CHAPTER 05

# Vectors

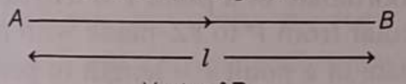
## Scalar Quantities and Vector Quantities

Those quantities which have magnitude but no direction, are called **scalar quantities** or **scalars**. e.g. length, mass, time, distance, speed, area, volume, temperature, work, density, voltage, resistance, etc.

Those quantities which have magnitude as well as direction are called **vector quantities** or **vectors**. e.g. force, displacement, velocity, acceleration, weight, momentum, electric field intensity, etc.

## Representation of a Vector

A vector is represented by a directed line segment having an initial and terminal point.




Vector **AB**

Here the length of line segment $l$ represents magnitude of vector and direction represent **direction** of vector.

Generally, vectors are denoted by **a**, **b**, **c** etc.

## Types of Vectors

There are following types of vectors as given below

(i) **Zero or Null Vector** A vector whose magnitude is zero, is called a null vector or zero vector. It is denoted by **0**.

(ii) **Unit Vector** A vector whose magnitude is one unit (i.e. unity) is called a unit vector. The unit vector in the direction of **a** is denoted by $\hat{\mathbf{a}}$ and read as '$a$ cap'.

(iii) **Coinitial Vectors** Two or more vectors having the same initial point are called coinitial vectors.

(iv) **Free Vector** If the value of a vector depends only on its magnitude and direction and is independent of its position in the space, it is called **free vector**.

(v) **Collinear or Parallel Vectors** Two or more vectors are said to be collinear, if they are parallel to same line, irrespective of their magnitudes and directions.

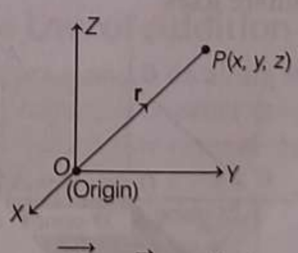
(vi) **Equal Vectors** Two vectors are said to be equal, if they have same magnitude and direction regardless of the positions of their initial points.
Symbolically if **a** and **b** are equal, then it is written as **a** = **b**.

(vii) **Negative of a Vector** A vector whose magnitude is same as that of a given vector but the direction is opposite to that of it, is called negative of the given vector.
e.g. Vector **BA** is negative of the vector **AB** and written as

$$\mathbf{BA} = -\mathbf{AB}$$

(viii) **Coplanar Vectors** Three or more vectors, which either lie in the same plane or are parallel to the same plane, are called coplanar vectors.
Generally, two vectors are always coplanar.

(ix) **Position Vector** The vector joining a point in the space to the origin is called position vector of the point.
Let $O(0, 0, 0)$ be the origin and $P$ be a point in space having coordinates $(x, y, z)$ with respect to the origin $O$.

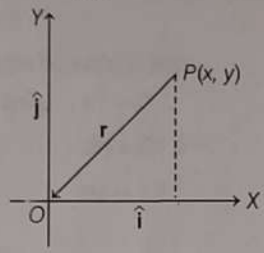



Then, the vector $\overrightarrow{OP}$ or $\vec{r}$ is called the position vector of the point $P$ with respect to $O$.

## Components of a Vector in Two Dimensions

Any vector r can be expressed as a linear combination of two unit vectors $\hat{i}$ and $\hat{j}$ at right angle.

i.e. $\mathbf{r} = x\hat{i} + y\hat{j}$




The vectors $x\hat{i}$ and $y\hat{j}$ are vector components of vector $\mathbf{r}$. The scalars $x$ and $y$ are called the scalar components of $\mathbf{r}$ in the direction of $X$ and $Y$-axes respectively.

