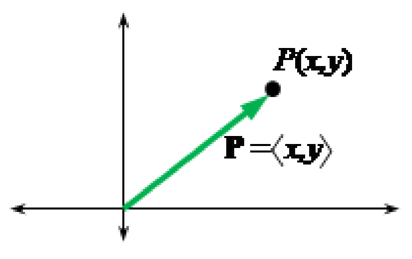
Lines and Planes

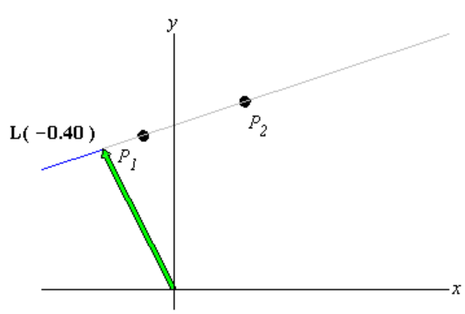
Part 1: Equations of Lines

In this section, we use vectors, the cross product, and the dot product to explore lines and planes in 3-dimensional space. We begin by exploring the *vector* equation of a line in the xy-plane.

To begin with, a *position vector* is a vector $\mathbf{P} = \langle x, y \rangle$ whose initial point is fixed at the origin, so that each point $P(x, y)$ in $\mathbb{R}^2$ corresponds to a position vector $\mathbf{P} = \langle x, y \rangle$.



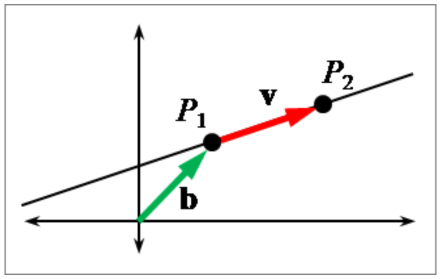
(note: because a position vector cannot be translated, it is not really a vector but should instead should be considered as a means of using the arithmetic of vectors with points in the plane). Thus, a line corresponds to the endpoints of a set of 2-dimensional position vectors.



Indeed, if we define $\mathbf{L}(t)$ to be a *vector-valued function*, which is a function that maps inputs t to output vectors $\mathbf{L}(t)$, then a line in 2-dimensions is a vector valued function of the form

$$\mathbf{L}(t) = \mathbf{v}t + \mathbf{b}$$

where $\mathbf{b}$ is a fixed point (= position vector) on the line and $\mathbf{v}$ is a constant "slope" vector for the line.



The variable t is called a *parameter*, and it can be thought of as the "label" or "index" of each point on the line. We use t for the parameter because in applications a point is often labeled (i.e., indexed) by the *time* at which an object is located at that point.

EXAMPLE 1 Find the vector form of the line through points $(1, 3)$ and $(4, 0)$, and then determine the slope-intercept form of the line *from the vector form.*

Solution: To begin with, the "slope vector" $\mathbf{v}$ is

$$\mathbf{v} = \overrightarrow{P_1P_2} = \langle 4-1, 0-3 \rangle = \langle 3, -3 \rangle$$

Let us let $\mathbf{b} = \langle 1, 3 \rangle$ (i.e., $\mathbf{b}$ is a *position vector* (= point) for a point on the line). Then the vector equation of the line is

$$\begin{aligned} \mathbf{L}(t) &= \mathbf{v}t + \mathbf{b} \\ &= \langle 3, -3 \rangle t + \langle 1, 3 \rangle \\ &= \langle 3t+1, -3t+3 \rangle \end{aligned}$$

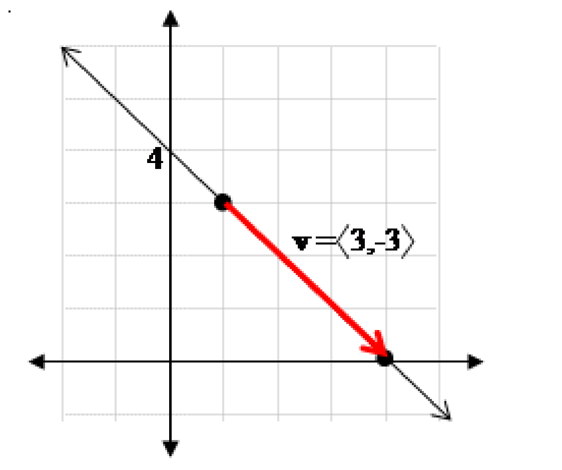
To find the slope-intercept form of the line, let us notice that the x-coordinates are given by $x = 3t+1$ and the y-coordinates are given by $y = -3t+3$. Solving for t in the first equation yields

$$x - 1 = 3t \quad \textit{and} \quad t = \frac{x-1}{3}$$

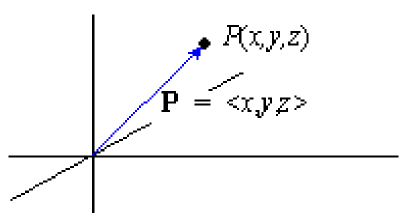
Substituting into $y = -3t+3$ yields

$$y = -3\left(\frac{x-1}{3}\right) + 3 = -(x-1) + 3 = -3x + 4$$

That is, $y = 4 - 3x$ is the slope-intercept form of the line.



