



TECHNICAL UNIVERSITY OF MUNICH

DEPARTMENT OF INFORMATICS

Master's Thesis in Informatics

# **Vehicle Localization and Tracking for Collision Avoidance System**

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## **Fahrzeuglokalisierung und -verfolgung für das Kollisionsvermeidungssystem**

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I confirm that this master's thesis is my own work and I have documented all sources and material used.

Ich versichere, dass ich diese Master's Thesis selbständig verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet habe.

Munich, 15.05.2020

Behtarin Ferdousi

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# Abstract

With the current rate of development in autonomous vehicles, the demand for a high-intelligent collision avoidance system is increasing. Due to the inability to determine the inner state of tracked vehicles from Lidar, GPS (Global Positioning System), and radar sensors, researchers have utilized state estimation methods to converge available measurements to the true state of the system. Set-based methods are used to enclose the true state of the system in a set, in contrast to stochastic methods which give a point-estimate close to the true state. Encapsulating the true state in a set is important to not allow any divergence from the true state for safety-critical tasks in autonomous vehicles. The purpose of this thesis is to review and implement different algorithms of set-based state estimation, using zonotopes as domain representation, on existing datasets of real traffic participants (approx. 10,518 entities). The algorithms implemented are segment intersection methods (using F-radius, P-radius, and volume) and an interval observer (using  $H-\infty$  observer). They are compared in terms of computation time, time to converge, tightness of bound and accuracy. The  $H-\infty$  interval observer performs better in terms of computation time but starts with a wider initial bound. Segment intersection minimization using P-radius is faster than using F-radius, but compromises on the bounds and accuracy. Of all the methods compared, segment minimization using F-radius gives the most desirable estimates for this use-case.

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# 1. Introduction

There is steep progress in the research and development of autonomous vehicles. The race to the top of the automobile industry, featuring companies like BMW (Bavarian Motor Works), Tesla, Waymo/Google, requires fast development and vigorous testing of novel technologies. One of the many challenges of this field is to ensure collision avoidance. With no human behind the wheel for Level 5 [1] cars, the vehicle must keep track of roads and surrounding traffic participants (like vehicles and pedestrians) in different circumstances, including rain and fog, to ensure the safety of its passengers. Current collision avoidance systems based on sensors, radar, and camera would be overwhelmed with high computation demands for this purpose. Tolerating error in such a system can cause accidents, including fatality <sup>1</sup>.

The collision avoidance system in a car consists of two parts: sensing and tracking and motion planning. The sensing and tracking part is achieved by applying sophisticated algorithms on signals from sensors like radar, camera, and GPS (Global Positioning System). With decline in cost of cameras and advancement in technologies in image processing, image analysis, and object detection, sensing and tracking is developing fast. Although cameras can classify vehicles, they cannot guarantee measurement in a low-light environment (e.g. night) [2]. In contrast, radar guarantees robustness to weather in exchange of a higher cost. Similarly, there are limitations in GPS, e.g. the inability to function in the urban canyon environment. Thus, one uses sensor fusion to compensate for shortcomings of specific sensors. After detecting all relevant elements in the environment, a motion planner has to find a collision-free path. Computation of such a path requires certain parameters to predict the tracked vehicle's trajectory. The sensors cannot solely measure all these parameters, hence researchers have turned to state estimation algorithms.

One of the widely-applied state estimation techniques is the Kalman filter [3], which can estimate target dynamics for measurement with additive Gaussian noise. Despite its simplicity, the filter is not suitable for vehicle localization for two reasons. Firstly, statistical noise with known covariance is, unfortunately, not practical. Secondly, the filter provides close point-estimation, relying on which can be safety-critical. These motivate to use set-based state estimation methods.

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<sup>1</sup><https://www.theguardian.com/technology/2018/mar/19/uber-self-driving-car-kills-woman-arizona-tempe>



The set-based state estimation technique computes a set of state enclosing the true state of the system as long as the dynamics are accurately modeled and the noise and perturbations have known bounds. The main steps are prediction step and correction step. The prediction step extrapolates prior estimate, while the correction step improves the extrapolation. There are multiple algorithms with varying approach for the correction step. Another differentiating factor is the choice of geometric shape to represent the estimated set. Zonotope is one of the popular choice, compared to ellipsoid and pollytopes, due to higher accuracy for a lower computation cost. Furthermore, zonotopes have gained fame for state estimation because of wrapping effect(i.e. not increasing in size due to accumulated noises over time) and Minkowski sum(i.e. the sum of zonotopes is also a zonotope). Therefore, we chose zonotopes to represent state for all the algorithms in this paper.

The first zonotope-based guaranteed state estimation is developed by Puig et al. in 2001 [4], when they used gain matrix to map input measurement to a set of estimation. Following in 2003, Combastel [5] used a singular value decomposition to overapproximate the estimate consistent with the input. Although aforementioned methods are computationally light, they did not focus on the size of the estimated region. In 2005, T. Alamo et al. [6] formulated convex optimization problem to minimize some size criterion. They focused on two main size criterion: F-radius and volume. F-radius resulted in fast but conservative estimates, whereas volume computation is heavy, but gave tight bounds. In 2011, T. Alamo et al. [7] optimized P-radius to obtain good accuracy for a reasonable computation load. Initially, the algorithms were developed for single-output linear discrete systems, and were later generalised to multi-output and non-linear systems.

Another classification of set-based estimation are interval observers, where the idea is to design observers such that the error in the estimation is minimal. Despite its high efficiency, construction of such observer is not very easy. Hence, the observer design requirements are relaxed in [8], [9]. The relaxation results in conservatism, which led to an interval observer based on  $H_\infty$  with reachability analysis [17].

The prerequisite step before applying state estimation algorithms is to model the tracked vehicle with a well-defined mathematical model. Although there are complex models that can be used to represent a vehicle state [10], not all can be used due to the unavailability of parameters like wheelbase, velocity, etc. accessible to the ego vehicle. Hence, the models used in this paper to compare are the simplest, yet complete enough to determine the properties of the tracked vehicle for trajectory prediction: Constant Velocity, Constant Acceleration, and the Point-Mass Model.

A high degree of accuracy and guarantee is the necessity of the collision avoidance system, hence we chose to compare the set based state estimation algorithms for different scenarios involving dynamic traffic participants from a dataset collected from

intersections using drones and fixed cameras. [11] has encouraged many sections in this paper and compares a superset of algorithms covered here; however, the algorithms were compared on simulated data, in contrast to this paper.

The paper is organized as follows. Chapter 2 builds up the vehicle localization problem to be solved by state estimation algorithms. The following chapter 3 discusses the zonotope-based state estimation algorithms to be compared. In chapter 4, we evaluated the algorithms and discussed the results. Finally, chapter 5 concludes with a summary and a discussion of possible future works.

## 2. Vehicle Localization : The Guaranteed Estimation Problem

### 2.1. Preliminaries

The following standard notations are maintained in this paper.

- $\mathcal{R}^n$  and  $\mathcal{R}^{n \times m}$  denote the  $n$  and  $n \times m$  dimensional Euclidean space, respectively.
- $I^n$  represents the  $n$ -identity matrix. If  $n$  is missing, then appropriate dimension is assumed.
- For a matrix  $A$ ,  $A^T$ ,  $A^{-1}$ ,  $A_i$  and  $A^j$  denote its transpose, inverse,  $i^{th}$  row, and  $j^{th}$  column, respectively.  $rs(A)$  is the row sum of  $A$ , and  $det(A)$  the determinant.
- $|\cdot|$  is the absolute value and  $\|\cdot\|_x$  is the  $x$ -norm.
- With a vector  $a \in \mathcal{R}^n$ ,  $diag(a)$  is a diagonal matrix of dimension  $n$ .
- With a vector  $a$ ,  $n_a$  is its dimension.
- For a real symmetric matrix,  $P \in \mathcal{R}^{n \times n}$ ,  $P \prec 0$  ( $P \succ 0$ ) implies  $P$  is a negative (positive) definite.

#### 2.1.1. Zonotopes: Definitions and Properties

The following definitions and properties are essential for this paper.

**Definition 1 Interval** An interval  $[a, b]$  is defined as the set  $\{x : a \leq x \leq b\}$ .

**Definition 1.1** The unitary interval, denoted by  $\mathbf{B}$ , is  $[-1, 1]$

**Definition 1.2** A box  $([a_1, b_1], \dots, [a_n, b_n])^T$  is an interval vector.

**Definition 1.3** A unitary box in  $\mathcal{R}^n$  is denoted by  $\mathbf{B}^n$  and is a box with  $n$  unitary intervals.

**Definition 2** The Minkowski sum of two sets,  $\mathcal{X}$  and  $\mathcal{Y}$ , is defined by:

$$\mathcal{X} \oplus \mathcal{Y} = \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\} \quad (2.1)$$

**Definition 3 Zonotope:** An affine transformation of a hypercube,  $\mathbf{B}^m$  is called an  $m$ -ordered zonotope, denoted by  $\mathcal{Z} \in \mathcal{R}^n$ :

$$\mathcal{Z} = \langle p, H \rangle = \{p + Hz : z \in \mathbf{B}^m\} \quad (2.2)$$

where  $p \in \mathcal{R}^n$  is the center of  $\mathcal{Z}$ , and  $H \in \mathcal{R}^{n \times m}$  is called the generator of  $\mathcal{Z}$ .

**Property 1** For two zonotopes,  $\mathcal{Z}_1 = \langle p_1, H_1 \rangle$  and  $\mathcal{Z}_2 = \langle p_2, H_2 \rangle$ , the following equations hold:

$$\begin{aligned} \mathcal{Z}_1 \oplus \mathcal{Z}_2 &= \langle p_1 + p_2, [H_1 \ H_2] \rangle \\ L\mathcal{Z}_1 &= \langle Lp_1, LH_1 \rangle \end{aligned} \quad (2.3)$$

**Property 2** [12] A box of an  $m$ -zonotope,  $\mathcal{Z} = \langle p, H \rangle \in \mathcal{R}^n$ , is an  $n$ -interval vector, over-approximating the zonotope such that:

$$[\mathcal{Z}] = \text{box}(\mathcal{Z}) = [p - \Delta H, p + \Delta H], \quad \Delta H = \sum_{i=1}^m |H^i| \quad (2.4)$$

**Property 3 Zonotope Reduction** [6], [5]: An  $m$ -zonotope ( $\mathcal{Z} = \langle p, H \rangle \in \mathcal{R}^n$ ) can be reduced to an  $s$ -zonotope, s.t  $n < s < m$ , by first sorting the columns of  $H$  in decreasing order of Euclidean norm ( $\hat{H} = [\hat{h}_1 \dots \hat{h}_m]$ , with  $\|\hat{h}_i\|_2 \geq \|\hat{h}_{i+1}\|_2$ ). Then let's construct  $\hat{H}_A$  from the first  $s - n$  columns of  $\hat{H}$ , and  $\hat{H}_B$  from the rest of the matrix  $\hat{H}$ . Then the following inclusion is obtained:

$$\mathcal{Z} \subseteq p \oplus [\hat{H}_A \ \text{rs}(\hat{H}_B)]\mathbf{B}^s \quad (2.5)$$

It is denoted by  $\mathcal{Z}_{\downarrow s}$  in this paper.

