Part-I (Chapters from **Class 11th** Syllabus)

CHAPTER 01

Motion in a Plane

Motion

If the position of an object is continuously changing with respect to its surrounding with time, then it is said to be in the state of motion. A bird flying in air, a moving train, a ship sailing on water are the examples of motion.

Types of Motion

On the basis of the number of coordinates required to define the motion, it is classified as

(i) **One-dimensional Motion** The motion of an object is considered as 1-D, if only one coordinate is needed to specify the position of the object.

In 1-D motion, the object moves along a straight line.

e.g. A boy running on a straight road.

(ii) **Two-dimensional Motion** The motion of an object is considered as 2-D if two coordinates (x, y) are needed to specify the position of the object.

In 2-D motion, the object moves in a plane.

e.g. A satellite revolving around the earth.

(iii) **Three-dimensional Motion** The motion of an object is considered as 3-D, if all the three coordinates (x, y, z) are needed to specify the position of the object. This type of motion takes place in three-dimensional space.

e.g. Butterfly flying in garden, the motion of water molecules, etc.

Motion in a Plane

(Two Dimensional Motion)

Motion of an object is called two dimensional motion when two of the coordinates $(x\text{-}y, y\text{-}z$ or $z\text{-}x)$ from the three coordinates (x, y, z) are required to specify the position of the object in space, changes with time.

In two dimensional motion, the object moves in XY-plane, YZ-plane or ZX-plane, therefore it is called motion in a plane.

e.g. Projectile motion, circular motion, etc.

When an object moves in a plane or in two dimensional motion, different physical quantities related with it changes with time.

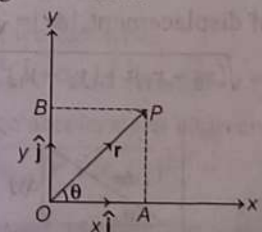
e.g. Position vector, displacement vector, velocity vector and acceleration vector.

In XY-plane, these physical quantities are studied in terms of their x and y-components.

Position Vector

A vector that extends from a reference point to the point at which particle is located is called position vector.

Let $\mathbf{r}$ be the position vector of a particle P located in a plane with reference to the origin O in XY-plane as shown in figure given below.



Representation of position vector

$$\mathbf{OP} = \mathbf{OA} + \mathbf{OB}$$

Position vector, $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$

Direction of this position vector $\mathbf{r}$ is given by the angle θ with X-axis, where, $\tan\theta = \left(\frac{y}{x}\right)$.

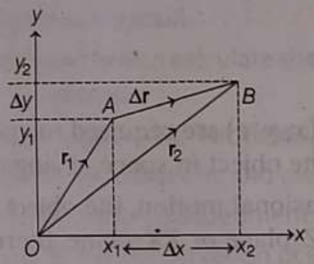
$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

In three dimensions, the position vector is represented as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Displacement Vector

Consider a particle moving in xy-plane with a uniform velocity $\mathbf{v}$ and point O as an origin for measuring time and position of the particle. Let the particle be at positions A and B at timings t_1 and t_2, respectively. The position vectors are $\mathbf{OA} = \mathbf{r}_1$ and $\mathbf{OB} = \mathbf{r}_2$.



Representation of displacement vector

Then, the displacement of the object in time interval $(t_2 - t_1)$ is AB. From triangle law of vector addition, we have

$$\mathrm{OA} + \mathrm{AB} = \mathrm{OB}$$

$$\Rightarrow \qquad \mathrm{AB} = \mathrm{OB} - \mathrm{OA}$$

$$\mathrm{AB} = \mathbf{r}_2 - \mathbf{r}_1 \qquad \text{...(i)}$$

If the coordinates of the particle at points A and B are (x_1, y_1) and (x_2, y_2), then

$$\therefore \qquad \mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$$

Substituting the values of $\mathbf{r}_1$ and $\mathbf{r}_2$ in Eq. (i), we have

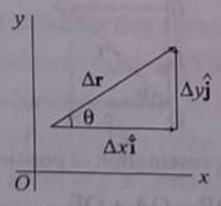
$$\mathrm{AB} = (x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}) - (x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}})$$

Displacement, $\mathrm{AB} = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\,\hat{\mathbf{j}}$

Displacement, $\Delta\mathbf{r} = \Delta x\hat{\mathbf{i}} + \Delta y\hat{\mathbf{j}}$

$\Rightarrow$ Magnitude of displacement, $|\Delta\mathbf{r}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Component of displacement

Direction of the displacement vector $\Delta\mathbf{r}$ is given by

$$\tan\theta = \frac{\Delta y}{\Delta x} \Rightarrow \theta = \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right)$$

where, θ = angle made by $\Delta\mathbf{r}$ with X-axis.

Similarly, in three dimensions the displacement can be represented as

$$\Delta\mathbf{r} = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\,\hat{\mathbf{j}} + (z_2 - z_1)\,\hat{\mathbf{k}}$$

Example 1. An object moves from position (3,4) to (6,5) in the xy-plane. The magnitude of displacement vector of the particle will be

a. $\sqrt{10}$ b. $\sqrt{20}$
c. $\sqrt{30}$ d. $\sqrt{5}$

Sol (*a*) Position vectors of the particle are

$$\mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} = 6\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$$

$\therefore$ Displacement vector, $\Delta\mathbf{r} = (x_2 - x_1)\,\hat{\mathbf{i}} + (y_2 - y_1)\,\hat{\mathbf{j}}$

$$= (6 - 3)\,\hat{\mathbf{i}} + (5 - 4)\,\hat{\mathbf{j}} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

$\therefore$ Magnitude of displacement vector, $|\Delta\mathbf{r}| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}$

Velocity Vector

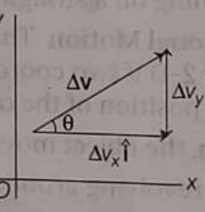
It is of two types, which are discussed below

Average Velocity

It is defined as the ratio of the displacement and the corresponding time interval.

