An Overview of
The Kruskal Wallis Test

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Contents

Origins of the test

- The **Kruskal–Wallis test** by ranks or **Kruskal–Wallis H test** named after **William Kruskal** and **W. Allen Wallis**, or **one-way ANOVA on ranks** is a non-parametric method for testing whether samples originate from the same distribution.

Origins of the test

- The **Kruskal–Wallis test** by ranks or **Kruskal–Wallis H test** named after **William Kruskal** and **W. Allen Wallis**, or **one-way ANOVA on ranks** is a non-parametric method for testing whether samples originate from the same distribution.
- It is used for comparing two or more independent samples of equal or different sample sizes. It extends the **Mann–Whitney U test**, which is used for comparing **only two** groups. The **parametric equivalent** of the Kruskal–Wallis test is the **one-way analysis of variance (ANOVA)**.

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- However the test **does not identify where** this **stochastic dominance** occurs or for **how many pairs of groups** stochastic dominance obtains.
- For this we can compare pairwise groups using **pairwise Mann–Whitney** or using asymptotic distributions.

Basic Assumptions of the test

- Since it is a nonparametric method, the Kruskal–Wallis test does not assume a **normal distribution** of the **residuals**, unlike the analogous one-way analysis of variance.

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- Here we only assume of an **identically shaped** and **scaled** distribution **for all groups**, **except** for any **difference** in **medians**.

Hypothesis for the Test

- Here we consider k mutually independent samples with n_i observations in each sample $i = 1, \ldots, k$ from continuous populations with the assumption that they are **identically shaped** and **scaled**. Let the median of the i^{th} sample be $\theta_i, i = 1, \ldots, k$.

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- Then the null hypothesis will be all the medians for individual groups are equal which we can write as :-

$$\mathscr{H}_0 : \theta_1 = \theta_2 = \cdots = \theta_k$$

- And naturally, the alternate hypothesis will be $\mathscr{H}_1$: At least two $\theta_i' s$ differ.

Testing Procedure

- Since under $\mathscr{H}_0$ we have essentially a single sample of size $N = \sum n_i$ from the common population, combine the N observations into a single ordered sequence from smallest to largest and assign the ranks $1, 2, .., N$.

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- Let us denote the rank of the j^{th} observation from the i^{th} sample as R_{ij}.

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- The deviation for each observed group rank sum from its expected value i.e., $R_i - \frac{n_i(N+1)}{2}$ can be thought as a measure of deviation from the null assumption.
- Hence, a reasonable test statistic could be based on a function of the all these deviations.

The S statistic

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- So we construct the statisitc $S = \sum_{i=1}^{k} \left[R_i - \frac{n_i(N+1)}{2} \right]^2$.
- Hence, the null hypotheis $\mathscr{H}_0$ should be rejected for large value of S.

Null Distribution of S (no tie case)

- In order to determine the null probability distribution of S, we first consider all the possible arrangements of ranks $1, 2, \ldots, N$ into k groups of size n_i each. This can be done in $\frac{N!}{\prod_{i=1}^{k} n_i!}$.

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- Then for each of these arrangements, we calculate the value of the S statistic and let us denote by $t(s)$ number of arrangements for which $S = s$.
- Finally, we can write, $P[S = s] = t(s) \frac{\prod_{i=1}^{k} n_i!}{N!}$.

Drawbacks of S Statistic

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n	$(n_1+n_2+n_3)!/n_1!n_2!n_3!$	n	$(n_1+n_2+n_3)!/n_1!n_2!n_3!$
2	90	9	227,873,431,500
3	1,680	10	5,550,996,791,340
4	34,650	11	136,526,995,463,040
5	756,756	12	3,384,731,762,521,200
6	17,153,136	13	84,478,098,072,866,400
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- Also there is no standard asymptotic distribution for S which can be used for large sample tests.
- Lasltly, S only consider the sum of square of deviations of R_i from its mean but it do not standarize the observations R_i.

Kruskal-Wallis H Statistic

- Due to all the drawbacks of the S statistic, William H. Kruskal & W. Allen Wallis (1952) proposed the following statistic for testing $\mathscr{H}_0$:-

$$H = \frac{12}{N\left(N+1\right)}\sum_{i=1}^{k}\frac{R_i^2}{n_i} - 3\left(N+1\right)$$

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- Here,

$$\begin{aligned} k &= \text{the number of samples} \\ n_i &= \text{the number of observations in the } i^{th}\text{sample} \\ N &= \sum_i n_i \text{ total number of observations in all samples} \\ R_i &= \text{ the sum of the ranks of the } i^{th} \text{ sample} \end{aligned}$$

Different Representations of H

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2. Secondly, $H = \frac{12}{N(N+1)} \sum_{i=1}^{k} \frac{1}{n_i} \left[R_i - \frac{n_i(N+1)}{2}\right]^2 = \frac{12}{N(N+1)} \sum_{i=1}^{k} n_i \left[\overline{R}_i - \frac{(N+1)}{2}\right]^2$ which is useful in further discussions since it uses the average rank sums $\overline{R}_i$.

