7. Differential Equations

- An equation is called a differential equation, if it involves variables as well as derivatives of dependent variable with respect to independent variable.

  For example:

  $$x\frac{d^4y}{dx^4}+y\left(\frac{d^2y}{dx^2}\right)^3-2x^2y\frac{dy}{dx}+3=0$$ is a differential equation.

  Sometimes, we may write $\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3},\frac{d^4y}{dx^4}$ etc. as $y'y'',y''',y''''$ etc. respectively. Also, note that we cannot say that $\tan(y')+x=0$ is a differential equation.

- Order of a differential equation is defined as the order of the highest order derivative of dependent variable with respect to independent variable involved in the given differential equation.

  For example: The highest order derivative present in the differential equation $x^3y^5y''''-3x^2y''+xyy'-5=0$ is y''''. Therefore, the order of this differential equation is 4.

- Degree of a differential equation is the highest power of the highest order derivative in it.

  For example: The degree of the differential equation $(y'')^2-2x(y'')^5-xy(y'')^2+y'=0$ is 2, since the highest power of the highest order derivative, y'', is 2.

  Note: The degree of the differential equation $\frac{d^2y}{dx^2}+\cos\left(\frac{dy}{dx}\right)=0$ is not defined since it is not a polynomial equation in $\frac{dy}{dx}$. However, its order is 2.

- If a differential equation is defined, then its order and degree are always positive integers.

- A function that satisfies the given differential equation is called a solution of a given differential equation.

Example: Verify whether $y = \sin x + \cos x - 5$ is a solution of the differential equation $y''+y'=0$ or not.

Solution:

We have, $y = \sin x + \cos x - 5$

$\because y' = \cos x - \sin x$

$y' = -\sin x - \cos x = -(\sin x + \cos x)$

$y'' = -(\cos x - \sin x) = -y'$

$\Rightarrow y''' + y' = 0$

Therefore, $y = \sin x + \cos x - 5$ is a solution of the differential equation $y''+y'=0$

- An equation is called a differential equation, if it involves variables as well as derivatives of dependent variable with respect to independent variable.

  For example:

  $$x\frac{d^4y}{dx^4}+y\left(\frac{d^2y}{dx^2}\right)^3-2x^2y\frac{dy}{dx}+3=0$$ is a differential equation.

  Sometimes, we may write $\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3},\frac{d^4y}{dx^4}$ etc. as $y'y'',y''',y''''$ etc. respectively. Also, note that we cannot say that $\tan(y')+x=0$ is a differential equation.

- Order of a differential equation is defined as the order of the highest order derivative of dependent variable with respect to independent variable involved in the given differential equation.
  For example: The highest order derivative present in the differential equation x3y5y""-3x2y"+xyy'-5=0 is y"". Therefore, the order of this differential equation is 4.

- Degree of a differential equation is the highest power of the highest order derivative in it.
  For example: The degree of the differential equation $(y'')^2 - 2x(y'')^5 - xy(y'')^2 + y' = 0$ is 2, since the highest power of the highest order derivative, y'', is 2.
  Note: The degree of the differential equation $\frac{d^2y}{dx^2} + \cos\left(\frac{dy}{dx}\right) = 0$ is not defined since it is not a polynomial equation in $\frac{dy}{dx}$. However, its order is 2.

- If a differential equation is defined, then its order and degree are always positive integers.

- A function that satisfies the given differential equation is called a solution of a given differential equation.

Example: Verify whether $y = \sin x + \cos x - 5$ is a solution of the differential equation $y'' + y' = 0$ or not.

Solution:
We have, $y = \sin x + \cos x - 5$
$\because y' = \cos x - \sin x$
$y' = -\sin x - \cos x = -(\sin x + \cos x)$
$y'' = -(\cos x - \sin x) = -y'$
$\Rightarrow y''' + y' = 0$
Therefore, $y = \sin x + \cos x - 5$ is a solution of the differential equation $y'' + y' = 0$

- To form a differential equation from a given function, we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.

Example: Form the differential equation, representing the family of circle $(x - a)^2 + (y - b)^2 = r^2$, where a and b are arbitrary constants.

