CHAPTER 01

Mathematical Logic

Logic deals with the principles of reasoning. It is sometimes defined as the Science of proof. Mathematics and other Science subjects deal with the reasoning and the arguments and hence every student of Mathematics and other Science should know the principles of logic.

Statement (Proposition)

We convey our daily views in the form of sentence which is a collection of words. This collection of words is called statements, if it has some sense.

Therefore, "a declarative sentence, whose truth or falsity can be decided is called a statement of logical sentence. Statements are denoted by $p, q, r, \ldots$, etc.

e.g. "Delhi is the capital of India" is a statement, while "Do your work" is not a statement.

The working nature of statement in logic is same as nature of switch in circuit.

i.e. Switch → ON (1) / OFF (0) and Statement → True (T) / False (F)

Types of Statements

There are various types of statement as follows

Simple Statement

A statement, which cannot be broken into two or more statements, is called a simple statement. The truth value of the simple statement does not explicitly depend on any other statement. Generally, small letters $p, q, r, \ldots$ denote simple statements.

e.g. (a) $\sqrt{5}$ is a real number (true statement)

(b) Sun set in the east (false statement)

Compound Statement

A statement which is formed by combining two or more simple statements by the words such as 'and', 'or', 'not', 'if then', 'if and only if', is called a compound statement.

e.g. (a) The school is open or a holiday is declared.

(b) A quadrilateral is a square, if and only if its four sides are equal and each adjacent angle is 90°.

Here, we see that in both examples there are two statements combined with 'or', if and only if.

Open Statement

A sentence which contains one or more variables such that when certain values are given to the variable, it becomes a statement, is called an open statement.

e.g. "He is a great man" is an open sentence because in this sentence, 'he' can be replaced by any person.

Important Points Related to Statements

- A true statement is known as a **valid statement**.
- A false statement is known as an **invalid statement**.
- Imperative, exclamatory, interrogative and optative sentences are not statements.
- Mathematical identities are considered to be statements because they can either be true or false but not both.

Truth Value of a Statement

As we know that each statement is either true or false. If a statement is true then its truth value is 'T' and if the statement is false, then its truth value is F.

e.g. (i) 'Delhi is capital of India'. Its truth value is T.

(ii) 'Kanpur is the capital of Uttar Pradesh'. Its truth value is F.

+ two or more statements

Truth Table

A table that shows the relationship between the truth value of compound statement, $S(p, q, r, \ldots)$ and its substatements $p, q, r, \ldots$, etc., is called the truth table of statement S.

If a compound statement has simply n substatements, then there are 2^n rows representing logical possibilities.

(i). For a single statement p, number of rows $= 2^1 = 2$

p
T
F

(ii) For two statements p and q, number of rows $= 2^2 = 4$

p	q
T	T
T	F
F	T
F	F

(iii) For three statements p, q and r, number of rows $= 2^3 = 8$

p	q	r
T	F	F
T	F	T
T	T	F
T	T	T
F	F	F
F	F	T
F	T	F
F	T	T

Logical Connectives

Two or more statements are combined to form a compound statement by using some connecting words.

These connecting words (or symbols) are called logical connectives and are given below

Logical Connectives

Words	Symbols
and	$\wedge$
or	$\vee$
implies that (if ..., then)	$\Rightarrow$ or $(\rightarrow)$
If and only if (implies and is implied by)	$\Leftrightarrow$ or $(\leftrightarrow)$

Elementary Operations of Logic

Formation of compound Statements from simple Statements using logical connectives are termed as elementary operation of logic.

There are five such operations discussed below

Negation (Inversion) of Statement

A statement which is formed by changing the truth value of the given statement by using word like 'no' or 'not', is called negation of a given statement. It is represented by the symbol '~'.

If p is a statement, then negation of p is denoted by '$\sim p$'.

e.g. Let p : Number 2 is greater than 7.

Then, $\sim p$: Number 2 is not greater than 7.

Truth table for 'not' Operation

p	$\sim p$
T	F
F	T

Conjunction

A compound statement formed by combining two simple statements p and q using connective 'and' is called the conjunction of p and q and is represented by $p \wedge q$.

e.g. Let p : Ramesh is a student and q : Ramesh belongs to Allahabad. Then,

$p \wedge q \equiv$ Ramesh is a student and he belongs to Allahabad.

