

## Revision Notes for Class 12 Mathematics

### Chapter 5 – Continuity and Differentiability

#### Continuity

##### 1. Definition:

A function  $f(x)$  is said to be continuous at  $x = a$ ; where  $a \in \text{domain of } f(x)$ , if

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

i.e., LHL = RHL = value of a function at  $x = a$

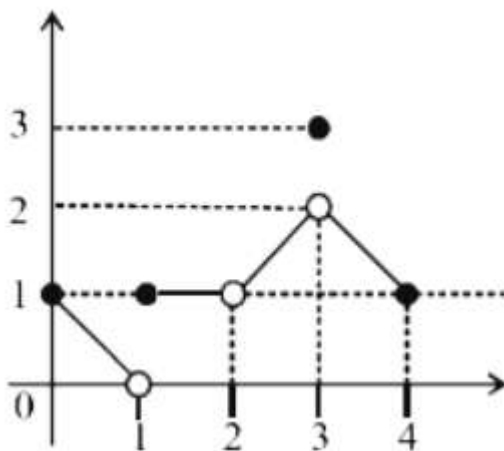
or 
$$\lim_{x \rightarrow a} f(x) = f(a)$$

##### 1.1 Reasons of discontinuity

If  $f(x)$  is not continuous at  $x = a$ , we say that  $f(x)$  is discontinuous at  $x = a$

There are following possibilities of discontinuity:

1.  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  exist but they are not equal.
2.  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  exists and are equal but not equal to  $f(a)$
3.  $f(a)$  is not defined.
4. At least one of the limits does not exist. The graph of the function will show a break at the location of discontinuity from a geometric standpoint.



The graph as shown is discontinuous at  $x=1, 2$  and  $3$ .

## 2. Properties of Continuous Functions

Let  $f(x)$  and  $g(x)$  be continuous functions at  $x = a$ . Then,

1.  $cf(x)$  is continuous at  $x = a$ , where  $c$  is any constant.
2.  $f(x) \pm g(x)$  is continuous at  $x = a$ .
3.  $f(x) \cdot g(x)$  is continuous at  $x = a$ .
4.  $f(x)/g(x)$  is continuous at  $x = a$ , provided  $g(a) \neq 0$ .
5. Assuming  $f(x)$  be continuous on  $[a, b]$  in such a way that the function  $f(a)$  and  $f(b)$  will be at opposite signs, then there will exist at least one solution of equation  $f(x) = 0$  in the open interval  $(a, b)$ .

## 3. The Intermediate Value Theorem

Suppose  $f(x)$  is continuous on an interval  $I$ , also  $a$  and  $b$  are any two points of  $I$ . Then if  $y_0$  is a number between  $f(a)$  and  $f(b)$ , there exists a number  $c$  between  $a$  and  $b$  such that  $f(c) = y_0$ .

The Function  $f$ , being continuous on  $(a,b)$  takes on every value between  $f(a)$  and  $f(b)$

Note:

That a function  $f$  which is continuous in  $[a, b]$  possesses the following properties:

- (i) If  $f(a)$  and  $f(b)$  possess opposite signs, then there exists at least one solution of the equation  $f(x) = 0$  in the open interval  $(a,b)$ .
- (ii) If  $K$  is any real number between  $f(a)$  and  $f(b)$ , then there exists at least one solution of the equation  $f(x) = K$  in the open interval  $(a,b)$ .

#### 4. Continuity in an Interval

(a) A function  $f$  is said to be continuous in  $(a, b)$  if  $f$  is continuous at each and every point  $\in (a,b)$

(b) A function  $f$  is said to be continuous in a closed interval  $[a,b]$  if :

- (1)  $f$  is continuous in the open interval  $(a,b)$  and
- (2)  $f$  is right continuous at 'a' i.e.  $\lim_{x \rightarrow a^+} f(x) = f(a) = a$  finite quantity
- (3)  $f$  is left continuous at 'b'; i.e.  $\lim_{x \rightarrow b^-} f(x) = f(b) = a$  finite quantity

## 5. A List of Continuous Functions

S.No	Function $f(x)$	Interval in which $f(x)$ is continuous
1	Constant (c)	$(-\infty, \infty)$
2	$x^n$ , n is an integer	$(-\infty, \infty)$
3	$x^{-n}$ , n is a positive integer	$(-\infty, \infty) - \{0\}$
4	$ x - a $	$(-\infty, \infty)$
5	$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$	$(-\infty, \infty)$
6	$\sin x$	$(-\infty, \infty)$
7	$\cos x$	$(-\infty, \infty)$

8	$\tan x$	$(-\infty, \infty) - \left\{ (2n+1)\frac{\pi}{2} : n \in I \right\}$
9	$\cot x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
10	$\sec x$	$(-\infty, \infty) - \{(2n+1)\pi : n \in I\}$
11	$\operatorname{cosec} x$	$\pi/2 : n \in I$
12	$e^x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
13	$\log_c x$	$(-\infty, \infty)$ & $(0, \infty)$

