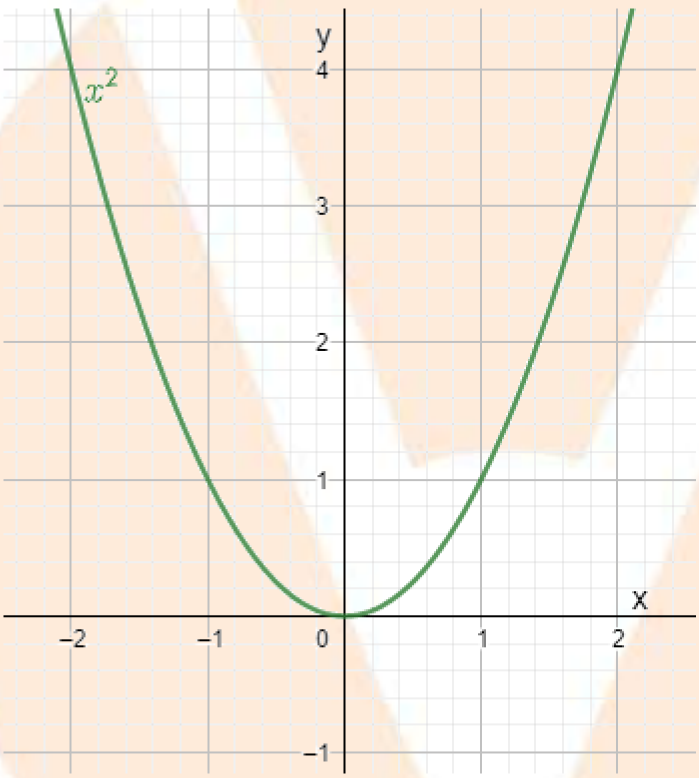
Revision Notes

Class 11 Maths

Chapter 13 Limits and Derivatives

Limits:

- Consider a function $f(x)=x^2$. Plotting it gives



Here, the value of x approaches 0 as the value of function f(x) moves to 0.

- In general as $x\to a$, $f\left(x\right)\to l$, then l is called limit of the function $f\left(x\right)$. This is written symbolically as $\lim\limits_{x\to a} f(x)=l$.
- Irrespective of the limits, the function should assume at a given point $x=a$
- There are two ways in which x can approach a number. It can either be from left or from right. This means that all the values of x near a could be less than a or could be greater than a.
    - Right hand limit - Value of $f\left(x\right)$ which is dictated by values of $f\left(x\right)$ when x tends to from the right. It is written as $\lim\limits_{x\to a^+} f(x)$.

- Left hand limit - Value of $f(x)$ which is dictated by values of $f(x)$ when x tends to from the left. It is written as $\lim\limits_{x \to a^-} f(x)$.

- Here, the right and left hand limits are different. So, the limit of $f(x)$ as x tends to zero does not exist (even though the function is defined at 0).
- If the right and left hand limits coincide then the common value is the limit and denoted by $\lim\limits_{x \to a} f(x)$.

Algebra of limits:

Theorem 1:

Let f and g be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then

- Limit of sum of two functions is sum of the limits of the functions, i.e. $\lim\limits_{x \to a}\left[f(x)+g(x)\right]=\lim\limits_{x \to a} f(x)+\lim\limits_{x \to a} g(x)$.
- Limit of difference of two functions is difference of the limits of the functions, i.e. $\lim\limits_{x \to a}\left[f(x)-g(x)\right]=\lim\limits_{x \to a} f(x)-\lim\limits_{x \to a} g(x)$
- Limit of product of two functions is product of the limits of the functions, i.e., $\lim\limits_{x \to a}\left[f(x).g(x)\right]=\lim\limits_{x \to a} f(x).\lim\limits_{x \to a} g(x)$
- Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e.,
$$\lim_{x \to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$$
- In particular as a special case of (iii), when g is the constant function such that $g(x)=\lambda$, for some real number λ , we have $\lim\limits_{x \to a}\left[(\lambda.f)(x)\right]=\lambda.\lim\limits_{x \to a} f(x)$

Limits of polynomials and rational functions:

- A function f is said to be a polynomial function if $f(x)$ is zero function or if $f(x)=a_0+a_1x+a_2x^2+...+a_nx^n$ where a_iS is are real numbers such that $a_n \neq 0$ for some natural number n.
- We know that $\lim\limits_{x \to a} x = a$

$$\lim_{x\to a} x^2 = \lim_{x\to a}(x.x) = \lim_{x\to a} x.\lim_{x\to a} x = a.a = a^2$$

Hence,

$$\lim_{x\to a} x^n = a^n$$

- Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial function

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left[a_0 + a_1x + a_2x^2 + \ldots + a_nx^n\right]$$
$$= \lim_{x\to a} a_0 + \lim_{x\to a} a_1x + \lim_{x\to a} a_2x^2 + \ldots + \lim_{x\to a} a_nx^n$$
$$= a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + \ldots + a_n\lim_{x\to a} x^n$$
$$= a_0 + a_1a + a_2a^2 + \ldots + a_na^n$$
$$= f(a)$$

- A function f is said to be a rational function, if $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$.

