CHAPTER 10

Indefinite Integration

If a function f is differentiable in an interval I, i.e. its derivative f' exists at each point of I, then functions that could have possibly given function as a derivative are called anti-derivatives of the function.

The formula that gives all these anti-derivatives is called the indefinite integral of the function and such process of finding anti-derivatives is called the **integration** or **anti-differentiation**.

Two forms of integral are indefinite and definite integral which together constitute integral calculus.

Integration as an Inverse Process of Differentiation

Let $F(x)$ and $f(x)$ be two functions connected together such that $\frac{d}{dx}F(x) = f(x)$, then $F(x)$ is called **integral** or **indefinite integral** or **anti-derivative of $f(x)$**.

If $\frac{d}{dx}F(x) = f(x)$, then for any arbitrary constant C,

$$\frac{d}{dx}[F(x)+C] = f(x).$$

Thus, $F(x)+C$ is also an anti-derivative of $f(x)$.

Hence, $\int f(x)dx = F(x)+C$, where C is an arbitrary.

constant (also called constant of integration) and symbol '$\int$' indicates the sign of integration.

Here, the symbol '$\int$' is integral sign, $f(x)$ is integrand (the function whose integral is to be find), x is variable of integration and dx is element of integration or differentiation of x.

It is clear that for different values of C, we get different integrals. Thus, we get many integrals for given $f(x)$. Hence, integral of any given function is not unique.

Elementary Integration Formulae

	Derivatives	Indefinite integrals (Anti-derivatives)
1.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
2.	$\frac{d}{dx}(x) = 1$	$\int dx = x + C$
3.	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
4.	$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\,dx = -\cos x + C$
5.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
6.	$\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$	$\int \text{cosec}^2 x\,dx = -\cot x + C$
7.	$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\int \sec x \cdot \tan x\,dx = \sec x + C$
8.	$\frac{d}{dx}(-\text{cosec}\,x) = \text{cosec}\,x \cdot \cot x$	$\int \text{cosec}\,x \cdot \cot x\,dx = -\text{cosec}\,x + C$
9.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
10.	$\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
11.	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
12.	$\frac{d}{dx}(-\cot^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

Derivatives	Indefinite integrals (Anti-derivatives)
13. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
14. $\frac{d}{dx}(-\operatorname{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$
15. $\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
16. $\frac{d}{dx}(\log\|x\|) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \log\|x\| + C$
17. $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x, a > 0, a \neq 1$	$\int a^x\,dx = \frac{a^x}{\log a} + C, a > 0, a \neq 1$

Properties of Integration

- $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$
- $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$
- $\int kf(x)dx = k\int f(x)dx$, where k is a constant.
- $\int f(x)dx = g(x) + C \Rightarrow \int f(ax+b)\,dx = \frac{g(ax+b)}{a} + C$

Method of Integration

Some integrals are not in standard form, to reduce them into standard forms, we use the following methods

(i) Integration by substitution

(ii) Integration by parts

(iii) Integration by partial fraction

1. Integration by Substitution

For integral $\int f'\{g(x)\}g'(x)dx$, we create a new variable $t = g(x)$, so that $g'(x) = \frac{dt}{dx}$ or $g'(x)dx = dt$.

Hence, $\int f'\{g(x)\}g'(x)dx = \int f'(t)dt = f(t) + C$

$$= f\{g(x)\} + C$$

Note *(i)* $\int \{f(x)\}^n \cdot f'(x)dx = \frac{\{f(x)\}^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C, f(x) \neq 0$

Integrals of Trigonometric Function

Some standard formulae for integrals involving trigonometric functions are given below. These formulae are obtained by using substitution technique.

(i) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$

(ii) $\int \cot x\,dx = \log|\sin x| + C$

(iii) $\int \sec x\,dx = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$

Some Important Deductions

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$ and n is a rational number.

(ii) $\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a} + C$

(iii) $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + C$

(iv) $\int \tan(ax+b)dx = -\frac{1}{a}\log|\cos(ax+b)| + C$

$= \frac{1}{a}\log|\sec(ax+b)| + C$

(v) $\int \cot(ax+b)\,dx = \frac{1}{a}\log|\sin(ax+b)| + C$

(vi) $\int \sec(ax+b)\,dx = \frac{1}{a}\log|\sec(ax+b) + \tan(ax+b)| + C$

(vii) $\int \operatorname{cosec}(ax+b)\,dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + C$

(viii) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$

(ix) $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$

(x) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$

(xi) $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$

(xii) $\int e^{(ax+b)}dx = \frac{e^{(ax+b)}}{a} + C$

(xiii) $\int a^{mx+b}\,dx = \frac{a^{mx+b}}{m\log_e a} + C$

Some Special Integrals

Here, we discussed some standard formulae with their proof and the methods to solve some other standard integrals with the help of these formulae.

