S. No.	Date	Title	Page No.	Teacher's Sign / Remarks

Introduction to Probability

Experiment: An activity that gives us a result is called experiment. It is a test, trials or procedure for the purpose of discovering something unknown.

Random Experiment: In each trial of an experiment conducted under identical conditions the outcomes is not unique but maybe any one of the possible outcomes. Such experiment is called random experiment.

Sample Space (S): The set of all possible outcomes of a random experiment is called sample space.

Event: The subset of sample space is called event
For example: Tossing two coins
Sample space $S = \{HH, HT, TH, TT\}$
Event A. Atleast one head $= \{HH, HT, TH\}$

Probability: Let S be the sample space and A be the event of a sample space S, then the probability of A is denoted by P(A) and is defined as:

$$P(A) = \frac{\text{number of outcomes of A}}{\text{number of outcomes of S}}$$

Note: ① $0 \leq P(A) \leq 1$
② $P(S) = 1$

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Addition Rule of Probability

Let A, B, C be events of sample space S then

1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

Mutually exclusive events: The events A and B of a sample space S are said to be mutually exclusive if A and B are disjoint i.e. ~~of~~ $A \cap B = \emptyset$ (empty) or $P(A \cap B) = 0$.

For example: Throw of a dice

Sample space $(S) = \{1, 2, 3, 4, 5, 6\}$

A = getting odd number = $\{1, 3, 5\}$

B = getting even number = $\{2, 4, 6\}$

Then $(A \cap B) = \emptyset$

$\Rightarrow P(A \cap B) = 0$

A, B are mutually exclusive.

18-07-24 **Mutually Exhaustive Events:** The events A, B of a sample space S is said to be mutually exhaustive if $A \cup B = S$

Independent events: The events A and B of a sample space S are said to be independent if the occurrence of event A ~~is~~ does not affect the occurence of event B and vice versa.

For example: Tossing one coin and one dice

$S = \{(H,1), (H,2), (H,3) \ldots\ldots\ldots (T,6)\}$

$|S| = 12$

A = getting head = $\{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)\}$

B = number less than 4 $= \{(H,1), (H,2), (H,3), (T,1), (T,2), (T,3)\} = |6|$

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Note: When events A and B are independent if $P(A \cap B) = P(A)\ P(B)$

$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap \ldots\ldots \cap A_n) = P(A_1)\ P(A_2) = P(A_3) \ldots\ P(A_n)$

Complement of event: The complement of event A of a sample space S is denoted by A^c or A' and is defined as $A' = S - A$

$P(A') = 1 - P(A)$

Conditional probability: If probability of event A provided that the event B has ~~alw~~ already occured is called the conditional ~~prop~~ probability and is defined as ~~P(A)~~ $P(A/B) = \frac{P(A \cap B)}{P(B)}$ $P(B) \neq 0$

Similarly, $P(B/A) = \frac{P(A \cap B)}{P(A)}$ $P(A) \neq 0$

Note: ① If A, B are independent then

$P(A/B) = \frac{P(A)\ \cancel{P(B)}}{\cancel{P(B)}} = P(A)$

$\therefore P(B/A) = P(B)$

② If A, B are mutually exclusive then

$P(A/B) = 0$ $P(B/A) = 0$

Multiplication rule of probability:

1) $P(A \cap B) = P(B)\ P(A/B) = P(A)\ P(B/A)$

A box contains 4 bad and 6 good tubes. 2 are drawn out ~~of~~ from the box at a time. One of them is tested and found to be good. What is the probability that other one is also good.

Let A = first drawn tube is good.
B = second drawn tube is good.

$P(A) = \frac{6}{10} = \frac{3}{5}$

$P(B) = \frac{5}{9}$

$P(A \cap B) = \frac{^6C_2}{^{10}C_2} =$

$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{^6C_2}{^{10}C_2}}{\frac{6}{10}} = \frac{5}{9}$

In a random experiment $P(A) = \frac{1}{12}$ and $P(B) = \frac{5}{12}$, $P(B/A) = \frac{1}{15}$. find $P(A \cup B)$

$P(B/A) = \frac{P(A \cap B)}{P(A)}$

$\frac{1}{15} \times \frac{1}{12} = P(A \cap B)$

$P(A \cap B) = \frac{1}{180}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{1}{12} + \frac{5}{12} - \frac{1}{180}$
$= \frac{15 + 75 - 1}{180} = \frac{89}{180}$

~~If $B_1, B_2, \ldots B_n$ are mutually exclusive and exhaustive event of a sample space S and A be associated with the events $B_1, B_2 \ldots B_n$ then $P(A) = \Sigma B_i\ \Sigma P(B_i)\ P(A$~~

Total Probability Theorem:

If $B_1, B_2, \ldots B_n$ are mutually exclusive and exhaustive event of a sample space S and A be associated with the events $B_1, B_2 \ldots B_n$ then
$P(A) = \sum_{i=1} P(B_i)\, P(A/B_i)$

~~Bayere~~ Baye's Theorem:

Let $A_1, A_2 \ldots A_n$ be the mutually exclusive and exhaustive event of a sample space S such that $P(A_i) \neq 0$ for every $i = 1, 2, \ldots n$ and ~~P(B)~~ B be any other event associated with the events $A_1, A_2 \ldots A_n$ such that $P(B) \neq 0$ then

$$P(A_i/B) = \frac{P(A_i) \times P(B/A_i)}{\Sigma P(A_i)\, P(B/A_i)}$$

19-07-24 A box contains 7 red and 13 blue balls. 2 balls are selected randomly and are discarded without their colours being seen. If a 3rd ball is drawn randomly and observed to be red, what is the probability that both the discarded ball were blue.

Let A_1 be both balls are red
A_2 be both balls are red and blue
A_3 be both balls are blue.
B = Third ball drawn is red.

$P(A_1) = \frac{{}^7C_2}{{}^{20}C_2} = \frac{21}{190}$

$P(A_2) = \frac{{}^7C_1 \cdot {}^{13}C_1}{{}^{20}C_2} = \frac{91}{190}$

$P(A_3) = \frac{{}^{13}C_2}{{}^{20}C_2} = \frac{78}{190}$

$P(B/A_1) = \frac{{}^5C_1}{{}^{18}C_1} = \frac{5}{18}$

$P(B/A_2) = \frac{{}^6C_1}{{}^{18}C_1} = \frac{6}{18}$

$P(B/A_3) = \frac{{}^7C_1}{{}^{18}C_1} = \frac{7}{18}$

∴ By Baye's theorem

$$P(A_3/B) = \frac{P(A_3) \times P(B/A_3)}{\sum_{i=1}^{3} P(A_i) \times P(B/A_i)} = \frac{P(A_3)\times P(B/A_3)}{\cancel{P(A_1)\times} + P(A_2)\times P(B/A_2) + P(A_3)\times P(B/A_3)}$$

$$= \frac{\frac{78}{190} \times \frac{7}{18}}{\frac{105}{3420} + \frac{546}{3420} + \frac{546}{3420}}$$

$$= \frac{546}{1197}$$

$$= \frac{26}{57}$$

In a bag 3 true coins and 1 false coin with head on both side. A coin is chosen randomly and tossed 4 times. If head occurs all the time what is the probability that false coin was chosen and used.

A_1 = True coin selected

A_2 = False coin selected

B = Head on 4 times toss

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$P(A_1) = \frac{3}{4}$

$P(A_2) = \frac{1}{4}$

$P(B/A_1) = \frac{{}^2C_1}{4}$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

$P(B/A_2) = 1$

$$P(A_2/B) = \frac{P(A_2) \times P(B/A_2)}{P(A_1) \times P(B/A_1) + P(A_2) + P(B/A_2)}$$

$$= \frac{\frac{1}{4} \times 1}{\frac{3}{4} \times \frac{1}{16} + \frac{1}{4}}$$

$$= \frac{\frac{1}{4}}{\frac{3+4}{16}}$$

~~$= \frac{1}{4} \times \frac{8}{5} = \frac{2}{5}$~~

~~$\frac{1}{4} \times \frac{16}{7}$~~

$$\frac{\frac{1}{4}}{\frac{19}{64}} = \frac{16}{19}$$

The chance that doctor A will diagnose a disease X correctly is 60%. The chance that a patient will die by his treatment after correct diagnosis is 40% and chance of death by wrong diagnosis is 70%. A patient of doctor A who had disease X, died. What is the chance that his ~~dea~~ disease was diagnosed correctly.

A_1 - correct diagnose

A_2 - wrong diagnose

B - death

$P(A_1) = 60\%$

$P(A_2) = 40\%$

$P(B/A_1) = 40\%$

$P(B/A_2) = 70\%$

$$P(A_1/B) = \frac{P(A_1) \times P(B/A_1)}{P(A_1) \times P(B/A_1) + P(A_2)\, P(B/A_2)}$$

$$= \frac{0.6 \times 0.4}{0.24 + 0.4 \times 0.7}$$

$$= \frac{0.24}{0.52}$$

$$= \frac{6}{13}$$

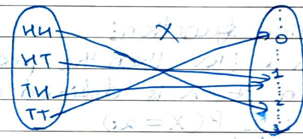
Random ~~Experiment~~ Variable and its Distribution

Random Variable: A random variable is a function that maps from the sample space of random experiment to the set of real number

For example: Tossing 2 coins

$S = \{HH, TH, HT, TT\}$

$X =$ Number of heads



Discrete random variable: ~~Available~~ A variable X is said to be if X takes finite or countably infinite values $x_0, x_1, \ldots\ldots, x_n, \ldots\ldots$

NOTE:

1) If X_1 and X_2 are two random variables associated with same sample S then for any constant c_1, c_2 c_1X_1 ~~is also~~ $+ c_2X_2$ is also random variable.

2) If X is random variable then $\frac{1}{X}$ and $|X|$ is also random variable.

Expectation [E(X)]: Let X with a discrete random variable the expectation of X is defined as

$$E(X) = \sum_i P(X_i) \times X_i$$

Mean: Mean of $X = E(X)$

Varience: Varience of X $(V(X)) = E(X^2) - [E(X)]^2$

Probability mass function (p.m.f)

Let capital X is discrete random variable (DRV) and $x_0, x_1, \ldots, x_n, \ldots$ are the values of X and $p(x_0), p(x_1), \ldots, p(x_n), \ldots$ are the corresponding probabilities then the function P is called probability mass function if ~~$P(x_i) \geq 0$ for all i~~

1) $P(X_i) \geq 0$ for all $i = 1, 2, 3, \ldots, n, \ldots$
2) $\sum P(X_i) = 1$

Probability distribution function:

~~If X~~ Let X with a DRV then the probability distribution function of X is defined as

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i)$$

24-07-24 **Example**

If X is D.R.V with the following probability distribution

x	1	2	3	4	5	6	7
$P(X < x)$	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

Find i) k ii) $P(x<5)$ iii) $P(x<5 / 2<X\leq 6)$

i) Note that $\sum P(X_i) = 1$

$$k + 2k + 3k + k^2 + k^2 + k + 2k^2 + 4k^2$$
$$8k^2 + 7k = 1$$
$$8k^2 + 7k - 1$$
$$8k^2 + 8k - k - 1$$
$$8k(k+1) - 1(k+1) = 0$$
$$k = \frac{1}{8} \text{ or } k = -1$$
Not possible

$$k = \frac{1}{8} \quad \because P(x_i) \geq 0$$

~~$\therefore P(X_i$~~

ii) $P(X < 5) = P(X = 1, 2, 3, 4)$
$$= k + 2k + 3k + k^2$$
$$= k^2 + 6k$$
$$= \frac{1}{64} + \frac{6}{8}$$
$$= \frac{49}{64}$$

iii) ~~$P(X<5 / 2<X\leq 6) = P($~~

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

$$\therefore P(X<5 / 2<X\leq 6) = \frac{P(X=3,4)}{P(2<X\leq 6)}$$
$$= \frac{3k+k^2}{3k+k^2+k^2+k+2k^2}$$
$$= \frac{k(k+3)}{4k^2+4k}$$
$$= \frac{k(k+3)}{4k(k+1)}$$
$$= \frac{\frac{1}{8}+3}{4\left(\frac{1}{8}+1\right)}$$
$$= \frac{\frac{25}{8}}{\frac{36}{8}} = \frac{25}{36}$$

The probability distribution of random variable X is

X	0	1	2	3
$P(X=x)$	0.1	0.3	0.5	0.1

If $Y = X^2 + 2X$ find $P(Y \leq 5)$ and variance of Y.

