CHAPTER 10

Continuity

Definition of Continuity

A function $f(x)$ is said to be continuous at a point $x = a$, if the following three conditions are satisfied:

(i) f is defined at every point on an open interval containing a.

(ii) $\lim_{x \to a} f(x)$ exists.

(iii) $\lim_{x \to a} f(x) = f(a)$.

Continuity from the Right and from the Left

(i) A function $f(x)$ is continuous at $x = a$ from right, if $\lim_{x \to a^+} f(x) = f(a)$.

(ii) A function $f(x)$ is continuous at $x = a$ from left, if $\lim_{x \to a^-} f(x) = f(a)$.

(iii) If a function is continuous on left hand limit, right hand limit and the value of the function at $x = a$ coincide.

In other words, a function $f(x)$ is said to be continuous at $x = a$,

if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

Examples of Continuous Functions

- Constant function, that is $f(x) = k$, is continuous at every point on R.
- Power functions, that is $f(x) = x^n$, with positive integral exponents are continuous at every point on R.
- Polynomial functions,
$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$
are continuous at every point on R.
- The trigonometric functions $\sin x$ and $\cos x$ are continuous at every point on R.
- The exponential function a^x $(a > 0)$ and logarithmic function $\log_b x$ (for $x, > 0$ and $b, b \neq 1$) are continuous on R.
- Rational functions are of the form
$$\frac{P(x)}{Q(x)}, Q(x) \neq 0.$$
They are continuous at every point a if $Q(a) \neq 0$.

Properties of Continuous Functions

Theorem 1 Let f and g be two real functions continuous at a real number c, then

(i) $(f + g)$ is continuous at $x = c$.

(ii) $(f - g)$ is continuous at $x = c$.

(iii) fg is continuous at $x = c$.

(iv) $\dfrac{f}{g}$ is continuous at $x = c$, provided that $g(c) \neq 0$.

(v) Suppose f and g are real valued functions such that (fog) is defined at c. If g is continuous at c and if f is continuous at g (c), then (fog) is continuous at c.

Continuity Over an Interval

A function $f(x)$ is said to be continuous in its domain, if it is continuous at every point in its domain.

(i) A function $f(x)$ is said to be continuous in open interval (a, b), if it is continuous for every value of x in the interval (a, b).

(ii) A function $f(x)$ is said to be continuous in closed interval $[a, b]$, if

(a) it is continuous for every value of x in the open interval (a, b).

(b) $f(x)$ is continuous at $x = a$ from right, i.e. $\lim_{x \to a^+} f(x) = f(a)$.

(c) $f(x)$ is continuous at $x = b$ from left, i.e. $\lim_{x \to b^-} f(x) = f(b)$.

Other important results

There are some functions which are always continuous in their respective domain. e.g.

(i) Every constant function is continuous.
(ii) Every identity function is continuous.
(iii) Every rational function is continuous.
(iv) Every polynomial function is continuous.
(v) Modulus function $f(x) = |x|$ is continuous.
(vi) All trigonometric functions are continuous.

Discontinuity

A function $f(x)$ is said to be **discontinuous** at $x = a$, if it is not continuous at $x = a$, i.e.

(i) $\lim_{x \to a} f(a)$ does not exist.
(ii) Left hand limit and right hand limit are not equal.
(iii) $\lim_{x \to a} f(x) \neq f(a)$

Types of Discontinuity

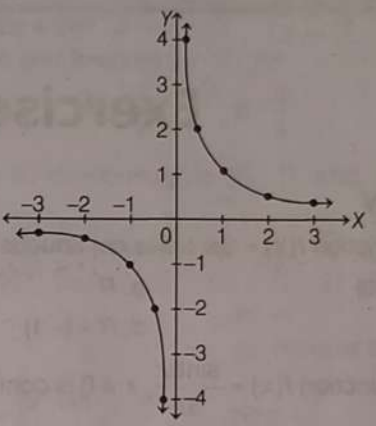
We have seen that discontinuities have several different types.

(i) **Jump Discontinuity** A function $f(x)$ has a Jump Discontinuity at $x = a$ if the left hand and right hand limits both exist but are different, that is

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

(ii) **Removable Discontinuity** $\lim_{x \to a} f(x)$ exists and either it is not equal defined, then the function $f(x)$ is said to discontinuity (missing point discontinuity) of this discontinuity can be removed by suitable of a.

(iii) **Infinite Discontinuity** Observe the graph of $xy = 1$, $y = f(x) = \frac{1}{x}$ is the function to be considered.



It is easy to see that $f(x) \to \infty$ as $x \to 0^+$ and $f(x) \to -\infty$ as $x \to 0^+$. $f(0)$ is not defined. Of course, this function is discontinuous at $x = 0$.

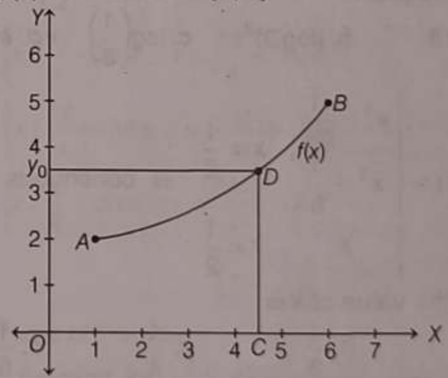
A function $f(x)$ is said to have an infinite discontinuity at $x = a$, if

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

above figure says, $f(x)$ has an infinite discontinuity.

Intermediate Value Theorem for Continuous Functions

Theorem If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$ then $y_0 = f(c)$ for some c in $[a, b]$.



Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the Y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$. The proof of the Intermediate Value Theorem depends on the completeness property of the real number system and can be found in more advanced texts. The continuity of f on the interval is essential. If f is discontinuous at even one point of the interval, the conclusion of the theorem may fail.