Then

$$\lim_{x\to a} f(x) = \lim_{x\to a}\frac{g(x)}{h(x)} = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{g(a)}{h(a)}$$

- However, if $h(a) = 0$, there are two scenarios –
    - when $g(a) \neq 0$
        - limit does not exist
    - When $g(a) = 0$.
        - $g(x) = (x-a)^k g_1(x)$, where k is the maximum of powers of $(x-a)$ in $g(x)$.
        - Similarly, $h(x) = (x-a)^l h_1(x)$ as $h(a) = 0$. Now, if $k \geq l$, we have

$$\lim_{x\to a} f(x) = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{\lim_{x\to a}(x-a)^k g_1(x)}{\lim_{x\to a}(x-a)^l h_1(x)}$$
$$= \frac{\lim_{x\to a}(x-a)^{(k-l)} g_1(x)}{\lim_{x\to a} h_1(x)} = \frac{0.g_1(a)}{h_1(a)} = 0$$

If $k < l$, the limit is not defined.

Theorem 2:

For any positive integer n, $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$.

The proof is shown below.

Dividing $\left(x^n-a^n\right)$ by $\left(x-a\right)$,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\left(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+xa^{n-2}+a^{n-1}\right)$$

$$=a^{n-1}+a\,a^{n-2}+\ldots+a^{n-2}(a)+a^{n-1}$$

$$=a^{n-1}+a^{n-1}+\ldots+a^{n-1}+a^{n-1}\left(n\text{ terms}\right)$$

$$=na^{n-1}$$

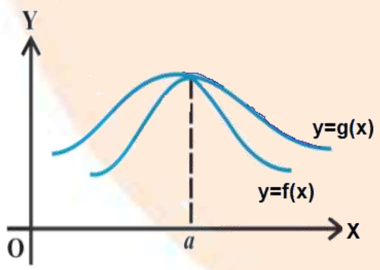
The expression in the above theorem for the limit is true even if n is any rational number and a is positive.

Limits of Trigonometric Functions:

Theorem 3:

Let f and g be two real valued functions with the same domain such that $f\left(x\right)\le g\left(x\right)$ for all x in the domain of definition,

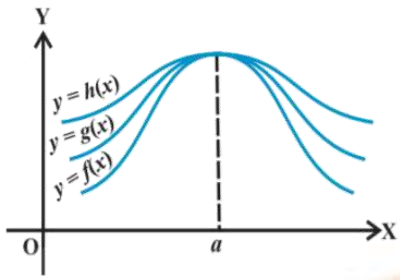
For some a, if both $\lim\limits_{x\to a}f(x)$ and $\lim\limits_{x\to a}g(x)$ exist, then $\lim\limits_{x\to a}f(x)\le\lim\limits_{x\to a}g(x)$



Theorem 4 (Sandwich Theorem):

Let f,g and h be real functions such that $f\left(x\right)\le g\left(x\right)\le h\left(x\right)$ for all x in the common domain of definition.

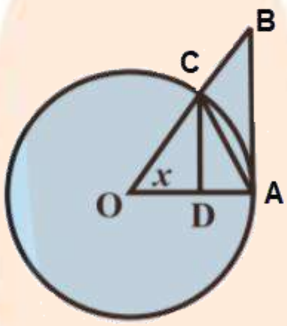
For some real number a, if $\lim\limits_{x\to a}f(x)=l=\lim\limits_{x\to a}g(x)$, then $\lim\limits_{x\to a}g(x)=l$.



To prove:

$\cos x < \frac{\sin x}{x} < 1$ for $0 < |x| < \frac{\pi}{2}$

Proof: Use known facts that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$



- From the figure, it is noted that O is the centre of the unit circle such that the angle AOC is x radians and $0 < x < \frac{\pi}{2}$.
- Two perpendiculars to OA are the line segments BA and CD.
- Now join AC and then,

Area of $\Delta OAC <$ Area of sector $OAC <$ Area of ΔOAB

i.e., $\frac{1}{2} OA.CD < \frac{x}{2\pi}.\pi.(OA)^2 < \frac{1}{2} OA.AB$

i.e., $CD < x.OA < AB$

From ΔOCD,

$\sin x = \frac{CD}{OA}$ (since $OC = OA$) and hence $CD = OA \sin x$.

Also $\tan x = \frac{AB}{OA}$ and hence $AB = OA \tan x$. Thus

$OA\sin x < OA.x < OA\tan x$

Since OA is positive,

$\sin x < x < \tan x$

Since $0 < x < \dfrac{\pi}{2}$, $\sin x$ is positive and thus by dividing throughout by $\sin x$,

$1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$.

Taking reciprocals throughout,

$\sin x < x < \tan x$.

Since $0 < x < \dfrac{\pi}{2}$, $\sin x$ is positive and thus by dividing throughout by $\sin x$,

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Taking reciprocals throughout,

$\cos x < \dfrac{\sin x}{x} < 1$

Hence, Proved.

The following are two important limits

i. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

ii. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$

The proof is given as below,

The function $\dfrac{\sin x}{x}$ is sandwiched between the function $\cos x$ and the constant function which takes value 1.