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3. For understanding the similarity of this test with the conventional ANOVA test we write H in the following form :-

$$H = (N-1)\,\frac{\sum_{i=1}^{k} n_i\left(\overline{R}_i - \overline{R}\right)^2}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(R_{ij} - \overline{R}\right)^2}$$

where $\overline{R} = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} R_{ij} = \frac{N+1}{2}$.

Kruskal-Wallis H Statistic

- For understanding the nature of H, a better formulation would be :-

$$H = \frac{N-1}{N} \sum_{i=1}^{k} \frac{n_i \left(\overline{R}_i - \frac{N+1}{2} \right)^2}{\left(N^2 - 1 \right) / 12}$$

where $\overline{R}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} r_{ij}$ is the mean of n_i ranks in the i^{th} sample.

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- If we ignore the factor $\frac{N-1}{N}$ and note that $\frac{1}{2}(N+1)$ is the mean and $\frac{1}{12}\left(N^2-1\right)$ is the variance of the uniform distribution over the first N integers, we see that H is essentially a sum of squared standardized deviations of random variables from their population mean.

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- In this respect H is similar to a χ^2 variate which is defined as a sum of square of standardized normal variates, subject to certain conditions on the relations among the terms of the sum.
- If the n_i's are not too small, the $\overline{R}_i$ jointly will be approximately normally distributed and the relations among them will meet the χ^2 conditions.
- We further investigate approximate distribution of H with more rigourous mathematical arguments.

Mean and Variance of $\overline{R}_i$

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$$\text{Population Variance}: \sigma^2 = \frac{N^2-1}{12}$$

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- And the mean rank of the i^{th} sample can be thought as a SRSWOR sample of size n_i hence,

$$E\left(\overline{R}_i\right) = \mu = \frac{N+1}{2}$$
$$V\left(\overline{R}_i\right) = \frac{\sigma^2}{n_i}\frac{N-n_i}{N-1} = \frac{N^2-1}{12n_i}\frac{N-n_i}{N-1} = \frac{(N+1)(N-n_i)}{12n_i}$$
$$\text{Cov}\left(\overline{R}_i, \overline{R}_j\right) = -\frac{N+1}{12}, \rho_{\overline{R}_i, \overline{R}_j} = -\sqrt{\frac{n_i n_j}{(N-n_i)(N-n_j)}}$$

Asymptotic Distribution of $\overline{R}_i$

- If n_i is large, the standarized random variables,

$$Z_i = \frac{\overline{R}_i - \frac{N+1}{2}}{\sqrt{\frac{(N+1)(N-n_i)}{12n_i}}} \overset{d}{\sim} N(0,1)$$

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- Hence, we can say,

$$Z_i^2 = \frac{12n_i}{N(N+1)(N-n_i)}\left(\overline{R}_i - \frac{N+1}{2}\right)^2 \overset{d}{\sim} \chi_1^2$$

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- But we can see there is a linear dependence between the quantities $\overline{R}_i$ as, the sum of all ranks is

$$\sum_{i=1}^{k} n_i \overline{R}_i = \sum_{i=1}^{k} R_i = \sum_{i=1}^{k}\sum_{j=1}^{n_i} r_{ij} = \frac{N\left(N+1\right)}{2}$$

  so all the k variates $Z_i, i = 1, \ldots, k$ can't be independent (Any $k-1$ of them are independent.)

Asymptotic Distribution of H

- Kruskal (1952) showed that under $\mathscr{H}_0$,

$$\sqrt{\frac{N-n_i}{N}} Z_i = \frac{\overline{R}_i - \frac{N+1}{2}}{\sqrt{\frac{N(N+1)}{12n_i}}} \overset{d}{\sim} N(0,1)$$

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  gives a better approximation (faster convergence) for asymptotic normality as it can be applied for individual sample sizes as small as 5.