**Property 4** [6] Given a zonotope  $\mathcal{Z} = p \oplus H\mathbf{B}^m \in \mathcal{R}^n$ , a strip  $\mathcal{S} = \{x \in \mathcal{R}^n : |cx - y| \leq \phi\}$ , there exists a vector  $\lambda \in \mathcal{R}^n$  such that a family of zonotopes parameterized by  $\lambda$  contains the intersection of a zonotope and a strip s.t.:

$$\mathcal{Z} \cap \mathcal{S} \subseteq \mathcal{Z}(\lambda) = p(\lambda) \oplus H(\lambda)\mathbf{B}^{m+1} \quad (2.6)$$

with  $p(\lambda) = p + \lambda(y - cp) \in \mathcal{R}^n$

and  $H(\lambda) = [(I - \lambda c)H \ \phi\lambda]$

The construction, reduction and interval calculation of zonotopes are implemented in the Matlab<sup>®</sup> toolbox CORA (COntinuous Reachability Analyzer) [13].

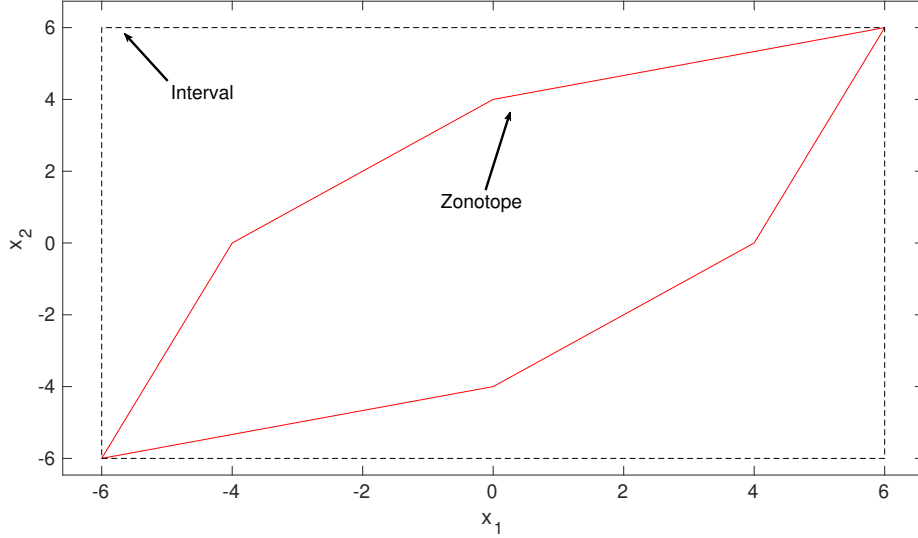


Figure 2.1.: An illustration of a zonotope and its interval hull in 2-D

## 2.2. Problem Formulation

Let us denote the state of the vehicle to be tracked at time  $k$  as  $x_k$  and the measured state as  $y_k$ . (2.7) formulates a multi-output discrete-time linear system for the tracked vehicle, where  $A, E \in \mathcal{R}^{n_x \times n_x}$ ,  $C \in \mathcal{R}^{n_y \times n_x}$ , and  $F \in \mathcal{R}^{n_y \times n_y}$  are matrices defined by the vehicle system model;  $w_k$  and  $v_k$  are process noise and measurement noise at time  $k$ , respectively.

$$\begin{aligned} x_k &= Ax_{k-1} + Ew_k \\ y_k &= Cx_k + Fv_k \end{aligned} \tag{2.7}$$

Assuming  $w_k$  and  $v_k$  have known bounds ( $\overline{w_k}$  and  $\overline{v_k}$ ), we can define  $\mathcal{W}$  and  $\mathcal{V}$  such that  $w_k \in \mathcal{W}$  and  $v_k \in \mathcal{V}$  as (2.8).

$$\mathcal{W} = \langle 0, H_w \rangle, \quad \mathcal{V} = \langle 0, H_v \rangle \tag{2.8}$$

The dimension of  $x_k$  ( $n_x$ ) varies across vehicle models; however, the measurement vector ( $y_k$ ) is fixed to position in x and y-direction (2.9).

$$y = [s_x \quad s_y]^T \tag{2.9}$$

Given a vehicle model (discussed in the next section), the problem of set-based state estimation is to compute an outer bound of the state ( $x_k$ ) containing all the possible

values of the true state of the system consistent with the uncertain vehicle model and the measurements.

## 2.3. Vehicle Model

A major performance-influencing factor is to choose the right model for the tracked vehicle. Three linear systems are implemented in this paper to compare the state estimation algorithms. Although there exists highly precise vehicle models for ego vehicles, the simplest models are used here to represent the tracked vehicle, because complex vehicle models require parameters which are non-acquirable for tracked vehicles. In particular, physical dimensions like wheelbase or side-slip, cannot be measured directly. Another reason is that adding steering angle and yaw rate makes the system non-linear and hence does not suit all the algorithms presented. Hence, the following models are investigated:

- **Constant Velocity Model**
- **Constant Acceleration Model**
- **Point Mass Model**

### 2.3.1. Constant Velocity Model

The vehicle is assumed to travel in constant velocity [14]. The state of the system ( $x_k$ ), state transition matrix ( $A$ ), and the measurement matrix( $C$ ) is shown in (2.10).

$$\begin{aligned}
 x &= [s_x \quad s_y \quad v_x \quad v_y]^T \\
 A &= \begin{bmatrix} 1 & 0 & \Delta T & 0 \\ 0 & 1 & 0 & \Delta T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
 \end{aligned} \tag{2.10}$$

### 2.3.2. Constant Acceleration Model

Although the constant velocity model is easy to implement, it is unrealistic to assume constant velocity. Acceleration model takes care of changing velocity and assumes constant acceleration [14]. Hence, the estimation error for position and velocity are expected to be relatively smaller when the velocity is constantly changing. The state of

the system ( $x_k$ ), state transition matrix ( $A$ ), and the measurement matrix( $C$ ) is shown in (2.11).

$$\begin{aligned}
 x &= [s_x \quad s_y \quad v_x \quad v_y \quad a_x \quad a_y]^T \\
 A &= \begin{bmatrix} 1 & 0 & \Delta T & 0 & \frac{1}{2}\Delta T^2 & 0 \\ 0 & 1 & 0 & \Delta T & 0 & \frac{1}{2}\Delta T^2 \\ 0 & 0 & 1 & 0 & \Delta T & 0 \\ 0 & 0 & 0 & 1 & 0 & \Delta T \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 C &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned} \tag{2.11}$$

### 2.3.3. Point-Mass Model

It is trivial to note that vehicles might have varying acceleration, which is not satisfied in the previous models. This brings us to the point-mass model [10], which is similar to the constant acceleration model, except that the acceleration here can strike up to a certain limit. This model treats the tracked vehicle as a point mass, ignoring wheel-base, slip-angle, etc. of the tracked vehicle. The state transition and measurement matrices are the same as the constant acceleration model. The acceleration bounds are set as  $11.5m/s^2$  [10] in both x and y-direction for this paper.

### 3. Zonotope-based guaranteed state estimation

With the essential knowledge on zonotope and vehicle models from the previous chapter, this chapter digs deeper into the state estimation algorithms in terms of vehicle dynamics. With the vehicle dynamics represented by (3.1) (refer to (2.7) for details), we look into segment intersection techniques (minimizing F-radius, volume and P-radius) and an H- $\infty$ -based interval observer.

$$\begin{aligned} x_{k+1} &= Ax_k + Ew_k \\ y_k &= Cx_k + Fv_k \end{aligned} \tag{3.1}$$

#### 3.1. Segment Intersection

---

**Algorithm 1** State estimation using segment intersection

---

**Input**  $y$

**Output**  $\bar{x}, \underline{x}$

- 1:  $\bar{\mathcal{X}}_k \leftarrow \text{PREDICT}(\hat{\mathcal{X}}_{k-1})$
  - 2:  $\hat{\mathcal{X}}_k \leftarrow \bar{\mathcal{X}}_k \cap \bar{\mathcal{X}}_y$
  - 3:  $[\bar{x}, \underline{x}] \leftarrow \text{INTERVAL}(\hat{\mathcal{X}}_k)$
  - 4:  $\hat{\mathcal{X}}_k \leftarrow \hat{\mathcal{X}}_{k \downarrow m}$
- 

Alg. 1 shows the overview of segment intersection technique. It starts by predicting the state at time  $k$  as  $\bar{\mathcal{X}}_k = \langle p, H \rangle$ , an  $r$ -zonotope. Line 2 finds the intersected segment between the prediction and the state consistent with the measurement input. One way to do this is to iteratively find the intersected segment for every measurement in a multi-output system. The set to represent the  $i^{\text{th}}$  input in measurement at time  $k$  is a strip, denoted by  $\mathcal{S}_i = \{x \in \mathbb{R} : |C_i x - y_{k/i}| \leq \bar{v}_{k/i}\}$ . With  $\hat{\mathcal{X}}_{k/1} = \bar{\mathcal{X}}_k \cap \bar{\mathcal{S}}_0$ ,  $\hat{\mathcal{X}}_{k/i}$  is intersected with  $\mathcal{S}_i$  for  $i = 2$  to  $n_y$ . Using Property 4, the  $\hat{\mathcal{X}}_{k/i}$  can be parameterized by



a vector  $\lambda_i \in \mathbb{R}^n$  s.t. (3.2) holds.

$$\begin{aligned} \hat{\mathcal{X}}_{k/i} &= \hat{p}(\lambda_i) + \hat{H}(\lambda_i) \mathbf{B}^{r+1} \\ \text{where } \hat{p}(\lambda_i) &= p + \lambda_i(y_{k/i} - C_i p) \\ \text{and } \hat{H}(\lambda_i) &= [(I - \lambda_i C_i)H \quad \bar{v}_{k/i} \lambda_i] \end{aligned} \quad (3.2)$$

The motive of all segment intersection methods is to find the value of  $\lambda$  such that the intersected segment is compact. Once the intersected segment is computed, the upper and lower bounds of the estimation can easily be derived using the Property 2.

For every iteration, the order of the zonotope, and hence computation overhead, increases. It can be prevented by reducing the zonotope to a maximum order of  $m$  as shown in Line 4.

### 3.1.1. F-radius

---

#### Algorithm 2 Segment minimization

---

**Input:**  $y_k$

**Output:**  $\bar{x}_k, x_k$

```

1:  $\bar{\mathcal{X}}_k \leftarrow \text{PREDICT}(\hat{\mathcal{X}}_{k-1})$ 
2:  $\langle p, H \rangle \leftarrow \bar{\mathcal{X}}_k$ 
3: for  $i \leftarrow 0$  to  $n_y$  do
4:    $\lambda_i \leftarrow \text{CALCULATE\_LAMBDA}(H, C, \bar{v}_{k/i})$ 
5:    $p \leftarrow p + \lambda_i(y_{k/i} - C_i p)$ 
6:    $H \leftarrow [(I - \lambda_i C_i)H \quad \bar{v}_{k/i} \lambda_i]$ 
7: end for
8:  $\hat{\mathcal{X}}_k \leftarrow \langle p, H \rangle$ 
9:  $[\bar{x}_k, x_k] \leftarrow \text{INTERVAL}(\hat{\mathcal{X}}_k)$ 
10:  $\hat{\mathcal{X}}_k \leftarrow \hat{\mathcal{X}}_{k \downarrow m}$ 
    
```

---

One approach to minimize the intersected segment is to minimize the F-radius of the resulted zonotope. The F-radius of a zonotope is the F-norm of its generators. Alg 2 presents the algorithm to implement this approach.

