CHAPTER 03

Trigonometric Functions

Trigonometric ratios are defined for acute angles as the ratio of the sides of a right angled triangle. The extension of trigonometric ratios to any angle in terms of radian measure (real numbers) are known as **trigonometric functions.**

Trigonometric Equation

An equation involving trigonometric functions of a variable is called a trigonometric equation.

e.g. $\cos^2\theta - \sin\theta = \frac{1}{2}$, $\tan m\theta = \cot n\theta$, etc. are trigonometric equations.

Solutions or Roots of a Trigonometric Equation

A value of an unknown angle which satisfies the given equation, is called the solution or root of the trigonometric equation.

As algebraic equation, the trigonometric equation may also have infinite number of solutions and can be classified as

Principal Solution

The least value of unknown angle $(0 \le \alpha < 2\pi)$ which satisfies the given equation, is called a principal solution of trigonometric equation.

General Solution

We know that, trigonometric functions are periodic and solutions of trigonometric equations can be generalised with the help of the periodicity of the trigonometric functions.

The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Principal Solution of Standard Trigonometic Equations

1.

sine	Reciprocal equations	Principal solutions
$\sin x = \frac{1}{2}$	$\operatorname{cosec} x = 2$	$\frac{\pi}{6}, \frac{5\pi}{6}$
$\sin x = \frac{1}{\sqrt{2}}$	$\operatorname{cosec} x = \sqrt{2}$	$\frac{\pi}{4}, \frac{3\pi}{4}$
$\sin x = \frac{\sqrt{3}}{2}$	$\operatorname{cosec} x = \frac{2}{\sqrt{3}}$	$\frac{\pi}{3}, \frac{2\pi}{3}$
$\sin x = -\frac{1}{2}$	$\operatorname{cosec} x = -2$	$\frac{7\pi}{6}, \frac{11\pi}{6}$
$\sin x = -\frac{1}{\sqrt{2}}$	$\operatorname{cosec} x = -\sqrt{2}$	$\frac{5\pi}{4}, \frac{7\pi}{4}$
$\sin x = -\frac{\sqrt{3}}{2}$	$\operatorname{cosec} x = -\frac{2}{\sqrt{3}}$	$\frac{4\pi}{3}, \frac{5\pi}{3}$

2.

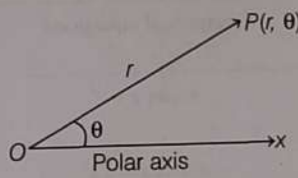
cosine		
$\cos x = \frac{1}{2}$	$\sec x = 2$	$\frac{\pi}{3}, \frac{5\pi}{3}$
$\cos x = \frac{1}{\sqrt{2}}$	$\sec x = \sqrt{2}$	$\frac{\pi}{4}, \frac{7\pi}{4}$
$\cos x = \frac{\sqrt{3}}{2}$	$\sec x = \frac{2}{\sqrt{3}}$	$\frac{\pi}{6}, \frac{11\pi}{6}$
$\cos x = -\frac{1}{2}$	$\sec x = -2$	$\frac{2\pi}{3}, \frac{4\pi}{3}$
$\cos x = -\frac{1}{\sqrt{2}}$	$\sec x = -\sqrt{2}$	$\frac{3\pi}{4}, \frac{5\pi}{4}$
$\cos x = -\frac{\sqrt{3}}{2}$	$\sec x = -\frac{2}{\sqrt{3}}$	$\frac{5\pi}{6}, \frac{7\pi}{6}$

3.

Tangent		
$\tan x = \frac{1}{\sqrt{3}}$	$\cot x = \sqrt{3}$	$\frac{\pi}{6}, \frac{7\pi}{6}$
$\tan x = 1$	$\cot x = 1$	$\frac{\pi}{4}, \frac{5\pi}{4}$
$\tan x = \sqrt{3}$	$\cot x = \frac{1}{\sqrt{3}}$	$\frac{\pi}{3}, \frac{4\pi}{3}$
$\tan x = -\frac{1}{\sqrt{3}}$	$\cot x = -\sqrt{3}$	$\frac{5\pi}{6}, \frac{11\pi}{6}$
$\tan x = -1$	$\cot x = -1$	$\frac{3\pi}{4}, \frac{7\pi}{4}$
$\tan x = -\sqrt{3}$	$\cot x = -\frac{1}{\sqrt{3}}$	$\frac{2\pi}{3}, \frac{5\pi}{3}$

Polar Coordinates

The polar coordinate system is two dimensional coordinate system in which each point P on a plane is determined by distance r from a fixed point O that is called **pole** (or origin) and an angle θ from a fixed direction.