The distance of a point $P$ from origin is

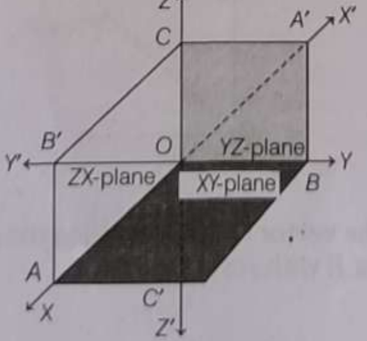
$$|\mathbf{r}| = \sqrt{x^2 + y^2} \Rightarrow \theta = \tan^{-1}\frac{y}{x}$$

## Three Dimension Coordinate System

We know that, the position of a point in a plane can be determined if the coordinates $(x, y)$ of the point with reference to two mutually perpendicular lines called $X$ and $Y$-axes, are known. In order to locate a point in space, two coordinate axes are insufficient. So, we need three coordinate axes called $X$, $Y$ and $Z$-axes having coordinates $(x, y, z)$. Hence, three dimensional geometry deals with the system of these three coordinate axes and their coordinates.

## Coordinate Axes and Coordinate Planes

Let $X'OX$, $Y'OY$ and $Z'OZ$ be three mutually perpendicular lines intersecting at $O$. The point $O$ is called the origin and the lines $X'OX$, $Y'OY$ and $Z'OZ$ are called $X$-axis $Y$-axis and $Z$-axis, respectively. These three lines are also called the **rectangular coordinate axes.**

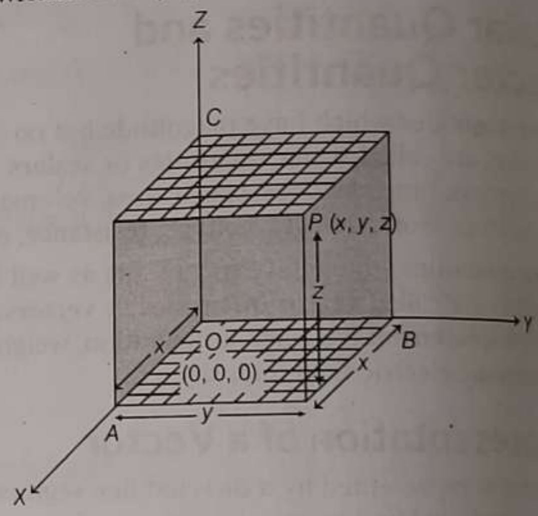



These lines constitute the rectangular coordinate system. These three axes, taken in pairs determine three mutually perpendicular planes, $XOY$, $YOZ$ and $ZOX$ or simply $XY$-plane, $YZ$-plane and $ZX$-plane called rectangular **coordinate planes** which divide the space into eight parts called **octants.**

## Coordinates of a Point in Space

Let $P$ be a point in space. Through $P$, draw three planes parallel to the coordinate axes to meet the axes in $A$, $B$ and $C$, respectively.

Let $OA = x$, $OB = y$ and $OC = z$. These three numbers taken in order are called **coordinates** of a point $P$ and written as $P(x, y, z)$.




Thus, $x$-coordinate of a point $P$ = length of perpendicular from $P$ to $YZ$-plane with proper sign.

$y$-coordinate of a point $P$ = length of perpendicular from $P$ to $ZX$-plane with proper sign.

$z$-coordinate of a point $P$ = length of perpendicular from $P$ to $XY$-plane with proper sign.

- The coordinates of any point on the $X$-axis, $Y$-axis and $Z$-axis will be considered as $A(x, 0, 0)$, $B(0, y, 0)$ and $C(0, 0, z)$ respectively.
- The coordinates of any point on the $XY$-plane, $YZ$-plane and $ZX$-planes are $L(x, y, 0)$, $M(0, y, z)$ and $N(x, 0, z)$ respectively.
- Distance of $P(x, y, z)$ from coordinate planes are given below
  (i) Distance from point $P(x, y, z)$ to the $XY$-plane is $|z|$.
  (ii) Distance from point $P(x, y, z)$ to the $YZ$-plane is $|x|$.
  (iii) Distance from point $P(x, y, z)$ to the $ZX$-plane is $|y|$.
- The distance from any point $P(x, y, z)$ to the origin is $\sqrt{x^2 + y^2 + z^2}$.