In line 4, the  $\lambda_i$ , corresponding to segment with minimum F-radius, is computed. To derive the  $\lambda_i$ , let us rewrite  $\hat{H}_i(\lambda_i)$  from Eq. 3.2 as  $A + \lambda b^T$  such that  $A = [H \quad 0]$  and  $b^T = [-C_i H \quad \bar{v}_{k/i}]$ . Thus, the Frobenius norm of the generators of a zonotope can be calculated using the formula (3.3). Refer to [6] for proof.

$$\begin{aligned} \|H\|_F^2 &= \|A + \lambda b^T\|_F^2 \\ &= 2\lambda^T A b + (b^T b) \lambda^T \lambda + \text{tr}(A^T A) \end{aligned} \quad (3.3)$$

$$\lambda^* = \frac{-Ab}{b^T b} = \frac{HH^T(C_i)^T}{C_i HH^T(C_i)^T + \bar{v}_{k/i}^2} \quad (3.4)$$

The  $\lambda^*$  that corresponds to the minimum F-radius of the intersected zonotope is calculated using the formula (3.4) for each measurement. With the  $\lambda$ , the intersected segment is computed as shown in Line 5 and 6 in Alg. 2. The remaining steps find the bounds, and reduce the zonotope order.

This approach is used when a fast real-time state estimation is needed. However, sharper bounds of estimation can be obtained by using zonotope volume.

### 3.1.2. Volume

The volume of  $\hat{\mathcal{X}}$  for the  $i^{th}$  measurement state is [6]:

$$\begin{aligned} Vol(\hat{\mathcal{X}}(\lambda)) = & 2^n \sum_{j=1}^{N(n,r)} |det[(I - \lambda C_i)A_j]| \\ & + 2^n \sum_{j=1}^{N(n-1,r)} |det[(I - \lambda C_i)B_j \quad \bar{v}_{k/i}\lambda]| \end{aligned} \quad (3.5)$$

where  $N(n, r)$  denotes the number of combinations of  $r$  elements from a set of  $n$  elements.  $A_j$  and  $B_j$  denote each of the different matrices generated by choosing  $n$  and  $n - 1$  columns from  $H$  respectively.

The algorithm is same as F-radius (Alg. 2), with the exception in the  $\lambda$  calculation in Line 4. The `zonotope.volume` function provided by CORA along with `fmincon` solver in Matlab® can be used to find the value of  $\lambda$ , parameterizing the intersected segment with minimum volume.

Although volume minimization significantly improves the intersected zonotope, the computations are extremely heavy. Therefore, it works best for use-cases that are not time-sensitive, e.g. fault diagnosis and fault-tolerant control systems [15].

### 3.1.3. P-radius

The P-radius of a zonotope can be calculated with the formula (3.6) where  $P$  is a positive definite matrix [6].

$$\max_{z \in Z} (||z - p||_P^2) = \max_{z \in Z} ((z - p)^T P (z - p)) \quad (3.6)$$

The  $\lambda$ , to parametrize non-increasing P-radius, can be computed off-line by solving the LMI (Linear Matrix Inequality) in Equation (3.7). The  $\beta \in (0, 1]$  can be found using

binary search, and  $\lambda$  can be solved using Mosek solver in Matlab®.

$$\begin{bmatrix} \beta P & 0 & 0 & A^T P - A^T C_i Y^T \\ * & F^T F & 0 & F^T P - F^T C_i Y^T \\ * & * & \bar{v}_{k/i}^2 & Y^T \bar{v}_{k/i} \\ * & * & * & P \end{bmatrix} \succeq 0, \text{ where } Y = P \lambda_i \quad (3.7)$$

Due to off-line computation, this method is expected to be substantially faster in comparison to the previous methods. In contrast, the over-approximation parameter,  $\lambda$ , does not correct itself with measurements. Therefore, it has been used in lower accuracy-prone systems like secure monitoring of cyber-physical systems against attacks [16].

## 3.2. Interval Observer

Interval observers require accurate design of observers to minimize the error in the estimation. For the system in (3.1), (3.8) defines the observer, where  $L$  is the observer gain to be designed. The design of such observers is not very easy. The following section discusses a method, which uses H- $\infty$  observer combined with reachability analysis.

$$x_{k+1} = Ax_k + L(y_k - Cx_k) \quad (3.8)$$

### 3.2.1. H- $\infty$ -based Observer

The interval observer, proposed in [17], computes the observer gain as  $L = P^{-1}Y$  with  $P$ , a positive definite matrix with dimension  $n_x \times n_x$ , and  $Y$ , a matrix with dimension  $n_x \times n_y$ , both solution to the optimization problem in (3.9).

$$\min_{\gamma_2} \text{ s.t. } (3.10) \quad (3.9)$$

$$\begin{bmatrix} I_{n_x} - P & * & * & * \\ 0 & -\gamma^2 I_{n_w} & * & * \\ 0 & 0 & -\gamma^2 I_{n_v} & * \\ PA - YC & PE & -YF & -P \end{bmatrix} \prec 0 \quad (3.10)$$

With  $L$  derived using a Mosek solver on (3.9) in Matlab®, the estimator is initialized with the following parameters:

$$\begin{aligned} \mathcal{W} &= \langle 0, H_w \rangle, & \mathcal{V} &= \langle 0, H_v \rangle \\ \mathcal{D}_w &= E\mathcal{W}, & \mathcal{D}_v &= -LF\mathcal{V} \\ \mathcal{S}_x &= \langle 0, H_0 \rangle, & \mathbf{S}_w &= \emptyset, & \mathbf{S}_v &= \emptyset \end{aligned} \quad (3.11)$$

where  $H_w = \text{diag}(\bar{w})$  and  $H_v = \text{diag}(\bar{v})$

For every measurement,  $y$ , the estimator estimates using the Alg. 3.

---

**Algorithm 3** Estimation using H- $\infty$  interval observer

---

**Input**  $y$

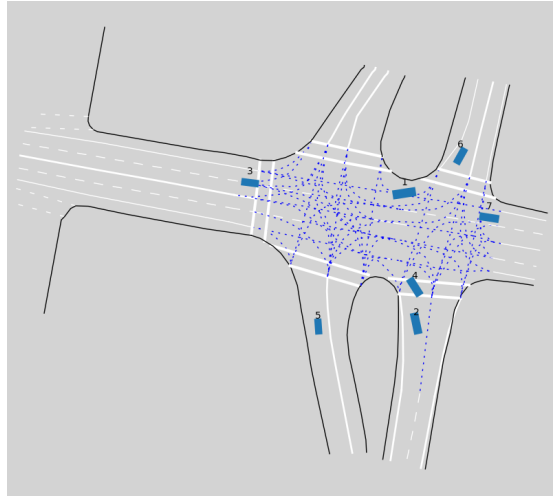
**Output**  $\bar{x}, \underline{x}$

- 1:  $[\bar{e}, \underline{e}] \leftarrow \text{INTERVAL}(\mathcal{S}_x) \oplus \mathbf{S}_w \oplus \mathbf{S}_v$
  - 2:  $\bar{x} \leftarrow \hat{x} + \bar{e}$
  - 3:  $\underline{x} \leftarrow \hat{x} + \underline{e}$
  - 4:  $\hat{x} \leftarrow A\hat{x} + L(y - C\hat{x})$
  - 5:  $\mathcal{S}_x \leftarrow (A - LC)\mathcal{S}_x$
  - 6:  $\mathbf{S}_w \leftarrow \mathbf{S}_w \oplus \text{INTERVAL}(\mathcal{D}_w)$
  - 7:  $\mathbf{S}_v \leftarrow \mathbf{S}_v \oplus \text{INTERVAL}(\mathcal{D}_v)$
  - 8:  $\mathcal{D}_w \leftarrow (A - LC)\mathcal{D}_w$
  - 9:  $\mathcal{D}_v \leftarrow (A - LC)\mathcal{D}_v$
- 

Interval observers have gained fame in control theory and can be used in stabilization, optimal control, fault detection and circuit systems [Huang2019].

## 4. Evaluations

Figure 4.1.: Simulation of the dataset



The INTERACTION Dataset <sup>1</sup> is used to evaluate the algorithms. The dataset contains multiple scenarios in different locations, captured using drones or fixed cameras over a variable amount of time. Each scenario consists of multiple traffic participants, identified by an ID, and each frame per 0.1s has a set of vehicles and their position and velocity in the x and y-direction. Over a choice of multiple videos, the location with all videos summing up to a total length of 259.43 minutes is chosen for this paper. A simulated scenario is demonstrated in Fig. 4.1. There are 60 recorded files in this location with a total of 10,518 vehicles. The position in the x and y-direction is used as a measurement input to the algorithms, whereas the velocity in the x and y-direction is used to calculate the error and evaluate the estimates.

With every participant modeled as (2.7), the initial state is set as (4.1). The matrices  $E$

---

<sup>1</sup><https://interaction-dataset.com/>

and  $F$  are  $I_{n_x}$  and  $I_{n_y}$  respectively. All zonotopes are limited to maximum order 20.

$$\begin{aligned}\mathcal{X}_0 &= \langle 0, \text{diag}([1000 \ 1000 \ 10 \ 10 \ 10 \ 10]^T) \rangle \\ \overline{w}_k &= [0.1 \ 0.1 \ 0.4 \ 0.4 \ 0.1 \ 0.1]^T \\ \overline{v}_k &= [0.1 \ 0.1]^T\end{aligned}\tag{4.1}$$

All the evaluation are carried out by single-threaded scripts run on an Intel(R) Core(TM) i7-7500U CPU @ 2.70GHz machine with MATLAB® 2019b. The CORA toolbox is used for set computations; the Mosek solver in YALMIP toolbox is used to solve the optimization problems.