- Finally we can say that under $\mathscr{H}_0$, if no n_i is very small, the random variable :-

$$H = \sum_{i=1}^{k} \frac{N-n_i}{N} Z_i^2 = \frac{12}{N(N+1)} \sum_{i=1}^{k} n_i \left(\overline{R}_i - \frac{N+1}{2} \right)^2$$

  is approximately distributed as a Chi-Squared Distribution with $k-1$ degrees of freedom $\left(\chi^2_{k-1}\right)$. Hence, we reject $\mathscr{H}_0$ at α level of significance if $H_{obs} \geq \chi^2_{\alpha;k-1}$.

Tie Case

- If ties to the extent t are present and are handled by the midrank method, the variance of the finite population is,

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- Then the Kruskal-Wallis Statistic becomes,

$$\begin{aligned} H^{'} &= \sum_{i=1}^{k} \frac{N - n_i}{N} \left\{ \frac{\left[\overline{R}_i - \frac{N+1}{2}\right]^2}{\frac{(N+1)(N-n_i)}{12 n_i} - \frac{N-n_i}{n_i(N-1)} \frac{\sum t(t^2-1)}{12}} \right\} \\ &= \sum_{i=1}^{k} \frac{\left[\overline{R}_i - \frac{N+1}{2}\right]^2}{\frac{N(N+1)}{12 n_i}\left[1 - \frac{\sum t(t^2-1)}{N(N^2-1)}\right]} = \frac{H}{1 - \frac{\sum t(t^2-1)}{N(N^2-1)}} \end{aligned}$$

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$$\sigma^2 = \frac{N^2-1}{12} - \frac{\sum t\left(t^2-1\right)}{12}$$

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$$\begin{aligned} H^{'} &= \sum_{i=1}^{k} \frac{N-n_i}{N} \left\{ \frac{\left[\overline{R}_i - \frac{N+1}{2}\right]^2}{\frac{(N+1)(N-n_i)}{12n_i} - \frac{N-n_i}{n_i(N-1)} \frac{\sum t(t^2-1)}{12}} \right\} \\ &= \sum_{i=1}^{k} \frac{\left[\overline{R}_i - \frac{N+1}{2}\right]^2}{\frac{N(N+1)}{12n_i}\left[1 - \frac{\sum t(t^2-1)}{N(N^2-1)}\right]} = \frac{H}{1 - \frac{\sum t(t^2-1)}{N(N^2-1)}} \end{aligned}$$

- So, we just need to divide the original H statistic by the factor $1 - \frac{\sum t\left(t^2-1\right)}{N(N^2-1)}$.

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- We reject our null hypothesis $\mathscr{H}_0^{ij} : \theta_i = \theta_j$ at α^* level of significance if,

$$\left|Z_{ij(obs)}\right| > \tau_{\alpha^*} \text{ where } \alpha^* = \frac{\alpha}{k(k-1)}$$

Pairwise Comparison

- Since, we are comparing ${}^{k(k-1)}/_{2}$ many pairs,

$$
\begin{aligned}
P\left(\mathscr{H}_0^{ij} \text{ accepted } \forall i,j\right) &= P\left(\bigcap_{1\le i<j\le k} \mathscr{H}_0^{ij} \text{ accepted }\right) \\
&\ge \sum_{1\le i<j\le k} P\left(\mathscr{H}_0^{ij} \text{ accepted }\right) - \frac{k\left(k-1\right)}{2} \\
&= \sum_{1\le i<j\le k} \left(1-\alpha^*\right) - \frac{k\left(k-1\right)}{2} = 1-\alpha
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$$

- The quantity α is called the experimentwise error rate or the overall significance level, which is the probability of at least one erroneous rejection among the $\binom{k}{2} = {}^{k(k-1)}/_{2}$ pairwise comparisons. Typically, one takes $\alpha \ge 0 \cdot 20$ because we are making such a large number of statements.

Sampling Distribution of H using Simulation

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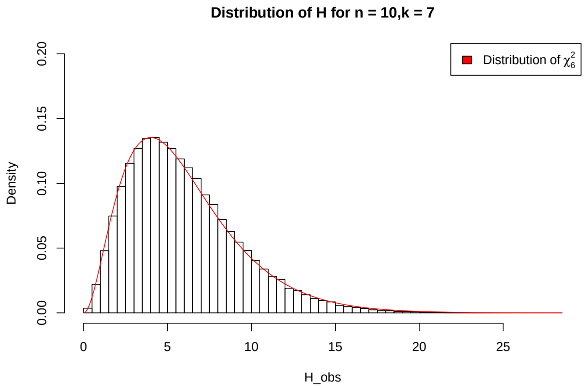
- We can see how accurately the asymptotic distribution fits the actual density function of χ^2_6:-



Demonstration with Example

- For demonstration purposes, we choose a dataset consisting of Mileage of 60 car models from 4 different manufacturing companies. Namely, we have the four companies **Apollo, Bridgestone, CEAT** and **Falken**.

