CHAPTER 08

Functions

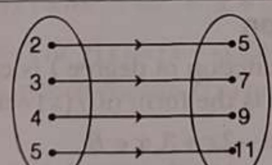
Let A and B be two non-empty sets, then a **function** f from set A to set B is a rule which associates each element of A to a unique element of B. It is represented as $f: A \rightarrow B$ and function is also called mapping.

Domain, Codomain and Range of a Function

The set A is called the **domain** of f, denoted by dom$f$ and the set B is called the **codomain** of f.

The set of all second elements of the pairs (a, b) of the function f is the **range** of the function f, so range is a subset of codomain.

Representation of Function

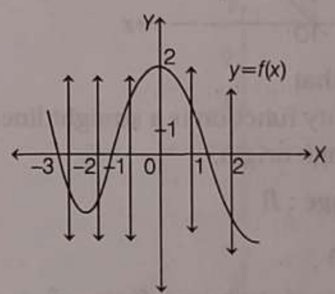
Verbal form	Output exceeds twice the input by 1 **Domain** : Set of inputs **Range** : Set of outputs
Arrow form on Venn Diagram	 **Domain** : Set of pre-images **Range** : Set of images
Ordered Pair (x, y)	$f = \{(2, 5), (3, 7), (4, 9), (5, 11)\}$ **Domain** : Set of 1st components from each ordered pair = $\{2, 3, 4, 5\}$ **Range** : Set of 2nd components from each ordered pair = $\{5, 7, 9, 11\}$

Graph of a Function

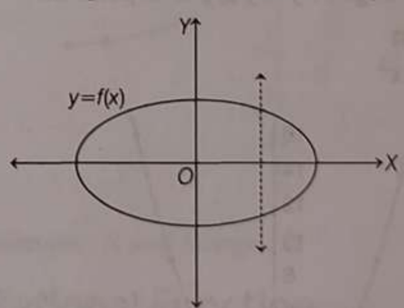
If the domain of function is in R, we can show the function by a graph in xy-plane. The graph consists of points (x, y), where $y = f(x)$.

Vertical Line Test

Given a graph, let us find if the graph represents a function of x, i.e. $f(x)$. A graph represents function of x, only if no vertical line intersects the curve in more than one point.



Since, every x has a unique associated value of y. It is a function.



This graph does not represent a function as vertical line intersects at more than one point some x has more than one values of y.

Value of function $f(a)$ is called the value of function $f(x)$ at $x = a$.

Some Basic Functions

Constant Function

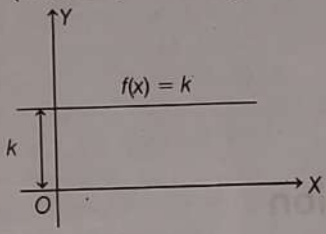
A function $f : R \to R$ is said to be a constant function, if there exists a **real number** k, such that

$$f(x) = k, \forall x \in R$$

Domain : R **and range :** $\{k\}$

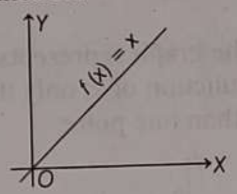
e.g. $\quad f(x) = 5, \forall x \in R$

Here, $f(x)$ is a constant function whose domain $= \{x : x \in R\}$ and range $= \{5\}$.



Identity Function

The real function $f : R \to R$ defined by $f(x) = x, \forall\, x \in R$ is called an identity function.



It may be observed that

(i) graph of identity function is a straight line.

(ii) it passes through origin.

Domain : R **and Range :** R

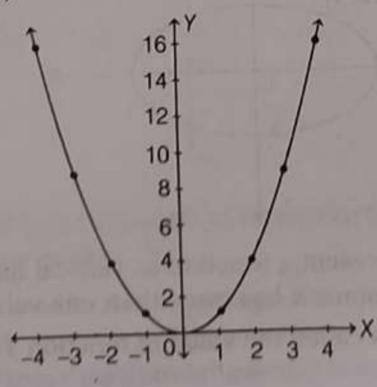
Power Function

The power function is given by $y = f(x) = x^n, n \in I, n \neq 1, 0$.

