A NOTE ON A MULTIVARIATE GENERALIZATION OF THE KRUSKAL-WALLIS TEST

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ABSTRACT

Marketers are often interested in testing whether the mean vectors of multivariate distributions are equal. The test usually applied, one-way MANOVA, assumes the distributions are multinormal. Unfortunately, this assumption is not supported in many studies. As an alternative, a nonparametric multivariate one-way analysis of variance procedure is presented.

INTRODUCTION

In statistical literature, reference is made to two types of errors, rejection of a true hypothesis (Type I error) and acceptance of a false hypothesis (Type II error). Not discussed formally is another type error, a Type III error, which is made when a person incorrectly assumes that his data conform with a set of assumptions. In most applied research, a Type III error is present since the data seldom conform exactly to the set of assumptions laid down for the purpose of deriving the mathematical theory. The important question is, how serious is the Type III error?

The seriousness of this error can be evaluated along two dimensions. First, does the error affect the probability statements associated with statistical inferences? i.e., Are the inferences valid? Second, does the error affect the amount of information extracted from the data? i.e., Are the inferences efficient? When a researcher selects a particular statistical technique to be used in analyzing data, his inferences are only as good as the efficiency of that technique in extracting information on the reality of interest.

Unfortunately, the actual measurement of validity and efficiency of statistical techniques is still in its infancy. Only in quite restricted situations have any results been published. All of these published results concern univariate statistical techniques. In the absence of such theoretical results, what should the practitioner do? A reasonable approach might be as follows:

1. Carefully examine the assumptions underlying the development of the statistical technique.
2. Carefully examine the reality giving rise to the data.
3. Choose the statistical technique with the set of assumptions which most closely matches the reality giving rise to the data.
4. In cases where there are several viable techniques, use them all and compare results.
5. Mitigate conclusions with respect to comparative results.

In order to follow the above or similar steps, the researcher must be aware of the alternative statistical analyses which could be applied to his data.

The following presentation makes available a test statistic which previously has been unavailable to practical users. The technique is a nonparametric alternative to one-way multivariate analysis of variance (i.e., one-way MANOVA). In the discussion of this test, the structure of the data is noted. It is this structure and its nonconformity with an assumption of the commonly used MANOVA test that accentuate the value of this nonparametric alternative. A marketing example which uses data of a typical structure is presented to create an intuitive understanding of the test statistic and to illustrate how and when it should be used. The reader should keep in mind that the emphasis of this paper is not on marketing implications, but is methodological.

COMMUNICATION EXPERIMENT

An experiment was conducted to determine whether or not an individual's attitude toward various product alternatives is influenced by the source of a message. Fifty-seven students in an undergraduate marketing class were randomly assigned to one of three treatment groups or to a control group. The members within each group were asked to examine four different pieces of luggage. In addition, the students in the three treatment groups were exposed to video tape messages which presented information on the luggage. The messages also attempted to persuade the viewers on the desirability of the various bags. All of the video tape messages were identical in content. The only difference in the communication presented to each treatment group was that the source of the message differed for each group, that is, different individuals presented the information. The source for Treatment Group I was the school's associate dean who was well known by the students and represented an authoritative figure. The captain of the university football team, who was a business major and a popular student, was the message source for Treatment Group II. Treatment Group III had as its message source a female typist, unknown to the students, who represented a housewife figure. At the completion of the experiment, the students in all treatment and control groups answered a questionnaire which measured the satisfaction which they associated with each product attribute, their perception of each product attribute as related to each piece of luggage, and a rank ordering of their brand preferences. All of these measures were ordinal in rank. A Fishbein type attitude model was then used to obtain attitude measures toward each of the bags.

