Chapter

1 Rotational Dynamics

Quick Review

Circular Motion

- It is an Accelerated Motion
- It is a Periodic Motion

Uniform circular Motion (UCM)

- Speed is constant
- Acceleration is radial and always directed towards the centre.

Centripetal force

Centripetal force is directed along the radius towards the centre of a circle.
In vector form, it is given by $\vec{F} = -\frac{mv^2}{r}\hat{r}_0$

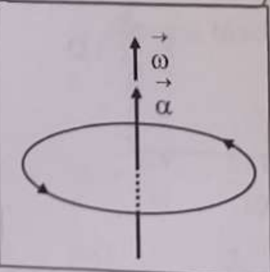
Centrifugal force

Centrifugal force is directed along the radius away from the centre of a circle.
In vector form, it is given by, $\vec{F} = +\frac{mv^2}{r}\hat{r}_0$

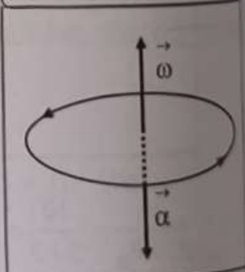
Non-uniform circular Motion (Non-UCM)

- Speed is not constant
- The acceleration of the particle is radial $(\vec{a})$ as well as tangential $(\vec{\alpha})$

Increasing speed



Decreasing speed



➤ Application of Uniform circular Motion (U.C.M):

1 Banking of Roads

- **Unbanked Road:** Vehicles moving on a circular horizontal road have to follow a speed limit considering the friction between the tyre and the road. The maximum possible speed, $v_s = \sqrt{\mu rg}$
- **Banked Road:** To avoid the skidding of vehicles travelling on a circular path, the roads are banked at certain angle (θ).

i. **Most safe speed:** The most safe speed a vehicle can move on a road banked at angle θ is, $v = \sqrt{rg\tan\theta}$

ii. **Banking angle:** The angle by which a road should be banked is, $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$

iii. **Speed limits:** Minimum speed a vehicle should maintain is, $v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1+\mu_s\tan\theta}\right)}$

Maximum speed a vehicle can attain is, $v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1-\mu_s\tan\theta}\right)}$

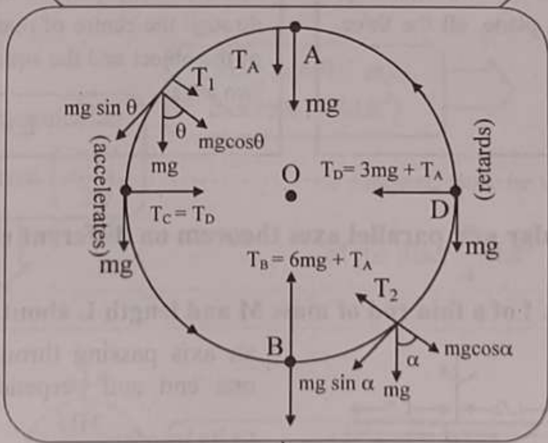
2 Well of death

- A person performing in a well of death has to maintain the minimum speed of his vehicle as $v_{min} = \sqrt{\frac{rg}{\mu_s}}$

3 Conical Pendulum

- When string of a pendulum is revolved the pendulum performs uniform circular motion.

Vertical Circular Motion (VCM)



Highest Point	Middle Point	Lowest Point
• Tension = minimum • Velocity = $\sqrt{rg}$ (minimum)	• Tension = (intermediate) • Velocity = $\sqrt{3rg}$	• Tension = (maximum) • Velocity = $\sqrt{5rg}$ (maximum)

➢ Terms involved in Rotational Motion:

Rotational Motion

Moment of Inertia (M.I.)

Moment of inertia of a rigid body about an axis of rotation is defined as the sum of product of the mass of each particle and the square of its perpendicular distance from the axis of rotation.

K.E. of rotation

Rotational K.E. of the object, is the sum of individual translational kinetic energies.

Torque (τ)

The turning effect of a force about the axis of rotation is called moment of force or torque due to that force. In case of rotational motion, torque is given as, $\tau = I\alpha$

Angular Momentum (L)

- The quantity in rotational mechanics, analogous to linear momentum is angular momentum.
- **Law of conservation of angular momentum:** Angular momentum of an isolated system is conserved in the absence of an external unbalanced torque.