- The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- The distance from point $P(x, y, z)$ to the coordinate axes, $X, Y$ and $Z$ are respectively

$$\sqrt{y^2 + z^2}, \sqrt{x^2 + z^2} \text{ and } \sqrt{x^2 + y^2}.$$

- The sign of coordinates of the points in the octants in which the space is divided are given in the following table

| Octants | I | II | III | IV | V | VI | VII | VIII |
|---|---|---|---|---|---|---|---|---|
| Coordinates | OXYZ | OX′YZ | OX′Y′Z | OXY′Z | OXYZ′ | OX′YZ′ | OX′Y′Z′ | OXY′Z′ |
| $x$ | + | − | − | + | + | − | − | + |
| $y$ | + | + | − | − | + | + | − | − |
| $z$ | + | + | + | + | − | − | − | − |

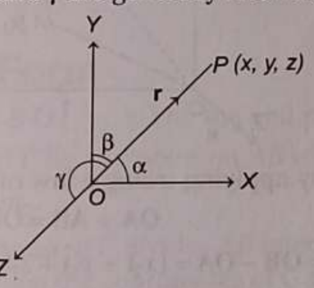
- **Components of a Vector in Three Dimensions** The position vector of $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. The vectors $x\hat{\mathbf{i}}, y\hat{\mathbf{j}}$ and $z\hat{\mathbf{k}}$ are vector components of $\mathbf{r}$. The scalars $x, y$ and $z$ are scalar components of r in the direction of $X$, $Y$ and $Z$-axes, respectively.

## Direction Angles and Direction Cosines

Suppose, the direction angles of a non-zero vector **OP** makes, $\alpha, \beta, \gamma$ with the coordinate axes $OX, OY$ and $OZ$ respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as the direction cosines of $OP$ and are generally denoted by the letters $l, m$ and $n$ respectively.

i.e. $\quad l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$

Here, the angles $\alpha$, $\beta$ and $\gamma$ are generally known as **direction angles**.




If $P(x, y, z)$ is at a distance **r** from the origin, then the direction angle of the line $OP$ are given by

$$\cos\alpha = \frac{x}{r}, \cos\beta = \frac{y}{r} \text{ and } \cos\gamma = \frac{z}{r}.$$

In vector form, direction cosine can be written as $l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$.

### Properties of Direction Cosines

(i) If the direction cosine of a line is $(l, m, n)$, then the direction cosine of the reverse (opposite) of a line is $(-l, -m, -n)$.

(ii) If $OP$ is a directed line segment with direction cosines $l, m$ and $n$ such that $OP = r$. Then, the coordinates of $P$ are $(lr, mr, nr)$.

(iii) Parallel lines have same direction cosines.

(iv) Direction cosines of a line are always unique.

(v) DC's of $X$-axis are $(1, 0, 0)$, $Y$-axis are $(0, 1, 0)$ and $Z$-axis are $(0, 0, 1)$.

(vi) The DC's of a line which is equally inclined to the coordinate axes are

$$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right).$$

## Direction Ratios

Let $l, m$ and $n$ be direction cosines of a line and $a, b$ and $c$ be three numbers such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

Then, direction ratios of the line are proportional to $a, b$ and $c$.

In vector form, direction ratios can be written as $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$.

(i) A line has infinite number of direction ratios.

(ii) The direction ratios of two parallel lines are proportional.

### Relation between Direction Cosines and Direction Ratios

If the direction ratios of a line are proportional to $a, b$ and $c$, then its direction cosines are

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

## Algebra of Vectors

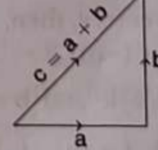
### Addition of Vectors

The addition of two vectors **a** and **b** is denoted by **a** + **b** and it is known as resultant of **a** and **b**.