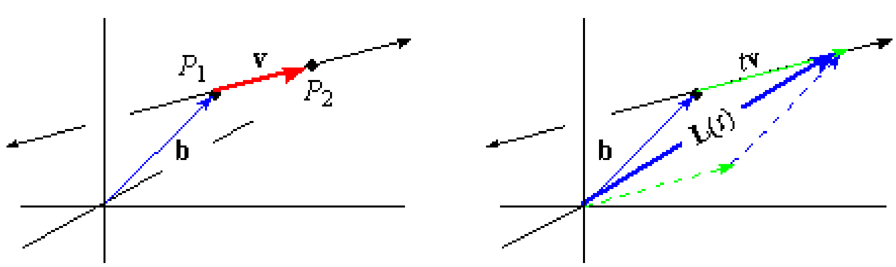
The advantage of a vector equation of a line over point-slope and slope-intercept forms of a line is that a vector equation generalizes to 3 or more dimensions, whereas the slope-based forms of a line do not. Indeed, if we notice that point $P(x, y, z)$ in $\mathbb{R}^3$ corresponds to a 3-dimensional *position vector* $\mathbf{P} = \langle x, y, z \rangle$,



then a line in 3-dimensional space through two points (= position vectors) $\mathbf{P}_1$ and $\mathbf{P}_2$ is of the form

$$\mathbf{L}(t) = \mathbf{b} + \mathbf{v}t \tag{1}$$

where $\mathbf{v}$ is the vector from $\mathbf{P}_1$ to $\mathbf{P}_2$ and $\mathbf{b}$ is the position vector of a point on the line (such as $\mathbf{b} = \mathbf{P}_1$).   .



A physical interpretation of (1) is that if an object moves along the line $\mathbf{L}(t)$ at a constant speed equal to the magnitude of $\mathbf{v}$, then t would be the *time* at which the object is located at point $\mathbf{L}(t)$ on the line.

EXAMPLE 2 Find the equation of the line which passes through the points $P_1(1,0,1)$ and $P_2(4,3,2)$, and interpret the result given that an object is located at point $\mathbf{P}_1$ at time $t=0$ sec and at point $\mathbf{P}_2$ at time $t=1$ sec.

Solution: The vector $\mathbf{v}$ is given by

$$\mathbf{v} = \overrightarrow{P_1P_2} = \langle 4-1, 3-0, 2-1\rangle = \langle 3,3,1\rangle$$

As a result, the equation of the line is

$$\mathbf{L}(t) = \mathbf{P}_1 + t\mathbf{v} = \langle 1,0,1\rangle + \ t\langle 3,3,1\rangle$$

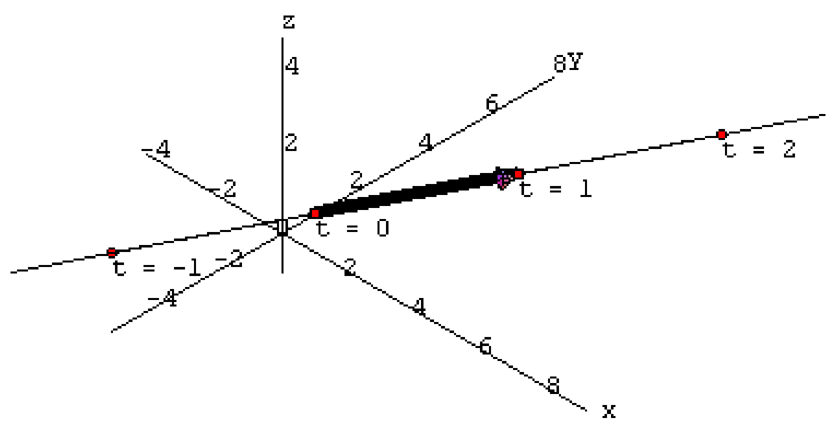
which reduces to $\mathbf{L}(t) = \langle 3t+1, 3t, t+1\rangle$. For example, $t=0$ and $t=1$ yields

$$\mathbf{L}(0) = \langle 1,0,1\rangle = \mathbf{P}_1 \quad \textit{and} \quad \mathbf{L}(1) = \langle 3\cdot 1+1, 3\cdot 1, 1+1\rangle = \langle 4,3,2\rangle = \mathbf{P}_2$$

Other points on the line follow from other choices of t. For example, when $t=2$, we obtain the position vector

$$\mathbf{L}(2) = \langle 3\cdot 2+1, 3\cdot 2, 2+1\rangle = \langle 7,6,3\rangle,$$

Likewise, when $t=-1$, we get $\mathbf{L}(-1) = \langle -2,-3,0\rangle$, which corresponds to the point $P_0\,(-2,-3,0)$.