The domain and range of $y = f(x)$, is depend on n.

Square Function

Suppose, $\quad f(x) = x^2$



Domain : R **or** $(-\infty, \infty)$ **and Range :** $[0, \infty)$

Properties

(i) Graph of $f(x) = x^2$ is a parabola opening upwards and with vertex at origin.

(ii) Graph is symmetric about Y-axis.

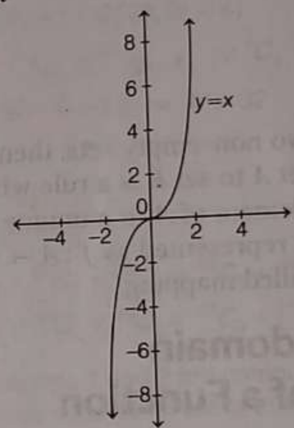
(iii) The graph of even powers of x looks similar to square function. (verify !) e.g. x^4, x^6.

(iv) $(y - k) = (x - h)^2$ represents parabola with vertex at (h, k).

(v) If $-2 \leq x \leq 2$ then $0 \leq x^2 \leq 4$ (see above figure) and if $-3 \leq x \leq 2$ then $0 \leq x^2 \leq 9$ (see above figure).

Cube Function

Suppose, $\quad f(x) = x^3$



Domain : R or $(-\infty, \infty)$ **and Range :** R **or** $(-\infty, \infty)$

Property

The graph of odd powers of x (more than 1) looks similar to cube function. e.g. x^5, x^7.

Polynomial Function

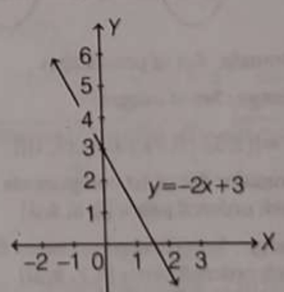
Suppose, $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n$

is polynomial function of degree n, if $a_0 \neq 0$ and a_i's are real.

Linear Function

A polynomial function of degree 1 is called a linear function, which is the form of $f(x) = ax + b \; (a \neq 0)$.

Suppose, $\quad f(x) = -2x + 3, x \in R$



Domain : R **or** $(-\infty, \infty)$ **and Range :** R **or** $(-\infty, \infty)$

Properties

(i) Graph of $f(x) = ax + b$ is a line with slope a, y-intercept b and x-intercept $\left(-\frac{b}{a}\right)$.

(ii) Function is increasing when slope is positive and decreasing when slop is negative.

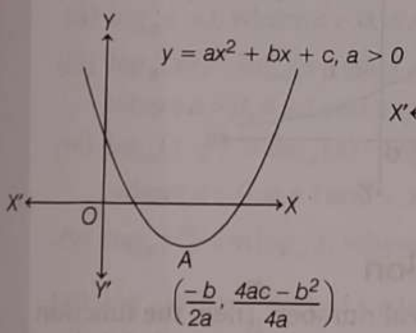
Quadratic Function

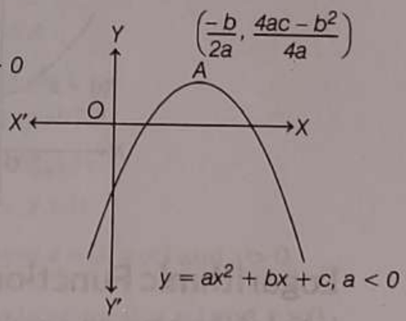
A polynomial function of degree 2 is called a quadratic function, which is the form of

$$y = f(x) = ax^2 + bx + c,\ a \neq 0$$

$$\Rightarrow \qquad y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

which represents a downward parabola, if $a < 0$ and upward parabola, if $a > 0$ and vertex of this parabola is at $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$.





Domain of $f(x) = R$

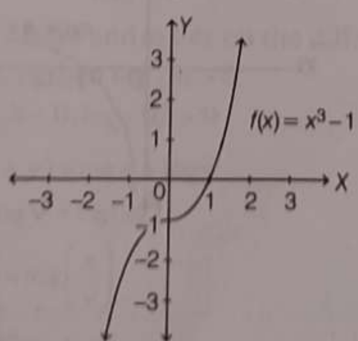
Range of $f(x)$ is $\left(-\infty, \frac{4ac - b^2}{4a}\right]$, if $a < 0$ and $\left[\frac{4ac - b^2}{4a}, \infty\right)$, if $a > 0$.

