PROD-1 (~~2021-22~~) (2023-24) - 2nd Sem
21/03/2024

Engg. Mechanics

Basic unit and Dimensions:

The subject Engg. Mech. may be divided into ~~two~~ Three main groups.

1) Statics 2) Dynamics 3) Hydromechanics

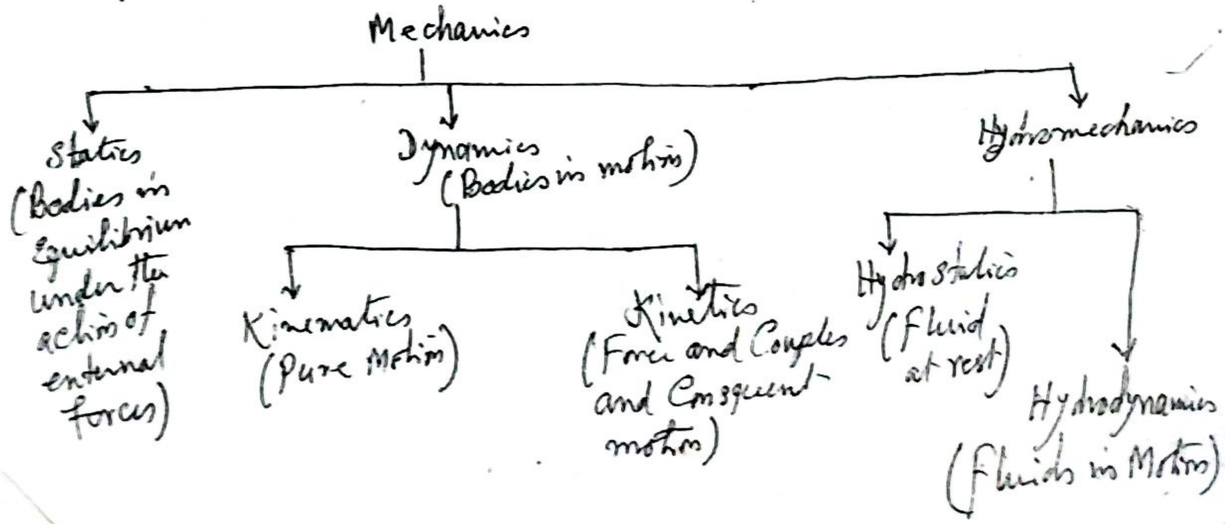
1) **Statics**:- It is that branch of Engg. Mech. which deals with the forces and their effects, while acting upon the bodies at rest.

2) **Dynamics**: It is that branch of Engg. Mech., which deals with forces and their effects, while acting upon the bodies in motion. The subject of Dynamics may be further divided into two branches.

(i) Kinematics (ii) Kinetics.

1) **Kinematics**: It is the branch of Dynamics, which deals with the bodies in motion, without any reference to the forces which are responsible for the motion.

ii) **Kinetics**:- It is the branch of Dynamics, which deals with the bodies in motion due to the application of forces.



(2)

Units: All the physical quantities, met with in
1) Fundamental unit:
Engg. Mech, are expressed in terms of three fundamental quantities

1) Length 2) Mass 3) ~~Force~~ Time

11) Derived units:- Sometimes, the units are also expressed in the other units (Which are derived from fundamental units) Known as derived units
e.g. area, Velocity, acceleration, pressure etc.

System of units:-
1) C.G.S. 2) F.P.S 3) M.K.S 4) S.I units.

S.I units

Density	Kg/m^3
Force	Newton (N)
Pressure	N/mm^2, N/m^2
workdone	J, N-m.
power	watt, J/second.

V-(2A)

Units Force $1\ KN = 10^3\ N$

$1\ MN = 10^6\ N.$

Kilo	10^3	Milli - 10^{-3}
Mega	10^6	Micro - 10^{-6}
Giga	10^9	Nano - 10^{-9}
Terra	10^{12}	Pica - 10^{-12}

Work, Energy.

1 Kilo joul $= 10^3$ joule

1 watt = 1 joule per Second = 1 Newton metre per Second

1 H.P = 736 watt

Density or Specific mass ρ $\Rightarrow$ units kg/m^3

Specific Weight 'W' $\Rightarrow$ Newton per m^3

Pressure intensity p $\Rightarrow$ Newton$/m^2$ = Pascal (Pa)

Stress, Modulus of Elasticity $\Rightarrow$ Newton$/m^2$ (Pascal)

Newton$/mm^2$

(V-2B)

Principal SI units used in MECHANICS

Quantity	units	Symbol	Formula
Length	metre	m	m
Area	Square metre		m^2
Volume(S)	cubic metre		m^3
Volum(L)	Litre	L	$10^{-3} m^3$
Angle	radian	rad	
Time	Second	S	
Velocity	metre/Sec		m/s
Angular Vel	radian/Sec		rad/s.
Acceleration	m/s/s		m/Sec^2
Angular accl	rad/Sec^2		rad/Sec^2
Mass	Kilogram	Kg.	
Density	$Kg./m^3$		Kg/m^3
Force	Newton	N	Kg
Moment of a Force	NeWton metr		~~rd~~ Nm.
Pressure	Pascal	Pa	N/m^2
Stress	Pascal	Pa	N/m^2
Freqeency	Hertz	Hz	S^{-1}
Impulse	Newton Second		Kg m/sec
Work	Joule	J	N m.
Energy	Joule	J	N m
Power	Watt	W	J/S.

<u>Vector</u>: The vector quantities are those quantities which have both magnitude and direction such as FORCE, DISPLACEMENT, VELOCITY, ACCELERATION, MOMENTUM etc.

i) <u>Important features of vector quantities</u>:

1) Representation of vector → arrowhead
^ Cap.

