LIVE ONLINE TUTORING

Revision Notes

Class 11 Maths

Chapter 3 – Trigonometric Functions

TRIGONOMETRIC RATIOS & IDENTITIES

1. The meaning of Trigonometry

Tri Gon Metron

$\downarrow$ $\downarrow$ $\downarrow$

3 sides Measure

As a result, this area of mathematics was established in the ancient past to measure a triangle's three sides, three angles, and six components. Time-trigonometric functions are utilised in a variety of ways nowadays. The sine and cosine of an angle in a right-angled triangle are the two fundamental functions, and there are four more derivative functions.

2. Basic Trigonometric Identities

(a) $\sin^2\theta+\cos^2\theta=1:-1\leqslant\sin\theta\leqslant1;-1\leqslant\cos\theta\leqslant1\,\forall\theta\in R$

(b) $\sec^2\theta-\tan^2\theta=1:|\sec\theta|\geqslant1\,\forall\theta\in R$

(c) $\operatorname{cosec}^2\theta-\cot^2\theta=1:|\operatorname{cosec}\theta|\geqslant1\,\forall\theta\in R$

Trigonometric Ratios of Standard Angles

Angles(In Degrees)	0°	30°	45°	60°	90°	180°	270°	360°

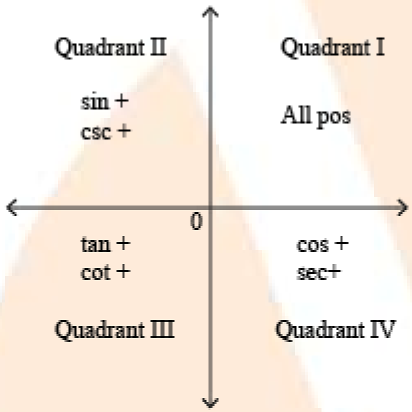
Angles(In radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin	**0**	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	**1**	**0**	**-1**	**0**
Cos	**1**	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	**0**	**-1**	**0**	**1**
Tan	**0**	$\frac{1}{\sqrt{3}}$	**1**	$\sqrt{3}$	**Not Defined**	**0**	**Not Defined**	**1**
Cot	**Not Defined**	$\sqrt{3}$	**1**	$\frac{1}{\sqrt{3}}$	**0**	**Not Defined**	**0**	**Not Defined**
Csc	**Not Defined**	**2**	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	**1**	**Not Defined**	**-1**	**Not Defined**
Sec	**1**	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	**2**	**Not Defined**	**-1**	**Not Defined**	**1**

The relation between these trigonometric identities with the sides of the triangles can be given as follows:

- Sine (theta) = Opposite/Hypotenuse
- Cos (theta) = Adjacent/Hypotenuse
- Tan (theta) = Opposite/Adjacent

- Cot (theta) = Adjacent/Opposite
- Cosec (theta) = Hypotenuse/Opposite
- Sec (theta) = Hypotenuse/Adjacent

The following are the signs of trigonometric ratios in different quadrants:



3. Trigonometric Ratios of Allied Angles

We might calculate the trigonometric ratios of angles of any value using the trigonometric ratio of allied angles.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \textit{ and } \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$\tan(-\theta) = -\tan\theta \textit{ and } \cot(-\theta) = -\cot\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta \textit{ and } \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$

$$\text{cosec}(-\theta) = -\text{cosec}\,\theta \textit{ and } \sec(-\theta) = \sec\theta$$

$$\sec\left(\frac{\pi}{2}-\theta\right) = \text{cosec}\,\theta \textit{ and } \text{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta$$

$$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta \quad and \quad \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$$

$$\sin(\pi-\theta)=\sin\theta \quad and \quad \cos(\pi-\theta)=-\cos\theta$$

$$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta \quad and \quad \cot\left(\frac{\pi}{2}+\theta\right)=-\tan\theta$$

$$\tan(\pi-\theta)=-\tan\theta \quad and \quad \cot(\pi-\theta)=-\cot\theta$$

$$\sec\left(\frac{\pi}{2}+\theta\right)=-\operatorname{cosec}\theta \quad and \quad \operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)=\sec\theta$$