Cubic Function

A polynomial function of degree 3 is called a quadratic function, which is the form of $f(x) = ax^3 + bx^2 + cx + d\ (a \neq 0)$.

The graph of $f(x) = x^3 - 1$ or $f(x) = (x - 1)\ (x^2 + x + 1)$ cuts X-axis at only one point $(1, 0)$, which means $f(x)$ has one real root and two complex roots.

Domain : R or $(-\infty, \infty)$ and Range : R or $(-\infty, \infty)$



Properties

Note that, any polynomial of odd degree must have at least one real root, because the complex roots appear in conjugate pairs.

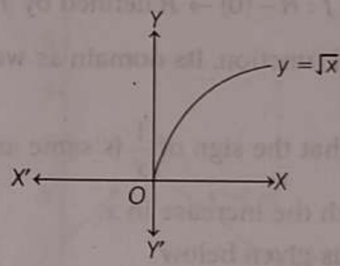
Radical Function

Radical function is defined by $f(x) = \sqrt[n]{x}, n \in N$.

Square Root Function

Square root function is defined by

$$y = f(x) = \sqrt{x}, x \geq 0$$

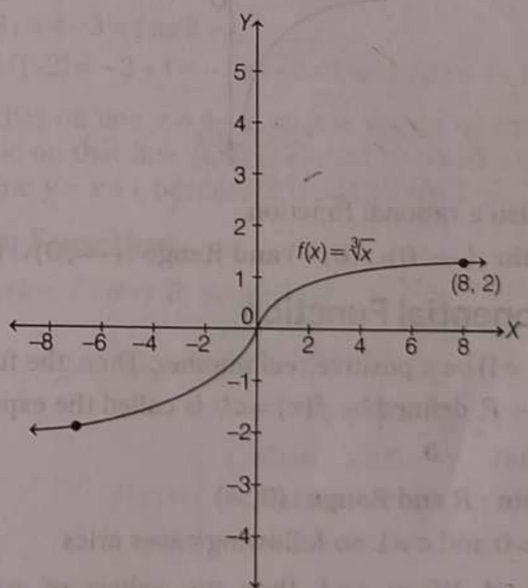


Domain : $[0, \infty)$ and Range : $[0, \infty)$

Cube Root Function

Cube root function is defined by

$$f(x) = \sqrt[3]{x}$$



Domain : R and Range : R

Rational Function

A function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions of x defined in a domain and $Q(x) \neq 0$, is called a rational function.

The domain of a rational function is $R-\{x : Q(x)=0\}$ and range depends on the expression representing the function.

e.g. $f(x)=\left(\dfrac{x^2+4}{x^3-6x+4}\right)$ is called a rational function, where $x^3-6x+4\neq 0$.

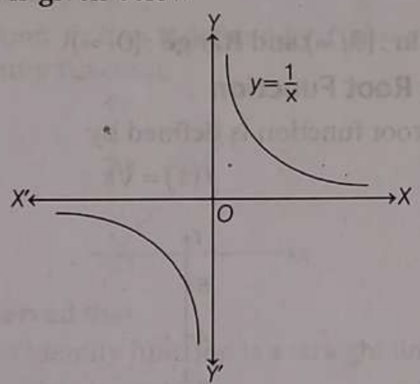
Its domain is $R-\{x : x^3-6x+4=0\}$.

Reciprocal Function

The function $f : R-\{0\}\to R$ defined by $f(x)=\dfrac{1}{x}$ is called the reciprocal function. Its domain as well as range is $R-\{0\}$.

We observe that the sign of $\dfrac{1}{x}$ is same as that of x and $\dfrac{1}{x}$ decreases with the increase in x.

So, its graph is given below



It is also a rational function.