Thus, average velocity $= \dfrac{\text{Displacement}}{\text{Time taken}}$

Average velocity, $\mathbf{v}_{\text{av}} = \dfrac{\Delta\mathbf{r}}{\Delta t} = \dfrac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$



Components of velocity

Velocity can be expressed in the component form as

$$\mathbf{v}_{\text{av}} = \frac{\Delta x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta y}{\Delta t}\hat{\mathbf{j}} = \Delta v_x\hat{\mathbf{i}} + \Delta v_y\hat{\mathbf{j}}$$

where, Δv_x and Δv_y are the components of average velocity along x-direction and y-direction, respectively.

The magnitude of $\mathbf{v}_{\text{av}}$ is given by

$$v_{\text{av}} = \sqrt{\Delta v_x^2 + \Delta v_y^2}$$

and the direction of $\mathbf{v}_{\text{av}}$ is given by $\tan\theta = \dfrac{\Delta v_y}{\Delta v_x}$

Example 2. A particle moves in XY-plane from position (1m, 2m) to (3m, 4m) in 2 s. The magnitude of average velocity will be

a. $\sqrt{3}\ \text{ms}^{-1}$ b. $\sqrt{5}\ \text{ms}^{-1}$

c. $\sqrt{2}\ \text{ms}^{-1}$ d. $\sqrt{8}\ \text{ms}^{-1}$

Sol (c) Given, positions of the particle are

$$\mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} \text{ and } \mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$$

Displacement, $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

$\therefore$ Average velocity, $\mathbf{v}_{av} = \dfrac{\Delta\mathbf{r}}{\Delta t} = \dfrac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}}{2} \Rightarrow \mathbf{v}_{av} = (\hat{\mathbf{i}} + \hat{\mathbf{j}})\ \text{ms}^{-1}$

$$\Rightarrow \quad |\mathbf{v}_{av}| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}\ \text{ms}^{-1}$$

Instantaneous Velocity

The velocity at an instant of time (t) is known as instantaneous velocity. The average velocity will become instantaneous, if Δt approaches to zero.

$\therefore$ Instantaneous velocity, $\mathbf{v} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \mathbf{r}}{\Delta t} = \dfrac{d\mathbf{r}}{dt}$

Now, we can write, $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$

$$\therefore \quad \mathbf{v} = \frac{dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} \Rightarrow \mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$$

Similarly in three dimensions, we can write

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

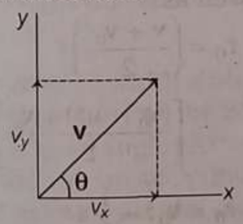
where, $v_x = \dfrac{dx}{dt}$ is magnitude of instantaneous velocity in x-direction,

$v_y = \dfrac{dy}{dt}$ is magnitude of instantaneous velocity in y-direction

and $v_z = \dfrac{dz}{dt}$ is magnitude of instantaneous velocity in z-direction.

Magnitude of instantaneous velocity, $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$

Angle θ made by $\mathbf{v}$ with X-axis



Direction of instantaneous velocity

$$\tan\theta = \frac{v_y}{v_x} \Rightarrow \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Example 3. Position vector of a particle is given as $\mathbf{r} = 2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}}$, where t is in seconds and the coefficients have the proper units, for r to be in metres.

The instantaneous velocity of the particle at $t = 2$s will be

a. $\sqrt{148}\ \text{ms}^{-1}$ b. $\sqrt{196}\ \text{ms}^{-1}$

c. $\sqrt{14}\ \text{ms}^{-1}$ d. $\sqrt{50}\ \text{ms}^{-1}$

Sol (a) Given, $\mathbf{r} = 2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}}$

Instantaneous velocity, $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt} = \dfrac{d}{dt}(2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}})$

$$\Rightarrow \quad \mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = 2\hat{\mathbf{i}} + 6t\hat{\mathbf{j}}$$

Magnitude of $\mathbf{v}(t)$, $|\mathbf{v}(t)| = \sqrt{v_x^2 + v_y^2} = \sqrt{(2)^2 + (6t)^2} = \sqrt{4 + 36t^2}$

At $t = 2$s, $|\mathbf{v}(t)| = \sqrt{4 + 36 \times 4} = \sqrt{148}\ \text{ms}^{-1}$

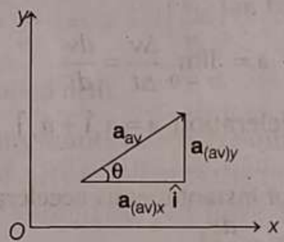
Acceleration Vector

It is of two types, which are discussed below

Average Acceleration

It is defined as the change in velocity ($\Delta\mathbf{v}$) divided by the corresponding time interval (Δt).