$$\sec(\pi-\theta)=-\sec\theta \quad and \quad \operatorname{cosec}(\pi-\theta)=\operatorname{cosec}\theta$$

$$\sin(\pi+\theta)=-\sin\theta \quad and \quad \cos(\pi+\theta)=-\cos\theta$$

$$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta \quad and \quad \cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$$

$$\tan(\pi+\theta)=\tan\theta \quad and \quad \cot(\pi+\theta)=\cot\theta$$

$$\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta \quad and \quad \cot\left(\frac{3\pi}{2}-\theta\right)=\tan\theta$$

$$\sec(\pi+\theta)=-\sec\theta \quad and \quad \operatorname{cosec}(\pi+\theta)=-\operatorname{cosec}\theta$$

$$\sec\left(\frac{3\pi}{2}-\theta\right)=-\operatorname{cosec}\theta \quad and \quad \operatorname{cosec}\left(\frac{3\pi}{2}-\theta\right)=-\sec\theta$$

$$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta \quad and \quad \cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$$

$$\sin(2\pi-\theta)=-\sin\theta \quad and \quad \cos(2\pi-\theta)=\cos\theta$$

$$\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta \quad and \quad \cot\left(\frac{3\pi}{2}+\theta\right)=-\tan\theta$$

$$\tan(2\pi-\theta)=-\tan\theta \quad and \quad \cot(2\pi-\theta)=-\cot\theta$$

$$\sec\left(\frac{3\pi}{2}+\theta\right)=\operatorname{cosec}\theta \quad and \quad \operatorname{cosec}\left(\frac{3\pi}{2}+\theta\right)=-\sec\theta$$

$\sec(2\pi-\theta)=\sec\theta$ *and* $\mathrm{cosec}(2\pi-\theta)=-\mathrm{cosec}\,\theta$

$\sin(2\pi+\theta)=\sin\theta$ *and* $\cos(2\pi+\theta)=\cos\theta$

$\tan(2\pi+\theta)=\tan\theta$ *and* $\cot(2\pi+\theta)=\cot\theta$

$\sec(2\pi+\theta)=\sec\theta$ *and* $\mathrm{cosec}(2\pi+\theta)=\mathrm{cosec}\,\theta$

4. Trigonometric Functions of Sum or Difference of Two Angles

(a) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(b) $\sin(A-B)=\sin A\cos B-\cos A\sin B$

(c) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

(d) $\cos(A-B)=\cos A\cos B+\sin A\sin B$

(e) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

(f) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

(g) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$

(f) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

(h) $\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A=\sin(A+B)\cdot\sin(A-B)$

(i) $\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A=\cos(A+B)\cdot\cos(A-B)$

(j) $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

5. Multiple Angles and Half Angles

(a) $\sin 2A = 2\sin A\cos A;\quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$2\cos^2\frac{\theta}{2} = 1 + \cos\theta, 2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

(c) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}; \tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(d) $\sin 2A = \frac{2\tan A}{1-\tan^2 A}; \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

6. Transformation of Products into Sum or Difference of Sines & Cosines

(a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(b) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(d) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

7. Factorisation of the Sum or Difference of Two Sines or Cosines

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(b) $\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(d) $\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

8. Important Trigonometric Ratios

(a) $\sin n\pi=0;\cos n\pi=(-1)^n;\tan n\pi=0$ where $n\in Z$

(b) $\sin 15^\circ$ or $\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}=\cos 75^\circ$ or $\cos\frac{5\pi}{12}$;

$\cos 15^\circ$ or $\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}=\sin 75^\circ$ or $\sin\frac{5\pi}{12}$

$\tan 15^\circ=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}=\cot 75^\circ$

$\tan 75^\circ=\frac{\sqrt{3}+1}{\sqrt{3}-1}=2+\sqrt{3}=\cot 15^\circ$

(c) $\sin\frac{\pi}{10}$ or $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$ & $\cos 36^\circ$ or $\cos\frac{\pi}{5}=\frac{\sqrt{5}+1}{4}$

9. Conditional Identities

If $A+B+C=\pi$ then :

(i) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(ii) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(iii) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

