CHAPTER 06

Complex Number

Suppose, we consider equation $x^2+5=0$, the solution is not possible, because in the given equation $x^2=-5$ or $x=\sqrt{-5}$ and there is no set of real numbers whose square is -5.

Hence, equations of the form $x^2+n=0$, $n \in N$ cannot be solved out.

But in the year 1777, mathematician Euler make i sign. By using this sign, equation $x^2+n=0, n \in N$ has the non-real or imaginary solution.

e.g. the solution of equation $x^2+5=0$ or $x=\sqrt{-5}$ or $x=\sqrt{5}i$ will be possible.

Complex Numbers

A complex number can be defined as a number of the form $a+ib$, where a and b are real numbers. Here, the symbol i is used to denote $\sqrt{-1}$ and it is called **iota**.

e.g. $6+9i$, $-3+4i$ etc., are complex numbers.

The complex number is generally denoted by z.

i.e. $z=a+ib$.

Here, a is called the **real part**, denoted by **Re**(z) and b is called the **imaginary part** denoted by **Im**(z).

e.g. If $z=2+3i$, then $\text{Re}(z)=2$ and $\text{Im}(z)=3$.

Complex number z can be represented in the form of order pair i.e. z can be represented as (a, b).

Purely Real and Purely Imaginary Complex Numbers

A complex number $z=a+ib$, is called purely real, if $b=0$

i.e. $\text{Im}(z)=0$ and is called purely imaginary, if $a=0$

i.e. $\text{Re}(z)=0$. e.g. $z=6$, purely real and $z=6i$ is purely imaginary.

Zero Complex Number

A complex number is said to be zero, if its both real and imaginary parts are zero.

In other words, $z=a+ib=0$, if and only if $a=0$ and $b=0$.

Set of Complex Numbers

The product set $R \times R$ consisting of the ordered pair of real number called the set of real number. The set of complex numbers is denoted by C and it is defined as

$$C=\{a+ib : a, b \in R\}$$

Equality of Complex Numbers

Two complex numbers $z_1=a+ib$ and $z_2=c+id$ are said to be equal, if $a=c$ and $b=d$.

Integral Power of i (iota)

Positive Integral Powers of i

As we have seen, $i=\sqrt{-1}$. So, we can write the higher powers of i as follows

(i) $i^2=-1$

(ii) $i^3=i^2 \cdot i=(-1)\cdot i=-i$

(iii) $i^4=(i^2)^2=(-1)^2=1$

(iv) $i^5=i^{4+1}=i^4 \cdot i=1\cdot i=i$

(v) $i^6=i^{4+2}=i^4 \cdot i^2=1\cdot i^2=-1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

While evaluating i^n for $n>4$, we are writing n as $4q+r$ for some $q, r \in N$ and $0 \le r \le 3$.

So, in order to compute i^n for $n>4$, write $i^n=i^{4q+r}$ for some $q, r \in N$ and $0 \le r \le 3$.

Then, $i^n = i^{4q} \cdot i^r = (i^4)^q \cdot i^r = (1)^q \cdot i^r = i^r$

e.g. $i^{17} = i^{4\times4+1} = i^{4\times4} \cdot i = (i^4)^4 \cdot i = 1 \cdot i = i$

In general for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$,

$$i^{4k+2} = -1 \text{ and } i^{4k+3} = -i$$

Negative integral powers of i

Negative integral powers of i can be evaluated as follows

(i) $i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i}$

[multiply numerator and denominator by i]

$= \frac{i}{i^2} = \frac{i}{(-1)} = -i \quad [\because i^2 = -1]$

(ii) $i^{-2} = \frac{1}{i^2} = \frac{1}{(-1)} = -1$

(iii) $i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i}$

[multiplying numerator and denominator by i]

$= \frac{i}{(i^4)} = \frac{i}{(1)} = i$

(iv) $i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$

In order to compute i^{-n} for $n > 4$, first write

$$i^{-n} = \frac{1}{i^n} = \frac{1}{i^{4q+r}} \text{ for some } q, r \in N \text{ and } 0 \le r \le 3.$$

Then, evaluate i^{4q+r}.