It can be expressed as



Components of acceleration

$$\text{Average acceleration}, \mathbf{a}_{av} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta v_x\hat{\mathbf{i}} + \Delta v_y\hat{\mathbf{j}}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}}$$

$$\text{Average acceleration} = \mathbf{a}_{(av)x}\hat{\mathbf{i}} + \mathbf{a}_{(av)y}\hat{\mathbf{j}}$$

which is expressed in component form.

Also, $\mathbf{a}_{(av)x} = \dfrac{\Delta v_x}{\Delta t}$ = average acceleration in x-direction

and $\mathbf{a}_{(av)y} = \dfrac{\Delta v_y}{\Delta t}$ = average acceleration in y-direction.

In three dimensions, we can write

$$\mathbf{a}_{av} = a_{(av)x}\hat{\mathbf{i}} + a_{(av)y}\hat{\mathbf{j}} + a_{(av)z}\hat{\mathbf{k}}$$

Magnitude of average acceleration is given by

$$|\mathbf{a}_{av}| = \sqrt{(a_{(av)x})^2 + (a_{(av)y})^2}$$

Angle θ made by average acceleration with X-axis,

$$\tan\theta = \frac{a_{(av)y}}{a_{(av)x}} \Rightarrow \theta = \tan^{-1}\left(\frac{a_{(av)y}}{a_{(av)x}}\right)$$

Example 4. Velocity of a particle changes from $(3\hat{i} + 4\hat{j})$ m/s to $(6\hat{i} + 5\hat{j})$ m/s is 2 s. The magnitude of average acceleration will be

a. $20\,\hat{i} + 3.0\,\hat{j}$ b. $3.0\,\hat{i} + 5.0\,\hat{j}$

c. $5.0\,\hat{i} + 4.0\,\hat{j}$ d. $1.5\,\hat{i} + 0.5\,\hat{j}$

Sol (*d*) Given, velocities of the particle,

$$\mathbf{v}_1 = 3\hat{i} + 4\hat{j} \quad \text{and} \quad \mathbf{v}_2 = 6\hat{i} + 5\hat{j}$$

Change in velocity, $\Delta \mathbf{v}_x = (v_{2x} - v_{1x})\hat{i} = (6-3)\hat{i} = 3\hat{i}$

$$\Delta \mathbf{v}_y = (v_{2y} - v_{1y})\hat{j} = (5-4)\hat{j} = \hat{j}$$

$\therefore$
$$\Delta \mathbf{v} = \Delta \mathbf{v}_x + \Delta \mathbf{v}_y = 3\hat{i} + \hat{j}$$

$\therefore$ Average acceleration,

$$\mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{3\hat{i} + \hat{j}}{2} \qquad (\because t = 2\text{ s})$$

$$= 1.5\hat{i} + 0.5\hat{j}$$

Instantaneous Acceleration

It is defined as the limiting value of the average acceleration as the time interval approaches to zero.

It can be expressed as

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

Instantaneous acceleration, $\mathbf{a} = a_x\hat{i} + a_y\hat{j}$

where,

a_x = magnitude of instantaneous acceleration in -direction $= \dfrac{dv_x}{dt}$

a_y = magnitude of instantaneous acceleration in y-direction $= \dfrac{dv_y}{dt}$

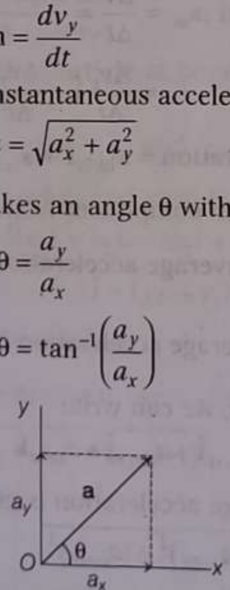
The magnitude of instantaneous acceleration is given by

$$a = \sqrt{a_x^2 + a_y^2}$$

If acceleration $\mathbf{a}$ makes an angle θ with X-axis, then

$$\tan\theta = \frac{a_y}{a_x}$$

$\Rightarrow$
$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right)$$



Direction of instantaneous acceleration

In three dimensions, we can write $\mathbf{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

Example 5. The position of a particle is given by

$$\mathbf{r} = 3t\hat{i} + 2t^2\hat{j} + 8\hat{k}$$

where, t is in seconds and the coefficients have the proper units for r to be in metres.

The magnitude of the acceleration will be

a. $4\,\hat{j}$ b. $5\,\hat{j}$

c. $6\,\hat{j}$ d. $8\,\hat{j}$

Sol (*a*) Position of particle, $\mathbf{r} = 3t\,\hat{i} + 2t^2\hat{j} + 8\,\hat{k}$

$\therefore$
$$\mathbf{v}(t) = \frac{d}{dt}(3t\,\hat{i} + 2t^2\,\hat{j} + 8\,\hat{k}) = 3\,\hat{i} + 4t\,\hat{j}$$

$$\mathbf{a}(t) = \frac{dv}{dt} = 4\,\hat{j}$$

Equations of Motion for an Object Travelling in a Plane with Uniform Acceleration

Let an object is moving in XY-plane and its acceleration $\mathbf{a}$ is constant. At time $t = 0$, the velocity of an object be $\mathbf{v}_0$ (say) and $\mathbf{v}$ be the velocity at time t.