If $Y = X^2 + 2X$ find $P(Y \le 5)$

y	0	3	8	15
x	0	1	2	3
$P(X=x)$	0.1	0.3	0.5	0.1

$P(Y \le 5) = \frac{0.1 + 0.3}{1}$

$= \frac{0.4}{1} = \frac{4}{10}$

$E(Y) = 3\times0.3 + 8\times0.5 + 15\times0.1$
$= 0.9 + 4 + 1.5$
$= 6.4$

Variance $= E(Y^2) - (E(Y))^2$
$= 9\times0.3 + 64\times0.5 + 225\times0.1 - (6.4)^2$
$= 2.7 + 32 + 22.5 - (6.4)^2$
$= 57.2 - 40.96$
$= \cancel{48.16}\ 16.24$

$57.2 - 40.96 = 16.24$

If random variable X takes the values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$ find the probability distribution and mean.

Let $k = P(X=3)$

X	1	2	3	4
$P(X)$	$\frac{k}{2}$	$\frac{k}{3}$	k	$\frac{k}{5}$

$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$

$15k + 10k + 30k + 6k = 30$

$61k = 30$

$k = \frac{30}{61}$

X	1	2	3	4
$P(X)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

$E(X) = \frac{15}{61} + 2\times\frac{10}{61} + \frac{30\times3}{61} + 4\times\frac{6}{61}$

$= \frac{15 + 20 + 90 + 24}{61}$

$= \frac{149}{61}$

Examples of Total Probability Theorem

In ~~an~~ a coin tossing experiment if coin shows head, one dice is is thrown and its number is recorded. If the coin shows tail, two dices are thrown and sum of numbers is recorded. What is the probability that the recorded number will be two

$\frac{1}{2}\times\frac{1}{6} + \frac{1}{2}\times\frac{1}{36}$

$\frac{1}{12} + \frac{1}{72} = \frac{1+6}{72} = \frac{7}{72}$

By total probability theorem, $P(A) = \sum P(B_i)\, P(A/B_i)$

26-07-24 **Continous Random Variable:** Random Variable X is said to be continous if X takes uncountably infinite values in the given interval

Probability Density Function (PDF)
Let X be the Continous Random Variable (CRV) and x be the continous function define on [a, b] then PDF is of X is defined as
$P(\alpha \le X \le \beta) = \int_{\alpha}^{\beta} f(x)\,dx$, $a < \alpha < \beta < b$

Expectation:
If X is CRV the $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

* Mean = $E(X)$
* Variance = $E(X^2) - (E(X)^2)$

Cumulative Distribution Function (CDF):
Let $f(x)$ is a continous function of $[a,b]$ and X is a CRV then cumulative distribution function of X with PDF $f(x)$ is defined as

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

1) A random variable X has PDF
$$f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$
~~f(x)~~ Find i) $P(X<\frac{1}{2})$ ii) $P(\frac{1}{4}<X<\frac{1}{2})$
iii) $P(X>\frac{1}{3} / X>\frac{1}{2})$ iv) CDF of X

$$P(\alpha < x < \beta) = \int_{\alpha}^{\beta} f(x)\,dx$$

i) $P(X<\frac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\frac{1}{2}} 2x\,dx$

$= [x^2]_0^{\frac{1}{2}}$

$= \frac{1}{4}$

ii) $P(\frac{1}{4} < X < \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} 2x\,dx$

$= [x^2]_{\frac{1}{4}}^{\frac{1}{2}}$

$= \left[\frac{4}{16} - \frac{1}{16}\right]$

$= \frac{3}{16}$

iii) $P(A/B) = \frac{P(A\cap B)}{P(B)}$

$= \frac{P(X>\frac{1}{3}) \cap P(X>\frac{1}{2})}{P(X>\frac{1}{2})}$

$= \frac{P(X>\frac{1}{2})}{P(X>\frac{1}{2})}$

$= \frac{\cancel{\int_{\frac{1}{2}}^{\infty} 2x\,dx}}{\cancel{\int_{\frac{1}{2}}^{\infty} 2x\,dx}} = 1$

iv) $x < 0$, $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \int_{-\infty}^{x} 0\,dx = 0$

$0 \le x < 1$ $\quad F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} 2x\,dx$

$= x^2$

$x \ge 1$ $\quad F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 2x\,dx + \int_{1}^{x} 0\,dx$

$= 0 + [x^2]_0^1 + 0$

$= 1$

$$F(x) = \begin{cases} 0, & x \le 0 \\ x^2, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$

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Ex② The PDF of R.V. X is $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & x > 2 \end{cases}$

Find CDF of X.

Note that CDF of X is

$F(x) = \int_{-\infty}^{x} f(x)\,dx$

if $x < 0$, $f(x)$ $\int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0\,dx = 0$

if $0 < x < 1$, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} x\,dx$

$= \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$

if $1 \le x < 2$; $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{x} 2-x\,dx$

$= 0 + \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$

$= \frac{1}{2} + 2x - \frac{x^2}{2} - \left[2 - \frac{1}{2}\right]$

$= 2x - \frac{x^2}{2} - 1$

if $x \ge 2$, $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} 2-x + \int_{2}^{x} 0\,dx$

$= 0 + \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 + 0$

$= \frac{1}{2} + 4 - 2 - \left[2 - \frac{1}{2}\right] + 0$

$= 1 + \cancel{4} - \cancel{2} - \cancel{2}$

$= 1$

The CDF of X is

$$F(x) = \begin{cases} 0 & x \le 0 \\ x^2/2 & 0 \le x < 1 \\ 1 - \frac{(2-x)^2}{2} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

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A R.V. X has a PDF $f(x) = k\,x(1-x)$, $0 \le x \le 1$

find i) k ii) determine the number 'b' such that $P(X \ge b) = P(X \le b)$

Note that $P(-\infty < x < \infty) = 1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} k\,x(1-x)\,dx + \int_{1}^{\infty} 0\,dx$$

$$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$k\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

$$k \cdot \frac{1}{6} = 1$$

$$k = 6$$

ii) $P(x \ge b) = P(x \le b)$

$P(b \le x < 1) = P(0 < x \le b)$

$$\int_b^1 f(x)\,dx = \int_0^b 6x(1-x)\,dx$$

$$\cancel{6}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_b^1 = \cancel{6}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b$$

$$\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right) = \left(\frac{b^2}{2} - \frac{b^3}{3}\right) - 0$$

$$\cancel{\frac{1}{2} - \frac{1}{3}}\ \frac{1}{6} = b^2 - \frac{2b^3}{3}$$

$$\frac{2b^3}{3} - b^2 + \frac{1}{6}$$

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$$4b^3 - 6b^2 + 1 = 0$$

$$b = \frac{1}{2}, \cancel{\frac{1+\sqrt{3}}{2}}, \cancel{\frac{1-\sqrt{3}}{2}}$$

$\because$ b is between $[0, 1]$

Two Dimensional Random Variable

Let X and Y be the random variable defined on same sample space then the function (X, Y) that maps S to the R^2 $(R \times R)$ is called two dimensional random variable.

31-07-24 Note: $\{X \le a, Y \le b\}$ denotes the event of all ~~element~~ elements $s \in S$ such that $X(s) \le a$ and $Y(s) \le b$

① ~~Probability~~ Discrete Random Variable:

Join probability distribution of (X, Y)

Let (X, Y) be 2-dimensional discrete random variable then the element $X(S) = (x_1, x_2, \ldots, x_n)$ and $Y(S) = (y_1, y_2, \ldots\ y_m)$ are image sets. Hence the function P on sets $X(S) \times Y(S)$ is defined by ~~$f(x_i, y_j)$~~ $P(x_i, y_j) = P(X = x_i, Y = y_j) = P_{ij}$ and is represented by using the following way.

X \ Y	y_1	y_2	......	y_m	Total
x_1	P_{11}	P_{12}	......	P_{1m}	P_{1*}
x_2	P_{21}	P_{22}	......	P_{2m}	P_{2*}
$\vdots$	$\vdots$	......	......	$\vdots$	$\vdots$
x_n	P_{n1}	P_{n2}	......	P_{nm}	P_{n*}
Total	P_{*1}	P_{*2}	......	P_{*m}	1

Two dimensional probability mass function

Let (X,Y) be 2 dimensional DRV then the function P is said to be 2 dimensional pmf if

① $P_{ij} \geq 0$ for all i,j

② $\sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij} = 1$

Cumulative Distribution Function (Two dimensional probability distribution function)

Let (X,Y) be 2dimensional DRV then the cumulative distribution function of (X,Y) is denoted by F(x,y) and is given by

$$F(x,y) = P(X\le x, Y\le y) = \cancel{\sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}} \sum_{x_i\le x}\sum_{y_j\le y} P(x_i,y_j)$$

Marginal probability distribution:

1) The marginal probability distribution of X is

$$P(X=x_i) = \sum_{j=1}^{m} P_{ij} = P_{i*} = P_{i1}+P_{i2}+P_{i3}+\dots+P_{im}$$

2) The marginal probability distribution of Y is

$$P(Y=y_j) = \sum_{i=1}^{n} P_{ij} = P_{*j} = P_{1j}+P_{2j}+P_{3j}+\dots+P_{nj}$$

Conditional Probability distribution:

1) The c.p. of X given that $Y=y_j$ is

$$P(X=x_i/Y=y_j) = \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)}$$

2) The c.p. of Y given that $X=x_i$ is

$$P(Y=y_j/X=x_i) = \frac{P(X=x_i, Y=y_j)}{P(X=x_i)}$$

③ Independent Random Variable

The variable X and Y are said to be independent if $P_{ij} = P_{i*}\cdot P_{*j}$ for all i,j

i.e. $P(X=x_i, Y=y_j) = P(X=x_i)\cdot P(Y=y_j)$ for all i,j

Example ①

The joint probability mass function (X,Y) is given by $P(x,y) = k(2x+3y)$, $x = 0,1,2$ and $y = 1,2,3$

Find marginal and condition probability distribution for

① $P(X=2, Y\le 2)$

② $P(X\le 2, Y=3)$

③ $P(X=2)$

④ $P(Y=2)$

⑤ $P(X\le 1/Y\le 2)$

⑥ $P(Y\le 2/X\le 2)$

Soln. The probability distribution for (X,Y) is

X \ Y	1	2	3	Total
0	3k	6k	9k	18k
1	5k	8k	11k	24k
2	7k	10k	13k	30k
Total	15k	24k	33k	

$$\sum_{i=1}^{3}\sum_{j=1}^{3} P_{ij} = 1$$

$$72k = 1$$

$$k = \frac{1}{72}$$

X \ Y	1	2	3	Total
0	$\frac{3}{72}$	$\frac{6}{72}$	$\frac{9}{72}$	$\frac{18}{72}$
1	$\frac{5}{72}$	$\frac{8}{72}$	$\frac{11}{72}$	$\frac{24}{72}$
2	$\frac{7}{72}$	$\frac{10}{72}$	$\frac{13}{72}$	$\frac{30}{72}$
Total	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$	

1) $P(X=2, Y\le 2) = P(X=2, Y=1) + P(X=2, Y=2)$
$= \frac{7}{72} + \frac{10}{72} = \frac{17}{72}$

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2) $P(X\le 2, Y=3) = P(X=0, Y=3) + P(X=1, Y=3) + P(X=2, Y=3)$
$= \frac{9+11+13}{72} = \frac{33}{72}$

3) $P(X=2) = \frac{7}{72} + \frac{10}{72} + \frac{13}{72} = \frac{30}{72} = \frac{5}{12}$

