CHAPTER 06

Line and Plane

Line

A line (or straight line) is a curve such that all the points on the line segment joining any two points of it lies on it.

A line can be determined uniquely, if

- its direction and the coordinates of a point on it are known.
- it passes through two given points.

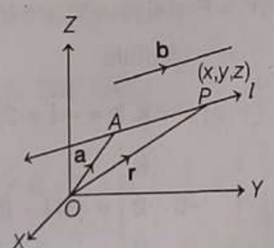
Equation of a Line Passing through a Given Point and Parallel to a Given Vector

Vector Form

The vector equation of a line l passing through a point A with position vector $\mathbf{a}$ and parallel to a given vector $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where, $\mathbf{r}$ is the position vector of any arbitrary point P and λ is a real number.



The vector equation of a straight line passing through the origin and parallel to given vector $\mathbf{b}$ is

$$\mathbf{r} = \lambda\mathbf{b}$$

Cartesian Form

Equation of a straight line passing through a point A with position vector $\mathbf{a}(x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}})$ and parallel to a vector $\mathbf{b}$ $(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

On putting the value of $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{b}$, we get

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}) + \lambda(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})$$

Equating the coefficient of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ on both sides, we get

$$x - x_1 = \lambda a,\ y - y_1 = \lambda b,\ z - z_1 = \lambda c \qquad \text{...(i)}$$

$$\therefore \quad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

which is the required equation of line in cartesian form and it is called the **symmetric form** of cartesian equation of line.

Here, a, b, c are direction ratios and with the help of Eq. (i) we can determined any point on the line is

$$(x_1 + \lambda a,\ y_1 + \lambda b,\ z_1 + \lambda c)$$

If l, m, n are direction cosines, then equation of straight line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Equation of Line Passing through Two Given Points

Vector Form

The vector equation of a line passing through two given points having position vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

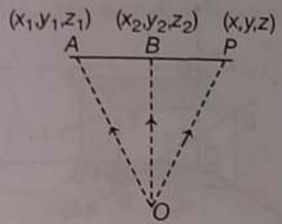
or $\quad \mathbf{r} = \mathbf{b} + \lambda(\mathbf{b} - \mathbf{a})$, where λ is a scalar.

Above equation can be rewritten is $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$, which is called the **non-parametric form** of vector equation of line.

Cartesian Form

Direction ratios of $AB = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Direction ratios of $AP = (x - x_1, y - y_1, z - z_1)$



Since, points A, B and P lie on a line, so they are proportional.

$$\therefore \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

This is required equation of line passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in cartesian form.

Point of Intersection of Line

Vector Form

Let the two lines be

$$\mathbf{r} = (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) + \lambda(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}) \quad \text{...(i)}$$

and $$\mathbf{r} = (a'_1\hat{\mathbf{i}} + a'_2\hat{\mathbf{j}} + a'_3\hat{\mathbf{k}}) + \mu(b'_1\hat{\mathbf{i}} + b'_2\hat{\mathbf{j}} + b'_3\hat{\mathbf{k}}) \quad \text{...(ii)}$$

If Eqs. (i) and (ii) intersect, then they have a common point.

So, we have

$$(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) + \lambda(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}) = (a'_1\hat{\mathbf{i}} + a'_2\hat{\mathbf{j}} + a'_3\hat{\mathbf{k}}) + \mu(b'_1\hat{\mathbf{i}} + b'_2\hat{\mathbf{j}} + b'_3\hat{\mathbf{k}})$$

$$\Rightarrow \quad (a_1 + \lambda b_1)\hat{\mathbf{i}} + (a_2 + \lambda b_2)\hat{\mathbf{j}} + (a_3 + \lambda b_3)\hat{\mathbf{k}} = (a'_1 + \mu b'_1)\hat{\mathbf{i}} + (a'_2 + \mu b'_2)\hat{\mathbf{j}} + (a'_3 + \mu b'_3)\hat{\mathbf{k}}$$

$\therefore \quad a_1 + \lambda b_1 = a'_1 + \mu b'_1, a_2 + \lambda b_2 = a'_2 + \mu b'_2$

and $a_3 + \lambda b_3 = a'_3 + \mu b'_3$

Now, find the value of λ and μ by solving any two of the above equations. If the values of λ and μ satisfy the third equation, then the two lines intersect, otherwise not. If they intersect, then the point of intersection can be obtained by substituting the value of λ (or μ) in Eq. (i) or Eq. (ii).