## 6. Types Of Discontinuities

### Type-1 : (Removable type of discontinuities)

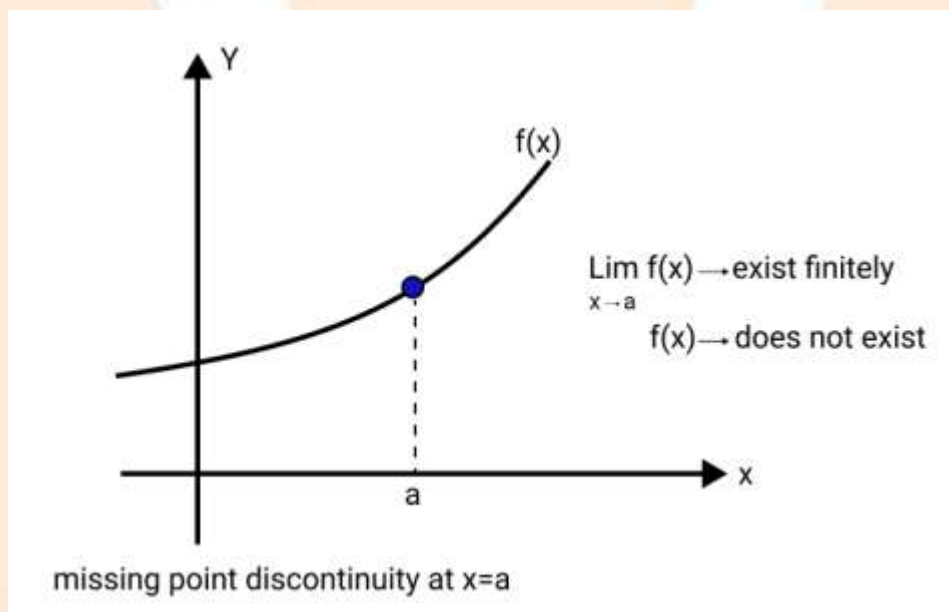
In this case,  $\lim_{x \rightarrow c} f(x)$  exists but it will not equal to  $f(c)$ . As a result, the function is said to have a removable discontinuity or discontinuity of the first kind. In such scenario, we can redefine the function such that  $\lim_{x \rightarrow c} f(x) = f(c)$  and make it continuous at  $x = c$ . It can be further categorised as:

### (a) Missing Point Discontinuity:

Where  $\lim_{x \rightarrow a} f(x)$  exists finitely but  $f(a)$  is not defined.

E.g.  $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$  will have a missing point discontinuity at  $x=1$ , and

$f(x) = \frac{\sin x}{x}$  will have a missing point discontinuity at  $x=0$



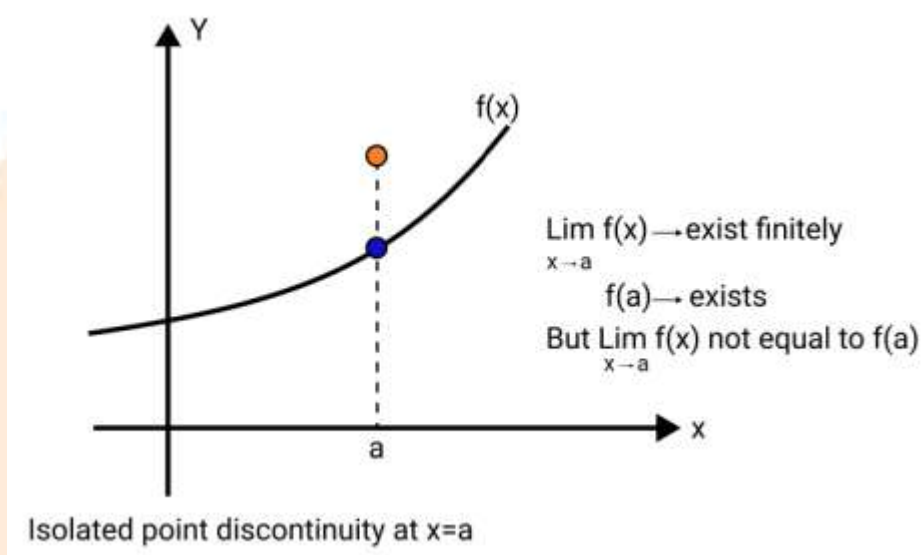
### (b) Isolated Point Discontinuity :

Where  $\lim_{x \rightarrow a} f(x) \neq f(a)$  exists  $f(a)$  also exists but.