4) $P(Y=2) = \frac{6}{72} + \frac{8}{72} + \frac{10}{72} = \frac{24}{72} = \frac{1}{3}$

5) $P(X\le 1 / Y\le 2) = \dfrac{P(X=0,Y=1) + P(X=0, Y=2) + P(X=1, Y=1) + P(X=1, Y=2)}{P(Y=1) + P(Y=2)}$
$= \dfrac{\frac{3}{72} + \frac{6}{72} + \frac{5}{72} + \frac{8}{72}}{\frac{15}{72} + \frac{24}{72}}$
$= \frac{22}{39}$

6) $P(Y\le 2 / X\le 2) = \dfrac{P(X=0, Y=1) + P(X=1, Y=1) + P(X=2, Y=1) + P(X=0, Y=2) + P(X=1, Y=2) + P(X=2, Y=2)}{P(X=0) + P(X=1) + P(X=2)}$
$= \dfrac{\frac{3}{72} + \frac{5}{72} + \frac{7}{72} + \frac{6}{72} + \frac{8}{72} + \frac{10}{72}}{\frac{18}{72} + \frac{24}{72} + \frac{30}{72}}$
$= \dfrac{\frac{39}{72}}{72} = \frac{39}{72}$

Example ②

Given the joint probability distribution

Y \ X	-1	0	1
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

a) find marginal probability distribution of X and Y
b) Find conditional probability distribution of Y given that $X=0$
c) check whether X, Y are independent

a) Marginal probability $P(X) = \sum_{i=-1} P_{ij}$
$= P_{-1}$

$P_{X=-1} = \frac{1}{15} + \frac{3}{15} + \frac{2}{15} = \frac{6}{15} = \frac{2}{5}$

$P(X=0) = \frac{2}{15} + \frac{2}{15} + \frac{1}{15} = \frac{1}{3}$

$P(X=1) = \frac{1}{15} + \frac{1}{15} + \frac{2}{15} = \frac{4}{15}$

b) $P(Y\le 2 / X=0) = \dfrac{P(Y=1, X=0) + P(Y=2, X=0) + P(Y=0, X=0)}{P(X=0)}$
$= \dfrac{\frac{2}{15} + \frac{1}{15} + \frac{2}{15}}{\frac{5}{15}}$
$= \cancel{\frac{5}{5}} \; 1$

c) $P(X=-1, Y=0) = \frac{1}{15}$, $P(X=-1) \cdot P(Y=0) = \frac{6}{15} \cdot \frac{4}{15} = \frac{24}{225}$

$P(X=-1, Y=0) \neq P(X=-1) \cdot P(Y=0)$

$P_{ij} \neq P_{i*} \cdot P_{*j}$ for all i,j

$\therefore$ X and Y are not independent random variables.

Ex ③ The joint probability distribution of (X,Y) is given by

X \ Y	1	3	5	
2	0.10	0.20	0.10	0.4
4	0.15	0.30	0.15	0.6
	0.25	0.5	0.25	

Check whether X and Y are independent

$P_{ij} = P_{i*} \cdot P_{*j}$

$P(X=2, Y=1) = 0.10$ $\qquad P(X=2) \cdot P(Y=1) = 0.4 \times 0.25 = 0.1$

$P(X=2, Y=3) = 0.2$ $\qquad P(X=2) \cdot P(Y=3) = 0.4 \times 0.5 = 0.2$

$P(X=2, Y=5) = 0.1$ $\qquad P(X=2)\ P(Y=5) = 0.4 \times 0.25 = 0.1$

$P(X=4, Y=1) = 0.15$ $\qquad P(X=4)\ P(Y=1) = 0.6 \times 0.25 = 0.15$

$P(X=4, Y=3) = 0.3$ $\qquad P(X=4)\ P(Y=3) = 0.6 \times 0.5 = 0.3$

$P(X=4, Y=5) = 0.15$ $\qquad P(X=4)\ P(Y=5) = 0.6 \times 0.25 = 0.15$

$\therefore P_{ij} = P_{i*} \times P_{*j}$ for all i,j

$\therefore$ It is independent event

02-08-24 Find probability distribution function for the following probability distribution of X, Y

X \ Y	1	3	5
2	0.1	0.2	0.1
4	0.15	0.3	0.15

Note that the probability distribution function of (x,y) is

$F(X,Y) = P(X \le x, Y \le y) = \sum_{x_i \le x} \sum_{y_j \le y} P(X=x_i, Y=y_j)$

if $x=2, y=1$, $F(x,y) = P(x \le 2, y \le 1) = 0.1$

~~$F(X,Y) = P(x \le x, Y \le y)$~~

if $x=2, y=3$, $F(x,y) = P(x \le 2, y \le 3) = 0.3$

if $x=2, y=5$, $F(x,y) = P(x \le 2, y \le 5) = 0.4$

if $x=4, y=1$, $F(x,y) = P(x \le 4, y \le 1) = \cancel{0.3}\ 0.25$

if $x=4, y=3$, $F(x,y) = P(x \le 4, y \le 3) = 0.75$

if $x=4, y=5$, $F(x,y) = P(x \le 4, y \le 5) = 1$

$\therefore$ The PDF

$$F(x,y) = \begin{cases} 0.10 & x=2, y=1 \\ 0.30 & x=2, y=3 \\ 0.40 & x=2, y=5 \\ 0.25 & x=4, y=1 \\ 0.75 & x=4, y=3 \\ 1 & x=4, y=5 \end{cases}$$

Continuous Random Variable (c.r.v.)

Two dimensional joint probability density function

Let (X,Y) be the 2 dimensional c.r.v. and $f(x,y)$ be the 2 variable (bivariate) function such that
$P(x-\frac{dx}{2} < X < x+\frac{dx}{2},\ y-\frac{dy}{2} < Y < y+\frac{dy}{2}) = d\,f(x,y)$
then $f(x,y)$ is said to be joint P.D.F. if

① $f(x,y) \geq 0$ for all x,y

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$

~~P(a<X<b, c<Y~~

$$P(a < X \leq b,\ c < Y < d) = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Marginal probability density function (m.p.d.f.)

m.p.d.f. of x is defined as $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Conditional probability density function (c.p.d.f.)

c.p.d.f. of X given that Y is defined as

$$f(x/y) = \frac{f(x,y)}{f_X(y)}$$

c.p.d.f. of Y given that X is defined as

$$f(y/x) = \frac{f(x,y)}{f_Y(x)}$$

Probability distribution function (c.d.f.) (cumulative distributive function)

Let X,Y be the 2 dimensional c.r.v. and $f(x,y)$ be the joint ~~p.d.f.~~ probability density function then c.d.f. is denoted by x,y

$$F(x,y) = P(X<x, Y<y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$

Independent Random Variables:

The random variable X and Y are said to be independent if $f(x,y) = f_X(x) \times f_Y(y)$

Note:

1) If X,Y are independent then
$f(x/y) = f_Y(y)$, $f(y/x) = f_X(x)$

Example ①. Find the value of k if joint probability density function is
$$f(x,y) = \begin{cases} k(1-x)(1-y), & 0<x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_1^5\int_0^4 k(1-x)(1-y) = 1$$

$$k\int_1^5\int_0^4 (1-y)\left[x-\frac{x^2}{2}\right]_0^4$$

$$-4k\int_1^5\left[y-\frac{y^2}{2}\right]_1^5$$

$$-4k\ \ 5-\frac{25}{2}-1+\frac{1}{2}$$

$$-4k\,[4-12] = 1$$

~~$32k$~~ $= 1$ $\qquad 32k = 1$

~~$k = \frac{1}{48}$~~ $\qquad k = \frac{1}{32}$

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Example ② $f(x,y) = kye^{-x}, \quad 0 < x < \infty$
$0 < y < 1$

$0 \quad$ otherwise

$$\int_0^1 \int_0^\infty k y e^{-x} = 1$$

$$k \int_0^1 y \left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$-k \int_0^1 y (0-1)$$

$$k \int_0^1 y = 1$$

$$k \left[\frac{y^2}{2}\right]_0^1 = 1$$

$$k = 2$$

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Ex ③ Given the join probability density function
$$f(x,y) = \begin{cases} xe^{-x(y+1)} & 0 < x < 4, \ 1 \leq y < 5 \\ 0 \end{cases}$$

ⓐ find marginal probability density function
ⓑ find conditional probability function
ⓒ check whether X and Y are independent

Marginal p.d.f. of X is
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-\infty}^{1} 0\, dy + \int_1^5 xe^{-x(y+1)}$$

$$= \cancel{x} \left[\frac{e^{-x(y+1)}}{-\cancel{x}}\right]_1^5$$

$$= \left[\frac{e^{-6x} - e^{-2x}}{-1}\right]$$

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$$e^{-2x} - e^{-6x}$$

Marginal p.d.f.
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^4 xe^{-x(y+1)}\, dx$$

$$= x \int_0^4 e^{-x(y+1)}\, dx - \int_0^4 \frac{dx}{dx} \int_0^4 e^{-x(y+1)}\, dx$$

$$= x \int_0^4 \left(\frac{e^{-x(y+1)}}{y+1}\right)_0^4 - \int_0^4 \left(\frac{e^{-x(y+1)}}{y+1}\right)_0^4 dx$$

$$= \frac{xe^{-x(y+1)}}{y+1} - \int_0^4 \frac{e^{-4(y+1)}}{y+1}\, dx - \frac{1}{y+1}\, dx$$

Conditional probability
$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{xe^{-x(y+1)}}{e^{-2x} - e^{-6x}}$$

$$f(x/y) = \frac{xe^{-x(y+1)} (y+1)^2}{1 - e^{-4(y+1)} - 4e^{-4(y+1)}(y+1)}$$

$$= \frac{xe^{-x(y+1)}(y+1)^2}{1 - \cancel{5e^{-4(y+1)}}\, e^{-4(y+1)}(1 - 4(y+1))}$$

If X and Y are independent if $f(x,y) = f_X(x) \cdot f_Y(y)$

$$(e^{-2x} - e^{-6x}) \cdot \left(\frac{1-e^{-4(y+1)}}{(y+1)^2} - \frac{4e^{-4(y+1)}}{(y+1)} \right)$$

$f_x(x)\cdot f_y(y) \neq f(x,y)$

$\therefore$ X and Y are independent

Ex ④ The joint p.d.f. of (x,y) is given by

$$f(x,y) = \begin{cases} kxy & 0<x<1 \quad 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

i) Find k   ii) Find $(x+y \leq 1)$

iii) Are x and y independent

$$\int_0^1 \int_0^1 kxy \, dx\, dy = 1$$

$$ky\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\frac{1}{2}k \int \left[\frac{y^2}{2}\right]_0^1 = 1$$

$$\frac{k}{4} = 1$$

$$k = 4$$

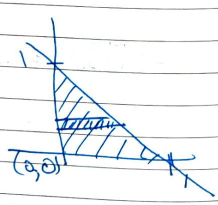
$x + y < 1$

$X = \cancel{0 \text{ to }} \; 0 \text{ to } 1-y$

$Y = 0 \text{ to } 1$

$$\int_0^1 \int_0^{1-y} 4xy$$

$$\int_0^1 y\left[2x^2\right]_0^{1-y}$$



$$\int_0^1 y \; 2(1+y^2-2y)$$

$$\int_0^1 2y + 2y^3 - 4y^2$$

$$\left[y^2 + \frac{y^4}{2} - \frac{4y^2}{3}\right]_0^1$$

$$1 + \frac{1}{2} - \frac{4}{3}$$

$$\frac{6+3-8}{6} = \frac{1}{6}$$

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Ex ⑤ The joint probability density function of (x,y) is given by

$f(x,y) = xy^2 + \frac{x^2}{8}, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 1$

