MathsStart

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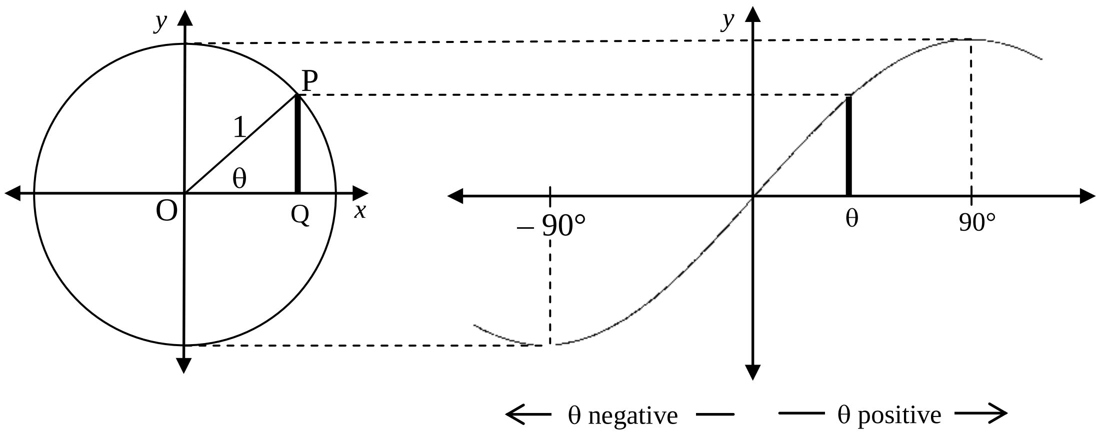
THE UNIVERSITY of ADELAIDE

Topic 6

Trigonometry II



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This Topic...

This topic introduces a new way of thinking about angles, and extends the definitions of sine, cosine and tangent to angles greater than 90°. It explores the properties and graphs of the trigonometric *functions* sin θ, cos θ and tan θ, and their applications.

Author of Topic 6: Paul Andrew.

Prerequisites

You will need a scientific calculator.
We also assume you have read Topic 5: Trigonometry I

Contents

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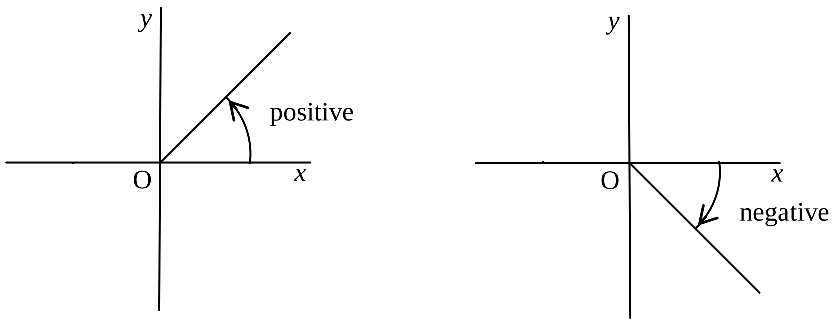
1
Measuring Angles

In module 1, we saw how the counting numbers **N** were extended to the real numbers **R**. It was necessary to invent new numbers because **N** was not closed under subtraction, for example the calculation 3 – 7 did not have an answer among the natural. We also saw how negative numbers only gained acceptance after they were freshly interpreted as being points on a number line, with the numbers on the right of the origin 0 being taken as positive and numbers on the left being taken as negative.

A similar problem occurs with angles. In Module 5, we calculated and interpreted angles like 30° + 30° and 90° – 45°, but what meaning should we give to the ‘angle’ 30° – 60°?

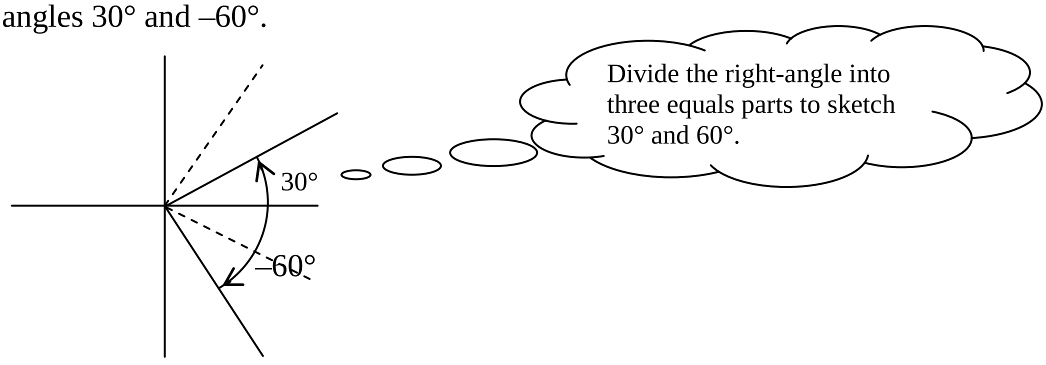
Angles can be freshly interpreted in a coordinate plane, as being *measured from the positive x-axis*. Angles measured in an *anti-clockwise direction* are thought of as *positive*, and angles measured in a *clockwise direction* as thought of as being *negative*. This seems very different to how we have been thinking about angles inside triangles, but you will find later that both interpretations give the same answers.



Example

Sketch the angles 30° and –60°.