10. Range of Trigonometric Expression

$E = a\sin\theta + b\cos\theta$

$E = \sqrt{a^2 + b^2}\sin(\theta + \alpha), \left(\text{ where } \tan\alpha = \frac{b}{a}\right)$

$E = \sqrt{a^2 + b^2}\cos(\theta - \beta), \left(\text{ where } \tan\beta = \frac{a}{b}\right)$

Hence for any real value of $\theta, -\sqrt{a^2 + b^2} \leqslant E \leqslant \sqrt{a^2 + b^2}$

The trigonometric functions are very important for studying triangles, light, sound or wave. The values of these trigonometric functions in different domains and ranges can be used from the following table:

Trigonometric Functions	**Domain**	**Range**
Sin x	R	$-1 \le \sin x \le 1$
Cos x	R	$-1 \le \cos x \le 1$
Tan x	$R - \{(2n+1)\pi/2, n \in I$	R

Cosec x	$R-\{(n\pi), n\in I$	$R-\{x:-1<x<1\}$
Sec x	$R-\{(2n+1)\pi/2, n\in I$	$R-\{x:-1<x<1\}$
Cot x	$R-\{(n\pi), n\in I$	R

11. Sine and Cosine Series

(a) $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots.+\sin(\alpha+\overline{n-1}\beta)$

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{n-1}{2}\beta\right)$$

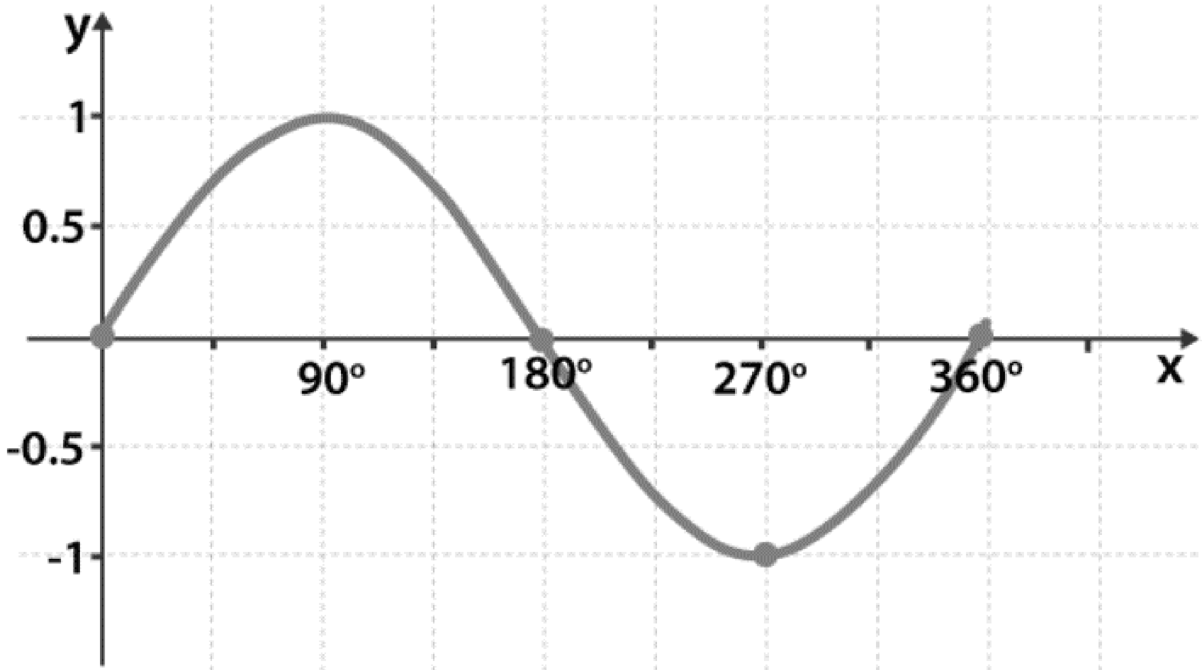
(b) $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos(\alpha+\overline{n-1}\beta)$

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha+\frac{n-1}{2}\beta\right)$$