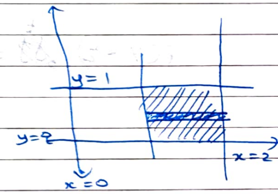
~~Find ① P(X>1)~~

Find ① $P(X>1)$ ② $P(2Y \leq 1)$ ③ $P(X>1 / Y<\frac{1}{2})$

④ $P(X<Y)$ ⑤ $P(X>Y \cap X<Y)$

① $1 \leq x \leq 2$

$0 \leq y \leq 1$



$P(X>1) = P(1 \leq x \leq 2, 0 \leq y \leq 1)$

$$= \int_0^1 \int_1^2 xy^2 + \frac{x^2}{8} \, dx\, dy$$

$$\int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_1^2 dy$$

$$\int_0^1 \left[2y^2 + \frac{1}{3} - \frac{y^2}{2} - \frac{1}{24}\right] dy$$

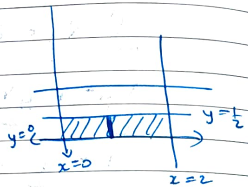
$$\int_0^1 \frac{3y^2}{2} + \frac{7}{24} \, dy$$

$$\left[\frac{y^2}{2} + \frac{7}{24}y\right]_0^1 = \frac{12}{24} + \frac{7}{24} = \frac{19}{24}$$

② $P(2Y \le 1)$

$P(Y < \frac{1}{2})$

$$\int_0^2\int_0^{\frac12} xy^2 + \frac{x^2}{8}\,dy\,dx$$



$$\int_0^2 \left[\frac{xy^3}{3} + \frac{x^2y}{8}\right]_0^{\frac12}$$

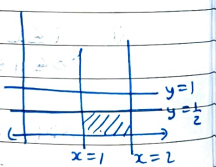
$$\int_0^2 \frac{x}{24} + \frac{x^2}{16}\,dx$$

$$\left[\frac{x^2}{48} + \frac{x^3}{48}\right]_0^2$$

$$\frac{4+8}{48} = \frac{1}{4}$$

③ $P(X>1 / Y<\frac12) = \frac{P(X>1,\ Y<\frac12)}{P(Y<\frac12)}$ $[, = \cap]$

$P(Y<\frac12) = \int_0^{\frac12}\int_0^2 xy^2 + \frac{x^2}{8}\,dx\,dy$



$= \int_0^{\frac12}\left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^2$

$= \int_0^{\frac12} 2y^2 + \frac{1}{3}$

$= \left[\frac{2y^3}{3} + \frac{y}{3}\right]_0^{\frac12}$

$= \frac{1}{12} + \frac{1}{6}$

$= \frac{1}{4}$

$P(X>1,\ Y<\frac12) = \int_0^{\frac12}\int_1^2 xy^2 + \frac{x^2}{8}\,dx\,dy$

$= \int_0^{\frac12}\left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_1^2$

$= \int_0^{\frac12} 2y^2 + \frac{1}{3} - \frac{y^2}{2} - \frac{1}{24}$

$= \int_0^{\frac12} \frac{3y^2}{2} + \frac{7}{24}$

$= \left[\frac{y^3}{2} + \frac{7}{24}y\right]_0^{\frac12}$

$= \frac{1}{16} + \frac{7}{48}$

$= \frac{10}{48} = \frac{5}{24}$

$P(X>1 / Y<\frac12) = \frac{P(X>1,\ Y<\frac12)}{P(Y<\frac12)}$

$= \frac{\frac{5}{24}}{\frac{1}{4}}$

$= \frac{5}{6}$

$P(X<Y) = P(0 \le x \le y,\ 0 \le y \le 1)$

$= \int_0^1\int_0^y xy^2 + \frac{x^2}{8}\,dx\,dy$

$= \int_0^1\left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^y$

$= \int_0^1 \frac{y^4}{2} + \frac{y^3}{24}$

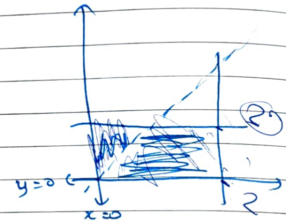
$= \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1 = \frac{1}{10} + \frac{1}{96} = \frac{48+5}{480}$

$= \frac{53}{480}$

$P(X > Y \cap X < Y)$

$P(\phi)$

$= 0$



Expectation of two dimensional Random variable

① For discrete random variable

* $E(X) = \sum_i P(x_i) \cdot x_i$

$= \sum_{x_i} P(X = x_i)\, x_i$

* $E(Y) = \sum_{y_j} P(y_j)\, y_j$

$= \sum_{y_j} P(Y = y_j)\, y_j$

* $E(XY) = \sum_j \sum_i P(X = x_i, Y = y_j)\, x_i\, y_j$

* $E(X/Y) = \sum_i P(X = x_i, Y = y_j) \cdot x_i$

* $E(Y/X) = \sum_j P(X = x_i, Y = y_j) \cdot y_j$

② For continous random variable

* $E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$

* $E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$

* $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

* $E(X/Y) = \int_{-\infty}^{\infty} y\, f(y/x)\, dy$

③ Mean of $X = E(X)$

④ Variance of $X = Var(X)$

$= E(X^2) - [E(X)]^2$

⑤ Mean of $(X/Y) = E(X/Y)$

⑥ Variance of $X/Y = E(X^2/Y) - [E(X/Y)]^2$

Covariance (X, Y)

$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

Properties

① If X and Y are independent then $Cov(X, Y) = 0$

② If $Cov(X, Y) = 0$, then X, Y are uncorrelated

③ $E(ax + by) = a E(x) + b E(x)$ where $a, b \in R$

④ $Var(ax + b) = \cancel{a^2 E(x)}\ a^2 Var(X)$ (Variance of $b = 0$, constant variance is zero)

⑤ Variance of $K = 0$ if $K \in R$.

⑥ $Cov(ax + by, cx + dy) = \cancel{E(ax+by}\ ac\, Var(x) + bd\, Var(y) + (ad + bc)Cov(X, Y)$

⑦ Variance of $(ax \pm by) = a^2 Var(X) + b^2 Var(Y) \pm 2ab\, Cov(X,Y)$

Ex ① Given the following bivariant probability distribution

Y \ X	-1	0	1
-1	0	0.1	0.1
0	0.2	0.2	0.2
1	0	0.1	0.1

Find $E(X)$, $E(Y)$, $E(XY)$, $Var(X)$ and prove that X and Y are uncorrelated

~~$E(X) = \Sigma\, x_i P(x_i)$~~
~~$= -1\times 0 + 0\times 0.1 + 0.1\times 1$~~
~~$= 0.1$~~

Y \ X	-1	0	1	$P(Y=y_j)$
-1	0	0.1	0.1	0.2
0	0.2	0.2	0.2	0.6
1	0	0.1	0.1	0.2
$P(X=x_i)$	0.2	0.4	0.4	

(1) $E(X) = \Sigma\, P(x_i)\, x_i$
$= -1\times 0.2 + 0 + 1\times 0.4$
$= 0.2$

My name is Shrey

(2) $E(Y) = -1\times 0.2 + 1\times 0.2$
$= 0$

(3) $E(XY) = \sum_j \sum_i P(X=x_i, Y=y_j)\, x_i y_j$

$= P(-1,-1)(-1\cdot -1) + P(-1,1)\cdot -1 + P(1,-1)\cdot -1$
$P(1,1)\ (1\cdot 1)$
$= 0\cdot 1 + 0 + (0.1)(-1) + (0.1)\,1$
$= 0$

(4) $E(X^2) = \Sigma\, P(X=x_i)\, x_i^2$
$= (-1)^2\times 0.2) + 1\times 0.4$
$= 0.6$

$Var(X) = E(X^2) - [E(X)]^2$
$= 0.6 - (0.2)^2$
$= 0.56$

(5) $Cov(X,Y) = E(XY) - E(X)\,E(Y)$
$= 0 - (0.2)\cdot 0$
$= 0$

$\therefore Cov(X,Y) = 0$
$\therefore$ They are uncorrelated

Ex (2) If $Y = -2x + 3$ find $Cov(X,Y)$

08-08-24 Consider

$Cov(X,Y) = Cov(X, -2x+3)$
$= E[\cancel{(x)}\,x\ (-2x+3)] - E(x)\cdot E(2x+3)$
$= E(-2x^2 + 3x) - E(x)[-2E(x) + E(3)]$
$= -2E(x^2) + \cancel{3E(x)} \cancel{E(x)} + 2(E(x))^2 - 3E(x)$
$= -2\,[E(x^2) - [E(x)]^2]$
$= -2\, Var(X)$

Ex (3) The joint probability density function of (X,Y) is
$f(x,y) = x+y,\ 0 \le x, y \le 1$ find $E(X), E(Y), E(xy)$

(1) $E(X) = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$

$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$= \int_0^1 x+y\, dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$

$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$

$= \int_0^1 x\left(x+\frac{1}{2}\right) dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1$

$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

~~Similarly,~~

$f_Y(y) = \int_{-\infty}^{\infty} \cancel{x+y} \, f_Y(y) \, dx$

$= \int_0^1 x + y \, dx$

$= \left[\frac{x^2}{2} + xy\right]_0^1$

$= \frac{1}{2} + y$

$E(Y) = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy = \int_0^1 y\left(y+\frac{1}{2}\right) dy$

$= \int_0^1 y^2 + \frac{y}{2}$

$= \left[\frac{y^3}{3} + \frac{y^2}{4}\right]_0^1$

$= \frac{1}{3} + \frac{1}{4}$

$= \frac{7}{12}$

S Palanimal
Kapoor & Gupta

$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f(x,y) \, dx \, dy$

$= \int_0^1 \int_0^1 xy(x+y)$

$= \int_0^1 \int_0^1 x^2y + xy^2 \, dx \, dy$

$= \int_0^1 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2}\right]_0^1 dy$

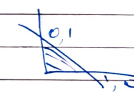
$= \int_0^1 \frac{y}{3} + \frac{y^2}{2} \, dy$

$= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$

$= \frac{1}{3}$

Ex(6) The join probability density function of (X,Y) is $f(x,y) = \begin{cases} 24xy & x>0,\ y>0,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$

Find conditional mean and variance of Y given X.



$E(Y/X) = \int_{-\infty}^{\infty} y \, f(y/x)$

$= \int_{-\infty}^{\infty} y \, \cancel{f(y/x)} \, \frac{f(x,y)}{f(x)}$

$\cancel{= y \cdot 24xy}$

$\cancel{f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\int_{-\infty}^{\infty}\int_0 24xy \, dy}{\int_0^{1-x} 24xy \, dy}}$

$\cancel{= 24x}$

$x+y=1$
$x=1-y$
$y=1-x$

$$f_X(x) = \int_0^{1-x} 24xy\,dy$$

$$= 24x \int_0^{1-x} \cancel{24}\,y \cdot dy$$

$$= 24x\left[\frac{y^2}{2}\right]_0^{1-x}$$

$$= \cancel{24}^{12}x\left[\frac{1+x^2-2x}{x}\right]$$

$$= \cancel{12+12x^2}\ 12x[1-x]^2$$

$$f(Y/x) = \frac{\int_0^{1-x} 24xy}{12x[1-x]^2}$$

$$E(X) = \int_{-\infty}^{\infty} y\,\frac{24xy}{12x[1-x^2]}\,dy$$

$$E(Y/X) = \int_{-\infty}^{\infty} y\,\frac{f(x,y)}{f_x(x)}$$

$$= \int_0^{1-x} y\cdot\frac{2y}{(1-x)^2}\,dy$$

$$= \frac{2}{(1-x)^2}\left[\frac{y^3}{3}\right]_0^{1-x}$$

$$= \frac{2}{3}\frac{(1-x)^3}{\cancel{(1-x)^2}}$$

$$= \frac{2}{3}(1-x)$$

$$E(Y^2/X) = \int_{-\infty}^{\infty} y^2 f(y/x)$$

$$= \int_0^{1-x} y^2\,\frac{2y}{(1-x)^2}$$

$$= \frac{\cancel{2}}{(1-x)^2}\left[\frac{y^4}{4\,2}\right]_0^{1-x}$$

$$= \frac{1}{2\cancel{(1-x)^2}}(1-x)^{\cancel{4}\,2}$$

$$= \frac{(1-x)^2}{2}$$

$$Var(Y/X) = E(Y^2/X) - [E(Y/X)]^2$$

$$= \frac{(1-x)^2}{2} - \frac{4}{9}(1-x)^2$$

$$= (1-x)^2\left[\frac{1}{2}-\frac{4}{9}\right]$$

$$= (1-x)^2\left[\frac{9-8}{18}\right]$$

$$= \frac{(1-x)^2}{18}$$ //

Ex ⑤ The joint p.d.f. of (x,y) is

$$f(x,y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \qquad 0\le x<\infty,\ 0\le y<\infty$$

