Chapter

5 Gravitation

Quick Review

Kepler's Laws

1 First law/ law of orbits:

All planets move in elliptical orbits with Sun at one of the foci.

2 Second law/ law of equal areas:

The line joining the planet to the Sun sweeps out equal areas in equal time intervals.

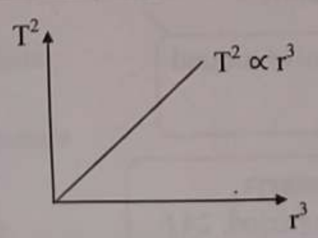
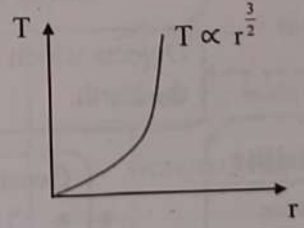
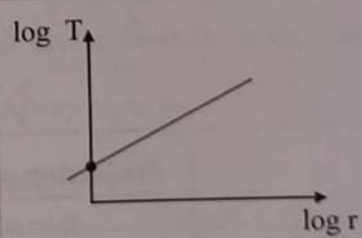
- It is outcome of conservation of angular momentum.
- $\frac{d\vec{A}}{dt} = \frac{\vec{L}}{2m}$ = constant

3 Third law/ law of periods:

The square of the time period of revolution of a planet around the Sun is proportional to the cube of the semi major axis of the ellipse traced by the planet.

$$T^2 \propto r^3 \Rightarrow \left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$$

➤ **Graphical representation of Kepler's third law:**

Graph of	T^2 vs r^3	T vs r	Log T vs log r
Plot			
Equation	$T^2 = (\text{constant})\, r^3$	$T = (\text{constant})\, r^{\frac{3}{2}}$	$T^2 \propto r^3 \Rightarrow T^2 = Kr^3$ Taking log on both sides, $\text{Log } T^2 = \log r^3 + \log K$ i.e., $2 \log T = 3 \log r + \log K$ $\log T = \left(\frac{3}{2}\right)\log r + \frac{\log K}{2}$

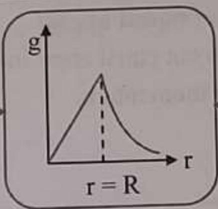
Comparing equation with Standard equation	$y = mx$	$y = (\text{constant})\ x^{\frac{3}{2}}$.	$y = mx + c$
Nature	Straight line passing through origin.	Parabola passing through origin.	Straight line with positive slope and positive intercept

Newton's law of gravitation: *Every particle of matter attracts every other particle in the universe with a force whose magnitude is directly proportional to the product of masses and inversely proportional to the square of distance between them.*

Acceleration due to gravity (g)

Altitude
As h↑, g↓

Depth
As d↑, g↓



Latitude
Equator ($\theta = 0$)
$g = g_{min}$

Shape
$R_{equator} > R_{pole}$
$g_{equator} < g_{pole}$

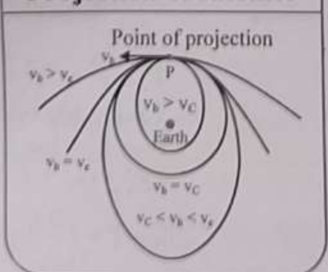
Potential and P.E.

Potential
- Gravitational P.E. per unit mass is gravitational potential
- $V_\infty = 0$
- Independent of mass of object.

Potential energy
The amount of work done against conservative forces is change in P.E.

Earth Satellite
Objects which revolve around the Earth.

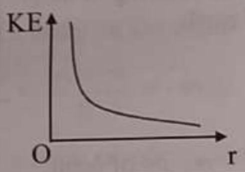
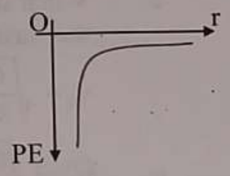
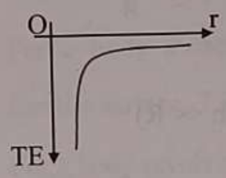
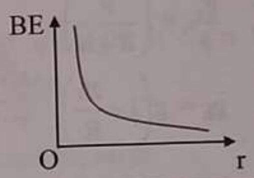
Projection of satellite



Geostationary:
- Time period: 24 h
- Equatorial plane

Polar:
- Time period: 85 min
- Low altitude
- Orbits in north-south direction.

