CHAPTER 07

Linear Programming

The term 'programming' means 'planning' and it refers to a particular plan of action from amongst several alternatives for maximising or minimising a function under given restrictions such as maximising profit or minimising cost, etc. The term 'linear' means that 'all inequations' or 'equations used' and the function to be maximised or minimised are linear. Thus, linear programming is a technique for resource utilisation.

Before starting Linear Programming Problems, we should know some terminology of Linear inequalition are given below.

Linear Inequalities

An inequality or inequation is said to be linear, if each variable occurs in first degree only and there is no term involving the product of the variables.

e.g. $ax + b \leq 0$, $ax + by + c > 0$, $ax \leq 4$, etc.

Linear Inequality in One Variable

A linear inequality or inequation, which has only one variable, is called linear inequality or inequation in one variable. e.g. $ax + b < 0$, where $a \neq 0$, $3x + 4 > 0$.

Linear Inequality in Two Variables

A linear inequality, which has two variables, is called linear inequality in two variables. e.g. $3x + 11y \leq 0$, $4t + 3s > 0$.

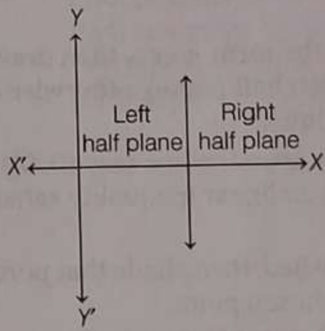
Concept of Half Planes

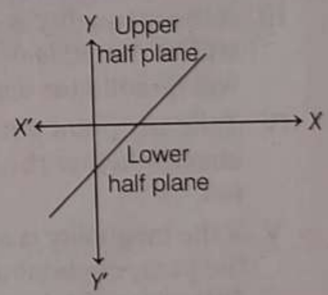
The graph of a line $ax + by + c = 0$ is a straight line which divides the cartesian plane or xy-plane into two parts. Each part is known as half plane.

Types of Half Planes

(i) **Left and right half planes** A vertical line will divide the xy-plane in two parts, left half plane and right half plane.

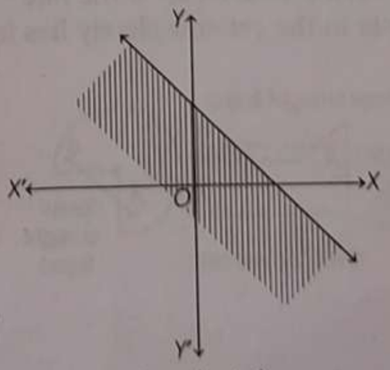
(ii) **Lower and upper half planes** A non-vertical line will divide the xy-plane into two parts, lower half plane and upper half plane.



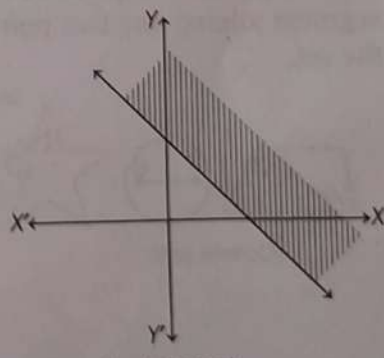


(iii) **Closed half plane** A half place in xy-plane is called a closed half plane, if the line separating the half plane is also included in the half plane.

Therefore, the graph of a linear inequality involving sign $\leq$ or $\geq$ is always closed half plane.

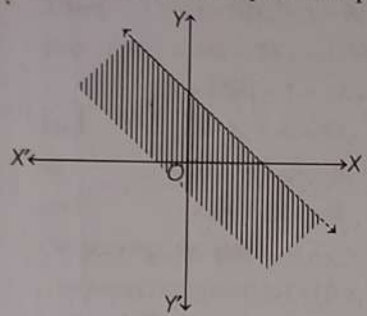


$(ax + by \leq c)$

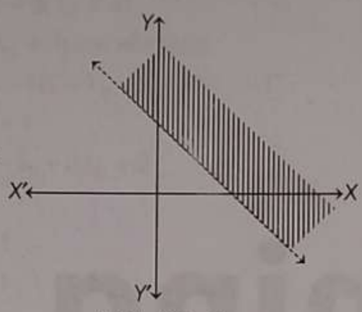


$(ax + by \geq c)$

(iv) **Open half plane** A half plane in xy-plane called an open half plane, if the line separating the half plane is not included in the half plane. Therefore, the graph of a linear inequality involving sign < or > is always an open half plane.



