CHAPTER 12

Applications of Definite Integrals

One of the important applications of integration is to find the area bounded by a curve. Often such an area can have a physical significance like the work done by a motor or the distance travelled by a vehicle.

Continuous curve which is bounded by under the certain conditions, the space which is occupied, is called area bounded by curves.

Tracing of Curve

For the evaluation of the area of bounded regions, it is very essential to know the rough sketch of the curves. The following points are very useful to draw the rough sketch of a curve.

Symmetry

(i) **Symmetry about X-axis** If all powers of y in the equation of the given curve are even, then it is symmetric about X-axis, i.e. the shape of the curve above X-axis is exactly identical to its shape below X-axis. e.g. $y^2 = 4ax$ is symmetric about X-axis.

(ii) **Symmetry about Y-axis** If all powers of x in the equation of the given curve are even, then it is symmetric about Y-axis. e.g. $x^2 = 4ay$ is symmetric about Y-axis.

(iii) **Symmetry in opposite quadrants** If the equation of curve remains same by putting $-x$ for x and $-y$ for y, then it is symmetric in opposite quadrants. e.g. $xy = c^2$ and $x^2 + y^2 = a^2$ are symmetric in opposite quadrants.

(iv) **Symmetric about the line $y = x$** If the equation of a given curve remains unaltered by interchanging x and y, then it is symmetric about the line $y = x$ which passes through the origin and makes an angle of $45°$ with positive direction of X-axis.

Origin and Tangents at the Origin

See whether the curve passes through origin or not. If the point $(0, 0)$ satisfies the equation of the curve, then it passes through the origin and in such a case to find the equations of the tangents at the origin, equate the lowest degree term to zero.

e.g. $y^2 = 4ax$ passes through the origin. The lowest degree term in this equation is $4ax$. Equating $4ax$ to zero, we get $x = 0$.

So, $x = 0$, i.e. Y-axis is tangent at the origin to $y^2 = 4ax$.

Points of Intersection of the Curve with the Coordinate Axes

By putting $y = 0$ in the equation of the given curve, we can find points where the curve crosses the X-axis. Similarly, by putting $x = 0$ in the equation of the given curve we can find points where the curve crosses the Y-axis.

e.g. To find the points where the curve $xy^2 = 4a^2(2a - x)$ meets X-axis, we put $y = 0$ in the equation which gives $4a^2(2a - x) = 0$ or $x = 2a$.

So, the curve $xy^2 = 4a^2(2a - x)$, meets X-axis at $(2a, 0)$. This curve does not intersect Y-axis, because by putting $x = 0$ in the equation of the given curve get an absurd result.

Asymptotes

A straight line is called an asympote of a curve $y = f(x)$, if it touches the curve at infinity.

In other words, an asymptote is a line whose distance from the curve tends to zero at point on curve moves towards infinity along branch of curve.

If $y = f(x) = \dfrac{u_1(x)}{u_2(x)}$ be a function, then asymptotes are given by $u_2(x) = 0$

e.g. The asymptotes of $y = \dfrac{2+x}{2-x}$ is $2 - x = 0$ or $x = 2$ and the asymptotes of $y = \dfrac{x^2 - a^2}{x^2 + a^2}$ do not exist as $x^2 + a^2 \neq 0$ for any value of x.

Regions Where the Curve does not Exist

To determine the regions in which the curve does not exist, find the value of y in terms of x from the equation of the curve and find the value of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y.

e.g. The values of y obtained from $y^2 = 4ax$ are imaginary for negative values of x. So, the curve does not exist on the left side of Y-axis.

Similarly, the curve $a^2y^2 = x^2(a - x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

Keeping the above facts in mind and plotting some points on the curve one can easily have a rough sketch of the curve.

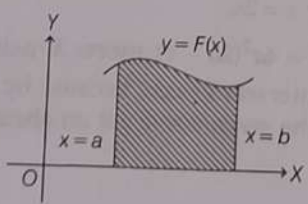
Area Under the Curve

For evaluation of area bounded by certain curves, we need to know the nature of the curves and their graphs. We should also be able to draw sketch of the curves.

While finding the area between a curve and an axis, we take the absolute value of the integration (where negative value is obtained).

