)$

$= P(-\infty \le Z \le 2) - P(-\infty \le Z \le -1.25)$



$= 0.9772 - 0.1056$

$= 0.8716$

② $P(x \le 12) = P(-\infty \le x \le 12)$

$x = 12 \Rightarrow \frac{12-10}{4} = \frac{2}{4} = 0.5$

$\therefore P(-\infty \le Z \le 0.5) = 0.6915$

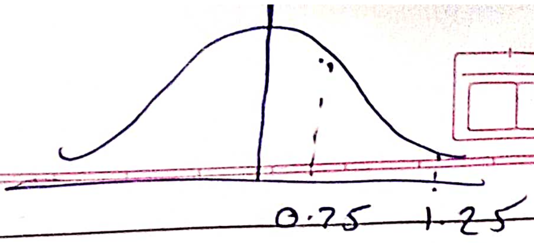
③ $P(|x-14| < 1) = P(-1 \le x - 14 \le 1)$

~~At x − 14, $z_1 = x - 14 - 10$~~

$= P(13 \le x \le 15)$

At $x = 13$, $z_1 = \frac{13-10}{4} = \frac{3}{4} = 0.75$

At $x = 15$, $z_1 = \frac{15-10}{4} = \frac{5}{4} = 1.25$



$\therefore P(0.75 \leq Z \leq \cancel{0}\,1.25)$

$= P(-\infty \leq Z \leq \overset{1.25}{\cancel{0.75}}) \;\cancel{+} - P(-\infty \leq Z \leq \overset{0.75}{\cancel{1.25}})$

$= \cancel{0.7734 =}\; 0.8944 - 0.7734$

$= 0.121$