### 4.1. Computation Time

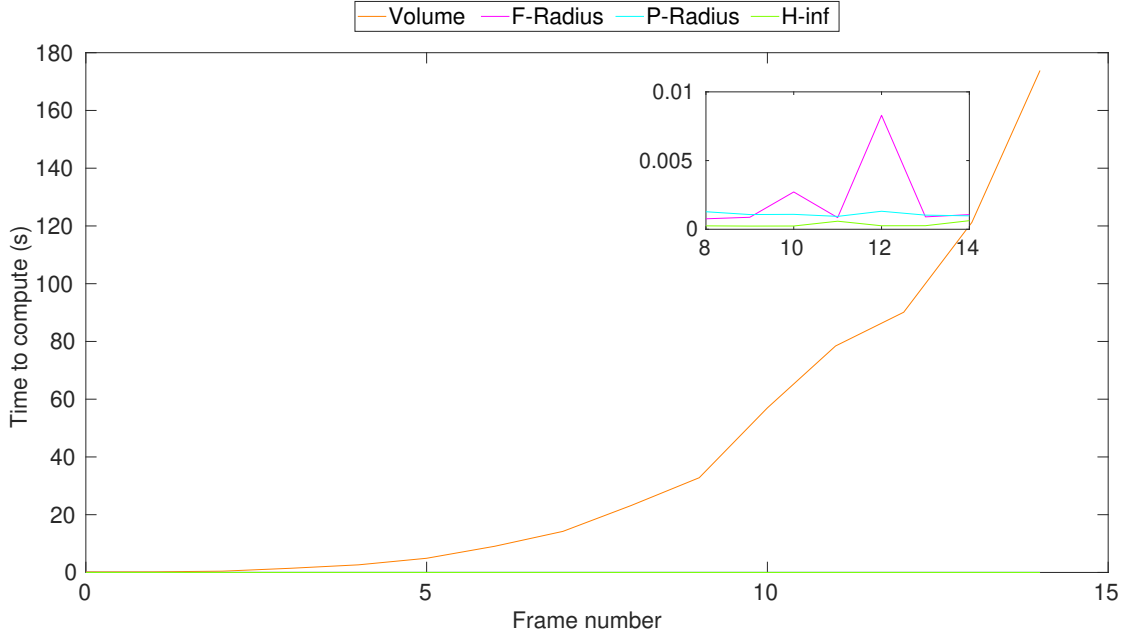


Figure 4.2.: Computation time for each method to estimate using the CV model

Fig. 4.2 illustrates that the computation time for volume minimization rises exponentially over time. Due to the choice of limiting zonotoper order to  $m \leq 20$ , the time for volume computation is expected to steeply rise till the 20<sup>th</sup> frame. Such load makes volume minimization futile for a collision-avoidance system; hence this method is avoided in the rest of the paper.

Table 4.1.: Comparison of computation time (ms)

Method	Average Computation Time (ms)		
	CV model	CA model	PM model
F-Radius	0.396	0.375	0.621
P-Radius	0.312	0.319	0.544
H- $\infty$ approximation	0.145	0.147	0.144

Tab. 4.1 shows that the computation time for the other methods are negligible compared to the frame rate, i.e 100ms. Furthermore, the interval observer using H- $\infty$  has almost half the time taken by the segment intersection methods, which can be explained by the absence of zonotope reduction operation in the former method.

In addition, Tab. 4.1 highlights how the choice of vehicle model affects the performance. The H- $\infty$  observer is not much affected by model, whereas, the segment minimization methods have significant decline in performance with point-mass model. This observation can be explained by how the constraints in point-mass model are applied. In segment minimization methods, the point-mass model applies constraints on the zonotope for the state, whereas, the H- $\infty$  observer only constraints the upper and lower bound.

## 4.2. Time to Converge

Table 4.2.: Comparison of approximate time (in s) to converge

Method	Velocity			Acceleration	
	CV model	CA model	PM model	CA model	PM model
F-Radius	0.2	0.2	0.2	0.2	2.8
P-Radius	2.3	2.47	2.45	0.2	5
H- $\infty$ approximation	1.4	0.9	0.8	0.8	3

The time to converge is an important performance metric, but is difficult to measure. For this paper, it is calculated as the time taken for the rate of change of estimated bounds to approach zero. Tab. 4.2 shows the approximate time for  $velocity_x$  and  $acceleration_x$  estimates to converge. It is trivial to note that the P-radius technique demonstrates the worst performance, which can be explained by its fixed pre-computed parameter for intersected segment. In contrast, the F-radius technique, which adjusts

the aforementioned parameter in run-time for every measurement, performs the best, taking only 2 time steps to converge both velocity and acceleration.

One interesting observation is how the point-mass model significantly delays the time to converge for all the techniques. This observation can be the result of the acceleration constraint in pm-model that changes the behaviour of the linear system of the vehicle from the expectation. An extended result of the rate of change of bounds of all the variables over time is available in appendix.

### 4.3. Bounds

Table 4.3.: Comparison of bounds of estimation

Method	Constant Velocity					
	$s_x$	$s_y$	$v_x$	$v_y$	$a_x$	$a_y$
F-Radius	.441	.441	5.686	5.686	-	-
P-Radius	.4606	.459	13.45	13.45	-	-
H- $\infty$ approximation	.9867	.937	6.177	6.177	-	-
	Constant Acceleration Model					
	$s_x$	$s_y$	$v_x$	$v_y$	$a_x$	$a_y$
F-Radius	.5713	.5075	8.461	8.461	15.86	15.97
P-Radius	.4598	.4523	16.39	16.39	16.61	17.42
H- $\infty$ approximation	1.5	1.5	9.414	9.414	16.42	16.35
	Point-Mass Model					
	$s_x$	$s_y$	$v_x$	$v_y$	$a_x$	$a_y$
F-Radius	.5713	.5075	8.461	8.461	15.79	15.78
P-Radius	.4598	.4523	16.39	16.39	16.43	16.18
H- $\infty$ approximation	1.5	1.5	9.414	9.414	16.11	16.24

Bounds using constant velocity model is tighter compared to constant acceleration and point-mass model as seen from Tab. 4.3. This is an indication that better results are obtained for model with fewer unmeasured state. Bounds from constant acceleration and point-mass are identical in all variables, except the acceleration where point-mass performs better. As expected, F-radius has better bounds for the unmeasured states.

As point-mass model provides estimation for both velocity and acceleration, and outperforms constant acceleration model, this model is preferred in this paper and used to compare the techniques in the next section. To see how the bounds enclose each state, refer to appendix.



#### 4.4. Accuracy

Accuracy is represented by the root mean square error (RMSE) of the estimation from the true state of the system. Since, the dataset does not have the measurement for acceleration, the accuracy of acceleration cannot be evaluated.

The initial estimation before convergence gives an extreme error which affects the result, hence the estimations after convergence (i.e. after 50 time-steps) are allowed in the evaluation. The RMSE is then computed as a percentage from the maximum measurement in the time frame.

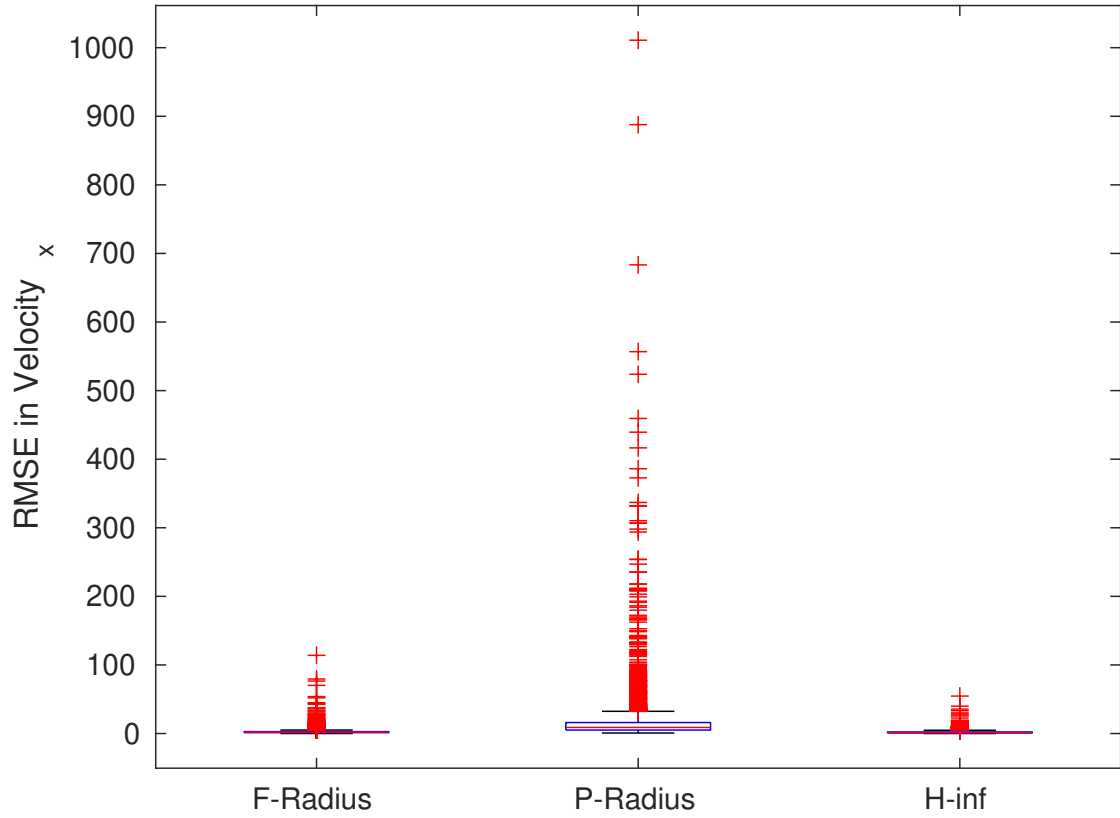


Figure 4.3.: RMSE(Root Mean Squared Error)

The boxplot of RMSE for  $velocity_x$  using the point-mass model for all the techniques are shown in Fig. 4.3. The segment minimization using P-radius has a high range of extremes and the mean error is also greater than the other methods. The range of error is the lowest for H- $\infty$  observer with mean 1.9048 and standard deviation of 1.9776 for  $velocity_x$ . A detailed report of the mean and standard deviation of each of the

methods can be found in Tab. 4.4. The error expectation from segment minimization from F-radius is similar to H- $\infty$  observer, however, the error in the measured state is slightly lesser in F-Radius compared to H- $\infty$  observer.

Table 4.4.: Comparison of RMSE

Method	Mean $\pm$ SD			
	$s_x$	$s_y$	$v_x$	$v_y$
F-Radius	$0.0007 \pm 0.0004$	$0.0004 \pm 0.0003$	$2.3412 \pm 2.7092$	$2.4184 \pm 1.5641$
P-Radius	$0.0016 \pm 0.0020$	$0.0008 \pm 0.0017$	$13.4470 \pm 26.2465$	$13.7642 \pm 54.6724$
H- $\infty$ approximation	$0.0007 \pm 0.0004$	$0.0006 \pm 0.0005$	$1.9048 \pm 1.9776$	$2.1230 \pm 2.1946$

## 4.5. Discussion

The results in the previous sections prove that the output of the techniques depend heavily on the vehicle model used. In this section, we try to choose an estimator and a model, best suited to the collision avoidance system.

Firstly, from the presented models, the point-mass model is the most accurate model for tracked vehicles, because it gives better estimates for acceleration. Secondly, choosing the algorithm is tricky. It seems that every algorithm has pros and cons. Comparing the estimator, the volume minimizer lost in computation time, whereas the P-radius minimizer gave terrible bounds. In contrast, the H- $\infty$ -based observer is the fastest observer, and F-radius segment minimizer gave the tightest bounds with fewer measurements. Although, H- $\infty$  gave estimates with higher confidence (lower RMSE mean and standard deviation), H- $\infty$  observer fails to enclose the true state in initial time steps if the initial estimation is not appropriate. In comparison, F-radius does not depend on the initial estimation. Particularly, enclosing true state with sharp bounds is the most desirable outcome for our use-case. For this reason, F-radius can be concluded to be the most suitable technique for collision avoidance system.