Cartesian Form

Let the two lines be $L_1 : \dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1} = \lambda$ [say]

and $L_2 : \dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2} = \mu$ [say]

Consider the coordinates of general points on L_1 and L_2,

i.e., $(x_1 + a_1\lambda, y_1 + b_1\lambda, z_1 + c_1\lambda)$ ...(i)

and $(x_2 + a_2\mu, y_2 + b_2\mu, z_2 + c_2\mu)$ ...(ii)

where λ and μ are some real constants. If the lines L_1 and L_2 intersect, then they have a common point.

$$\therefore \quad (x_1 + a_1\lambda, y_1 + b_1\lambda, z_1 + c_1\lambda) = (x_2 + a_2\mu, y_2 + b_2\mu, z_2 + c_2\mu)$$

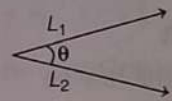
for some constants λ and μ.

$$\Rightarrow \quad x_1 + a_1\lambda = x_2 + a_2\mu,$$
$$y_1 + b_1\lambda = y_2 + b_2\mu$$
and $$z_1 + c_1\lambda = z_2 + c_2\mu$$

Now, find the values of λ and μ by solving any two of above equations. If the values of λ and μ satisfy the third equation, then the two lines intersect, otherwise not. If they intersect, then the point of intersection can be obtain by substituting the value of λ (or μ) in Eq. (i) or Eq. (ii).

Angle between Two Lines

Let L_1 and L_2 be two lines and θ be the acute angle between them as shown in the figure



Vector Form

Let the vector equations of lines L_1 and L_2 be $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$, then the angle between these two lines is given by

$$\cos\theta = \left|\frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}\right|, \text{ where } \lambda \text{ and } \mu \text{ are scalars.}$$

(i) If two lines are perpendicular, then $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$

(ii) If two lines are parallel, then $\mathbf{b}_1 = \lambda\mathbf{b}_2$.

Cartesian Form

Let the cartesian equations of lines L_1 and L_2 be

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

Then, the angle between the lines L_1 and L_2 is given by

$$\cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

The angle between the lines in terms of $\sin\theta$ is given by

$$\sin\theta = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If (l_1, m_1, n_1) and (l_2, m_2, n_2) are direction cosines of lines L_1 and L_2, then angle between the lines is given by

$$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$$
$$[\because l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2]$$

and $\sin\theta = \sqrt{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2}$

There are always two angles i.e. θ and $\pi - \theta$ between two lines.

(i) If two lines are perpendicular, then
$a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) If two lines are parallel, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

Perpendicular Distance of a Point from a Line

Vector Form

The length of the perpendicular from a point $P(\mathbf{a})$ on the line $\mathbf{r} = \mathbf{b} + \mu\,\mathbf{c}$ is given by

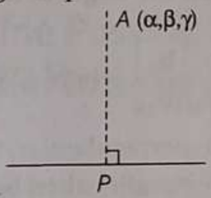
$$\sqrt{|\mathbf{a}-\mathbf{b}|^2 - \left\{\frac{(\mathbf{a}-\mathbf{b})\cdot\mathbf{c}}{|\mathbf{c}|}\right\}^2}$$

Cartesian Form

Let the equation of the line be

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda \quad \text{[say]}$$

and $A(\alpha, \beta, \gamma)$ be the given point.



Then, length of perpendicular (AP) is

$$\sqrt{\begin{aligned}&(\alpha-x_1)^2 + (\beta-y_1)^2 + (\gamma-z_1)^2\\ &\quad -[a(\alpha-x_1)+b(\beta-y_1)+c(\gamma-z_1)]\end{aligned}}$$

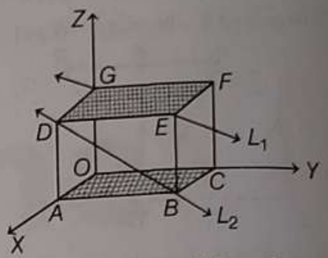
Shortest Distance between Two Lines

The shortest distance between two lines can be determined in two conditions, i.e. when the lines intersect and when they are parallel to each other.

Skew-Lines

If two lines in space intersect at a point, then the shortest distance between them is zero.
If two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from a point on one line onto the other line. If two lines are neither intersecting nor parallel, then such pair of lines are **non-coplanar** and are called **skew-lines**.

In the given figure, line GE (lies in ceiling $DEFG$) and BO (lies in wall $ABED$) are skew-lines, since they are not parallel and also never meet.



Representation of skew-lines

Distance between Two Skew-Lines

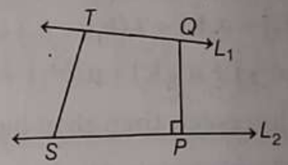
For skew-lines, the line of the shortest distance will be perpendicular to both the lines.

Vector Form

Let L_1 and L_2 be two skew-lines with equations

$$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$$

and
$$\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$$



Then, the shortest distance between these two skew-lines PQ is

$$\text{SD} = \left|\frac{(\mathbf{a}_2 - \mathbf{a}_1)\cdot(\mathbf{b}_1 \times \mathbf{b}_2)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$$

If two lines are intersecting, then shortest distance is zero, i.e. $(\mathbf{a}_2 - \mathbf{a}_1)\cdot(\mathbf{b}_1 \times \mathbf{b}_2) = 0$

Cartesian Form

Let the two skew-lines be $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$

and $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$.