Finally, we can obtain an especially useful form of a line if we notice that $\mathbf{v} = \mathbf{P}_2 - \mathbf{P}_1$. Substituting into (1) leads to $\mathbf{L}\,(t) = \mathbf{P}_1 + t\,(\mathbf{P}_1 - \mathbf{P}_2)$, which reduces to

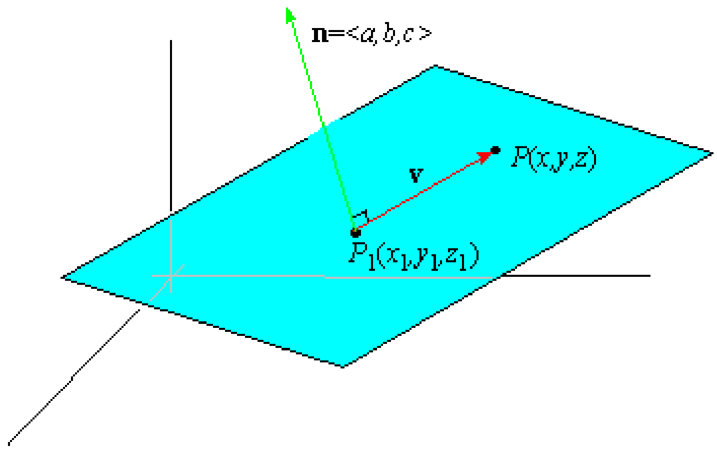
$$\mathbf{L}\,(t) = (1-t)\,\mathbf{P}_1 + t\mathbf{P}_2 \tag{2}$$

Clearly, $\mathbf{L}\,(0) = \mathbf{P}_1$ and $\mathbf{L}\,(1) = \mathbf{P}_2$, so that $\mathbf{L}\,(t)$ with $0 \le t \le 1$ is the *line segment* with endpoints $\mathbf{P}_1$ and $\mathbf{P}_2$. Moreover, (2) shows that the order of the points is not important in the equation of a line.

Check your Reading: What is the parameter for the line with vector equation $K\,(s) = \mathbf{m}s$?

Equation of a Plane

Given any plane, there must be at least one nonzero vector $\mathbf{n} = \langle a, b, c \rangle$ that is perpendicular to every vector **v** parallel to the plane.



In particular, the plane through the point $P_1(x_1, y_1, z_1)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is the set of points $P(x, y, z)$ such that the vectors

$$\mathbf{v} = \overrightarrow{P_1 P} = \langle x - x_1, y - y_1, z - z_1 \rangle$$

are perpendicular to **n**. That is, the plane is the set of all points such that

$$\mathbf{n} \cdot \mathbf{v} = 0 \qquad or \qquad \langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$

Computing the inner product then leads us to the following definition:

> **Definition 4.3:** The equation of the plane with normal $\mathbf{n} = \langle a, b, c \rangle$ through the point $\langle x_1, y_1, z_1 \rangle$ is
>
> $$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \tag{3}$$

If $c \neq 0$, then we can transform (3) into *functional form*, which is

$$z = mx + ny + d$$

where $m = -a/c$, $n = -b/c$ and $d = (ax_1 + by_1 + cz_1)/c$.

> *EXAMPLE 3* Find the equation of the plane with normal $\mathbf{n} = \langle 1, 2, 7 \rangle$ which contains the point $P_1(5, 3, 4)$.

Solution: To do so, we simply substitute into the equation (3). The result is that

$$1\,(x-5)+2\,(y-3)+7\,(z-4)=0$$

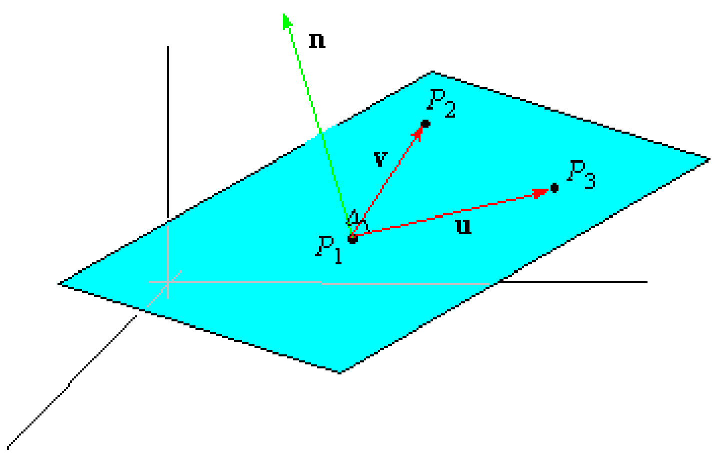
We then solve for z to obtain the *functional form*:

$$z=-\frac{1}{7}x-\frac{2}{7}y+\frac{39}{7}$$

To find the equation of the plane through three non-collinear points P_1, P_2, and P_3, we first form the two vectors

$$\mathbf{u}=\overrightarrow{P_1P_2}\qquad\text{and}\qquad\mathbf{v}=\overrightarrow{P_1P_3}$$

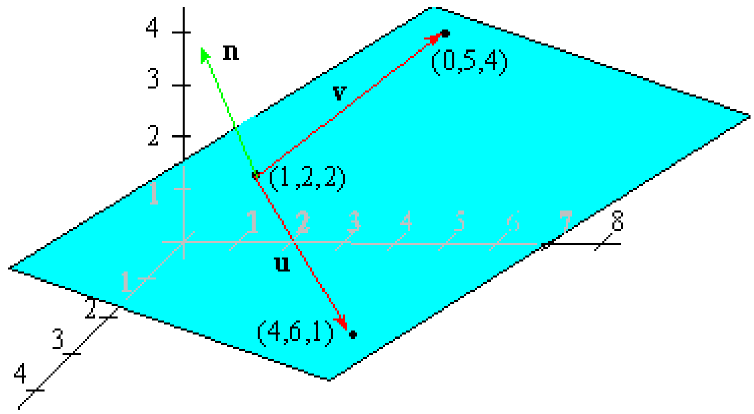
Since $\mathbf{u}$ and $\mathbf{v}$ are both parallel to the plane, the cross product $\mathbf{u}\times\mathbf{v}$ is perpendicular to the plane.



That is, $\mathbf{n}=\mathbf{u}\times\mathbf{v}$ is a normal to the plane, which allows us to use Definition 4.3 to write the equation of the plane.

EXAMPLE 4 Find the equation of the plane passing through $P_1\,(1,2,2)$,

$P_2\,(4,6,1)$ and $P_3\,(0,5,4)$.



Solution: We form the vectors

$$\begin{aligned}\mathbf{u} &= \overrightarrow{P_1P_2} = \langle 4-1, 6-2, 1-2\rangle = \langle 3,4,-1\rangle \\ \mathbf{v} &= \overrightarrow{P_1P_3} = \langle 0-1, 5-2, 4-2\rangle = \langle -1,3,2\rangle\end{aligned}$$

The cross product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \langle 11,-5,13\rangle$$

Thus, the equation of the plane through P_1, P_2, and P_3 is

$$11\,(x-1) - 5\,(y-2) + 13\,(z-2) = 0$$

which in functional form is

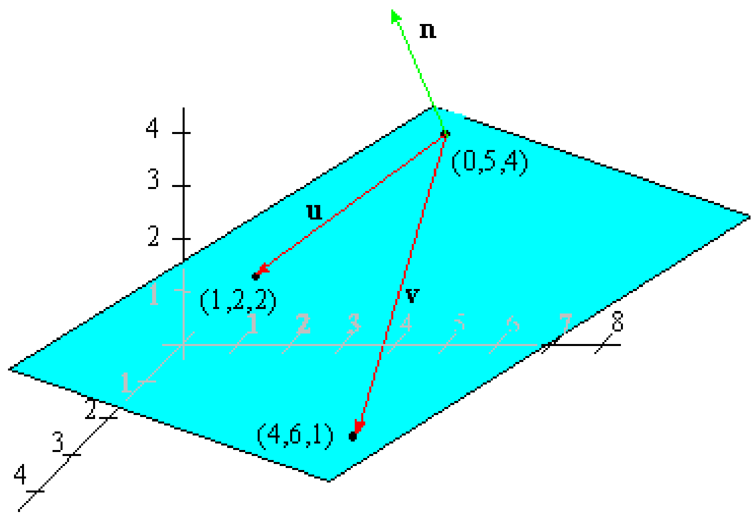
$$z = -\frac{11}{13}x + \frac{5}{13}y + \frac{27}{13} \tag{4}$$

Check your Reading: Explain why $P_2\,(1,2,2)$ satisfies (4).

More with Equations of Planes

Using the points in a different order may result in a different normal, and (3) may also appear to be different. However, the functional form will be the same regardless of how the points are labeled.

