CHAPTER 09

# Applications of Derivatives

## Tangent

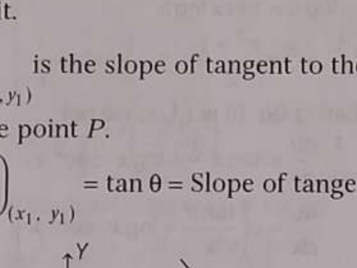
A tangent is a straight line, which touches the curve $y = f(x)$ at a point.

## Slope of Tangent

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be the point on it.

Then, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is the slope of tangent to the curve $y = f(x)$ at the point $P$.

i.e. $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \tan\theta =$ Slope of tangent at $P$,




where, $\theta$ is the angle which the tangent at $P(x_1, y_1)$ makes with the positive direction of $X$-axis as shown in the above figure.

**Particular Cases**

***Case* I** If the slope of the tangent line is zero,

i.e. $\frac{dy}{dx} = 0$, then $\tan\theta = 0 \Rightarrow \theta = 0$

It means the tangent line is parallel to $X$-axis.

***Case* II** If $\theta = \frac{\pi}{2}$, then $\frac{dy}{dx} = \tan\theta = \infty$, which means the tangent line is perpendicular to $X$-axis i.e. parallel to $Y$-axis.

## Equation of Tangent

Let $y = f(x)$ be a curve and $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ be the slope at point $P(x_1, y_1)$, then equation of tangent is

$$y - y_1 = m(x - x_1)$$

## Normal

The normal to the curve at any point $P$ on it is the straight line which passes through $P$ and is perpendicular to the tangent to the curve at $P$.

## Slope of Normal

We know that, normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ and passing through $P$.

- Slope of the normal at $P$

$$= -\frac{1}{\text{Slope of the tangent at } P}$$

- Slope of normal at $P(x_1, y_1)$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

## Equation of Normal

Let $y = f(x)$ be a curve and $m$ be the slope of tangent at point $P(x_1, y_1)$, then equation of normal at $P$ is

$y - y_1 = m_1(x - x_1)$ where, $m_1 = \frac{-1}{m}$

## Important Points Related to Tangent and Normal

- If equation of the curve is in parametric form, i.e. $x = f(t)$ and $y = g(t)$.

  Then, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)}$

  (a) Equation of tangent is
  $$y - g(t) = \frac{g'(t)}{f'(t)}\{x - f(t)\}$$
  (b) Equation of normal is
  $$y - g(t) = \frac{-f'(t)}{g'(t)}\{x - f(t)\}$$

- If the tangent at any point on the curve is equally inclined to both the axes.

  Then, $\dfrac{dy}{dx} = \pm 1$

- If the tangent at any point makes an equal intercept on the coordinate axes.

  Then, $\dfrac{dy}{dx} = -1$

## Derivative as the Rate Measure

The derivative $\dfrac{dy}{dx}$ represents the rate of change of variable $y$ with respect to $x$.

So, the rate of change of any physical quantitiy at any time is obtained by differentiating the physical quantity with respect to time.

e.g. Let $s$ be the distance measured from a fixed point after time $t$, then $\dfrac{ds}{dt}$ represents the rate of change of distance $(s)$ with respect to time $(t)$.

i.e. $\dfrac{ds}{dt} = \text{speed}$

- If two variables are varying with respect to another variable $t$,

  i.e. $y = f(t)$ and $x = g(t)$.

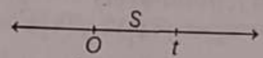
  Then, rate of change of $y$ with respect to $x$ is
  $$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}, \text{ provided } \frac{dx}{dt} \neq 0$$
  or $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$ [by chain rule of derivative]

- If $y$ increases as $x$ increase,then we take $\dfrac{dy}{dx}$ is positive and if $y$ increase as $x$ decrease, then we take $\dfrac{dy}{dx}$ as negative.

## Velocity

The rate of change of displacement $s$ of a particle with respect to time $t$ is called the velocity of the particle is denoted by $v$.

Thus, $v = \dfrac{ds}{dt}$




velocity at $t = 0$ is called **initial velocity**.

(i) If $v > 0$, then the particle is moving to the right of $O$ (or upwards).

(ii) If $v < 0$, then the particle is moving to the left of $O$ (or downwards).

(iii) If $v = 0$, then the particle stops i.e., particle is in the state of rest.

**Speed** It is the absolute value of velocity.

i.e. $v = \left|\dfrac{ds}{dt}\right|$

## Acceleration

The rate of change of velocity $v$ with respect to time $t$ is called the acceleration and is denoted by $a$.

