CHAPTER 08

Differentiation

Let $f(x)$ be a function differentiable in an interval $[a, b]$. That is at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define a new function $g:[a, b] \to R$, such that,

$$g(x) = f'(x), \forall\, x \in [a, b].$$

This new function is said to be **differentiation** (differential coefficient) of the function $f(x)$ with respect to x and it is denoted by $\dfrac{df(x)}{dx}$ or $Df(x)$ or $f'(x)$.

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Algebra of Differentiation

Algebra of differentiation is defined in the following ways

(i) **Differentiation of the sum of two functions** Let $f(x)$ and $g(x)$ be two real valued functions. Then,

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

(ii) **Differentiation of the difference of two functions** Let $f(x)$ and $g(x)$ be two real valued functions. Then,

$$[f(x) - g(x)]' = f'(x) - g'(x)$$

(iii) **Differentiation of the product of two or more functions**

(a) Let $f(x)$ and $g(x)$ be two real valued functions. Then,

$$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

(b) Let $f(x)$, $g(x)$ and $h(x)$ are three real valued functions are given, then

$$\frac{d}{dx}[f(x) \cdot g(x) \cdot h(x)] = f(x) \cdot g(x) \cdot h'(x) + f(x) \cdot g'(x) \cdot h(x) + f'(x) \cdot g(x) \cdot h(x)$$

(iv) **Differentiation of the quotient of two functions** Let $f(x)$ and $g(x)$ be two real valued functions.

Then,

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{[g(x)]^2},$$

provided $g(x) \neq 0$

Differentiation of Composite Function (Chain Rule)

If y is a differentiable function of t and t is a differentiable function of x, i.e. $y = f(t)$ and $t = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Similarly, if $y = f(u)$, where $u = g(v)$ and $v = h(x)$ and so on

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \text{......}$$

Some Standard Derivatives

Some common standard derivatives used in differentiation are given below

Algebraic Functions

(i) $\dfrac{d}{dx}(x^n) = nx^{n-1}$, $\dfrac{d}{dx}\{f(x)\}^n = n\{f(x)\}^{n-1}\dfrac{d}{dx} f(x$

(ii) $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -nx^{-n-1}$,

$$\frac{d}{dx}\left\{\frac{1}{f(x)}\right\}^n = \frac{-n}{\{f(x)\}^{n+1}}\frac{d}{dx}(f(x))$$

(iii) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{1}{(f(x))^2}\frac{d}{dx}(f(x))$

(iv) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \frac{d}{dx}(\sqrt{f(x)}) = \frac{1}{2\sqrt{f(x)}}\frac{d}{dx}f(x)$

Exponential Functions

(i) $\frac{d}{dx}(e^x) = e^x, \frac{d}{dx}(e^{\{f(x)\}}) = [e^{\{f(x)\}}]\frac{d}{dx}f(x)$

(ii) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0,$

$\frac{d}{dx}a^{\{f(x)\}} = [a^{\{f(x)\}} \log_e a]\frac{d}{dx}(f(x)), a > 0$

Logarithmic Functions

(i) $\frac{d}{dx}(\log_e x) = \frac{1}{x}, \frac{d}{dx}\{\log_e f(x)\} = \frac{1}{f(x)}\frac{d}{dx}(f(x))$

(ii) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}, \ a > 0, \ a \neq 1$

$\frac{d}{dx}(\log_a f(x)) = \frac{1}{f(x)\log_e a}\frac{d}{dx}(f(x)), a > 0, a \neq 1$

Trigonometric Functions

(i) $\frac{d}{dx}(\sin x) = \cos x,$

$\frac{d}{dx}\sin(f(x)) = \cos f(x)\frac{d}{dx}(f(x))$

(ii) $\frac{d}{dx}(\cos x) = -\sin x,$

$\frac{d}{dx}\cos(f(x)) = -\sin f(x)\frac{d}{dx}(f(x))$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x,$

$\frac{d}{dx}\tan(f(x)) = \sec^2 f(x)\frac{d}{dx}f(x)$

(iv) $\frac{d}{dx}(\cot x) = -\text{cosec}^2(x),$

$\frac{d}{dx}\cot(f(x)) = -\text{cosec}^2 f(x)\frac{d}{dx}(f(x))$

(v) $\frac{d}{dx}(\sec x) = \sec x \tan x,$

$\frac{d}{dx}\sec(f(x)) = \sec(f(x))\tan f(x)\frac{d}{dx}(f(x))$

(vi) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x,$

$\frac{d}{dx}\text{cosec}(f(x)) = -\text{cosec}(f(x)) \cot(f(x))\frac{d}{dx}(f(x))$

Derivative of Inverse Functions

If $y = f(x)$ is one-one onto function, then $x = f^{-1}(y)$ exists.