EXAMPLE 5 Redo example 4 with $P_2\,(1,2,2)$, $P_3\,(4,6,1)$ and $P_1\,(0,5,4)$.



Solution: To do so, we form the vectors

$$\begin{aligned}\mathbf{u} &= \overrightarrow{P_1P_2} = \langle 1-0, 2-5, 2-4\rangle = \langle 1,-3,-2\rangle \\ \mathbf{v} &= \overrightarrow{P_1P_3} = \langle 4-0, 6-5, 1-4\rangle = \langle 4,1,-3\rangle\end{aligned}$$

Once again, the cross product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \langle 11,-5,13\rangle$$

Thus, the equation of the plane through P_1, P_2, and P_3 is

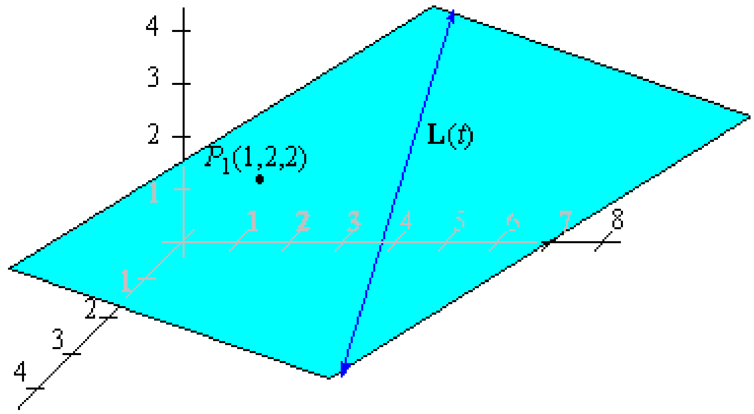
$$11\,(x-0) - 5\,(y-5) + 13\,(z-4) = 0$$

Reducing to functional form then yields the equation

$$z = -\frac{11}{13}x + \frac{5}{13}y + \frac{27}{13}$$

A plane can also be determined by a line and a point not on that line, or by two intersecting lines. In particular, if given the vector equation of a line $\mathbf{L}\,(t)$, points in the plane can be obtained by choosing different values of the parameter t.

EXAMPLE 6 Find the equation of the plane containing the point $P_1(1,2,2)$ and the line $\mathbf{L}(t) = (4t+8, t+7, -3t-2)$



Solution: If we let $t = 0$, then $\mathbf{L}(0) = (8, 7, -2)$. If we let $t = 1$, then $\mathbf{L}(1) = (12, 8, -5)$. Thus, we need to find the equation of the plane through $P_1(1,2,2)$, $P_2(8,7,-2)$, and $P_3(12,8,-5)$. We first form two vectors

$$\mathbf{u} = \overrightarrow{P_1P_2} = \langle 7, 5, -4 \rangle, \qquad \mathbf{v} = \overrightarrow{P_1P_3} = \langle 11, 6, -7 \rangle$$

Thus, a normal to the plane is

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle -11, 5, -13 \rangle$$

so that the equation of the plane is

$$-11(x-1) + 5(y-2) - 13(z-2) = 0$$

Solving for z then yields

$$z = -\frac{11}{13}x + \frac{5}{13}y + \frac{27}{13}$$

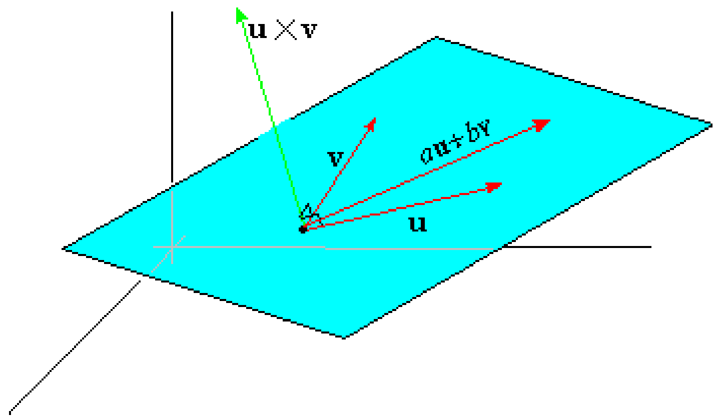
Check your Reading: Why are the planes in examples 4, 5, and 6 all the same?

The Span of Two Non-parallel Vectors

A *linear combination* of non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector of the form

$$a\mathbf{u}+b\mathbf{v}$$

where a and b are numbers. The set of all linear combinations of 2 non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ is called the *span* of $\mathbf{u}$ and $\mathbf{v}$.



Moreover, if $\mathbf{u}$ and $\mathbf{v}$ are parallel to given plane P, then the plane P is said to be *spanned* by $\mathbf{u}$ and $\mathbf{v}$.

