CHAPTER 09

Limits

Limit of a Function

If value of a function cannot be determined at a point, then with the help of limit, we try to find the value of function at nearest points.

To understand the limit, we consider the function,

$$f(x)=\frac{x^2-4}{x-2}$$

Clearly at $x=2$, $f(2)=\frac{4-4}{2-2}=\frac{0}{0}$, which is meaningless.

Thus, $f(x)$ is not defined at $x=2$.

If $x\neq 2$, then $f(x)=\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=(x+2)$

Now, if value of x taken slightly less than 2 then clearly value of $f(x)$ will be slightly less than 4.

This shows as below

When $x=1.9$, then $f(x)=1.9+2=3.9$

When $x=1.99$, then $f(x)=1.99+2=3.99$

When $x=1.999$, then $f(x)=1.999+2=3.999$

When $x=1.9999$, then $f(x)=1.9999+2=3.9999$

$\vdots \qquad \vdots$

When $x=1.9999999$, then

$f(x)=1.9999999+2=3.9999999$

Thus, we can say that as x approaches 2, then value of $f(x)$ approaches 4. We can write it as

if $x\to 2$, then $f(x)\to 4$

Similarly, if value of taken slightly more than x, then value of $f(x)$ will slightly more than 4.

When $x=2.1$, then $f(x)=2.1+2=4.1$

When $x=2.01$, then $f(x)=2.01+2=4.01$

When $x=2.001$, then $f(x)=2.001+2=4.001$

When $x=2.0001$, then $f(x)=2.0001+2=4.0001$

$\vdots \qquad \vdots$

When $x=2.00000001$, then $f(x)=4.00000001$

We can write it as

if $x\to 2$, then $f(x)\to 4$

From above discussion we conclude that if x approaches to 2 from left or x approaches to 2 from right then values of $f(x)$ approaches 4. This definite 4 is called limit of $f(x)$ at $x\to 2$. In symbols we shall write as $\lim\limits_{x\to 2} f(x)=4$, where $x\to 2$ means x approaches to 2.

Definition of Limit

If $f(x)$ approaches to a real number l, when x approaches to a (through lesser or greater values to a) i.e. if $f(x)\to l$ when $x\to a$, then l is called limit of the function $f(x)$. In symbolic form, it can be written as $\lim\limits_{x\to a} f(x)=l$.

Left Hand Limit

If given $\epsilon>0$, there exists $\delta>0$ such that for $|f(x)-l|<\epsilon$ for all x with $a-\delta<x<a$, then $\lim\limits_{x\to a^-} f(x)=l$.

Right Hand Limit

If given $\epsilon>0$ there exists $\delta>0$ such that for $|f(x)-l|<\epsilon$ for all x with $a<x<a+\delta$, then $\lim\limits_{x\to a^+} f(x)=l$.

One Sided Limit

$\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$; if they exist are called one sided limits.

Existence of a Limit of a Function at a Point $x = a$

If the right hand limit and left hand limit coincide (i.e. same), then we say that limit exists and their common value is called the limit of $f(x)$ at $x = a$ and denoted by $\lim_{x \to a} f(x)$.

Algebra of Limits

Let f and g be two real functions with common domain D, such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists. Then,

(i) Limit of sum of two functions is sum of the limits of the functions. i.e.
$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

(ii) Limit of difference of two functions is difference of the limits of the function. i.e.
$$\lim_{x \to a} (f - g)(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

(iii) Limit of product of a constant and one function is the product of that constant and limit of a function, i.e.
$$\lim_{x \to a} [c \cdot f(x)] = c \lim_{x \to a} f(x), \text{ where } c \text{ is a constant.}$$

(iv) Limit of product of two functions is product of the limits of the function, i.e.
$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

(v) Limit of quotient of two functions is quotient of the limits of the functions, i.e.
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ where } \lim_{x \to a} g(x) \neq 0$$

Note *(i)* $\lim_{x \to a} k = k$, *where k is a constant*

(ii) $\lim_{x \to a} x = a$

(iii) $\lim_{x \to a} x^n = a^n$

(iv) $\lim_{x \to a} \sqrt[r]{x} = \sqrt[r]{a}$

(v) If $p(x)$ *is a polynomial, then* $\lim_{x \to a} p(x) = p(a)$

While evaluating limits, we must always check whether the denominator tends to zero, and if it does, then whether the numerator also tends to zero. In case both tend to zero we have to study the function in detail.

Limits of Rational Functions

A function f is said to be a rational function, if $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial functions such that $h(x) \neq 0$.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)}$
$$= \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h(a) = 0$, then there are two cases arise,
(i) $g(a) \neq 0$ (ii) $g(a) = 0$.