Further, use above four negative integral powers of i.

e.g. $i^{-15} = \frac{1}{i^{15}} = \frac{1}{i^{4\times3+3}} = \frac{1}{i^3} \quad [\because i^{4q+3} = i^3]$

$= \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i \quad [\because i^4 = 1]$

Algebra of Complex Numbers

Here, we shall study how to add, subtract, multiply and divide the complex numbers.

Addition of Two Complex Numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers, then their addition is defined as

$$z = z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

Properties of Addition of Complex Numbers

- **Closure Law** If z_1 and z_2 are any two complex numbers, then $z_1 + z_2$ is also a complex number.
- **Commutative Law** If z_1 and z_2 are two complex numbers, then $z_1 + z_2 = z_2 + z_1$.
- **Associative Law** If z_1, z_2 and z_3 are any three complex numbers, then $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- **Existence of Additive Identity** There exists the complex number $0 = 0 + 0\,i$ called the identity element for addition.
  i.e. $z + 0 = z = 0 + z$ for all $z \in C$.
- **Existence of Additive Inverse** For every complex number $z = a + ib$, there exists $-z = (-a) + i(-b)$ such that $z + (-z) = 0 = (-z) + z$.
  Here, complex number $(-z)$, is called the **additive inverse of** z.

Subtraction of Two Complex Numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, their subtraction $z_1 - z_2$ is defined as the addition of z_1 and $(-z_2)$.

Thus, $z_1 - z_2 = z_1 + (-z_2) = (a_1 + ib_1) + (-a_2 - ib_2)$

$$z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

Multiplication of Complex Numbers

The product of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ can be as follow

$$z_1 z_2 = (a + ib)(c + id) = ac + iad + ibc + i^2 bd$$

$$= ac + i(ad + bc) + (-1)bd \quad [\because i^2 = -1]$$

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

Properties of Multiplication of Complex Numbers

- **Closure Law** If z_1 and z_2 are any two complex numbers, then $z_1 z_2$ is also a complex number.
- **Commutative Law** If z_1 and z_2 are any two complex numbers, then $z_1 z_2 = z_2 z_1$.
- **Associative Law** If z_1, z_2 and z_3 are any three complex numbers, then $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
- **Existence of Multiplicative Identity** There exists the complex number $1 = 1 + 0 \cdot i$ is the identity element for multiplication i.e. for every complex number z, we have $z \cdot 1 = 1 \cdot z = z$.
- **Existence of Multiplicative Inverse** (or Reciprocal) Corresponding to every non-zero complex number $z = a + ib$, there exists a complex number $z_1 = x + iy$ such that $z \cdot z_1 = 1 = z_1 \cdot z$, where

$$x = \frac{a}{a^2 + b^2} \text{ and } y = \frac{-b}{a^2 + b^2}.$$

Then, z_1 is called multiplicative inverse of z and it is denoted by $\frac{1}{z}$ or z^{-1}. We also called z_1, the **reciprocal of** z.

- **Distributive Law** If z_1, z_2 and z_3 are any three complex numbers.
  Then, $z_1(z_2+z_3)=z_1z_2+z_1z_3$ [left distributive law]
  and $(z_1+z_2)z_3=z_1z_3+z_2z_3$ [right distributive law]

Division of Two Complex Numbers

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

Therefore, $\frac{z_1}{z_2}=z_1\cdot z_2^{-1}=z_1\cdot\left(\frac{1}{z_2}\right)$

Conjugate of a Complex Number

The conjugate of a complex number z, is the complex number, obtained by changing the sign of imaginary part of z. It is denoted by $\bar{z}$.

e.g. If $z=2+3i$, then $\bar{z}=2-3i$
and if $z=-4-3i$, then $\bar{z}=-4+3i$

Properties of Conjugate of a Complex Number

- $\overline{(\bar{z})}=z$
- $z+\bar{z}=2\text{Re}(z)$
- $z-\bar{z}=2i\,\text{Im}\,(2)$
- $z=\bar{z}\Leftrightarrow z$ is purely real
- $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$
- $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$
- $\overline{z_1z_2}=\overline{z_1}\;\overline{z_2}$
- $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\overline{z_1}}{\overline{z_2}},\ z_2\neq 0$

Modulus (Absolute Value) of Complex Numbers

The modulus (or absolute value) of a complex number, $z=a+ib$ is defined as the non-negative real number $\sqrt{a^2+b^2}$.