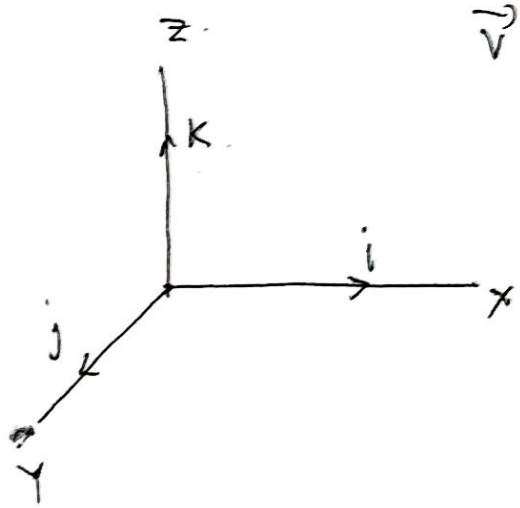
ii) <u>unit vector</u>: A unit vector is defined as a vector having a unit magnitude.

$$\vec{A} = \hat{e}.A \qquad \hat{e} = \frac{\vec{A}}{A} = \text{unit vector}$$

$$\vec{V} = \hat{e}V \qquad \hat{e} = \frac{\vec{V}}{V}$$

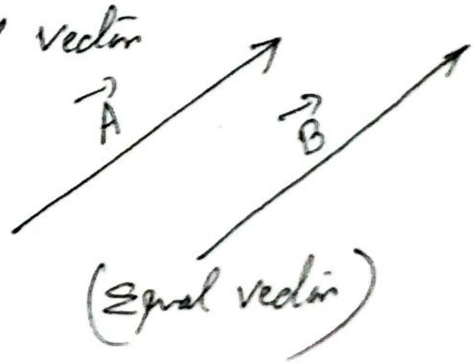
$$\hat{i} = \frac{\vec{X}}{X}, \quad \hat{j} = \frac{\vec{Y}}{Y}, \quad \hat{k} = \frac{\vec{Z}}{Z}$$

$$\left.\vec{X} = \hat{i}X, \quad \vec{Y} = \hat{j}Y, \quad \vec{Z} = \hat{k}.Z\right\}$$



iii) <u>Equal vector</u>: Two vectors are said to be equal, if only if have same magnitude, same directions and same sense.

$\vec{A}$ and $\vec{B}$ are two equal vector
The equation $\vec{A} = \vec{B}$



(Equal vector)

iii) Like Vector: The vectors, which are parallel to each other and have same sense but unequal magnitude, are known as like Vector.

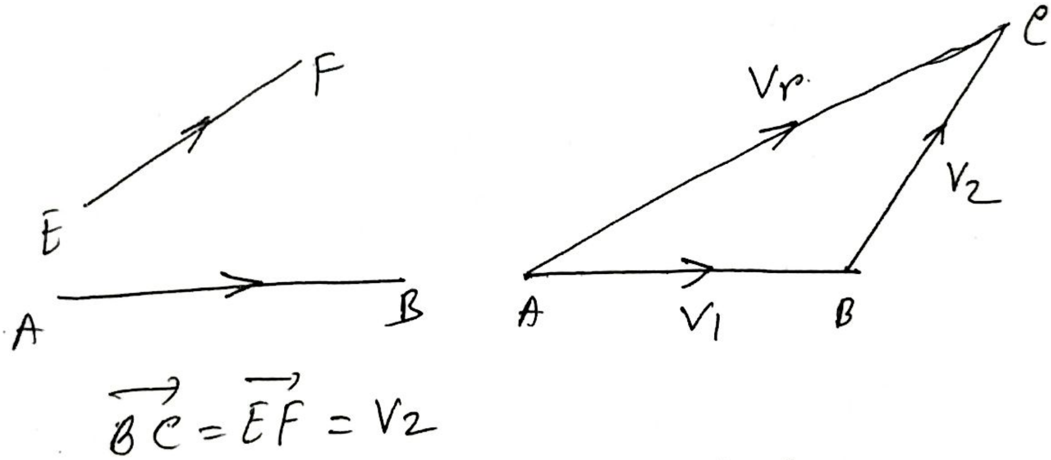
iv) Addition and Subtraction of vectors :→

The displacement Vectors V_1 and V_2 remaining in the same plane.

V_1 is displacement from A to B.

V_2 " E to F

$\therefore \overrightarrow{AB} = V_1 \; , \; \overrightarrow{EF} = V_2$



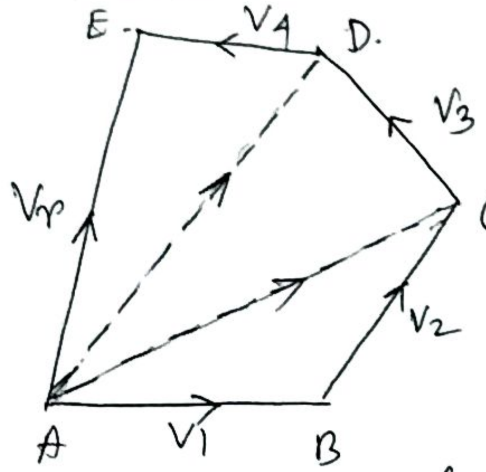
$\overrightarrow{BC} = \overrightarrow{EF} = V_2$

Now the net displacement i.e resultant displacement Vector is

$$\boxed{V_r = \overrightarrow{AC}}$$

$$\therefore \boxed{V_r = V_1 + V_2}$$

A number of Coplanar Vectors may also be added in a similar manner.