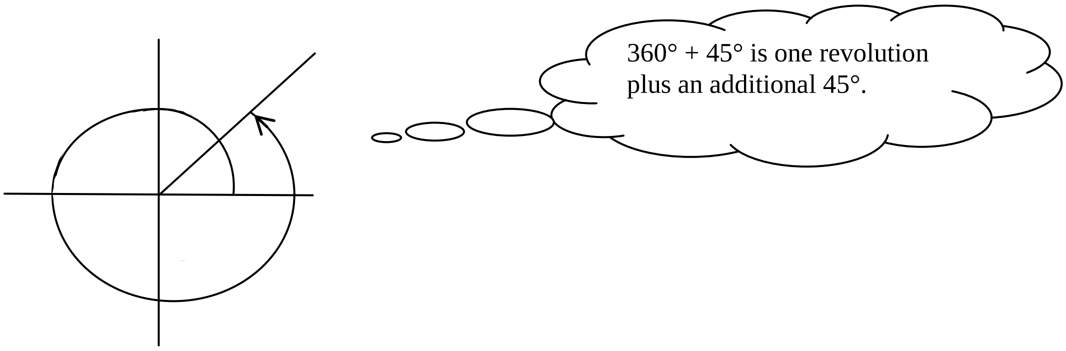
Answer



Example

Sketch the angle 405° = 360° + 45°

Answer



1. Problems

Sketch the following angles

(a) ±120°	(b) ±135°	(c) ±150°	(d) ±210°	(e) ±225°
(f) ±240°	(g) ±300°	(h) ±315°	(i) ±330°	(j) ±390°

2

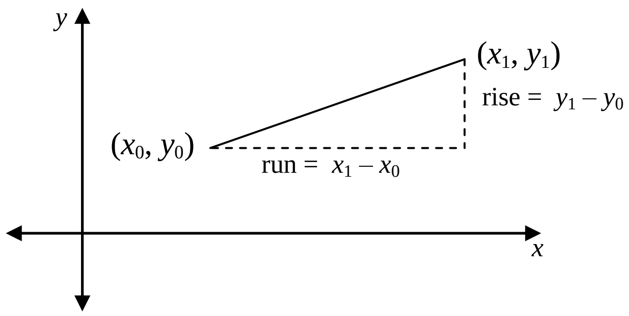
Sine, Cosine and Tangent

2.1 The Unit Circle.

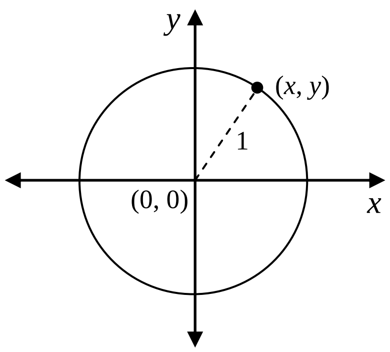
The distance between two points (x_0, y_0) and (x_1, y_1) in the coordinate plane is

$$\boxed{\text{distance } = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}}$$

This can be deduced from the diagram below using Pythagoras' Theorem, and is true wherever the points (x_0, y_0) and (x_1, y_1) are in the plane.

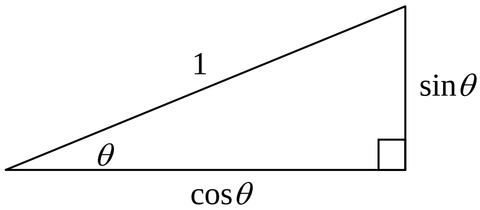


The *unit circle* is a circle with centre at the origin (0, 0) and radius 1. As the distance of each point (x, y) on the circle from the origin is equal to 1, the coordinates of the points on the unit circle satisfy the equation $\sqrt{(x-0)^2 + (y-0)^2} = 1$ or $x^2 + y^2 = 1$.

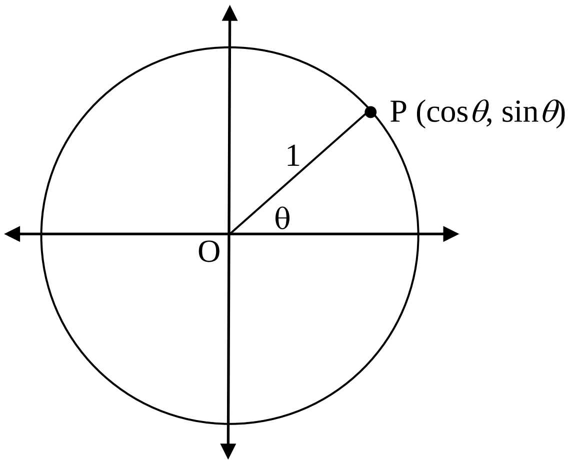


2.2 The Sine and Cosine Functions

In Topic 5, we saw that $(\sin\theta)^2 + (\cos\theta)^2 = 1$ for any angle θ in a right-angled triangle. The diagram below shows that this is another way of interpreting Pythagoras' Theorem.



You can see that if P is a point in the first quadrant of the unit circle, and if the angle between OP and the positive *x*-axis is θ, then P must have coordinates $(\cos\theta, \sin\theta)$.



This is only true when θ is an angle in a right-angled triangle, ie. $0° < \theta < 90°$, because $\sin\theta$ and $\cos\theta$ are only defined for angles in right-angled triangles. However . . . we can use this idea to define $\sin\theta$ and $\cos\theta$ for *any* angle θ.

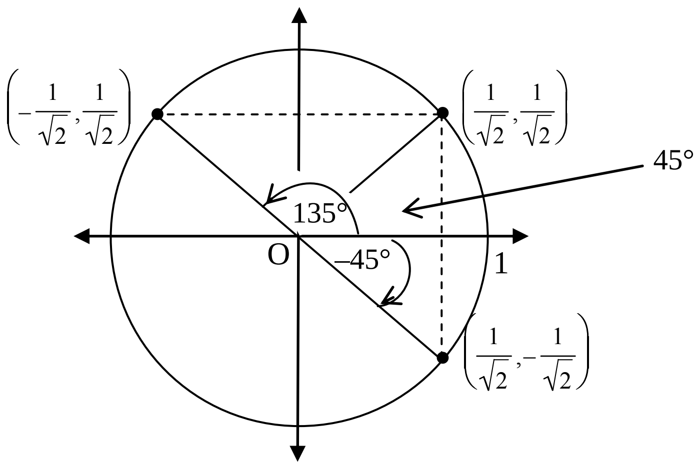
We define $\sin\theta$ and $\cos\theta$ for *any* angle θ as being *the coordinates of the point P on the unit circle, when OP has angle θ with the positive x-axis*. Remember: positive angles are measured in an anticlockwise direction and negative angles are measured in a clockwise direction.