Radius of Gyration (K)

The radius of gyration of a rigid body about a given axis of rotation is defined as the distance between the axis of rotation and a point at which the entire mass of the body can be supposed to be concentrated so as to possess the same moment of inertia as that of the body about that axis.

Moment of inertia

Parallel Axis Theorem

Statement: The moment of inertia (I_z) of a laminar object about an axis (z) perpendicular to its plane is the sum of its moment of inertias about two mutually perpendicular axes (x and y) in its plane, all the three axes being concurrent,

$$I_z = I_x + I_y$$

Perpendicular Axis Theorem

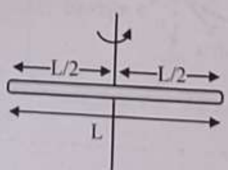
Statement: The moment of inertia (I_o) of an object about any axis is the sum of its moment of inertia (I_c) about an axis parallel to the given axis, and passing through the centre of mass and the product of the mass of the object and the square of the distance between the two axes,

$$I_o = I_c + Mh^2$$

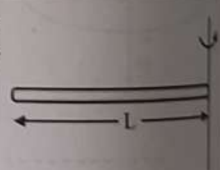
➢ Application of perpendicular and parallel axes theorem on different regular bodies:

M. I of a thin rod of mass M and length L about

an axis passing through its centre perpendicular to its length: $\frac{ML^2}{12}$



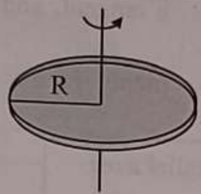
an axis passing through its one end and perpendicular to its length: $\frac{ML^2}{3}$



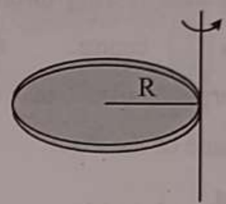
Using parallel axes theorem $\left[+M\left(\frac{L}{2}\right)^2\right]$

M. I of a circular ring of mass M and radius R about

an axis passing through its centre and perpendicular to the plane of the ring: $\mathbf{MR^2}$



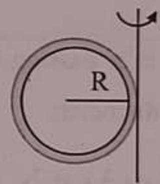
a tangent, and perpendicular to the plane of the ring: $\mathbf{2MR^2}$



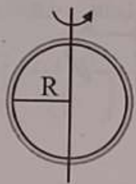
Using parallel axes theorem ($+ MR^2$)

Using perpendicular axes theorem $\left(\times\frac{1}{2}\right)$

a tangent, and in the plane of the ring: $\mathbf{\frac{3}{2}MR^2}$



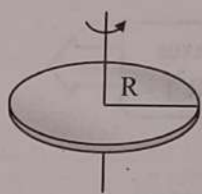
an axis passing through its diameter: $\mathbf{\frac{1}{2}MR^2}$



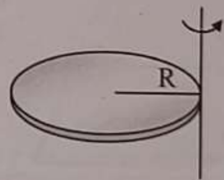
Using parallel axes theorem ($+ MR^2$)

M. I of a circular disc of mass M and radius R about

an axis passing through its centre and perpendicular to the plane of the disc: $\mathbf{\frac{1}{2}MR^2}$



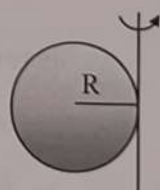
a tangent, and perpendicular to the plane of the disc: $\mathbf{\frac{3}{2}MR^2}$



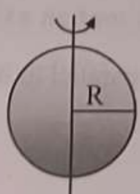
Using parallel axes theorem ($+ MR^2$)

Using perpendicular axes theorem $\left(\times\frac{1}{2}\right)$

a tangent, and in the plane of the disc: $\mathbf{\frac{5}{4}MR^2}$



an axis passing through its diameter: $\mathbf{\frac{1}{4}MR^2}$

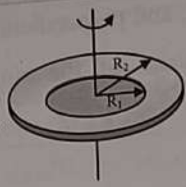


Using parallel axes theorem ($+ MR^2$)

M. I of a flat annular disc of mass M and inner and outer radii R_1 and R_2 about

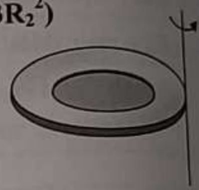
an axis passing through its centre and perpendicular to the plane of the disc:

$\frac{M}{2}(R_1^2 + R_2^2)$



Using parallel axes theorem $(+\ MR_2^2)$

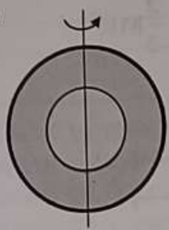
a tangent, and perpendicular to the plane of the disc: $\frac{M}{2}(R_1^2 + 3R_2^2)$