## 5. Conclusion

A demand for intelligent collision avoidance system is timeless. To take the load off sensors and hardware of a vehicle, state estimation algorithms can be used to track vehicles and estimate properties required for collision-free path prediction. On comparing multiple techniques using different models to represent the tracked vehicle, it can be concluded that the segment intersection minimizing F-radius ensures convergence to more accurate and tighter bounded estimation with lesser data.  $H-\infty$  observer and P-radius minimizer carry out off-line computation and hence are ahead in terms of run-time computation cost; nonetheless, as a consequence, these methods over-approximate and might not improve estimation significantly for each measurement. The choice of a model to represent the state of the system also has a significant effect on the performance. Higher the number of measured state, the more tuned the estimate gets. This paper can be a starting point to implement higher-defined models of the tracked vehicle and compare the performance of state estimation methods. The state estimation methods can further be evaluated on the effect of the initial estimated state on the algorithms. Further developments can be to use non-linear state-estimation algorithms on complex vehicle models and compare the performance.

## **A. Extended Result**

### **A.1. Set Estimation**

Estimation using the techniques with different models are illustrated using data for one particular vehicle in the dataset in this chapter. Results show that the true state is always bounded by the set of estimation. For acceleration, there is no true measurement because the acceleration of the tracked vehicle is absent in the dataset.