Example

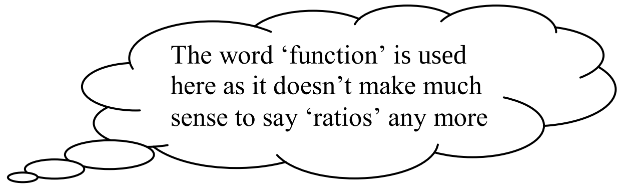
What are the exact values of sin 135°, cos 135°, sin (–45°) and cos (–45°)?

Answer

When $\theta = 45°$, $\sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{1}{\sqrt{2}}$, so the point on the unit circle corresponding to the angle 45° is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.



So $\sin 135° = \frac{1}{\sqrt{2}}$, $\cos 135° = -\frac{1}{\sqrt{2}}$, $\sin(-45°) = -\frac{1}{\sqrt{2}}$ and $\cos(-45°) = \frac{1}{\sqrt{2}}$.



Problems 2.2

Find the exact values of the trigonometric functions below, and check you answers using a calculator.

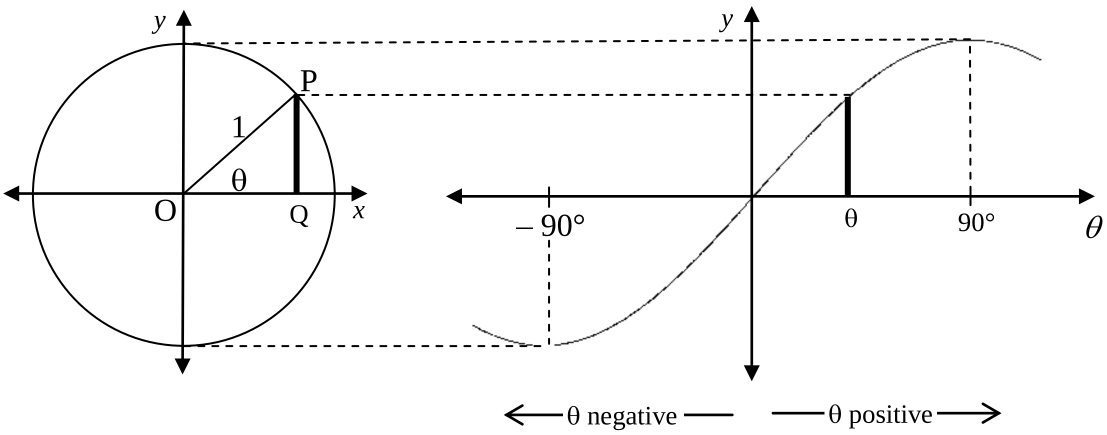
(a) sin 120° and cos 120° (b) sin 150° and cos 150°

(c) sin 210° and cos 210° (d) sin 330° and cos 330°

(e) sin(–30°) and cos(–30°) (f) sin(–150°) and cos(–150°)

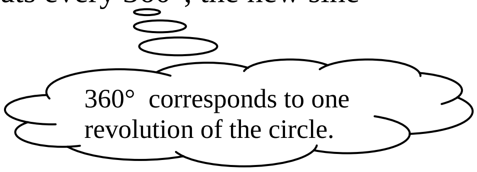
2.3 Graphs of the Sine and Cosine Functions

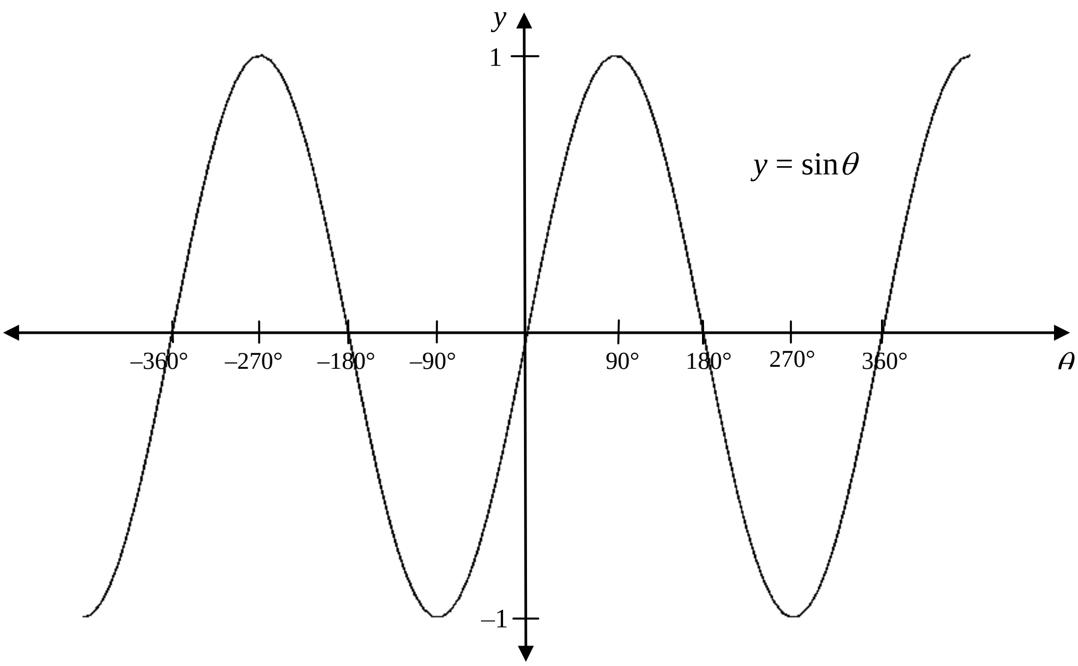
The unit circle can be used to draw the graph of $y = \sin\theta$, using the idea that the length of the line PQ is equal to sin θ.