THE NONPARAMETRIC TEST

Consider the vectors

$$\begin{bmatrix} A_{11}^{(1)} \\ \vdots \\ A_{41}^{(1)} \end{bmatrix}, \ldots, \begin{bmatrix} A_{1n_1}^{(1)} \\ \vdots \\ A_{4n_1}^{(1)} \end{bmatrix}, \begin{bmatrix} A_{11}^{(2)} \\ \vdots \\ A_{41}^{(2)} \end{bmatrix}, \ldots, \begin{bmatrix} A_{1n_2}^{(2)} \\ \vdots \\ A_{4n_2}^{(2)} \end{bmatrix}, \ldots, \begin{bmatrix} A_{11}^{(4)} \\ \vdots \\ A_{41}^{(4)} \end{bmatrix}, \ldots, \begin{bmatrix} A_{1n_4}^{(4)} \\ \vdots \\ A_{4n_4}^{(4)} \end{bmatrix} \tag{1}$$

where $A_{ij}^{(k)}$ is the attitude of individual j in group k toward luggage i and where n_k is the sample size of the k^{th} group. Next consider the question: did the source of communication influence brand attitude? With reference to the above vectors this question can be rephrased as follows: are the multivariate distributions of the three samples identical? Letting $F_k(\underline{A})$ represent the multivariate distribution of the vectors of the k^{th} group of sample, we are interested in the hypothesis.

$$H_0\colon\ F_1(\underline{A}) \equiv F_2(\underline{A}) \equiv F_3(\underline{A}).$$

Since the subjects were in a controlled experimental setting, we can alter H_0 to a form that considers the location differences in F_k; i.e.,

$$H_0'\colon\ F_1(\underline{A}) = F_2(\underline{A} + \underline{\delta}_2) = F_3(\underline{A} + \underline{\delta}_3).$$

In most experiments this hypothesis becomes

$$H_0''\colon\ \underline{\mu}_1 = \underline{\mu}_2 = \underline{\mu}_3$$

where $\underline{\mu}_k$ is the location parameter of the distribution of the k^{th} sample or experimental group. $\underline{\mu}_k$ is a vector parameter, and the test for H_0'' is the usual multivariate one-way analysis of variance. It is important to note that the multivariate one-way analysis of variance test makes the following assumptions about the vector measurements in each sample:

(A) The distribution of the vectors of each sample is multivariate normal. (This assumption implies the measurements of the vector are at least interval scaled.)

(B) The vectors of each sample are independently and identically distributed.

Given the nature of the questionnaire, the data obtained were not interval scaled, nor were the distributions of the vectors normal. Consequently, the usual multivariate one-way analysis of variance is based on an assumption that is not consistent with the nature of the data to be analyzed. A form of analysis is needed which is more consistent with the structure of the data.

An intuitive development and outline of a nonparametric multivariate alternative to the "normal" one-way MANOVA is now given. A more formal and theoretical presentation of this alternative can be found in Puri and Sen [1]. It should be noted that both analyses require assumption (B) for their theoretical development and for their proper application.

Let $\{x_\alpha^{(k)} = (x_{1\alpha}^{(k)} \ldots, x_{p\alpha}^{(k)})', \alpha = 1, \ldots, n_k, k = 1, \ldots, c\}$ be a set of independent vector-valued random values. The notation indicates a natural partitioning of the set into c groups. Each of these groups represents a set of vectors that have a common "treatment" effect. The set of vector observations can be arranged into a $p \times N$ matrix where $N = \sum_{k=1}^{c} n_k$. In the matrix presented below the vertical lines interior to the matrix indicate the partitioning into c groups.

$$\left[\begin{array}{ccc|ccc|c|ccc} x_{11}^{(1)} & \ldots & x_{1n_1}^{(1)} & x_{11}^{(2)} & \ldots & x_{1n_2}^{(2)} & & x_{11}^{(c)} & \ldots & x_{1n_c}^{(c)} \\ x_{21}^{(1)} & \ldots & x_{2n_1}^{(1)} & x_{21}^{(2)} & \ldots & x_{2n_2}^{(2)} & & x_{21}^{(c)} & \ldots & x_{2n_c}^{(c)} \\ \vdots & & \vdots & \vdots & & \vdots & \ldots & \vdots & & \vdots \\ x_{p1}^{(1)} & \ldots & x_{pn_1}^{(1)} & x_{p1}^{(2)} & \ldots & x_{pn_2}^{(2)} & & x_{p1}^{(c)} & \ldots & x_{pn_c}^{(c)} \end{array}\right]. \tag{2}$$