Domain : $(-\infty, 0)\cup(0,\infty)$ and Range : $(-\infty,0)\cup(0,\infty)$

Exponential Function

Let $a(\neq 1)$ be a positive real number. Then, the function $f : R\to R$, defined by $f(x)=a^x$, is called the exponential function.

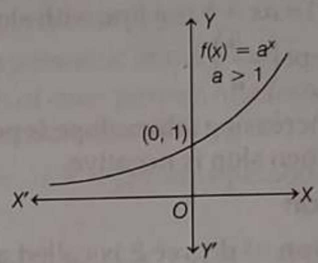
Domain : R and Range : $(0,\infty)$

As, $a>0$ and $a\neq 1$. So following cases aries

***Case* I** When $a>1$, then the values of $y=f(x)=a^x$ increase as the values of x increase.

Thus, $f(x)=a^x=\begin{cases} <1, \text{ for } x<0 \\ 1, \text{ for } x=0 \\ >1, \text{ for } x>0 \end{cases}$

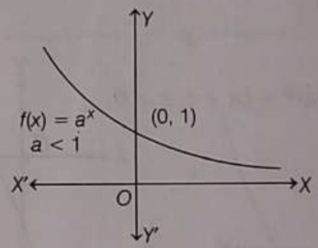
Its graph is given below



***Case* II** When $0<a<1$, then the values of $y=f(x)=a^x$ decrease with the increase in x and $y>0$ for all $x\in R$.

Thus, $f(x)=a^x=\begin{cases} >1, \text{ for } x<0 \\ 1, \text{ for } x=0 \\ <1, \text{ for } x>0 \end{cases}$

Its graph is given below



Logarithmic Function

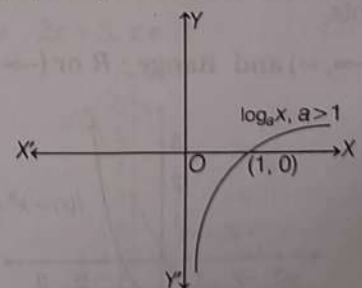
Let $a(\neq 1)$ be a positive real number. Then, the function $f : (0,\infty)\to R$, defined by $f(x)=\log_a x$, is called the logarithmic function.

Domain : $(0,\infty)$ and range : R

As, $a>0$ and $a\neq 1$. So following cases aries

***Case* I** When $a>1$, then values of y increases with the increase in x. So, $y=\log_a x=\begin{cases} <0, \text{ for } x<1 \\ 0, \text{ for } x=1 \\ >0, \text{ for } x>1 \end{cases}$

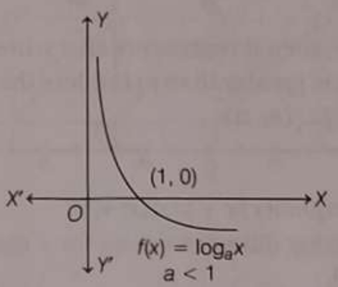
Its graph is given below



***Case* II** When $0 < a < 1$, then values of y decreases with the increase in x.

$$\text{So, } y = \log_a x = \begin{cases} >0, & \text{for } 0 < x < 1 \\ 0, & \text{for } x = 1 \\ <0, & \text{for } x > 1 \end{cases}$$

Its graph is given below



Properties

(i) $\log_a 1 = 0$, where $a > 0, a \neq 1$

(ii) $\log_a a = 1$, where $a > 0, a \neq 1$

(iii) $\log_a(xy) = \log_a(x) + \log_a(y)$, where $a > 0, a \neq 1$ and $x, y > 0$.

(iv) $\log_a(x/y) = \log_a(x) - \log_a(y)$, where $a > 0, a \neq 1$ and $x, y > 0$.

(v) $\log_a x^m = m \log_a x$, where $a > 0, a \neq 1$ and $x > 0$.

(vi) $\log_{a^n} x^m = \dfrac{m}{n} \log_a(x)$, where $a > 0, a \neq 1$ and $x > 0$.