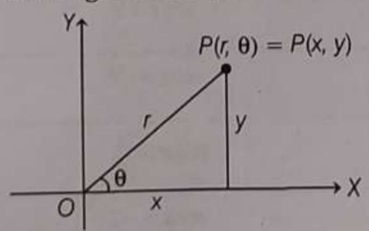
The point P is represented by the ordered pair (r, θ) and r, θ are called polar coordinates.

Relationship between Cartesian and Polar Coordinates

(i) If (r, θ) are the polar coordinates of a point P in a plane, then its cartesian coordinates (x, y) are given by

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$

where r is called the radius vector and θ is called the vectorial angle of the point P and $0 \le \theta < 2\pi$.



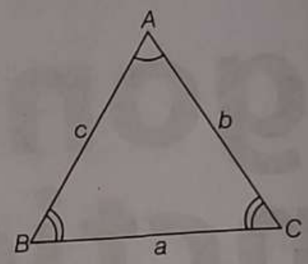
(ii) If (x, y) are the cartesian coordinates of the point P whose polar coordinates are (r, θ), then

$$\tan\theta = \frac{y}{x} \text{ and } r = \sqrt{x^2 + y^2},$$

where $0 \le \theta < 2\pi$.

Solution of Triangle

If three elements including atleast one side is known and another three elements including atleast one angle is know then remaining other elements can be uniquely determined. This is known as solution of triangle.



The angles $\angle A, \angle B, \angle C$ of a ΔABC are denoted by A, B, C respectively.

The lengths of sides of ΔABC are given by $AB = c$, $BC = a$ and $CA = b$.

For any ΔABC,

(i) Sum of all angles of a triangle is 180°.
i.e. $A + B + C = \pi$.

(ii) Sum of any two sides of a triangle is greater than third side,
i.e. $a + b > c, b + c > a, c + a > b$.

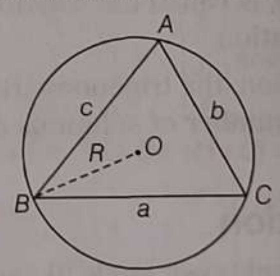
(iii) Difference of any two sides of a triangle is less than third side,
i.e. $|a - b| < c, |b - c| < a, |c - a| < b$.

Sine Rule

In any ΔABC, the lengths of the sides are proportional to the sines of the opposite angle.

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where, R is the radius of circumcircle of the triangle.



In different ways, we can use the sine rules are given below

(i) $a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$

(ii) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R} = k$ (say)

(iii) $a\sin B = b\sin A, b\sin C = c\sin B, a\sin C = c\sin A$.
or $\frac{a}{b} = \frac{\sin A}{\sin B}, \frac{b}{c} = \frac{\sin B}{\sin C}, \frac{a}{c} = \frac{\sin A}{\sin C}$

Cosine Rule

The square of one side of a triangle equals the sum of the squares of the other two sides and subtract twice their product and cosine of their included angle.

In any ΔABC,

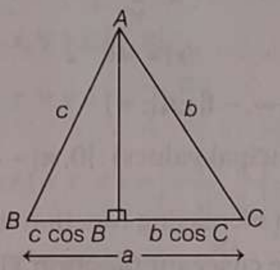
$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = c^2 + a^2 - 2ca\cos B \quad \text{or} \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Projection Rule

In any ΔABC, the length of one side of a triangle is equal to the sum of the product of another side and the cosine between one side and the other side.



