CHAPTER 02

Matrices

In Mathematics, a **matrix** (plural matrices) is a rectangular array of numbers, symbols or expressions, arranged in rows and columns. The individuals in a matrix are called its **elements** or **entries.** A matrix is represented by a capital letters $A, B, X, \dots$ etc and elements of a matrix are represented by the small letters $a, b, c, \dots$ etc.

Generally, matrix is written in the following way

$$A=\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \vdots \ddots & \vdots \\ a_{m1} & a_{m2}\cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n}, 1\le i\le n, 1\le j\le n, i, j\in N$$

where, element a_{ij} is the entry at ith row and jth column.

The order of a matrix A is $m\times n$, where m is the number of rows and n is the number of columns.

Elementary Operations (Transformation)

There are six operations (or transformation) in a matrix, three of which are due to rows and three are due to columns. These operations are known as elementary operations or transformation.

Following elementary operations are given below

(i) Interchanging any two rows (or columns) is indicated as

$$R_i \leftrightarrow R_j \text{ (or } C_i \leftrightarrow C_j)$$

(ii) Multiplication of the elements of any row (or column) by a non-zero scalar quantity and indicated as

$$R_i \leftrightarrow kR_i \text{ (or } C_i \leftrightarrow kC_i)$$

(iii) Addition of constant multiple of the elements of any row (or column) to the corresponding element of any other row (or column), indicated as

$$R_i \to R_i + kR_j \text{ (or } C_i \to C_i + kC_j)$$

Inverse of a Matrix

If A is a square matrix of order m and if there exists another square matrix B of the same order such that $AB = BA = I$, where I is the identity matrix of order m, then B is called as the inverse of A and is denoted by A^{-1}.

Theorem (Uniqueness of inverse) Prove that if A is a square matrix and its inverse exists, then it is unique.

Inverse of a Non-singular Matrix by Elementary Transformation

Let A be any square matrix, then to find A^{-1}, using elementary row operations, write

$$A = IA$$

and apply a sequence of elementary row operations on $A = IA$ till, we get

$$I = BA$$

Hence, B will be the A^{-1}.

Similarly, to find A^{-1} using elementary column operation, write

$$A = AI$$

and apply a sequence of elementary column operations on $A = AI$ till we get

$$I = AB$$

Hence, B will be the A^{-1}.

Note that sometimes, after applying one or more elementary row (column) operations on $A = IA$ $(A = AI)$, we obtain all zeroes in one or more rows (columns) of the matrix A on LHS, then A^{-1} does not exist.

Determinant

Every square matrix (i.e. rows and columns are equal) A is associated with a number, called its determinant and it is denoted by $\det(A)$ or $|A|$.

(i) Expansion of first order determinant of matrix $A=[a]$ is $|A|=a$

(ii) Expansion of second order determinant of matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $|A|=a_{11}a_{22}-a_{21}a_{12}$

(iii) Expansion of third order determinant of matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$|A|=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$=a_{11}(a_{22}a_{33}-a_{32}a_{23})-a_{12}(a_{21}a_{33}-a_{31}a_{23})+a_{13}(a_{21}a_{32}-a_{31}a_{22})$$

Minors

Let $A=[a_{ij}]$ be a square matrix of order n. Then, the minor M_{ij} of a_{ij} in A is the determinant obtained by deleting ith row and jth column in which element a_{ij} lies. It is denoted by M_{ij} of A.

Consider a matrix of order 3, such that $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then, the minor of a_{11} is $M_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=a_{22}a_{33}-a_{32}a_{23}$

Also, the minor a_{12} is $M_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=a_{21}a_{33}-a_{31}a_{23}$ and similarly, we can find other minors of elements of A.

Cofactors

Let $A=[a_{ij}]$ be a square matrix of order n. Then, the cofactor C_{ij} (or A_{ij}) of a_{ij} in A is $(-1)^{i+j}$ times M_{ij}, where M_{ij} is the minor of a_{ij} in A.

$$\therefore \quad C_{ij}=(-1)^{i+j}M_{ij}$$

Consider the matrix, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then, cofactor of a_{11} is $C_{11}=(-1)^{1+1}M_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

Also, the cofactor of a_{12} is $C_{12}=(-1)^{1+2}M_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and similar other cases.

(i) The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) is Δ.

i.e. $\Delta=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$.

(ii) The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero.

i.e. $a_{11}C_{31}+a_{12}C_{32}+a_{13}C_{33}=0$.