Find marginal distribution of X and the conditional distribution of Y given that $X=x$ and expected value of conditional distribution

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Marginal p.d.f. of x

$f_X(x) = \int_{-\infty} f(x,y)\,dy$

$\int_0^\infty \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}\,dy$

$\frac{9}{2(1+x)^4} \int_0^\infty \frac{\cancel{9}(1+x+y)}{(1+y)^4}\,dy$

$\cancel{\int_0^\infty}$ Let $1+y = t$

$dy = dt$

$y=0 \quad t \to 1$

$y = \infty \quad t \to \infty$

$\frac{9}{2(1+x)^4} \int \left(\frac{t+x}{t^4}\right) dt$

$\frac{9}{2(1+x)^4} \left[\frac{t^{-2}}{-2} + \frac{x\,t^{-3}}{-3}\right]_1^\infty$

$\frac{9}{2(1+x)^4} \left[0+0\right] - \left[\frac{1}{-2} + \frac{x}{-3}\right]$

$\frac{9}{2(1+x)^4} \left[\frac{+1}{2} + \frac{x}{3}\right]$

$\frac{\cancel{9}\,3}{2(1+x)^4} \left[\frac{+3+2x}{\cancel{6}\,2}\right]$

$\cancel{\frac{9\;6x}{4(1+x)^4}}$ $\quad \frac{9+6x}{4(1+x)^4} \quad 6x+9$

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Marginal p.d.f. of Y

$f_Y(y) = \int_{-\infty}^\infty f(x,y)\,dx = \frac{9+6y}{4(1+y)^4}$

$f(Y/x) = \frac{f(x,y)}{f(x)}$

$= \int_0^\infty \frac{\cancel{9}(1+x+y)}{2\cancel{(1+x)^4}(1+y)^4} \cdot \frac{2\,\cancel{4(1+x)^4}}{\cancel{3}(2x+3)}\,dy$

$= \frac{6(1+x+y)}{(1+y)^4(2x+3)}$

$E(Y/x) \Rightarrow \int_{-\infty}^\infty f(Y/x)\cdot y\,dy$

$= 6\int_0^y \frac{y(1+x+y)}{(1+y)^4(2x+3)}\,dy$

$1+y = t$

$dy = dt$

$1-\infty$

$\frac{6}{(2x+3)} \int_1^\infty \frac{(t-1)(t+x)}{t^4\,\cancel{(2x+3)}}\,dt$

$\int_1^\infty \frac{t^2+xt-t-x}{t^4}\,dt$

$\frac{6}{2x+3} \int_1^\infty \frac{1}{t^2} + \frac{x}{t^3} - \frac{1}{t^3} - \frac{x}{t^4}\,dt$

$\frac{6}{2x+3} \left[\frac{1}{-t} + \frac{x}{-2t^2} + \frac{1}{2t^2} + \frac{x}{3t^3}\right]_1^\infty$

$\frac{6}{2x+3} \; \cancel{0} - \left[-1 - \frac{x}{2} + \frac{1}{2} + \frac{x}{3}\right]$

$\frac{6}{2x+3} \; -\left[-\frac{1}{2} - \frac{x}{6}\right] \Rightarrow \frac{6}{2x+3}\left[\frac{1}{2} + \frac{x}{6}\right]$

16-08-24 **Raw and Central Moments:**

Let X be the random variable then

① The r^{th} moment of X about origin is

$\mu_r = E(X^r)$ $r = 1, 2, 3, \ldots\ldots$

② The r^{th} moment of X about mean $(\bar{x})$ (central moment)

$\mu_r' = E[(x-\bar{x})^r]$

③ The r^{th} moment of X about the point A (raw moment) is $\mu_r = E[(X-A)^r]$

Relation between moments about origin and central moment

Consider $\mu_r = E(X-\bar{x})^r]$

$= \cancel{E[{}^rC_0 x^r - {}^rC_1 x^{r-1} + {}^rC_2 x^{r-2}}$

$= E[{}^rC_0 x^r - {}^rC_1 x^{r-1}\bar{x} - {}^rC_2 x^{r-2}\bar{x}^2$

$- \ldots\ldots + (-1)^r\, {}^rC_r\, \bar{x}^r + \ldots\ldots]$

$= E[x^r - r x^{r-1}\bar{x} + \frac{r(r-1)}{2!} x^{r-2} \bar{x}^2 \ldots + (-1)^r \bar{x}^r]$

$\cancel{= E(x^r) - rE(x^{r-1}}$

$= E(x^r) - rE(x^{r-1})\bar{x} + \frac{r(r-1)}{2!} E(x^{r-2})(\bar{x})^2$

$- \ldots\ldots + (-1)^r \bar{x}^r]$

$[\because E(\bar{x}) = \bar{x}]$

$= \mu_r' - r\mu'_{r-1}\mu_1' + \frac{r(r-1)}{2!}\mu'_{r-2}\mu_1'^2$

$- \ldots\ldots + (-1)^r \mu_1'^r$

if $r = 1 \Rightarrow \mu_1 = \mu_1'$

$r = 2 \Rightarrow \mu_2 = \mu_2' - 2\mu_1'\mu_1' + \frac{2(2-1)}{2}\mu_0'\mu_1'^2$

$= \mu_2' - 2\mu_1'^2 + \mu_1'^2$

$= \mu_2' - \mu_1'^2$

$r = 3 \Rightarrow \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$r = 4 \Rightarrow \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_1'^2\mu_2' - 3\mu_1'^4$

Rotation between Raw moment and Central moment

here $\mu_r' = E[(X-A)^r]$

$\bar{x} = \mu_1' + A$

$\mu_2' = \mu_2 + \mu_1'^2$

$\mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3$

$\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1'^2 + 4\mu_1'^4$

and so on.....

19-08-24 **Moment Generating Function: (MGF)**

The MGF of random variable X about origin is

$M_x(t) = E(e^{tx})$

Note: ① If X is discrete R.V. then $M_x(t) = E(e^{tx})$

$= \sum_x e^{tx} . P(x)$

② If X is continous R.V. $M_x(t) = E(e^{tx})$

$= \int_{-\infty}^{\infty} e^{tx} . f(x)\, dx$

③ Moments using power series:

$M_x(t) = E(e^{tx}) = E\left[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots + \frac{t^rx^r}{r!} + \ldots\right]$

$\cancel{= 1 + tE(x)}$

$= 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \ldots + \frac{t^r}{r!}E(x^r) + \ldots$

$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots$

$\boxed{\mu_r' = \text{coefficient of } \frac{t^r}{r!}}$

④ Moments using derivative

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

in general

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$$

⑤ MGF ~~for~~ will not always exist

Example ① Find first 4 moments of the distribution about $X=4$ are 1, 4, 10 and 45 respectively. Show that mean is 5, variance is 3 $\mu_3 = 0$, ~~before~~ $\mu_4 = 26$.

Let $\mu_1' = 1$, $\mu_2' = 4$, $\mu_3' = 10$, $\mu_4' = 45$ and $A = 4$

$$\text{Mean} = \mu_1' + A = 1 + 4 = 5$$

$$\text{Variance} = E(X^2) - [E(X)]^2 = \mu_2' - (\mu_1')^2 = 4 - 1^2 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 10 - 3(4)(1) + 2(1)^3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4 = 26$$

1
① 1
1 ② 1
1 3 ③ 1
1 4 6 ④ 1
1 5 10 10 ⑤ 1

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$\mu_5 = \mu_5' - 5\mu_4'\mu_1' + 10\mu_3'\mu_1'^2 - 10\mu_2'\mu_1'^3 + 4\mu_1'^5$$

Ex ② $\begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

is p.d.f. of r.v. X, using MGF find first four moments about origin and first ~~for~~ four central moments.

The MGF of x about origin is

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \cdot 3e^{-3x}\,dx$$

$$= 3\int_0^{\infty} e^{x(t-3)}\,dx$$

$$= 3\left[\frac{e^{x(t-3)}}{t-3}\right]_0^{\infty}$$ Considering $(t-3)$ is -ve

$$= 3\left[\frac{e^{-\infty}}{t-3} - \frac{e^{0}}{t-3}\right], \quad t < 3$$

$$M_x(t) = \frac{3}{3-t}$$
$$= \frac{3}{3\left[1-\frac{t}{3}\right]}$$
$$= \frac{1}{1-\frac{t}{3}}$$

$$M_x(t) = 1 + \frac{t}{3} + \frac{t^2}{3^2} + \frac{t^3}{3^3} + \dots\dots + \frac{t^r}{3^r} + \dots \qquad \left|\frac{t}{3}\right|<1$$
$$= 1 + \frac{t}{1!}\left(\frac{1!}{3}\right) + \frac{t^2}{2!}\left(\frac{2!}{3^2}\right) + \frac{t^3}{3!}\left(\frac{3!}{3^3}\right) + \dots\dots$$
$$+ \frac{t^r}{r!}\left(\frac{r!}{3^r}\right) + \dots\dots$$

We know, μ_r' = coefficients of $\frac{t^r}{r!}$

μ_1' = coefficient of $\frac{t}{1!} = \frac{1}{3}$

μ_2' = coefficient of $\frac{t^2}{2!} = \frac{2!}{3^2} = \frac{2}{9}$

μ_3' = coefficient of $\frac{t^3}{3!} = \frac{3!}{3^3} = \frac{6}{27} = \frac{2}{9}$

μ_4' = coefficient of $\frac{t^4}{4!} = \frac{4!}{3^4} = \frac{24}{81} = \frac{8}{27}$

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1'^2$
$= \frac{2}{9} - \frac{1}{3}$
$= \frac{1}{9}$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$
$= \frac{2}{9} - \cancel{3} \times \frac{2}{9} \times \frac{1}{\cancel{3}} + 2 \times \frac{1}{27} = \frac{2}{27}$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$
$= \frac{8}{27} - 4 \times \frac{2}{9} \times \frac{1}{3} + 6 \times \frac{2}{9} \times \frac{1}{9} - 3 \times \frac{1}{81}$
$= \frac{8}{27} - \frac{8}{27} + \frac{4}{27} - \frac{1}{27} = \frac{3}{27} = \frac{1}{9}$

Special Probability Distribution

* Discrete Probability Distribution

1) Binomial distribution

Let n be the number of trials and p be the probability of success of each trial and q be the probability of failure, then the random variable X is said to follow binomial distribution if it has the probability ~~P(X=X)~~ $P(X=x) = {}^nC_x\, p^x \cdot q^{n-x}$

$x = 0, 1, \dots, n$

Note

1. n must be finite
2. mean $= E(X) = \sum_x p(x)\cdot x = \sum_{x=0}^{n} {}^nC_x \cdot p^x \cdot q^{n-x} \cdot x$

$$\sum_{x=0}^{n} \frac{n!}{(n-x)!\,x!}\, p^x \cdot q^{n-x} \cdot x$$

$$\sum_{x=1}^{n} \frac{n!}{(n-x)!\,(x-1)!}\, p^x \cdot q^{n-x}$$

$$\sum_{x=1}^{n} \frac{n(n-1)!}{(n-x)!\,(x-1)!}\, p \cdot p^{x-1} \cdot q^{n-x}$$

$$np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!\,(x-1)!}\, p^{x-1}\, q^{n-x}$$

$$= np\,(p+q)^{n-1}$$
$$= np\,(1)^{n-1}$$
$$= np$$

③ Variance $= npq$
④ Mean $\geq$ Variance (Mean = Variance when $p=0$)
⑤ MGF
$$M_x(t) = E(e^{tx}) = e^{-npt}(q+pe^{t})^n$$

⑥ $M_{(x_1+x_2)}(t) = M_{x_1}(t) + M_{x_2}(t)$

Ex ① If the mean and variance of binomial distribution are 16 and 8, find $P(x \geq 3)$

Mean $= np$ Variance $= npq$

$np = 16$ $npq = 8$

$16 \cdot q = 8$

$q = 0.5$ or $\frac{1}{2}$

$p = 1-q$
$= \frac{1}{2}$

$n \cdot \frac{1}{2} \cdot \frac{1}{2} = 8$

$n = 32$

The binomial distribution for X has
$P(X=x) = {}^nC_x p^x q^{n-x}$

$= {}^{32}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{32-x}$

$x = 0$ $P(X=0) = \left(\frac{1}{2}\right)^{32}$

$x = 1$ $P(X=1) = {}^{32}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{31}$
$= 32\left(\frac{1}{2}\right)^{32}$

$x=2$ $P(X=2) = {}^{32}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{32-2}$
$= 496\left(\frac{1}{2}\right)^{32}$

$P(x \geq 3) = 1 - P(x < 3)$
$= 1 - [P[x=0) + P(x=1) + P(x=2)]$
$= 1 - \left[\left(\frac{1}{2}\right)^{32} + 32\left(\frac{1}{2}\right)^{32} + 496\left(\frac{1}{2}\right)^{32}\right]$
$= 1 - 529\left(\frac{1}{2}\right)^{32}$

Ex ② In 256 sets of 12 tosses of fair coin. In how many cases may one expects 8 heads and 4 tails?