If sides $a = BC$, $b = CA$ and $c = AB$, then the projection formula is

$$a = c\cos B + b\cos C$$
$$b = a\cos C + c\cos A$$
$$c = a\cos B + b\cos A$$

Inverse Trigonometric Functions

Consider the sine function with domain R and range $[-1, 1]$. It is clear that, this is not an one-one and onto (bijection), so it is not invertible. Therefore, we have to restrict the domain of it, in such a way that it becomes bijective (one-one and onto) then it would become invertible. Suppose we consider sine as a function with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and codomain $[-1, 1]$, then it is a bijection and therefore invertible. Such types of trigonometric functions, are called inverse trigonometric functions.

Principal Value

The value of an inverse trigonometric function which is numerically least, is called principal value and the set of least values for which the domain is uniquely defined, is called the principal value branch of the function.

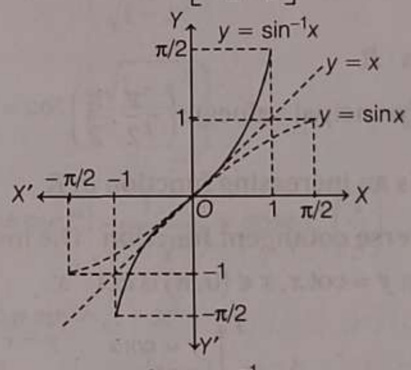
Domain and Range of Inverse Trigonometric Functions

Function	Domain	Range Principal value
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left[\frac{\pi}{2}\right]$
$\cot^{-1} x$	R	$(0, \pi)$

Graphs of Inverse Trigonometric Functions

In the following graphs, dotted curves show trigonometric functions and solid curves show their corresponding inverse trigonometric functions.

(i) **The inverse sine function** The inverse of the function $y = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is $\sin^{-1} x$.

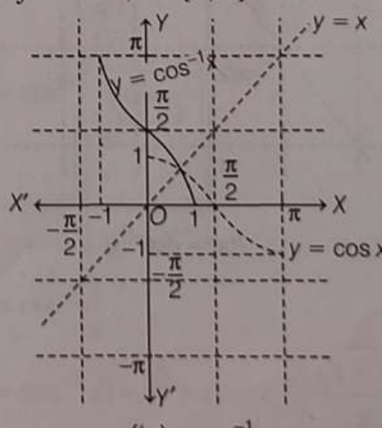


$f(x) = \sin^{-1} x$

Domain : $[-1, 1]$, Range (principal values) : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1} x$ is an increasing function in $[-1, 1]$.

(ii) **The inverse cosine function** The inverse of the function $y = \cos x$, $x \in [0, \pi]$ is $\cos^{-1} x$.



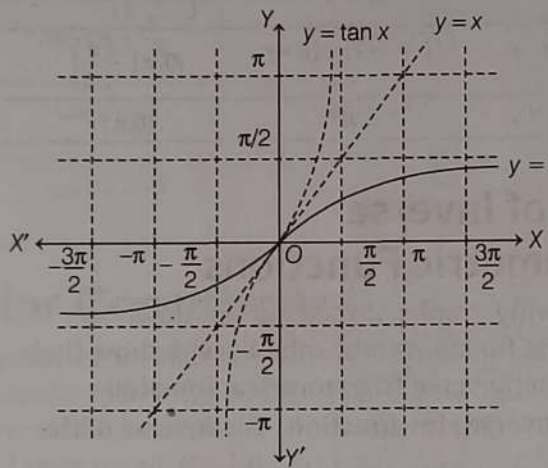
$f(x) = \cos^{-1} x$

Domain : $[-1, 1]$,

Range (principal values) : $[0, \pi]$

$\cos^{-1} x$ is a decreasing function in $[-1, 1]$.

(iii) **The inverse tangent function** The inverse of the function $y = \tan x, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is $\tan^{-1} x$.