If $y = f(x)$ is a differentiable function of x such that inverse function $x = f^{-1}(y)$ exists, then x is differentiable function of y and $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, where $\frac{dy}{dx} \neq 0$.

Derivative of Inverse Trigonometric Functions

As we know that inverse trigonometric functions are multivalued functions, so their derivative determine the derivative of inverse trigonometric function, it is necessary to give more attention or domain, and range of that function.

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; |x| < 1$

$\frac{d}{dx}(\sin^{-1} f(x)) = \frac{1}{\sqrt{1-\{f(x)\}^2}}\frac{d}{dx}f(x); |f(x)| < 1$

(ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}; |x| < 1$

$\frac{d}{dx}(\cos^{-1} f(x)) = -\frac{1}{\sqrt{1-\{f(x)\}^2}}\frac{d}{dx}f(x); \ |f(x)| < 1$

(iii) $\frac{d}{dx}\tan^{-1} x = \left(\frac{1}{1+x^2}\right); x \in R$

$\frac{d}{dx}(\tan^{-1} f(x)) = \frac{1}{1+\{f(x)\}^2}\frac{d}{dx}f(x)$

(iv) $\frac{d}{dx}\cot^{-1} x = -\left(\frac{1}{1+x^2}\right); x \in R$

$\frac{d}{dx}(\cot^{-1} f(x)) = -\frac{1}{1+\{f(x)\}^2}\frac{d}{dx}f(x)$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}; |x| > 1$

$\frac{d}{dx}(\sec^{-1} f(x)) = \frac{1}{|f(x)|\sqrt{\{f(x)\}^2-1}}\frac{d}{dx}f(x);$ $|f(x)| > 1$

(vi) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\left(\frac{1}{|x|\sqrt{x^2-1}}\right); |x| > 1$

$\frac{d}{dx}(\text{cosec}^{-1} f(x)) = -\frac{1}{|f(x)|\sqrt{\{f(x)\}^2-1}}\frac{d}{dx}f(x);$ $|f(x)| > 1$

Differentiation by Substitution

Substitution is useful to reduce the function into simple form. For problems involving inverse trigonometric functions, first try for a suitable substitution to simplify it and then differentiate it. If no such substitution is found, then differentiate directly.

Some Standard Substitutions

Expressions	Substitutions
$1-x^2$	$x=\sin\theta$ or $x=\cos\theta$
$1+x^2$	$x=\tan\theta$ or $x=\cot\theta$
x^2-1	$x=\sec\theta$ or $x=\operatorname{cosec}\theta$
$2x\sqrt{1-x^2}$	$x=\sin\theta$ or $x=\cos\theta$
$\dfrac{1-x^2}{1+x^2}$	$x=\tan\theta$
$3x-4x^3$	$x=\sin\theta$
$4x^3-3x^2$	$x=\cos\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$ or $x=a\cot\theta$
$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $x=a\cos\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$ or $x=a\operatorname{cosec}\theta$
$\dfrac{a+x}{a-x}$ or $\dfrac{a-x}{a+x}$	$x=a\tan^2\theta$ or $x=a\cot^2\theta$
$\sqrt{\dfrac{a+x}{a-x}}$ or $\sqrt{\dfrac{a-x}{a+x}}$	$x=a\cos\theta$ or $x=a\cos 2\theta$
$\dfrac{2x}{1+x^2}$ or $\dfrac{2x}{1-x^2}$	$x=\tan\theta$ or $x=\cot\theta$
$a\sin x+b\cos x$	$a=r.\cos\alpha$ and $b=r\sin\alpha$
$\sqrt{x-\alpha}$ and $\sqrt{\beta-x}$	$x=\alpha\sin^2\theta+\beta\cos^2\theta$
$\sqrt{2ax-x^2}$	$x=a(1-\cos\theta)$

Differential Coefficient Using Inverse Trigonometrical Substitutions

Sometimes, the given function can be deducted with the help of inverse trigonometrical substitution and then find the differential coefficient is very easily.