Note *Functions $\log_a x$ and a^x are inverse of each other. So, their graphs are mirror images of each other in the line $y = x$.*

Logarithmic Inequalities

(i) If $a > 1, 0 < m < n$, then $\log_a m < \log_a n$.
e.g. $\log_{10} 20 < \log_{10} 30$

(ii) If $0 > a < 1$, $0 < m < n$, then $\log_a m > \log_a n$.
e.g. $\log_{0.1} 20 > \log_{0.1} 30$

(iii) For $a, m > 0$ if a and m lies on the same side of unity (i.e. 1), then $\log_a m > 0$.
e.g. $\log_2 3 > 0, \log_{0.3} 0.5 > 0$

(iv) For $a, m > 0$ if a and m lies on the different sides of unity (i.e. 1), then $\log_a m > 0$.
e.g. $\log_{0.2} 3 < 0, \log_3 0.5 > 0$

Note *(i)* $\log(x + y) \neq \log x + \log y$

(ii) $\log x \log y \neq \log(xy)$

(iii) $\dfrac{\log x}{\log y} \neq \log\left(\dfrac{x}{y}\right)$

(iv) $(\log x)^n \neq n \log^n$

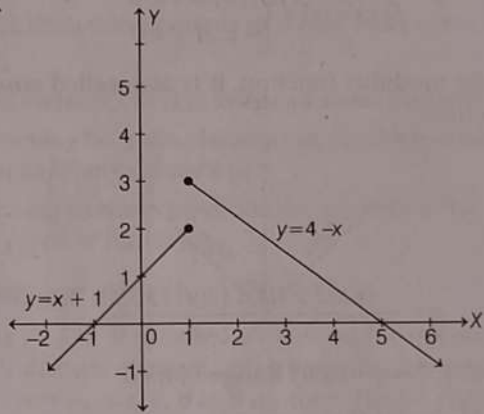
Change of Base Formula

For $a, x, b > 0$ and $a, b \neq 1$, $\log_a x = \dfrac{\log_b x}{\log_b a}$

Note $\log_a x = \dfrac{\log_x x}{\log_x a} = \dfrac{1}{\log_x a}$ *(Verify!)*

Piecewise Defined Functions

A function defined by two or more equations on different parts of the given domain is called piecewise defind function.



e.g. If $f(x) = \begin{cases} x + 1, & \text{if } x < 1 \\ 4 - x, & \text{if } x \geq 1 \end{cases}$

Here $f(3) = 4 - 3 = 1$ as $3 > 1$,
whereas $f(-2) = -2 + 1 = -1$ as $-2 < 1$ and $f(1) = 4 - 1 = 3$.

As (1, 3) lies on line $y = 4 - x$, so it is shown by small black disc on that line. (1, 2) is shown by small white disc on the line $y = x + 1$, because it is not on the line.

Signum Function

The function $f : R \to R$ defined by

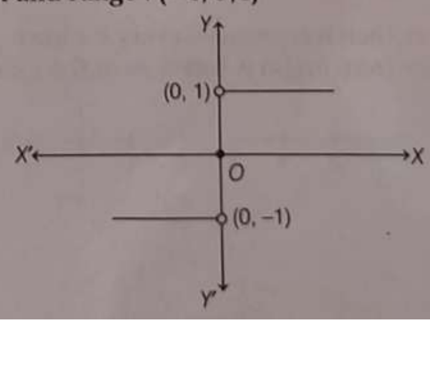
$$f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

or

$$f(x) = \begin{cases} 1, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -1, & \text{when } x < 0 \end{cases}$$

is called the signum function.

Domain : R and range : $\{-1, 0, 1\}$



Properties

(i) For $x > 0$, the graph is line $y = 1$ and for $x < 0$, the graph is line $y = -1$.

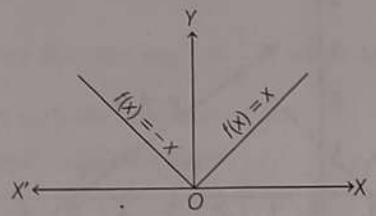
(ii) For $f(0) = 0$, so point $(0, 0)$ is shown by black disc, whereas points $(0, -1)$ and $(0, 1)$ are shown by white disc.

Absolute Value Function (Modulus Function)

The function $f : R \rightarrow R$ defined by

$$f(x) = |x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

is called the modulus function. It is also called absolute value function.