➢ Energy associated with satellite:

Sr. No.	Kinetic energy $= \frac{GMm}{2r}$	Potential energy $= \frac{-GMm}{r}$	Total energy $= \frac{-GMm}{2r}$	Binding energy $= -TE = \frac{GMm}{2r}$
i.	Decreases with increase in r	Increases with increase in r	Increases with increase in r	Decreases with increase in r
ii.	$KE \propto \frac{1}{r}$	$PE \propto -\frac{1}{r}$	$TE \propto -\frac{1}{r}$	$BE \propto \frac{1}{r}$
iii.				

➢ Different cases for mass m as a satellite:

For object of mass 'm'	K.E.	P.E.	T.E. = K.E. + P.E.	B.E. = – T.E.
Stationary at $r = (R+h)$	0	$-\frac{GMm}{(R+h)}$	$-\frac{GMm}{(R+h)}$	$\frac{GMm}{(R+h)}$
Stationary at surface of the Earth $(r = R)$	0	$-\frac{GMm}{R}$	$-\frac{GMm}{R}$	$\frac{GMm}{R}$
Revolving at height h $(r = R+h)$	$\frac{GMm}{2(R+h)}$	$-\frac{GMm}{(R+h)}$	$-\frac{GMm}{2(R+h)}$	$\frac{GMm}{2(R+h)}$
Revolving close to earth's surface $(r = R)$	$\frac{GMm}{2R}$	$-\frac{GMm}{R}$	$-\frac{GMm}{2R}$	$\frac{GMm}{2R}$

Formulae

1. Kepler's formula for planetary motion:

i. Kepler's law of equal areas:

Areal velocity, $\frac{\overrightarrow{\Delta A}}{\Delta t} = \frac{\vec{L}}{2m} = \text{constant}$

ii. Kepler's law of periods:

a. $T^2 \propto r^3$ or $\frac{T^2}{r^3} = \text{constant}$

b. Ratio of time periods of two planets, $\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{3/2}$

Where, r = average distance of planet from the Sun.

2. Gravitational force:

i. Scalar form: $F = \frac{Gm_1m_2}{r^2}$

ii. Gravitational force between two equal masses,

$F = \frac{Gm^2}{r^2}$

iii. Vector form: $\vec{F}_{21} = G\frac{m_1m_2}{r^2}(-\hat{r}_{21})$

where, $\hat{r}_{21}$ = the unit vector from m_1 to m_2 and force $\vec{F}_{21}$ is directed from m_2 to m_1.

3. Gravitational constant: $G = \frac{Fr^2}{m_1m_2}$

4. Measurement of G using Cavendish balance:

Restoring torque. $\tau = G\frac{mM}{r^2}L = K\theta$

Where, r = initial distance of separation between the centres of the large (with mass M) and the neighbouring small (with mass m) sphere,
K = restoring torque per unit angle and θ = the angle of twist.

5. Acceleration due to gravity:

i. On the surface of the earth, $g = \frac{GM}{r^2}$

ii. In terms of density, $g = \frac{4}{3}\pi\rho GR$

6. Variation in acceleration due to gravity:

i. At a height above the earth, $g_h = \frac{GM}{(R+h)^2} = \frac{GM}{r^2}$

where $r = R + h$

ii. $\frac{g_h}{g} = \left(\frac{R}{R+h}\right)^2$

iii. $g_h = g\left(1 - \frac{2h}{R}\right)$ ....(For $h << R$)

iv. At a depth below the earth, $g_d = g\left(1 - \frac{d}{R}\right)$

v. At latitude θ, $g' = g - R\omega^2\cos^2\theta$

a. At equator, $\theta = 0$

Hence $g' = g - R\omega^2$

b. At poles, $\theta = 90°$

Hence $g' = g$

c. If earth stops rotating, then the value of g at equator increases by $R\omega^2$