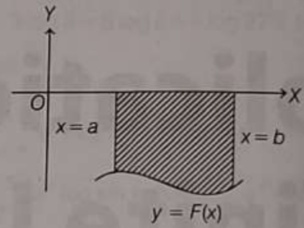
We will study in details about it in the following points

(i) The area bounded by the curve $y = F(x)$ above the X-axis and between the lines $x = a$, $x = b$ is given by $\int_a^b y\,dx = \int_a^b F(x)\,dx$



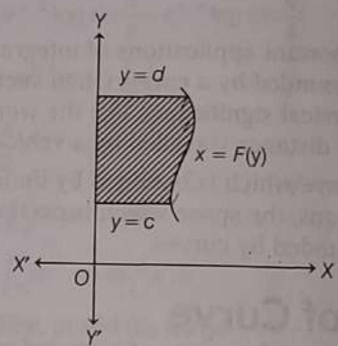
(ii) If the curve between the lines $x = a$, $x = b$ lies below the X-axis, then the required area is given by

$$\left|\int_a^b (-y)\,dx\right| = \left|-\int_a^b y\,dx\right|$$
$$= \left|-\int_a^b F(x)\,dx\right|$$



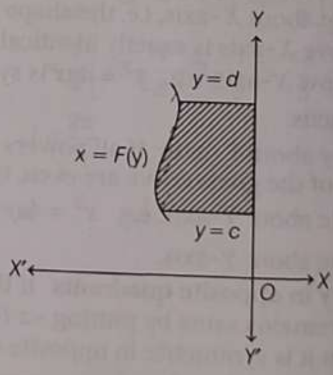
(iii) The area bounded by the curve $x = F(y)$ right to the Y-axis and the lines $y = c$, $y = d$ is given by

$$\int_c^d x\,dy = \int_c^d F(y)\,dy$$



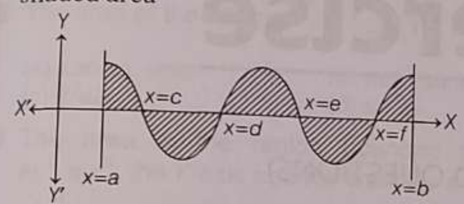
(iv) If the curve between the lines $y = c$, $y = d$ left to the Y-axis, then the area is given by

$$\left|\int_c^d (-x)\,dy\right| = \left|-\int_c^d x\,dy\right|$$
$$= \left|-\int_c^d F(y)\,dy\right|$$



(v) If $f(x)$ change its sign a number of times in the interval $a \le x \le b$, then we divide the region $[a, b]$ in such a way that we elearly get the points lying between $[a, b]$, where $f(x)$ changs it's sign.

If $y = f(x)$ is the curve shown in the figure, then shaded area



$$A = \int_a^c f(x)dx + \left|\int_c^d f(x)dx\right| + \int_d^e f(x)dx$$
$$+\left|\int_e^f f(x)dx\right| + \int_f^b f(x)dx$$

Whenever we solve this type of question, generally, we follow the steps given below

(i) Firstly we sketch the given curve.

(ii) Now, we find the intersection of curve with axis and given line.

(iii) We select the bounded region in the figure and take interval between bounded region.

(iv) Now, we apply the appropriate formula to calculate the area of bounded region.

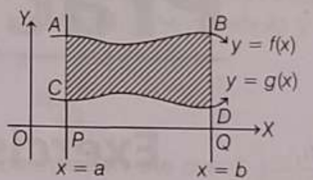
Area between two Curves

(i) **Area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $y = b$.**

Let the curves $y = f(x)$ and $y = g(x)$ be represented by AB and CD, respectively. We assume that the two curves do not intersect each other in the interval $[a, b]$. Then, the shaded area

$$= \text{Area of curve } APQB - \text{Area of curve } CPQD$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \{f(x) - g(x)\}\,dx$$



(ii) **When $f(x)$ and $g(x)$ intersect each other in the interval $[a, b]$.**

First of all we should find the intersection point of $y = f(x)$ and $y = g(x)$.

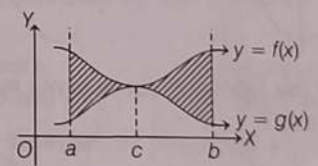
For that we solve $f(x) = g(x)$.

Let the root be $x = c$.

(we consider only one intersection point)

Thus, the required (shaded) area

$$= \int_a^c \{f(x) - g(x)\}\,dx + \int_c^b \{g(x) - f(x)\}\,dx$$



Hence, area between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is always given by

$$\int_a^b \{f(x) - g(x)\}\,dx$$

provided $f(x) > g(x)$ in $[a, b]$.