#### **A.1.1. Segment Minimization using F-Radius**

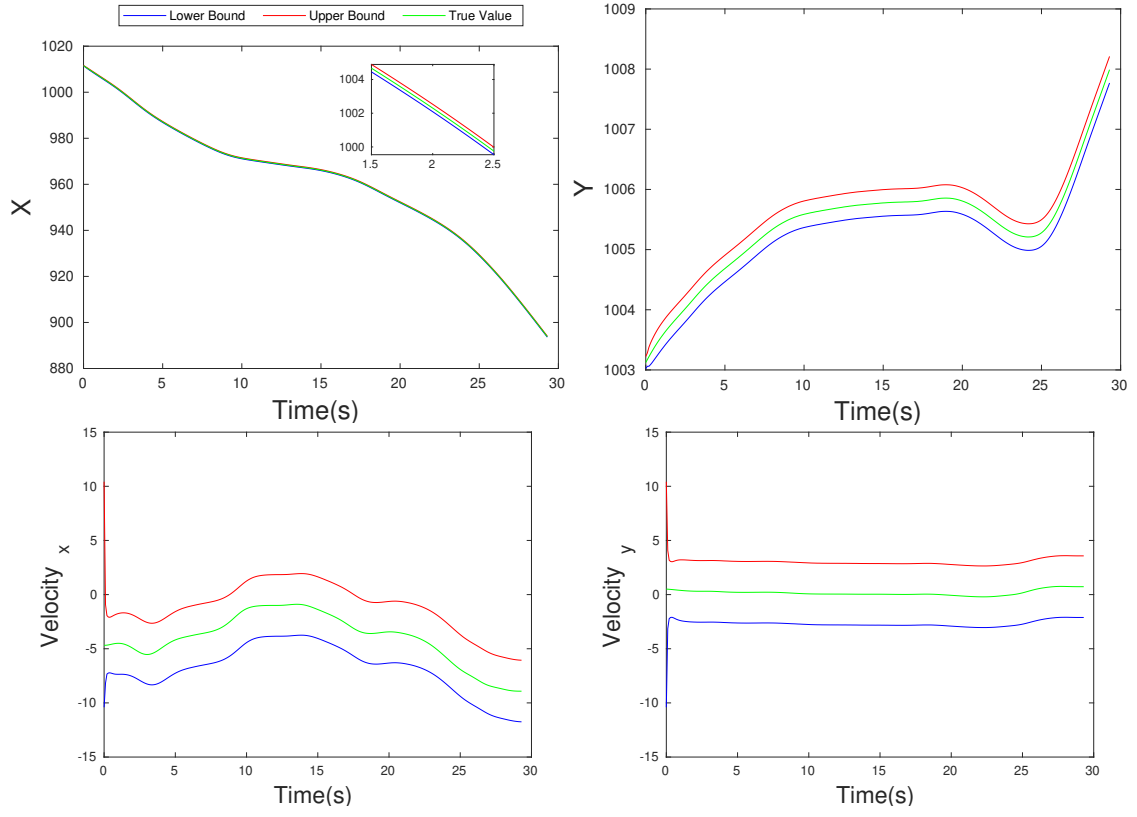


Figure A.1.: Estimation using Constant Velocity

### A. Extended Result

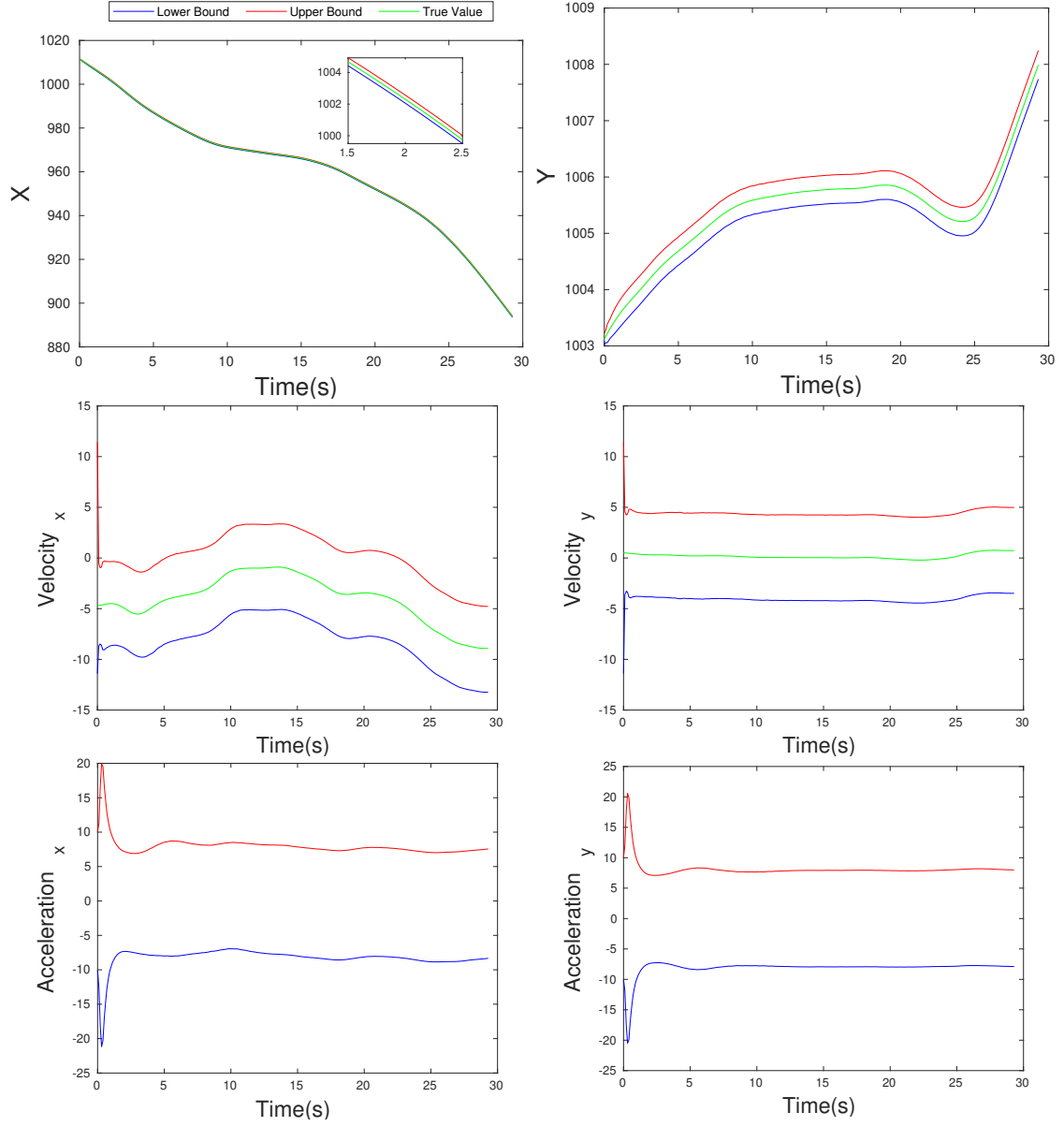


Figure A.2.: Estimation using Constant Acceleration

### A. Extended Result

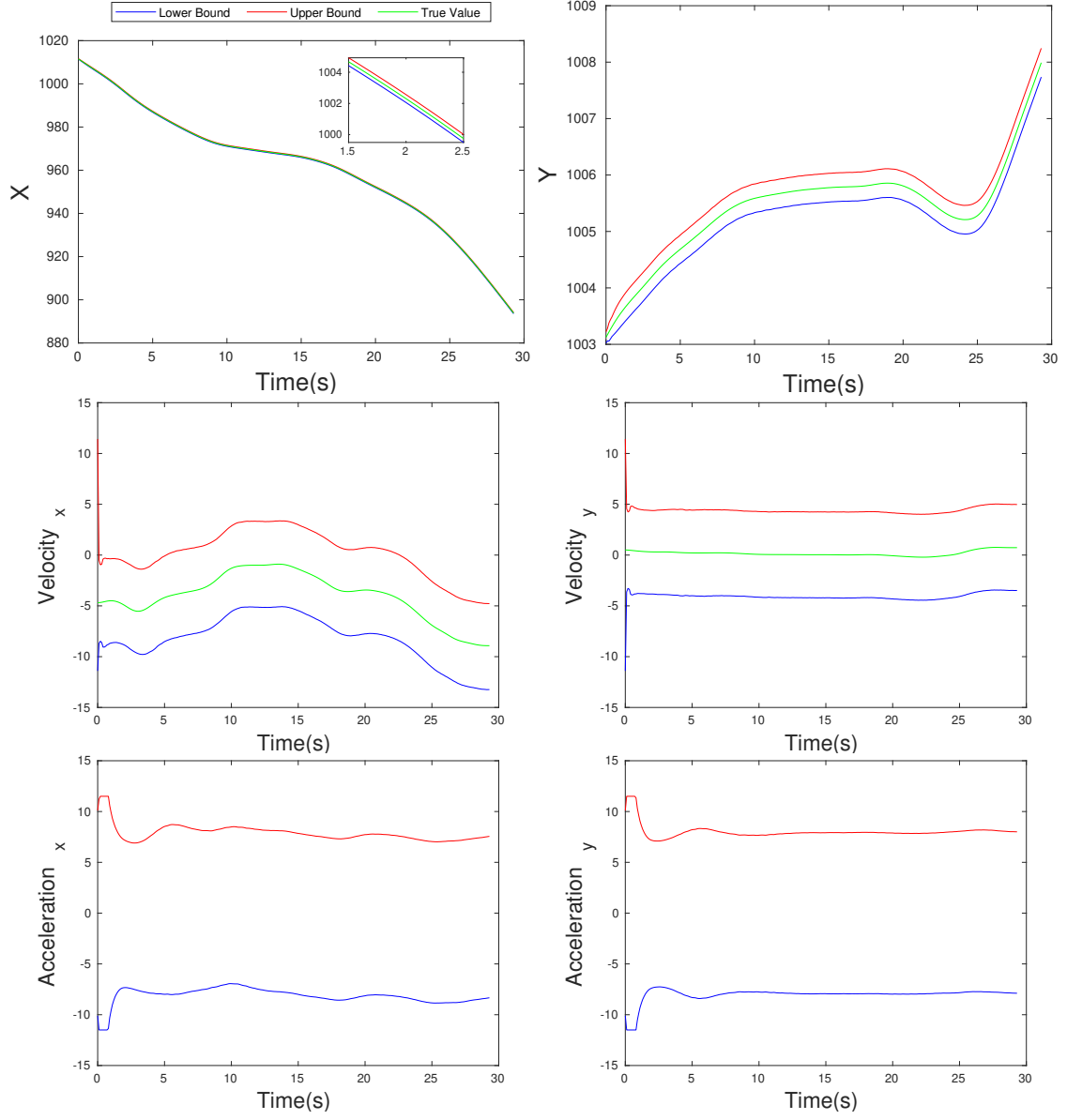


Figure A.3.: Estimation using Point Mass Model

### A.1.2. Segment Minimization using P-Radius

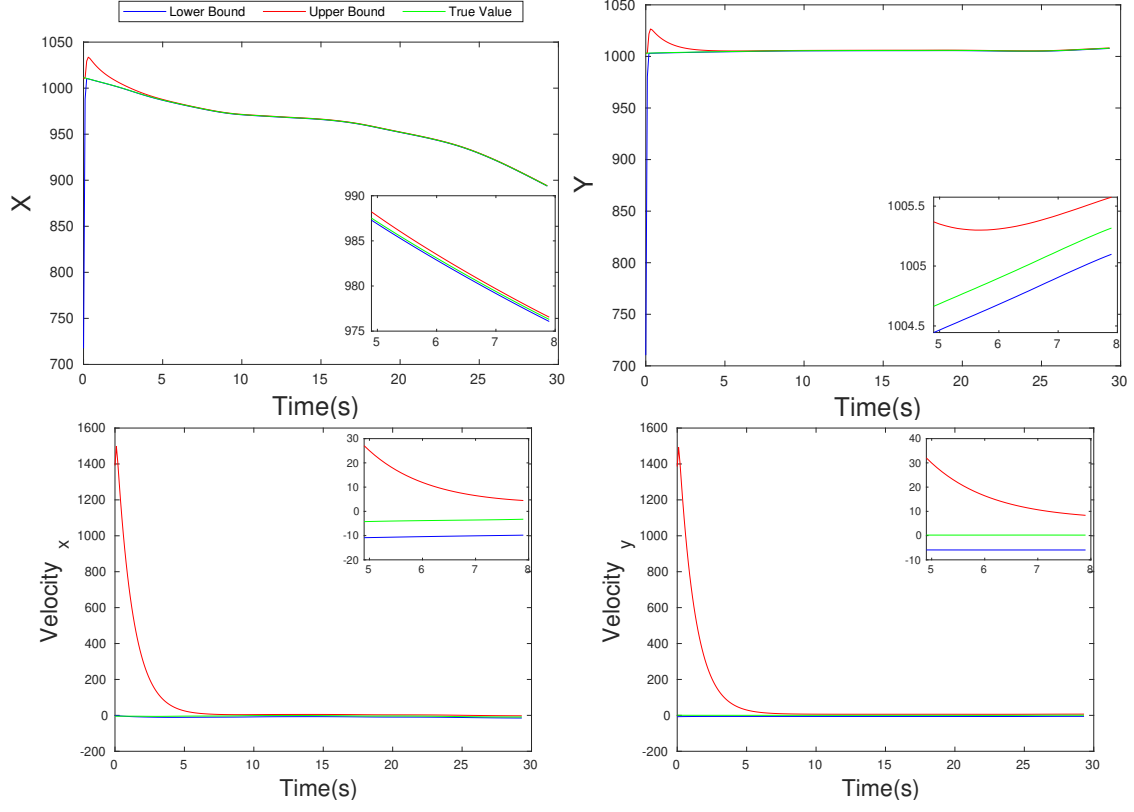


Figure A.4.: Estimation using Constant Velocity