$n = 12$

$p = 0.5$ $q = 0.5$

$P(X=x) = {}^{12}C_8 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^8$

$= \frac{495}{4096}$

for 256 sets $= 256 \times \frac{495}{4096}$ $\left(\frac{495}{16}\right)$

$= 30.9375$

22-08-24 Poisson Distribution

The discrete random variable X is said to follow the poisson distribution if it has the probability
$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$, $\lambda > 0$

Note ① Here n is infinitely large
② Mean $= E(X) = \sum_x P(x) \cdot x$

$$= \sum_x \frac{e^{-\lambda}\lambda^x}{x!}\cdot x = e^{-\lambda} \sum_{x=1} \frac{\lambda\,\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\dots\right]$$

$$= \lambda e^{-\lambda}[e^{\lambda}]$$

$$= \lambda$$

i.e. mean $= \lambda$

③ Variance $= \lambda$

④ Mean = Variance

⑤ MGF $M_X(t) = e^{\lambda(e^t - 1)}$

Ex ① If X and Y are independent poisson variable such that $P(X=1) = P(X=2)$
$P(Y=1) = P(Y=3)$. find variance of $(X-2Y)$

$P(X=1) = P(X=2)$

$$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$$

$$\frac{\cancel{\lambda}}{1} = \frac{\lambda^{\cancel{2}}}{2!} \qquad \frac{\lambda^2}{2} = \frac{\lambda}{1}$$

$$\lambda = 2 \qquad \lambda^2 = 2\lambda$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\lambda = 2, \cancel{\lambda = 0}$$

$\lambda \neq 0$

$P(Y=1) = P(Y=3)$

$$\frac{e^{-\lambda}\lambda_1^1}{1!} = \frac{e^{-\lambda}\lambda_1^3}{3!}$$

$$\lambda_1^3 = 6\lambda_1$$

$$\lambda_1(\lambda_1^2 - 6) = 0$$

$$\lambda_1^2 = 6$$

$$\lambda_1 = \pm\sqrt{6}$$

$$\lambda_1 = \sqrt{6}$$

$\because \lambda > 0$

$$\text{Variance}(X-2Y) = Var(X) + (-2)^2 Var(Y)$$

$$= \cancel{2 + 4\sqrt{6}}$$

$$= \lambda + (-2)^2 \lambda_1$$

$$= \cancel{\lambda + 2 + 4\sqrt{6}}\ 2 + 4\sqrt{6}$$

Ex ② In a component manufacturing industry there is a small chance of $\frac{1}{500}$ for any component to be defective. The component are supplied in packet of 10. Use Poisson distribution to calculate the approximate no. of packet containing

1) No defective
2) 1 defective
3) 2 defective comp. resp. In a consignment of 10000 packets.

$n = 10$

p = probability of defective component $= \frac{1}{500}$

$$\lambda = np = 10 \times \frac{1}{500} = \frac{1}{50} = 0.02$$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X=0) = \frac{e^{-0.02}\cdot 1}{1}$$

$$= 9.8 \times 10^{-1}$$

No. of defective component in 10000 $= 9.8 \times 10^{-1} \times 10^4$

$$= \cancel{0.98}\ 9800$$

$$P(X=1) = \frac{e^{-0.02}\cdot 0.02}{1!} = 1.96 \times 10^{-2}$$

No. of def comp $= 1.96 \times 10^{-2} \times 10^4$

$$= 196$$

$$P(X=2) = \frac{e^{-0.02}\cdot(0.02)^2}{2!} = 0.000196$$

$\therefore$ Def comp $= 1.96$
≈ 2

classmate
Date 22-08-24
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Ex ③ An insurance company ensure 4000 people against lose of both eyes in a car accident. Based on the previous data the rates were computed on assumption that on the average of 10 persons in 1 lakh will have car accident each year. ~~The~~ that result in this type of injury. What is the probability that more than 3 of insured will collect on their policy in a given year.

$\cancel{P(}P = \frac{10}{100000} = 10^{-4}$

$P(X>3) = 1-[P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$N = 4000$

$P(\text{Person will die}) = 10^{-4}$

$\lambda = nP$

$= 0.4$

$P(X=0) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

$= e^{-0.4} = 0.6703$

$P(X=1) = e^{-0.4} \cdot 0.4 = 0.268$

$P(X=2) = e^{-0.4} \cdot 0.08 = 0.0536$

$P(X=3) = \frac{e^{-0.4} \cdot 0.064}{6} = 7.15 \times 10^{-3}$

$P[X>3] = 1 - [0.6703 + 0.268 + 0.0536 + 7.15 \times 10^{-3}]$

$= 0.00095$

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23-08-24 Ex ④ Fit a poisson distribution to the following data.

No. of deaths	0	1	2	3	4
Frequencies	123	59	14	3	1

Note that the poisson distribution has the probability

$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$

$\text{Mean} = \lambda = \frac{\sum f_i x_i}{\sum f_i}$

$= \frac{1\times 59 + 2\times 14 + 3\times 3 + 4\times 1}{123 + 59 + 14 + 3 + 1}$

$= \frac{100}{200}$

$\lambda = 0.5$

$P(X=x) = \frac{e^{-0.5} (0.5)^{x}}{x!}$

$P(X=0) = e^{-0.5} = 0.606$

$P(X=1) = e^{-0.5} \cdot 0.5 = 0.303$

$P(X=2) = \frac{e^{-0.5} \cdot 0.25}{2} = 0.0758$

$P(X=3) = \frac{e^{-0.5} \cdot 0.125}{6} = 0.0126$

$P(X=4) = \frac{e^{-0.5} \cdot 0.0625}{24} = 1.58 \times 10^{-3}$

Exp frequency $x=0$: $200 \times 0.606 = 121$

$x=1 = 200 \times 0.303 = 61$

$x=2 = 200 \times 0.0758 = 15$

$x=3 = 200 \times 0.0126 = 3$

$x=4 = 200 \times 1.58 \times 10^{-3} = 0$

3
59
28
9
4
123
59
14
3
1
200

26-08-24 **Continous Distribution**

① **Normal Distribution**: The random variable X is said to follow the normal distribution if X has the probability density function $x = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$

Note:

① Mean of $x = \bar{x} = m$

② Variance $= Var(x) = \sigma^2$

③ mean = median = mode

④ $MGF = M_x(t) = e^{\left(mt + \frac{t^2\sigma^2}{2}\right)}$

* **Standard Normal Variable**:

A random variable X with the parameter m and σ

$\boxed{Z = \frac{x-m}{\sigma}}$ is said to standard normal variable

$\sigma \rightarrow$ standard deviation

Area property

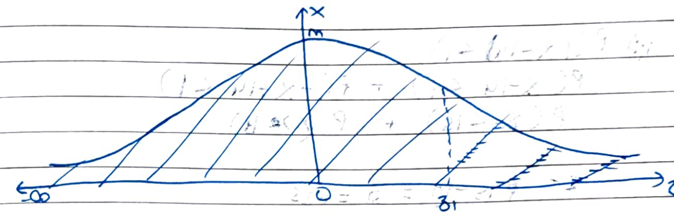
If $x = m \Rightarrow Z = \frac{m-m}{\sigma} = 0$

If $x = x_1 \Rightarrow Z = \frac{x_1 - m}{\sigma} = z_1$ (say)

The area under the normal ~~variables~~ variates X between $X = m$ and $X = x_1$ is equals to the area under the standard normal curve Z between $Z = 0$ to $Z = z_1$.

i.e. $P(m \le X \le x_1) = P(0 \le Z \le z_1) = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{1}{2}x^2} dx$

Note that $p(z_1 \le Z \le z_2) = p(0 \le Z \le z_1) + P(0 \le Z \le z_2)$



Ex ① If X is normal variates with mean 10 and standard deviation 4 find

i) $P(5 \le X \le 18)$ ii) $P(X \le 12)$

iii) $P(|X - 14| < 1)$

$m = 10, \quad \sigma = 4$

Standard normal variable is $Z = \frac{X-m}{\sigma}$

i) To find $P(5 \le x \le 18)$

if $X = 5 \Rightarrow Z = \frac{X-m}{\sigma} = \frac{5-10}{4} = -1.25$

if $X = 18 \Rightarrow Z = \frac{18-10}{4} = 2$

$P(5 \le X \le 18) = P(-1.25 \le Z \le 2)$

$= P(-\infty \le Z \le 2) - P(-\infty \le Z \le -1.25)$

$= 0.9772 - 0.1056$

$= 0.8716$

ii) $P(X \le 12)$

if $x = 12 \Rightarrow Z = \frac{12-10}{4} = 0.5$

$P(-\infty \le Z \le 0.5) = 0.6915$

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iii) $P(|x-14|<1)$

$P(x-14<1) + P(-x+14<1)$

$P(x<15) + P(x>13)$

$z = \frac{15-10}{4} = \frac{5}{4} = 1.25$

$z = \frac{13-10}{4} = 0.75$

$P(z<1.25) + P(z>0.75)$

$0.8944 + 0.7734$

~~1.1678~~

$= 0.121$

29-08-24 The marks obtained by 1000 students in an examination are found to be normally distributed with mean 70 and standard deviation 5. Estimate the no. of students whose marks will be
i) between 60 and 75
ii) More than 75

$m = 70 \qquad \sigma = 5 \qquad N = 1000$

$60 = \frac{x-75}{5}$ note that SNV $z = \frac{x-m}{\sigma}$

$x = 375$ i) First we find $P(60 \le x \le 75)$

if $x = 60$

$z = \frac{60-70}{5} = -2$

if $x = 75$

$z = \frac{75-70}{5} = 1$

$P(60 \le x \le 75) = P(-2 \le z \le 1)$

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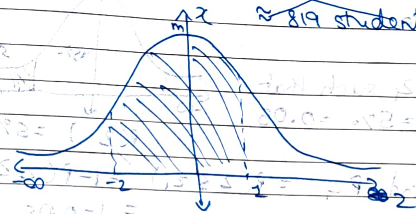
$= P(-\infty < z < 1) - P(-\infty < z < -2)$

~~$= 0.0228 + 0.8413$~~

$= 0.8413 - 0.0228$

$= 0.8185$

~~Total no. of students $= 0.8185 \times 1000$~~
~~$= 818.5$~~
~~≈ 819 students~~



The no. of students whose marks 60 and 75 is $N \times P = 1000 \times 0.8185 = 818.5$

≈ 819 students

ii) $P(x \ge 75) =$

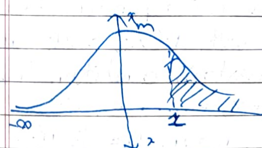
$x = 75$

$z = \frac{75-70}{5} = 1$

$P(x \ge 75) = P(1 \le z < \infty)$

$= 1 - 0.8413$

$= 0.1587$



$\therefore$ The no. of students whose marks $> 75 = 0.1587 \times 1000$

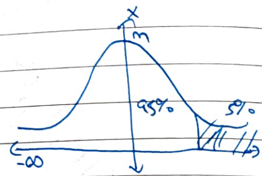
$= 158.7$

≈ 159 students.