In the diagram above, you can see that as θ increases from 0° to 90° (in an anticlockwise direction) the graph of y = sinθ increases until it reaches a maximum at (90°, 1), and then begins to decrease. Also, if θ decreases from 0° to –90° (in a clockwise direction), the graph of y = sinθ decreases until it reaches a minium at (–90°, –1), then it begins to increase again.

The full graph of y = sinθ is shown below. As the shape repeats every 360°, the new sine function is described as being *periodic* with *period* 360°.





The graph of $y = \sin\theta$ extends infinitely in both directions, it

- is periodic with period 360°
- has x-intercepts at 0°, ±180°, ±360°, etc
- has *turning points* at θ = ±90°, θ = ±270°, etc
- is symmetric about the lines θ = ±90°, θ = ±270°, etc.

The function sinθ has

- natural domain **R** and range [–1, 1]
- a maximum value of +1 in the interval [0°, 360°] at θ = 90°
- a minimum value of –1 in the interval [0°, 360°] at θ = 270°.

The graph of $y = \sin\theta$ is very useful for solving trigonometric equations when there is more than one solution.

Example

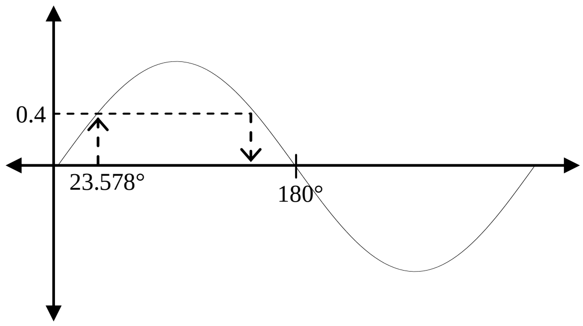
Solve the equation $\sin\theta = 0.4$ for $0 \le \theta \le 360°$.

Find one solution using a calculator, and the rest from the graph.

Answer

Calculator: $\sin\theta = 0.4 \Rightarrow \theta = 23.578°$

Graph: There are two solutions between 0° and 360°.



This can be checked on your calculator

This

By symmetry, the second solution is 180° – 23.578° = 156.422°.

The solutions are 23.6° and 156.4°.

Leave rounding off until the final answers.

Problems 2.3A

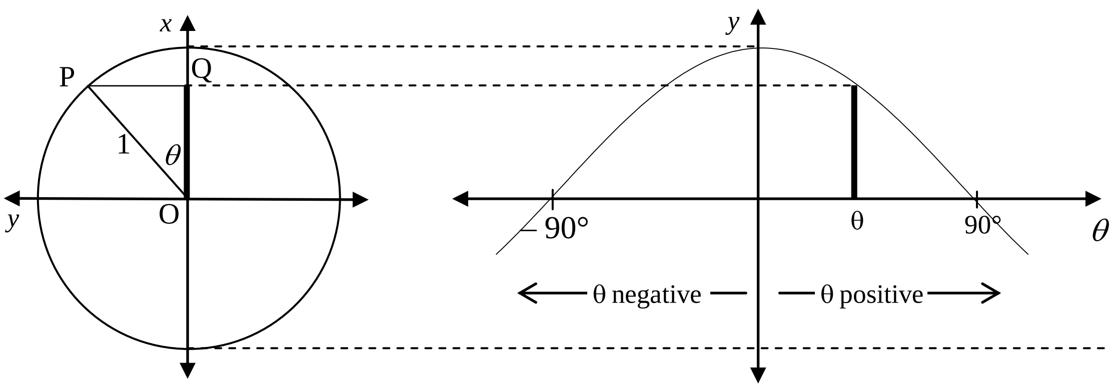
1. Solve the equation $\sin\theta = 0.6$ for

(a) $0 \le \theta \le 360°$ (b) $-360° \le \theta \le 0°$

2. Solve the equation $\sin\theta = -0.5$ for

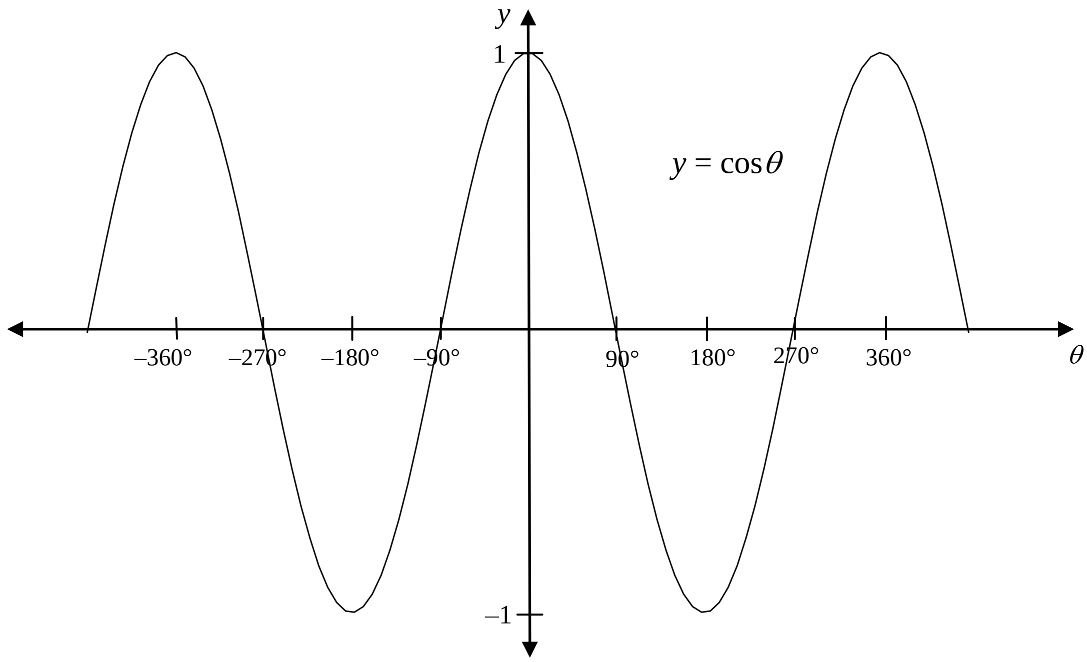
(a) $-180° \le \theta \le 180°$ (b) $0° \le \theta \le 720°$

