CHAPTER 01

# Trigonometry-II

## Trigonometric Functions of Allied Angles

Two angles are said to be allied when their sum or difference is either 0 (zero) or an integral multiple of $\pi/2$.

The angles $-\theta, \frac{\pi}{2} \pm \theta, \pi \pm \theta, \frac{3\pi}{2} \pm \theta, 2\pi \pm \theta$ etc., are angles allied to the angle $\theta$, if $\theta$ is measured in radians.

The trigonometric functions changes at allied angles which are given in the following table

| Allied angle/ Trigonometric function | $\sin\theta$ | $\operatorname{cosec}\theta$ | $\cos\theta$ | $\sec\theta$ | $\tan\theta$ | $\cot\theta$ |
|---|---|---|---|---|---|---|
| $-\theta$ | $-\sin\theta$ | $-\operatorname{cosec}\theta$ | $\cos\theta$ | $\sec\theta$ | $-\tan\theta$ | $-\cot\theta$ |
| $\pi/2-\theta$ | $\cos\theta$ | $\sec\theta$ | $\sin\theta$ | $\operatorname{cosec}\theta$ | $\cot\theta$ | $\tan\theta$ |
| $\pi/2+\theta$ | $\cos\theta$ | $\sec\theta$ | $-\sin\theta$ | $-\operatorname{cosec}\theta$ | $-\cot\theta$ | $-\tan\theta$ |
| $\pi-\theta$ | $\sin\theta$ | $\operatorname{cosec}\theta$ | $-\cos\theta$ | $-\sec\theta$ | $-\tan\theta$ | $-\cot\theta$ |
| $\pi+\theta$ | $-\sin\theta$ | $-\operatorname{cosec}\theta$ | $-\cos\theta$ | $-\sec\theta$ | $\tan\theta$ | $\cot\theta$ |
| $3\pi/2-\theta$ | $-\cos\theta$ | $-\sec\theta$ | $-\sin\theta$ | $-\operatorname{cosec}\theta$ | $\cot\theta$ | $\tan\theta$ |
| $3\pi/2+\theta$ | $-\cos\theta$ | $-\sec\theta$ | $\sin\theta$ | $\operatorname{cosec}\theta$ | $-\cot\theta$ | $-\tan\theta$ |
| $2\pi-\theta$ | $-\sin\theta$ | $-\operatorname{cosec}\theta$ | $\cos\theta$ | $\sec\theta$ | $-\tan\theta$ | $-\cot\theta$ |
| $2\pi+\theta$ | $\sin\theta$ | $\operatorname{cosec}\theta$ | $\cos\theta$ | $\sec\theta$ | $\tan\theta$ | $\cot\theta$ |

- Trigonometric functions for $(2\pi-\theta)$ and $(2n\pi-\theta)$ are same as those for $(-\theta)$, where $n \in Z$.
- Trigonometric functions for $(2\pi+\theta)$ and $(2n\pi+\theta)$ are same as those for $\theta$, where $n \in Z$.

Trigonometric Functions of Some Useful Angles

| Angle | $0°/0$ | $30°/\frac{\pi}{6}$ | $45°/\frac{\pi}{4}$ | $60°/\frac{\pi}{3}$ | $90°/\frac{\pi}{2}$ | $180°/\pi$ | $270°/\frac{3\pi}{2}$ | $360°/2\pi$ |
|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | $-1$ | 0 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | $-1$ | 0 | 1 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ | 0 | $\infty$ | 0 |
| $\cot\theta$ | $\infty$ | $\sqrt{3}$ | 1 | $\frac{1}{\sqrt{3}}$ | 0 | $\infty$ | 0 | $\infty$ |
| $\sec\theta$ | 1 | $\frac{2}{\sqrt{3}}$ | $\sqrt{2}$ | 2 | $\infty$ | $-1$ | $\infty$ | 1 |
| $\operatorname{cosec}\theta$ | $\infty$ | 2 | $\sqrt{2}$ | $\frac{2}{\sqrt{3}}$ | 1 | $\infty$ | $-1$ | $\infty$ |

## Trigonometric Functions of Compound Angles

Compound angles are sum or difference of given angles. For any two angles $A$ and $B$.

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$

(viii) $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

## Trigonometric Functions of Multiple Angles

Angles of the form $2A, 3A, 4A$ etc. are integral multiple of $A$, these angles are called multiple angles and angles of the form $\frac{A}{2}, \frac{3A}{2}$ etc. are called submultiple angles of $A$.

### Trigonometric Functions of Double Angles (2A)

For any angle $A$,

(i) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A} = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

(iii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

### Trigonometric Functions of Triple Angle (3A)

For any angle $A$,

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

### Trigonometric Functions of Half Angles

For any angle $A$,

(i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 1 - 2\sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1 = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(iii) $\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

(iv) $\cot A = \frac{\cot^2\frac{A}{2} - 1}{2\cot\frac{A}{2}}$

(v) $\sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$

(vi) $\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$

(vii) $\tan A = \frac{3\tan\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1-3\tan^2\left(\frac{A}{3}\right)}$

## Conversion of Sum or Difference into Product

For any angles $C$ and $D$,

(i) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(ii) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(iii) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(iv) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

## Conversion of Product into sum or Difference

For any angles $A$ and $B$,

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

## Trigonometric Functions of Angles of a Triangle

In $\Delta ABC$, $m\angle BAC = A, m\angle ABC = B$ and $m\angle ACB = C$

$\therefore \quad A + B + C = \pi$

(i) $\sin(A+B) = \sin C$

(ii) $\sin(B+C) = \sin A$

(iii) $\sin(C+A) = \sin B$

(iv) $\cos(A+B) = -\cos C$

(v) $\cos(B+C) = -\cos A$

(vi) $\cos(C+A) = -\cos B$

(vii) $\sin\frac{(B+C)}{2} = \cos\frac{A}{2}$

(viii) $\sin\frac{(C+A)}{2} = \cos\frac{B}{2}$

(ix) $\sin\frac{(A+B)}{2} = \cos\frac{C}{2}$

(x) $\cos\frac{(A+B)}{2} = \sin\frac{C}{2}$

(xi) $\cos\frac{(B+C)}{2} = \sin\frac{A}{2}$

(xii) $\cos\frac{(C+A)}{2} = \sin\frac{B}{2}$