There are following three methods of addition of vectors

### Triangle Law of Addition

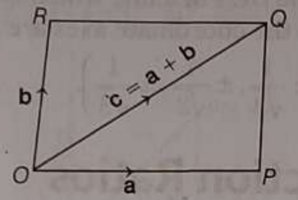
If two vectors **a** and **b** lie along the two sides of a triangle in consecutive order (as shown in the figure), then third side represents the sum (resultant) **a** + **b**. i.e. $\quad \mathbf{c} = \mathbf{a} + \mathbf{b}$



## Parallelogram Law of Addition

If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram.

$$OQ = OP + PQ \Rightarrow \mathbf{c} = \mathbf{a} + \mathbf{b}$$




## Addition in Component Form

If the vectors are defined in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$,

i.e. if $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

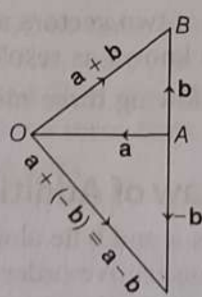
Then, $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{\mathbf{i}} + (a_2 + b_2)\hat{\mathbf{j}} + (a_3 + b_3)\hat{\mathbf{k}}$

## Properties of Vector Addition

- **Closure property** The sum of two vectors is always a vector.
- **Commutativity** For any two vectors **a** and **b**, we have $$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$
- **Associativity** For any three vectors **a**, **b** and **c**, we have $$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$
- **Additive identity** For any vector **a**, we have $$\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0}$$
- **Additive inverse** For every vector **a**, $(-\mathbf{a})$ is the additive inverse of the vector **a**.
  i.e. $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$

# Subtraction of Vectors

If **a** and **b** are two vectors, then subtraction of two vectors in defined as $\mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}$, when $-\mathbf{b}$ is the negative of vector **b**.




## Subtraction in Component Form

If **a** and **b** are two vectors, then, subtraction of two vectors is $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Then, $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\hat{\mathbf{i}} + (a_2 - b_2)\hat{\mathbf{j}} + (a_3 - b_3)\hat{\mathbf{k}}$

## Properties of Vector Subtraction

- $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$
- $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$

# Multiplication of a Vector by a Scalar

If **a** is a vector and $m$ is a scalar, then $m\mathbf{a}$ is a vector whose magnitude is $m$ times the magnitude of **a**.

## Scalar Multiplication in Component Form

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $m$ is scalar then scalar multiplication of vector is given by

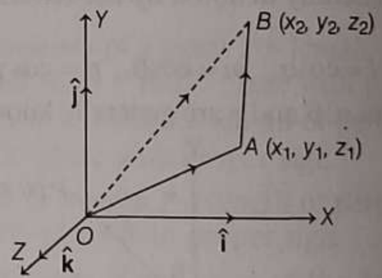
$$m\mathbf{a} = m(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) = ma_1\hat{\mathbf{i}} + ma_2\hat{\mathbf{j}} + 3a_3\hat{\mathbf{k}}$$

## Properties of Scalar Multiplication

- $m(-\mathbf{a}) = -m\mathbf{a}$
- $(-m)(-\mathbf{a}) = m\mathbf{a}$
- $m(n\mathbf{a}) = (mn)\mathbf{a} = n(m\mathbf{a})$
- $(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$
- $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$ {Distributive}

# Vector Joining Two Points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points on the plane. Then, position vectors of $A$ and $B$ with respect to the origin $O$ are $\mathbf{OA} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$ and $\mathbf{OB} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$, respectively.




In $\Delta OAB$, by applying triangle law of addition, we get

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB}$$

$$\therefore \quad \mathbf{AB} = \mathbf{OB} - \mathbf{OA} = (x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}) - (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}})$$

$$= (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$$

and $|\mathbf{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

# Section Formula

## In Vector Form

Let **a** and **b** be two vectors represented by $OA$ and $OB$ and the point $P$ divides $AB$ in the ratio $m:n$.