Then, shortest distance between two skew-lines is

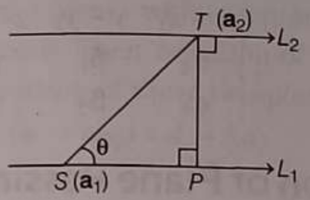
$$\text{SD} = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(a_1b_2-b_1a_2)^2 + (b_1c_2-c_1b_2)^2 + (c_1a_2-a_1c_2)^2}}$$

If two lines are intersecting, then distance is zero.

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Distance between Two Parallel Lines

If two lines L_1 and L_2 are parallel, then they are coplanar.



The shortest distance TP between parallel lines

$L_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}$ and $L_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}$ is

$$\text{SD} = \left|\frac{(\mathbf{a}_2 - \mathbf{a}_1)\times\mathbf{b}}{|\mathbf{b}|}\right|$$

Plane

A plane is a surface such that a line segment joining any two points on it lies wholly on it. A straight line, which is perpendicular to every line lying on a plane is called a normal to the plane.

Equation of Plane

Every equation of first degree of the form $ax + by + cz + d = 0$ represents the general equation of a plane. The coefficients of x, y and z i.e. a, b and c are the direction ratios of the normal to the plane.

Equation of Coordinate Planes

(i) Equation of the XY plane is $z = 0$

(ii) Equation of the YZ plane is $x = 0$

(iii) Equation of the ZX plane is $y = 0$

The plane $ax + by + cz + d = 0$ is parallel to

(i) X-axis, iff coefficient of $x = 0$

(ii) Y-axis, iff coefficient of $y = 0$

(iii) Z-axis, iff coefficient of $z = 0$

Equation of a Plane in Normal Form

Suppose ABC is a plane and ON is a normal line, to the given plane.

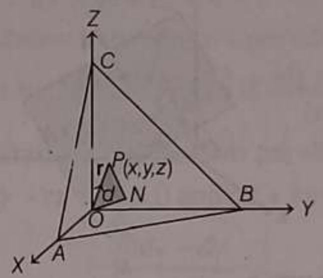
Vector Form

The equation of plane having normal unit vector $\hat{\mathbf{n}}$ to the plane is $\mathbf{r}\cdot\hat{\mathbf{n}} = d$ ...(i)

where, d is a perpendicular distance of the plane from origin, $\mathbf{r}$ is the position vector of any point P on the plane and $\hat{\mathbf{n}}$ is the unit normal vector $\left(\text{i.e. } \hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}\right)$.

If a, b and c are the direction ratios of normal to the plane, then the vector equation of plane is

$$\mathbf{r}\cdot(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}) = d$$



Cartesian Form

Let $P(x, y, z)$ be any point on the plane. Then,

$$\text{OP} = \mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

Again, let l, m and n be the direction cosines of normal unit vector $\hat{\mathbf{n}}$.

Then, $\hat{\mathbf{n}} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$

Therefore, Eq. (i) gives

$$(x\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})\cdot(l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}) = d$$

i.e. $$lx + my + nz = d \quad \text{...(ii)}$$

which is the cartesian equation of plane in the normal form.

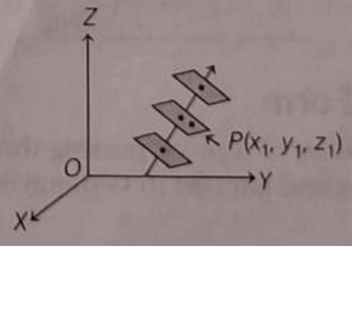
If a, b and c are the direction ratios of normal to the plane, then the cartesian equation of plane is

$$ax + by + cz = d$$

If d is the distance from the origin to the plane and l, m and n are the direction cosines of normal to the plane through the origin, then the foot of the perpendicular is (ld, md, nd).

Equation of Plane Passing Through a Given Point and Perpendicular to a given Vector

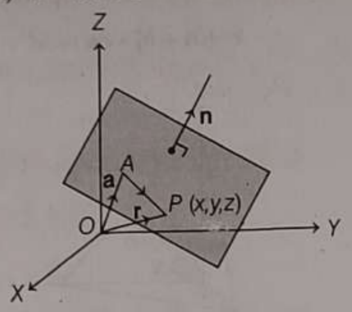
In the space, there can be many planes that are perpendicular to the given vector but through a given point $P(x_1, y_1, z_1)$ only one such plane exists.