The ~~mot~~ monthly salary x in an big organization is normally distributed with the mean ₹3000 and standard deviation of ₹250. what should be the minimum salary of the worker in this organization so that the probability ~~of~~ that he belongs to top 5% workers.

$m = ₹3000$

$\sigma = ₹250$



To find z_1 such that

$P(Z = Z_1) = 5\% = 0.05$

$P(Z \quad) = 5\%$

i.e. to find $P z_1 = P(Z \leq z_1) = 1 - P(Z \geq z_1)$

$= 1 - 0.05$

$= 0.95$

The corresponding value of 0.95 in statistical table of z

$z_1 = 1.65$

We know $z = \frac{x - m}{\sigma}$

$z_1 = \frac{x_1 - m}{\sigma}$

$1.65 \times 250 = x - 3000$

$x = ₹3412.5$

12-09-24 **Central Limit Theorem :-**

Let $X_1, X_2, \ldots, X_n$ be the n independent identically distributed random variable with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$ then under certain condition the random variable $S_n = X_1 + X_2 + \ldots + X_n$ is a asymptotically normal with mean $\mu = \sum_{i=1}^{n} \mu_i$ and variance $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$

Note:

The normal variate for the random variable S_n is

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

where $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$

Ex ① A coin is tossed 10 times using central limit theorem, find the probability of getting 3 or 4 or 5 heads

Given $n = 10$

$p = 0.5$

$q = 0.5$

Mean $= \mu = np = 10 \times 0.5 = 5$

$V(X) = \sigma^2 = npq = 2.5$

$\sigma = \sqrt{2.5} = 1.581$

$P(3 \leq x \leq 5) = ?$

~~if $x = 3 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{3-5}{1.581} = -1.$~~

~~$x = 4 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{4-5}{1.581} =$~~

Here x is discrete random variable

$P(2.5 \leq x \leq 5.5)$

if $x = 2.5$ then $z = \frac{x-\mu}{\sigma} = \frac{2.5-5}{1.58} = -1.58$

$x = 5.5$ then $z = \frac{x-\mu}{\sigma} = \frac{5.5-5}{1.58} = 0.316$

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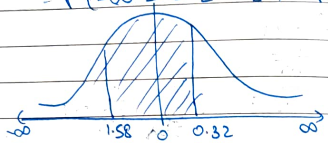
~~if X~~ $P(2.5 \le X \le 5.5) = P(-1.58 \le Z \le 0.32$

$= P(-\infty \le Z \le 0.32) - P(-\infty \le Z \le -1.58)$



~~= 0.625~~

$= 0.6255 - 0.0571$

$= 0.5684$

Ex ② A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using c.l.t (central limit theorem), find the probability such that mean ~~of~~ of sample will not differ not from $\mu = 60$ by more than 4.

$n = 100$

$\mu = 60$

$\sigma^2 = 400 \quad \therefore \sigma = 20$

~~$\sigma =$~~

$\bar{x}$ does not differ from 60 by more than 4

$P(|X - 60| \le 4)$

$P(|X-60| \le 4) = P\left(\frac{|X-60|}{\sigma/\sqrt{n}} \le \frac{4}{\sigma/\sqrt{n}}\right)$

$= P(|Z| \le 2)$

$= P(-2 \le Z \le 2)$

$= P(-\infty \le Z \le 2) - P(-\infty \le Z \le -2)$

$= 0.9772 - 0.0228$

$= 0.9544$

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13-09-24 Ex ③ The lifetime of certain brand of electric bulbs maybe consider as random variable with the mean 1200 hours and standard deviation 250 hours. Find the probability using ~~std~~ c.l.t that the avg lifetime of 60 bulbs exceeds ~~120~~ 1250 hrs.

$n = 60$

$\mu = 1200$ hrs

$\sigma = 250$ hrs $\quad \sigma^2 = 62500$

$P(\bar{x} > 1250) = \cancel{P\left(\frac{1250-1200}{250/\sqrt{60}} \le 1250\right)}$

$= P(|\bar{x} - \mu| \ge 1250 - \mu)$

$= P\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \ge \frac{1250-\mu}{\sigma/\sqrt{n}}\right)$

$= P(|Z| \ge 1.549)$

$= 1 - P(|Z| < 1.549)$

$= 1 - 0.9394$

$= 0.0606$

Ex ④ A distribution with unknown mean μ has variance 1.5. ~~Using~~ Using clt find how large a sample should be taken from the distribution in order that the probability will be at least 0.95 such that this probability sample mean will be with 0.5 of the population mean.

$\sigma^2 = 1.5 \qquad \sigma = 1.22$

$P(\mu - 0.5 \le \bar{x} \le \mu + 0.5) \ge 0.95$

$P(-0.5 \le \bar{x} - \mu \le 0.5) \ge 0.95$

$P(|\bar{x} - \mu| \le 0.5) \ge 0.95$

$P\left(\frac{|\bar{x} - \mu|}{\sigma/\sqrt{n}} \le \frac{0.5}{\sigma/\sqrt{n}}\right) \ge 0.95$

$P(Z \le 0.408\sqrt{n}) \ge 0.95$

The corresponding entry in z-table of 0.95 is 1.65

$0.4082\sqrt{n} = 1.65$

$\sqrt{n} = \frac{1.65}{0.4082}$

$n = (4.042)^2$

$n = 16.33$

$n \approx 17$

Because if we take 17 then its above $P = 0.95$

Ex ⑤ The burning time of certain type of lamp is an experimental random variable with mean 30 hrs. What is the probability that 144 of this lamp will provide more than 4500 hrs of burning time.

$n = 144$ $\mu = 30$

$np = \mu$

$p = \frac{30}{144} = \frac{10}{48} = \frac{5}{24} = 0.208$

$q = 0.7916$

$\sigma^2 = npq$

$= 144 \times 0.208 \times 0.7916$

$= 23.75$

$\sigma = 4.873$

$z =$

Sampling Distribution

Population: The group of individuals under study is called population or universe.

Sampling: A part selected from the population is called sample and the process of selection of the sample is called sampling.

Notation:

1) μ = Mean of population
2) $\bar{x}$ = Mean of sample
3) N = Size of population
4) n = Size of sample
5) σ = Standard deviation of population
6) s = Standard deviation of sample

Testing of Hypothesis

On the basis of sample information we make certain decision about population. In taking such decision we make assumption. This assumption is called statistical hypothesis.

1) Null hypothesis (H_0): In this hypothesis we shall ~~pursue~~ presume that there is no significant difference between observed value and expected value.

2) Alternate hypothesis (H_a): It specifies the range of values rather than one value.

Level of Significance:

It is usually applied in the tested of hypothesis are 5% or 1%. It is always fixed in advance before collecting the sample information.

Critical Region:

The level marked by probability is 0.05 or 0.01 which decide the significance of an event and the corresponding region are called critical region.

A region ~~R~~ in the sample space S which amounts to rejection of H_0 is called critical region of rejection.

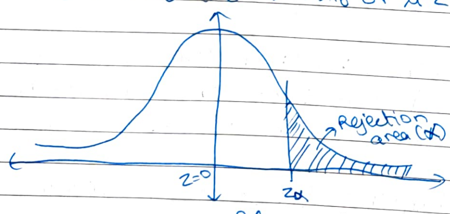
Error in sampling:

1) Type I error: Reject H_0 when it is true. $\boxed{z > z_\alpha}$
2) Type II error: Accept H_0 when it is false.

One tailed and two tailed test

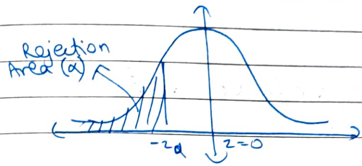
In alternate hypothesis H_a if ~~$\mu > \mu_0$~~

1) $\mu > \mu_0 \Rightarrow$ one tailed (right tailed)
   $\mu < \mu_0 \Rightarrow$ one tailed (left tailed)
   ~~$\mu > \mu_0$~~ $\mu \neq \mu_0$ (i.e. $\mu > \mu_0$ or $\mu < \mu_0$) $\Rightarrow$ two tailed

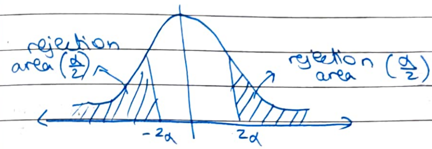


right tailed

$$P(z > z_\alpha) = \alpha$$



left tailed



two tailed test

LOS (α)	One tailed (right tailed)	One tailed (left tailed)	Two tailed
1%	2.33	-2.33	2.58
5%	1.65	-1.65	1.96

19-09-23 **Test the significance for single mean (Large Samples):**

Note that standard normal variable is

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \quad \text{OR} \quad z = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

Example ① A random sample of 400 members is found to have a mean of 4.45 cms. Can it be reasonably regarded as a sample from a large population whose mean is 5 cms and variance is 4 cms

$n = 400$

$\bar{X} = 4.45$ cms

$\mu = 5$ cms

$\sigma^2 = 4$ cms

$\sigma = 2$ cms

H_0 : $\mu = 5$ cms

H_a : $\mu \neq 5$ cms ($\mu > 5$ or $\mu < 5$)

Two-tail test

$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

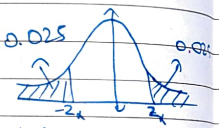
$$= \frac{4.45 - 5}{2/\sqrt{400}}$$

$$= \frac{-0.55}{\cancel{2/20}\ 0.1}$$

$$= -5.5$$

$|Z| = 5.5$

0.025+0.025
= 0.05



(α)

~~Lo~~ Level of Significance (LOS) = 5% (assume 5% since it is not given)

Critical value : The value corresponding to $\alpha = 0.05$ is $|Z_\alpha| = |-1.96| = 1.96$

Decision : $|Z| > |Z_\alpha|$ ($5.5 > 1.96$)

∴ Null hypothesis (H_0) is accepted

Result :

There is no significant difference between μ and $\bar{x}$

Example ② A sample of 900 members has a mean of 3.4 cms and standard deviation ~~2.61~~ 2.61 cms is the sample from large population of mean 3.25 cms ~~and~~ and ~~sta~~ standard deviation is 2.61 cms (LOS = 2%)

$n = 900$

$\bar{x} = 3.4$ cms

$\mu = 3.25$ cms

$\sigma = s = 2.61$

H_0 : $\mu = 3.25$ cms

H_a : $\mu \neq 3.25$ ($\mu > 3.25$ or $\mu < 3.25$)

Two tailed test

$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$= \frac{3.4 - 3.25}{2.61/30}$$

$$= 1.724$$

LOS = 2% = 0.02 = α

Critical value $\alpha = 0.02$

$Z_\alpha = \cancel{3.08}\ -2.33$

$|Z_\alpha| = 2.33$

Decision $|Z| < |Z_\alpha|$ ($1.724 < 2.33$)

H_0 is rejected

∴ There is a significant difference between $\underset{\text{population mean}}{\mu}$ and $\underset{\text{sample mean}}{\bar{x}}$

Example ③ It is hoped that a newly developed pain reliever will be more quickly produced perceptible reduction in pain to patient after ~~major~~ minor surgeries than a standard pain reliever. The standard pain reliever is known to bring relief in the average of 3.5 minutes with standard deviation 2.1 minute. To test whether new pain reliever works more quickly then the standard one, 50 patients with minor surgeries were given new pain reliever and their time to relief were recorded. Experiment in sample mean $\bar{x} = 3.1$ min and standard deviation $s = 1.5$ min (LOS = 5%)

$n = 50$, $\bar{x} = 3.1$ min, $\mu = 3.5$ min, $s = 1.5$ min

$\sigma = 2.1$ min

H_0: $\mu = 3.5$ min

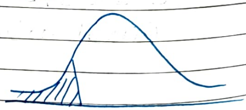
H_a: $\mu < 3.5$ min

One tail test

$$z = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{3.1 - 3.5}{1.5/\sqrt{50}} = \frac{-0.4}{1.5} \times \sqrt{50} = 1.885$$

$\alpha = 0.05$

$z_\alpha = -1.64$



z_α

Decision $z < z_\alpha$ $(-1.885 < -1.64)$

H_0 is rejected (H_a is accepted)

IV

SKIP

23-08-24 Example ③ The average of the various hourly wages of the sample of 150 worker in the plant A was ₹2.56 with std deviation of ₹1.08. The hourly wage of 200 sample of plant B is ₹2.87 with std deviation ₹1.28. Can an applicant safely assume that hourly wages paid by plant B are higher than those paid by plant A.