According to definition of average acceleration, we have,

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v}_0}{t - 0} = \frac{\mathbf{v} - \mathbf{v}_0}{t}$$

$\Rightarrow$
$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t$$

In terms of rectangular components, we can express it as

$$v_x = v_{0x} + a_x t \quad \text{and} \quad v_y = v_{0y} + a_y t$$

Path of Particle under Constant Acceleration

Now, we can also find the position vector $(\mathbf{r})$. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of the particle at time $t = 0$ and $t = t$ and their velocities at these instants be $\mathbf{v}_0$ and $\mathbf{v}$, respectively. Then, the average velocity is given by

$$\mathbf{v}_{av} = \frac{\mathbf{v}_0 + \mathbf{v}}{2}$$

Displacement is the product of average velocity and time interval. It is expressed as

$$\mathbf{r} - \mathbf{r}_0 = \left(\frac{\mathbf{v} + \mathbf{v}_0}{2}\right)t$$

$$= \left[\frac{(\mathbf{v}_0 + \mathbf{a}t) + \mathbf{v}_0}{2}\right]t$$

$\Rightarrow$
$$\mathbf{r} - \mathbf{r}_0 = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

$\Rightarrow$
$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

In terms of rectangular components, we have

$$x\hat{i} + y\hat{j} = x_0\hat{i} + y_0\hat{j} + (v_{0x}\hat{i} + v_{0y}\hat{j}) \times t + \frac{1}{2}(a_x\hat{i} + a_y\hat{j})t^2$$

Now, equating the coefficients of $\hat{i}$ and $\hat{j}$,

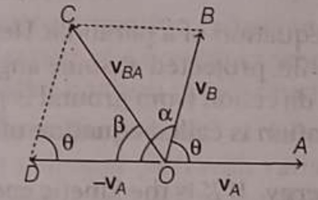
$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \text{ .... along } X\text{-axis}$$

and $$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \text{....... along } Y\text{-axis}$$

Relative Velocity in a Plane

The time rate of change of relative position of one object with respect to another is called relative velocity.

If $\mathbf{v}_B$ and $\mathbf{v}_A$ are inclined to each other at an angle θ, as shown in figure given below, then the magnitude of vector $\mathbf{v}_{BA}$, (*i.e.* $\mathbf{v}_{BA}$ is velocity of B w.r.t. A) is given by



Representation of relative velocity of $\mathbf{v}_A$ and $\mathbf{v}_B$

$$v_{BA} = [v_A^2 + v_B^2 - 2v_A v_B \cos\theta]^{1/2} \quad (\because \alpha = 180° - \theta)$$

If $\mathbf{v}_A$ and $\mathbf{v}_B$ are in the **same directions** and $\theta = 0°$, then

$$v_{BA} = v_B - v_A$$

If $\mathbf{v}_A$ and $\mathbf{v}_B$ are in the **opposite directions** and $\theta = 180°$, then

$$v_{BA} = v_B + v_A$$

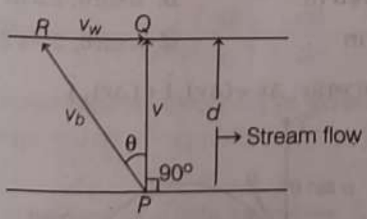
If $\mathbf{v}_{BA}$ makes an angle β with the direction of $-\mathbf{v}_A$, then

$$\tan\beta = \frac{\mathbf{v}_B \sin\theta}{\mathbf{v}_A - \mathbf{v}_B \cos\theta}$$

Applications Based on the Relative Velocity in a Plane

Some applications based on relative velocity are described below

(i) **To cross the river of width d along the shortest path** Shortest path is PQ, the boat must move along PR making an angle $(90° + \theta)$ with the direction of the stream such that the direction of the resultant velocity v is along PQ (see figure).

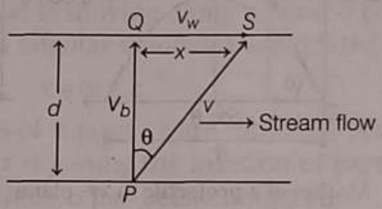


So, time taken to cross the river along the shortest path PQ is given by

$$t = \frac{d}{v} = \frac{d}{\sqrt{v_b^2 - v_w^2}}$$

where, v_w = speed of stream flow
and v_b = speed of the boat.

(ii) **To cross the river in the shortest time** The shortest time is given by $t = \dfrac{d}{v_b}$. At this time, boat will reach point S on the opposite bank of the river at a distance x from Q.

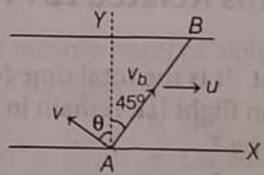


We can write, $x = d\tan\theta$

But $\tan\theta = \dfrac{v_w}{v_b} \Rightarrow x = d\left(\dfrac{v_w}{v_b}\right)$

x is also called **drift.**

Example 6. A boy wants to reach point B on the opposite bank of a river. What is the minimum speed relative to water should the boy have, so that he can reach point B?



a. $v_{\min} = \dfrac{u}{2\sqrt{2}}$ **b.** $v_{\min} = \dfrac{u}{\sqrt{6}}$ **c.** $v_{\min} = \dfrac{2u}{\sqrt{6}}$ **d.** $v_{\min} = \dfrac{u}{\sqrt{2}}$