The unit circle can be used to draw the graph of $y = \cos\theta$, using the idea that the length of the line OQ is equal to $\cos\theta$. To draw this graph, we need to turn the unit circle on its side.



In the diagram above, you can see that as θ increases from 0° to 90° (in an anticlockwise direction) the graph of $y = \cos\theta$ decreases from (0°, 1) until it reaches (90°, 0), and then continues to decrease. Also, if θ decreases from 0° to –90° (in a clockwise direction), the graph of $y = \cos\theta$ decreases from (0°, 1) until it reaches (–90°, 0), then continues to decrease.

The full graph of $y = \cos\theta$ is shown below. It has the same shape as the graph of $y = \sin\theta$ but *translated to the left by 90°*. As the shape repeats every 360°, the new cosine function is described as being *periodic* with *period* 360°.



The graph of $y = \cos\theta$ extends infinitely in both directions, it

- is periodic with period 360°
- has x-intercepts at ±90°, ±270°, etc
- has *turning points* at $\theta = 0°$, $\theta = ±180°$, etc
- is symmetric about the lines $\theta = 0°$, $\theta = ±180°$, $\theta = ±360°$, etc.
- is the translation of the graph of $y = \sin\theta$ to the left by 90°.

The function $\cos\theta$ has

- natural domain **R** and range [–1, 1]
- a maximum value of +1 in the interval [0°, 360°] at $\theta = 0°$ and 360°.
- a minimum value of –1 in the interval [0°, 360°] at $\theta = 180°$.

Example

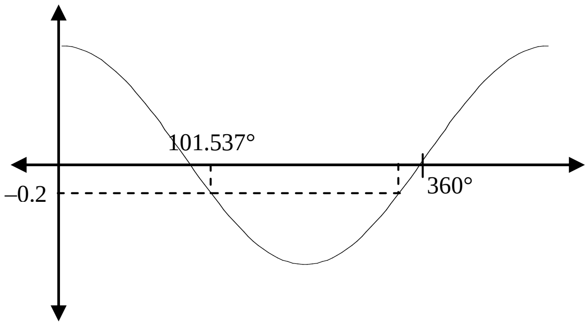
Solve the equation $\cos\theta = -0.2$ for $0 \le \theta \le 360°$.

Answer

Calculator: $\cos\theta = -0.2 \Rightarrow \theta = 101.537°$

Find one solution using a calculator, and the rest from the graph.

Graph: There are two solutions between 0° and 360°.



By symmetry, the second solution is 360° – 101.537° = 258.463°.

The solutions are –101.5° and 258.5°.

Leave rounding off until the final answers.

Problems 2.3B

1. Solve the equation $\cos\theta = 0.6$ for

(a) $0 \le \theta \le 360°$ (b) $-360° \le \theta \le 0°$

2. Solve the equation $\cos\theta = -0.5$ for

(a) $-180° \le \theta \le 180°$ (b) $0° \le \theta \le 720°$

2.4 The Tan Function and its Graph

In Module 5, we saw that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for any angle θ in a right-angled triangle. We can use this relationship to define $\tan\theta$ for any angle θ, provided that $\cos\theta \ne 0$ ie. $\theta \ne \pm 90°$, etc.

Example

What are the exact value of tan 135°?

Answer

The first example in section 2.2 shows $\sin 135° = \dfrac{1}{\sqrt{2}}$ and $\cos 135° = -\dfrac{1}{\sqrt{2}}$.

so $\tan 135° = \dfrac{\sin 135°}{\cos 135°}$

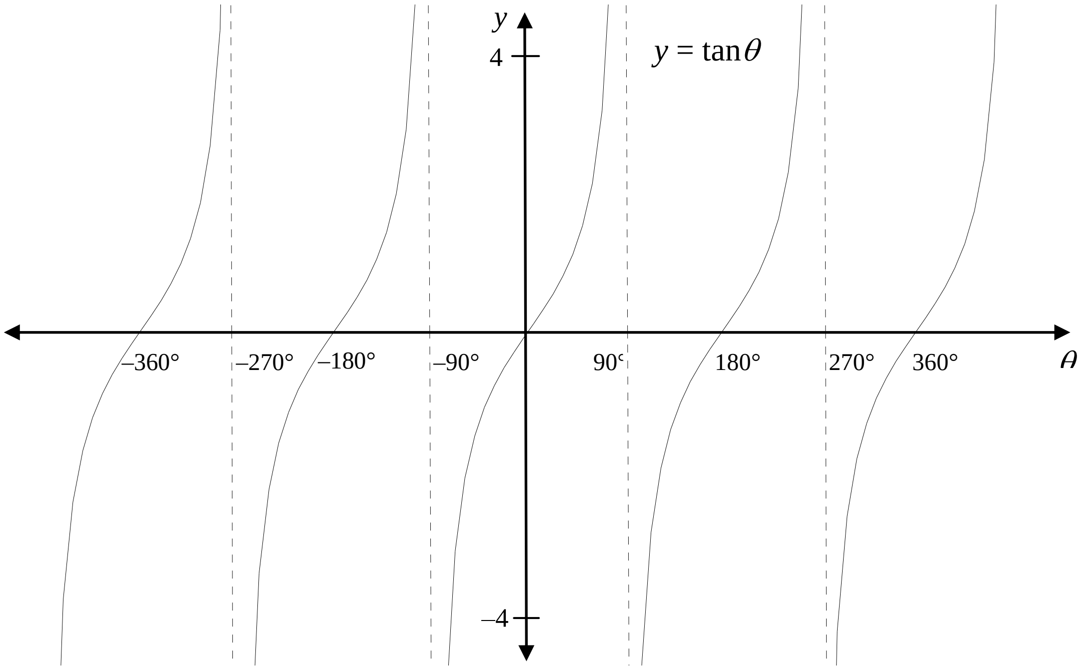
$$= \dfrac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}$$