(i) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(ii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(iii) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(iv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x + \sqrt{x^2-a^2}| + C$

(v) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + C$

Some Standard Substitutions which are Useful in Evaluating Integrals

	Expression	Substitution
1.	a^2-x^2 or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $a\cos\theta$
2.	a^2+x^2 or $\sqrt{a^2+x^2}$	$x = a\tan\theta$ or $a\cot\theta$
3.	x^2-a^2 or $\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
4.	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
5.	$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
6.	$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
7.	$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$

Important Forms to be Converted into Special Integrals

(i) **Form I** $\int \frac{1}{ax^2+bx+c}dx$ or $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

Express ax^2+bx+c as sum or difference of two squares.
For this, write

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

(ii) **Form II** $\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Put $px+q = \lambda\cdot\frac{d}{dx}(ax^2+bx+c)+\mu$.

Now, find values of λ and μ and then integrate it.

(iii) **Form III** $\int \frac{dx}{a\sin^2 x + b\cos^2 x + c}$

To evaluate the above type of integrals, we proceed as follows

(a) Divide numerator and denominator by $\cos^2 x$.

(b) Replace $\sec^2 x$, if any in denominator by $1+\tan^2 x$.

(c) Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

(iv) **Form IV** $\int \frac{dx}{a\sin x + b\cos x + c}$

To evaluate the above type of integrals, we proceed as follows

(a) Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

(b) Replace $1+\tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$.

(c) Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

(v) **Form V** $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$,

To evaluate the above type of integrals, we proceed as follows

Write numerator

= λ (differentiation of denominator)

+ μ (denominator)

i.e. $a\sin x + b\cos x = \lambda(c\cos x - d\sin x) + \mu(c\sin x + d\cos x)$

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx = \lambda\int \frac{c\cos x - d\sin x}{c\sin x + d\cos x}dx + \mu\int \frac{c\sin x + d\cos x}{c\sin x + d\cos x}dx$$

$$= \lambda\log|c\sin x + d\cos x| + \mu x + C$$

2. Integration by Partial Fractions

If integrand is a rational function i.e. of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are the polynomial functions of x and $g(x) \neq 0$, then use method of partial fractions, if degree of $f(x)$ is less than the degree of $g(x)$. Otherwise first reduced to the proper rational function by long division process.

The following table indicates the types of simpler partial fractions that are associated with various kinds of rational functions.

Form of the rational function	Form of the partial fraction
$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$ where, x^2+bx+c cannot be factorised further.

Here, A, B and C are the real constants and these can be determined by reducing both sides of the equation as identity in polynomial form and by comparing the coefficients of like powers.

3. Integration by Parts

If integrand $f(x)$ can be expressed as product of two functions i.e. $f(x) = f_1(x) \cdot f_2(x)$, then we use the following formula:

$$\int f_1(x) \cdot f_2(x)dx = f_1(x)\int f_2(x)dx - \int f_1'(x)[\int f_2(x)dx]dx,$$

where $f_1(x)$ and $f_2(x)$ are known as first and second functions respectively.

i.e. The integral of the product of two functions
= (First function) × (Integral of the second function)
− Integral of {(Differentiation of first function)
× (Integral of second function)}

Important Points Related to Integration by Parts

- We do not put constant of integration in 1st integral. We put this only once at the end.
- Order of $f_1(x)$ and $f_2(x)$ is normally decided by the rule ILATE, where
  - I = Inverse function
  - L = Logarithmic function
  - A = Algebraic function
  - T = Trigonometric function
  - E = Exponential function
- Normally, we have to suppose function as a 2nd function, which can be integrate easily.
- If the integral contains a single logarithmic or single inverse trigonometric function, take unity as the second function.

e.g. $\int \underset{\text{II}}{1} \underset{\text{I}}{\log x}\, dx = x\log x - \int \frac{1}{x} \times x\, dx = x\log x - x + C$

- If the integrals of both the functions are known, the function which is easy to integrate is taken as the second function.
- In some cases, integration by parts will lead to a simple equation involving the integral. Solve the equation and determine the integral.

Integration of the Form $\int e^x \{g(x) + g'(x)\}\, dx$

If integrand is of the form $e^x\{g(x) + g'(x)\}$, then

$$\int e^x \cdot [g(x) + g'(x)]\, dx = e^x g(x) + C$$

Integral of the Form $\int e^{ax}\sin(bx+c)dx$ or $\int e^{ax}\cos(bx+c)dx$

(i) $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin(bx+c) - b\cos(bx+c)\} + k$

(ii) $\int e^{ax}\cos(bx+c)dx = \frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c) + b\sin(bx+c)\} + k$

Some More Special Integral based on Integration by Parts

(i) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log|x+\sqrt{x^2+a^2}|\right] + C$

(ii) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + C$

(iii) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2} - a^2\log|x+\sqrt{x^2-a^2}|\right] + C$