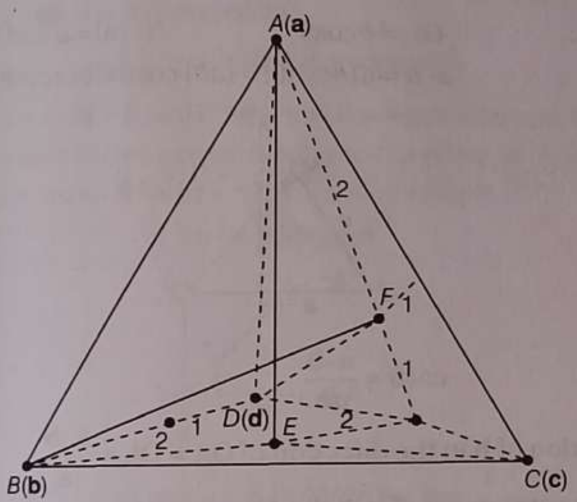
(i) **For Internal Ratio** If $P$ divides $AB$ in the ratio $m:n$ internally, then $\mathbf{r} = \dfrac{m\mathbf{b} + n\mathbf{a}}{m+n}$.

(ii) **For External Ratio** If $P$ divides $AB$ in the ratio $m:n$ externally, then $\mathbf{r} = \dfrac{m\mathbf{b} - n\mathbf{a}}{m-n}$.

(iii) **Mid-point Formula** If $C(\mathbf{c})$ is the mid-point of $AB$, then, $\mathbf{c} = \dfrac{\mathbf{a}+\mathbf{b}}{2}$.

(iv) **Centroid of a Triangle** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of the vertices with respect to origin $O$, then centroid of $\Delta ABC = \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$.

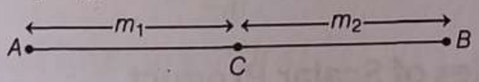
(v) In tetrahedron $ABCD$, centroid $G$ divides the line joining the vertices of tetrahedron (i.e. $A(\mathbf{a})$, $B(\mathbf{b})$, $C(\mathbf{c})$, $D(\mathbf{d})$) to centroid of opposite triangle in the ratio 3 : 1 and it is given by $\bar{g} = \dfrac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$.




## In Cartesian Form

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the end points of a line segment $AB$ and $C$ be any point on $AB$ which divide $AB$ in the ratio $m_1 : m_2$.

(i) **For Internal Ratio** If $C$ divides $AB$ internally in the ratio $m_1 : m_2$, then the coordinates of $C$ are




$$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$$

(ii) **For External Ratio** If $C$ divides $AB$ externally in the ratio $m_1 : m_2$, then the coordinates of $C$ are

$$\left(\frac{m_1x_2+(-m_2)x_1}{m_1+(-m_2)}, \frac{m_1y_2+(-m_2)y_1}{m_1+(-m_2)}, \frac{m_1z_2+(-m_2)z_1}{m_1+(-m_2)}\right)$$
$$= \left(\frac{m_1x_2-m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}, \frac{m_1z_2-m_2z_1}{m_1-m_2}\right)$$

(iii) **Mid-point Formula** If $C$ is the mid-point of $A$ and $B$, then coordinates of $C$ are
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

(iv) **Centroid of a Triangle** If vertices of a triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, then centroid of a triangle is
$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

## Coplanar Vectors

Two or more vectors are coplanar, if they lie in the same plane or in parallel plane.

Suppose, $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear vectors. A vector $\mathbf{r}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$ if and only if there exists unique scalar $\lambda_1$ and $\lambda_2$ such that
$$\mathbf{r} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}.$$

## Linear Combinations

(i) **Fundamental Theorem** Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero, non-collinear vectors. Then, any vector $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be expressed uniquely as a linear combination of $\mathbf{a}$, $\mathbf{b}$ i.e. there exists some unique $x, y \in R$ such that
$$x\mathbf{a} + y\mathbf{b} = \mathbf{r}.$$

(ii) If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero, non-coplanar vectors, then
$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c}$$
$$\Rightarrow \quad x = x', y = y', z = z'$$

(iii) **Fundamental Theorem in Space** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be non-zero, non-coplanar vectors in space. Then, any vector $\mathbf{r}$, can be uniquely expressed as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ i.e. there exists some unique $x, y, z \in R$ such that
$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{r}.$$
If three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, then each of them can be uniquely expressed as linear combination of the other two.

