Linear Combination

Definition:

A vector w in a vector space V is called a linear combination of the vectors $u_1, u_2, \ldots u_n$ in V if w can be expressed in the form $w = k_1u_1 + k_2u_2 + \ldots + k_nu_n$ where $k_1, k_2, \ldots k_n$ are scalars

Example

Every vector in R^3 can be expressed as a linear combination of the standard vectors

$i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$

$$[(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)]$$

Problems

1. Consider the vectors $u = (1,2,-1)$ and $v = (6,4,2)$ in R^3. Show that $w = (9,2,7)$ is a linear combination of u and v.

Solution:

If w is a linear combination of u and v

then $w = k_1u + k_2v$ ——— I

$$(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$$

To check whether k_1, k_2 exists

$$(9, 2, 7) = (K_1, 2K_1, -K_1) + (6K_2, 4K_2, 2K_2)$$

$$= (K_1 + 6K_2,\ 2K_1 + 4K_2,\ -K_1 + 2K_2)$$

$\therefore$ $K_1 + 6K_2 = 9$, $2K_1 + 4K_2 = 2$, $-K_1 + 2K_2 = 7$

Solve the above three eqns by Gauss Elimination method.

Rewrite in matrix form

$$\begin{pmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix}$$

[Coefficient matrix is denoted by A & RHS by B]

The augmented matrix is

$$(A, B) \sim \begin{array}{c} \begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left(\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 8 & 16 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1'' \\ R_2'' \\ R_3'' = R_2' - R_3' \end{array}$$

$R(A) = 2$, $R(A, B) = 2$

$R(A) = R(A, B) =$ No. of unknowns

$\therefore$ System of equations have unique solutions.

$$\therefore \begin{pmatrix} 1 & 6 \\ 0 & 8 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 16 \\ 0 \end{pmatrix}$$

$K_1 + 6K_2 = 9$ — (1)

$8K_2 = 16 \Rightarrow K_2 = 2$

Substituting $K_2 = 2$ in (1),

$K_1 + 12 = 9 \Rightarrow K_1 = -3$

Substitute K_1 and K_2 in I

$\therefore W = -3U + 2V$

2. Express $-9-7x-15x^2$ as a linear combination of $P_1 = 2+x+4x^2$, $P_2 = 1-x+3x^2$, $P_3 = 3+2x+5x^2$

Let $P = -9-7x-15x^2$

If P is a linear combination of P_1, P_2, P_3 then

$$P = K_1P_1 + K_2P_2 + K_3P_3$$

$$-9-7x-15x^2 = K_1(2+x+4x^2) + K_2(1-x+3x^2) + K_3(3+2x+5x^2)$$

$$-9 = 2K_1 + K_2 + 3K_3$$
$$-7 = K_1 - K_2 + 2K_3$$
$$-15 = 4K_1 + 3K_2 + 5K_3$$

Solve above 3 equs using Gauss Elimination method

Rewritting in matrix form,

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} -9 \\ -7 \\ -15 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -9 \\ -7 \\ -15 \end{pmatrix}$$

The augmented matrix is

$$(A, B) \sim \begin{array}{c} \left(\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right) \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 0 & 3 & -1 & 5 \\ 0 & 7 & -3 & 13 \end{array}\right) \begin{array}{l} R_1'' \\ R_2'' = R_1 - 2R_2 \\ R_3'' = R_3 - 4R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & +2 & -4 \end{array}\right) \begin{array}{l} R_1''' \\ R_2''' \\ R_3''' = 7R_2'' - 3R_3'' \end{array}$$

$R(A) = 3$, $R(A,B) = 3$

Rewrite in matrix form

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \\ -4 \end{pmatrix}$$

$$2K_1 + K_2 + 3K_3 = -9 \quad \text{--- } (1)$$

$$3K_2 - K_3 = 5 \quad \text{--- } (2)$$

$$2K_3 = -4 \Rightarrow K_3 = -2$$

Subs $K_3 = -2$ in (2),

$$3K_2 + 2 = 5 \Rightarrow 3K_2 = 3 \Rightarrow K_2 = 1$$

from (1), $2K_1 + 1 - 6 = -9$

$$2K_1 = -4 \Rightarrow K_1 = -2$$

$$\therefore P = -2P_1 + P_2 - 2P_3$$

3) Check whether the vector $w = (1, -2, 2)$ is a linear combination of vectors in the set $S = \{\underset{u_1}{(1,2,3)}, \underset{u_2}{(0,1,2)}, \underset{u_3}{(-1,0,1)}\}$