> *EXAMPLE 7* Find the equation of the plane through the point $P_1\,(0,0,0)$ spanned by the vectors $\mathbf{u} = \langle 1,2,1\rangle$ and $\mathbf{v} = \langle 3,1,-2\rangle$
>
> **Solution:** The normal to the plane is $\mathbf{n} = \mathbf{u}\times\mathbf{v}$, which is
>
> $$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \langle 1,2,1\rangle \times \langle 3,1,-2\rangle = \langle -5,5,-5\rangle$$
>
> Definition 4.1 then implies that the equation of the plane is
>
> $$-5\,(x-0) + 5\,(y-0) - 5\,(z-0) = 0$$
>
> Solving for z then produces the functional form $z = y - x$.

Finally, since $\mathbf{r}\times(\mathbf{u}\times\mathbf{v})$ is orthogonal to $\mathbf{u}\times\mathbf{v}$, the vector $\mathbf{r}\times(\mathbf{u}\times\mathbf{v})$ must be in the span of $\mathbf{u}$ and $\mathbf{v}$. That is, there must be scalars a and b such that $\mathbf{r}\times(\mathbf{u}\times\mathbf{v}) = a\mathbf{u} + b\mathbf{v}$. And since $\mathbf{r}\times(\mathbf{u}\times\mathbf{v})$ is orthogonal to $\mathbf{r}$, we also have

$$\mathbf{r}\cdot(a\mathbf{u}+b\mathbf{v}) = 0 \qquad \textit{or} \qquad a\mathbf{r}\cdot\mathbf{u} = -\,b\mathbf{r}\cdot\mathbf{v}$$

Indeed, it is straightforward to show that $a = \mathbf{r}\cdot\mathbf{v}$ and $b = -\mathbf{r}\cdot\mathbf{u}$, thus leading to the *triple vector product identity:*

$$\mathbf{r}\times(\mathbf{u}\times\mathbf{v}) = (\mathbf{r}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})\,\mathbf{v} \tag{5}$$

Exercises 41-44 will provide additional insights into the interpretation and application of (5).

EXAMPLE 8 Verify (5) for the vectors

$$\mathbf{u} = \langle 1,2,0\rangle, \quad \mathbf{v} = \langle 1,1,0\rangle, \quad \textit{and} \quad \mathbf{r} = \langle 3,2,1\rangle$$

Solution: Since $\mathbf{u}\times\mathbf{v} = \langle 0,0,-1\rangle$, the left side of (5) is

$$\mathbf{r}\times(\mathbf{u}\times\mathbf{v}) = \langle 3,2,1\rangle \times \langle 0,0,-1\rangle = \langle -2,3,0\rangle$$

Notice that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{r}\times(\mathbf{u}\times\mathbf{v})$ are parallel to the xy-plane. The right side of (5) is

$$\begin{aligned}(\mathbf{r}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})\,\mathbf{v} &= (\langle 3,2,1\rangle\cdot\langle 1,1,0\rangle)\,\mathbf{u} - (\langle 3,2,1\rangle\cdot\langle 1,2,0\rangle)\,\mathbf{v} \\ &= 5\mathbf{u} - 7\mathbf{v} \\ &= 5\langle 1,2,0\rangle - 7\langle 1,1,0\rangle \\ &= \langle 5-7, 10-7, 0\rangle \\ &= \langle -2,3,0\rangle\end{aligned}$$

Exercises

Find the vector equation of the line through the given points. If the points are in $\mathbb{R}^2$, use the vector equation to find the slope-intercept equation for the line. If the points are in R^3, then determine where an object moving on the line would be at time $t=2$ sec given that it is at P_1 at $t=0$ sec and at P_2 at $t=1$ sec.

1. $P_1(0,7)$, $P_2(1,2)$
2. $P_1(0,3)$, $P_2(1,-3)$
3. $P_1(-1,-1)$, $P_2(5,2)$
4. $P_1(-2,3)$, $P_2(5,3)$
5. $P_1(7,9,2)$, $P_2(3,7,0)$
6. $P_1(0,0,0)$, $P_2(1,3,1)$
7. $P_1(\pi,e,2)$, $P_2(\pi-e,\pi+e,0)$
8. $P_1\left(\pi,e^{-1},\ln(2)\right)$, $P_2(\tan(1),\sin(3),e)$

Find the equation of the plane through the three given points.