(iv) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are $n$ non-zero vectors and $k_1, k_2, \ldots, k_n$ are $n$ scalars and if the linear combination $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \ldots + k_n\mathbf{x}_n = 0$ $\Leftrightarrow k_1 = 0, k_2 = 0 \ldots k_n = 0$, then we say that vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent vectors.

(v) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are not linearly independent, then they are said to be linearly dependent vectors, i.e. if $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \ldots + k_n\mathbf{x}_n = 0$, if there exists atleast one $k_r \neq 0\ (r = 1, 2, \ldots, n)$, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are said to be linearly dependent.

## Collinearity and Coplanarity of Vectors

### Test of Collinearity of Three Points

The three points $A$, $B$ and $C$ with position vectors **a**, **b** and **c**, respectively are collinear, if and only if there exist scalars $x, y$ and $z$ not all zero such that

$$x\,\mathbf{a} + y\,\mathbf{b} + z\,\mathbf{c} = 0 \text{ and } x + y + z = 0$$

The vectors AB and AC are collinear, if there exists a linear relation between the vectors, such that

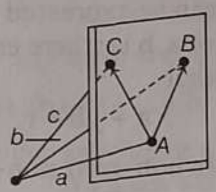
$$\mathbf{AB} = \lambda\,\mathbf{AC}$$

### Coplanarity of Three Points

Three points $A$, $B$ and $C$ represented by position vectors **a**, **b** and **c** respectively represent two vectors AB and AC. From the figure, two vectors are always coplanar, i.e. two vectors always form their own plane.

Thus, **a**, **b** and **c** will be coplanar, if we can find two scalars $\lambda$ and $\mu$ such that

$$\boxed{\mathbf{a} = \lambda\mathbf{b} + \mu\mathbf{c}.}$$




Three vectors $a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

and $c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ are coplanar, if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

### Coplanarity of Four Points

The necessary and sufficient condition that four points with position vectors **a**, **b**, **c** and **d** should be coplanar is that there exist four scalars $x$, $y$, $z$ and $t$ not all zero, such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + t\mathbf{d} = 0,$$
$$x + y + z + t = 0$$

## Product of Two Vectors

There are two types of product of two vectors

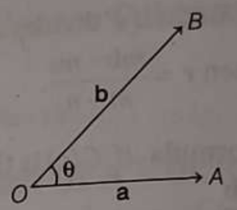
### Scalar or Dot Product of Two Vectors

The scalar product of two vectors **a** and **b** is expressed as

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Where, $\quad 0 \le \theta \le \pi.$

$\therefore \quad \mathbf{a}\cdot\mathbf{b} \le |\mathbf{a}||\mathbf{b}|$




- $\mathbf{a}\cdot\mathbf{b} > 0$, then angle between **a** and **b** is acute.
- $\mathbf{a}\cdot\mathbf{b} < 0$, then angle between **a** and **b** is obtuse.

### Angle between Two Vectors

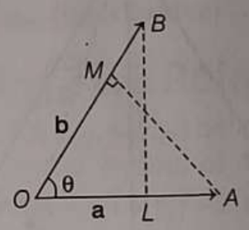
The angle between two non-zero vectors **a** and **b** is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

### Projection of Vector

Let $OL$ is the projection of vector **b** in the direction of vector **a**. Then,

$$OL = b\cos\theta \qquad [\because |\mathbf{a}| = a \text{ and } |\mathbf{b}| = b]$$
$$\mathbf{a}\cdot\mathbf{b} = a(b\cos\theta) = (ab)\cos\theta = b(a\cos\theta)$$




$$\Rightarrow \qquad \cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab}$$

Projection of **b** in the direction of $\mathrm{OA} = \mathrm{OL} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector component of **b** in the direction of

$$\mathrm{OA} = \mathrm{OL} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\hat{\mathbf{a}} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}.$$