Solution

Let $w = K_1 u_1 + K_2 u_2 + K_3 u_3$

$$(1, -2, 2) = K_1(1,2,3) + K_2(0,1,2) + K_3(-1,0,+1)$$

$$K_1 - K_3 = 1 \; ; \; 2K_1 + K_2 = -2 \; ; \; 3K_1 + 2K_2 + K_3 = 2$$

Matrix form is

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

Augmented matrix (A, B) is

$$(A, B) \sim \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right) \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 2 & 4 & -1 \end{array}\right) \begin{array}{l} R_1' \\ R_2' = R_2 - 2R_1 \\ R_3'' = R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right) \begin{array}{l} R_1'' \\ R_2'' \\ R_3'' = R_3' - 2R_2' \end{array}$$

$R(A) = 2, \; R(A,B) = 3$

$R(A) \neq R(A,B)$

$\therefore$ The System has no Solution

Hence, K_1, K_2, K_3 will not exist

$\therefore$ W is not a linear Combination of u_1, u_2, u_3

4. Write the vector $w = (1,1,1)$ as a linear Combination of vectors in the Set $S = \{\underset{u_1}{(1,2,3)}, \underset{u_2}{(0,1,2)}, \underset{u_3}{(-1,0,1)}\}$

Solution:

Let $W = K_1 u_1 + K_2 u_2 + K_3 u_3$

$(1,1,1) = K_1(1,2,3) + K_2(0,1,2) + K_3(-1,0,1)$

$K_1 - K_3 = 1 \; ; \; 2K_1 + K_2 = 1 \; ; \; 3K_1 + 2K_2 + K_3 = 1$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Augmented matrix is

$$(A, B) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{array}\right)\begin{array}{l} R_1'' \\ R_2'' = R_2 - 2R_1 \\ R_3'' = R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l} R_1''' \\ R_2''' \\ R_3''' = R_3'' - 2R_2'' \end{array}$$

$R(A) = R(A, B) = 2 < \text{no. of unknowns}$

$\therefore$ The system of equations have infinite no of solutions.

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$K_1 - K_3 = 1 \quad \text{——} \ (1)$$
$$K_2 + 2K_3 = -1 \quad \text{——} \ (2)$$

Let $K_3 = 1$

from (1), $K_1 = 2$
from (2), $K_2 = -3$

$\therefore K_1 = 2,\ K_2 = -3,\ K_3 = 1$

Hence $W = 2u_1 - 3u_2 + u_3 = 0$

Note:

(1) If $R(A) = R(A, B) =$ no. of unknowns, then the system of eqns is consistent and have unique solution.

(2) If $R(A) = R(A, B) <$ no. of unknowns, then the system is consistent and have infinite no. of solutions

(3) If $R(A) \neq R(A, B)$, then the system is inconsistent and have no solution.

Linear Dependence and Independence

Definition: Linearly Dependent

A Set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is said to be linearly dependent, if there exists scalars $K_1, K_2, \ldots K_n$ not all zero such that $K_1 V_1 + K_2 V_2 + \ldots + K_n V_n = 0$

Linearly Independent

A Set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is said to be linearly Independent if there exists scalars $K_1 = 0$, $K_2 = 0, \ldots, K_n = 0$ such that $K_1 v_1 + \ldots + K_n v_n = 0$.

Note

In a system of homogeneous linear equations

(1) the system is always consistent if $R(A) = R(A, B)$

(2) if $R(A) = R(A, B) =$ no. of unknowns, then the system will have trivial solution (ie all unknowns $= 0$)

(3) If $|A| \neq 0$, the system has unique solution which is the trivial solution

(4) If $|A| = 0$, the system has an infinite number of non-zero solutions

Problems.

1. Determine whether the vectors $V_1=(1,-2,3)$ $V_2=(5,6,-1)$, $V_3=(3,2,1)$ form a linearly dependent set (or) a linearly independent set.

Solution:

Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 1(6+2)+2(5+3)+3(10-18)$$

$$= 8+16-24 = 0$$

$|A| = 0$

$\therefore$ The system has an infinite no. of non-zero solutions. i.e K_1, K_2, K_3 excists. not all zero

$\therefore \{V_1, V_2, V_3\}$ form a linearly dependent set.

2. Determine whether the set of vectors $S=\{(1,2,3), (0,1,2), (-2,0,1)\}$ in R^3 is linearly independent (or) linearly dependent.

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Solution.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{vmatrix} = 1(1-0)-2(0+4)+3(0+2)$$
$$= 1-8+6 = -1 \neq 0$$

∴ The System has unique Solution (ie) trivial Solution $K_1 = K_2 = K_3 = 0$

∴ S is linearly independent.

3. Determine whether the set of vectors in P_2 is linearly independent where the Set is $\{1+x+2x^2, 2+5x-x^2, x+x^2\}$

Solution

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 5 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 1(5+1)-1(2)+2(2)$$
$$= 6-2+4 = 8 \neq 0$$

∴ The Set Vectors are linearly independent