Using perpendicular axes theorem $\left(\times\frac{1}{2}\right)$

an axis passing through its diameter:

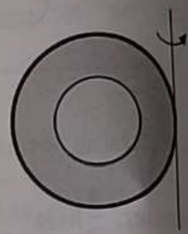
$\frac{M}{4}(R_1^2 + R_2^2)$



Using parallel axes theorem $(+\ MR_2^2)$

a tangent, and in the plane of the disc:

$\frac{M}{4}(R_1^2 + 5R_2^2)$



M. I of a solid sphere of mass M and radius R about an axis passing through

its diameter $\frac{2}{5}MR^2$

Using parallel axes theorem $(+\ MR^2)$

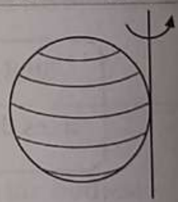
its tangent $\frac{7}{5}MR^2$

M. I of a hollow sphere of mass M and radius R about an axis passing through

its diameter $\frac{2}{3}MR^2$

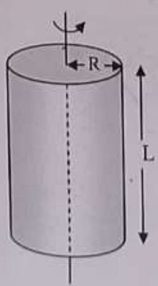
Using parallel axes theorem $(+\ MR^2)$

its tangent $\frac{5}{3}MR^2$



M. I of a cylinder of mass M, radius R and length L about an axis passing through

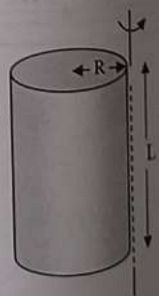
its own axis: $\frac{1}{2}MR^2$



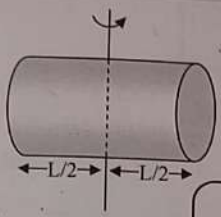
Using parallel axes theorem $(+\ MR^2)$

a tangent parallel to its length:

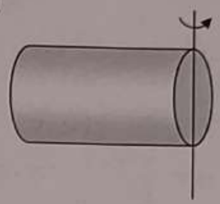
$\frac{3}{2}MR^2$



centre and perpendicular to length: $M\left(\frac{R^2}{4}+\frac{L^2}{12}\right)$



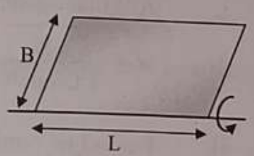
one of its end face and perpendicular to length: $M\left(\frac{R^2}{4}+\frac{L^2}{3}\right)$



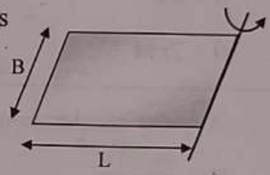
Using parallel axes theorem $\left[+M\left(\frac{L}{2}\right)^2\right]$

M. I of a rectangular lamina of mass M, length L and breadth B about an axis passing through

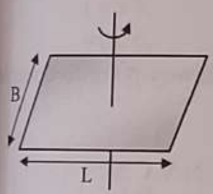
length of lamina and in its plane $\frac{MB^2}{3}$



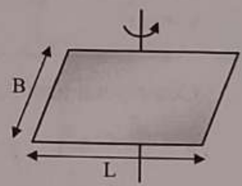
breadth of lamina and in its plane $\frac{ML^2}{3}$



centre of mass of lamina and perpendicular to its plane $M\left(\frac{L^2+B^2}{12}\right)$

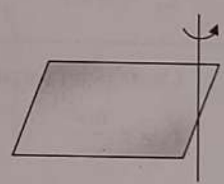


centre of length and perpendicular to its plane $M\left(\frac{L^2}{12}+\frac{B^2}{3}\right)$



Using parallel axes theorem $\left[+M\left(\frac{B}{2}\right)^2\right]$

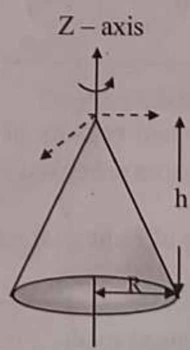
centre of breadth and perpendicular to its plane $M\left(\frac{L^2}{3}+\frac{B^2}{12}\right)$