Since $\lim\limits_{x\to 0} \cos x = 1$ and $1-\cos x = 2\sin^2\left(\dfrac{x}{2}\right)$,

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\sin\left(\frac{x}{2}\right)$$

$$= \lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)}{\frac{x}{2}}.\lim_{x\to 0}\sin\left(\frac{x}{2}\right) = 1.0 = 0$$

Using the fact that $x \to 0$ is equivalent to $\frac{x}{2} \to 0$. This may be justified by putting $y = \frac{x}{2}$.

Derivatives:

Derivative of a function at a given point in its domain of definition.

- **Definition 1 -** Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at a is defined by $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided this limit exists. $f'(a)$ is used to denoted the derivative of $f(x)$ at a.
- **Definition 2 -** Suppose f is a real valued function, the function defined by $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ wherever limit exists is defined to be derivative of f at x denoted by $f'(x)$. This definition of derivative is also called the **first principle of derivative**.

  Thus $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The derivative of function f(x) with respect to x can be denoted in two ways: $f'(x)$ is denoted by $\frac{d}{dx}(f(x))$ or if $y = f(x)$, it is denoted by $\frac{dy}{dx}$.

Another notation is $D(f(x))$.

Further, derivative of f at $x = a$ is also denoted by $\left.\frac{d}{dx} f(x)\right|_a$ or $\left.\frac{df}{dx}\right|_a$ or even $\left(\frac{df}{dx}\right)_{x=a}$.

Theorem 5:

- Let f and g be two functions such that their derivatives are defined in a common domain. Then
  - Derivative of sum of two functions is sum of the derivatives of the functions.

    $$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$
  - Derivative of difference of two functions is difference of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

- Derivative of product of two functions is given by following product rule.

$$\frac{d}{dx}\left[f(x).g(x)\right]=\frac{d}{dx}f(x).g(x)+f(x).\frac{d}{dx}g(x)$$

- Derivative of quotient of two functions is given by the following quotient rule (whenever the denominator is non–zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}f(x).g(x)-f(x)\frac{d}{dx}g(x)}{(g(x))^2}$$

- Let $u=f(x)$ and $v=g(x)$.
  - **Product Rule:**
    - $(uv)'=u'v+uv'$.
    - Also referred as Leibnitz rule for differentiating product of functions
  - **Quotient Rule:**
    - $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$
- Derivative of the function $f(x)=x$ is the constant.

Theorem 6:

Derivative of $f(x)=x^n$ is nx^{n-1} for any positive integer n.

Proof

- By definition of the derivative function, we have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h} .$$

Binomial theorem tells that

$(x+h)^n=\left({}^nC_0\right)x^n+\left({}^nC_1\right)x^{n-1}h+...+\left({}^nC_n\right)h^n$ and

$(x+h)^n-x^n=h\left(nx^{n-1}+...+h^{n-1}\right)$. Thus

$$\frac{df(x)}{dx}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{h\left(nx^{n-1}+...+h^{n-1}\right)}{h}$$

$$=\lim_{h\to 0}\left(nx^{n-1}+\ldots+h^{n-1}\right)=nx^{n-1}$$

- This can be proved as below alternatively

$$\frac{d}{dx}\left(x^n\right)=\frac{d}{dx}\left(x.x^{n-1}\right)$$

$$=\frac{d}{dx}(x).\left(x^{n-1}\right)+x.\frac{d}{dx}\left(x^{n-1}\right) \text{ (By product rule)}$$

$$=1.x^{n-1}+x.\left((n-1)x^{n-2}\right) \text{ (By induction hypothesis)}$$

$$=x^{n-1}+(n-1)x^{n-1}=nx^{n-1}$$

Theorem 7:

- Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+....+a_1x+a_0$ be a polynomial function, where a_i s are all real numbers and $a_n\neq 0$. Then, the derivative function is given by

$$\frac{df(x)}{dx}=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+....+2a_2x+a_1$$

Quick Reference:

- **For functions f and g the following holds:**

$$\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x)$$

$$\lim_{x\to a}\left[f(x).g(x)\right]=\lim_{x\to a}f(x).\lim_{x\to a}g(x)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}$$

- **Following are some of the standard limits**

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\lim_{x\to a}\frac{\sin\left(x-a\right)}{x-a}=1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

$$\lim_{x\to 0}\frac{\tan x}{x}=1,\lim_{x\to a}\frac{\tan\left(x-a\right)}{x-a}=1$$

$$\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1,\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a, a>0, a\neq 1$$

- **Derivatives**
  - The derivative of a function f at a is defined by

    $$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

  - Derivative of a function f at any point x is defined by

    $$f'(x)=\frac{df(x)}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

- **For functions u and v the following holds:**

$$(u\pm v)'=u'\pm v'$$

$$(uv)'=u'v+uv'\Rightarrow\frac{d}{dx}(uv)=u.\frac{dv}{dx}+v.\frac{du}{dx}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}\Rightarrow\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v.\frac{du}{dx}-u.\frac{dv}{dx}}{v^2}$$

- **Following are some of the standard derivatives**

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

$$\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x)=\sec x.\tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x.\cot x$$