$$\lim_{x \rightarrow a} f(x) \neq f(a)$$

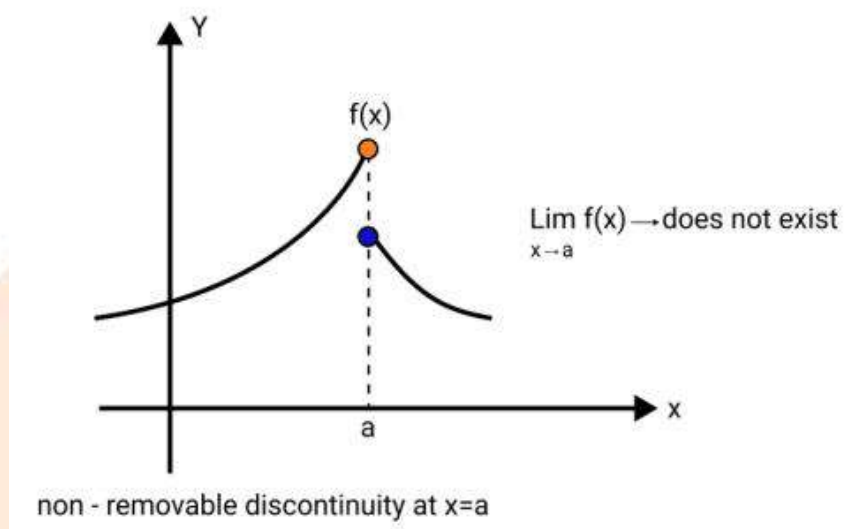
E.g.  $f(x) = \frac{x^2-16}{x-4}, x \neq 4$  and  $f(4) = 9$  will have an isolated point discontinuity at  $x=4$

In the same way,  $f(x) = [x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$  will have an isolated point discontinuity at all  $x \in I$ .



### Type-2 : (non-removable type of discontinuities)

In case,  $\lim_{x \rightarrow a} f(x)$  does not exist, then it is not possible to make the function continuous by redefining it. Such discontinuities are known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as:



**(a) Finite Discontinuity:**

E.g.,  $f(x) = x - [x]$  at all integral  $x$ ;  $f(x) = \tan^{-1} \frac{1}{x}$  at  $x = 0$  and  $f(x) = \frac{1}{1 + 2^{\frac{1}{x}}}$  at  $x = 0$  (note that  $f(0^+) = 0$ ;  $f(0^-) = 1$ )

**(b) Infinite Discontinuity:**

E.g.,  $f(x) = \frac{1}{x-4}$  or  $g(x) = \frac{1}{(x-4)^2}$  at  $x = 4$ ;  $f(x) = 2^{\tan x}$

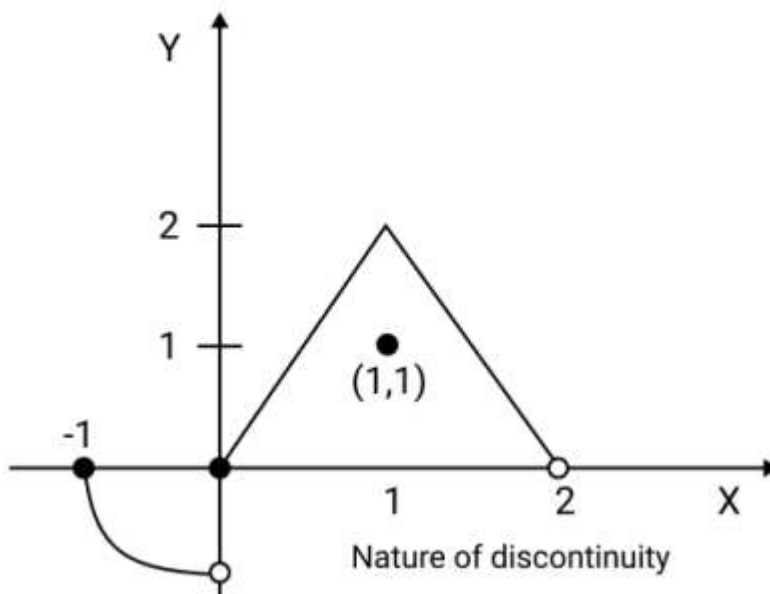
at  $x = \frac{\pi}{2}$  and  $f(x) = \frac{\cos x}{x}$  at  $x = 0$

**(c) Oscillatory Discontinuity:**

E.g.,  $f(x) = \sin \frac{1}{x}$  at  $x = 0$

In all these cases the value of  $f(a)$  of the function at  $x = a$  (point of discontinuity) may or may not exist but  $\lim_{x \rightarrow a}$  does not exist.





From the adjacent graph note that

$f$  is continuous at  $x = -1$

$f$  has isolated discontinuity at  $x = 1$

$f$  has missing point discontinuity at  $x = 2$

$f$  has non-removable (finite type) discontinuity at the origin.

**Note:**

(a) In case of dis-continuity of the second kind the nonnegative difference between the value of the RHL at  $x = a$  and LHL at  $x = a$  is called the jump of discontinuity. A function having a finite number of jumps in a given interval  $I$  is called a piece wise continuous or sectionally continuous function in this interval.

(b) All Polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their domains.