(i) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

(ii) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$ or $\cos^{-1}(1-2x^2)$

(iii) $2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)\\ \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)\\ \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)\end{cases}$

(iv) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$

(v) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

(vi) $3\tan^{-1}x=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

(vii) $\cos^{-1}x+\sin^{-1}x=\pi/2$

(viii) $\tan^{-1}x+\cot^{-1}x=\pi/2$

(ix) $\sec^{-1}x+\operatorname{cosec}^{-1}x=\dfrac{\pi}{2}$

(x) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}]$

(xi) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}[xy\mp\sqrt{(1-x^2)(1-y^2)}]$

(xii) $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\left[\dfrac{x\pm y}{1\mp xy}\right]$

Important Trigonometrical Formulae

The following formulae will be helpful to solve the problems involving inverse trigonometric function, etc.

(i) $\sin 2x=2\sin x\cos x=\dfrac{2\tan x}{1+\tan^2 x}$

(ii) $\cos 2x=2\cos^2 x-1=1-2\sin^2 x=\cos^2 x-\sin^2 x$
$=\dfrac{1-\tan^2 x}{1+\tan^2 x}$

(iii) $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$

(iv) $\dfrac{1\pm\tan x}{1\mp\tan x}=\tan\left(\dfrac{\pi}{4}\pm x\right)$

Logarithmic Differentiation

When a function consists product or quotient of number of functions then, we take the logarithm and then differentiate.

The functions which can be evaluated by using this method are of following types:

(i) $y=\{f_1(x)\}^{\{f_2(x)\}}$ (ii) $y=f_1(x)\cdot f_2(x)\cdot f_3(x),\ldots$

(iii) $y=\dfrac{f_1(x)\cdot f_2(x)\cdot f_3(x)\ldots}{\phi_1(x)\cdot\phi_2(x)\cdot\phi_3(x)\ldots}$

Alternate Method

If $y=\{f(x)\}^{g(x)}$,

then $\dfrac{dy}{dx}=$ Differentiation of $\{f(x)\}^{g(x)}$ w.r.t. x

[taking $g(x)$ as a constant]

+ Differentiation of $\{f(x)\}^{g(x)}$ w.r.t. x

[taking $f(x)$ as a constant]

$$\Rightarrow \frac{dy}{dx}=g(x)\cdot\{f(x)\}^{g(x)-1}\cdot\frac{d}{dx}f(x)+\{f(x)\}^{g(x)}\cdot\log f(x)\cdot\frac{d}{dx}g(x)$$

Derivative of Implicit Function

If the relation between the variables x and y are given by an equation containing both the variables and this equation is not immediately solvable for y, then y is called an implicit function of x.

Implicit functions are given by $f(x, y) = 0$.

To find the differentiation of implicit function, we differentiate each term w.r.t. x considering y as a function of x and then collect the terms of $\frac{dy}{dx}$ together on left hand side and remaining terms on right hand side and find $\frac{dy}{dx}$.

Derivative of Parametric Functions

A relation expressed between two variables x and y in the form $x = f(t)$ and $y = g(t)$ are said to be parametric form, where t is a parameter. Then, the derivative $\frac{dy}{dx}$ of such function is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{g'(t)}{f'(t)}, \text{ provided } f'(t) \neq 0.$$

Differentiation of a Function w.r.t. Another Function

If $y = f(x)$ and $z = g(x)$ be two functions of x. Then, to find the derivative of $f(x)$ w.r.t. $g(x)$ i.e. to find $\frac{dy}{dz}$, we use the following formula.

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$$

(i) If y and z are functions of x, then

$$\frac{d}{dx}(y^z) = zy^{z-1}\frac{dy}{dx} + (y^z \log y)\frac{dz}{dx}$$

(ii) If $y = z^n$ and $z = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = nz^{n-1}\frac{dz}{dx}$$

Higher Order Derivatives

(Second Order Derivatives)

For given $y = f(x)$, the process of finding its higher derivatives is called successive differentiation.

Suppose $y = f(x)$ is a differentiable function of x, then $\frac{dy}{dx} = f'(x)$ is the first order derivative of y on $f(x)$ w.r.t. x.

The second derivative of above differentiable function is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x).$$

Similarly, this process is going on nth derivative.

i.e. $$\frac{d}{dx}\left(\frac{d^{n-1}(x)}{dx^{n-1}}\right) = f^n(x).$$

In order to find second order derivative of parametric function, we can also use the following formula.

Let $$x = \phi(t), y = \psi(t)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{\psi'(t)}{\phi'(t)}\right) = \frac{\frac{d}{dt}\left(\frac{\psi'(t)}{\phi'(t)}\right)}{\frac{dx}{dt}}$$