Solution:
We have $(x - a)^2 + (y - b)^2 = r^2$ (1)
Differentiating with respect to x, we obtain

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0 \qquad \ldots(2)$$

Again differentiating with respect to x, we obtain

$$2 + 2\left[\left(\frac{dy}{dx}\right)^2 + (y-b)\frac{d^2y}{dx^2}\right] = 0$$

$$\Rightarrow (y-b)\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow y - b = -\left[\frac{1+(y')^2}{y''}\right]$$

Substituting $y - b = -\frac{[1+(y')^2]}{y''}$ in equation (2), we obtain

$$2(x-a)-2\frac{[1+(y')^2]}{y''}\times y'=0$$
$$\Rightarrow x-a=\frac{y'+(y')^3}{y''}$$

Substituting the values of $(x - a)$ and $(y - b)$ in equation (1), we obtain

$$\left[\frac{y'+(y')^3}{y''}\right]^2+\left[-\frac{1+(y')^2}{y''}\right]^2=r^2$$

$$\Rightarrow (y')^2\left[1+(y')^2\right]^2+\left[1+(y')^2\right]^2=r^2(y'')^2$$

$$\Rightarrow \left[1+(y')^2\right]^2\left[1+(y')^2\right]=r^2(y'')^2$$

$$\Rightarrow \left[1+(y')^2\right]^3-r^2(y'')^2=0$$

This is the required differential equation of the given circle.

- The three methods of solving first order, first degree differential equations are given as follows:
  - **Variable separable method:** This method is used to solve such an equation in which variables can be separated completely i.e., terms containing *y* should remain with *dy* and terms containing *x* should remain with *dx*.

Example: Solve the differential equation: $x(1+y^2)dx+y(4+x^2)dy=0$

Solution:

$$x(1+y^2)dx+y(4+x^2)dy=0$$

$$\Rightarrow \frac{x}{4+x^2}dx+\frac{y}{1+y^2}dy=0$$

$$\Rightarrow \frac{1}{2}\cdot\left(\frac{2x}{4+x^2}\right)dx+\frac{1}{2}\cdot\left(\frac{2y}{1+y^2}\right)dy=0$$

$$\Rightarrow \int\frac{2x}{4+x^2}dx=-\int\frac{2y}{1+y^2}dy$$

$$\Rightarrow \log(4+x^2)=-\log(1+y^2)+\log C$$

$$\Rightarrow \log(4+x^2)(1+y^2)=\log C$$

$$\Rightarrow (4+x^2)(1+y^2)=C$$

This is the required solution of the given differential equation.

- **Homogeneous differential equation:**

A differential equation which can be expressed as $\frac{dy}{dx} = f(x,y)$ or $\frac{dx}{dy} = g(x,y)$ **,** where $f(x, y)$ and $g(x, y)$ are homogenous functions of degree zero is called a homogenous differential equation. To solve such an equation, we have to substitute $y = vx$ in the given differential equation and then solve it by variable separable method.

Example: Solve the differential equation: $xydy - (x^2 - 3y^2)dx = 0$

Solution:

$xydy - (x^2 - 3y^2)dx = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 3y^2}{xy} = F(x,y)$, … (1)

Now,

$F(\lambda x, \lambda y) = \frac{\lambda^2 x^2 - 3\lambda^2 y^2}{\lambda^2 xy} = \frac{x^2 - 3y^2}{xy}$

$= \lambda^\circ f(x,y)$

F is homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

Let $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (1) becomes

$v + x\frac{dv}{dx} = \frac{x^2 - 3v^2x^2}{vx^2} = \frac{1-3v^2}{v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2}{v} - v = \frac{1-4v^2}{v}$

$\Rightarrow \int \frac{v}{1-4v^2} dv = \int \frac{dx}{x}$

$\Rightarrow -\frac{1}{8}\int \frac{-8v}{1-4v^2} dv = \int \frac{dx}{x}$

$\Rightarrow -\frac{1}{8}\log(1-4v^2) = \log(x) - \log C_1$

$\Rightarrow \log\left[x(1-4v^2)^{\frac{1}{8}}\right] = \log C_1$

$\Rightarrow x(1-4v^2)^{\frac{1}{8}} = C_1$

$\Rightarrow x^8(1-4v^2) = C_1^8 = C(\text{say})$

$\Rightarrow x^8\left(1 - 4\times\frac{y^2}{x^2}\right) = C$

$\Rightarrow x^6(x^2 - 4y^2) = C$

This is the required solution of the given differential equation.