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- Here is a glimpse of the raw dataset we have in R :-

```
   Brands  Mileage
44   CEAT 32.16845
33   CEAT 33.41499
35   CEAT 36.97277
6  Apollo 35.91500
48 Falken 36.12400
2  Apollo 36.43500
55 Falken 37.38200
11 Apollo 36.43000
5  Apollo 36.30400
47 Falken 38.93700
```

Demonstration with Example

- So for exploratory data analysis, we first plot the mileage values for the four different companies(factor levels) :-

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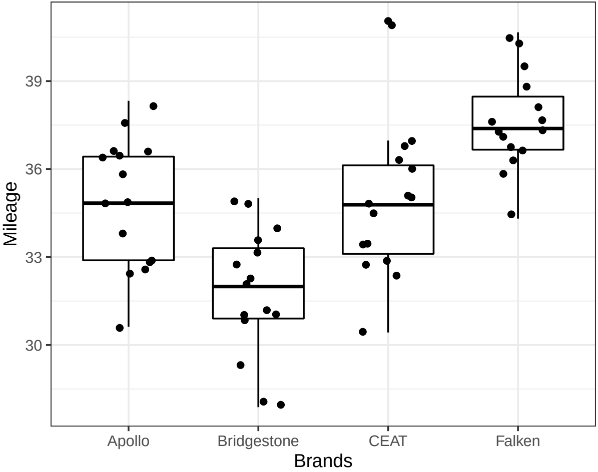
- Since here we have one factor variable with four levels (brands) hence we use boxplot for demonstration :-



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- We shall verify our claims using the Kruskal-Wallis testing procedures stated so far.

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- For this we firstly, assign rank for each observed mileage in the data set.

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- After assigning, the data would look like :-

```
        Brands  Mileage rank
3       Apollo 32.77700   17
42        CEAT 36.11675   37
50      Falken 36.58600   44
54      Falken 40.25200   58
43        CEAT 41.05000   60
37        CEAT 34.95412   32
52      Falken 36.73700   45
14      Apollo 30.62300    5
25 Bridgestone 30.88100    6
26 Bridgestone 28.14400    2
```

Demonstration with Example

- For a better understanding, we plot the mileages along with the assigned rank for each observation in the following plot :-

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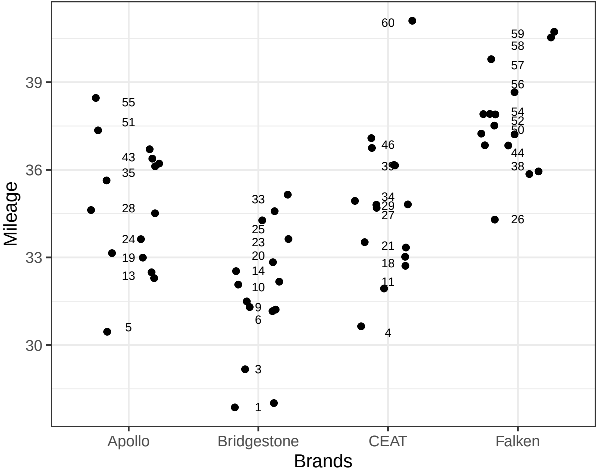
- Here we use the jittered plot for different car brands along different vertical axes and also corresponding rank in the pooled sample :-



Demonstration with Example

- Now, we calculate the rank sum values for each level as :-

```
Apollo Bridgestone        CEAT      Falken
   458         204         443         725
```

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Apollo	Bridgestone	CEAT	Falken
458	204	443	725

- Which means that $R_1 = 458, R_2 = 204, R_3 = 443, R_4 = 725$ and $n_1 = n_2 = n_3 = n_4 = 15$ since we have 15 observations from each brand(company) so $N = \sum n_i = 60$.

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- Hence, the observed value of the Kruskal-Wallis H Statistic is :-

$$\begin{aligned} H_{obs} &= \frac{12}{N\left(N+1\right)} \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3\left(N+1\right) \\ &= \frac{12}{60 \times 61} 64883 \cdot 6 - 3 \times 61 = 29 \cdot 73311 \end{aligned}$$

Demonstration with Example

- Since, $H_{obs} = 29.73311 > \chi^2_{4-1;0.05} = 7.814728$, so we can reject the null hypothesis :-

$$\mathscr{H}_0 : \theta_{\text{Apollo}} = \theta_{\text{Bridgestone}} = \theta_{\text{CEAT}} = \theta_{\text{Falken}}$$

at 5% level of significance.