12. Graphs of Trigonometric Functions

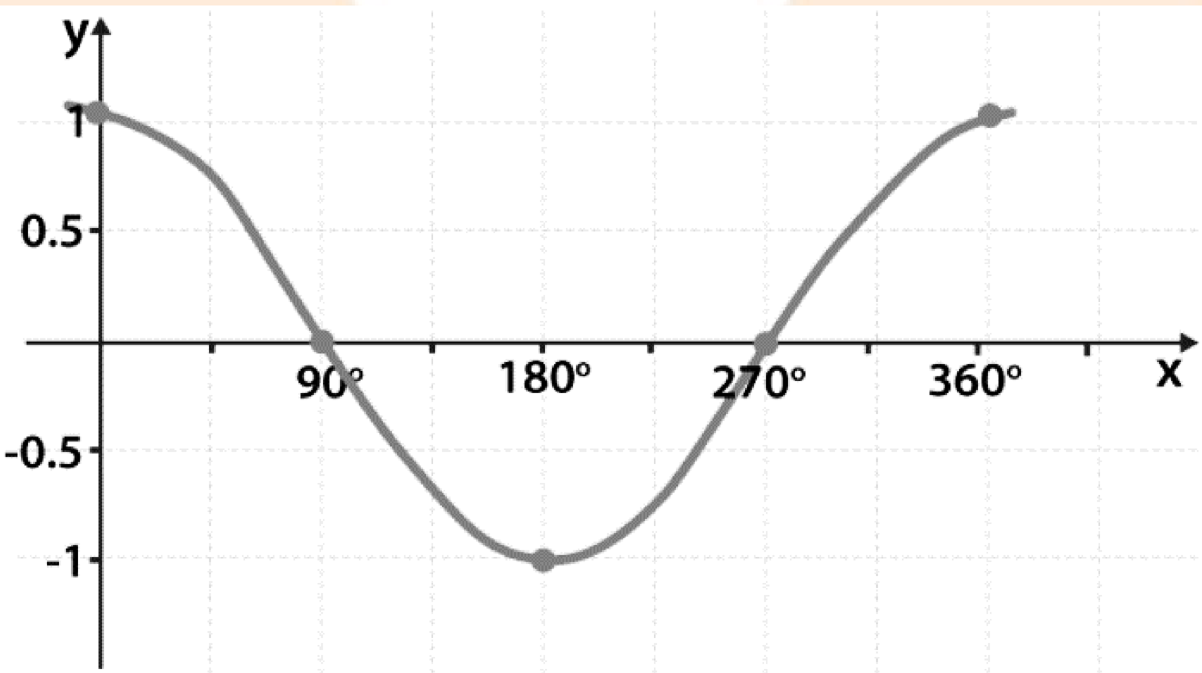
a. $y=\sin x,$

$x\in R; y\in[-1,1]$



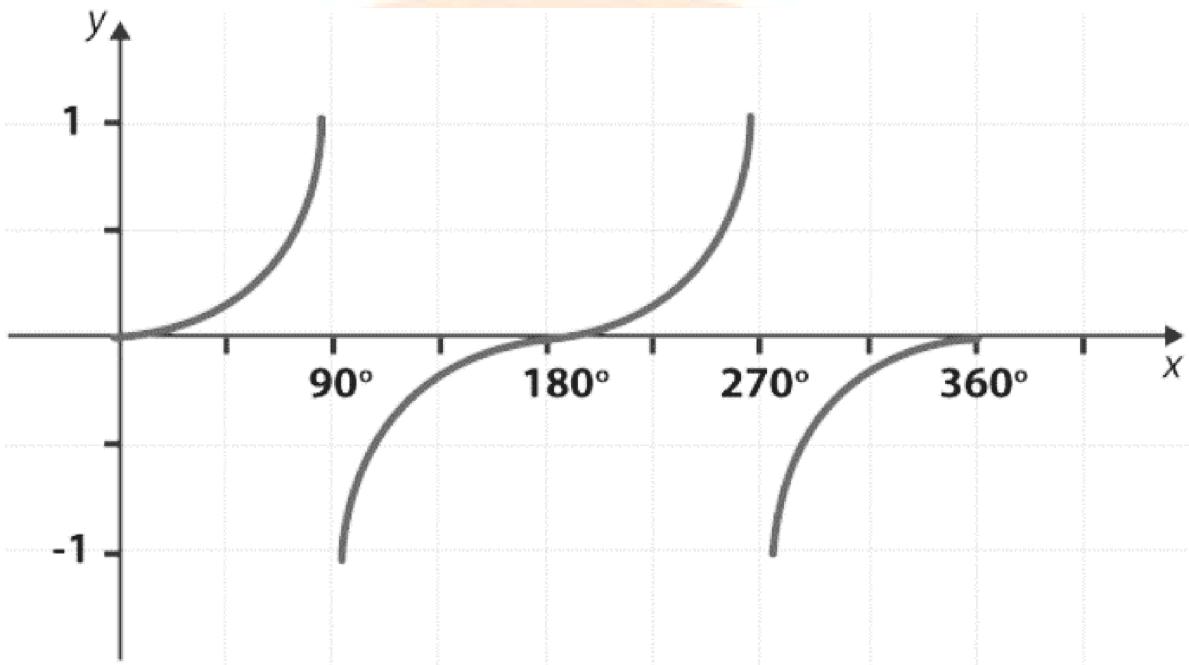
b. $y = \cos x$

$x \in R; y \in [-1,1]$



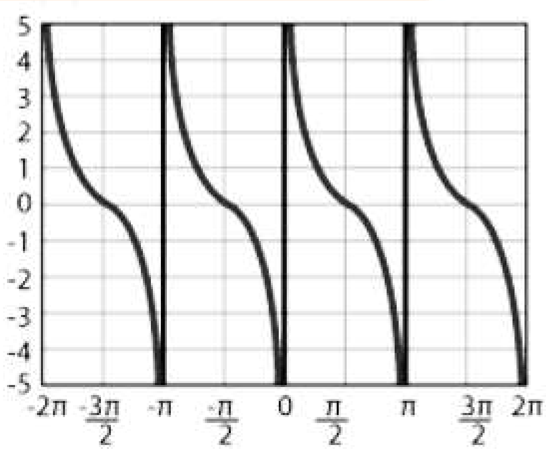
(c) $y = \tan x$

$$x \in R - \left\{ (2n+1)\frac{\pi}{2}; n \in Z \right\}; y \in R$$