Thus, $a = \dfrac{dv}{dt}$ or $a = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2s}{dt^2}$

(i) If $a > 0$, then the velocity increases.

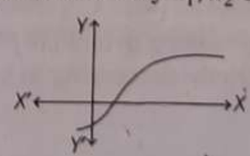
(ii) If $a < 0$, then the velocity decreases.

(iii) If $a = 0$, then the velocity is constant i.e. uniform.

## Increasing and Decreasing Functions

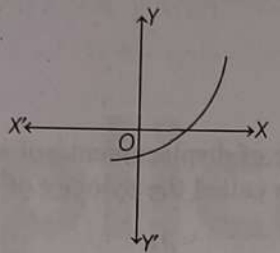
Let $I$ be an open interval contained in the domain of a real valued function $f$.

(i) **Increasing function** A function $f(x)$ is said to be increasing in $I$ iff for every $x_1, x_2 \in I$,

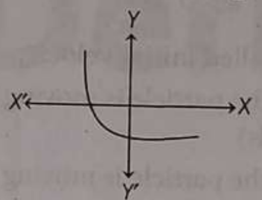



$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

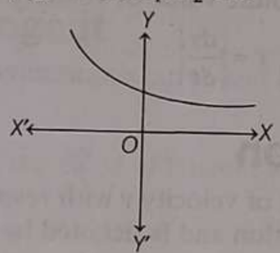
(ii) **Strictly increasing function** A function $f(x)$ is said to be strictly increasing function in $I$, iff $x_1 < x_2$ $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.




(iii) **Decreasing function** A function $f(x)$ is said to be derceasing in $I$, iff for every $x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.




(iv) **Strictly Decreasing Function** A function $f(x)$ is said to be strictly decreasing function in $I$, iff $x_1 < x_2$ $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.




(v) A function is said to be monotonic if it is either continuously increasing or continuously decreasing.

## First Derivative Test for Increasing and Decreasing Functions

Let $f$ be continuous on $[a, b]$ and differentiable on an open interval $(a, b)$. Then,

(i) $f$ is increasing in $[a, b]$, if $f'(x) \geq 0$ for each $x \in (a, b)$.

(ii) $f$ is decreasing in $[a, b]$, if $f(x) \leq 0$ for each $x \in (a, b)$.

(iii) $f$ is a constant function in $[a, b]$, if $f(x) = 0$ for each $x \in (a, b)$.

Also, $f$ is strictly increasing in $(a, b)$ if $f'(x) > 0$ for each $x \in (a, b)$ and $f$ is strictly decreasing in $(a, b)$ if $f'(x) < 0$ for each $x \in (a, b)$.

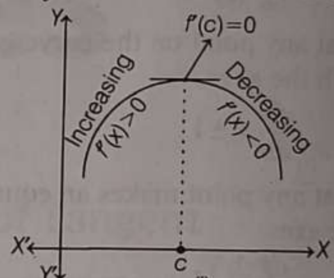
## Maxima and Minima

Let $f$ be a real valued function and $c$ be an interior point in the domain of $f$, then

- $c$ is called a point of local maxima, if there exist $h > 0$ such that $f(c) > f(x)$ for all $x$ in $(c - h, c + h)$. Here, value $f(c)$ is called the local maximum value of $f$.
- $c$ is called a point of local minima, if there exist $h > 0$ such that $f(c) < f(x)$ for all $x$ in $(c - h, c + h)$. Here, value $f(c)$ is called the local minimum value of $f$.

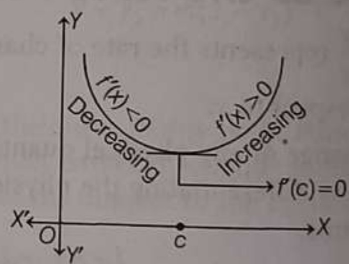
## Geometrical Interpretation

(i) Suppose $x = c$ is a point of local maxima of $f$, then the graph of $f$ around $c$ will be as shown in the figure.




(i)

(ii) Similarly, if $c$ is a point of local minima of $f$, then the graph of $f$ around $c$ will be as shown in the figure.




(ii)

Thus, the nature of $f$ in intervals is given below :

| Interval | $f$ in figure (i) | $f$ in figure (ii) |
|---|---|---|
| $(c - h, c)$ | increasing (i.e. $f'(x) > 0$) | decreasing (i.e. $f'(x) < 0$) |
| $(c, c + h)$ | decreasing (i.e. $f'(x) < 0$) | increasing (i.e. $f'(x) > 0$) |

Thus, either local maxima or local minima, $f'(c)$ must be zero.