Using parallel axes theorem $\left[+M\left(\frac{L}{2}\right)^2\right]$

M. I. of solid cone of mass M, Radius R and height h about an axis passing through

its central axis along Z – axis is $\frac{3}{10}MR^2$



Formulae

1. Angular velocity:

i. $\omega = \frac{v}{r}$ ii. $\omega = \frac{\theta}{t}$

iii. $\omega = 2\pi n$ iv. $\omega = \frac{2\pi}{T}$

2. Angular displacement:

i. $\theta = \omega t$ ii. $\theta = \frac{2\pi t}{T}$

iii. $\theta = 2\pi nt$

3. Angular acceleration:

i. $\alpha = \frac{\omega_2 - \omega_1}{t}$ ii. $\alpha = \frac{2\pi}{t}(n_2 - n_1)$

4. Linear velocity:

i. $v = r\omega$ ii. $v = 2\pi nr$

5. Centripetal acceleration or radial acceleration:

$a = \frac{v^2}{r} = \omega^2 r$

6. Tangential acceleration:

$\vec{a}_T = \vec{\alpha} \times \vec{r}$

7. Centripetal force:

i. $F_{CP} = \frac{mv^2}{r}$ ii. $F_{CP} = mr\omega^2$

iii. $F_{CP} = mr4\pi^2 n^2$ iv. $F_{CP} = \frac{4\pi^2 mr}{T^2}$

8. Centrifugal force: $F_{CF} = -F_{CP}$

9. Inclination of banked road:

$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$

10. On unbanked road:

i. Maximum velocity of vehicle to avoid skidding on a curve unbanked road: $v_{max} = \sqrt{\mu rg}$

ii. Angle of leaning: $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$

11. On banked road:

i. Upper speed limit: $v_{max} = \sqrt{rg\left[\frac{\mu_s + \tan\theta}{1 - \mu_s \tan\theta}\right]}$

ii. Lower speed limit: $v_{min} = \sqrt{rg\left[\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta}\right]}$

iii. $v_{max} = \sqrt{rg\tan\theta}$ (in absence of friction)

12. Height of inclined road: $h = l \sin\theta$

13. Conical Pendulum:

i. Angular velocity of the bob of conical pendulum

$\omega = \sqrt{\frac{g}{L\cos\theta}}$

ii. Period of conical pendulum, $T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$

14. For mass tied to string:

i. Minimum velocity at lowest point to complete V.C.M: $v_L = \sqrt{5rg}$

ii. Minimum velocity at highest point to complete V.C.M: $v_H = \sqrt{rg}$

iii. Minimum velocity at midway point to complete in V.C.M: $v_M = \sqrt{3rg}$

iv. Tension at highest point in V.C.M:

$T_H = \frac{mv_H^2}{r} - mg$

v. Tension at midway point in V.C.M: $T_M = \frac{mv_M^2}{r}$

vi. Tension at lowest point in V.C.M:

$T_L = \frac{mv_L^2}{r} + mg$

vii. Difference between tension at lower most and uppermost point: $T_L - T_H = 6\ mg$

15. Moment of Inertia: $I = \sum_{i=1}^{n} m_i r_i^2 = \int dmr^2$

16. Radius of gyration: $K = \sqrt{\frac{I}{M}}$

17. Kinetic energy:

i. $K.E_{rotational} = \frac{1}{2} I\omega^2 = \frac{1}{2} I(2\pi n)^2$

ii. $K.E_{translational} = \frac{1}{2} Mv^2$

iii. $K.E_{rolling} = \frac{1}{2}[Mv^2 + I\omega^2] = \frac{1}{2} Mv^2\left[1 + \frac{K^2}{R^2}\right]$

18. From principle of parallel axes:
$I_o = I_c + Mh^2$

19. From principle of perpendicular axes:
$I_Z = I_X + I_Y$

20. Angular momentum of a body: $L = I\omega = I(2\pi n)$

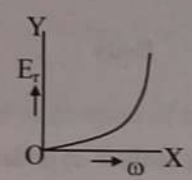
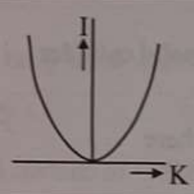
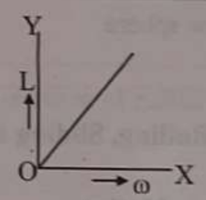
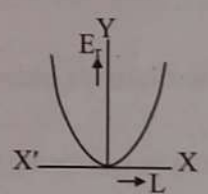
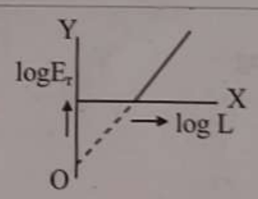
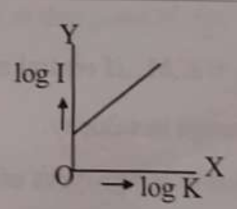
21. From principle of conservation of angular momentum:
i. $I_1\omega_1 = I_2\omega_2$ ii. $I_1n_1 = I_2n_2$