$n_1 = 150$ $\qquad$ $n_2 = 200$

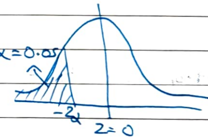
$\bar{x}_1 = 2.56$ $\qquad$ $\bar{x}_2 = 2.87$

$S_1 = 1.08$ $\qquad$ $S_2 = 1.28$

$H_0 : \mu_1 = \mu_2$

$H_a : \mu_1 < \mu_2$ $\qquad$ (Left-tailed test)

$$S.E. \; z = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$



$$= \sqrt{\frac{1.1664}{150} + \frac{1.6384}{200}}$$

$$= 0.126$$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{SE}$$

$$= \frac{2.56 - 2.87}{0.126}$$

$$= -2.46031746$$

L.O.S. $\alpha = 0.05 = 5\%$ (considered)

Critical Value : The corresponding entry in z-table of $\alpha = 0.05$ is $z_\alpha = 1.64$

$z_\alpha = -1.64$ (left tailed test)

Decision : $z < z_\alpha$

$\therefore H_0$ is accepted

Therefore, the hourly wages paid by plant B is not higher than those paid by plant A.

Student - t - distribution (t-test)

This test is useful when the size of sample is 30 or less.

1) ~~In~~ Test the significance for difference between sample mean and population mean.

Formula

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$$

$$S^2 = \frac{1}{(n-1)} \sum_{i=1} (x_i - \bar{x})^2$$

$$\bar{x} = \frac{1}{n} \sum_{i=1} x_i$$

26-09-24 In $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$ if s is not directly provided

$t = \dfrac{\bar{x} - \mu}{S/\sqrt{n-1}}$ if s is directly provided in data

Degree of freedom (ν):

The number of independent parameters required to specify the location of particle in the space is called ~~dg~~ degree of freedom.

Note: for single mean dof $= n-1$

The mean of weekly sales of soap bars in departmental store was 146.3 bars per soap. After an advertising campaign the mean in 22 stores for a typical week increased to 153.7 and showed a standard deviation of 17.2. Was the advertising campaign successful.

$n = 22$ $\bar{x} = 153.7$ $s = 17.2$ $\mu = 146.3$

$H_0 = 146.3$ $H_a = \mu > 146.3$ (Right tailed test)

$$t = \frac{7.4 \times \sqrt{21}}{17.2} \quad \left[t = \frac{\bar{x} - \mu}{S/\sqrt{n-1}} \right]$$

$t = 1.971$

LOS: $\alpha = 5\% = 0.05$

$d.o.f = n-1$
$= 21$

Critical value the corresponding entry in t-table for $\alpha = 0.05$ & dof $= 21$, $t_\alpha = 1.72$

Decision $t > t_\alpha$

H_0 is rejected

H_a is accepted

The advertising campaign is successful

A random sample of 10 boys have the IQ: 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Do this data supports the assumption of a population mean of IQ 100

$n = 10$ $\mu = 100$

$$\bar{x} = \frac{70 + 120 + 110 + 101 + 88 + 83 + 95 + 98 + 107 + 100}{10}$$

$= 97.2$

IQ (x_i)	70	120	110	101	88	83	95	98	107	100
$x_i - \bar{x}$	-27.2	22.8	12.8	3.8	-9.2 ~~3.8~~	-14.2	-2.2	0.8	9.8	2.8
$(x_i - \bar{x})^2$										

$\sum (x_i - \bar{x})^2 = 1833.60$

$$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2 = \frac{1833.60}{9} = 203.73$$

$\therefore s = 14.27$

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$H_0 : \mu = 100$

$H_a : \mu \neq 100$ (Two tail test)

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{97.2 - 100}{14.27/\sqrt{10}} = -0.62$$

$|t| = 0.62$

DOF $= n - 1 = 9$

LOS $= \alpha = 5\% = 0.05$

Critical value at $\alpha = 0.05$ and $\nu = 9$ $\therefore$ C.V. $= t_{\alpha\nu} =$ ~~1.833~~ 2.262

Decision $|t| < |t_{\alpha\nu}|$

H_0 is accepted

Therefore data supports the assumption that $\mu = 100$.

27-09-24 Test the significance for difference between two sample means (Dependent samples)

Formula:

- $t = \dfrac{\bar{x} - \bar{y}}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$
- $s^2 = \dfrac{1}{n_1 + n_2 - 2}\left[\sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{i=1}^{n_2}(y_i - \bar{y})^2\right]$
- $s^2 = \dfrac{1}{n_1 + n_2 - 2}\left[n_1 s_1^2 + n_2 s_2^2\right]$ ...... for Unbiased condition
- $s^2 = \dfrac{1}{n_1 + n_2 - 2}\left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right]$ ..... for biased condition
- Degree of freedom $(V) = n_1 + n_2 - 2$

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Ex ①: Sample of two types of electric bulbs were tested for length of life and the following data were obtained

Sample no.	Type I	Type II
Sample size	$n_1 = 8$	$n_2 = 7$
Sample mean	$\bar{x}_1 = 1234$ hrs	$\bar{x}_2 = 1036$ hrs
Sample S.D's	$S_1 = 36$ hrs	S_2 40hrs

Is the difference in the mean sufficient to warrant that Type I is superior than Type II regarding length of life.

$H_0 = \mu_1 = \mu_2$

$H_a = \mu_1 > \mu_2$

$$S^2 = \frac{1}{n_1 + n_2 - 2}\left[n_1 s_1^2 + n_2 s_2^2\right]$$

$$= \frac{1}{8 + 7 - 2}\left[8 \times 36^2 + 7 \times 40^2\right]$$

$$\frac{1}{13}\left[10368 + 11200\right]$$

$= 1659.0769$ hrs

$S = 40.73$

$$t = \frac{\bar{X} - \bar{Y}}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{198}{\sqrt{1659.0769\left(\frac{1}{8} + \frac{1}{7}\right)}}$$

$t = 9.39$

LOS $= 0.05$

DOF $= \nu = n_1 + n_2 - 2 = 13$

$\alpha = 0.05$ $\nu = 13$

$t_{\alpha\nu} = 1.771$

$t > t_{\alpha\nu}$

H_0 is rejected

H_a is accepted

Decision

Type 2 is superior.

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Height of 6 randomly chosen sailors (in inches) 63, 65, 68, 69, 71 and 72. Those of 10 randomly chosen soldiers are 61, 62, 65, 66, 69, 69, 70, 71, 72 and 73. Discuss the like that this data throw on the suggestion that sailors are on the avg taller than soldiers (LOS = 1%)

~~Mean sailors =~~

$n_1 = 6$ $n_2 = 10$

$H_0 : \mu_1 = \mu_2$

$H_a : \mu_1 < \mu_2$ (Left tailed test)

Mean sailors = $\frac{63+65+68+69+71+72}{6} = 68$

Mean soldier = $\frac{61+62+65+66+69+69+70+71+72+73}{10}$

$= 67.8$

$(X_i - \bar{x})^2$

$x_1 = 25$, $x_2 = 9$, $x_3 = 0$, $x_4 = 1$, $x_5 = 9$, $x_6 = 16$

$(Y_i - \bar{y})^2$

$Y_1 = 46.24$, $Y_2 = 33.64$, $Y_3 = 7.84$, $Y_4 = 3.24$, $Y_5 = 1.44$, $Y_6 = 1.44$

$Y_7 = 4.84$, $Y_8 = 10.24$, $Y_9 = 17.64$, $Y_{10} = 27.04$

$\Sigma(x_i - \bar{x})^2 = 60$

$\Sigma(Y_i - \bar{Y})^2 = 153.60$

$s^2 = \frac{1}{14}[213.60]$

$s^2 = 15.2571$

$t = \frac{\bar{X} - \bar{Y}}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.2}{\sqrt{4.06857}}$

$t = 0.099$

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LOS = 1% = 0.01

DOF = $n_1 + n_2 - 2 = 14$

$t_{\alpha v} = 2.624$

$t < t_{\alpha v}$

H_0 is Accepted

Decision: There is no significant height difference between sailor and soldiers. Therefore sailor is not taller than soldiers

∴ Data are inconsistent with the suggestion that sailors are on the avg taller than soldier.

30-09-24 Test the significance for difference between the two samples mean (Independent samples)

Formula

$$t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

To test the claim that the resistance of a wire can be reduced by at least 0.05 Ω by alloying, 25 values obtained for each alloying wire and standard wire produced the following results

	Mean	S.D.
Alloyed wire	0.083 Ω	0.003 Ω
Standard wire	0.136 Ω	0.002 Ω

Test at 5% LOS whether or not the claim is substantiated.

$n_1 = 25$ $n_2 = 25$
$\bar{x}_1 = 0.083\Omega$ $\bar{x}_2 = 0.136\Omega$
$S_1 = 0.003\Omega$ $S_2 = 0.002\Omega$
$(\mu_x - \mu_y) \geq 0.05\Omega$

$H_0 : \mu_x - \mu_y \geq 0.05$
$H_a : \mu_{x_1} - \mu_{x_2} < 0.05$ (Left tailed test)

$$s^2 = \frac{1}{n_1+n_2-2}\left[n_1 s_1^2 + n_2 s_2^2\right]$$
$$= \frac{1}{25+25-2}\left[25(0.003)^2 + 25(0.002)^2\right]$$
$$= 6.77\times10^{-6}$$

$$t = \frac{(0.083-0.136)-0.05}{\sqrt{6.77\times10^{-6}\left(\frac{1}{25}+\frac{1}{25}\right)}} = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_{x_1}-\mu_{x_2})}{\sqrt{s^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$$

$\boxed{t = -140}$

LOS : $\alpha = 5\% = 0.05$ DOF $= v = n_1+n_2-2 = 48$

$t_{\alpha v} = -1.684$

$t < t_{\alpha v}$
H_0 is accepted
Hence the claim is substantiated.

In a random experiment to compare two types of animal foods A and B, the following results of increase in weights are observed in the animals.

Animal no		1	2	3	4	5	6	7	8
Increase in wt	Food A	49	53	51	52	47	50	52	53
	Food B	52	55	52	53	50	54	54	53

Assuming that the two samples of animals are independent, can we conclude that food B is better than food A

$\bar{X}_1 = (49 + 53 + 51 + 52 + 47 + 50 + 52 + 53)/8$
$= 50.125$

$\bar{X}_2 = 52.875$

$n_1 = 8$ $n_2 = 8$
$\bar{X}_1 = 50.125$ $\bar{X}_2 = 52.875$

$(x_i - \bar{x}_1)^2 = 0.765 + 8.265 + 0.765 + 3.515 + 9.765 + 0.015 + 3.515 + 8.265$

$s_1^2 = \cancel{84.875}$ 35.374

s_2^2

$(x_i - \bar{x}_2) = 0.765 + 4.515 + 0.765 + 0.015 + 8.265 + 1.265 + 1.265 + 0.015$
$= 16.875$

$s^2 = \frac{1}{14}(50.875$