## Critical Stationary Point

A point $c$ in the domain of a function $f$ at which either $f'(c) = 0$ or $f$ is not differentiable, is called a critical point of $f$. If $f$ is continuous at point $c$ and $f'(c) = 0$, then there exists $h > 0$ such that $f$ is differentiable in the interval $(c - h, c + h)$.

## Method of Finding Maxima and Minima

### First Derivative Test

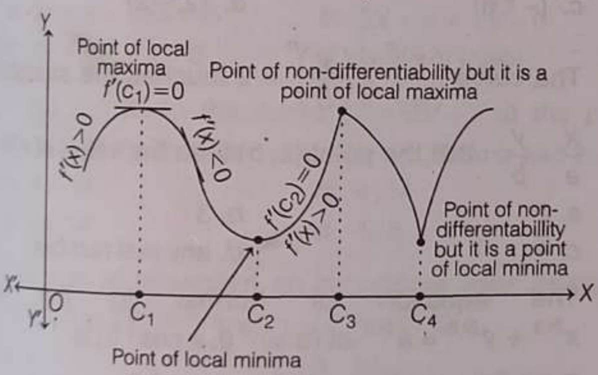
Let $f$ be a function defined on an open interval $I$ and $f$ be continuous at a critical point $c$ in $I$.

(i) If $f'(x)$ changes sign from positive to negative as $x$ increases through point $c$, i.e. if $f'(x) > 0$ at every point sufficient close to and to the left of $c$ and $f'(x) < 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of local maxima. The value of $f(x)$ at $x = c$ is local maximum value.

(ii) If $f'(x)$ changes sign from negative to positive as $x$ increases through point $c$, i.e. if $f'(x) < 0$ at every point sufficiently close to and to the left of $c$ and $f'(x) > 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of local minima. The value of $f(x)$ at $x = c$ is local minimum value.

(iii) If $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima.

Infact, such a point is called point of inflection.




### Second Derivative Test

Let $f$ be a function defined on an interval $I$ and $c \in I$ and $f$ be twice differentiable at $c$. Then,

(i) $x = c$ is a point of local maxima, if

$$f'(c) = 0 \text{ and } f''(c) < 0$$

The value $f(c)$ is local maximum value of $f$.

(ii) $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$.

The value $f(c)$ is local minimum value of $f$.

(iii) The test fails, if $f'(c) = 0$ and $f''(c) = 0$.

Then, further determine $f'''(x)$.

If $f'''(c) \neq 0$, then $f(x)$ has neither maximum nor minimum (inflexion point) at $x = c$.

But if $f'''(c) = 0$, then find $f^{\text{iv}}(c) = 0$.

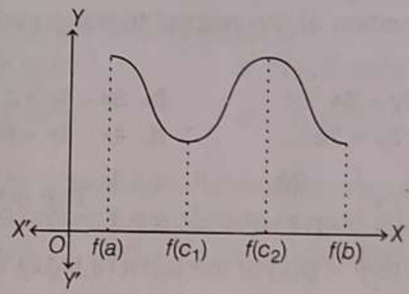
If $f^{\text{iv}}(c) =$ positive, then $f(x)$ is minimum at $x = c$.

If $f^{\text{iv}}(c) =$ negative, then $f(x)$ is maximum at $x = a$.

This process is going on until the point is discussed.

### Maximum and Minimum Values of a Function in a Closed Interval

A function may have a number of local maxima or local minima in a given interval. A local maximum value may not be the greatest and a local minimum value may not be the least value of the function in any given interval.




If a function $f(x)$ is continuous on a closed interval, $[a, b]$, then it attains the absolute maximum value or global maximum (absolute minimum value or global minimum) at critical points or at the end points of the interval $[a, b]$. Thus, to find the absolute maximum and absolute minimum value of the function, we choose the largest and smallest amongst the numbers $f(a)$, $f(c_1)$, $f(c_2)$, $f(b)$, where $c_1$ and $c_2$ are the critical points.

### Method to Find Absolute Maximum or Absolute Minimum Values in an Interval $[a, b]$

Suppose $f(x)$ be the given function. Then, to find absolute maximum and absolute minimum values in the given interval, *we use the following steps*

(i) Find all critical points of $f$ in the interval, i.e. find points at which either $f'(x) = 0$ or $f$ is not differentiable.

(ii) Calculate the values of $f$ at all critical points and end points of an interval.

(iii) Identify the maximum and minimum values of $f$ out of the values calculated in Step II. This maximum value will be the absolute maximum (greatest) value of $f$ and the minimum value will be the absolute minimum (least) value of $f$.