$V_1+V_2=\vec{AC}$

$\vec{AC}+V_3=\vec{AD}$

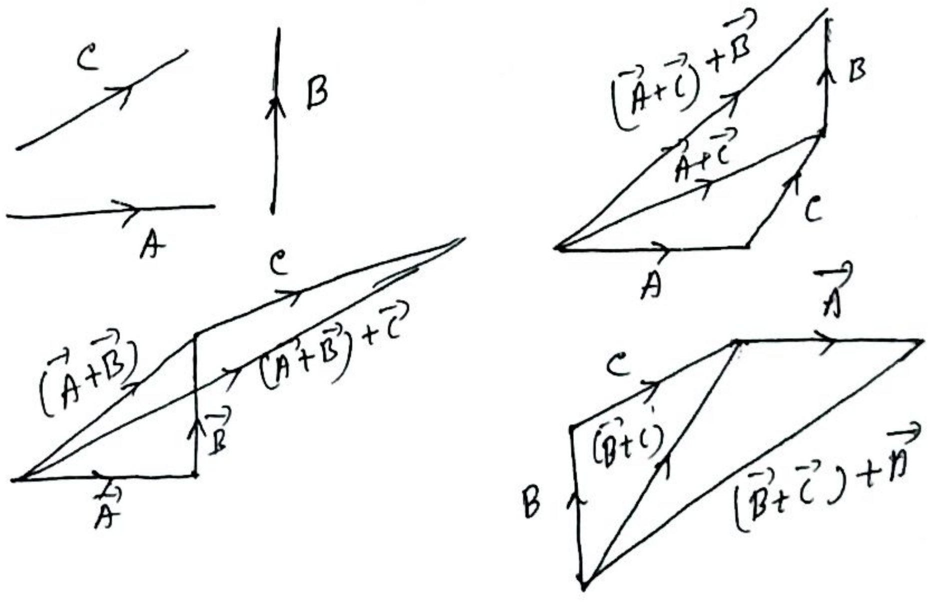
$\vec{AD}+V_4=\vec{AE}$

$=V_r$

The method of addition of vectors explained above applies also to 3-D vectors.

<u>Addition of Three Vectors:</u> To add three Vectors $\vec{A}, \vec{B}, \vec{C}$

$$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{C})+\vec{B}$$
$$=(\vec{B}+\vec{C})+\vec{A}$$

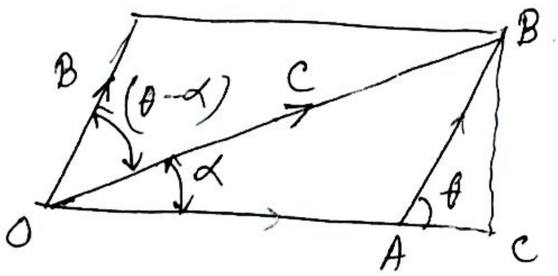


Addition of Two Vectors by Trigonometric Relation

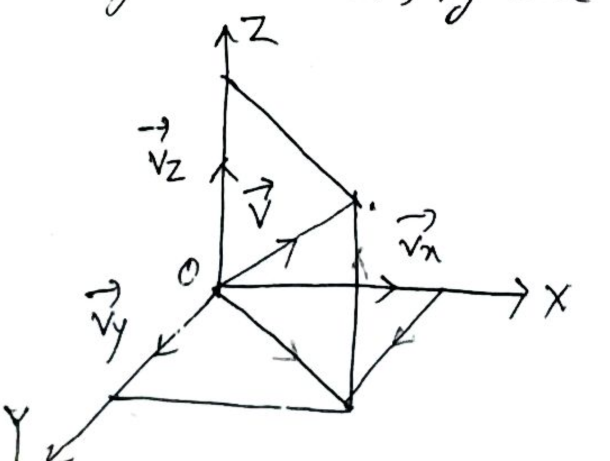
Let us consider the addition of two vectors $\vec{A}$ and $\vec{B}$ as shown in Fig, the resultant is given by $\vec{C}$. Let the angle between the two vectors be θ and ~~that~~ made be $\vec{C}$ with $\vec{A}$ be α, the from the obtuse angled tringle, Cab, the magnitude of $\vec{C}$ is given by

$$C = \sqrt{A^2 + B^2 + 2AB.\cos\theta} \quad \text{----} \quad (1)$$

$$\tan\alpha = \frac{B.\sin\theta}{(A + B.\cos\theta)} \quad \text{----} \quad (2)$$



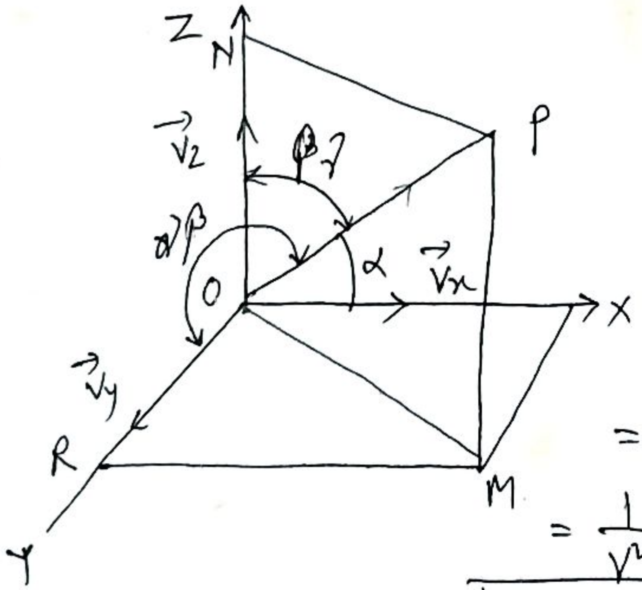
Resolution of a Vector :→ A vector $\vec{V}$ can always be resolved along three mutually $\perp^r$ direction into three components $\vec{V_1}$, $\vec{V_2}$ and $\vec{V_3}$ by constructing two parallelograms. In case of rectangular Cartesian Co-ordinates, three components are $\vec{V_x}$, $\vec{V_y}$, $\vec{V_z}$. The process of resolution of a vector $\vec{V}$ into three components $\vec{V_x}$, $\vec{V_y}$ and $\vec{V_z}$



Direction Cosines of a Vector If the angles made by a vector $\vec{V}$ with the axes of reference x, y and z be α, β, and γ respectively, the direction cosine l, m, n of the vector $\vec{V}$ are define as

$$l = \cos\alpha = \cos(\vec{V}\cdot x)$$
$$m = \cos\beta = \cos(\vec{V}, y)$$
$$n = \cos\gamma = \cos(\vec{V}, z)$$

$$\cos\alpha = \frac{V_x}{V}$$
$$\cos\beta = \frac{V_y}{V}$$
$$\cos\gamma = \frac{V_z}{V}$$



$$V^2 = V_x^2 + V_y^2 + V_z^2$$

$$l^2 + m^2 + n^2$$
$$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$
$$= \frac{1}{V^2}\left(V_x^2 + V_y^2 + V_z^2\right) = \frac{V^2}{V^2} = 1$$

Thus $\boxed{l^2 + m^2 + n^2 = 1}$

In particular, if the vector lies on a plane, say xy-plane, then $\beta = (90^\circ - \alpha)$ and $\gamma = 90^\circ$.