### A. Extended Result

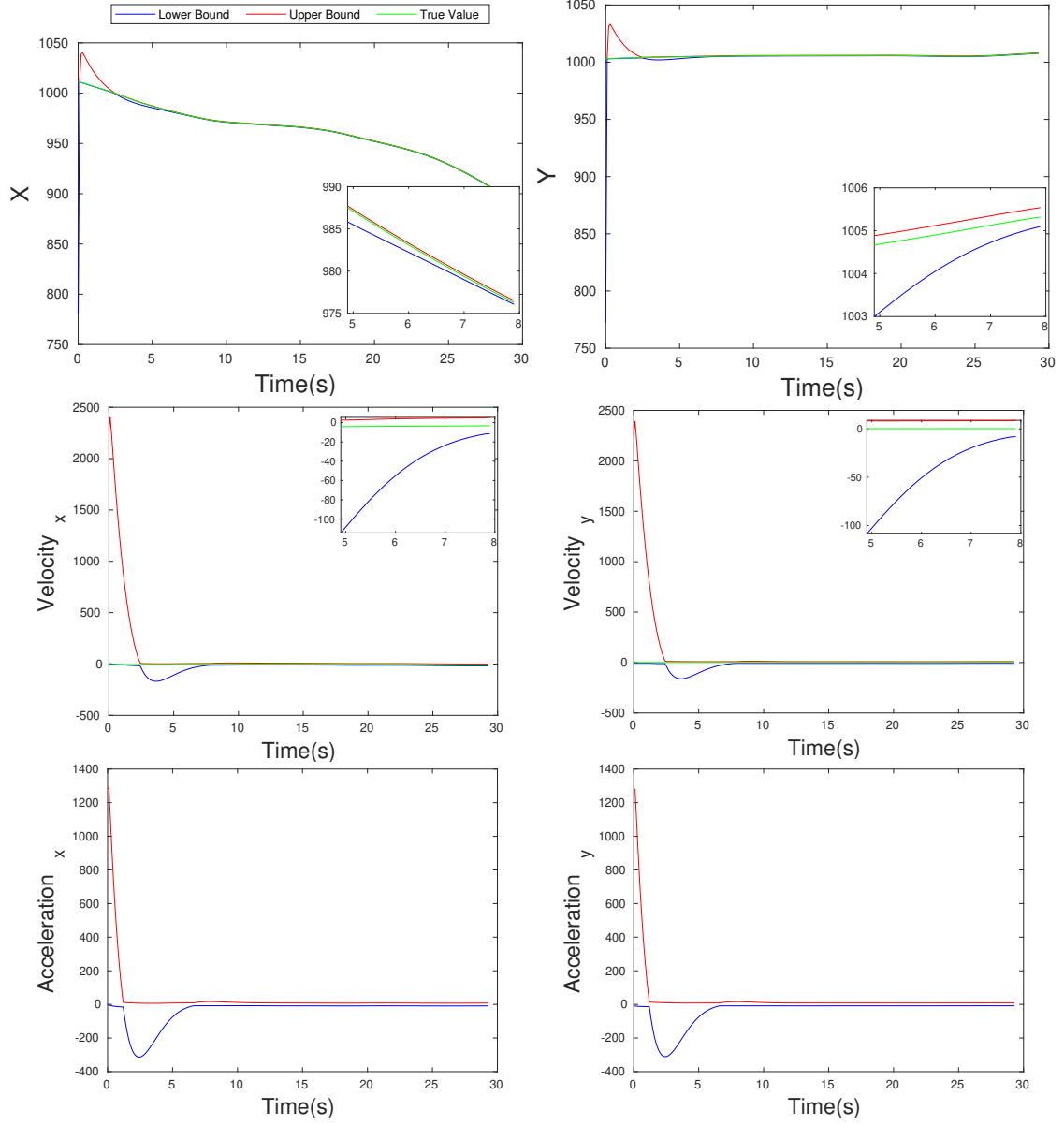


Figure A.5.: Estimation using Constant Acceleration

### A. Extended Result

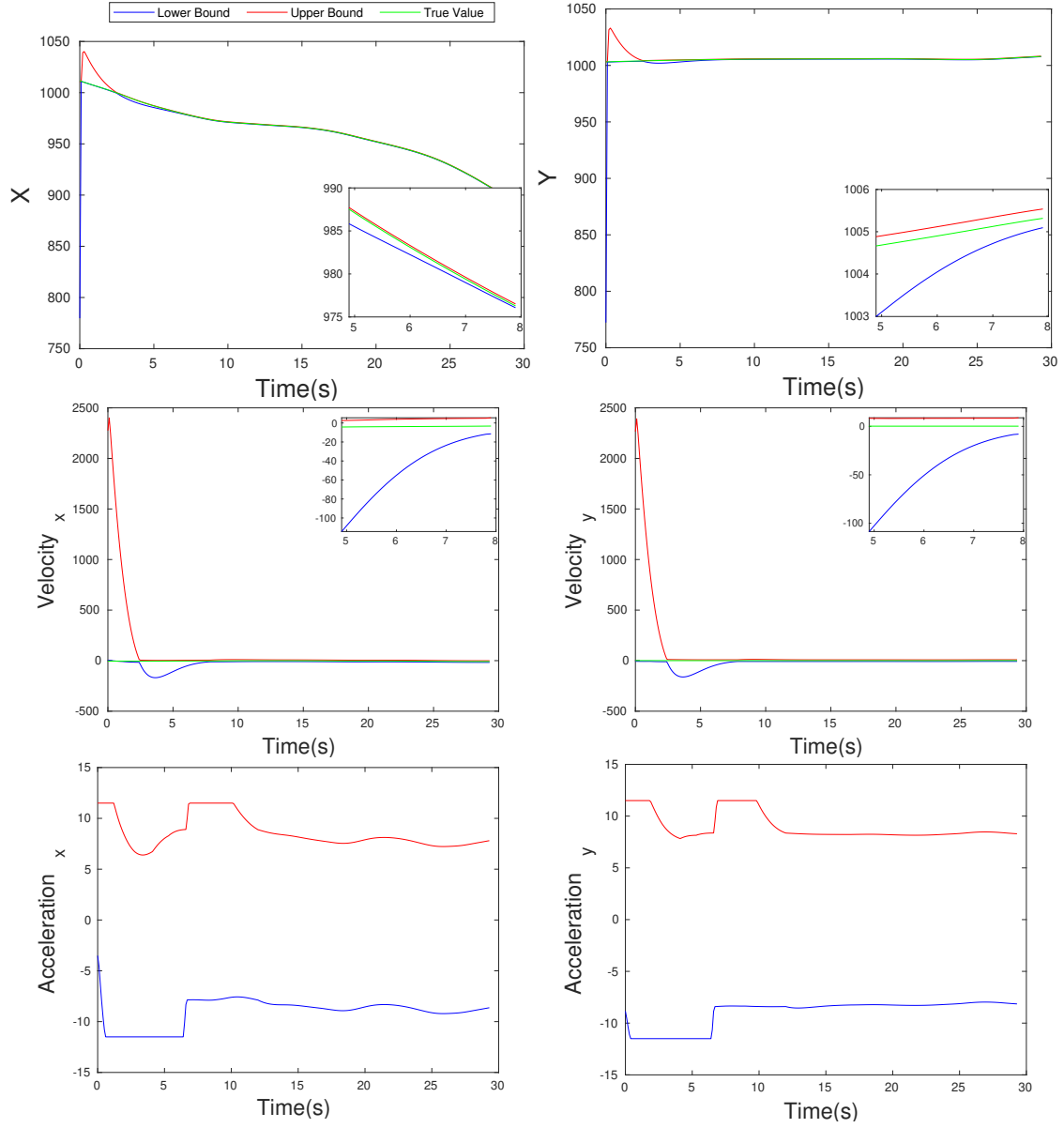


Figure A.6.: Estimation using Point Mass Model

### A.1.3. Interval Observer using $H_\infty$

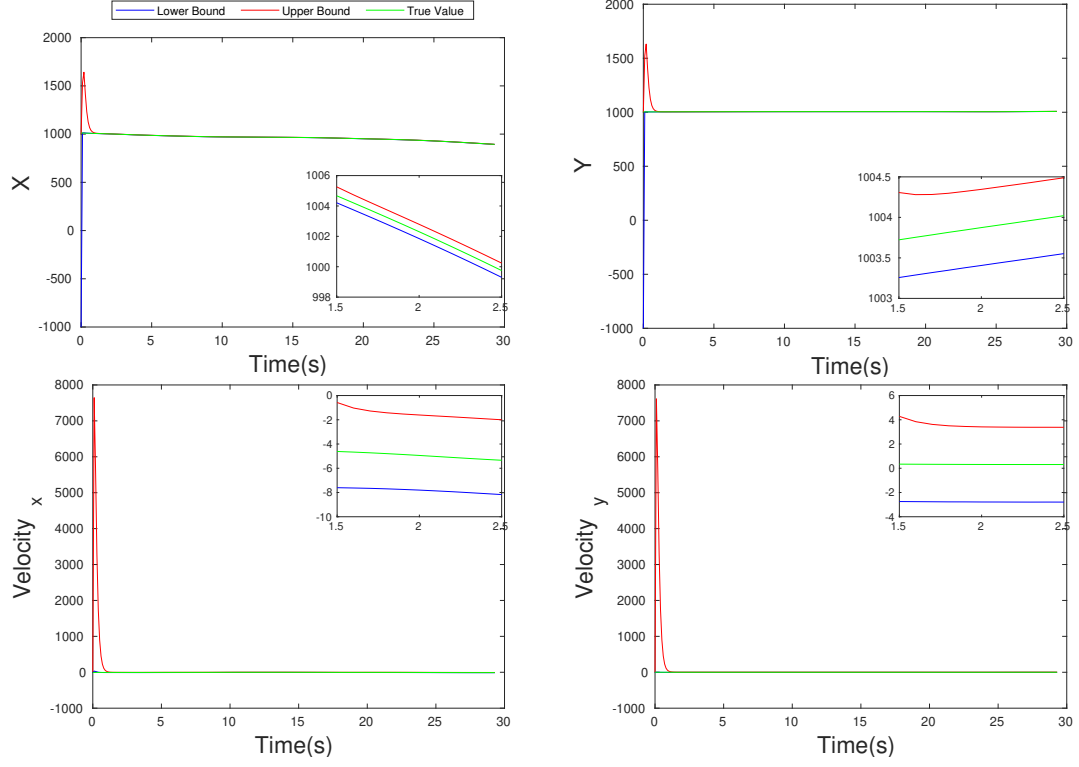


Figure A.7.: Estimation using Constant Velocity

### A. Extended Result

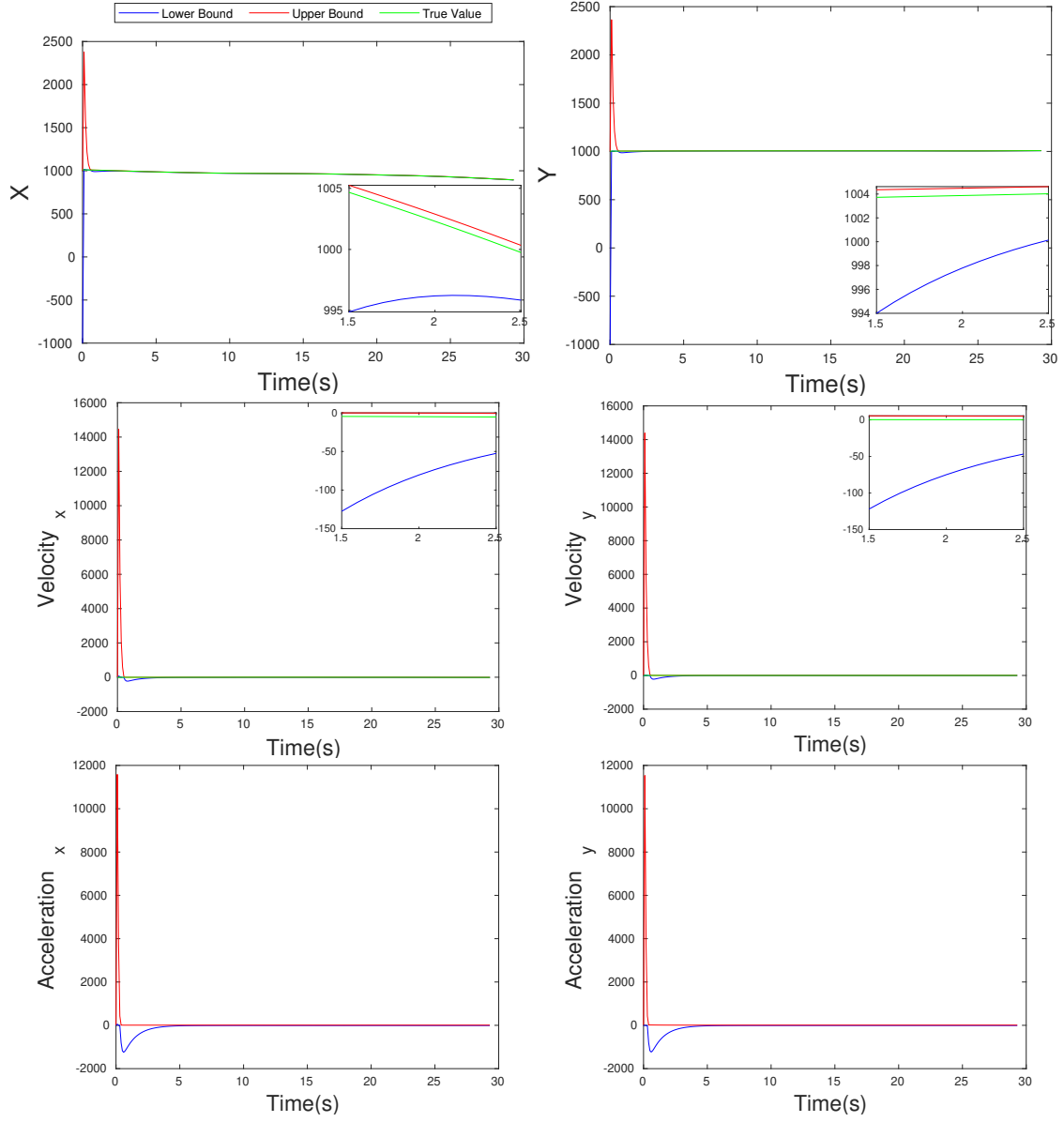


Figure A.8.: Estimation using Constant Acceleration

### A. Extended Result

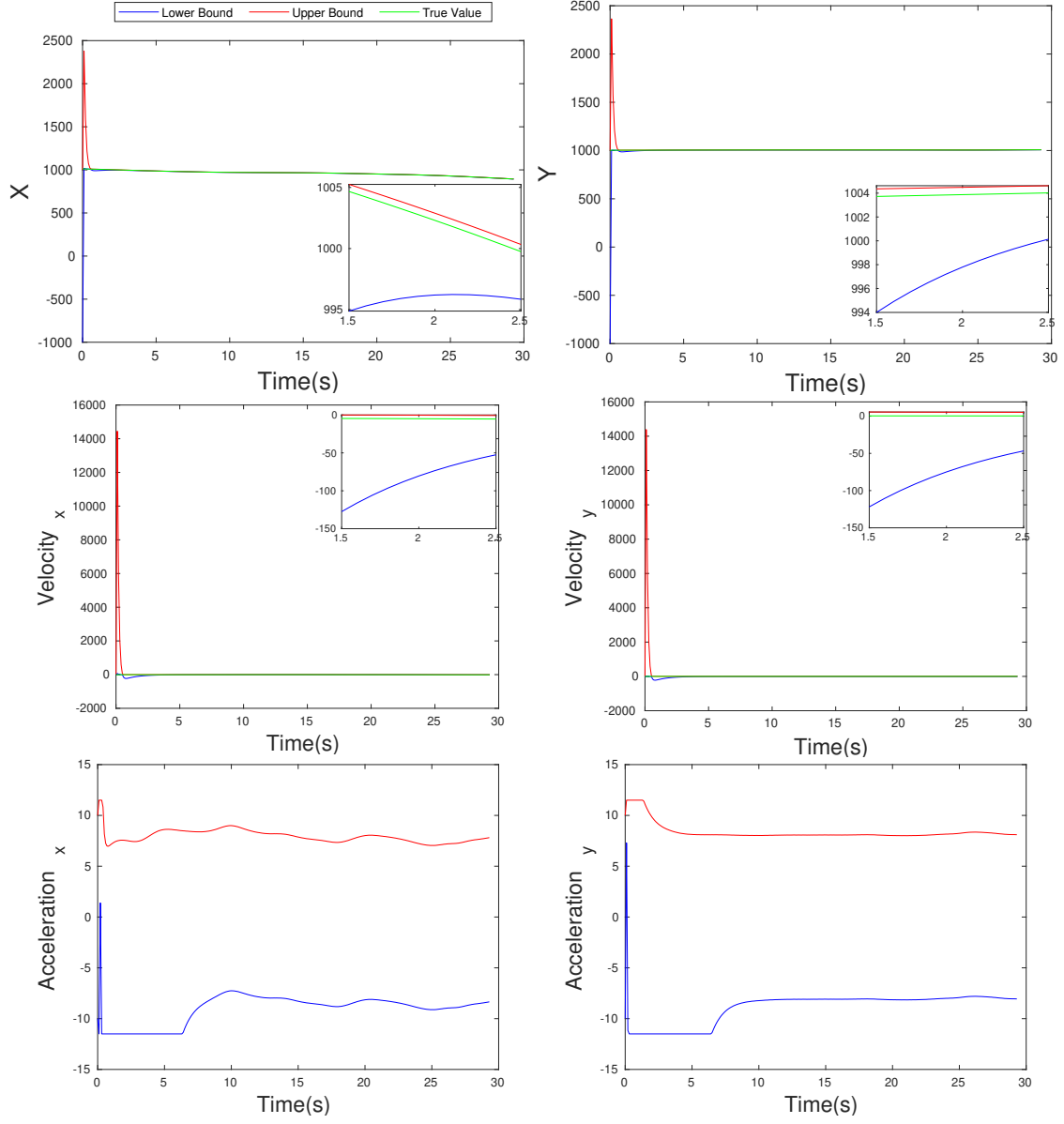