Sol (*d*) Resultant of relative velocity of boy w.r.t. water v and velocity of stream u should be along AB. Component of v_b (absolute velocity of boy) along x and y-directions are $v_x = u - v\sin\theta$ and $v_y = v\cos\theta$.

$$\because \quad \tan 45° = \frac{v_x}{v_y}$$

$$\Rightarrow \quad 1 = \frac{u - v\sin\theta}{v\cos\theta}$$

$$v = \frac{u}{\sin\theta + \cos\theta} = \frac{u}{\sqrt{2}(\sin\theta\cos 45° + \cos\theta\sin 45°)}$$

$$v = \frac{u}{\sqrt{2}\sin(\theta + 45°)}$$

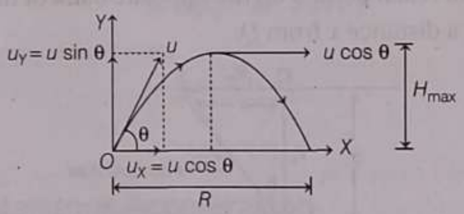
For v to be minimum, $\theta + 45° = 90° \Rightarrow \theta = 45°$

$$\therefore \quad v_{\min} = \frac{u}{\sqrt{2}\sin 90°} = \frac{u}{\sqrt{2}}$$

Projectile Motion

Projectile is the name given to a body thrown with some velocity (u) at an angle (θ) with the horizontal direction and then allowed to move in two-dimensions under the action of gravity only without being propelled by any engine or fuel. *e.g.* A bullet fired from a rifle, a tennis ball, etc.

Then, the path followed by the particle (or trajectory) is a parabola and the motion of the particle is called projectile motion.



Motion of a projectile in xy- plane

Assumptions

Assumptions made while considering parabolic path of a projectile are given below

(i) No frictional resistance of air.

(ii) Effect due to the rotation of the earth and curvature of the earth is considered as negligible.

(iii) Acceleration due to gravity is constant in magnitude and direction, at all points of the motion of projectile.

Important Terms Related to Projectile Motion

(i) **Time of flight** It is the total time for which the projectile is in flight (*i.e.* remain in air).
It is denoted by T.

$$T = \frac{2u\sin\theta}{g}$$

(ii) **Maximum height** It is the maximum vertical height attained by the projectile above the point of projection during its flight. It is denoted by H_{max}.

$$H_{max} = \frac{u^2\sin^2\theta}{2g}$$

(iii) **Horizontal range** It is the horizontal distance covered by the projectile between its points of projection and the point of hitting the ground.
It is denoted by R.

$$R = \frac{u^2\sin 2\theta}{g}$$

Horizontal range (R) will be maximum, if

$$\sin 2\theta = \text{maximum} = 1 = \sin 90^\circ$$

or $\quad 2\theta = 90^\circ$

$\Rightarrow \quad \theta = 45^\circ$

So, maximum horizontal range,

$$\Rightarrow \quad R_{max} = \frac{u^2}{g}$$

The maximum height attained by the projectile is equal to one-fourth of its maximum range.
($R_{max} = 4H_{max}$)

(iv) **Equations of motion for a projectile** Displacement of a projectile in horizontal direction is given by

$$x = u\cos\theta \cdot t$$

Displacement of a projectile in vertical direction is given by

$$y = x\tan\theta - \left(\frac{1}{2}\frac{g}{u^2\cos^2\theta}\right)x^2$$

This is an equation of a parabola. Hence, the path of a projectile, projected at some angle with the horizontal direction from ground is parabolic. The above equation is called **equation of trajectory.**

(v) **Kinetic energy** If K is the kinetic energy at the point of launch, then kinetic energy at the highest point is

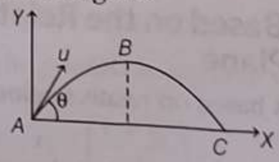
$$K' = \frac{1}{2}mv^2 = \frac{1}{2}mu^2\cos^2\theta \quad (\because v = u\cos\theta)$$

$$\Rightarrow \quad K' = K\cos^2\theta \qquad \left(\because K = \frac{1}{2}mu^2\right)$$

During projectile motion, total mechanical energy (PE + KE) of the projectile will remain constant (conserved).

Range of the projectile will be same for two angles of projections θ and $90^\circ - \theta$.

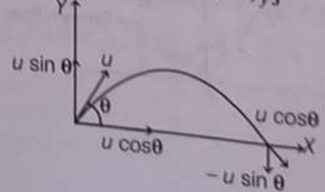
Example 7. A projectile is thrown at an angle θ with a speed u is as shown in figure.



The magnitude of change in velocity between the points A and C and the value of x-component of displacement, when the value of y-component of displacement is H_{max} are respectively (Take, $u = 10$ m/s and $\theta = 30^\circ$)

a. $2u\sin\theta$, $2.5\sqrt{3}$ m
b. $u\sin\theta$, 2.5 m
c. $2v\sin\theta$, $\sqrt{3}$ m
d. $u\sin\theta$, $2.5\sqrt{2}$ m

Sol (*a*) We can write, $\Delta v = (\Delta v)_x \hat{i} + (\Delta v)_y \hat{j}$



As, $(\Delta v)_x = 0$

$(\Delta v)_y = -u\sin\theta - u\sin\theta = -2u\sin\theta$

$|(\Delta v)_y| = 2u\sin\theta \Rightarrow |\Delta v| = |\Delta v_y| = 2u\sin\theta$

When $s_y = H_{max}$, then $s_x = R/2$

So, $s_x = \frac{R}{2} = \frac{u^2\sin 2\theta}{2g} = \frac{100\times\sqrt{3}}{2\times2\times10} = 2.5\sqrt{3}$ m

Circular Motion

Motion is the change in position of an object with time. When a particle moves in a circular path, then its motion is said to be **circular motion**. Circular motion is two dimensional motion.