Projection of **a** in the direction of $OB = OM = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

Vector component of **a** in the direction of

$$\mathrm{OA} = \mathrm{OM} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\cdot\mathbf{b}$$

### Properties of Scalar Product

- $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ [commutativity]
- $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ [distributivity]
- $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$
- $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{i}} = 0,\ \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{j}} = 0,\ \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = 0$
- For any two vectors **a** and **b**
  (a) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| \Rightarrow \mathbf{a} \parallel \mathbf{b}$
  (b) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \Rightarrow \mathbf{a} \perp \mathbf{b}$
  (c) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Rightarrow \mathbf{a} \perp \mathbf{b}$

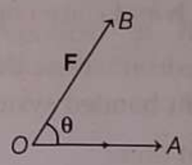
- If dot product of two vectors is zero, then atleast one of the vectors is a zero vector or they are perpendicular.

## Application of Scalar Product

Let a particle be placed at $O$ and a force $\mathbf{F}$ represented by $OB$ be acting on the particle at $O$.

Then, Work done = (Force)·(Displacement)

i.e. $$W = \mathbf{F}\cdot\mathbf{d} = |\vec{F}||\vec{d}|\cos\theta = Fd\cos\theta$$



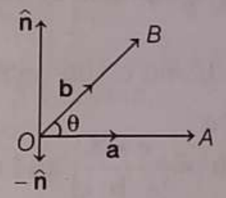

## Vector Product of Two Vectors

The vector product of two non-null and non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ is expressed as

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = ab\sin\theta\,\hat{\mathbf{n}}$$

Where, $|\mathbf{a}| = a$, $|\mathbf{b}| = b$ and $0 \le \theta < \pi$ is the angle between $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ form a right handed system.

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$




The unit vector $\hat{\mathbf{n}}$ along $\mathbf{a}\times\mathbf{b}$ is given by $\hat{\mathbf{n}} = \dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$.

## Vector Product in Terms of Components

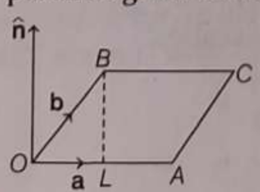
If $\quad \mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Then, $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

## Geometrical Interpretation of Vector Product

Modulus of $\mathbf{a}\times\mathbf{b}$ is the area of the parallelogram whose adjacent sides are represented by $\mathbf{a}$ and $\mathbf{b}$.

i.e. $|\mathbf{a}\times\mathbf{b}|$ = Area of parallelogram $OACB$.




## Properties of Vector Product

- $\mathbf{a}\times\mathbf{b} \ne \mathbf{b}\times\mathbf{a}$
  but $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$ (not commutative)
- $\mathbf{a}\times\mathbf{b} = 0$
  $\Leftrightarrow$ $\mathbf{a} \parallel \mathbf{b}$ or collinear or $\mathbf{a} = 0$ or $\mathbf{b} = 0$
- Lagrange's identity $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$
- $(m\mathbf{a})\times\mathbf{b} = m(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(m\mathbf{b})$
- $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$ [left distribution law]
- $(\mathbf{b}+\mathbf{c})\times\mathbf{a} = \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a}$ [right distribution law]
- $\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = 0$
- $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}, \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}}, \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$
- $\hat{\mathbf{j}}\times\hat{\mathbf{i}} = -\hat{\mathbf{k}}, \hat{\mathbf{k}}\times\hat{\mathbf{j}} = -\hat{\mathbf{i}}, \hat{\mathbf{i}}\times\hat{\mathbf{k}} = -\hat{\mathbf{j}}$

## Angle between Two Vectors

If $\theta$ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{ab},$$

where, $\quad \mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

$$\Rightarrow \quad \sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

## Vector Normal to the Plane of Two Given Vectors

The vectors of magnitude $\lambda$ normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ are

$$\pm\frac{\lambda(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$$

## Condition for Vectors to be Parallel

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ are parallel, then $\mathbf{a}\times\mathbf{b} = 0$ or $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$.