$$= -1$$

Problems 2.4A

Find the exact values of (a) tan 225° (b) tan 120° (b) tan(–30°)

The graph of $y = \tan\theta$ is shown below. You can see that the shape repeats every 180°, so tan function *is periodic with period 180°.*



Notice that the tan graph has asymptotes at ±90°, ± 270°, etc. This is because of the $\cos\theta$ in the denominator of $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. When $\cos\theta$ is small, the value of $\tan\theta$ is large.

Example

When θ = 89°, sin 89° = 0.9998 and cos 89° = 0.01745, so $\tan 89° = \dfrac{\sin 89°}{\cos 89°} = 57.23$.

The graph of $y = \tan\theta$ extends infinitely in both directions, it

- is periodic with period **180°**
- has x-intercepts at 0°, ±180°, ±360°, etc

The function $\tan\theta$ has natural domain **R** and range **R.**

Problems 2.4B

1. Solve the equation $\tan\theta = 0.6$ for

(a) $0 \leq \theta \leq 360°$ (b) $-360° \leq \theta \leq 0°$

2. Solve the equation $\tan\theta = -0.5$ for

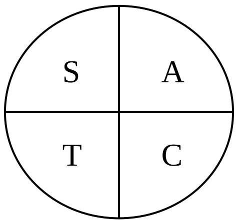
(a) $-180° \leq \theta \leq 180°$ (b) $0° \leq \theta \leq 720°$

3
Applications

3.1 All Stops To City

The new definitions of the sin, cos and tan functions were based upon a new way of thinking about angles, where positive angles are measured in an anticlockwise direction and negative angles are measured in a clockwise direction. We need to relate this the previous definitions of sin, cos and tan ratios based on right-angled triangles.

The sin function is positive between 0° and 180°, whereas the cos function is positive between 0° and 90°, and negative between 90° and 180°. The following diagram summarises this information for all angles:



The letters A, S, T and C are interpreted as:

A $\Rightarrow$ **<u>A</u>**ll of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive in the 1st quadrant of the coordinate plane.

S $\Rightarrow$ **<u>S</u>**inθ is positive in the 2nd quadrant; $\cos\theta$ and $\tan\theta$ are negative.

T $\Rightarrow$ **<u>T</u>**anθ is positive in the 3rd quadrant; $\sin\theta$ and $\cos\theta$ are negative.

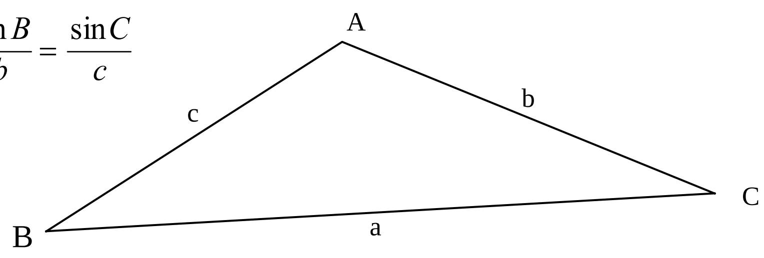
C $\Rightarrow$ **<u>C</u>**osθ is positive in the 4th quadrant; $\sin\theta$ and $\tan\theta$ are negative.

The two angles in a right-angled triangle (other than the right angle) are between 0° and 90°. As the new definitions of sin, cos and tan are all positive for angles in the first quadrant, there is no conflict with the previous definitions of the trigonometric ratios.

3.1 Trigonometry in Triangles Without Right-angles

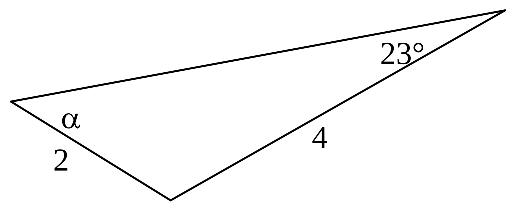
The **sine rule** is very useful in solving problems in triangles:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Example

Find the angle in the triangle below.



Answer

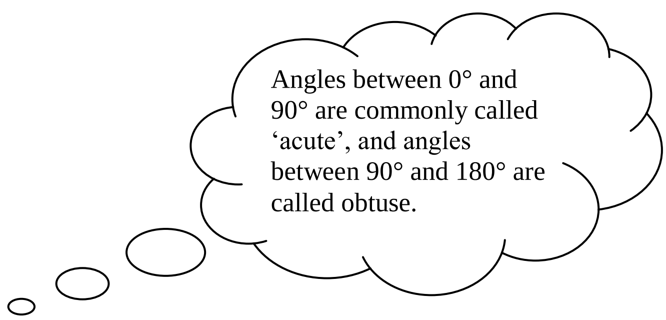
By the sine rule,

$$\frac{\sin\alpha}{4} = \frac{\sin 23°}{2}$$

$$\sin\alpha = \frac{\sin 23°}{2} \times 4$$

$$= 0.7815$$

$$\alpha = 51.398°$$

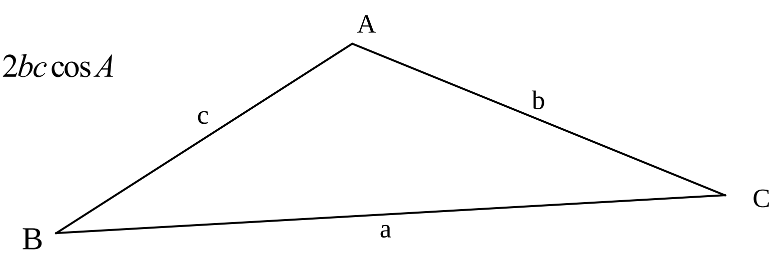


A second solution is 180° – 51.398° = 128.602°

The angle in the diagram is an acute angle, so α = 51.4°.