Domain : R or $(-\infty, \infty)$ and **Range** : $[0, \infty)$

It may be observed that

(i) The graph is symmetrical with respect to Y-axis.

(ii) Graph lies above the X-axis.

(iii) It passes through the origin.

(iv) In the first quadrant, it coincides with the graph of the identity function.

Properties

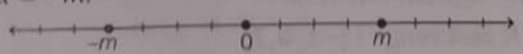
(i) Graph of $f(x) = |x|$ is union of line $y = x$ from quadrant I with the line $y = -x$ from quadrant II. As origin marks the change of directions of the two lines, we call it a critical point.

(ii) Graph is symmetric about Y-axis.

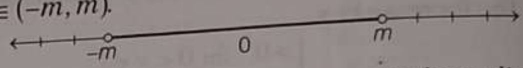
(iii) Graph of $f(x) = |x - 3|$ is the graph of $|x|$ shifted 3 units right and the critical point is $(3, 0)$.

(iv) $f(x) = |x|$, represents the distance of x from origin.

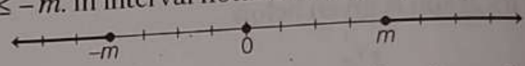
(v) If $|x| = m$, then it represents every x whose distance from origin is m, that is $x = +m$ or $x = -m$.



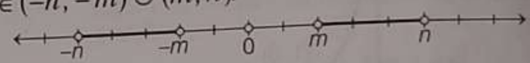
(vi) If $|x| < m$, then it represents every x whose distance from origin is less than m, $0 \leq x < m$ and $0 \geq x > -m$ that is $-m < x < m$. In interval notation $x \in (-m, m)$.



(vii) If $|x| \geq m$, then it represents every x whose distance from origin is greater than or equal to m, so $x \geq m$ and $x \leq -m$. In interval notation $x \in (-\infty, -m] \cup [m, \infty)$.



(viii) If $m < |x| < n$, then it represents all x whose distance from origin is greater than m but less than n, that is $x \in (-n, -m) \cup (m, n)$.



(ix) Triangle inequality $|x + y| \leq |x| + |y|$.
Verify by taking different values for x and y (positive or negative).

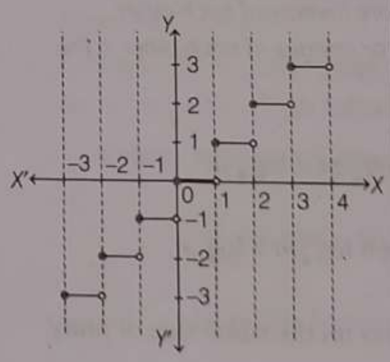
(x) $|x|$ can also be defined as $|x| = \sqrt{x^2} = \max\{x, -x\}$.

Greatest Integer Function (Step Function)

The function $f : R \rightarrow R$ defined by $f(x) = [x]$ is called the greatest integer function, where $[x]$ = integral part of x or greatest integer less than or equal to x.

Domain : R **and range** : I (I = **set of integer**).

e.g., $[7.4] = [7 + 0.4] = 7$
$[0.6] = [0 + 0.6] = 0$
$[-1.5] = [-2 + 0.5] = -2$

Graph	Values of x	$f(x) = [x]$
	⋮	⋮
	$-3 \leq x < -2$	-3
	$-2 \leq x < -1$	-2
	$-1 \leq x < 0$	-1
	$0 \leq x < 1$	0
	$1 \leq x < 2$	1
	$2 \leq x < 3$	2
	$3 \leq x < 4$	3
	⋮	⋮

It may be observed that

(i) $[x]$ = [Integer + Proper positive fraction] = Integer

(ii) It passes through origin.

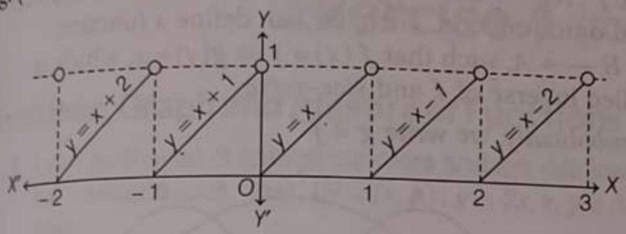
(iii) It is symmetrical in the opposite quadrant.