22. Torque acting on a body:
i. $\tau = I\alpha = \frac{dL}{dt}$ ii. $\tau = I\frac{d\omega}{dt} = 2\pi I\left(\frac{n_2 - n_1}{t}\right)$

23. Velocity of rolling body: $v = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}}$

24. Acceleration of rolling body: $a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}}$

> **Table representing the graphs of different parameters of rotational motion**

Sr. No.	Graph of	Formula	Graph
1.	K.E.$_{\text{rotational}}$ v/s ω where, ω = angular velocity	$K.E._{rot} = \frac{1}{2}I\omega^2$ i.e. $K.E._{rot} \propto \omega^2$ if I is constant	
2.	I v/s K where, K = radius of gyration	$I = MK^2$ i.e. $I \propto K^2$	
3.	L v/s ω where, L = angular momentum	$L = I\omega$ i.e. $L \propto \omega$	
4.	K.E.$_{\text{rotational}}$ v/s L	$K.E._{rot} = \frac{L^2}{2I}$ i.e. $K.E._{rot} \propto L^2$ if I is constant	
5.	log (K.E.$_{rot}$) v/s log (L)	$K.E._{rot} = \frac{L^2}{2I}$ i.e. $\log(K.E._{rot}) = 2\log(L) - \log(2I)$	
6.	log (I) v/s log (K)	$I = MK^2$ i.e. $\log(I) = \log(M) + 2\log(K)$	

➤ **Kinetic energy distribution table for different rolling bodies**

Body	$\frac{K^2}{R^2}$	Translational $(K_T) = \frac{1}{2}mv^2$	Rotational $(K_R) = \frac{1}{2}mv^2\frac{K^2}{R^2}$	Rolling $(K_{Roll}) = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right)$
Ring and Cylindrical shell	1	$\frac{1}{2}mv^2$	$\frac{1}{2}mv^2$	mv^2
Disc and solid cylinder	$\frac{1}{2}$	$\frac{1}{2}mv^2$	$\frac{1}{4}mv^2$	$\frac{3}{4}mv^2$
Solid sphere	$\frac{2}{5}$	$\frac{1}{2}mv^2$	$\frac{1}{5}mv^2$	$\frac{7}{10}mv^2$
Hollow sphere	$\frac{2}{3}$	$\frac{1}{2}mv^2$	$\frac{1}{3}mv^2$	$\frac{5}{6}mv^2$

➤ **Velocity, Acceleration and Time of descent for Different Bodies**

Body	Velocity $v = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}}$	Acceleration $a = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$	Time of descent $t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)}$
Ring or Hollow cylinder	$\sqrt{gh}$	$\frac{1}{2}g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{4h}{g}}$
Disc or solid cylinder	$\sqrt{\frac{4gh}{3}}$	$\frac{2}{3}g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}$
Solid sphere	$\sqrt{\frac{10}{7}gh}$	$\frac{5}{7}g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{14}{5}\frac{h}{g}}$
Hollow sphere	$\sqrt{\frac{6}{5}gh}$	$\frac{3}{5}g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{10}{3}\frac{h}{g}}$

➤ **Rolling, Sliding and Falling bodies**

Motion	Velocity	Acceleration	Time
Rolling	$\sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}}$	$\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$	$\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)}$
Sliding	$\sqrt{2gh}$	$g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$
Falling	$\sqrt{2gh}$	g	$\sqrt{\frac{2h}{g}}$

Shortcuts

1. In U.C.M., if central angle or angular displacement is given, then simply apply $dv = 2v\sin\frac{\theta}{2}$ to determine change in velocity.
2. There are two types of acceleration; a_r (radial) and a_t (tangential) acceleration.
   Formula for $a_r = \omega^2 r$ and $a_t = \frac{dv}{dt} = r\alpha$

3. To find out number of revolutions, always apply the formula,
Number of revolutions $= \frac{\theta}{2\pi} = \frac{\omega t}{2\pi} = \frac{2\pi nt}{2\pi} = nt$