7. Gravitational potential energy of a body:

i. For body stationary on the earth:

$U = \frac{-GMm}{R} = -mgR$

ii. For body revolving around the earth:

$U = \frac{-GMm}{r}$

8. Gravitational Potential:

i. In terms of potential energy: $V = \frac{U}{m}$

ii. On surface of the earth: $V = -\frac{GM}{R}$

iii. At certain height above the surface of the earth:

$V = -\frac{GM}{r}$

9. Escape Velocity:

i. For a body stationary on the earth's surface

a. $v_e = \sqrt{\frac{2GM}{R}}$

b. $v_e = \sqrt{2gR}$

c. $v_e = R\sqrt{\frac{8\pi\rho G}{3}}$

ii. For a body revolving around the Earth's surf at a height h

a. $v_e = \sqrt{\frac{2GM}{r}}$ b. $v_e = \sqrt{2gr}$

c. $v_e = R\sqrt{\frac{g}{r}}$

10. Critical (orbital) velocity:

i. a. When satellite is orbiting close to surface of earth, $v_c = \sqrt{\frac{GM}{R}}$

b. When satellite is orbiting at height 'h' from the surface of earth,

$v_c = \sqrt{\frac{GM}{R+h}}$

c. $v_c = \sqrt{\frac{GM}{r}}$, r = radius of orbit.

ii. In terms of acceleration due to gravity,

a. $v_c = \sqrt{gR}$ (when satellite is orbiting close to the surface of earth)

b. $v_c = \sqrt{g_h(R+h)}$ where

g_h = acceleration due to gravity at height

c. $v_c = \sqrt{\frac{gR^2}{R+h}}$ (if g_h is not known)

iii. In terms of density of earth (only close to surface of earth),

$v_c = 2R\sqrt{G\frac{\pi}{3}\rho}$

where ρ = Density of earth

iv. In terms of escape velocity, $v_c = \frac{v_e}{\sqrt{2}}$

11. Weightlessness:

i. when lift moves upward with acceleration 'a'

$W = m(g + a)$

ii. when lift moves downward with acceleration 'a'

$W = m(g - a)$

iii. when lift moves as free fall.

$W = 0$

where, W = Weight of the body in the lift

12. Time Period of a satellite:

At height h from surface of earth,

i. $T = \sqrt{\frac{4\pi^2}{GM}(R+h)^3}$ ii. $T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$

iii. $T = 2\pi\sqrt{\frac{R+h}{g_h}}$

iv. Close to surface of earth, $T = 2\pi\sqrt{\frac{R}{g}}$

v. Ratio of time periods of two satellites at different radius of orbits,

$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{3/2} = \left(\frac{R+h_1}{R+h_2}\right)^{3/2}$

13. Binding energy:

i. For a body stationary on the Earth,

B.E. = $\frac{GMm}{R}$

ii. For a body revolving around the Earth at a height h, B.E. = $\frac{GMm}{2(R+h)} = \frac{GMm}{2r}$

14. Potential energy:

i. For a body on the earth's surface,

P.E. = $\frac{-GMm}{R} = -mgR$

ii. For a body revolving around the Earth,

P.E. = $\frac{-GMm}{r}$

15. Kinetic Energy:

i. For a body stationary on the Earth, K.E. = 0

ii. For a body revolving around the Earth at a height h,

K.E. = $\frac{GMm}{2r}$

where r = R + h

16. Total Energy:

i. For a body on the earth's surface,

T.E. = $\frac{-GMm}{R} = -mgR$

ii. For a body stationary at a height h from the Earth's surface, T.E. = $\frac{-GMm}{r}$

iii. For a body revolving around the earth,

T.E. = $\frac{-GMm}{2r}$

Shortcuts

1. Mutual Whenever comparison between mass and radius of earth and other planets/ moons is given in the question, then acceleration due to gravity on that planet/moon can easily be calculated by multiplying the g on earth by $\frac{M'}{(R')^2}$. [Where, $M' = M_{planet}/M_{earth}$ and $R' = R_{planet}/R_{earth}$]

2.