(c) If  $f(x)$  is continuous and  $g(x)$  is discontinuous at  $x = a$  then the product function  $\phi(x) = f(x) \cdot g(x)$  is not necessarily be discontinuous at  $x = a$ . e.g.

$$f(x) = x \text{ and } g(x) = \begin{cases} \sin \frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(d) If  $f(x)$  and  $g(x)$  both are discontinuous at  $x = a$  then the product function  $\phi(x) = f(x) \cdot g(x)$  is not necessarily be discontinuous at  $x = a$ . e.g

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(e) Point functions are to be treated as discontinuous

e.g.  $f(x) = \sqrt{1-x} + \sqrt{x-1}$  is not continuous at  $x = 1$

(f) A continuous function whose domain is closed must have a range also in the closed interval.

(g) If  $f$  is continuous at  $x = a$  and  $g$  is continuous at  $x = f(a)$  then the composite  $g[f(x)]$  is continuous at  $x = a$

E.g.  $f(x) = \frac{x \sin x}{x^2 + 2}$  and  $g(x) = |x|$  are continuous at  $x = 0$ , hence the composite

$(g \circ f)(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$  will also be continuous at  $x = 0$ .

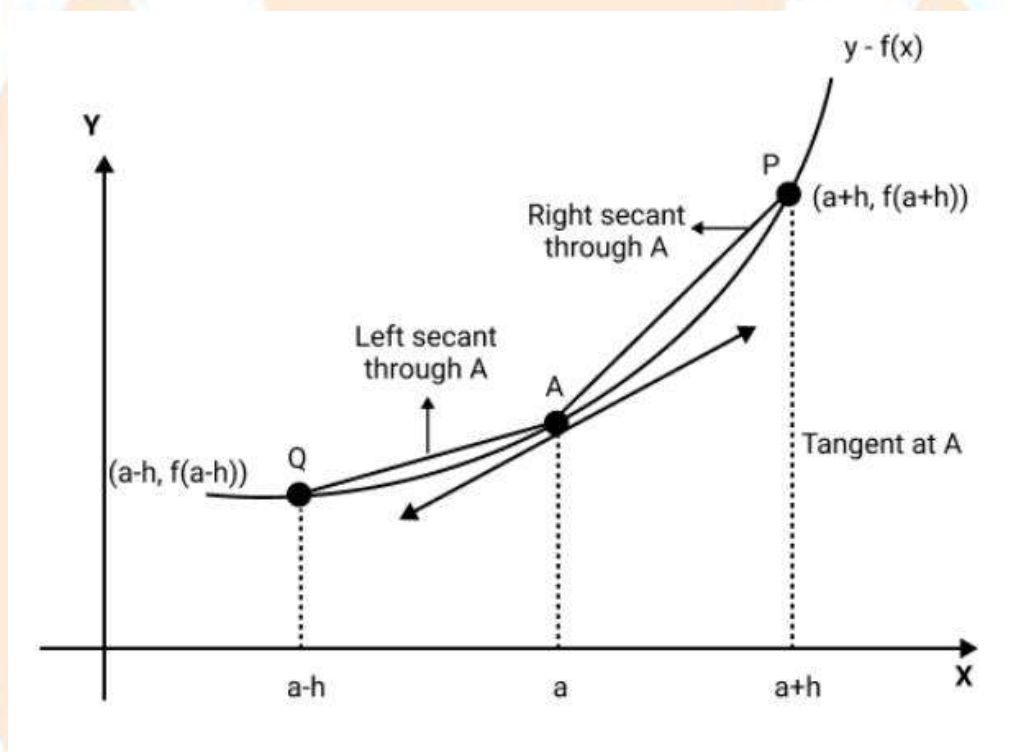
## Differentiability

### 1. Definition

Let  $f(x)$  be a real valued function defined on an open interval  $(a, b)$  where  $c \in (a, b)$ . Then  $f(x)$  is said to be differentiable or derivable at  $x = c$

if,  $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{(x - c)}$  exists finitely.

This limit is called the derivative or differentiable coefficient of the function  $f(x)$  at  $x = c$ , and is denoted by  $f'(c)$  or  $\frac{d}{dx}(f(x))_{x=c}$



- Slope of Right hand secant =  $\frac{f(a+h) - f(a)}{h}$  as  $h \rightarrow 0, P \rightarrow A$  and secant (AP)  $\rightarrow$  tangent at A

$$\Rightarrow \text{Right hand derivative} = \lim_{h \rightarrow 0} \left( \frac{f(a+h) - f(a)}{h} \right)$$

= Slope of tangent at A (when approached from right)  $f'(a^+)$

- Slope of Left hand secant =  $\frac{f(a-h)-f(a)}{-h}$  as  $h \rightarrow 0, Q \rightarrow A$  and secant  $AQ \rightarrow$  tangent at  $A$

$$\Rightarrow \text{Left hand derivative} = \lim_{h \rightarrow 0} \left( \frac{f(a-h)-f(a)}{-h} \right)$$

= Slope of tangent at  $A$  (when approached from left)  $f'(a^-)$

Thus,  $f(x)$  is differentiable at  $x = c$ .

