4. Determine whether the set of vectors in $R^3$ $\left\{\begin{pmatrix}2\\6\\-2\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}, \begin{pmatrix}8\\16\\-3\end{pmatrix}\right\}$ is linearly independent.

Solution:

$$A = \begin{pmatrix} 2 & 3 & 8 \\ 6 & 1 & 16 \\ -2 & 2 & -3 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 8 \\ 6 & 1 & 16 \\ -2 & 2 & -3 \end{vmatrix} = 2(-3-32) - 3(-18+32) + 8(12+2)$$
$$= -70 - 42 + 112 = 0$$

$\therefore$ Set of vectors are linearly dependent.

5. Determine whether the set 'S' of vectors in $M_{22}$ is linearly independent where

$S = \left\{ \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \right\}$

Let $K_1V_1 + K_2V_2 + K_3V_3 = 0$

$$K_1\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + K_2\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} + K_3\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Scanned with CamScanner

$$\begin{pmatrix} 2K_1 & K_1 \\ 0 & K_1 \end{pmatrix} + \begin{pmatrix} 3K_2 & 0 \\ 2K_2 & K_2 \end{pmatrix} + \begin{pmatrix} K_3 & 0 \\ 2K_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2K_1+3K_2+K_3 & K_1 \\ 2K_2+2K_3 & K_1+K_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$2K_1+3K_2+K_3 = 0$ — (1) ; $K_1 = 0$ — (2) ; $2K_2+K_3=0$ — (3) ; $K_1+K_2=0$ — (4)

Substituting (2) in (4),

$K_2 = 0$

Substituting $K_2=0$ in (3),

$K_3 = 0$

Substituting $K_2=0$, $K_3=0$ in (1),

$K_1 = 0$

$K_1 = K_2 = K_3 = 0$

$\therefore$ S is linearly independent.

6. Determine whether the set of vectors is linearly independent where

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

Solution:

Let $K_1V_1 + K_2V_2 + K_3V_3 + K_4V_4 = 0$

$$K_1\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} + K_2\begin{pmatrix}1\\1\\0\\2\end{pmatrix} + K_3\begin{pmatrix}0\\3\\1\\-2\end{pmatrix} + K_4\begin{pmatrix}0\\1\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}K_1\\0\\-K_1\\0\end{pmatrix} + \begin{pmatrix}K_2\\K_2\\0\\2K_2\end{pmatrix} + \begin{pmatrix}0\\3K_3\\K_3\\-2K_3\end{pmatrix} + \begin{pmatrix}0\\K_4\\-K_4\\2K_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$K_1 + K_2 = 0$; $K_2 + 3K_3 + K_4 = 0$; $-K_1 + K_3 - K_4 = 0$

$2K_2 - 2K_3 + 2K_4 = 0$

matrix form is

$$\begin{pmatrix}1 & 1 & 0 & 0\\0 & 1 & 3 & 1\\-1 & 0 & 1 & -1\\0 & 2 & -2 & 2\end{pmatrix}\begin{pmatrix}K_1\\K_2\\K_3\\K_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$A = \begin{pmatrix}1 & 1 & 0 & 0\\0 & 1 & 3 & 1\\-1 & 0 & 1 & -1\\0 & 2 & -2 & 2\end{pmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 2 & -2 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 & 1 \\ -1 & 1 & -1 \\ 0 & -2 & 2 \end{vmatrix}$$

$$= 1(2-2) - 3(0+2) + 1(0-2) - [-3(-2) + 1(2)]$$

$$= -6 - 2 - (6+2) = -16 \neq 0$$

$\therefore$ Set of vectors are linearly independent.

Note: If $A$ is not a square matrix then check whether set of vectors are LD (or) LI by finding $R(A)$ and $R(A,B)$. find $R(A)$ and $R(A,B)$ by considering augmented matrix and converting to row echelon form.

Theorem:

If the functions $f_1, f_2, \ldots f_n$ have $(n-1)$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty, \infty)$

Note:

Wronskian of $f_1, f_2, \ldots f_n$ is

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Problems:

7. Show that the functions $f_1 = x$ and $f_2 = \sin x$ form a linearly independent set of vectors in $C'(-\infty, \infty)$

Solution

$$f_1 = x, \quad f_2 = \sin x$$

$$f_1' = 1, \quad f_2' = \cos x$$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$

$$= x \cos x - \sin x \neq 0$$

$\therefore f_1, f_2$ are linearly independent

Note:

If the Wronskian of $f_1, f_2, \ldots f_n$ is Identically zero on $(-\infty, \infty)$, then the Set of vectors may be linearly independent (or) linearly dependent.

8. Check whether $\sin^2 x$, $\cos^2 x$, $5$ form a linearly dependent (or) linearly independent Set.

Solution:

To find $K_1, K_2, K_3$ such that

$$K_1 f_1 + K_2 f_2 + K_3 f_3 = 0$$

$$K_1 \sin^2 x + K_2 \cos^2 x + 5 K_3 = 0$$

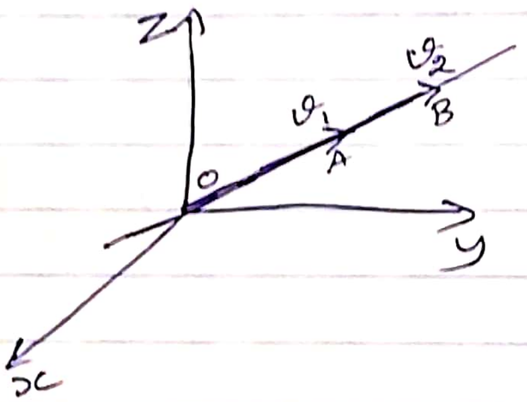
This is possible only when $K_1 = 5$, $K_2 = 5$, $K_3 = -1$

Hence the Set of Vectors are linearly dependent, as all scalars $K_1, K_2, K_3$ not all zero.

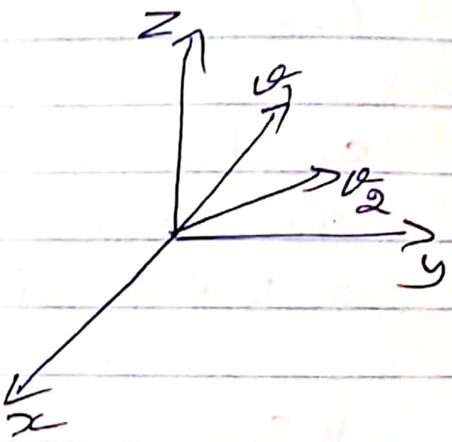
Geometric Interpretation of Linear Independence

## Geometric Interpretation of Linear Independence

In $R^2$, a set of two vectors is linearly independent if and only if the vectors do not lie on the same line when they are placed with their initial points at the origin




Linearly dependent




Linearly independent

Theorem 1

A set S with two or more vectors is

(a) linearly dependent iff atleast one of the vectors in S is expressible as a linear combination of the other vectors in S

(b) linearly independent iff no vector in S is expressible as a linear combination of the other vectors in S.

Theorem 2

(1) A finite set of vectors that contains the zero vector is linearly dependent

(2) A set with exactly two vectors is linearly independent iff neither vector is a scalar multiple of the other.

[Note: (2, 4, 6) is a scalar multiple of (1, 2, 3)]

Theorem 3

Let $S = \{v_1, v_2, \dots v_r\}$ be a set of vectors in $R^n$. If $r > n$, then $S$ is linearly dependent.

From the previous theorem we can say that a set in $R^2$ with more than 2 vectors is linearly dependent and a set in $R^3$ with more than three vectors is linearly dependent.