4. The minimum safe velocity for not overturning is $v = \sqrt{\frac{gdr}{2h}}$

5. While rounding a curve on a level road, centripetal force required by the vehicle is provided by force of friction between the tyres and the road.
$\frac{mv^2}{r} = F = \mu R = \mu mg$

6. The maximum velocity with which a vehicle can go without toppling is given by $v = \sqrt{rg\frac{d}{2h}} = \sqrt{rg\tan\theta}$
where, $\tan\theta = \frac{d}{2h}$
d = distance between the wheels; h = height of centre of gravity from the road; g = acceleration due to gravity

7. Skidding of an object placed on a rotating platform:
The maximum angular velocity of rotation of the platform so that object will not skid on it is $\omega_{max} = \sqrt{(\mu g/r)}$

8. If earth suddenly contracts to $\left(\frac{1}{n}\right)^{th}$ of its present size without changes in its mass, then duration of new day $= \frac{24}{n^2}$ hours.

9. If an inclined plane ends into a circular loop of radius r, then height from which a body must start from rest to complete the loop is given by $h = \frac{5}{2}r$. Hence h is independent of mass of the body.

10. When a small body of mass m slides down from the top of a smooth hemispherical surface of radius R, then height at which the body loses the contact with surface, $h = \frac{2R}{3}$

11. A cyclist has to bend through an angle θ from his vertical position while rounding a curve of radius r with velocity v such that $\tan\theta = \frac{v^2}{rg}$
If θ is very very small, then $\tan\theta = \sin\theta = \frac{v^2}{rg} = \frac{h}{l}$
Where, h is height of the outer edge from the inner edge and l is the distance between the tracks or width of the road.

12. Always remember the formulae for velocity of the body at the top, bottom and at the middle of a circle with two distinct cases:
i. path is convex : $\frac{mv^2}{r} = mg - N$
ii. path is concave : $\frac{mv^2}{r} = N - mg$
where N is normal reaction.
Remember if in the question, it is given that body falls from a certain point then at that point N = 0.

13. In case of V.C.M.,
i. $v_L = \sqrt{2gr}$, the body moves in a vertical semicircle about the lowest point L,
ii. $v_L < \sqrt{2gr}$, then the body oscillates in a circular arc smaller than the semicircle.

14. When a bucket full of water is rotated in a vertical circle, water will not spill only if velocity of bucket at the highest point is $\geq \sqrt{gr}$.

15. If velocity imparted to body at the lowest position is equal to $\sqrt{2rg}$, then it will oscillate in a semicircle.

16. The distance travelled by the particle performing uniform circular motion in t seconds is given by formula, $d = \frac{2\pi r}{T} t$.

17. If a rod falls, apply the formula,
$\frac{1}{2} I\omega^2 = mg \times \left(\frac{L}{2}\right)$ where L is the length of the rod because when the rod falls, centre of mass travels vertical distance of $\frac{L}{2}$ and I will be equal to $\frac{mL^2}{3}$.

18. If there is a change in mass or distribution of mass for example, for a piece of wax falling on rotating apply the formula, $I_1\omega_1 = I_2\omega_2$.

19. Whenever the body falls from an inclined plane, apply $mgh = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$ and always remember, acceleration of a rolling body is given by, $\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$. Therefore, body for which $\left(\frac{K^2}{R^2}\right)$ is smallest, will fall first.

20. The condition for a body to roll down the inclined plane without slipping: $\mu \geq \left[\frac{K^2}{K^2 + R^2}\right]\tan\theta$
where μ = coefficient of limiting friction (μ)

21. A body cannot roll down the inclined plane when the friction is absent.
For this situation, the relative values of μ for rolling without slipping down the inclined plane are:
$\mu_{ring} > \mu_{shell} > \mu_{disc} > \mu_{solid\ sphere}$

22. The ratio of moments of inertia of two discs of the same mass and same thickness but of different densities given by $\frac{I_1}{I_2} = \frac{R_1^2}{R_2^2} = \frac{d_2}{d_1}$

23. To find ratios of different K.E., use

i. $\frac{\text{Rotational K.E.}}{\text{Total K.E.}} = \frac{\frac{K^2}{R^2}}{\left(1+\frac{K^2}{R^2}\right)}$

ii. $\frac{\text{Linear K.E.}}{\text{Total K.E.}} = \frac{1}{\left(1+\frac{K^2}{R^2}\right)}$