$$\Leftrightarrow \lim_{x \rightarrow c} \frac{f(x)-f(c)}{(x-c)} \text{ exists finitely}$$

$$\Leftrightarrow \lim_{x \rightarrow c} \frac{f(x)-f(c)}{(x-c)} = \lim_{h \rightarrow 0} \frac{f(c-h)-f(c)}{-h}$$

$$\Leftrightarrow \lim_{h \rightarrow 0} \frac{f(c-h)-f(c)}{-h} = \lim_{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}$$

Hence,  $\lim_{x \rightarrow c^-} \frac{f(x)-f(c)}{(x-c)} = \lim_{h \rightarrow 0} \frac{f(c-h)-f(c)}{-h}$  is called the left hand derivative of  $f(x)$  at

$x = c$  and is denoted by  $f'(c^-)$  or  $Lf'(c)$  While,  $\lim_{x \rightarrow c^+} \frac{f(x)-f(c)}{(x-c)} = \lim_{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}$  is called the right hand derivative of  $f(x)$  at  $x = c$  and is denoted by  $f'(c^+)$  or  $Rf'(c)$

If  $f'(c^-) \neq f'(c^+)$ , we say that  $f(x)$  is not differentiable at  $x = c$ .

## 2. Differentiability in a Set

1. A function  $f(x)$  defined on an open interval  $(a, b)$  is said to be differentiable or derivable in open interval  $(a, b)$ , if it is differentiable at each point of  $(a, b)$

2. A function  $f(x)$  defined on closed interval  $[a, b]$  is said to be differentiable or derivable.  
"If  $f$  is derivable in the open interval  $(a, b)$  and also the end points  $a$  and  $b$ , then  $f$  is said to be derivable in the closed interval  $[a, b]$ "

i.e.,  $\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}$  and  $\lim_{x \rightarrow b^-} \frac{f(x) - f(b)}{x - b}$ , both exist.

A function  $f$  is said to be a differentiable function if it is differentiable at every point of its domain.

**Note:**

1. If  $f(x)$  and  $g(x)$  are derivable at  $x = a$  then the functions  $f(x) + g(x)$ ,  $f(x) - g(x)$ ,  $f(x) \cdot g(x)$  will also be derivable at  $x = a$  and if  $g(a) \neq 0$  then the function  $f(x)/g(x)$  will also be derivable at  $x = a$
2. If  $f(x)$  is differentiable at  $x = a$  and  $g(x)$  is not differentiable at  $x = a$ , then the product function  $F(x) = f(x) \cdot g(x)$  can still be differentiable at  $x = a$ . E.g.  $f(x) = x$  and  $g(x) = |x|$
3. If  $f(x)$  and  $g(x)$  both are not differentiable at  $x = a$  then the product function;  $F(x) = f(x) \cdot g(x)$  can still be differentiable at  $x = a$ . E.g.  $f(x) = |x|$  and  $g(x) = |x|$
4. If  $f(x)$  and  $g(x)$  both are not differentiable at  $x = a$  then the sum function  $F(x) = f(x) + g(x)$  may be a differentiable function. E.g.,  $f(x) = |x|$  and  $g(x) = -|x|$
5. If  $f(x)$  is derivable at  $x = a$

$\Rightarrow f'(x)$  is continuous at  $x = a$ .

e.g.

$$f(x) = \begin{cases} 2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

### 3. Relation Between Continuity and Differentiability

We learned in the last section that if a function is differentiable at a point, it must also be continuous at that point, and therefore a discontinuous function cannot be differentiable. The following theorem establishes this fact.

**Theorem:** If a function is differentiable at a given point, it must be continuous at that same point. However, the inverse is not always true.

or  $f(x)$  is differentiable at  $x = c$

$\Rightarrow f(x)$  is continuous at  $x = c$

**Converse:** The reverse of the preceding theorem is not always true, i.e., a function might be continuous but not differentiable at a given point.

E.g., The function  $f(x) = |x|$  is continuous at  $x = 0$  but it is not differentiable at  $x = 0$ .

**Note:**

(a) Let  $f'^+(a) = p; f'^-(a) = q$  where  $p, q$  are finite then

$\Rightarrow f$  is derivable at  $x = a$

$\Rightarrow f$  is continuous at  $x = a$

(ii)  $p \neq q \Rightarrow f$  is not derivable at  $x = a$ .

It is very important to note that  $f$  may be still continuous at  $x = a$

In short, for a function  $f$ :

Differentiable  $\Rightarrow$  Continuous.

Not Differentiable  $\neq$  Not Continuous

(i.e., function may be continuous)

But,

Not Continuous  $\Rightarrow$  Not Differentiable.

(b) If a function  $f$  is not differentiable but is continuous at  $x = a$  it geometrically implies a sharp corner at  $x = a$

Theorem 2: Let  $f$  and  $g$  be real functions such that  $fg$  is defined if  $g$  is continuous at  $x = a$  and  $f$  is continuous at  $g$ .