The **cosine rule** is also very useful:

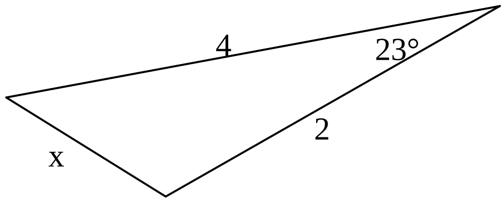
$$a^2 = b^2 + c^2 - 2bc\cos A$$



The cosine rule is a generalisation of Pythagoras' Theorem. You can choose any side to have length 'a', not just the longest side.

Example

Find x in the triangle below.



Answer

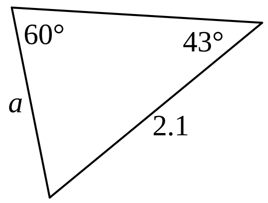
By the cosine rule,

$$\begin{aligned} x^2 &= 2^2 + 4^2 - 2 \times 4 \times \cos 23° \\ &= 12.636 \\ x &= 3.6 \end{aligned}$$

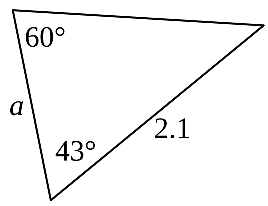
Problems 3.2

Find the unknown side or angle in the triangles below.

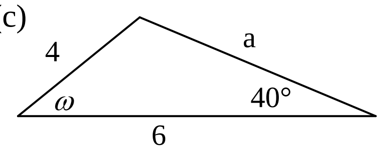
(a)



(b)



(c)

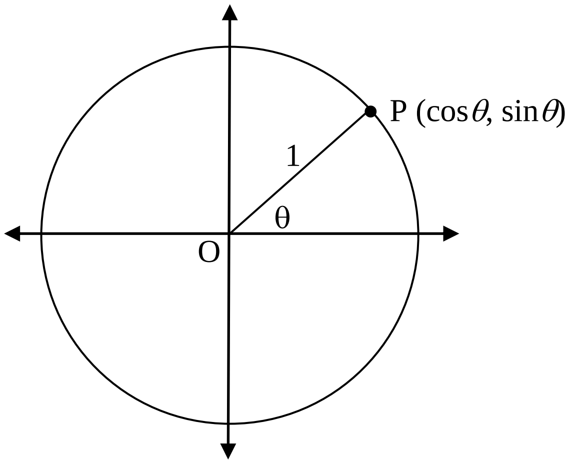


4
Identities

A trigonometric identity is a relationship between the trigonometric functions which is true for all angles.

4.1 $(\sin\theta)^2 + (\cos\theta)^2 = 1$

This is the most famous of all trigonometric identities. It is true because the equation of the unit circle is $x^2 + y^2 = 1$.

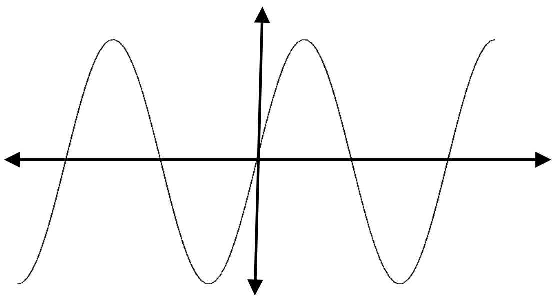


4.2 $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, when $\cos\theta \neq 0$

This is the definition of $\tan\theta$.

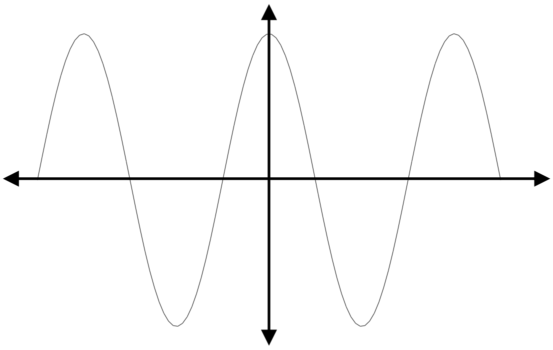
4.3 $\sin(-\theta) = -\sin\theta$

This is because the graph of $y = \sin\theta$ is symmetric across the origin.



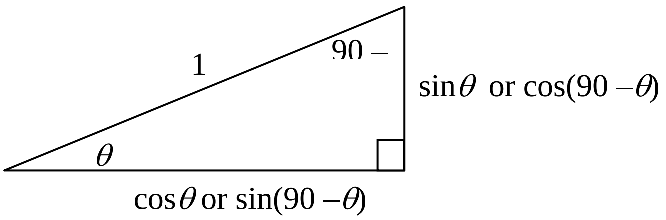
4.4 $\cos(-\theta) = \cos\theta$

This is because the graph of $y = \cos\theta$ is symmetric about the y-axis.



4.5 $\cos(90-\theta) = \sin\theta$; $\sin(90-\theta) = \cos\theta$

These identities come from the right-angled triangle.