Hence $V_x = V\cos\alpha$

$V_y = V\cos\beta = V\cos(90^\circ - \alpha) = V\sin\alpha$.

$V_z = V\cos 90^\circ = 0$.

Hence $l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2(90^\circ - \alpha) + \cos^2 90^\circ$

$= \cos^2\alpha + \sin^2\alpha = 1$.

$$\boxed{l^2 + m^2 + n^2 = 1}$$

Components of Vectors

If a vector $\vec{V}$ is resolved into three Components $\vec{V_x}$, $\vec{V_y}$, $\vec{V_z}$ along the three Coordinates axes, in Cartessian reference, then it is known that

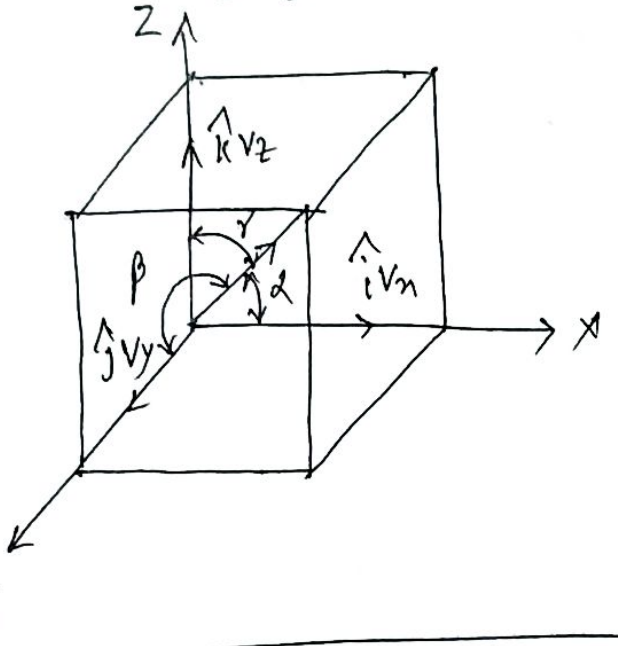
$$\vec{V} = \vec{V_x} + \vec{V_y} + \vec{V_z}$$

we know $\vec{V_x} = \hat{i} V_x$

$\vec{V_y} = \hat{j} V_y$

$\vec{V_z} = \hat{k} V_z$

$$\therefore \vec{V} = \hat{i} V_x + \hat{j} V_y + \hat{k} V_z$$



Unit vector in the Component form

The expression for the vector $\vec{V}$ in Component form with unit Coordinates vectors is given by

$$\vec{V} = \vec{V_x} + \vec{V_y} + \vec{V_z}$$

The unit vector $\hat{e}_V$ for the vector $\vec{V}$ is given by

$$\hat{e}_V = \frac{\vec{V}}{V} = \frac{1}{V}\left(\hat{i} V_x + \hat{j} V_y + \hat{k} V_z\right)$$

$$= \hat{i}\frac{V_x}{V} + \hat{j}\frac{V_y}{V} + \hat{k}\frac{V_z}{V}$$

$$\hat{e}_V = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$$

$$\boxed{\hat{e}_v = l\hat{i} + m\hat{j} + n\hat{k}}$$

Vector Algebra in Components Form

1) **Addition:** To add two vectors

$$\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z, \quad \vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z$$

$$\vec{A} + \vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) + (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z)$$

$$= \hat{i}(A_x + B_x) + \hat{j}(A_y + B_y) + \hat{k}(A_z + B_z)$$

$$= \hat{i}C_x + \hat{j}C_y + \hat{k}C_z = \vec{C}$$

$$\vec{A} + \vec{B} = \hat{i}C_x + \hat{j}C_y + \hat{k}C_z$$

The result of addition, where

$$\left.\begin{aligned} C_x &= A_x + B_x \\ C_y &= A_y + B_y \\ C_z &= A_z + B_z \end{aligned}\right\}$$

ii) **Subtraction:** → If a vector $\vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z$ is subtracted from another vector $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$.

$$\vec{A} - \vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) - (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z)$$

$$= \hat{i}(A_x - B_x) + \hat{j}(A_y - B_y) + \hat{k}(A_z - B_z)$$

$$\vec{A} - \vec{B} = \hat{i}C_x + \hat{j}C_y + \hat{k}C_z$$

the result of Subtraction

$$\left.\begin{aligned} C_x &= A_x - B_x \\ C_y &= A_y - B_y \\ C_z &= A_z - B_z \end{aligned}\right\}$$

iii) $\boxed{\text{Multiplication of Vector by a Scalar}}$

If a vector $\vec{A}$ is multiplied by a Scalar n, the result is another Vector $\vec{B}$

i.e $\vec{B} = n\vec{A}$

$\Rightarrow \hat{i}B_x + \hat{j}B_y + \hat{k}B_z = n(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$

$\Rightarrow B_x = nA_x$, $B_y = nA_y$, and $B_z = nA_z$

[★ When $n > 0$
In this Case the direction and Sense of the vector remain unaltered.

If $\boxed{n > 1}$, the magnitude of the vector is magnified n times.

If $\boxed{0 < n < 1}$ The magnitude of the vector is decreased n times.

If ~~nα~~ $n = 1$. The magnitude of vector remain unaltered.

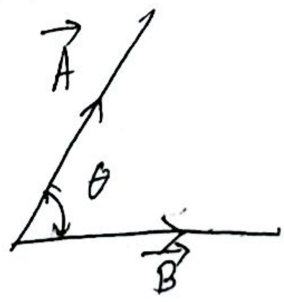
When $n = 0$; In this case the vector becomes a null vector.

When $n < 0$. In this case the direction of the vector remains unaltered but the vector becomes oppositely sensed.