## Differentiation:

### 1. Definition

(a) Let us consider a function  $y = f(x)$  defined in a certain interval. It has a definite value for each value of the independent variable  $x$  in this interval.

Now, the ratio of the function's increment to the independent variable's increment,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Now, as  $\Delta x \rightarrow 0, \Delta y \rightarrow 0$  and  $\frac{\Delta y}{\Delta x} \rightarrow$  finite quantity, then derivative  $f'(x)$  exists and is

denoted by  $y'$  or  $f'(x)$  or  $\frac{dy}{dx}$ . Thus,  $f'(x) = \lim_{\Delta x \rightarrow 0} \left( \frac{\Delta y}{\Delta x} \right) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$  (if it exists) for the limit to exist,

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x - h) - f(x)}{-h}$$

(Right Hand derivative)      (Left Hand derivative)

(b) The derivative of a given function  $f$  at a point  $x = a$  of its domain is defined as:

$$\text{Limit}_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}, \text{ provided the limit exists is denoted by } f'(a)$$

Note that alternatively, we can define

$$f'(a) = \text{Limit}_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}, \text{ provided the limit exists.}$$

This method is called first principle of finding the derivative of  $f(x)$

## 2. Derivative of Standard Function

$$(i) \frac{d}{dx}(x^n) = n \cdot x^{n-1}; x \in \mathbb{R}, n \in \mathbb{R}, x > 0$$

$$(ii) \frac{d}{dx}(e^x) = e^x$$

$$(iii) \frac{d}{dx}(a^x) = a^x \cdot \ln a (a > 0)$$

$$(iv) \frac{d}{dx}(\ln |x|) = \frac{1}{x}$$

$$(v) \frac{d}{dx}(\log_a |x|) = \frac{1}{x} \log_a e$$

$$(vi) \frac{d}{dx}(\sin x) = \cos x$$

$$(vii) \frac{d}{dx}(\cos x) = -\sin x$$



$$(viii) \frac{d}{dx}(\tan x) = \sec^2 x$$

$$(ix) \frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$(x) \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$$

$$(xi) \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$(xii) \frac{d}{dx}(\text{constant}) = 0$$

$$(xiii) \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$(xiv) \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$(xv) \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$(xvi) \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, \quad x \in \mathbb{R}$$

$$(xvii) \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

$$(xviii) \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

(xix) Results:

If the inverse functions  $f(g)$  are defined by  $y = f(x); x = g(y)$ . Then

$$g(f(x)) = x \Rightarrow g'(f(x)) \cdot f'(x) = 1$$

This result can also be written as, if  $\frac{dy}{dx}$  exists and  $\frac{dy}{dx} \neq 0$ , then  $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$  or  $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$  or

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right) \left[\frac{dx}{dy} \neq 0\right]}$$

### 3. Theorems On Derivatives

If  $u$  and  $v$  are derivable functions of  $x$ , then,

(i) Term by term differentiation :  $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) Multiplication by a constant  $\frac{d}{dx}(Ku) = K \frac{du}{dx}$ , where  $K$  is any constant

(iii) "Product Rule"  $\frac{d}{dx}(u.v) = u \frac{dv}{dx} + v \frac{du}{dx}$  known as In general,

(a) If  $u_1, u_2, u_3, u_4, \dots, u_n$  are the functions of  $x$ , then

$$\begin{aligned} & \frac{d}{dx}(u_1 \cdot u_2 \cdot u_3 \cdot u_4 \dots u_n) \\ &= \left(\frac{du_1}{dx}\right)(u_2 u_3 u_4 \dots u_n) + \left(\frac{du_2}{dx}\right)(u_1 u_3 u_4 \dots u_n) \\ &+ \left(\frac{du_3}{dx}\right)(u_1 u_2 u_4 \dots u_n) + \left(\frac{du_4}{dx}\right)(u_1 u_2 u_3 u_5 \dots u_n) \\ &+ \dots + \left(\frac{du_n}{dx}\right)(u_1 u_2 u_3 \dots u_{n-1}) \end{aligned}$$

(iv) Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2} \text{ where } v \neq 0 \text{ known as}$$

(b) Chain Rule : If  $y = f(u)$ ,  $u = g(w)$ ,  $w = h(x)$  then  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$

$$\text{or } \frac{dy}{dx} = f'(u) \cdot g'(w) \cdot h'(x)$$

Note:

