CHAPTER 04

Pair of Straight Lines

We know that, a linear equation in two variables $ax + by + c = 0, a, b, c \in R$ and $a, b \neq 0$ represents a straight line.

Suppose, we have equation of two lines $u_1 \equiv a_1x + b_1y + c_1 = 0$ and $u_2 \equiv a_2x + b_2y + c_2 = 0$, then $u_1 + \lambda u_2 = 0$ represents family of lines. If we combine these two equations, i.e. $u_1 \cdot u_2 = 0$, this represents a **pair of straight lines.**

Combined Equation of Pair of Lines

To find the combined (joint) equation of two lines, make RHS of two lines equal to zero and then multiply the two equations.

The combined equation of straight lines $(a_1x + b_1y) = 0$ and $(a_2x + b_2y) = 0$ is

$$(a_1x + b_1y)(a_2x + b_2y) = 0$$

Separate Equation of Two Lines

In order to find the separate equation of two lines when their joint equation is given, first of all make RHS equal to zero and then resolve LHS into two linear factors or use Sri Dharacharya method.

The two factors equated to zero will give the separate equations of lines.

e.g. The separate equation of the pair of lines

$12x^2 - 2y^2 + 5xy = 0$ is $(3x + 2y)(4x - y) = 0$

Homogeneous Equation of Second Degree

An equation of the form $ax^2 + 2hxy + by^2 = 0$ in which the sum of powers of x and y in every term is the same (here 2) is called homogeneous equation of second degree.

e.g. $2x^2 - xy - y^2 = 0$ and $6x^2 + 5xy - 4y^2 = 0$

Above all are homogeneous equations of second degree, because each term have dagree equals to 2.

Homogeneous equation of degree 2 represents two straight lines through origin, if

$$h^2 - ab \geq 0$$

Nature of Lines

The lines represented by $ax^2 + 2hxy + by^2 = 0$ are

(i) real and distinct, if $h^2 - ab > 0$

(ii) coincident, if $h^2 - ab = 0$

(iii) imaginary, if $h^2 - ab < 0$

(iv) If $b = 0$, then one line is the Y-axis and the other line having slope $-\frac{a}{2h}$

(v) If $h^2 - ab \geq 0$ and $b \neq 0$, then slope of the lines are

$$m_1 = \frac{-h - \sqrt{h^2 - ab}}{b} \text{ and } m_2 = \frac{-h + \sqrt{h^2 - ab}}{b}$$

(vi) The auxiliary equation of given homogeneous equation is $bm^2 + 2hm + a = 0$.

Sum and Product of Slopes

When lines represented by $ax^2+2hxy+by^2=0$ pass through the origin, let their equations be

$$y=m_1x \quad \text{and } y=m_2x,$$

then, $(y-m_1x)$ and $(y-m_2x)$ must be factors of $ax^2+2hxy+by^2=0$,

then $ax^2+2hxy+by^2=b(y-m_1x)(y-m_2x)$

Now, comparing the coefficient of xy and x^2 both sides, we get

$$2h=-b(m_1+m_2)$$

and

$$a=bm_1m_2$$

$$\therefore \quad \boldsymbol{m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b}}$$

Acute Angle between the Lines Represented by $ax^2+2hxy+by^2=0$

The acute angle θ between the pair of lines represented by

$$ax^2+2hxy+by^2=0$$

is given by

$$\boldsymbol{\theta=\tan^{-1}\left\{\frac{2\sqrt{(h^2-ab)}}{|a+b|}\right\}}$$

or

$$\theta=\sin^{-1}\left\{\frac{2\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}\right\}$$

$$\theta=\cos^{-1}\left\{\frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}\right\}$$

For acute angle, we take $\tan\theta$ as positive and for obtuse angle, we take $\tan\theta$ as negative.

Corollary 1

Condition for the lines to be perpendicular

The lines are perpendicular if the angle between them is $\frac{\pi}{2}$. i.e. $\theta=\frac{\pi}{2}$.

$$\Rightarrow \quad \cot\theta=\cot\frac{\pi}{2} \Rightarrow \cot\theta=0$$

$$\Rightarrow \quad \frac{|a+b|}{2\sqrt{(h^2-ab)}}=0 \Rightarrow a+b=0$$

$\Rightarrow$ Coefficient of x^2 + Coefficient of $y^2=0$

Corollary 2

Pair of any two perpendicular lines through the origin/point

Pair of lines perpendicular to the lines

$$ax^2+2hxy+by^2=0$$

(i) through origin is $bx^2-2hxy+ay^2=0$

(ii) through point (x_1, y_1) is

$$b(x-x_1)^2-2h(x-x_1)(y-y_1)+a(y-y_1)^2=0$$

Corollary 3

Condition for the lines to be Parallel (Coincident)

The lines are parallel coincident if the angle between them is 0° (or π)

i.e.

$$\theta=0 \quad (\text{or } \pi)$$

$$\therefore \quad \tan\theta=0^\circ$$

$$\Rightarrow \quad \frac{2\sqrt{(h^2-ab)}}{|a+b|}=0$$

$$\Rightarrow \quad h^2-ab=0$$

$$\Rightarrow \quad h^2=ab$$

Hence, the lines represented by $ax^2+2hxy+by^2=0$ are parallel (coincident) if $h^2=ab$, then $ax^2+2hxy+by^2$ is a perfect square.

Bisectors of the Angle between the Lines Given by a Homogeneous Equation

The joint equation of bisectors of the angles between the lines represented by the equation $ax^2+2hxy+by^2=0$ is

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$