$\boxed{\text{Product of two Vectors}}$ There are two different product of two vectors.

i) Scalar or Dot product.
ii) Vector or Cross product.

i) Scalar or Dot product :→ if the two vectors are $\vec{A}$ and $\vec{B}$ and the angle included between them is θ.



$$\therefore \vec{A}\cdot\vec{B} = A.B\cos\theta.$$

According to definition
$$\vec{e}_a \cdot \vec{e}_b = \cos(\vec{e}_a, \vec{e}_b).$$

In particular, for unit co-ordinate vectors
$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = \cos 0^\circ = 1.$$

and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \cos 90^\circ = 0.$

Two vectors $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$
$\vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z.$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

$$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2 = A^2$$

$$\boxed{\vec{A}\cdot\vec{A} = A^2}$$

$$\boxed{A = \sqrt{\vec{A} \cdot \vec{A}}}$$

Also
$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{A_x B_x + A_y \cdot B_y + A_z B_z}{AB}$$

$$= \frac{A_x B_x}{AB} + \frac{A_y B_y}{AB} + \frac{A_z B_z}{AB}$$

$$= \cos\alpha_A \cos\alpha_B + \cos\beta_A \cos\beta_B + \cos\gamma_A \cos\gamma_B .$$

$$\boxed{\cos\theta = l_A l_B + m_A m_B + n_A n_B}$$

l_A, m_A, n_A and l_B, m_B, n_B are the direction Cosine of the two vectors A and B.

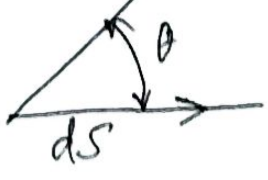
From the definition of Dot Product the relation Can be easily proved

i) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

ii) $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

iii) $m\vec{A} \cdot n\vec{B} = mn(\vec{A} \cdot \vec{B})$.

<u>Work done</u> - Force 'F' acting on a Particle Produces a displacement 'ds'
The elementary work done

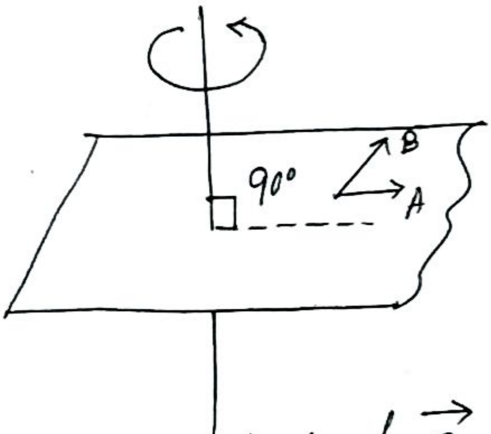


$$dw = F\cos\theta \cdot ds$$
$$= \vec{F} \cdot d\vec{s}$$
$$W = \int \vec{F} \cdot d\vec{s}$$

Therefore work done is a Scalar quantity.

11) **Vector Product or Cross Product of two Vectors**

The vector product or cross product of two vectors $\vec{A}$ and $\vec{B}$ generates another vector $\vec{C}$ i,e $\vec{A} \times \vec{B} = \vec{C}$, such that:



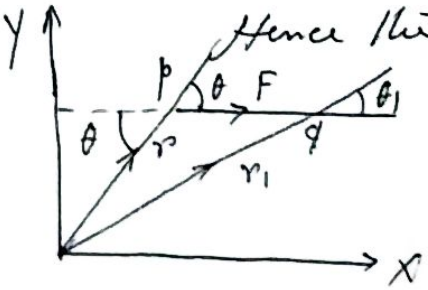
(a) The magnitude of $\vec{C}$ is given by ~~ABCosθ~~ $AB\sin\theta$

(b) The direction of $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$

(c) The sense of $\vec{C}$ is determined by the well known right hand cork screw rule when rotation of $\vec{C}$ is considered through the smaller angle θ from $\vec{A}$ to $\vec{B}$

As an illustration of cross product, the moment of a force $\vec{F}$ about a point O as shown in Fig. Let us consider any point p on the line of action of the force and hence the line segment op is represented by the vector $\vec{r}$. Magnitude of moment $\vec{M}$

$$\vec{M} = F r \sin\theta = r F \sin\theta.$$



Hence the moment vector

$$\boxed{\vec{M} = \vec{r} \times \vec{F}}$$

Since the position of the point p is not fixed, It could be anywhere on the line of action of the force. If we consider another point Q on the line of action of the force, then

$$\vec{M} = \vec{r}_1 \times \vec{F}$$

Thus $\vec{M} = \vec{r} \times \vec{F} = \vec{r}_1 \times \vec{F} = \vec{r}_2 \times \vec{F}$.

From the definition of cross product

(i) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ but $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

ii) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

i.e vector product is distributive

iii) $\vec{a} \times \vec{a} = a^2 \sin 0 = 0$.

iv) For unit vector

$\cancel{i \times j = j \times k = k}$

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$.

$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i} \quad \hat{i} \times \hat{k} = -\hat{j}$

Vector Product of two Vectors when the vectors are expressed in Component Form

Let two vectors $\vec{a} = \hat{i} a_x + \hat{j} a_y + \hat{k} a_z$

and $\vec{b} = \hat{i} b_x + \hat{j} b_y + \hat{k} b_z$

Then $\vec{a} \times \vec{b} = (\hat{i} a_x + \hat{j} a_y + \hat{k} a_z) \times (\hat{i} b_x + \hat{j} b_y + \hat{k} b_z)$

$$\vec{a} \times \vec{b} = \hat{i} \times \hat{i}\, a_x b_x + \hat{i} \times \hat{j}\, a_x b_y + \hat{i} \times \hat{k}\, a_x b_z$$
$$+ \hat{j} \times \hat{i}\, a_y b_x + \hat{j} \times \hat{j}\, a_y b_y + \hat{j} \times \hat{k}\, a_y b_z$$
$$+ \hat{k} \times \hat{i}\, a_z b_x + \hat{k} \times \hat{j}\, a_z b_y + \hat{k} \times \hat{k}\, a_z b_z$$

$$= \hat{k}\, a_x b_y - \hat{j}\, a_x b_z - \hat{k}\, a_y b_x + \hat{i}\, a_y b_z + \hat{j}\, a_z b_x - \hat{i}\, a_z b_y.$$

$$= \hat{i}(a_y b_z - a_z b_y) + \hat{j}(a_z b_x - a_x b_z) + \hat{k}(a_x b_y - a_y b_x)$$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

if $\vec{a} \times \vec{b} = 0$ either the vectors are parallel or they have same line of action