The first step in the analysis is to rank the N i-variate observations ($x_{i\alpha}^{(k)}$ where $\alpha = 1, \ldots n_k$, $k = 1, \ldots, c$) in ascending order of magnitude. Let $R_{i\alpha}^{(k)}$ denote the rank of $x_{i\alpha}^{(k)}$ in this set. The observation vector $\underline{x}_{\alpha}^{(k)} = (x_{1\alpha}^{(k)}, \ldots, x_{p\alpha}^{(k)})'$ then gives rise to the rank vector $\underline{R}_{\alpha}^{(k)} = (R_{1\alpha}^{(k)}, \ldots, R_{p\alpha}^{(k)})'$ where $\alpha = 1, \ldots, n_k$, $k = 1, \ldots, c$. The N rank vectors corresponding to the N observation vectors can be represented by the $p \times N$ rank matrix

$$\underline{R}_N = \left[\begin{array}{ccc|c|ccc} R_{11}^{(1)} & \ldots & R_{1n_1}^{(1)} & & R_{11}^{(c)} & \ldots & R_{1n_c}^{(c)} \\ \vdots & & \vdots & \ldots & \vdots & & \vdots \\ R_{p1}^{(1)} & \ldots & R_{pn_1}^{(1)} & & R_{p1}^{(c)} & \ldots & R_{pn_c}^{(c)} \end{array}\right] \tag{3}$$

Next consider the average rank scores $T_{Ni}^{(k)}$ for each $i = 1, \ldots, p$ of the c samples. $T_{Ni}^{(k)}$ can be defined as

$$T_{Ni}^{(k)} = \frac{1}{N+1} \cdot \frac{1}{n_k} \sum_{\alpha=1}^{n_k} R_{i\alpha}^{(k)} \quad , \tag{4}$$

$k = 1, \ldots, c$
$i = 1, \ldots, p.$

Further consider the matrix

$$\underline{V}(\underline{R}_N^*) = [v_{ij}(\underline{R}_N^*)] \tag{5}$$

where

$$v_{ij}(\underline{R}_N^*) = \frac{1}{N(N+1)^2} \sum_{k=1}^{c} \sum_{\alpha=1}^{n_k} R_{i\alpha}^{(k)} R_{j\alpha}^{(k)} - 1/4 \; . \tag{6}$$

The nonparametric multivariate test statistic for the null hypothesis

$$H_0': \ F_1(\underline{x}) = F_2(\underline{x} + \underline{\delta}_2) = \ldots = F_c(\underline{x} + \underline{\delta}_c)$$

is given by

$$L_N = \sum_{k=1}^{c} n_k [\underline{T}_N^{(k)} - (1/2)\underline{j}] \, \underline{V}^{-1}(\underline{R}_N^*) \, [\underline{T}_N^{(k)} - (1/2)\underline{j}]' \tag{7}$$

where $\underline{V}^{-1}(\underline{R}_N^*)$ is the inverse of the matrix $\underline{V}(\underline{R}_N^*)$, $\underline{j}$ is a vector of ones: i.e., $\underline{j} = [1, 1, \ldots, 1]$ and $\underline{T}_N^{(k)} = [T_{N1}^{(k)}, T_{N2}^{(k)}, \ldots, T_{Np}^{(k)}]$ with $T_{Ni}^{(k)}$ defined as above.

Puri and Sen indicate that L_N is approximately chi-square distributed with $p(c\text{-}1)$ degrees of freedom. Consequently, a test for H_0' having approximate significance level ϵ is to reject H_0' if $L_N > X^2_{1-\epsilon}(pc\text{-}p)$.

APPLICATION

The nonparametric alternative to one-way MANOVA was applied to the data obtained from the marketing experiment previously outlined. In this experiment the number of observations N equaled 57 and the number of groups c was 4 (three treatment groups and one control group). The three treatment groups and the control group consisted of 16, 14, 15, and 12 students, respectively. Thus, $n_1 = 16$, $n_2 = 14$, $n_3 = 15$, and $n_4 = 12$. Since four variables were observed on each observation (attitude toward each of the four bags), the number of variates p equaled 4. Equation (2) can be expressed for this study as

$$\left[\begin{array}{c|c|c} x_{11}^{(1)} \ldots x_{1,16}^{(1)} & & x_{11}^{(4)} \ldots x_{1,12}^{(4)} \\ x_{21}^{(1)} \ldots x_{2,16}^{(1)} & \ldots & x_{21}^{(4)} \ldots x_{2,12}^{(4)} \\ x_{31}^{(1)} \ldots x_{3,16}^{(1)} & & x_{31}^{(4)} \ldots x_{3,12}^{(4)} \\ x_{41}^{(1)} \ldots x_{4,16}^{(1)} & & x_{41}^{(4)} \ldots x_{4,12}^{(4)} \end{array}\right] \tag{2'}$$

