CHAPTER 13

Differential Equation

An equation involving independent variable(s), dependent variable, derivative(s) of dependent variable with respect to independent variable(s) and constant, is called a **differential equation**.

e.g. $x\dfrac{dy}{dx}+xy\dfrac{d^2y}{dx^2}+4=0,\ \dfrac{dy}{dx}+y\cos x=\sin x$

are differential equations.

Order and Degree of the Differential Equation

The order of highest differential coefficient (or highest order derivative) appearing in a differential equation is the order of differential equation.

e.g. (i) $\dfrac{d^3y}{dx^3}+x\dfrac{dy}{dx}+xy\dfrac{d^2y}{dx^2}+4=0$, so order is 3.

(ii) $\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^3=0$, so order is 2.

The highest exponent of the highest derivative is called the degree of a differential equation provided exponents of each derivative and the unknown variable appearing in the differential equation are non-negative integer,

e.g. (i) $\dfrac{d^2y}{dx^2}+\sqrt{\dfrac{dy}{dx}}+x=0 \Rightarrow \dfrac{dy}{dx}=\left(\dfrac{d^2y}{dx^2}+x\right)^2$

$\Rightarrow \left(\dfrac{d^2y}{dx^2}\right)^2-\left(\dfrac{dy}{dx}\right)+2x\dfrac{d^2y}{dx^2}+x^2=0$

So, degree is 2.

(ii) $\left(\dfrac{d^3y}{dx^3}\right)^{2/3}+x+y=0 \Rightarrow \left(\dfrac{d^3y}{dx^3}\right)^2=(-x-y)^3$

So, degree is 2.

Formation of Differential Equation

If an equation in independent and dependent variables involving some arbitrary constants is given, then a differential equation is obtained as follows:

(i) Differentiate the given equation with respect to the independent variable (say x) as many times as the number of arbitrary constants in it.

(ii) Eliminate the arbitrary constants.

(iii) The obtained equation is the required differential equation i.e. if we have an equation $f(x, y, c_1, c_2, \dots, c_n)=0$ containing n arbitrary constants $c_1, c_2, \dots, c_n$, then by differentiating this n times, we shall get n-equations.

Now, among these n equations and the given equation, in all $(n+1)$ equations, if the n arbitrary constants $c_1, c_2, \dots, c_n$ are eliminated, we shall evidently get a differential equation of the nth order. For there being n differentiation, the resulting equation must contain a derivative of the nth order.

Solution of Differential Equation

The solution of differential equation is a relation between the variables of the equation not containing the derivatives but satisfying the given differential equation. The process of finding the solution of a differential equations is called integrating the differential equation.

General Solution

A solution of a differential equation, containing independent arbitrary constants equal in number to the order of differential equation is called general solution (complete primitive). The general solution of a differential equation of the nth order must contain n and only n independent arbitrary constants.

Particular Solution

A solution obtained by giving particular values to the arbitrary constants in the general solution is called particular solution of differential equation or particular integral.

Variable Separable Method

The equation $\frac{dy}{dx} = f(x,y)$ is to be in variable separable form if it can be expressed as $h(x)dx = g(y)dy$.

The solution to this equation is obtained by integrating $h(x)$ and $g(y)$ with respect to x and y, respectively.

Differential Equation Reducible to Variable Separable Method

Sometimes the given differential equation of the first order cannot be solved directly by variable separable method. But by some substitution, we can reduce it to a differential equation with separable variable.

Let the differential equation is of the form

$$\frac{dy}{dx} = f(ax+by+c)$$

It can be reduced to variable separable form by the substitution $ax + by + c = z$

$$\therefore \quad a + b\frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore \quad \left(\frac{dz}{dx} - a\right)\frac{1}{b} = f(z)$$

$$\Rightarrow \quad \frac{dz}{dx} = a + bf(z)$$

Now, apply variable separable method.

Homogeneous Differential Equation

A differential equation of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$, where $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of x and y of the same degree, is called a homogeneous differential equation.

To solve a homogeneous differential equation of the type $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)} = g\left(\frac{y}{x}\right)$, we put

$$y = vx \text{ and } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

These substitutions transform the given equation into an equation of the form,

$$v + x\frac{dv}{dx} = f(v) \text{ i.e. } x\frac{dv}{dx} = f(v) - v$$

Now, it is in variable separable form. Separating the variables and integrating, we get

$\int \frac{dv}{f(v)-v} = \log x + C$, where C is an arbitrary constant.

Now, replacing v by $\left(\frac{y}{x}\right)$ in integration, we get the required solution.

If the homogeneous differential equation is in the form

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)} = h\left(\frac{x}{y}\right),$$

then we put $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$

and then we proceed as discussed above.

Applications of Differential Equation

There are many situations where the relation in the rate of change of a function is known. This gives a differential equation of the function and we may be able to solve it.

Population Growth

If the population P increases at time t, then the rate of change of P is proportional to the population present at that time.

$$\therefore \quad \frac{dP}{dt} \propto P$$

$$\Rightarrow \quad \frac{dP}{dt} = k \cdot P \qquad (k>0)$$

$$\Rightarrow \quad \int \frac{dP}{P} = \int k\,dt$$

$$\Rightarrow \quad \log P = kt + c_1$$

$$\Rightarrow \quad P = c \cdot e^{kt},$$

where $c = e^{c_1}$.

which gives the population at any time t.

Radio Active Decay

We know that the radio active substances (elements) like radium, cesium etc. disintegrate with time.

It means the mass of the substance decreases with time. The rate of disintegration of such elements is always proportional to the amount present at that time.

If x is the amount of any material present at time t, then

$$\frac{dx}{dt} = -k \cdot x$$

where, k is the constant of proportionality and $k > 0$. The negative sign appears because x decreases as t increases.

Solving this differential equation, we get

$$x = a \cdot e^{kt}, \text{ where } a = e^c \qquad \text{...(i)}$$

If x_0 is the initial amount of radio active substance at time $t=0$, then from Eq. (i), we get

$$x_0 = a \cdot 1$$

$$\Rightarrow \quad a = x_0$$

$$\Rightarrow \quad x = x_0 e^{-kt} \quad \text{...(ii)}$$

This expression gives the amount of radio active substance at any time t.

Half-life Period

Half life period of a radio active substance is defined as the time it takes for half the amount/mass of the substance to disintegrate.

Newton's Law of Cooling

Newton's law of cooling states that the rate of change of cooling heated body at any time is proportional to the difference between the temperature of a body and that of its surrounding medium.

Let θ be the temperature of a body at time t and θ_0 be the temperature of the medium.

Then $\frac{d\theta}{dt}$ is the rate of change of temperature with respect to time t and $\theta - \theta_0$ is the difference of temperature at time t. According to Newton's law of cooling,

$$\therefore \quad \frac{d\theta}{dt} \propto (\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_0) \quad \text{...(i)}$$

where, k is constant of proportionality and negative sign indicates that difference of temperature is decreasing.

$$\text{Now,} \quad \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\therefore \quad \frac{d\theta}{(\theta - \theta_0)} = -k\,dt$$

Integrating and using the initial condition *viz.* $\theta = \theta_1$ when $t = 0$, we get

$$\therefore \quad \theta = \theta_0 + (\theta_1 - \theta_0)e^{-kt} \quad \text{...(ii)}$$

Thus, Eq. (ii) gives the temperature of a body at any time t.

Surface Area

Knowledge of a differential equation is also used to solve problems related to the surface area.