Product of three vectors

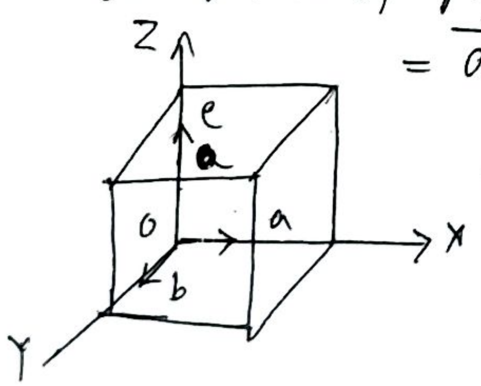
If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the scalar product of $\vec{a}$ and $\vec{b} \times \vec{c}$ is a scalar and it is called scalar product of $\vec{a}, \vec{b}, \vec{c}$

It is denoted by $\vec{a} \cdot (\vec{b} \times \vec{c})$ or $[\vec{a}\, \vec{b}\, \vec{c}]$

The volume of Parallelopiped

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) = b \cdot (\vec{c} \times \vec{a}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\Rightarrow [\vec{a}\, \vec{b}\, \vec{c}] = [\vec{b}\, \vec{c}\, \vec{a}] = [\vec{c}\, \vec{a}\, \vec{b}]$$



$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{a}\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

similarly $(\vec{a}\times\vec{b})\cdot c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\therefore \boxed{\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot c.}$$

Condition of Co-planarity of three vectors

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ or $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

Differentiation of a Vector with respect to a Scalar

If $\vec{a}(t)$ is a vector function, where t is a Scalar, say time, then derivative of $\vec{a}(t)$ is defined as

i) $\frac{d}{dt}\{\vec{a}(t)\} = \lim_{\Delta t \to 0} \frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t}$ - - (1)

ii) $\frac{d}{dt}(\vec{a}+\vec{b}) = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}$ (iii) $\frac{d}{dt}(\vec{a}.\vec{b}) = \vec{a}.\frac{d\vec{b}}{dt} + \vec{b}.\frac{d\vec{a}}{dt}$

iv) $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$

v) $\frac{d}{dt}\{c\vec{a}(t)\} = c\frac{d\vec{a}}{dt}$ (vi) $\frac{d}{dt}\{f(t)\vec{c}\} = \frac{df(t)}{dt}\vec{c}$.

where c = vector constant.

vii) $\frac{d}{dt}\vec{a}(t) = \frac{d}{dt}(a_x\hat{i} + a_y\hat{j} + a_z\hat{k})$

$$= \left(\frac{da_x}{dt}\hat{i} + \frac{da_y}{dt}\hat{j} + \frac{da_z}{dt}\hat{k}\right)$$

viii) $\frac{d}{dt}\vec{a}(s) = \frac{d\vec{a}(s)}{ds}.\frac{ds}{dt}$

(ix) $\frac{d}{dt}(\vec{e}.\vec{e}) = \vec{e}\frac{d\vec{e}}{dt} + \vec{e}\frac{d\vec{e}}{dt} = 2\vec{e}\frac{d\vec{e}}{dt}$

where $\vec{e}$ is a unit vector

$\vec{e}.\vec{e} = 1$. $\frac{d}{dt}(\vec{e}.\vec{e}) = 0$.

Integration of a vector with respect to a Scalar

Given a vector function $\vec{a}(t)$, if there exists another vector function $\vec{b}(t)$ such that

$$\frac{d}{dt}\vec{b}(t) = \vec{a}(t)$$

$$\int \vec{a}(t)\,dt = \vec{b}(t)$$

According to the definition, we arrive at the following results.

i) $\int (\vec{a} + \vec{b})\,dt = \int \vec{a}\,dt + \int \vec{b}\,dt$

ii) $\int c\,\vec{a}(t)\,dt = c\int \vec{a}(t)\,dt$

iii) $\int \vec{c}\cdot F(t)\,dt = \vec{c}\int F(t)\,dt$

iv) $\int \vec{a}\cdot\vec{b}(t)\,dt = \vec{a}\cdot\int \vec{b}(t)\,dt$

v) $\int \vec{a}\times\vec{b}(t)\,dt = \vec{a}\times\int \vec{b}(t)\,dt$

vi) $\int \vec{a}\,dt = \int (a_x\hat{i} + a_y\hat{j} + a_z\hat{k})\,dt$

$$= \hat{i}\int a_x\,dt + \hat{j}\int a_y\,dt + \hat{k}\int a_z\,dt.$$

Important Vector Quantities

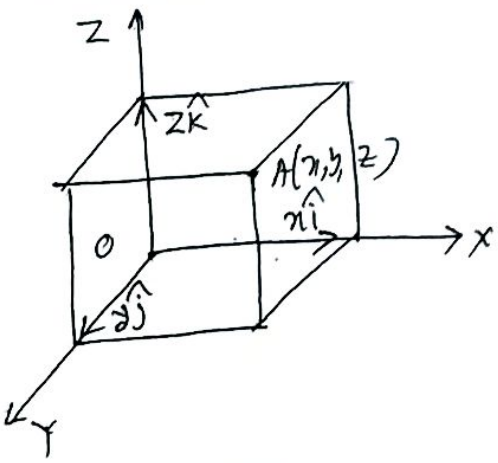
1) Position vector 2) Moment of a force about a point.

3) Moment of a Force about a line.

4) Varignon's Theorem. 5) Moment of a Couple.