Figure A.9.: Estimation using Point Mass Model

## A.2. Rate of Change of Bounds

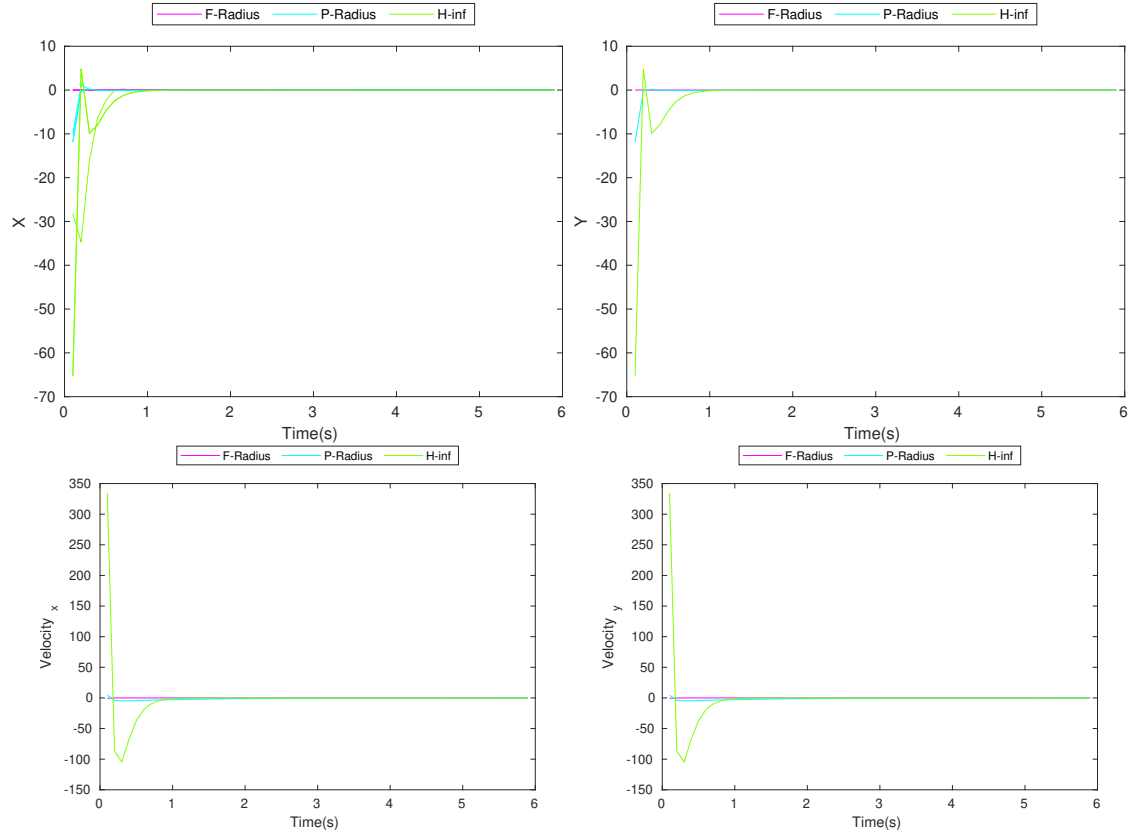


Figure A.10.: Rate of change of bounds using Constant Velocity Model

## A. Extended Result

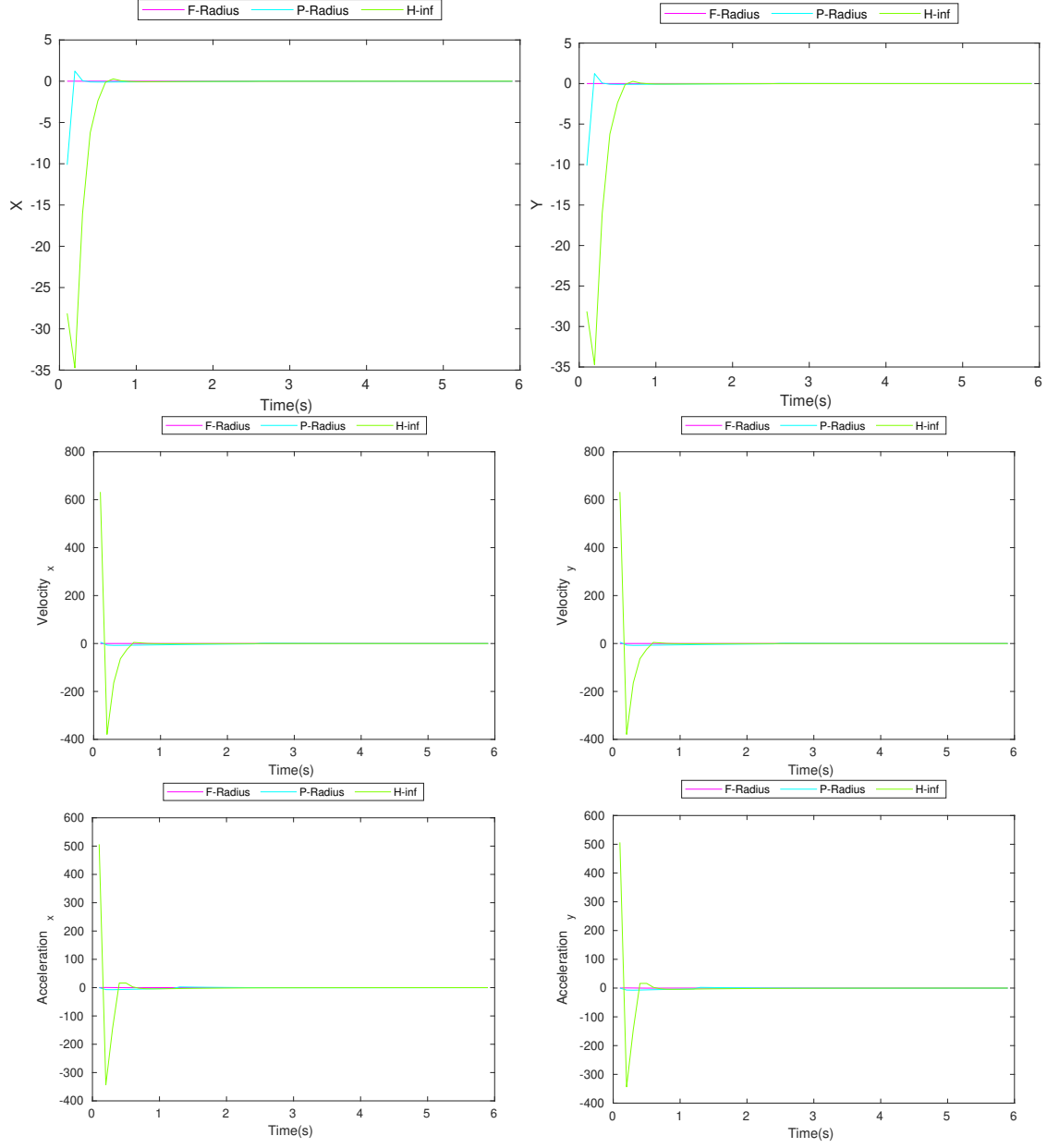


Figure A.11.: Rate of change of bounds using Constant Acceleration Model

## A. Extended Result

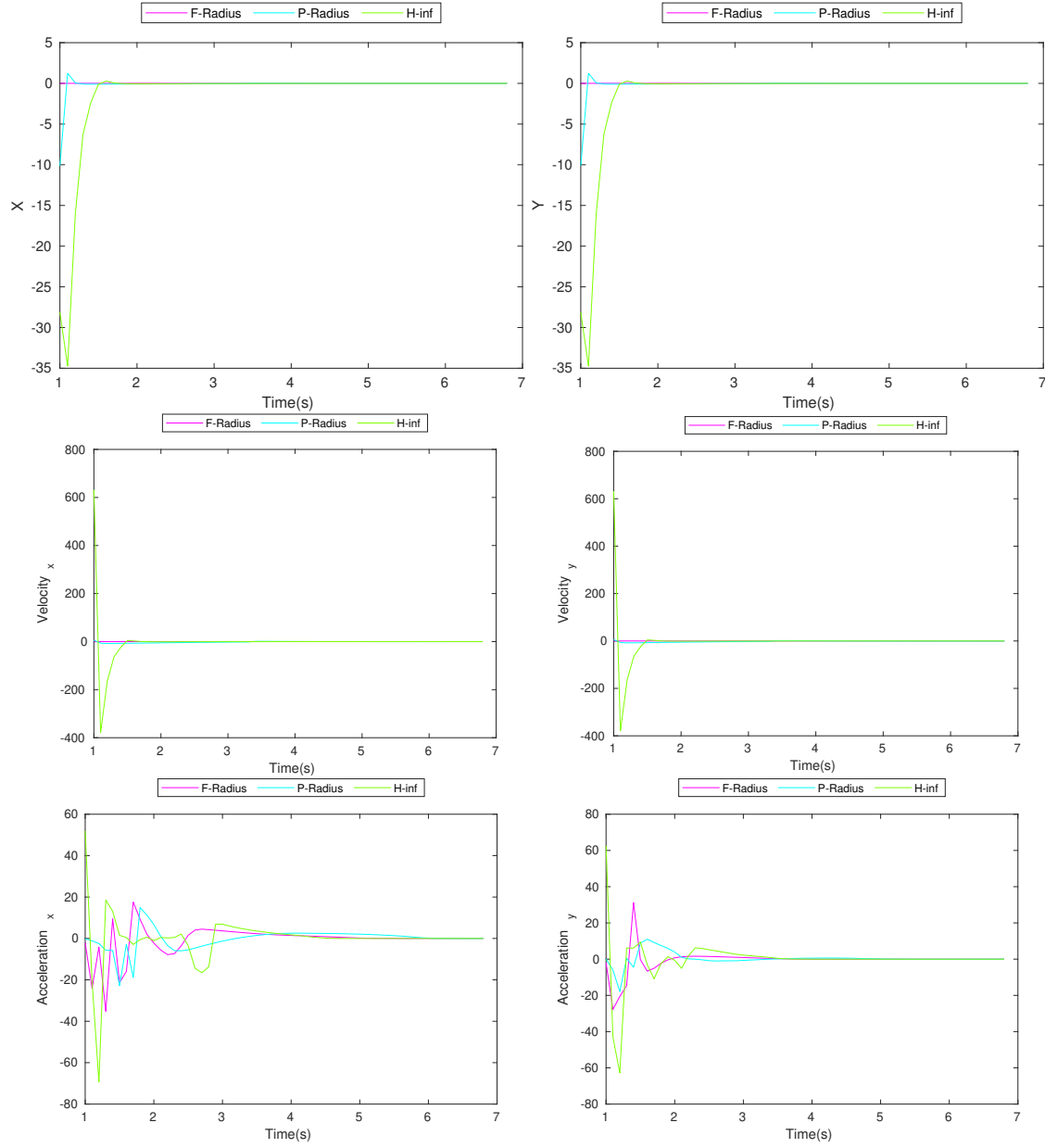


Figure A.12.: Rate of change of bounds using Point Mass Model



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