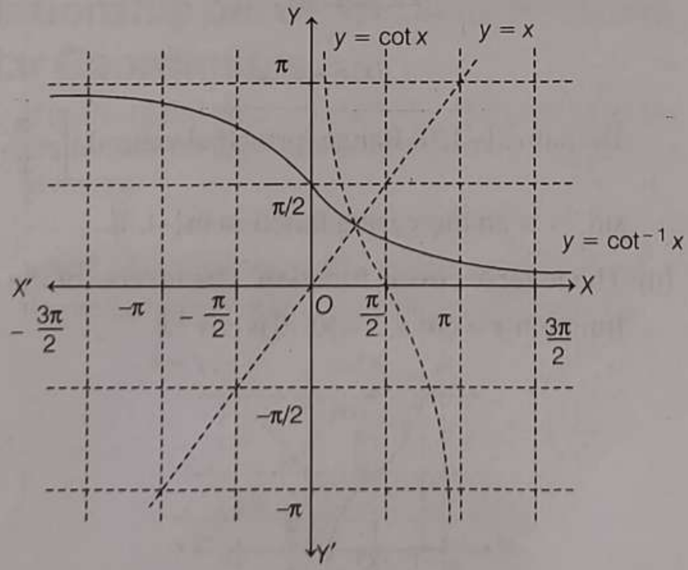
$f(x) = \tan^{-1} x$

Domain : R,

Range (principal values) : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan^{-1} x$ is an increasing function in R.

(iv) **The inverse cotangent function** The inverse of the function $y = \cot x, x \in (0, \pi)$ is $\cot^{-1} x$.



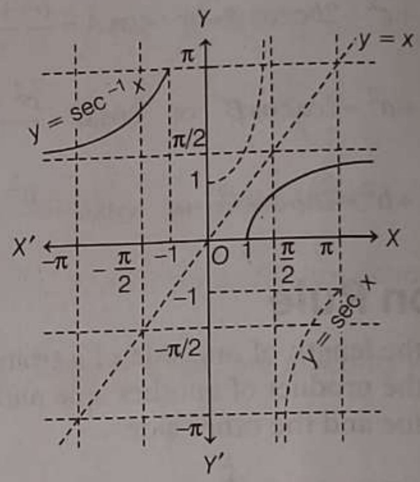
$f(x) = \cot^{-1} x$

Domain : R,

Range (principal values) : $(0, \pi)$

$\cot^{-1} x$ is a decreasing function in R.

(v) **The inverse secant function** The inverse of the function $y = \sec x, x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$ is $\sec^{-1} x$.



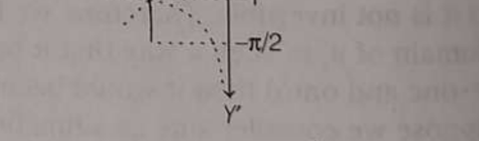
$f(x) = \sec^{-1} x$

Domain : $(-\infty, -1] \cup [1, \infty)$

Range (principal values) : $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\sec^{-1} x$ is an increasing function in $(-\infty, -1]$ and $[1, \infty)$.

(vi) **The inverse cosecant function** The inverse of function $y = \operatorname{cosec} x, x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is $\operatorname{cosec}^{-1} x$.



$f(x) = \operatorname{cosec}^{-1} x$

Domain : $(-\infty, -1] \cup [1, \infty)$

Range (principal values) : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\operatorname{cosec}^{-1} x$ is a decreasing function in $(-\infty, -1]$ and $[1, \infty)$.

Properties of Inverse Trigonometric Functions

Some properties of inverse trigonometric functions are given below:

Property 1

Self Adjusting Properties

(i) $\sin(\sin^{-1}x) = x, \forall x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x) = x, \forall x \in [-1, 1]$

(iii) $\tan(\tan^{-1}x) = x, \forall x \in R$

(iv) $\text{cosec}(\text{cosec}^{-1}x) = x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec(\sec^{-1}x) = x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot(\cot^{-1}x) = x, \forall x \in R$

(vii) $\sin^{-1}(\sin x) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(viii) $\cos^{-1}(\cos x) = x, \forall x \in [0, \pi]$

(ix) $\tan^{-1}(\tan x) = x, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(x) $\text{cosec}^{-1}(\text{cosec}\, x) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(xi) $\sec^{-1}(\sec x) = x, \forall x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(xii) $\cot^{-1}(\cot x) = x, \forall x \in (0, \pi)$

Property 2

Negative Arguments

(i) $\sin^{-1}(-x) = -\sin^{-1}x, \forall x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x, \forall x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x, \forall x \in R$

(iv) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x, \forall x \in R$

Property 3

Reciprocal Arguments

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\text{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x, x \neq 0$

(iii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(iv) $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x, x \neq 0$