- **Linear differential equation:**

- A differential equation which can be expressed in the form of $\frac{dy}{dx}+Py=Q$, where P and Q are constants or functions of x only, is called a first order linear differential equation.

In this case, we find integrating factor (I.F.) by using the formula:
$\text{I.F.}=e^{\int P\,dx}$
Then, the solution of the differential equation is given by,
$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dx+\text{C}$

- A linear differential equation can also be of the form $\frac{dx}{dy}+\text{P}_1x=\text{Q}$, where P_1 and Q_1 are constants or functions of y only.

In this case, $\text{I.F.}=e^{\int \text{P}_1 dy}$
And the solution of the differential equation is given by,
$x(\text{I.F.})=\int(\text{Q}_1\times\text{I.F.})\,dy+\text{C}$

Example: Find the solution of the differential equation $\sin y\,dx=\cos y(\sin y-x)\,dy$, satisfying the condition that $x = 5$ when $y=\frac{\pi}{2}$.

Solution:
We have,
$\sin y\,dx=\cos y(\sin y-x)\,dy$
$\Rightarrow \frac{dx}{dy}=\frac{\cos y(\sin y-x)}{\sin y}=\cos y-x\cot y$
$\Rightarrow \frac{dx}{dy}+x\cot y=\cos y$
This is a linear differential equation of the form $\frac{dx}{dy}+\text{P}_1x=\text{Q}$ where $\text{P}_1 = \cot y$ and $\text{Q}_1 = \cos y$

Now, I.F = $=e^{\int \text{P}_1 dy}=e^{\int \cot y\,dy}=e^{\int \log \sin y\,dy}=\sin y$

Therefore, the general solution of the given differential equation is

$x\times\sin y=\int\cos y\times\sin y\,dy+\text{C}$
$\Rightarrow x\sin y=\frac{1}{2}\int\sin 2y\,dy+\text{C}$
$\Rightarrow x\sin y=-\frac{1}{4}\cos 2y+\text{C}$
Substituting $y=\frac{\pi}{2}$ and $x = 5$ in this equation, we obtain

$5\sin\left(\frac{\pi}{2}\right)=-\frac{1}{4}\cos\left(2\times\frac{\pi}{2}\right)+\text{C}$
$5\sin\left(\frac{\pi}{2}\right)=-\frac{1}{4}\cos\pi+\text{C}$
$5\times1=-\frac{1}{4}\times(-1)+\text{C}$
$\Rightarrow \text{C}=\frac{19}{4}$
Therefore, the required solution is

$$x\sin y = -\frac{1}{4}\cos 2y + \frac{19}{4}$$
$$\Rightarrow x\sin y + \frac{1}{4}\cos 2y = \frac{19}{4}$$
$$\Rightarrow 4x\sin y + \cos 2y = 19$$

Application of Differential Equations

1) **Population growth and growth of bacteria:**

If the population *P* increases with time *t*, then the rate of change of *P* is generally proportional to the population present at that time.

i.e., dPdt = kP, where *k* is the constant of proportionality.

⇒ P = ekt + c = λ ekt , where ec = λ, which gives the population at any time *t*.

2) **Radio Active Decay:**

Radio active substances disintegrate with time. It means that the mass of the substance decreases with time. The rate of disintegration of such an element is always proportional to the amount present at that time.

If *x* is the amount of any material present at time *t,* then we have:

dxdt = - *kx*, where *k* is the constant of proportionality and *k* > 0.

Note: The negative sign appears because *x* decreases as *t* increases.

If x0 is the initial amount at time *t* = 0, then x = x0 e-kt is the amount of radio active substance at any time *t*.

Half-life period: Half-life period of a radio active substance is defined as the time taken for half of the substance to disintegrate.

3) **Newton's law of cooling :**

It states that the rate of cooling of a heated body at any time is proportional to the difference between the temperatures of the body and its surrounding medium.

If θ is the temperature of a body at time *t* and θ0 is the temperature of the medium, then according to Newton's law of cooling, we have:

dθdt = -kθ - θ0, where *k* is the constant of proportionality and the negative sign indicates that the difference in temperature is decreasing.

Integrating both sides, we get:

θ = θ0 + θ1 - θ0e-kt

This gives the temperature of a body at any time *t*.

4) **Surface Area:**

We also use differential equations to solve problems related to the surface area.