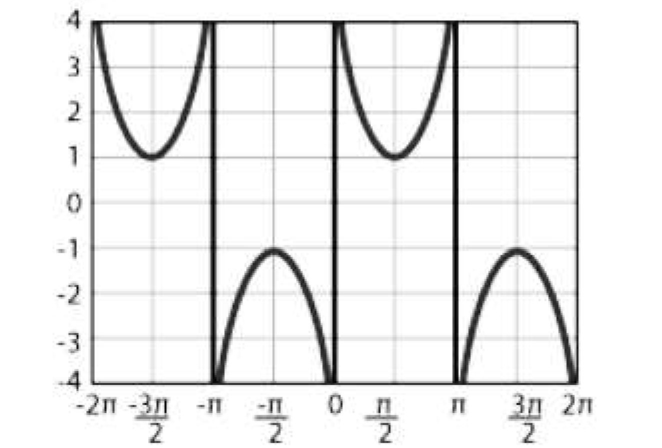
(d) $y = \cot x$

$$x \in R - \{n\pi; n \in z\}; y \in R$$



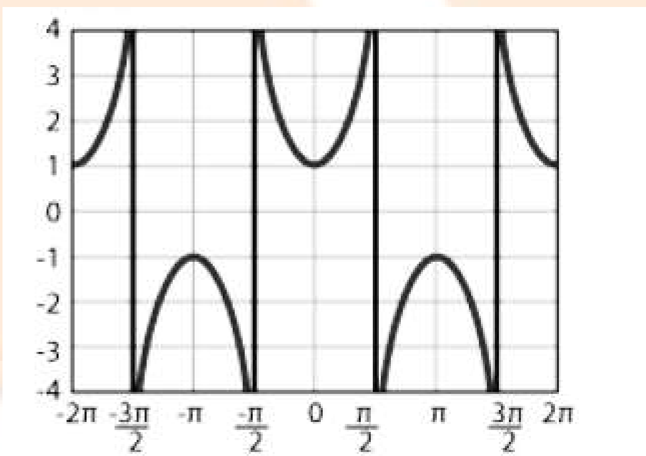
(e) $y = \operatorname{cosec} x$

$$x \in R - \{n\pi; n \in Z\}; y \in (-\infty, -1] \cup [1, \infty)$$



(f) $y = \sec x$

$x \in R - \left\{(2n+1)\dfrac{\pi}{2}; n \in Z\right\}; y \in (-\infty, -1] \cup [1, \infty)$



