CHAPTER 05

Probability

Probability is the measure of chance of occurrence of an event and it is quantified as a number between 0 and 1.

Here, we are introducing some important terminology, which are used in probability.

- **Random experiment** An experiment, where the result may not be same, when they are repeated under identical condition.
- **Outcomes** A possible result of a random experiment is called its outcome.
- **Sample space** The set of all possible outcomes of an experiment, is called the sample space, it is denoted by S.
- **Favourable outcome** An outcome that belongs to the specified event is called a favourable outcome.

Event

Every subset of a sample space is defined as the event.

Types of Events

According to possibility of occurrence and sample points, events are defined as follows

(i) **Elementary event** It has only one sample point. On throwing a die, the event of getting prime even number, i.e. $E = \{2\}$, is a simple event.

(ii) **Certain Event** The sample space is called the certain event, if all possible outcomes are favourable outcomes. i.e. the event consists of the whole sample space.

(iii) **Impossible event** It has no element and is denoted by ϕ. On throwing a die, the event of getting number 8 is an impossible event.

(iv) **Exhaustive Events** A set of events are said to be exhaustive, if one of them necessarily occurs whenever the experiment is performed.

Let $E_1, E_2, \dots E_n$ be subsets of sample space S. Then, events $E_1, E_2, \dots E_n$ are exhaustive events, if $E_1 \cup E_2 \cup \dots \cup E_n = S$.

(v) **Mutually Exclusive Events** Two or more events are said to be mutually exclusive, if the happening of one excludes the happening of the other i.e. if no two of them can occur together. If A and B are mutually exclusive events, then $(A \cap B) = \phi$.

e.g. In throwing a die, all the 6 faces numbered 1 to 6 are mutually exclusive, if anyone of these faces comes, the possibility of others in the same trial is ruled out.

If two events A and B are mutually exclusive and exhaustive, then they are called complementary events.

Symbolically, A and B are complementary events, if $A \cup B = S$ and $A \cap B = \phi$.

Algebra of Events

Events are subsets of the sample space. Algebra of events uses operations in set theory to define new events in terms of known events.

Union of Two Events

Let A and B be two events in the sample space S. The union of A and B is denoted by $A \cup B$ and is the set of all possible outcomes that belong to at least one of A and B.

Intersection of Two Events

Let A and B be two events in the sample space S. The intersection of A and B is the event consisting of outcomes that belong to the events A and B.

Complement of An Event

Complement of an event A is denoted by $A', \overline{A}$ or A^c. The following table shows how the operations of complement, union and intersection can be combined to define more events.

Operation	Interpretation
$A', \overline{A}$ or A^c	Not A
$A \cup B$	At least, one of A and B
$A \cap B$	Both A and B
$(A' \cap B) \cup (A \cap B')$	Exactly one of A and B
$(A' \cap B') = (A \cup B)'$	Neither A nor B

Concept of Probability

Probability is based on the assumption that all possible outcomes of an experiment are equally likely.

Equally Likely Outcomes

All possible outcomes of a random experiment are said to be equally likely, if none of them can be preferred over others.

Probability of an Event

Let E be an event and $P(E)$ denotes the probability of occurrence of the event E. Then, we have

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{\text{Number of outcomes favourable to event } E}{\text{Total number of outcomes in the sample space}}$$

It must be clearly understood that all the outcomes considered in the above definition should be equally likely.

Elementary Properties of Probability

- A' is complement of A and therefore $P(A') = 1 - P(A)$
- For any event A in $S, 0 \le P(A) \le 1$
- For the impossible event ϕ, $P(\phi) = 0$
- For the certain event S, $P(S) = 1$.
- If A_1 and A_2 two mutually exclusive events then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$
- If $A \subseteq B$, then $P(A) \le P(B)$ and $P(A' \cap B) = P(B) - P(A)$
- For any two events A and B. $P(A \cap B') = P(A) - P(A \cap B)$
- If $A_1, A_2, \ldots, A_m$ are mutually exclusive events in S, then $P(A_1 \cup A_2, \cup \ldots \cup A_m)$
  $= P(A_1) + P(A_2) + \ldots + P(A_m)$

Addition Theorem for Two Events

Theorems which express the probability of an event in terms of the probabilities of those events whose union is the given event are known as addition theorems on probability.

In this section, we shall discuss addition theorems for two events, which are given below

(i) **When events are not mutually exclusive** If A and B are two events which are not mutually exclusive, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(ii) **When events are mutually exclusive** If A and B are mutually exclusive events, then
$$n(A \cap B) = 0 \Rightarrow P(A \cap B) = 0$$
$$\therefore \quad P(A \cup B) = P(A) + P(B)$$

(iii) **Some other theorems** Let A and B be two events associated with a random experiment, then

(a) Probability of occurrence of neither A nor B is
$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$

(b) P (exactly one of A, B occurs)
$$= P(A \cap B) + P(A \cap B)$$
$$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) - 2P(A \cap B)$$

Conditional Probability

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of A under the condition that B has already occurred and $P(B) \ne 0$, is called the conditional probability and it is denoted by $P\left(\frac{A}{B}\right)$.

Thus, $P\left(\frac{A}{B}\right)$ = Probability of occurrence of A, given that B has already happened $= \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}$

Similarly, $P\left(\frac{B}{A}\right)$ = Probability of occurrence of B, given that A has already happened

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)}$$

Properties of Conditional Probability

Let A and B be two events of a sample space S of an experiment, then

(i) $P\left(\frac{S}{A}\right) = P\left(\frac{A}{A}\right) = 1$

(ii) $P\left(\frac{A'}{B}\right) = 1 - P\left(\frac{A}{B}\right)$, where A' is the complement of A.

Multiplication Theorems

If A and B are two independent events associated with a random experiment, then

$$P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right), \text{ if } P(A) \neq 0$$

or $$P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right), \text{ if } P(B) \neq 0$$

Independent Events

Two events are said to be independent, if the occurrence of one does not depend upon the other. If A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B)$$

not that $P(A \cap B) = P(A) \cdot (B/A) = P(A) \cdot P(B)$

If $A_1, A_2, \ldots, A_n$ are independent events, then

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2) \ldots P(A_n)$

Theorem

If A and B are two independent events, then

(i) A and B' are also independent events.

(ii) A' and B' are also independent events.

Bayes' Theorem

Let $E_1, E_2, E_3, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment.

If A is an event, which occurs together with E_i's. Then,

$$P(E_i / A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{j=1}^{n} P(E_j) \cdot P(A/E_j)}$$

Here, events $E_1, E_2, \ldots, E_n$ are called **hypothesis.** The probability $P(E_i)$ is called the **priori probability** of the hypothesis E_i and the conditional probability $P(E_i/A)$ is called a **posteriori probability** of the hypothesis E_i.

Baye's theorem is also called the formula for the probability of causes.

ODDS (Ratio of Two Complementary Probabilities)

If in a random experiment, total number of outcomes is n out of which m are favourable to an event A, then

$$\text{odds in favour of } A = \frac{m}{n-m}$$

and $$\text{odds against of } A = \frac{n-m}{m}$$

$\therefore$ $$P(A) = \frac{\text{Number of favourable cases to } A}{\text{Total number of cases}} = \frac{m}{n}$$

and $$P(\bar{A}) = 1 - P(A) = 1 - \frac{m}{n} = \frac{n-m}{n}$$