9. $P_1(0,0,0)$, $P_2(1,2,1)$, $P_3(2,1,1)$
10. $P_1(0,0,0)$, $P_2(2,3.2)$, $P_3(1,1,1)$
11. $P_1(1,3,2)$, $P_2(-2,5,7)$, $P_3(2,1,4)$
12. $P_1(-1,4,3)$, $P_2(3,4,6)$, $P_3(0,-3,2)$
13. $P_1(0,0,0)$, $P_2(1,2,0)$, $P_3(2,1,0)$
14. $P_1(1,3,2)$, $P_2(2,-7,9)$, $P_3(-2,1,5)$
15. $P_1(0,0,0)$, $P_2(1,1,0)$, $P_3(1,1,1)$
16. $P_1(0,0,0)$, $P_2(0,0,1)$, $P_3(1,1,1)$
17. $P_1(1,2,5)$, $P_2(0,-3,3)$, $P_3(-1,-2,1)$
18. $P_1(1,0,0)$, $P_2(0,1,10)$, $P_3(0,01)$

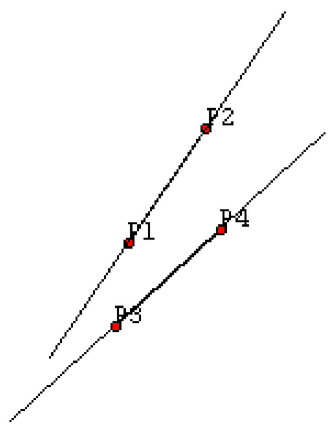
Find the equation of the plane with the given description.

19. Through $(0,0,0)$ and $\mathbf{L}(t) = (2t,3t,4)$
20. Through $(1,3,2)$ and $\mathbf{L}(t) = (2t-1, 1-4t, 7-2t)$
21. Through $\mathbf{K}(s) = (s,0,0)$ and $\mathbf{L}(t) = (0,t,0)$
22. Through $\mathbf{K}(s) = (s,0,0)$ and $\mathbf{L}(t) = (0,t,0)$
23. Through $(0,0,0)$ and spanned by $\mathbf{u} = \langle 2,1,2\rangle$, $\mathbf{v} = \langle -2,5,4\rangle$
24. Through $(2,1,3)$ and spanned by $\mathbf{u} = \mathbf{i}-\mathbf{j}$, $\mathbf{v} = \mathbf{i}+\mathbf{k}$

Verify the triple scalar product and the triple vector product for the following triples of vectors.

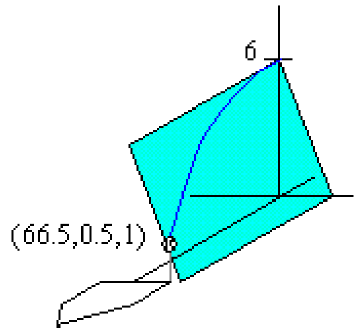
25. $\mathbf{i}, \mathbf{j}, \mathbf{k}$
26. $\mathbf{u} = \mathbf{i} + \mathbf{j}, \mathbf{v} = \mathbf{i} - \mathbf{j}, \ \mathbf{w} = \mathbf{k}$
27. $\mathbf{u} = \langle 1, 3, 2 \rangle, \mathbf{v} = \langle 2, 1, 3 \rangle, \ \mathbf{w} = \langle -1, 3, 4 \rangle$
28. $\mathbf{u} = a\mathbf{j} + b\mathbf{k}, \mathbf{v} = a\mathbf{k} - b\mathbf{j}, \ \mathbf{w} = \mathbf{i}$

29. Show that the three points $P_1(1, 3, 2)$, $P_2(3, 7, 5)$, and $P_3(5, 11, 8)$ all lie on a straight line. (i.e. vectors formed are all parallel). Then attempt to find the equation of the plane through the three points. What happens?
30. Find the equation of the plane through $(0, 0, 0)$ spanned by the position vectors $\mathbf{u} = \langle 1, 3, 2 \rangle$ and $\mathbf{v} = \langle 1, 1, 7 \rangle$, and then show that the plane contains both $\mathbf{u}$ and $\mathbf{v}$ when $\mathbf{u}$ and $\mathbf{v}$ are position vectors.
31. Find the equation of the line through $P_1(0, 1, 3)$ and $P_2(2, 1, 6)$. Then find the equation of the line through $P_3(-1, 1, 1)$ and $P_4(2, 1, -2)$ (use s as the parameter of this second line). Do the two lines intersect?