(v) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & \text{for } x > 0 \\ -\pi + \cot^{-1}x, & \text{for } x < 0 \end{cases}$

(vi) $\cot^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1}x, & x > 0 \\ \pi + \tan^{-1}x, & x < 0 \end{cases}$

Property 4

Inverse Sum Identities

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \forall x \in [-1, 1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \forall x \in R$

(iii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, \forall x \in (-\infty, -1] \cup [1, \infty)$

Property 5

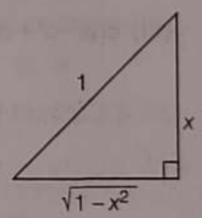
Conversion of a Inverse trigonometric function to Another Inverse Trigonometric Functions

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

$= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

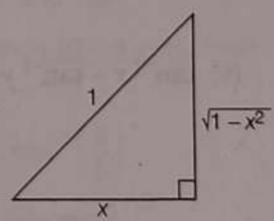


(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$

$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$= \sec^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

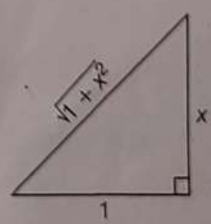


(iii) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \cot^{-1}\frac{1}{x}$

$= \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$



Property 6

Sum and Difference of Inverse Trigonometric Functions

(i) $\sin^{-1}x+\sin^{-1}y$

$$=\begin{cases}\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}), \text{ if } x\geq 0,\ y\geq 0 \\ \qquad\qquad \text{and } x^2+y^2\leq 1 \\ \pi-\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}), \text{ if } x\geq 0,\ y\geq 0 \\ \qquad\qquad \text{and } x^2+y^2>1\end{cases}$$

(ii) $\sin^{-1}x-\sin^{-1}y$

$$=\begin{cases}\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}, \text{ if } xy>0 \\ \qquad\qquad \text{and } x^2+y^2>1 \\ \pi-\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}, \text{ if } 0<x\leq 1 \\ \qquad -1\leq y\leq 0 \text{ and } x^2+y^2>1\end{cases}$$

(iii) $$\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}(xy-\sqrt{1-x^2}\cdot\sqrt{1-y^2}), \\ \quad \text{if} -1\leq x,\ y\leq 1 \text{ and } x+y\geq 0 \\ 2\pi-\cos^{-1}(xy-\sqrt{1-x^2}\cdot\sqrt{1-y^2}), \\ \quad \text{if} -1\leq x,\ y\leq 1 \text{ and } x+y\leq 0\end{cases}$$

(iv) $$\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}(xy+\sqrt{1-x^2}\sqrt{1-y^2}), \\ \quad \text{if } -1\leq x, y\leq 1 \text{ and } x\leq y \\ -\cos^{-1}(xy+\sqrt{1-x^2}\sqrt{1-y^2}), \\ \quad \text{if} -1\leq y\leq 0, 0<x\leq 1 \text{ and } x\geq y\end{cases}$$

(v) $$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)x,\ y>0 \text{ and } xy<1 \\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{ if } x>0,\ y>0 \\ \text{and } xy>1\end{cases}$$

(vi) $\tan^{-1}x-\tan^{-1}y$

$$=\left\{\tan^{-1}\left(\frac{x-y}{1+xy}\right), \text{ if } x>0,\ y>0 \text{ and } xy>-1\right.$$

(vii) $\tan^{-1}x-\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\dfrac{x+y+z-xyz}{1-zy-yz-zx}\right)$

Property 7

Inverse Trigonometric Ratios of Multiple Angles

(i) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$,

if $-\dfrac{1}{\sqrt{2}}\leq x\leq\dfrac{1}{\sqrt{2}}$

(ii) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$,

if $0\leq x\leq 1$

(iii) $2\tan^{-1}x=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$,

if $-1<x<1$

(iv) $2\tan^{-1}x=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$,

if $-1\leq x\leq 1$

(v) $2\tan^{-1}x=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$,

if $0\leq x<\infty$

(vi) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$,

if $\dfrac{-1}{2}\leq x\leq\dfrac{1}{2}$

(vii) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$,

if $\dfrac{1}{2}\leq x\leq 1$

(viii) $3\tan^{-1}x=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$, if $-\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}$