## Condition for Three Points to be Collinear

The condition that three points A, B and C are collinear, if $\quad \mathbf{AB}\times\mathbf{BC} = 0 \quad$ or $\quad \mathbf{AB} = k\mathbf{BC}$

where, $k$ is any scalar.

## Area of Parallelogram and Triangle

- The area of a parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$.
- The area of a parallelogram with diagonals $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{1}{2}|\mathbf{a}\times\mathbf{b}|$.
- The area of a plane quadrilateral $ABCD$ is $\dfrac{1}{2}|\mathbf{AC}\times\mathbf{BD}|$, where $AC$ and $BD$ are diagonals.

- The area of a triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.
- The area of a $\Delta ABC$ is $\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|$.
- If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are position vectors of vertices of $\Delta ABC$, then area $=\frac{1}{2}|(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})|$.

  If $(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})=0$, then three points with position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are collinear.

# Scalar Triple Product

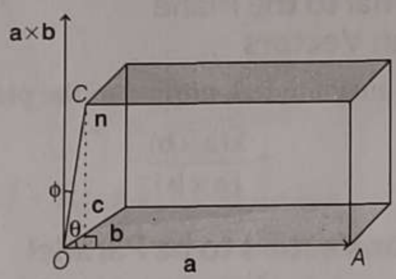
The scalar triple product of three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is defined as

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=|\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|\sin\theta\cos\phi$$

where, $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\phi$ is the angle between $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}$. It is also defined as $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

## Geometrical Interpretation of a Scalar Triple Product

The scalar triple product $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ represents the volume of the parallelopiped whose coterminous edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right handed system of vectors.




If $A, B, C$ and $D$ are the vertices of parallelopiped with AB, AC are concurrent edges, then volume of parallelopiped $=[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}]$

## Properties of Scalar Triple Product

- If $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}, \mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$ and $\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$.

  Then, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$
- $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]=[\mathbf{c}\,\mathbf{a}\,\mathbf{b}]$
- $[\mathbf{abc}]=-[\mathbf{bac}]$
- $[k\,\mathbf{abc}]=k[\mathbf{abc}]$
- $[\mathbf{a}+\mathbf{b}\ \mathbf{c}\ \mathbf{d}]=[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]+[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$
- $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]=2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
- $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
- $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2=\begin{vmatrix} \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$
- If $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$, then $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar.
- Volume of a tetrahedron whose three coterminous edges are in the right handed system are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is given by $\frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ cu units.
- If $A, B, C$ and $D$ are vertices of tetrahedron then volume of tetrahedron $=\frac{1}{6}[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}]$ cu units.

# Reciprocal System of Vectors

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be a system of three non-coplanar vectors. Then, the system of vectors $\mathbf{a}', \mathbf{b}'$ and $\mathbf{c}'$ which satisfies $\mathbf{a}\cdot\mathbf{a}'=\mathbf{b}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{c}=1$ and $\mathbf{a}\cdot\mathbf{b}'=\mathbf{a}\cdot\mathbf{c}'=\mathbf{b}\cdot\mathbf{a}'=\mathbf{b}\cdot\mathbf{c}'=\mathbf{c}\cdot\mathbf{a}'=\mathbf{c}\cdot\mathbf{b}'=0$ is called the reciprocal system to the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

In terms of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are given by

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \text{ and } \mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

# Vector Triple Product

The vector triple product of three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is defined as

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

## Properties of Vector Triple Product

- $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=-(\mathbf{b}\times\mathbf{c})\times\mathbf{a}$
- $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$
- $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$
- $\hat{\mathbf{i}}\times(\hat{\mathbf{j}}\times\hat{\mathbf{k}})=0$
- $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is linear combination of $\mathbf{b}$ and $\mathbf{c}$, hence it is coplanar with $\mathbf{b}$ and $\mathbf{c}$,

  i.e. $\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=x\mathbf{b}+y\mathbf{c}$,

  where $x$ and $y$ are scalars.