In the first case, we say that the limit does not exist.

In the second case, we can find limit.

Limit of a rational function can be find with the help of following methods

Substitution Method

In this method, we substitute the point, to which the variable tends to in the given limit. If it give us a real number, then the number so obtained is the limit of the function and if it does not give us a real number, then use other methods.

Method of Factorisation

Let $\lim_{x \to a} \frac{f(x)}{g(x)}$ reduces to the form $\frac{0}{0}$, when we substitute $x = a$. Then, we factorise $f(x)$ and $g(x)$ and then cancel out the common factor to evaluate the limit.

- If the given limit is of the form $\lim_{x \to a} \frac{x^n - a^n}{x - a}$, then we can find the limit directly by using the following theorem

Theorem *Let n be any positive integer. Then,*
$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

- The above theorem can also be verified, if n is a fraction say $n = \frac{p}{q}$ where $q \neq 0$. Then,
$$\lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = \lim_{x \to a} \left(\frac{x^{\frac{p}{q}} - a^{\frac{p}{q}}}{x - a} \right) = \frac{p}{q} a^{\frac{p}{q} - 1}$$

Method of Rationalisation

If we get $\frac{0}{0}$ form and numerator or denominator or both have radical sign, then we rationalise the numerator or denominator or both by multiplying their conjugate to remove $\frac{0}{0}$ form and then find limit by direct substitution method.

Substitution Method

In trigonometric function, if $x \to a$, we put $x - a = t$, so that $x \to a$ becomes $t \to 0$ and then solve them.

Squeeze Theorem (Sandwich Theorem)

Suppose $f(x)$, $g(x)$ and $h(x)$ are given functions such that $f(x) \le g(x) \le h(x)$ for all x in an open interval about a.

Suppose, $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$

So, $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x) \le \lim_{x \to a} h(x)$

$\Rightarrow$ $L \le \lim_{x \to a} g(x) \le L$

$\therefore$ $\lim_{x \to a} g(x) = L$

Limit of a Trigonometric Function

(i) $\lim_{x \to a} \sin x = \sin a$

(ii) $\lim_{x \to a} \cos x = \cos a$

(iii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iv) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

(v) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Corollaries

(i) $\lim_{\theta \to 0} \left(\frac{\theta}{\sin\theta}\right) = 1$

(ii) $\lim_{\theta \to 0} \left(\frac{\tan\theta}{\theta}\right) = 1$

(iii) $\lim_{\theta \to 0} \left(\frac{\theta}{\tan\theta}\right) = 1$

(iv) $\lim_{\theta \to 0} \left(\frac{\sin p\theta}{p\theta}\right) = 1$, ($p$ constant)

(v) $\lim_{\theta \to 0} \left(\frac{\tan p\theta}{p\theta}\right) = 1$, ($p$ constant)

(vi) $\lim_{\theta \to 0} \left(\frac{p\theta}{\sin p\theta}\right) = 1$, ($p$ constant)

(vii) $\lim_{\theta \to 0} \left(\frac{p\theta}{\tan p\theta}\right) = 1$, ($p$ constant)

Limits of Exponential Functions and Logarithmic Functions

(i) $\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = \log e = 1$

(ii) $\lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log a \ (a > 0, a \ne 0)$

(iii) $\lim_{x \to 0} [1 + x]^{\frac{1}{x}} = e$

(iv) $\lim_{x \to 0} \left(\frac{\log(1 + x)}{x}\right) = 1$

(v) $\lim_{x \to 0} \left(\frac{e^{px} - 1}{px}\right) = 1$, ($p$ constant)

(vi) $\lim_{x \to 0} \left(\frac{a^{px} - 1}{px}\right) = \log a$, ($p$ constant)

(vii) $\lim_{x \to 0} \left(\frac{\log(1 + px)}{px}\right) = 1$, ($p$ constant)

(viii) $\lim_{x \to 0} (1 + px)^{\frac{1}{px}} = e$, ($p$ constant)

Limit at Infinity (Function Tending to Infinity)

A function f is said to tend to limit 'l' as x tends to $-\infty$ if for given $\epsilon > 0$, there exists a positive number M such that $|f(x) - l| < \epsilon$, for all $x > M$.

$$\therefore \quad \lim_{x \to \infty} f(x) = l$$

Whenever expression is of the form $\frac{\infty}{\infty}$, then divide, by suitable power of x to get finite limits of numerator as well as denominator.

Infinite Limits

Let us consider the function $f(x) = \frac{1}{x}$. Observe the behavior of $f(x)$ as x approaches zero from right and from left.