Circular motion are of following two types

(i) **Uniform circular motion** When an object is moving in a circular path at a constant speed, then motion of object is called uniform circular motion.

(ii) **Non-uniform circular motion** When an object is moving in a circular path with variable speed, then motion of object is called non-uniform circular motion.

Terms Related to Circular Motion

The important terms used in circular motion are given as

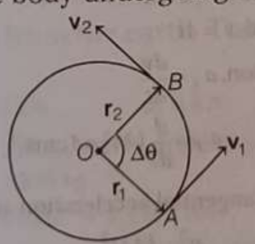
Angular Displacement

It is defined as the angle turned by the particle from some reference line. Angular displacement $\Delta\theta$ is usually measured in radians. It is dimensionless quantity.

Finite angular displacement $\Delta\theta$ is a scalar but an infinitesimally displacement is a vector. Since, it does not obey the commutative law of vector addition.

Angular Velocity

It is defined as the rate of change of the angular displacement of the body undergoing circular motion.



A body executing circular motion

$\therefore$ Angular velocity, $\omega = \lim_{\Delta t \to 0}\left(\frac{\Delta\theta}{\Delta t}\right) = \frac{d\theta}{dt}$

It is an axial vector whose direction is given by the right hand rule. Its unit is rad s^{-1} and dimension is $[T^{-1}]$.

Note

- *If a body complete one revolution in time T, then its angular velocity,* $\omega = 2\pi/T$
- *If a body complete n revolution in one second, then its angular velocity,* $\omega = 2\pi n$

Example 8. The angular velocity of the earth will be

a. $\frac{2\pi}{86400}$ rad s^{-1} b. $\frac{3\pi}{86400}$ rad s^{-1}

c. $\frac{\pi}{86400}$ rad s^{-1} d. $\frac{2\pi}{43200}$ rad s^{-1}

Sol (*a*) For earth, $T = 24$ h

$\Rightarrow \quad T = 24\times60\times60 = 86400$ s

$\therefore$ Angular velocity of the earth,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86400}\text{ rad s}^{-1}$$

Relation between Linear and Angular Velocities

When a particle is moving along a curved path, then its tangential and angular velocities are related by

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$$

The directions of **v**, ω and **r** are mutually perpendicular, where vector **r** is joining the location of particle and the point about which ω has been computed.

Example 9. A particle moves in a circular path of radius 0.5 m with a linear speed of 2 ms^{-1}, its angular speed is

a. 2 rad s^{-1} b. 3 rad s^{-1}

c. 4 rad s^{-1} d. None of these

Sol (*c*) Given, radius, $r = 0.5$ m

and linear speed, $v = 2$ ms^{-1}

Angular speed, $\omega = \frac{v}{r} = \frac{2}{0.5} = 4$ rad s^{-1}

Example 10. The minute hand of a clock is 10 cm long. The linear speed of its tip is

a. 1.643×10^{-4} ms^{-1} b. 1.876×10^{-4} ms^{-1}

c. 1.744×10^{-4} ms^{-1} d. 1.502×10^{-4} ms^{-1}

Sol (*c*) The tip of the minute hand completes 1 revolution in 1 h.

$\therefore \quad T = 1\text{ h} = 60\text{ min} = 60\times60$ s and $r = 10$ cm $= 0.1$ m

So, linear speed, $v = r\omega = r\times\frac{2\pi}{T}$ $\left(\because \omega = \frac{2\pi}{T}\right)$

$$= \frac{0.1\times2\times3.14}{60\times60}$$

$$= 1.744\times10^{-4}\text{ ms}^{-1}$$

Angular Acceleration

It is defined as the rate of change of angular velocity of a body. Let ω_1 and ω_2 be the instantaneous angular speeds. At times t_1 and t_2 respectively, then the average angular acceleration is defined as

$$\alpha = \frac{\omega_2-\omega_1}{t_2-t_1} = \frac{\Delta\omega}{\Delta t}$$

It is a vector quantity. Its unit is rad s^{-2} and dimensional formula is $[T^{-2}]$.

Example 11. A grindstone is found to have a angular speed of 150 rpm, 10 s after starting from rest. The angular acceleration of the grindstone is

a. $\frac{\pi}{3}$ rads^{-2} b. $\frac{\pi}{2}$ rads^{-2} c. $\frac{\pi}{6}$ rads^{-2} d. $\frac{\pi}{5}$ rads^{-2}

Sol (b) $\omega_1 = 0$, starting from the rest.

$$\omega_2 = \frac{2\pi \times 150}{60} \text{ rad s}^{-1} = 5\pi \text{ rad s}^{-1}$$

$\because$ Time, $\Delta t = 10$ s

$\therefore$ Angular acceleration, $\alpha = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{5\pi - 0}{10} = \frac{\pi}{2}$ rads^{-2}

Instantaneous Angular Acceleration

The angular acceleration of a body at a given instant of time or at a given point of its motion is called its instantaneous angular acceleration.