The first student in Treatment Group I had attitude scores of 38,-21,-2, and 6 toward the four bags. Thus, the vector $[x_{11}^{(1)}, x_{21}^{(1)}, x_{31}^{(1)}, x_{41}^{(1)}]'$ is $[38,-21,-2,6]'$. The (4 x 57) rank matrix $\underline{R}_N$ is found by ranking in ascending order the values within each of the four rows of (2′). By substituting rows of (2′) into (4), values for $T_{Ni}^{(k)}$ are obtained.

For example, the third row of $\underline{R}_N$ is (13, 39, . . . , 1 | 52, 2, . . . , 45 | 50, 17, . . . , 29 | 49, 28, . . . , 34).
thus

$$T_{N3}^{(1)} = \frac{(13 + 39 + \ldots + 1)}{(57 + 1)16} = .425,$$

$$T_{N3}^{(2)} = \frac{(52 + 2 + \ldots + 45)}{(57 + 1)14} = .560,$$

$$T_{N3}^{(3)} = \frac{(50 + 17 + \ldots + 29)}{(57 + 1)15} = .501,$$

$$T_{N3}^{(4)} = \frac{(49 + 28 + \ldots + 34)}{(57 + 1)12} = .529.$$

For group k, $\underline{T}_N^{(k)} = [T_{N1}^{(k)}, T_{N2}^{(k)}, \ldots, T_{N4}^{(k)}]$. The vector $[\underline{T}_N^{(k)} - (1/2)\underline{j}]$ is found by subtracting 1/2 from each value $T_{Ni}^{(k)}$.

To illustrate how the values of v_{ij} in the matrix $\underline{V}(\underline{R}_N^*)$ are found, consider the fourth row of the rank matrix $\underline{R}_N$ which is

(8, 17, . . . , 56 | 53, 1, . . . , 45 | 29, 5, . . . , 10 | 50, 14, . . . , 30).

Using equation (6) it can be seen that the value of v_{34} in matrix $\underline{V}(\underline{R}_N^*)$ is

$$v_{34}(\underline{R}_N^*) = \frac{[(13)(8) + (39)(17) + \ldots + (34)(30)]}{57(57 + 1)^2} - 1/4 \ .$$

Applying equation (7) to the matrices derived from the above operations, one finds that the test statistic L_n for the example problem is 20.398 with $p(c\text{-}1)$, or 12, degrees of freedom. The null hypothesis for this problem is that the four groups are the same in terms of their attitudes toward the four pieces of luggage. The alternative hypothesis is that the groups differ in their luggage attitudes. If the null hypothesis is rejected, it must be concluded that the source of the message has influenced attitude. With 12 degrees of freedom and an ϵ level of 0.05, the critical chi-square level is 21.0. At an ϵ level of 0.10 the critical level is 18.55. Thus, the null hypothesis is accepted at the 0.05 level of significance.

SUMMARY

When testing the identity of c ($\geqslant 2$) independent samples of multivariate distributions $F_1, \ldots, F_c$ all of which are normal, the distributions can differ only in location (mean) vectors and dispersion matrices. For many situations the dispersion matrices are assumed to be equal and interest revolves around testing whether the mean vectors are all equal. The analysis generally used is MANOVA. The distribution theory associated with the test statistic in MANOVA is heavily dependent upon the often unrealistic assumption of normality. Thus, the question of the existence of a more appropriate analysis arises. Such an alternative was presented in this paper. The specific procedure discussed is a multivariate extension of the Kruskal-Wallis test and can be thought of as a nonparametric alternative to one-way multivariate analysis of variance.

REFERENCES

[1] Puri, Madan Lal and Pranab Kumar Sen, *Nonparametric Methods in Multivariate Analysis,* New York: John Wiley & Sons, 1971.