Truth table for 'and' Operation is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The statement $p \wedge q$ is true, if both p and q are true.

The statement $p \wedge q$ is false, if atleast one of p and q or both are false.

Disjunction (Alternation)

at least 1 true

A compound statement formed by two simple statements p and q using connective 'or' is called the disjunction of p and q and is represented by $p \vee q$.

e.g. Let p : Bus left early and q : My watch is going slow.

Then, $p \vee q \equiv$ Bus left early or my watch is going slow.

Truth table for 'or' Operation is

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The statement $p \vee q$ is true, if atleast one of p and q or both are true. The statement $p \wedge q$ is false, if both p and q are false.

Implication (Conditional)

A compound statement formed by two simple statements p and q using connective 'if ..., then ... is called the implication of p and q and is represented by $p \Rightarrow q$ or $p \rightarrow q$ which is read as 'p implies q'. Here, p is called antecedent or hypothesis and q is called consequent or conclusion. e.g.

Let p: Train reaches in time

and q: I can attend the meeting. Then,

$p \Rightarrow q \equiv$ If train reaches in time, then I can attend the meeting.

Truth table for 'if ..., then'

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

It is clear from the truth table that Column III is equal to Column V i.e., '$p \Rightarrow q$' is equivalent to '$\sim p \vee q$.'

The statement $p \Rightarrow q$ is true if both are true or both are false or first is false and second is true.

The statement $p \Rightarrow q$ in false, if first (p) is true and second (q) is false.

Contrapositive of Implication

Contrapositive is a statement which can be formed from a given statement with 'if-then'. Contrapositive of the statement 'if p, then q' is 'if $\sim q$, then $\sim p$'.

e.g. Let p : You are born in India.

q : You are a citizen of India.

Then, the statement "If you are born in India, then you are a citizen of India" is written in symbolic form as $p \Rightarrow q$.

$\because$ The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.

The contrapositive of give statement is "If you are not a citizen of India, then you were not born in India.

Converse of Implication

If p and q are two statements, then the converse of the implication 'if p, then q' is 'if q, then p', i.e., $q \Rightarrow p$

e.g. Let p : Δ ABC is an isosceles triangle.

q : Δ ABC is an equilateral triangle.

Then, the statement "If Δ ABC is an equilateral triangle, then Δ ABC is an isosceles triangle" is written in symbolic form as $q \Rightarrow p$.

$\because$ The converse of $p \Rightarrow q$ is $q \Rightarrow p$.

The converse of give statement is, if ΔABC is an isosceles triangle, then Δ ABC is an equilateral triangle.

Inverse of Implication

If p and q are two statements, then the inverse of 'if p, then q' is 'if $\sim p$, then $\sim q$',

i.e. $\sim p \Rightarrow \sim q$

e.g. Let p : You get a job.

q : Your credentials are good.

Then, the statement "If you get a job, then your credentials are good" is written in symbolic form as $p \Rightarrow q$.

$\because$ The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

The inverse of give statement is "If you do not get a job, then your credentials are not good."

Biconditional Statement (Double Implication)

Two simple statements p and q connected by the phrase 'if and only if', is called the biconditional statement and is represented by the symbol '$\Leftrightarrow$' or $\leftrightarrow$.

e.g. Let p : ΔABC is an isosceles triangle

and q : Two sides of a triangle are equal.

Then, $p \Leftrightarrow q$: ΔABC is an isosceles triangle if and only if two sides of a triangle are equal.

Truth table for 'if and only if'

p	q	$p \Leftrightarrow q$	$\sim p$	$\sim q$	$\sim p \vee q$	$p \vee \sim q$	$(\sim p \vee q) \wedge (p \vee \sim q)$
T	T	T	F	F	T	T	T
T	F	F	F	T	F	T	F
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T

It is clear from the truth table that Column III is equal to Column VIII, i.e., statement $p \Leftrightarrow q$ is equivalent to $(\sim p \vee q) \wedge (p \vee \sim q)$.

The statement $p \Leftrightarrow q$ is true, if either both are true or both are false.

The statement $p \Leftrightarrow q$ is false, if exactly one of them is false.

