CHAPTER 07

Permutations and Combinations

Fundamental Principles of Counting

Tree Diagram

We have learnt in set theory that subsets of a set can be represented in the form of a Venn diagram. An alternative method is to draw a tree diagram, if the subsets are disjoint.

Addition Principle

Suppose one operation can be done in m ways and another operation can be done in n ways, with no common way among them. If one of these operations is to be performed, then there are $m+n$ ways to do it.

Multiplication Principle

If one operation can be carried out in m ways, followed by the second operation that can be carried out in n ways, and these two operations are independent, then the two operations can be carried out in $m \times n$ ways.

Extended Addition Principle

Suppose there are three possible choices with no common outcome for any two, the first choice can be made in m ways, second in n ways and third in r ways. If only one of the choices is to be made, it has $m+n+r$ possible ways.

Extended Multiplication Principle

Suppose an experiment consists of three independent activities, where first activity has m possible outcomes, second has n possible outcomes, third has r possible outcomes. Then, the total number of different possible outcomes of the experiment is $m \times n \times r$

Invariance Principle

The result of counting objects in a set does not depend on the order in which these objects are counted or on the method used for counting these objects.

Factorial Notation

n Factorial

Many times, we come across the products of the form $1\times 2, 1\times 2\times 3, 1\times 2\times 3\times 4, \ldots$. For our convenience, we use a special notation instead of writing all the factors of such a product. We write

$$\begin{aligned}1! &= 1\\ 2! &= 2\times 1\\ 3! &= 3\times 2\times 1\\ &\vdots\\ n! &= n(n-1)(n-2)\ldots 3\cdot 2\cdot 1\end{aligned}$$

Thus, the notation $n!$ represent the product of first n natural numbers. We read this notation as 'n factorial' and it is also denoted by $n\,!$

Clearly, for a natural number n,

$$\begin{aligned}n\,! &= n\,(n-1)!\\ &= n\,(n-1)\,(n-2)! && [\text{provided } n \ge 2]\\ &= n\,(n-1)\,(n-2)\,(n-3)! && [\text{provided } n \ge 3]\\ &\vdots\\ &= n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1\end{aligned}$$

e.g. $5! = 5\times 4\times 3\times 2\times 1 = 120$

and $8! = 8\times(7)! = 8\times 7\times(6!)$

Zero Factorial

It does not make any sense to define 0! as product of the integers from 1 to 0. So, we define $0! = 1$.

Note *Factorial of proper fractions or negative integers are not define. n! is defined only for whole numbers, i.e. for non-negative integers.*

Properties of the Factorial Notation

For any positive integers m, n

- $n! = n \times (n-1)!$
- $n > 1, n! = n \times (n-1) \times (n-2)!$
- $n > 2, n! = n \times (n-1) \times (n-2) \times (n-3)!$
- $(m+n)!$ is always divisible by $m!$ as well as by $n!$
- $(m \times n)! \neq m! \times n!$
- $(m+n)! \neq m! + n!$
- $m > n, (m-n)! \neq m! - n!$ but $m!$ is divisible by $n!$
- $(m \div n)! \neq m! \div n!$

Permutations

A permutation is an arrangement of objects in a definite order. Arrangement can be made by taking some or all objects at a time.

e.g. If there are three objects say A, B and C, then the permutations of these three objects taking two at a time are AB, BA, AC, CA BC, CB and the permutations of these three objects taking all at a time are $ABC, ACB, BAC, BCA, CAB, CBA$.

Here, in each case, number of permutations is 6.

Note that the order of arrangement is important. Because when the order is changed, then different permutation is obtained.

Permutations, when all the Objects are Distinct

When all given objects are distinct, then we can find the number of permutations with the help of following theorem

Theorem The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat, is $n(n-1)(n-2) \dots (n-r+1)$, which is denoted by ${}^nP_r$ or $P(n,r)$.

i.e. $$P(n,r) = {}^nP_r = \frac{n!}{(n-r)!}, 0 \leq r \leq n$$

(i) When $r = 0$, then ${}^nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

(ii) When $r = n$, then ${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ $\quad [\because 0! = 1]$

Permutations when Repetitions are Allowed

When repetition of objects is allowed, then number of permutations can be obtained with the help of following theorem.

Theorem 1 The number of permutation of n different objects taken r at a time, when each may be repeated any number of times in each arrangement, is n^r (permutation with repetitions).

Theorem 2 The number of permutations taken all at a time, when m specified objects among n always come together, is $m!(n-m+1)!$

Some Important Conditions for Permutation

(i) The number of permutations of n different objects, all at a time, when each may be repeated any number of times in each arrangement, is n^n.