$(ax + by < c)$



$(ax + by > c)$

Solution of a Linear Inequality in Two Variables by Graphical Method

Suppose, given linear inequality is $ax + by \leq c$ or $ax + by \geq c$ or $ax + by < c$ or $ax + by > c$, then to find its solution by graphical method, we use the following steps

I. Consider the inequality as equation $ax + by = c$ in which represents a straight line in XY-plane.

II. Determine the point on coordinate axes by putting $x = 0$ and $y = 0$ respectively and joining these points through line (i.e. line divide the plane in two half planes).

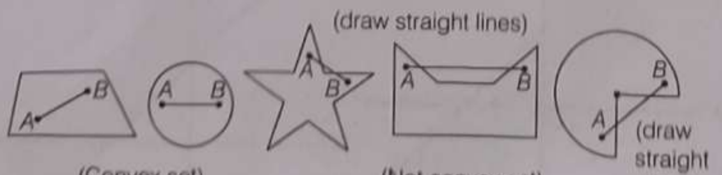
III. If the inequality is of the form < or >, then drawn lie will be dotted (say open half plane), otherwise line will be solid (or dark line).

IV. Take any point not lying on the line (say (0, 0)) and check whether the given linear inequality satisfies or not.

V. If the inequality is satisfied, then shade that portion of the plane containing chosen point.
Otherwise, shade unchosen point portion.
Hence, the shaded region represents the required solution set.

Convex Set and Convex Polygon

A set of points in a plane is called a convex set if the line segment joining any two points in the set completely lies in the set.



(Convex set) (Not convex set)

The solution set of a system of linear inequalities in two variables x and y (say or x_1 and x_2) is called a polygonal convex set and a bounded polygonal convex set is called convex polygon.

Note *The point of intersection of any two boundaries of the half planes determined by a system of linear inequalities is called an extreme point or corner point.*

Linear Programming Problem

A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum value) of a linear function of several variables, subject to the constraints that the variables are non-negative and satisfy a set of linear inequalities.

General or Mathematical Form of LPP

The general mathematical form of a linear programming problem is as follows:

Objective function, $Z = c_1x + c_2y$

Subject to constraints are $a_1x + b_1y \leq d_1$, $a_2x + b_2y \leq d_2$ etc.

and non-negative restrictions are $x \geq 0$, $y \geq 0$.

Important Terms Related to LPP

- **Constraints** The linear inequations or inequalities or restrictions on the variables of a linear programming problem are called constraints.
  The conditions $x \geq 0$, $y \geq 0$ are called **non-negative restrictions.**
- **Optimisation Problem** A problem which seeks to maximise or minimise a linear function subject to certain constraints as determined by a set of linear inequalities is called an optimisation problem. Linear programming problems are special type of optimisation problems.
- **Objective Function** A linear function of two or more variables which has to be maximised or minimised under the given restrictions is called an objective function.
- **Decision Variables** The variables used in the objective function are called decision variables.
- **Optimal Value** The maximum or minimum value of an objective function is known as the optimal value of LPP.

- **Feasible and Infeasible Region** The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a linear programming problem is called the feasible region or solution region. Each point in this region represents a feasible choice. The region other than feasible region is called an infeasible region.
- **Bounded and Unbounded Region** A feasible region of a system of linear inequalities is said to be bounded, if it can be enclosed within a circle. Otherwise, it is said to be unbounded region i.e. the feasible region does extend indefinitely in any direction.
- **Feasible and Infeasible Solution** Points within and on the boundary of the feasible region represents feasible solution of the constraints. Any point outside the feasible region is called an infeasible solution.
- **Optimal Feasible Solution** A feasible solution at which the objective function has optimal value (maximum or minimum) is called the optimal solution or optimal feasible solution of the linear programming problem.
- **Optimisation Technique** The process of obtaining the optimal solution of the linear programming problem is called optimisation technique.
- **Corner Point** A corner point of a feasible region is a point of intersection of two boundary lines in the region.

Graphical Method for Solving Linear Programming Problem

This method of solving linear programming problem is referred as **corner point method**. The process of this method is as follows

I. Draw the graph of each linear equation.

II. Find the feasible region of the LPP and determine its corner points (vertices).

III. Determine the value of objective function $Z = ax + by$ at each corner point.

(a) When the feasible region is **bounded**, then find maximum (M) and minimum (m) values of Z.

(b) When the feasible region is **unbounded**, then

- The maximum (M) value of Z exist, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.
- The minimum (m) value of Z exist, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

(c) It two corner points of the feasible region are both optimal solution of the same type, i.e. both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.