It is denoted by $|z|$ i.e. $|z|=\sqrt{a^2+b^2}$

Properties of Modulus of Complex Numbers

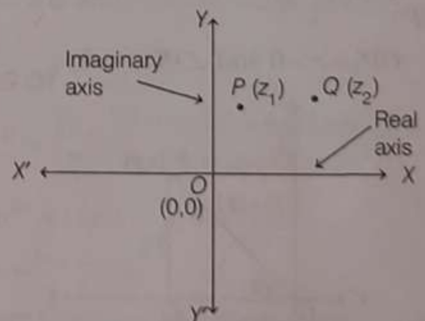
- $|z|=|\bar{z}|$
- $|z|=0\Leftrightarrow z=0$ i.e. $\text{Re}(z)=\text{Im}(z)=0$
- $-|z|\leq\text{Re}(z)\leq|z|;\ -|z|\leq\text{Im}(z)\leq|z|$
- $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\text{Re}(z_1\bar{z}_2)$
- $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\text{Re}(z_1\bar{z}_2)$
- $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$
- $|z_1z_2|=|z_1||z_2|$
  In general, if $z_1, z_2, \ldots, z_n$ are any complex numbers, then $|z_1z_2\ldots z_n|=|z_1||z_2|\ldots|z_n|$ ...(i)
  So, if $z_1=z_2=\ldots=z_n$, then from Eq. (i), we have $|z_1^n|=|z_1|^n$. Thus, we have $|z^n|=|z|^n$.
- $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$, provided $z_2\neq 0$
- $|z_1+z_2|\leq|z_1|+|z_2|$ [triangle inequality]
- $|z_1-z_2|\geq|z_1|-|z_2|$ [triangle inequality]
- $z\,\bar{z}=|z|^2$

Argand Plane

A complex number $z=a+ib$ can be represented by a unique point $P(a,b)$ in the cartesian plane.

A purely real number a, i.e. $(a+0i)$ is represented by the point $(a,0)$ on X-axis. Therefore, X-axis is called **real axis.**

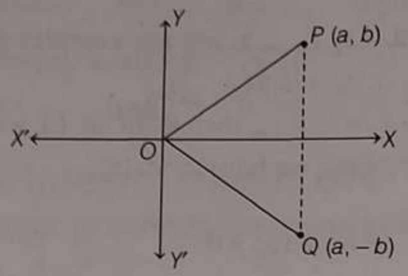
- A purely imaginary number ib i.e. $(0+ib)$ is represented by the point $(0,b)$ on Y-axis. Therefore, Y-axis is called **imaginary axis.**
- The intersection (common) of two axes is called **zero complex number** i.e. $z=0+0i$.
- Similarly, the representation of complex numbers as points in the plane is known as **Argand diagram.** The plane representing complex numbers as points, is called **Complex plane** or **Argand plane** or **Gaussian plane.**
- If two complex numbers z_1 and z_2 are represented by the points P and Q in the complex plane, then
  $|z_1-z_2|=PQ=$ Distance between P and Q



Representation of Conjugate of z on Argand Plane

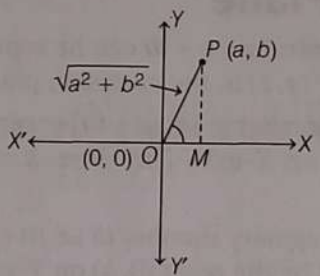
Geometrically, the mirror image of the complex number $z=a+ib$ [represented by the ordered pair (a,b)] about the X-axis is called **conjugate of z** represented by the ordered pair $(a,-b)$.

If $z = a + ib$, then $\bar{z} = a - ib$.