32. Does the line through the points $P_1(2, -3, 1)$ and $P_2(-3, 1, 3)$ intersect the line through the points $P_3(2, 1, -4)$ and $P_4(-1, -1, -1)$?
33. For what value of k does the line through the points $P_1(2, 1, 1)$ and $P_2(-2, 0, -1)$ intersect the line through the points $P_3(3, 5, 4)$ and $P_4(2, 3, k)$
34. Show that if the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ does not intersect the line through $P_3(x_3, y_3)$ and $P_4(x_4, y_4)$, then $\overrightarrow{P_1P_2}$ is parallel to $\overrightarrow{P_3P_4}$.
35. Are the points $P_1(0, 1, 2)$, $P_2(3, 2, 5)$, $P_3(1, 3, 7)$, and $P_4(5, 1, 3)$ all in the same plane? Explain.
36. Are the points $P_1(0, 2, 4)$, $P_2(2, 5, 1)$, $P_3(1, 1, 4)$, and $P_4(2, 7, 5)$ all in the same plane? Explain.
37. A baseball thrown from a height of 6 feet drops 5 feet and curves 0.5 feet

to the left once it reaches the plate 66.5 feet away.



Assuming motion of the ball is in a plane, what plane contains the trajectory of the ball?

38. What would the equation of the plane in exercise 37 be if the ball broke to the right instead of to the left?

39. Suppose that $\mathbf{u}$ is orthogonal to a unit vector $\mathbf{n}$ and that

$$\mathbf{u}_{\perp} = \mathbf{n} \times \mathbf{u}$$

which is called "u perp." What is the magnitude of $\mathbf{u}_{\perp}$? Why are $\mathbf{u}$ and $\mathbf{u}_{\perp}$ both in the plane with normal $\mathbf{n}$? What is the angle between $\mathbf{u}$ and $\mathbf{u}_{\perp}$?

40. Suppose that $\mathbf{u}$ is orthogonal to a unit vector $\mathbf{n}$ and that $\mathbf{u}_{\perp} = \mathbf{n} \times \mathbf{u}$. Show that

$$\mathbf{u}_{\theta} = \cos(\theta)\,\mathbf{u} + \sin(\theta)\,\mathbf{u}_{\perp}$$

is a vector in the plane spanned by $\mathbf{u}$ and $\mathbf{u}_{\perp}$ that is the same length as $\mathbf{u}$ and $\mathbf{u}_{\perp}$ and that forms an angle of θ with $\mathbf{u}$.

Exercises 41-44 explore interpretations and applications of the triple vector product.

41. Use the triple vector product to show that cross product multiplication satisfies

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} = \mathbf{0}$$

42. Suppose that $\mathcal{P}$ is a plane with normal vector $\mathbf{n}$ and containing point $P(x_1, y_1, z_1)$ and suppose that $\mathbf{n}$ is a unit vector. Then the projection of a vector $\mathbf{w}$ *into* P is defined to be

$$proj_{\mathcal{P}}(\mathbf{w}) = \mathbf{n} \times (\mathbf{w} \times \mathbf{n})$$

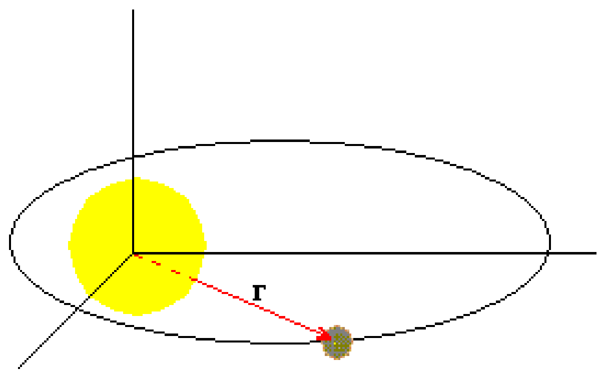
Show that if $\mathbf{w}$ is parallel to $\mathcal{P}$, which is to say that the endpoint of $\mathbf{w}$ is in the plane if the initial point of $\mathbf{w}$ is at $P(x_1, y_1, z_1)$, then

$$proj_{\mathcal{P}}(\mathbf{w}) = \mathbf{w}$$

43. Write to Learn: Write an essay in which you use the triple scalar product and the triple vector product to show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

44. Write to Learn: Suppose a planet is located at the tip of a vector $\mathbf{r}$ at time t as it orbits a sun located at the origin of a 3-dimensional coordinate system.



Then the acceleration of the planet about the sun is given by the inverse square law

$$\mathbf{a} = \frac{-GM}{r^2}\mathbf{u}$$

where $r = \|\mathbf{r}\|$ is the distance of the planet from the sun, M is the mass of the sun, G is the universal gravitational constant, and $\mathbf{u}$ is the direction vector of $\mathbf{r}$. Asssuming that the planet's orbit is in the xy-plane and its angular velocity vector is given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{v}$$

where $\mathbf{v}$ is in the same plane as is the planet's orbit, use the triple vector product to calculate $\mathbf{a} \times \mathbf{L}$ and explain the significance of the result, such as the direction that $\mathbf{a} \times \mathbf{L}$ is pointing in and what the magnitude of $\mathbf{a} \times \mathbf{L}$ would represent if $GM = 1$.