Thus, $\alpha_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \Rightarrow \alpha_{\text{inst}} = \frac{d^2\theta}{dt^2} \quad \left(\because \omega = \frac{d\theta}{dt}\right)$

Example 12. The angular velocity of a particle is given by the equation $\omega = (2t^2 + 5)$ rads^{-1}. The instantaneous angular acceleration at $t = 4$ s is

a. 16 rad s^{-2} b. 19 rad s^{-2} c. 10 rad s^{-2} d. 12 rad s^{-2}

Sol (a) Given, $\omega = 2t^2 + 5$

We know that, $\alpha_{\text{inst}} = \frac{d}{dt}\omega$

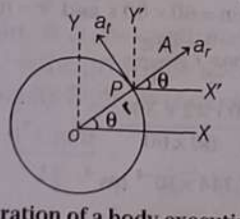
On putting the value of ω, we get

$\therefore \quad \alpha_{\text{inst}} = \frac{d}{dt}(2t^2 + 5) = 4t + 0$

At $t = t = 4$ s, $\alpha_{\text{inst}} = 4 \times 4 = 16$ rad s^{-2}

Components of Acceleration in Circular Motion

Acceleration of a particle in circular motion has following components



Two kinds of acceleration of a body executing circular motion

(i) **Tangential Acceleration** (a_t) It is the component of acceleration a in the direction of velocity.

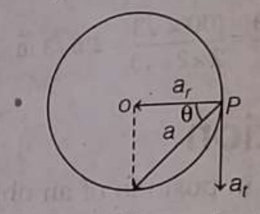
a_t = component of a along $\mathbf{v} = \frac{dv}{dt} = \frac{d|\mathbf{v}|}{dt}$

(ii) **Radial Acceleration** (a_r) It is the component of a, towards the centre of the circular motion. This is responsible for a change in the direction of velocity.

$$a_r = \frac{v^2}{r} = r\omega^2 = 4\pi^2 n^2 r \quad (\because \omega = 2\pi n)$$

Radial acceleration is also called **centripetal acceleration**.

As, a_t and a_r are perpendicular to each other as shown in the figure



Resultant of the acceleration

$\therefore$ Net acceleration, $a = \sqrt{a_t^2 + a_r^2}$; $\tan\theta = \frac{a_t}{a_r}$

If a particle is moving on a circular path with a constant speed, then $a_t = \frac{dv}{dt} = 0$ but $a_r \neq 0$.

Example 13. A particle is moving on a circular path of radius 0.3 m and rotating at 1200 rpm. The centripetal acceleration of the particle

a. 4732.60 ms^{-2} b. 47.3260 ms^{-2}
c. 4.73260 ms^{-2} d. 473.260 ms^{-2}

Sol (a) Given, angular speed, $\omega = \frac{2\pi \times 1200}{60}$ rads^{-1}

$= 40\pi$ rads^{-1}

Now, centripetal acceleration,

$$a = r\omega^2 = 0.3 \times (40\pi)^2 = 4732.60 \text{ ms}^{-2}$$

Example 14. A particle moves in a circle of radius 2 cm at a speed given by $v = 4t$, where v is in cms^{-1} and t is in second. The tangential acceleration at $t = 1$ s and total acceleration at $t = 1$ s are respectively

a. 6 cms^{-2} and 5 cms^{-2} b. 4 cms^{-2} and $4\sqrt{5}$ cms^{-2}
c. 5 cms^{-2} and $5\sqrt{5}$ cms^{-2} d. None of these

Sol (b) Given, radius, $r = 2$ cm

and speed, $v = 4t$

Tangential acceleration, $a_t = \frac{dv}{dt}$

or $a_t = \frac{d}{dt}(4t) = 4$ cms^{-2}

i.e. a_t is constant or tangential acceleration at $t = 1$ s is 4 cms^{-2}.

Radial acceleration, $a_r = \frac{v^2}{r} = \frac{(4t)^2}{r}$

or $a_r = \frac{16t^2}{2.0} = 8.0t^2$

At $t = 1$ s, $a_r = 8.0$ cms^{-2}

$\therefore$ Total acceleration, $a = \sqrt{a_t^2 + a_r^2}$

or $\Rightarrow a = \sqrt{(4)^2 + (8)^2} = \sqrt{80}$

$a = 4\sqrt{5}$ cms^{-2}

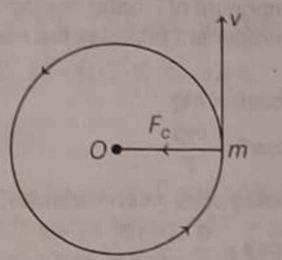
Force in Circular Motion

When a body executes circular motion, the force acting on it is as follows

Centripetal Force

The centripetal force is the force required to move a body along a circular path with a constant speed. Centripetal force never acts by itself. It is to be provided by some agency in order to maintain the uniform circular motion of an object.

The direction of the centripetal force is along the radius, acting towards the centre of the circle, on which the given body is moving.



Centripetal force in case of circular motion

Centripetal force, $F = \frac{mv^2}{r} = mr\omega^2$

$$F = mr\,4\pi^2\nu^2 = mr\frac{4\pi^2}{T^2}$$

where, m = mass of body,
v = speed of the body,
r = radius of circular path
and ω = angular speed.