Representation of Modulus of z on Argand Plane

Geometrically, the distance of the complex number $z = a + ib$ [represented by the ordered pair (a, b)] from origin, is called the modulus of z.



$$\therefore \; OP = \sqrt{(a-0)^2 + (b-0)^2}$$
$$= \sqrt{a^2 + b^2}$$
$$= \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2} = |a + ib|$$

Polar Form of a Complex Number

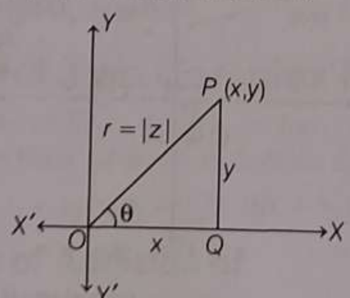
Let O be the origin and OX and OY be the real axis and imaginary axis, respectively.

Let $z = x + iy$ be represented by a point $P(x, y)$.

Draw $PQ \perp OX$. Then, $OQ = x$ and $PQ = y$.

Now, join OP.

Let $\qquad |OP| = r > 0$ and $\angle POX = \theta$



Then, in ΔPQO,

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \qquad \text{...(i)}$$

and

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta \qquad \text{...(ii)}$$

On squaring Eqs. (i) and (ii) and then adding, we get

$$r^2\cos^2\theta + r^2\sin^2\theta = x^2 + y^2$$
$$\Rightarrow \; r^2(\cos^2\theta + \sin^2\theta) = x^2 + y^2$$
$$\Rightarrow \; r^2 = x^2 + y^2 \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$
$$\Rightarrow \; r = \sqrt{x^2 + y^2} = |z|$$

On dividing Eq. (ii) by Eq. (i), we get

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) = \arg(z)$$

Thus, the polar form of a complex number z is $z = r(\cos\theta + i\sin\theta)$ and (r, θ) is called **polar coordinates** of the point P.

Here, r or $|z|$ is called the modulus of complex number z.

Angle θ is known as **argument** or **amplitude** of $z = x + iy$ and is written as 'arg(z)'. The **argument** of a complex number is not defined **uniquely**.

Thus, $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$

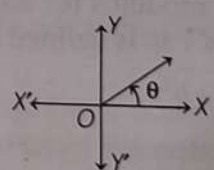
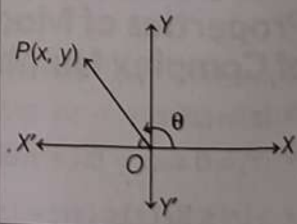
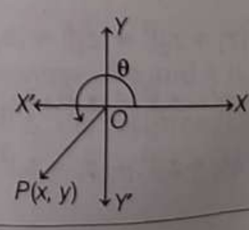
Principal Argument

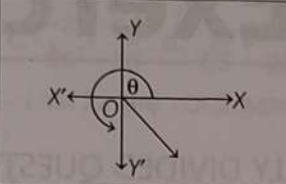
The unique value of θ such that $-\pi < \theta \le \pi$ is called the **principal value of the argument** or **amplitude** or **principal argument**.

The argument of a complex number depends upon the quadrant in which the point P lies.

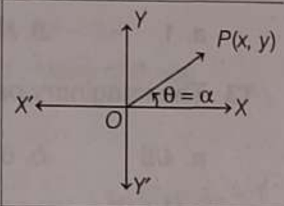
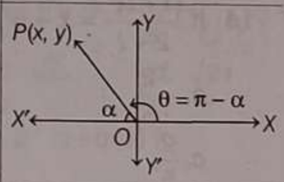
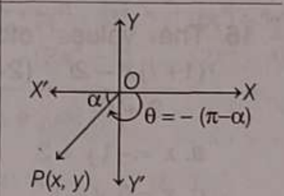
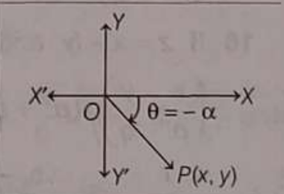
In different quadrant, the signs of real and imaginary parts of a complex number $z = x + iy$, its argument and graph are given in the tables on the next page.