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- which is significant upto $\alpha = 0.001$ or in other words the difference between the means are highly significant.

Demonstration with Example

- Now, since the null hypotheis is rejected, our natural tendency would be to compare the pairwise mean values for different levels as :-

$$Z_{ij} = \frac{\overline{R}_i - \overline{R}_j}{\sqrt{\frac{N(N+1)}{12}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}}$$

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- In R, all these pairwise differences can be evaluated simultaneously as :-

```
                Apollo Bridgestone       CEAT    Falken
Apollo       0.0000000    2.655359  0.1568125 -2.791263
Bridgestone -2.6553585    0.000000 -2.4985460 -5.446621
CEAT        -0.1568125    2.498546  0.0000000 -2.948075
Falken       2.7912627    5.446621  2.9480752  0.000000
```

Demonstration with Example

- Now, we consider two groups significantly different if

$$\left|Z_{ij(obs)}\right| > \tau_{\alpha^*} \text{ where } \alpha^* = \frac{\alpha}{k\,(k-1)} = \frac{0 \cdot 20}{4 \times 3} \approx 0 \cdot 0166$$

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```

- And the cut off point τ_{α^*} as :-

```
[1] 2.128045
```

Demonstration with Example

- So, now we compare if the $\left|Z_{ij(obs)}\right|$ values exceed τ_{α^*} and thus get the following TRUE-FALSE matrix :-

	Apollo	Bridgestone	CEAT	Falken
Apollo	FALSE	TRUE	FALSE	TRUE
Bridgestone	TRUE	FALSE	TRUE	TRUE
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Bridgestone	TRUE	FALSE	TRUE	TRUE
CEAT	FALSE	TRUE	FALSE	TRUE
Falken	TRUE	TRUE	TRUE	FALSE

- Since, only for the group $($Apollo, CEAT$)$ the outcome is FALSE hence their mean differences are not significant at $\alpha = 20\%$ and we can conclude that all other groups have significantly different mean mileage values and this can also be verified from the boxplot given before.

Some other approximations to the exact distribution of the kruskal-wallis test statistic

- **(Wallace Approximation)** Given by **Wallace (1959)**, this approximation is very similar to the F statistic we use in ordinary analysis of variance that can be written by :-

$$F = \frac{H/k-1}{(N-H-1)/N-k} = \frac{(N-k)\,H}{(k-1)\,(N-H-1)}$$

which approximately follows a $F_{k-1,N-k}$ distribution where H is the ordinary Kruskal-Wallis Statistic.

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- **(Iman Approximation)** This interesting approximation is based on techniques given by **Iman (1974,1976)** where a test statistic is formed by the linear combination of the χ^2 and F approximations already stated as :-

$$J = \frac{(k-1)\,F + H}{2} = \frac{H}{2}\left(\frac{N-k}{N-H-1} + 1\right)$$

  The approximate critical values are given by,

$$J_\alpha \approx \frac{(k-1)\,F_{k-1,N-k;\alpha} + \chi^2_{k-1;\alpha}}{2}.$$

Some other approximations to the exact distribution of the kruskal-wallis test statistic

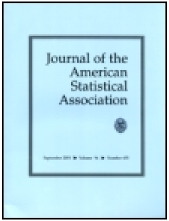
- **(Satterthwaite Approximation)** A more powerful test using the concept of Welch–Satterthwaite equation where the degrees of freedom of the F statistic previously stated is approximated in other words :-

$$F = \frac{H/k-1}{(N-H-1)/N-k} = \frac{N-k}{k-1} \frac{\sum_{i=1}^{k} n_i \left(\overline{R}_i - \overline{R}\right)^2}{\sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(R_{ij} - \overline{R}_i\right)^2} \overset{a}{\sim} F_{k-1,\widehat{f}}$$

where $\widehat{f} = \frac{\left(\sum_{i=1}^{k} (n_i-1)v_i\right)^2}{\sum_{i=1}^{k} (n_i-1)v_i^2}$ and $v_i = \frac{1}{n_i-1} \sum_{j=1}^{n_i} \left(R_{ij} - \overline{R}_i\right)^2$.

References

- Use of Ranks in One- Criterion Variance Analysis William H. Kruskal a & W. Allen Wallis a a University of Chicago Published online: 11 Apr 2012.



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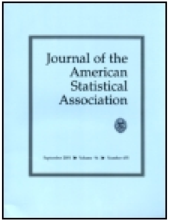

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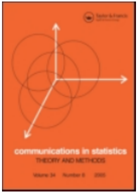

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