It is a vector quantity. Its unit is Newton and the dimensional formula is $[MLT^{-2}]$.

Example 15. An artificial satellite of mass 2500 kg is orbiting around the earth with a speed of 4 kms^{-1} at a distance of 10^4 km from the earth. The centripetal force acting on it is

a. 4 kN b. 6 kN c. 8 kN d. 2 kN

Sol (a) Given, $r = 10^4\text{ km} = 10^4 \times 1000\text{ m} = 10^7\text{ m}$,
$m = 2500$ kg
and $v = 4\text{ kms}^{-1} = 4 \times 10^3\text{ ms}^{-1}$

Now, centripetal force, $F = \frac{mv^2}{r} = \frac{2500 \times (4 \times 10^3)^2}{10^7}$

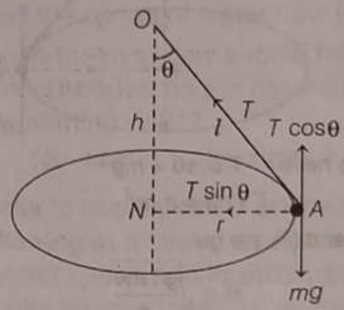
$$= \frac{2500 \times 16 \times 10^6}{10^7} = \frac{250 \times 16 \times 10^7}{10^7}$$

$\Rightarrow$ $F = 4000\text{ N} = 4\text{ kN}$

Conical Pendulum

Conical pendulum is a simple pendulum, which is given with such a motion that bob describes a horizontal circle and the string describes a cone. It consists of a string OA, whose upper end O is fixed and bob is tied at the other free end A.

Let the bob is moving in a horizontal circle of radius r with an uniform angular velocity ω in such a way that the string always makes an angle θ with the vertical. As the string traces the surface of the cone, the arrangement is called a **conical pendulum**.



Forces on a conical pendulum executing circular motion

Let T be the tension in the string of length l and r be the radius of circular path. The vertical component of tension T balances the weight of the bob and horizontal component provides the necessary centripetal force.

Thus, $T\cos\theta = mg$ ...(i)

and $T\sin\theta = mr\omega^2$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get

$$\tan\theta = \frac{r\omega^2}{g}$$

i.e., $\omega = \sqrt{\frac{g\tan\theta}{r}}$ ...(iii)

But, $r = l\sin\theta$ and $\omega = \frac{2\pi}{T}$

where, T being the period of completing one revolution,

$\therefore$ $\frac{2\pi}{T} = \sqrt{\frac{g\tan\theta}{l\sin\theta}}$

This gives, $T = 2\pi\sqrt{\frac{l\sin\theta}{g(\sin\theta/\cos\theta)}}$

or Period, $T = 2\pi\sqrt{\frac{l\cos\theta}{g}} = 2\pi\sqrt{\left(\frac{h}{g}\right)}$

where, $h = l\cos\theta$

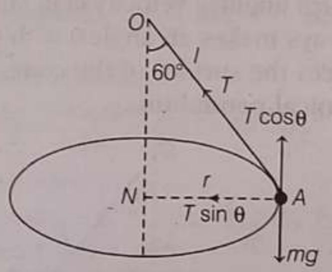
Example 16. Consider a bob whose upper end is fixed with length of 2 m is given with a horizontal push through angular displacement of 60°. What is the angular velocity of the bob, if it is of 2 kg? (Take, $g = 10\text{ ms}^{-2}$)

a. $\sqrt{10}$ rad s^{-1} b. $\sqrt{20}$ rad s^{-1}
c. $\sqrt{8}$ rad s^{-1} d. None of these

Sol. (*a*) Given, mass, $m = 2\,\text{kg}$

angular displacement, $\theta = 60^\circ$

length, $l = 2\,\text{m}$ and $g = 10\,\text{ms}^{-2}$



From FBD, we have $\quad T\cos\theta = mg \quad$ ...(i)

$$T\sin\theta = mr\omega^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\omega = \sqrt{\frac{g\tan\theta}{r}}$$

Also, $\quad r = l\sin\theta$

$$\therefore \quad \omega = \sqrt{\frac{g\sin\theta}{\cos\theta \times l\sin\theta}}$$

$$\omega = \sqrt{\frac{g}{l\cos\theta}} = \sqrt{\frac{10}{2\times\cos 60^\circ}} = \sqrt{\frac{10}{1}}$$

$$\Rightarrow \quad \omega = \sqrt{10}\,\text{rads}^{-1}$$

Example 17. Length of a simple pendulum is 2 m and mass of its bob is 0.2 kg. If the tension in the string exceeds 4 N, it wll break. If $g = 10\,\text{ms}^{-2}$ and the bob is whirled in a horizontal plane, the maximum angle through which the string can make with vertical durin rotation is

a. 40° b. 60°

c. 50° d. 180°

Sol. (*b*) Given, tension in the string,

$$T_{\max} = 4\text{N}$$

Length of pendulum, $l = 2\text{m}$

and mass of the bob $= 0.2\,\text{kg}$

The vertical component of T balances the height of the bob and the horizontal component provides the necessary centripetal force. Thus,

$$T\cos\theta = mg$$

$$\therefore \quad \cos\theta = \frac{mg}{T}$$

On putting the value of all given variables, we get

$$\cos\theta = \frac{0.2\times 10}{4} = \frac{1}{2}$$

$$\therefore \quad \theta = 60^\circ$$