1) **Position Vector** The location of a point $A(x, y, z)$ in a certain reference frame is designated by a vector $\vec{r}$, starting at the origine and terminating at the point A as shown in Fig.

Thus the position vector of the point $A(x, y, z)$ is given by $\boxed{\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}}$



2) **Moment of a Force about a point**

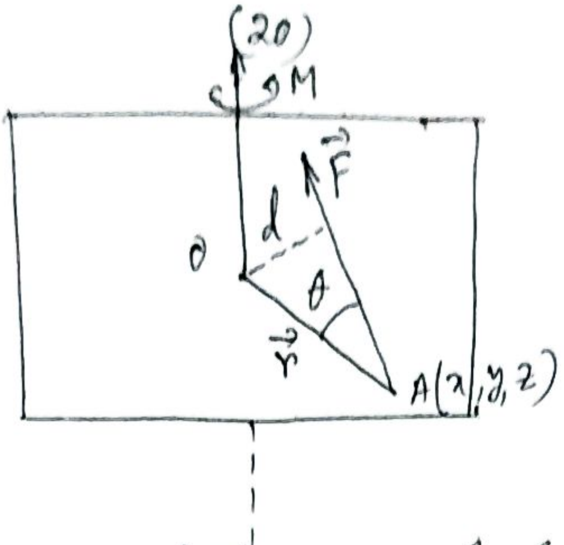
Let us consider a force F in space whose point of application is at A. The position vector of A is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

The moment of the Force is given by $\boxed{M = \vec{r} \times \vec{F}}$

$$M = Fr\sin\theta = rF\sin\theta.$$



The direction of $\vec{M}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ and the sense of is obtained by the right-hand ~~clock~~ crok screw rule.

Now $\vec{M} = M_x\hat{i} + M_y\hat{j} + M_z\hat{k} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

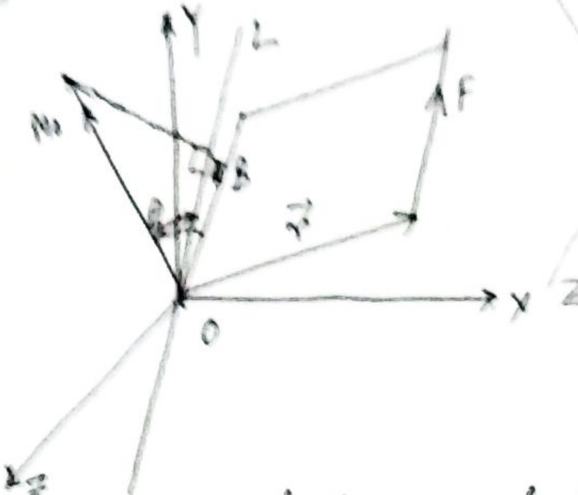
$$\Rightarrow \vec{M} = M_x\hat{i} + M_y\hat{j} + M_z\hat{k} = (yF_z - zF_y)\hat{i} + (zF_x - xF_z)\hat{j} + (xF_y - yF_x)\hat{k}$$

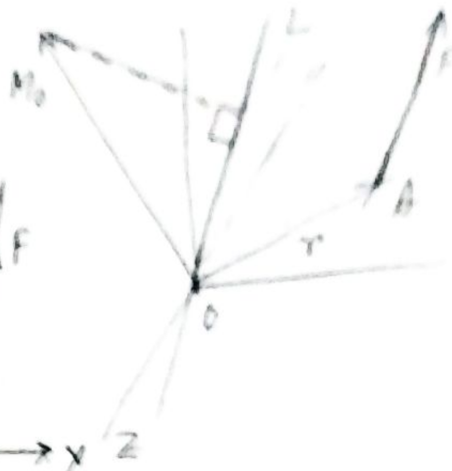
Thus the Component of $\vec{M}$ along three axes X, Y, and Z are

$$\boxed{\begin{aligned} M_x &= (yF_z - zF_y) \\ M_y &= (zF_x - xF_z) \\ M_z &= (xF_y - yF_x) \end{aligned}}$$

(3) **Moment of a Force about a line** Let us consider the force $\vec{F}$ in space shown in Fig. We shall consider the moment of the force about the line 'OL'. The position vector of the point of application of the force is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Let the moment of the force $\vec{F}$ about O be $\vec{M}_o$, such that $\vec{M}_o = \vec{r} \times \vec{F}$

and its magnitude is M_0





Now the magnitude of the moment of the force $\vec{F}$ about the line OL is given by [$\because \vec{M}_0 = M_0 \cos\theta$ & $\vec{M}_0 = \vec{r} \times \vec{F}$]

$$M_{OL} = \vec{M}_0 \cos\theta$$
$$= |\vec{e}\, M_0 \cos\theta|$$

where $\vec{e}$ is the unit vector along the line OL

then $M_{OL} = \vec{e}\cdot\vec{M}_0 = \vec{e}\cdot(\vec{r}\times\vec{F})$

$$= (e_x\hat{i} + e_y\hat{j} + e_z\hat{k})\cdot\begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{M}_{OL} = \begin{vmatrix} e_x & e_y & e_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

In particular, if the line OL ~~Consider~~ coincides with the axis of X, then $e_x = 1$, $e_y = e_z = 0$. and $M_{OL} = M_x = (yF_z - zF_y)$

But these are the Components of the moment about O on the three axes, thus

$$\vec{M}_0 = \vec{M}_{ox} + \vec{M}_{oy} + \vec{M}_{oz}$$

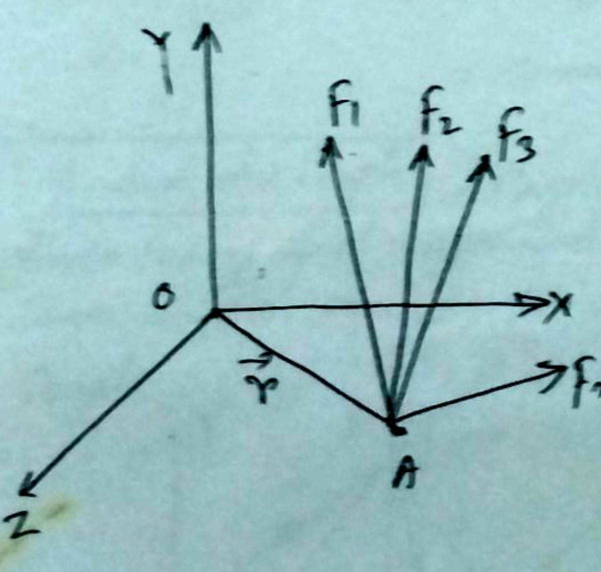
and $\boxed{\vec{M}_{ox} = (\vec{M}_0\cdot i)\, i}$

(4) $\boxed{\text{Varignon's Theorem}}$

The net moment of a system of forces about a point is equal to the moment of the resultant of the system about the same point.