Common Table for all Logical Connectives

Table for all Logical Connectives

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Statement Pattern

Let p, q and r be simple statements. Then, a statement formed from these simple statements and one or more connectives $\sim, \wedge, \vee, \Rightarrow, \Leftrightarrow$ is called statement pattern.

e.g. $p \wedge \sim q$, $p \vee (p \wedge q)$, $p \wedge (q \Rightarrow r)$, etc.

Logical Equivalence

Two statement patterns, say S_1 and S_2 are said to be logically equivalent, if they have same truth value.

If S_1 and S_2 are equivalent, then it is written as $S_1 \equiv S_2$.

e.g. $\sim (p \vee q) \equiv \sim p \wedge \sim q$

Tautology

A compound statement is called a tautology if it has truth value true (T) whatever may be the truth value of its components.

e.g. Statement $(p \Rightarrow q) \wedge p \Rightarrow q$ is a tautology.

Truth table for this is given below

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$(p \Rightarrow q) \wedge p \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Contradiction (Fallacy)

A compound statement is called contradiction, if its truth value is false (F) whatever may be the truth value of its components.

e.g. Statement $\sim p \wedge p$ is a contradiction.

Truth table for this is given below

p	$\sim p$	$\sim p \wedge p$
T	F	F
F	T	F

Contingency

A statement is said to be a contingency, if it is neither a tautology nor a contradiction. So, the truth value of a contingency is neither always true (T) nor always false (F).

e.g. Statement $(p \wedge q) \vee p \sim q$ is a contingency.

Truth table for this is given below

p	q	$\sim q$	$p \wedge q$	$(p \wedge q) \vee \sim q$
T	T	F	T	T
T	F	T	F	T
F	T	F	F	F
F	F	T	F	T

Quantifier and Quantified Statements

Quantifier Statements

The mathematical statements containing phrases like, 'there exists' and 'for every' or 'for all' are known as quantifier statements.

e.g.(i) There exist a number which is equal to its square.

(ii) For every real number x, x is less than $x+1$.

There are two types of quantifiers

(i) **Universal quantifier** Phrase 'for every' or 'for all', which is denoted by the symbol '$\forall$', are called universal quantifier.

(ii) **Existential quantifier** Phrase 'there exists', which is denoted by the symbol '$\exists$', is called Existential quantifier.

Quantified Statements

Open statement with a quantifier becomes a statement which is called a quantified statement.

Negation of Simple and Compound Statements

If p is any statement, then the denial of statement p is called the negation of statement p and is written as $\sim p$.

e.g. Suppose statement

p : I went to my class yesterday.

The negation of this statement is $\sim p$: I did not go to my class yesterday.

The negation of negation of a statement is the statement itself. i.e. $\sim (\sim p) \equiv p$.

Negations of compound statements having different connectives or quantifire are given below

Negation of a Conjunction

The negation of a conjunction $p \wedge q$, is the disjunction of the negation of its component statement p and the negation of its component statement q.

i.e. $$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

Negation of a Disjunction

The negation of a disjunction $p \vee q$, is the conjunction of the negation of its component statement p or the negation of its component statement q.

i.e. $$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

Negation of Implication

The negation of $p \rightarrow q$ is the conjunction of p and negation of q.

i.e. $$\sim (p \rightarrow q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$$

Negation of Biconditional

The negation of $p \leftrightarrow q$ is

$$\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$$

Negation of a Quantified Statement

'For every' is interchange by some or there exists at least one and *vice-versa*.

Algebra of Statement

Some important algebra of statement are given below:

(i) (a) $p \vee p \equiv p$
(b) $p \wedge p \equiv p$ [Idempotent law]

(ii) (a) $p \vee q \equiv q \vee p$
(b) $p \wedge q \equiv q \wedge p$ [Commutative law]

(iii) (a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
(b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ [Associative law]

(iv) (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ [Distributive law]

(v) (a) $p \vee \text{F} \equiv p$
(b) $p \wedge \text{T} \equiv p$
(c) $p \vee \text{T} \equiv \text{T}$
(d) $p \wedge \text{F} \equiv \text{F}$ [Identity law]

(vi) (a) $p \vee (\sim p) \equiv \text{T}$
(b) $p \wedge (\sim p) \equiv \text{F}$ [Complement law]

(vii) (a) $p \vee (p \wedge p) \equiv p$
(b) $p \wedge (p \vee p) \equiv p$ [Absorption law]

(viii) (a) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$
(b) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$ [De-Morgan's law]