TRIGONOMETRIC EQUATIONS

13. Trigonometric Equations

Trigonometric equations are equations using trigonometric functions with unknown angles.

e.g., $\cos\theta = 0, \cos^2\theta - 4\cos\theta = 1$.

The value of the unknown angle that satisfies a trigonometric equation is called a solution.

e.g., $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$ or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots$

As a result, the trigonometric equation can have an unlimited number of solutions and is categorised as follows:

Principal solution

As we know, the values of $\sin x$ and $\cos x$ will get repeated after an interval of 2π. In the same way, the values of $\tan x$ will get repeated after an interval of π.

So, if the equation has a variable $0 \le x < 2\pi$, then the solutions will be termed as principal solutions.

Example:

Find the principal solutions of the equation $\sin x = \frac{\sqrt{3}}{2}$.

Solution: We know that, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Also, $\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right)$

Now, we know that $\sin(\pi - x) = \sin x$.

Hence, $\sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Therefore, the principal solutions of $\sin x = \frac{\sqrt{3}}{2}$ are $x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

General solution

A general solution is one that involves the integer 'n' and yields all trigonometric equation solutions. Also, the character ' Z ' is used to denote the set of integers.

Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.

Solution: We know that $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$. Therefore, $\sin x=-\frac{\sqrt{3}}{2}=-\sin\frac{\pi}{3}$

Using the unit circle properties, we get $\sin x=-\sin\frac{\pi}{3}=\sin\left(\pi+\frac{\pi}{3}\right)=\sin\frac{4\pi}{3}$ Hence, $\sin x=\sin\frac{4\pi}{3}$

Since, we know that for any real numbers x and $y, \sin x=\sin y$ implies $x=n\pi+(-1)^n y$, where $n\in Z$.

So, we get, $x=n\pi+(-1)^n\left(\frac{4\pi}{3}\right)$

14.1 Results

1. $\sin\theta=0\Leftrightarrow\theta=n\pi$

2. $\cos\theta=0\Leftrightarrow\theta(2n+1)\frac{\pi}{2}$

3. $\tan\theta=0\Leftrightarrow\theta=n\pi$

4. $\sin\theta=\sin\alpha\Leftrightarrow\theta=n\pi+(-1)^n\alpha$, where $\alpha\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

5. $\cos\theta=\cos\alpha\Leftrightarrow\theta=2n\pi\pm\alpha$, where $\alpha\in[0,\pi]$

6. $\tan\theta=\tan\alpha\Leftrightarrow\theta=n\pi+\alpha$, where $\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

7. $\sin^2\theta=\sin^2\alpha\Leftrightarrow\theta=n\pi\pm\alpha$.

8. $\cos^2\theta=\cos^2\alpha\Leftrightarrow\theta=n\ \pi\pm\alpha$.

9. $\tan^2\theta=\tan^2\alpha\Leftrightarrow\theta=n\pi\pm\alpha$.

10. $\sin\theta=1\Leftrightarrow\theta=(4n+1)\frac{\pi}{2}$

11. $\cos\theta=1\Leftrightarrow\theta=2n\pi$

12. $\cos\theta = -1 \Leftrightarrow \theta = (2n+1)\pi$.

13. $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi + \alpha$

Steps to Solve trigonometric functions:

The following are the stages of solving trigonometric equations:

Step 1: Decompose the trigonometric equation into a single trigonometric ratio, preferably the sine or cos function.

Step 2: Factor the trigonometric polynomial given in terms of the ratio.

Step 3: Write down the general solution after solving for each factor.

Note:

1. Unless otherwise stated, is treated as an integer throughout this chapter.

2. Unless the answer is required in a specific interval or range, the general solution should be supplied.

3. The angle's main value is regarded as α. (The main value is the angle with the least numerical value.)