$\boxed{\text{Concurrent Forces}}$
Assume the a system of force $\vec{F_1}, \vec{F_2}, \vec{F_3} \cdots \vec{F_n}$ are concurrent at the point A, whose resultant is R_R such that

$$\vec{F_1} + \vec{F_2} + \vec{F_3} + \cdots + \vec{F_n} = \vec{R_R}$$



position vector of A is $\vec{r}$

$$\vec{M_0} = \left(\vec{r} \times \vec{F_1} + \vec{r} \times \vec{F_2} + \vec{r} \times \vec{F_3} + \cdots + \vec{r} \times \vec{F_n}\right)$$

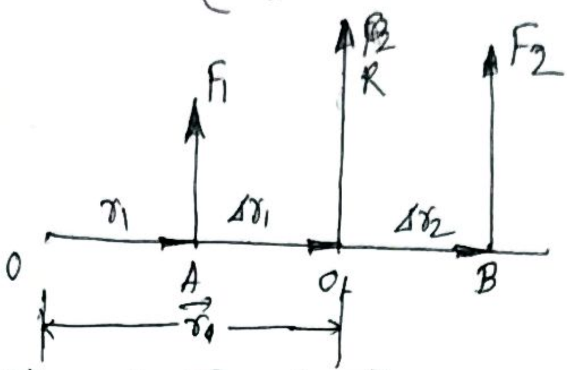
$$\vec{M_0} = \vec{r} \times \vec{R_R}$$

= Moment of the resultant.

$\boxed{\text{Parallel Forces}}$ For simplicity let us consider two like parallel forces $\vec{F_1}$ acting at A and $\vec{F_2}$ acting at B having a resultant $\vec{R}$ acting at O_1 as shown in fig.

Let us assume O, the origine of the reference to be moment centre.

Let $\vec{OA} = \vec{r_1}$, $\vec{OB} = \vec{r_2}$ and $O_1B = \vec{\delta r_2}$ and $\vec{OO_1} = \vec{r}$



Hence $\vec{M_0} = \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2}$

$$= (\vec{r} - \Delta\vec{r_1}) \times \vec{F_1} + (\vec{r} + \Delta\vec{r_2}) \times \vec{F_2}$$

$$= \vec{r} \times \vec{F_1} + \vec{r} \times \vec{F_2} + \Delta\vec{r_2} \times \vec{F_2} - \Delta\vec{r_1} \times \vec{F_1}$$

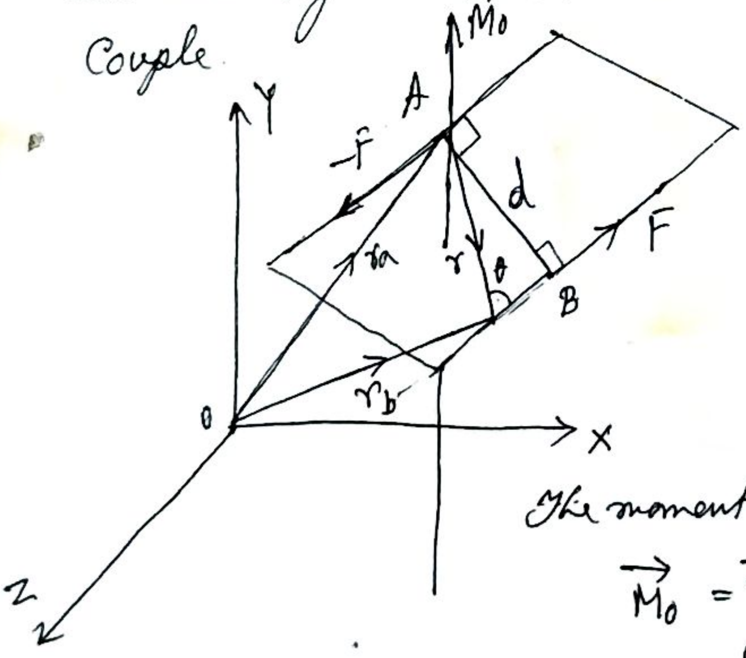
$$\vec{M_0} = \vec{r} \times (\vec{F_1} + \vec{F_2}) + (\Delta\vec{r_2} \times \vec{F_2} - \Delta\vec{r_1} \times \vec{F_1})$$

Since $\Delta\vec{r_2} \times \vec{F_2} = \Delta\vec{r_1} \times \vec{F_1}$

$$\therefore \vec{M_0} = \vec{r} \times (\vec{F_1} \times \vec{F_2}) = \vec{r} \times \vec{R}$$

= Moment of resultant about O.

5) $\boxed{\text{Moment of a Couple}}$ A Couple is formed by two parallel forces having equal magnitude and opposite sense as Shown in Fig. where the force F and −F have formed a Couple



Let two points A and B on the line of action of $-\vec{F}$ and $\vec{F}$ whose position vectors are $\vec{r_a}$ and $\vec{r_b}$ respectively. Then

The moment of Couple

$$\vec{M_0} = \vec{r_b} \times \vec{F} - \vec{r_a} \times \vec{F}$$

$$= (\vec{r_b} - \vec{r_a}) \times \vec{F} = \vec{r} \times \vec{F}$$

$$\therefore \vec{M_0} = F r \sin\theta = Fd.$$

where d is the perpendicular distance between the lines of action of the two forces.