Adjoint of a Matrix

Let $A=[a_{ij}]_{m\times n}$ be a square matrix of order n and C_{ij} be the cofactor of a_{ij}. Then, the adjoint of A is defined as the transpose (i.e. interchange rows and columns) of the cofactor matrix and it is denoted by adj (A).

Consider the matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then, cofactors of A are

$$C_{11}=(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ C_{12}=(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Similarly $C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}$

$$\therefore \quad \text{adj}(A)=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T=\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Note that if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $adj\,(A)=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Properties of Adjoint of a Matrix

Let A and B be a matrix of order n, then

(i) $\text{adj}(AB)=(\text{adj}\,B)(\text{adj}\,A)$

(ii) $(\text{adj}\,A)A=A(\text{adj}\,A)=|A|\cdot I_n$

(iii) (a) $|\text{adj}\,A|=|A|^{n-1}$, if $|A|\neq 0$

(b) $|\text{adj}\,A|=0$, if $|A|=0$

(iv) If $|A|=0$, then $(\text{adj}\,A)A=A(\text{adj}\,A)=O$

(v) $\text{adj}(A^m)=(\text{adj}\,A)^m, m\in N$

(vi) $\text{adj}(kA)=k^{n-1}(\text{adj}\,A), k\in R$

(vii) $|\text{adj}(\text{adj}\,A)|=|A|^{(n-1)^2}$

(viii) $\text{adj}(\text{adj}\,A)=|A|^{n-2}A$

Singular Matrix and Non-singular Matrix

If A is a square matrix and $|A| = 0$, then A is known as singular matrix.

If A is a square matrix and $|A| \neq 0$, then A is known as non-singular matrix.

Inverse of a Square Matrix by Adjoint Method

If A is non-singular matrix (i.e. $|A| \neq 0$), then its inverse exist and it is determined by the formula, $A^{-1} = \frac{\text{adj } A}{|A|}$.

Note that If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| \neq 0$,

then $\quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Properties of Inverse of a Matrix

If A, B and C are three matrices of same order and $|A| \neq 0, |B| \neq 0$ and $|C| \neq 0$, then

(i) (a) $AB = AC \Rightarrow B = C$ [left cancellation law]
(b) $BA = CA \Rightarrow B = C$ [right cancellation law]

(ii) (a) $(AB)^{-1} = B^{-1}A^{-1}$ (b) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(iii) (a) $(A^{-1})^{-1} = A$ (b) $(A^k)^{-1} = (A^{-1})^k$

(iv) $(kA)^{-1} = \frac{1}{k}A^{-1}$, if $k \neq 0$

(v) If A is a non-singular matrix, then

$$|A^{-1}| = |A|^{-1} \Rightarrow |A^{-1}| = \frac{1}{|A|}$$

(vi) The inverse of an identity matrix is again an identity matrix.

(vii) A square matrix is invertible, if it is non-singular and every invertible matrix possesses a unique inverse.

Inverse by using Algebraic Equation

Sometimes, a matrix A and an algebraic equation in matrix A is in the form of $aA^2 + bA + C = 0$.

Here, we have to find the A^{-1} by using given equation.

Firstly, we multiply the given equation by A^{-1} and simplify it to get inverse matrix

$$A^{-1} = \frac{1}{C}[-aA + bI]$$

Further, put the matrix A in right hand side and simplify it to the require A^{-1}.

Applications of Matrices

(Solution of a System of Linear Equations)

Let system of linear equations in three variables be

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$

and $\quad a_3x + b_3y + c_3z = d_3$

The given system of equations can be written in the matrix form as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

Firstly simplify the right hand side and then equate the matrix. Finally, we get the required solution.

This is known as **inverse method** to solve the system of equation.

Non-homogeneous Equations ($B \neq O$)

(i) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ and $(\text{adj} A)\cdot B \neq O$, then the system of equations is inconsistent and has no solution.

(iii) If $|A| = 0$ and $(\text{adj} A)\cdot B = O$, then the system of equations may be consistent and has an infinite number of solutions or inconsistent.

Homogeneous Equations ($B = O$)

(i) A homogeneous system of equations is consistent, if $|A| = 0$.

(ii) If $|A| \neq 0$, then system of equations have only trivial solution and it has one solution.

(iii) If $|A| = 0$, then system of equations has non-trivial solution and it has infinite number of solutions.

(iv) If number of equations is less than number of unknowns, then it has non-trivial solution.