Fractional Part Function

It is defined as $f(x) = \{x\}$, where $\{x\}$ represents the fractional part of x, i.e. if $x = n + f$, where $n \in I$ and $0 \le f < 1$, then $\{x\} = f$.

Domain : R and Range : $[0, 1)$

e.g. $\{0.7\} = 0.7$, $\{3\} = 0$, $\{-3.6\} = 0.4$



Properties

(i) $\{x\} = x - [x]$

(ii) $\{x\} = x$, if $0 \le x < 1$

(iii) $\{x\} = 0$, if $x \in I$

(iv) $\{-x\} = 1 - \{x\}$, if $x \notin I$

Algebra of Functions

Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions with domain D_1 and D_2, respectively. Then, algebraic operations such as addition, subtraction, multiplication, division and scalar multiplication on two real functions are given below

(i) **Addition of two real functions** The sum function $(f+g)$ is defined by

$$(f+g)(x) = f(x) + g(x), \forall\, x \in D_1 \cap D_2$$

The domain of $(f+g)$ is $D_1 \cap D_2$.

(ii) **Subtraction of two real functions** The difference function $(f-g)$ is defined by

$$(f-g)(x) = f(x) - g(x), \forall\, x \in D_1 \cap D_2$$

The domain of $(f-g)$ is $D_1 \cap D_2$.

(iii) **Multiplication of two real functions** The product function (fg) is defined by

$$(fg)(x) = f(x) \cdot g(x), \forall\, x \in D_1 \cap D_2$$

The domain of (fg) is $D_1 \cap D_2$.

(iv) **Quotient of two real functions** The quotient function is defined by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \forall\, x \in D_1 \cap D_2 - \{x : g(x) = 0\}$$

The domain of $\left(\frac{f}{g}\right)$ is $D_1 \cap D_2 - \{x : g(x) = 0\}$.

(v) **Multiplication of a real function by a scalar** The scalar multiple function cf is defined by

$$(cf)(x) = c \cdot f(x), \forall\, x \in D_1$$

where, c is a scalar (real number).

The domain of cf is D_1.

Note *For any real function $f : D \to R$ and $n \in N$, we define*

$$\underbrace{(fff \ldots f)}_{n \text{ times}}(x) = \underbrace{f(x)\, f(x) \ldots f(x)}_{n \text{ times}} = \{f(x)\}^n, \forall\, x \in D$$

Types of Function

As we know that, if $f : A \to B$ is a function, then f associates all the elements of set A to the elements in set B, such that an element of set A is associated to a unique element of set B. But there are some more possibilities, which may occur in a function, such as

(i) more than one elements of A may have same image in B.

(ii) each elements of B is image of some elements of A.

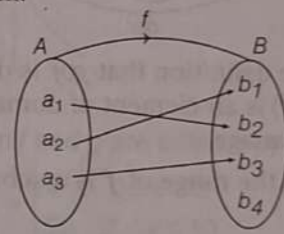
(iii) there may be some elements in B, which are not the images of any element of A.

Corresponding to these possibilities, we define the following types of functions :

One-One (or Injective) Function

A function $f : A \to B$ is called a one-one (or injective) function, if distinct elements of A have distinct images in B, i.e. for every $a_1, a_2 \in A$, if $a_1 \ne a_2$, then $f(a_1) \ne f(a_2)$ and if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

e.g. Let $f : A \to B$ be a function represented by the following diagram.

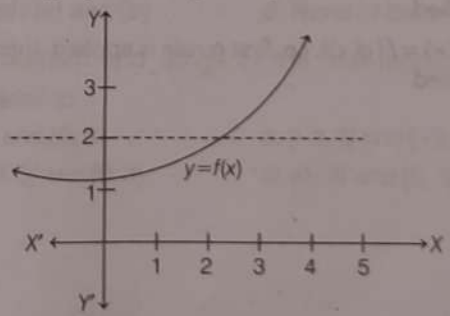


Here, f is a one-one function, because each element have distinct image.