In general, if  $y = f(u)$  then  $\frac{dy}{dx} = f'(u) \cdot \frac{du}{dx}$

## 4. Methods of Differentiation

### 4.1 Derivative by using Trigonometrical Substitution

The use of trigonometrical transforms before differentiation greatly reduces the amount of labour required. The following are some of the most significant findings:

$$(i) \sin 2x = 2 \sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$$

$$(ii) \cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$(iii) \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

$$(iv) \sin 3x = 3 \sin x - 4 \sin^3 x$$

$$(v) \quad \cos 3x = 4 \cos^3 x - 3 \cos x$$

$$(vi) \quad \tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

$$(vii) \quad \tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$$

$$(viii) \quad \tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$$

$$(ix) \quad \sqrt{1 \pm \sin x} = \left| \cos \frac{x}{2} \pm \sin \frac{x}{2} \right|$$

$$(x) \quad \tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left( \frac{x \pm y}{1 \mp xy} \right)$$

$$(xi) \quad \sin^{-1} x \pm \sin^{-1} y = \sin^{-1} \left\{ x\sqrt{1-y^2} \pm y\sqrt{1-x^2} \right\}$$

$$(xii) \quad \cos^{-1} x \pm \cos^{-1} y = \cos^{-1} \left\{ xy \mp \sqrt{1-x^2} \sqrt{1-y^2} \right\}$$

$$(xiii) \quad \sin^{-1} x + \cos^{-1} x = \tan^{-1} x + \cot^{-1} x = \sec^{-1} x + \operatorname{cosec}^{-1} x = \pi / 2$$

$$(xiv) \quad \sin^{-1} x = \operatorname{cosec}^{-1}(1/x); \cos^{-1} x = \sec^{-1}(1/x); \tan^{-1} x = \cot^{-1}(1/x)$$

Note:

### Some standard substitutions:

Expressions

Substitutions

$$(i) \quad \sqrt{a^2 - x^2}$$

$$x = a \sin \theta \text{ or } a \cos \theta$$

(ii)  $\sqrt{(a^2 + x^2)}$

$x = a \tan \theta$  or  $a \cot \theta$

(iii)  $\sqrt{(x^2 - a^2)}$

$x = a \sec \theta$  or  $a \operatorname{cosec} \theta$

(iv)  $\sqrt{\left(\frac{a+x}{a-x}\right)}$  or  $\sqrt{\left(\frac{a-x}{a+x}\right)}$

$x = a \cos \theta$  or  $a \cos 2\theta$

(v)  $\sqrt{(a-x)(x-b)}$  or

$x = a \cos^2 \theta + b \sin^2 \theta$

$\sqrt{\left(\frac{a-x}{x-b}\right)}$  or  $\sqrt{\left(\frac{x-a}{a-x}\right)}$

(vii)  $\sqrt{(x-a)(x-b)}$  or

$x = a \sec^2 \theta - b \tan^2 \theta$

$\sqrt{\left(\frac{x-a}{x-b}\right)}$  or  $\sqrt{\left(\frac{x-a}{x-a}\right)}$

(ix)  $\sqrt{(2ax - x^2)}$

$x = a(1 - \cos \theta)$

## 4.2 Logarithmic Differentiation

To find the derivative of:

If  $y = \{f_1(x)\}^{f_2(x)}$  or  $y = f_1(x) \cdot f_2(x) \cdot f_3(x) \dots$

or  $y = \frac{f_1(x) \cdot f_2(x) \cdot f_3(x) \dots}{g_1(x) \cdot g_2(x) \cdot g_3(x) \dots}$

then it's easier to take the function's logarithm first and then differentiate. This is referred to as the logarithmic function's derivative.

### Important Notes (Alternate methods)

1. If  $y = \{f(x)\}^{g(x)} = e^{g(x) \ln f(x)} \left( (\text{variable})^{\text{variable}} \right) \left\{ \because x = e^{\ln x} \right\}$

$$\therefore \frac{dy}{dx} = e^{g(x) \ln f(x)} \cdot \left\{ g(x) \cdot \frac{d}{dx} \ln f(x) + \ln f(x) \cdot \frac{d}{dx} g(x) \right\}$$

$$= \{f(x)\}^{g(x)} \cdot \left\{ g(x) \cdot \frac{f'(x)}{f(x)} + \ln f(x) \cdot g'(x) \right\}$$

2. If  $y = \{f(x)\}^{g(x)}$

$$\therefore \frac{dy}{dx} =$$

Derivative of y treating  $f(x)$  as constant + Derivative of y treating  $g(x)$  as constant

$$= \{f(x)\}^{g(x)} \cdot \ln f(x) \cdot \frac{d}{dx} g(x) + g(x) \{f(x)\}^{g(x)-1} \cdot \frac{d}{dx} f(x)$$

$$= \{f(x)\}^{g(x)} \cdot \ln f(x) \cdot g'(x) + g(x) \cdot \{f(x)\}^{g(x)-1} \cdot f'(x)$$

### 4.3 Implicit Differentiation:

$$\phi(x, y) = 0$$

(i) To get  $dy/dx$  with the use of implicit function, we differentiate each term w.r.t.  $x$ , regarding  $y$  as a function of  $x$  & then collect terms in  $dy/dx$  together on one side to finally find  $dy/dx$ .

(ii) In answers of  $dy/dx$  in the case of implicit function, both  $x$  and  $y$  are present.