Vector Form

The vector equation of a plane passing through a point having position vector **a** and normal to the vector **n** is

$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0 \text{ or } \mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$$



Cartesian Form

The cartesian form of equation of the plane passing through the point (x_1, y_1, z_1) {i.e. $\mathbf{a}=x_1\hat{\mathbf{i}}+y_1\hat{\mathbf{j}}+z_1\hat{\mathbf{k}}$} and perpendicular to a vector whose direction ratios are a, b, c is

$$(x-x_1)a+(y-y_1)b+(z-z_1)c=0$$

If l, m and n are direction cosines of normal to the plane, then cartesian equation of the plane passing through given point (x_1, y_1, z_1) and perpendicular to given vector is

$$(x-x_1)l+(y-y_1)m+(z-z_1)n=0$$

Equation of a Plane Passing through a Point and Parallel to Two Non-zero Vectors

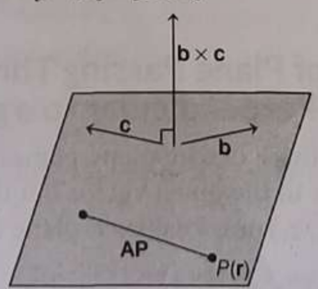
Vector Form

The equation of the plane passing through a point having position vector **a** and parallel to two non-zero vectors **b** and **c** is

$$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c})=0$$

or $\quad \mathbf{r}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

or $\quad [\mathbf{r}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$



Parametric Form

The vector equation of a plane passing through a point having position vector **a** and parallel to two non-zero vectors **b** and **c** is

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\,\mathbf{c}$$

where λ and μ are scalars.

Cartesian Form

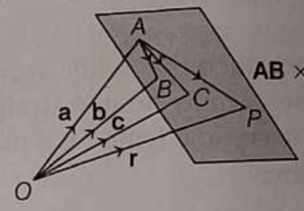
The cartesian equation of a plane passing through the point (x_1, y_1, z_1) and parallel to lines whose DR's are $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$, is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}=0$$

Equation of Plane Passing Through the Three Non-collinear Points

Vector Form

Let A, B and C be three points in a plane such that they are non-collinear. Let position vectors of three points A, B and C be **a**, **b**, **c** respectively and P be a point in the plane with position vector **r**.



Then, equation of plane is

$$(\mathbf{r}-\mathbf{a})\cdot[(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})]=0$$

or $\quad [\mathbf{r}-\mathbf{a}\ \ \mathbf{b}-\mathbf{a}\ \ \mathbf{c}-\mathbf{a}]=0$

or $\quad [\mathbf{r}\ \mathbf{b}\ \mathbf{c}]+[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]+[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Cartesian Form

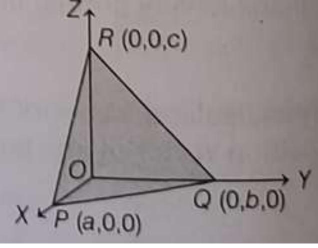
Equation of a plane passing through the three non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

Equation of a Plane in Intercept Form

The equation of a plane having intercepts of length a, b and c with coordinate axes X, Y and Z, respectively is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$



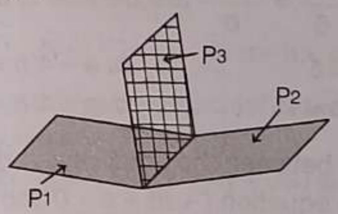
Equation of a Plane Passing through the Intersection of Two Planes

Vector Form

Let P_1 and P_2 be two planes with equations $\mathbf{r}\cdot\mathbf{n}_1 = d_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = d_2$, respectively. Then, equation of plane passing through the intersection of these two planes is

$$\mathbf{r}\cdot(\mathbf{n}_1 + \lambda\mathbf{n}_2) = d_1 + \lambda d_2$$

where λ is a scalar.



Cartesian Form

Let the two equations of plane be

$$P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$$

and $P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$

then $P_1 + \lambda P_2 = 0$ (where, λ is a parameter) represents family of planes passing through line of intersection of the planes $P_1 = 0$ and $P_2 = 0$.

Important Results

(i) (a) The equation of a plane parallel to the given plane

$$\mathbf{r}\cdot\mathbf{n} = d \text{ is } \mathbf{r}\cdot\mathbf{n} = \lambda$$

(b) The equation of a plane parallel to the given plane

$$ax + by + cz + d = 0 \text{ is}$$
$$ax + by + cz + \lambda = 0$$

(ii) (a) The distance between two parallel planes

$$\mathbf{r}\cdot\mathbf{n} = d_1 \text{ and } \mathbf{r}\cdot\mathbf{n} = d_2 \text{ is } \frac{|d_2 - d_1|}{|\mathbf{n}|}$$

(b) The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\left|\frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}\right|$$

(iii) The foot (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

(iv) The image (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$