4.6 $\cos(\theta - 90°) = \sin\theta$; $\sin(\theta + 90°) = \cos\theta$

The first identity comes from the fact that translating the graph of $y = \cos\theta$ to the right by 90° gives the graph of $y = \sin\theta$ □

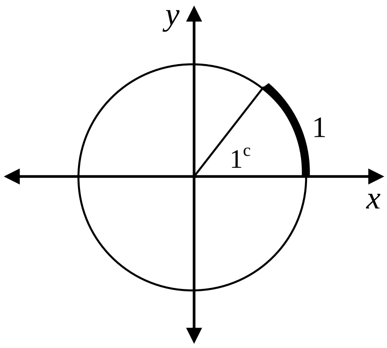
Problems 4

Use the graphs of $y = \sin\theta$ and $y = \cos\theta$ to explain the identity $\sin(\theta + 90°) = \cos\theta$.

5
Radian Measure

The use of degrees to measure angles has its source in the astronomy of ancient times – a degree being approximately the angle moved in one day by the earth in its journey around the sun. This is not the best unit of measurement in mathematics, and a more convenient unit is needed for subjects like Calculus.

The most natural unit for measuring angles is the *radian*. One radian is the angle in a unit circle which *subtends* an arc of length 1 unit.



The arc length around the whole unit circle is equal to its circumference ie. $2\pi \times 1$ units, so the angle in a whole revolution is 2π radians (pronounced "2 pie radians"). This tells us that 2π radians is equal to 360°, and that

$$\boxed{\pi \text{ radians} = 180°}$$

In mathematics we always assume that if no unit of measurement is mentioned, then the size of an angle is in radians. Hence an angle in degrees must always have a degree symbol. If a symbol to mean radians is necessary, many use the symbol ^{c} to mean "radians" (the c referring to circumference).

Example
Express an angle of 1 in terms of degrees.

Answer

$$\begin{aligned}\pi &= 180° \\ 1 &= \frac{180°}{\pi} \\ &= 57.3°\end{aligned}$$

Example

Express an angle of $\frac{\pi}{12}$ in terms of degrees.

Answer

$$\pi = 180°$$
$$\frac{\pi}{12} = \frac{180°}{12}$$
$$= 15°$$

Example
Express 15° in terms of radians.

Answer

$$180° = \pi$$
$$1° = \frac{\pi}{180}$$
$$15° = 15 \times \frac{\pi}{180} = \frac{\pi}{12}$$

Example
Express 37.9° in terms of radians.

Answer

$$180° = \pi$$
$$1° = \frac{\pi}{180}$$
$$37.9° = 37.9 \times \frac{\pi}{180}$$
$$= 0.21\pi \text{ or } 0.66$$

It is a good practice to write answers as multiples of π radians. This helps you understand the actual size of the angle.

In the example to the right, 0.21π can be quickly seen to be about a fifth of 180°, whereas the answer 0.66 is much harder to visualise.

Problems 5

1. Express the following angles in degrees.

(a) π (b) 2π (c) -2π (d) $\frac{\pi}{2}$ (e) $\frac{2\pi}{3}$ (f) $-\frac{3\pi}{8}$

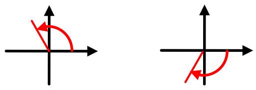
2. Express the following angles in radians.

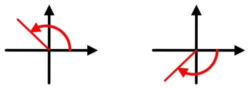
(a) 30° (b) 45° (c) 60° (d) –150° (e) 540° (f) 26.5°

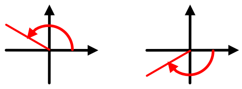
A

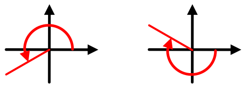
Answers

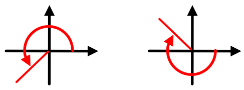
Section 1.1

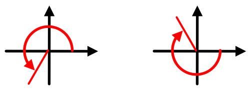
(a) 

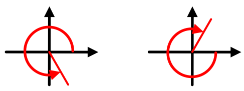
(b) 

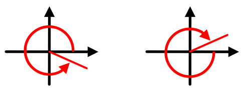
(c) 

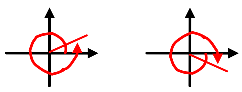
(d) 

(e) 

(f) 

(g) 

(h) 

(i) 

Section 2.2

(a) $\frac{\sqrt{3}}{2}, -\frac{1}{2}$ (b) $\frac{1}{2}, -\frac{\sqrt{3}}{2}$ (c) $-\frac{1}{2}, -\frac{\sqrt{3}}{2}$

(d) $-\frac{1}{2}, \frac{\sqrt{3}}{2}$ (e) $-\frac{1}{2}, \frac{\sqrt{3}}{2}$ (f) $-\frac{1}{2}, -\frac{\sqrt{3}}{2}$

Section 2.3A

1(a) 36.87°, 143.13° (b) –216.87°, –323.13°

2(a) –150°, –30° (b) 210°, 330°, 570°, 690°

Section 2.3B

1(a) 53.13°, 306.87° (b) –306.87°, –53.13°

2(a) –120°, 120° (b) 120°, 240°, 480°, 600°

Section 2.4A

1(a) 1 (b) $-\sqrt{3}$ (c) $-\frac{1}{\sqrt{3}}$

Section 2.4B

1(a) 30.96°, 210.96° (b) –329.04°, -149.04°

2(a) –26.57°, 153.43° (b) 153.43°, 333.43°, 513.43°, 693.43°

Section 3.2

(a) 1.7 (b) 2.4 (c) 65.4°, 6.5

Section 5

1(a) 180° (b) 360° (c) –360° (d) 90° (e) 120° (f) –67.5°

2(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$

(d) $-\frac{5\pi}{6}$ (e) 3π (f) 0.15π