Horizontal Line Test

If no horizontal line intersects the graph of a function in more than one point, then the function is one-one function.

The graph is one-one function as a horizontal line intersects the graph at only one point.



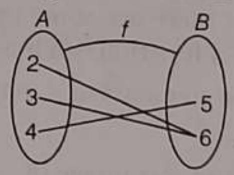
Onto (or Surjective) Function

A function $f: A \to B$ is said to be **onto** (or surjective) function, if every element of B is the image of some elements of A under f, i.e. for every $b \in B$, there exists an element a in A such that $f(a) = b$.

In other words, $f: A \to B$ is onto if and only if

Range of $f = B$

i.e. Range = Codomain

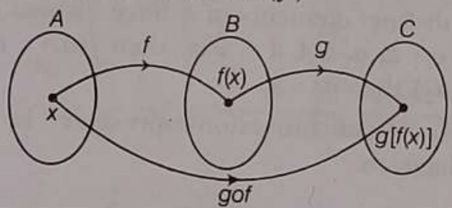


Composition of Functions

Let $f: A \to B$ and $g: B \to C$ be any two functions. Then, the composition of f and g, denoted by gof is defined as function $gof: A \to C$ given by

$$gof(x) = g[f(x)], \forall x \in A$$

Clearly, domain (gof) = domain (f)



It is clear from the definition that gof is defined only, if for each $x \in A$, $f(x)$ is an element of domain of g, so that we can take its g-image.

Thus, gof exists, if the range of f is a subset of domain of g.

Similarly, fog exists, if range of g is a subset of domain of f.

e.g. If $f(x) = 3x + 5$ and $g(x) = x^2$,

then $fog(3) = f[g(3)] = f(3^2) = f(9) = 3 \times 9 + 5 = 32$

Note

(i) $gof(x) = g\{f(x)\}$, i.e. first f rule is applied, then g rule is applied.

(ii) $fog(x) = f\{g(x)\}$, i.e. first g rule is applied, then f rule is applied.

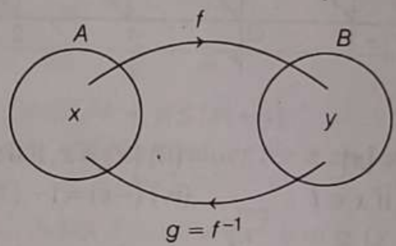
(iii) Generally, the composition of functions is not commutative, i.e. $fog \neq gof$.

(iv) *If $f: R \to R$ and $g: R \to R$ are real functions, then both fog and gof exists.*

Inverse Function

Let $f: A \longrightarrow B$ is a bijective function, i.e. it is one-one and onto function. Then, we can define a function $g: B \longrightarrow A$, such that $f(x) = y \Rightarrow g(y) = x$, which is called inverse of f and *vice-versa*.

Symbolically, we write $g = f^{-1}$



A function whose inverse exists, is called an **invertible function** or **inversible.**

(i) Domain (f^{-1}) = Range (f)

(ii) Range (f^{-1}) = Domain (f)

(iii) If $f(x) = y$, then $f^{-1}(y) = x$ and *vice-varse.*

Note

(i) As f is one-one and onto every element $y \in B$ has a unique element $x \in A$, such that $y = f(x)$.

(ii) If f and g are one-one and onto functions such that $f[g(x)] = x$ for every $x \in$ Domain of g and $g[f(x)] = x$ for every $x \in$ Domain of f, then g is called inverse of function f.

Function g is denoted by f^{-1} (read as f inverse).

i.e. $f[g(x)] = g[f(x)] = x$ then $g = f^{-1}$ which moreover this means $f[f^{-1}(x)] = f^{-1}[f(x)] = x$.

(iii) $f^{-1}(x) \neq [f(x)]^{-1}$, because $[f(x)]^{-1} = \dfrac{1}{f(x)}$

$[f(x)]^{-1}$ is reciprocal of function $f(x)$ where as $f^{-1}(x)$ is the inverse function of $f(x)$.

e.g. If f is one-one onto function with $f(3) = 7$, then $f^{-1}(7) = 3$.