General Argument when $0 \le \theta \le 2\pi$

Quadrant	Signs of x and y	Argument	Graph
In I quadrant	$x > 0, y > 0$	$0 < \theta < \frac{\pi}{2}$	
In II quadrant	$x < 0, y > 0$	$\frac{\pi}{2} < \theta < \pi$	
In III quadrant	$x < 0, y < 0$	$\pi < \theta < \frac{3\pi}{2}$	

Quadrant	Signs of x and y	Argument	Graph
In IV quadrant	$x>0, y<0$	$\frac{3\pi}{2}<\theta<2\pi$	

Principal Argument when $-\pi<\theta\leq\pi$

Quadrant	Signs of x and y	Argument	Graph
In I quadrant	$x>0, y>0$	$\theta=\alpha$ and $0<\theta<\frac{\pi}{2}$	
In II quadrant	$x<0, y>0$	$\theta=\pi-\alpha$ and $\frac{\pi}{2}<\theta<\pi$	
In III quadrant	$x<0, y<0$	$\theta=-(\pi-\alpha)$ $=\alpha-\pi$ and $-\pi<\theta<-\frac{\pi}{2}$	
In IV quadrant	$x>0, y<0$	$\theta=-\alpha$ and $-\frac{\pi}{2}<\theta<0$	

Properties of Argument of Complex Numbers

If z_1, z_2 and z_3 are three complex numbers, then

- $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$
- In general, $\arg(z_1z_2z_3\ldots z_n)=\arg(z_1)+\arg(z_2)+\arg(z_3)+\ldots+\arg(z_n)$
- $\arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$
- $\arg\left(\frac{z}{\bar{z}}\right)=2\arg(z)$
- $\arg(z^n)=n\arg(z)$
- If $\arg\left(\frac{z_2}{z_1}\right)=\theta$, then $\arg\left(\frac{z_1}{z_2}\right)=-\theta$
- $\arg(\bar{z})=-\arg(z)$
- If $\arg(z)=0 \Rightarrow z$ is real.
- $\arg(z_1\bar{z}_2)=\arg(z_1)-\arg(z_2)$
- $|z_1+z_2|=|z_1-z_2| \Rightarrow \arg(z_1)-\arg(z_2)=\frac{\pi}{2}$
- $|z_1+z_2|=|z_1|+|z_2| \Rightarrow \arg(z_1)=\arg(z_2)$
- $|z_1+z_2|^2=|z_1|^2+|z_2|^2 \Rightarrow \frac{z_1}{z_2}$ is purely imaginary.

Exponential form

It is known and can be proved using special series that $e^{i\theta}=\cos\theta+i\sin\theta$

$\therefore z=a+ib=r(\cos\theta+i\sin\theta)=re^{i\theta}$

where, $r=|z|$ and $\theta=\arg z$ is called an exponential form of complex number.

Cube Roots of Unity

Let $z=1^{1/3}$, then $z^3=1$ [on cubing both sides]

$$\Rightarrow z^3-1=0$$

$$\Rightarrow (z-1)(z^2+z+1)=0$$

$$\Rightarrow z-1=0 \text{ or } z^2+z+1=0$$

$$\Rightarrow z=1 \text{ or } z=\frac{-1\pm\sqrt{1-4}}{2}$$

$$\Rightarrow z=1 \text{ or } z=\frac{-1\pm i\sqrt{3}}{2}$$

So, the cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$.

Clearly, one of the cube roots of unity is real and the other two are complex.

Here, $\frac{-1+i\sqrt{3}}{2}=\omega$ and $\frac{-1-i\sqrt{3}}{2}=\omega^2$

Properties of 1, ω, ω^2

- $1+\omega^r+\omega^{2r}=\begin{cases}0, & \text{if } r \text{ is not a multiple of } 3\\ 3, & \text{if } r \text{ is multiple of } 3\end{cases}$
- $\omega^3=1$ or $\omega^{3r}=1$ and $1+\omega+\omega^2=0$
- $\omega^{3r+1}=\omega, \omega^{3r+2}=\omega^2$
- ω and ω^2 are the roots of the equation $z^2+z+1=0$.
- $\arg(\omega)=\frac{2\pi}{3}, \arg(\omega^2)=\frac{4\pi}{3}$
- Cube roots of -1 are $-1, -\omega, -\omega^2$.
- Cube roots of unity lie on the unit circle $|z|=1$ and divides its circumference into three equal parts.