(ii) The number of permutations of n distinct objects taken all at a time, when 2 specified objects are always together, is
$$2 \times (n-1)!$$

(iii) The number of permutations of n distinct objects taken r at a time, when a specified object is always to be included, is
$$r \times {}^{(n-1)}P_{(r-1)}$$
First keep the specified object aside. Arrange $(r-1)$ from the remaining $(n-1)$ objects in ${}^{(n-1)}P_{(r-1)}$ ways. Then, place the specified object in r possible ways. Hence, the total number of arrangements is
$$r \times {}^{(n-1)}P_{(r-1)}$$

(iv) The number of permutation of n distinct objects taken r at a time, when a specified object is not be included in any permutation, is
$${}^{(n-1)}P_r$$

Permutations When Some Objects are Identical

When all the objects are not distinct, i.e. some objects are of same kind, then we can find the number of permutations with the help of following theorems (without proof).

Theorem The number of permutations of n objects, where p objects are of the same kind or identical and other are distinct, is given by $\frac{n!}{p!}$.

The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ... p_k are of kth kind and the rest if any, are of different kind is $\frac{n!}{p_1! p_2! \dots p_k!}$.

Properties of Permutations

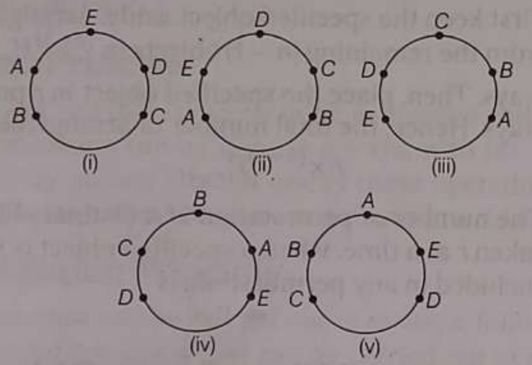
- ${}^nP_n = n!$ • ${}^nP_0 = 1$ • ${}^nP_1 = n$
- ${}^nP_r = n \times {}^{(n-1)}P_{(r-1)} = n(n-1) \times {}^{(n-2)}P_{(r-2)}$
  $= n(n-1)(n-2) \times {}^{(n-3)}P_{(r-3)}$ and so on.
- $\dfrac{{}^nP_r}{{}^nP_{(r-1)}} = n - r + 1$

Circular Permutation

If objects are arranged along a closed curve, then permutation is known as circular permutation.

In other words, the permutation in a row has a beginnings and end but there is no beginning and end in circular permutation. So, we need to consider one object is fixed and the remaining objects are arranged in $(n-1)!$ ways.

e.g. Consider five persons A, B, C, D and E to be seated on the circumference of a circular table in order (which has no head). Now, shifting A, B, C, D and E one position in anti-clockwise direction, we will get arrangements as follows



We see that arrangements in all figures are same.

$\therefore$ The number of circular permutations of n different things taken all at a time is $\dfrac{{}^nP_n}{n} = (n-1)!$, if clockwise and anti-clockwise orders are taken as different.

In a circular permutation, if the position are given by number, then it is treated as a **linear arrangement**.

Important Results on Circular Permutation

(i) Number of circular permutations of n different things taken all at a time $= (n-1)!$. If clockwise and anti-clockwise orders are taken as different.

(ii) The number of circular permutations of n different things taken all at a time $= \frac{1}{2}(n-1)!$. If clockwise and anti-clockwise orders are taken as not different.

(iii) The number of circular arrangements of n objects, of which m objects are alike (identical), is given by $\dfrac{(n-1)!}{m!}$.

(iv) The number of circular permutations of r objects taken from n distinct objects can be found under two different conditions as follows

(a) when clockwise and anti-clockwise arrangements are considered to be different, then the required number of circular arrangements is given by $\dfrac{{}^nP_r}{r}$.

(b) When clockwise and anti-clockwise arrangements are not to be considered different, then the required number of circular arrangements is given by $\dfrac{{}^nP_r}{2r}$.

Combinations

Each of the different groups or selections which can be made by some or all of a number of given things without reference to the order of the things in each group is called a combination.

Mathematically, the number of combinations of n different things taken r at a time is

$$C(n,r) \text{ or } {}^nC_r \text{ or } \binom{n}{r}$$

i.e. $${}^nC_r = \frac{n!}{r!(n-r)!}, \; 0 \le r \le n$$

Properties of Combinations

- Consider ${}^nC_{n-r} = \dfrac{n!}{(n-r)!\,[n-(n-r)]!} = \dfrac{n!}{(n-r)!\,r!} = {}^nC_r$

  Thus, ${}^nC_{n-r} = {}^nC_r$ for $0 \le r \le n$
- ${}^nC_0 = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{n!} = 1$ because $0! = 1$ as has been stated earlier.
- If ${}^nC_r = {}^nC_s$, then either $s = r$ or $s = n - r$
- ${}^nC_r = \dfrac{{}^nP_r}{r!}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_0 + {}^nC_1 + \dots {}^nC_n = 2^n$
- ${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{(n-1)}$
- ${}^nC_r = \left(\dfrac{n}{r}\right){}^{(n-1)}C_{(r-1)} = \left(\dfrac{n}{r}\right)\left(\dfrac{n-1}{r-1}\right){}^{(n-2)}C_{(r-2)} = \dots$
- ${}^nC_r$ has maximum value, if

  (a) $r = \dfrac{n}{2}$ when n is even.

  (b) $r = \dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$ when n is odd.