	Quantity changed by x%	Quantity kept constant	gravitational acceleration (g) changes by
i.	mass	radius	x%
ii.	density	radius	x%
iii.	radius	density	x%
iv.	radius	mass	2x%

3. At height $\left(h = \frac{R}{n}\right)$, the value of acceleration due to gravity is given by, $g_h = \left(\frac{n}{n+1}\right)^2 g$

4. If h << R, then decrease in the value of g with height:

i. $\Delta g = g - g' = \frac{2hg}{R}$ ii. $\frac{\Delta g}{g} = \frac{g-g'}{g} = \frac{2h}{R}$ iii. $\frac{\Delta g}{g} \times 100\% = \frac{2h}{R} \times 100\%$

5. For acceleration due to gravity at height h above the Earth's surface, if $g_h = \frac{g}{n}$ then, $h = \left(\sqrt{n} - 1\right) R$.

6. The depth d at which the value of acceleration due to gravity becomes $\frac{1}{n}$ times the value at the surface is $d = \left(\frac{n-1}{n}\right)R$

7. When a particle of mass m is taken from the Earth's surface to a height h = nR, then the change in P.E. can be calculated as, $\Delta U = mgR\left(\frac{n}{n+1}\right)$

8. If the radius of the earth is made n times keeping the mass unchanged, the escape velocity will become $(1/\sqrt{n})$ times the present value. Thus, in percentage change, the escape velocity changes as, $\Delta v_e = -\frac{1}{2} \times \Delta R\%$

9. If ratio of the radii of two planets is r and the ratio of the acceleration due to gravity on their surface is a, then ratio of escape velocities is $\sqrt{a\,r}$.

10. If the altitude of the satellite is n times the radius of the earth, then the orbital velocity will be $(1/\sqrt{1+n})$ times the orbital velocity near the surface of the earth.

11. If the radius of the orbit of a satellite is n times the radius of the earth, then its orbital velocity will be $(1/\sqrt{n})$ times the orbital velocity near the surface of the earth.

12. When a body is projected horizontally with velocity v, from any height from the surface of earth, then there are following possibilities:

Cases	Orbit	K.E. and P.E.	T.E.
$v < v_c$	Spiral	K.E. < P.E.	–ve
$v = v_c$	Circular	K.E. < P.E.	–ve
$v_c < v < v_e$	Elliptical	K.E. < P.E.	–ve
$v = v_e$	Parabolic	K.E. = P.E.	0
$v > v_e$	Hyperbolic	K.E. > P.E.	+ve

13. If the gravitational force is inversely proportional to the n^{th} power of distance r, then the orbital velocity of satellite, $v_c \propto r^{n/2}$ and time period is $T \propto r^{(n+2)/2}$.

14. If body is projected with velocity v $(v < v_e)$ then height up to which it will rise, $h = \dfrac{R}{\left(\dfrac{v_e^2}{v^2} - 1\right)}$

15. Work done in sending a body from surface to height h,

i. $W = mgR\left(\dfrac{h}{R+h}\right)$ (when h is comparable to R)

ii. If $h = nR$, then $W = mgR\left(\dfrac{n}{n+1}\right)$

iii. If $h = R$, then $W = \dfrac{1}{2}mgR$

iv. If h is very small as compared to radius of the earth, then term $\dfrac{h}{R}$ can be neglected,

i.e., $W = \dfrac{mgh}{\left(1+\dfrac{h}{R}\right)} = mgh$ $\quad\ldots\left[\text{As}\ \dfrac{h}{R} \to 0\right]$

16. Work done in taking a satellite from surface to height h and putting it in orbit $= \dfrac{mgR(R+2h)}{2(R+h)}$.