Alternate Method: If  $f(x, y) = 0$

$$\text{then } \frac{dy}{dx} = - \frac{\left( \frac{\partial f}{\partial x} \right)}{\left( \frac{\partial f}{\partial y} \right)} = - \frac{\text{diff of } f \text{ w.r.t. } x \text{ treating } y \text{ as constant}}{\text{diff. of } f \text{ w.r.t. } y \text{ treating } x \text{ as constant}}$$

#### 4.4 Parametric Differentiation

If  $y = f(t); x = g(t)$  where  $t$  is a Parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

**Note:**

$$1. \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$2. \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dt}{dx} \left( \because \frac{dy}{dx} \text{ in terms of } t \right)$$

$$= \frac{d}{dt} \left( \frac{f'(t)}{g'(t)} \right) \cdot \frac{1}{f'(t)} \{ \text{From (1)} \}$$

$$= \frac{f''(t)g'(t) - g''(t)f'(t)}{\{f'(t)\}^2}$$

#### 4.5 Derivative of a Function w.r.t. another Function

$$\text{Let } y = f(x); z = g(x) \text{ then } \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$$

#### 4.6 Derivative of Infinite Series

When one or more terms are removed from an infinite series, the series stays unaltered. as a result.

(A) If  $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots \infty}}}$

then  $y = \sqrt{f(x) + y} \Rightarrow (y^2 - y) = f(x)$

Differentiating both sides w.r.t.  $x$ , we get  $(2y - 1) \frac{dy}{dx} = f'(x)$

(B) If  $y = \{f(x)\}^{\{f(x)\}^{\{f(x)\}^{-1}}}$ , then  $y = \{f(x)\}^y \Rightarrow y = e^{y \ln f(x)}$

Differentiating both sides w.r.t.  $x$ , we get

$$\frac{dy}{dx} = \frac{y\{f(x)\}^{y-1} \cdot f'(x)}{1 - \{f(x)\}^y \cdot \ln f(x)} = \frac{y^2 f'(x)}{f(x)\{1 - y \ln f(x)\}}$$

#### 5. Derivative of Order Two & Three

Let us assume a function  $y = f(x)$  be defined on an open interval  $(a, b)$ . It's derivative, if it exists on  $(a, b)$ , is a certain function  $f'(x)$  [ or  $(dy/dx)$  or  $y'$  ] is called the first derivative of  $y$  w.r.t.  $x$ . If it occurs that the first derivative has a derivative on  $(a, b)$  then this derivative is called the second derivative of  $y$  w.r.t.  $x$  is denoted by  $f''(x)$  or  $(d^2y/dx^2)$  or  $y''$ .

Similarly, the 3<sup>rd</sup> order derivative of  $y$  w.r.t.  $x$ , if it exists, is defined by  $\frac{d^3y}{dx^3} = \frac{d}{dx} \left( \frac{d^2y}{dx^2} \right)$  it is also denoted by  $f'''(x)$  or  $y'''$  Some Standard Results :



$$(i) \quad \frac{d^n}{dx^n} (ax + b)^m = \frac{m!}{(m-n)!} \cdot a^n \cdot (ax + b)^{m-n}, m \geq n$$

$$(ii) \quad \frac{d^n}{dx^n} x^n = n!$$

$$(iii) \quad \frac{d^n}{dx^n} (e^{mx}) = m^n \cdot e^{mx}, m \in \mathbb{R}$$

$$(iv) \quad \frac{d^n}{dx^n} (\sin(ax + b)) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right), n \in \mathbb{N}$$

$$(v) \quad \frac{d^n}{dx^n} (\cos(ax + b)) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right), n \in \mathbb{N}$$

$$(vi) \quad \frac{d^n}{dx^n} \{e^{ax} \sin(bx + c)\} = r^n \cdot e^{ax} \cdot \sin(bx + c + n\phi), n \in \mathbb{N}$$

$$\text{where } r = \sqrt{a^2 + b^2}, \phi = \tan^{-1}(b/a)$$

$$(vii) \quad \frac{d^n}{dx^n} \{e^{ax} \cdot \cos(bx + c)\} = r^n \cdot e^{ax} \cdot \cos(bx + c + n\phi), n \in \mathbb{N}$$

$$\text{where } r = \sqrt{a^2 + b^2}, \phi = \tan^{-1}(b/a)$$

## 6. Differentiation Of Continuity and Differentiability

$$\text{If } F(X) = \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

where  $f, g, h, \ell, m, n, u, v, w$  are differentiable function of  $x$  then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

## 7. L' Hospital's Rule

If  $f(x)$  and  $g(x)$  are functions of  $x$  such that :

(i)  $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$  or  $\lim_{x \rightarrow a} f(x) = \infty = \lim_{x \rightarrow a} g(x)$   $f(x)$  and

(ii) Both  $f(x)$  and  $g(x)$  are continuous at  $x = a$  and

(iii) Both  $f(x)$  and  $g(x)$  are differentiable at  $x = a$  and

(iv) Both  $f(x)$  and  $g(x)$  are continuous at  $x = a$ , Then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}$  & so on, till determinant form vanishes.

■■.