ME 543: Computational Fluid Dynamics

COMPUTER ASSIGNMENT - 1

Stream and Temperature problem solved using five iterative methods and study of their relative performance.

Iterative Methods:

Jacobian
Point Gauss-Seidel
Point Successive Over Relaxation Method
Line Gauss-Seidel
Alternating Direction Implicit Method

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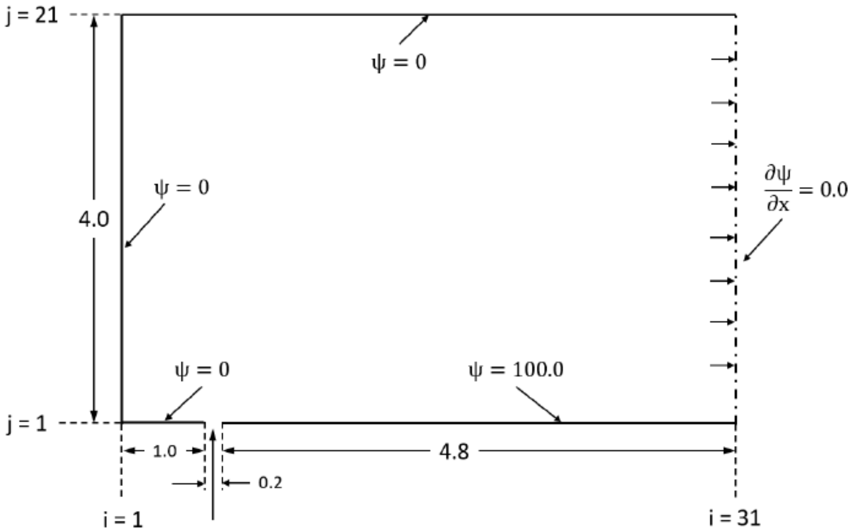
1. **First Problem: (Stream)**



Differential Equation: $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

Discretized equation:

$$\beta^2\psi_{i,j-1} + \psi_{i-1,j} - 2(1+\beta^2)\psi_{i,j} + \psi_{i+1,j} + \beta^2\psi_{i,j+1} = 0$$

where, $\beta = \frac{\Delta x}{\Delta y}$

- Coding on python using *Jupyter Notebook*, and using the results to plot the contours in *Techplot*.

- Further plotting the number of iterations to converge upto $\epsilon = 10^{-6}$ for each iterative method on a graph using python.

- Tabulating the time taken for convergence through each iteration using python.

Taking the following as the inputs for the code:
$X = 6$
$Y = 4$
$\Delta x = 0.2$
$\Delta y = 0.2$

1.1.Jacobi Iterative Method

Discretized equation:

$$\psi_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}\left[\beta^2\,\psi_{i,j-1}^{k} + \psi_{i-1,j}^{k} + \psi_{i+1,j}^{k} + \beta^2\,\psi_{i,j+1}^{k}\right]$$

Contour:

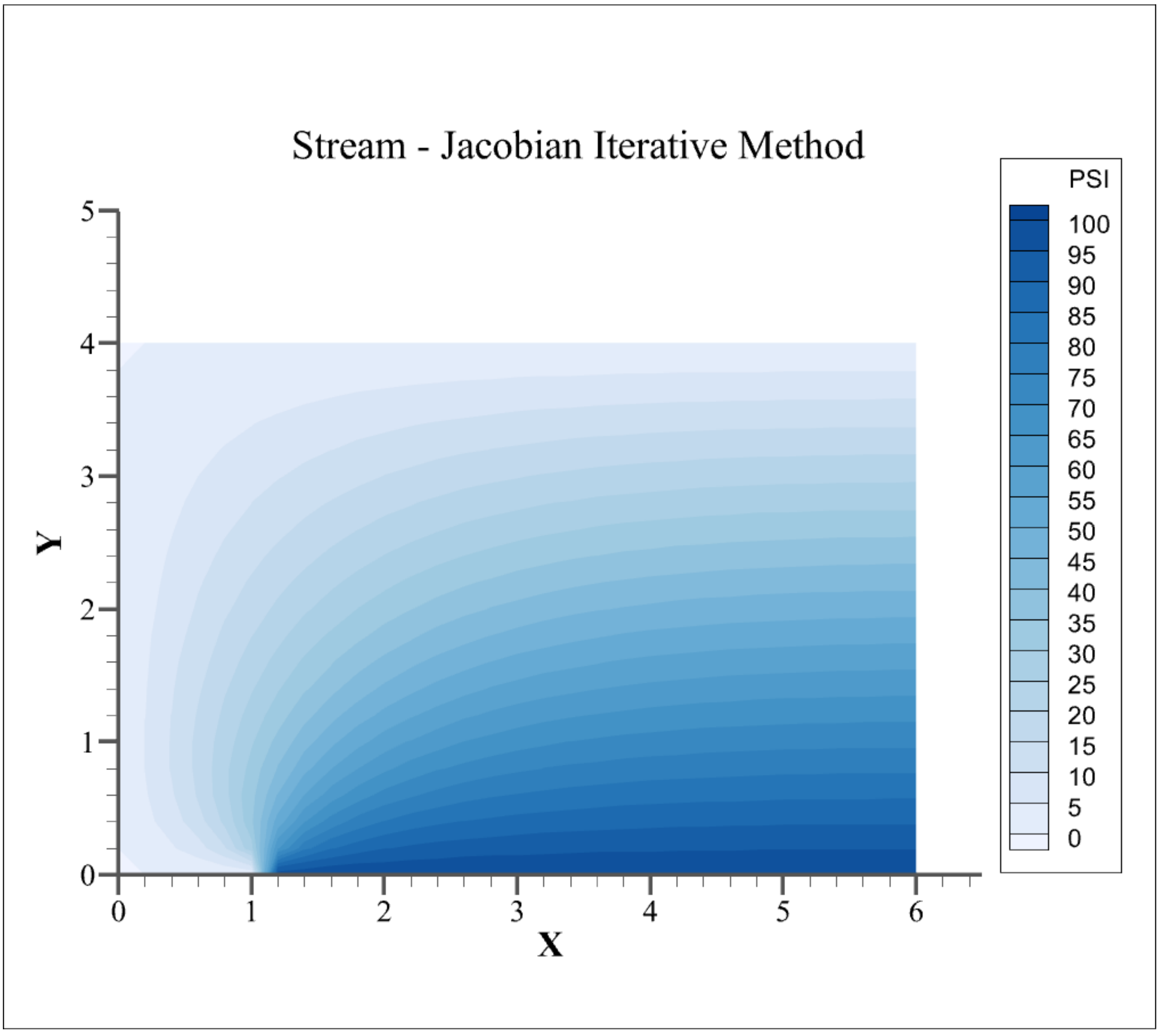


Fig 1.1: Stream contour using Jacobian Iterative method

Number of Iterations: 1808

Time taken to converge to $\epsilon < 10^{-6}$: 5,135 msec

1.2. Point Gauss-Seidel Iterative Method

Discretized equation:

$$\psi_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}\left[\beta^2\, \psi_{i,j-1}^{k+1} + \psi_{i-1,j}^{k+1} + \psi_{i+1,j}^{k} + \beta^2\, \psi_{i,j+1}^{k}\right]$$

Contour:

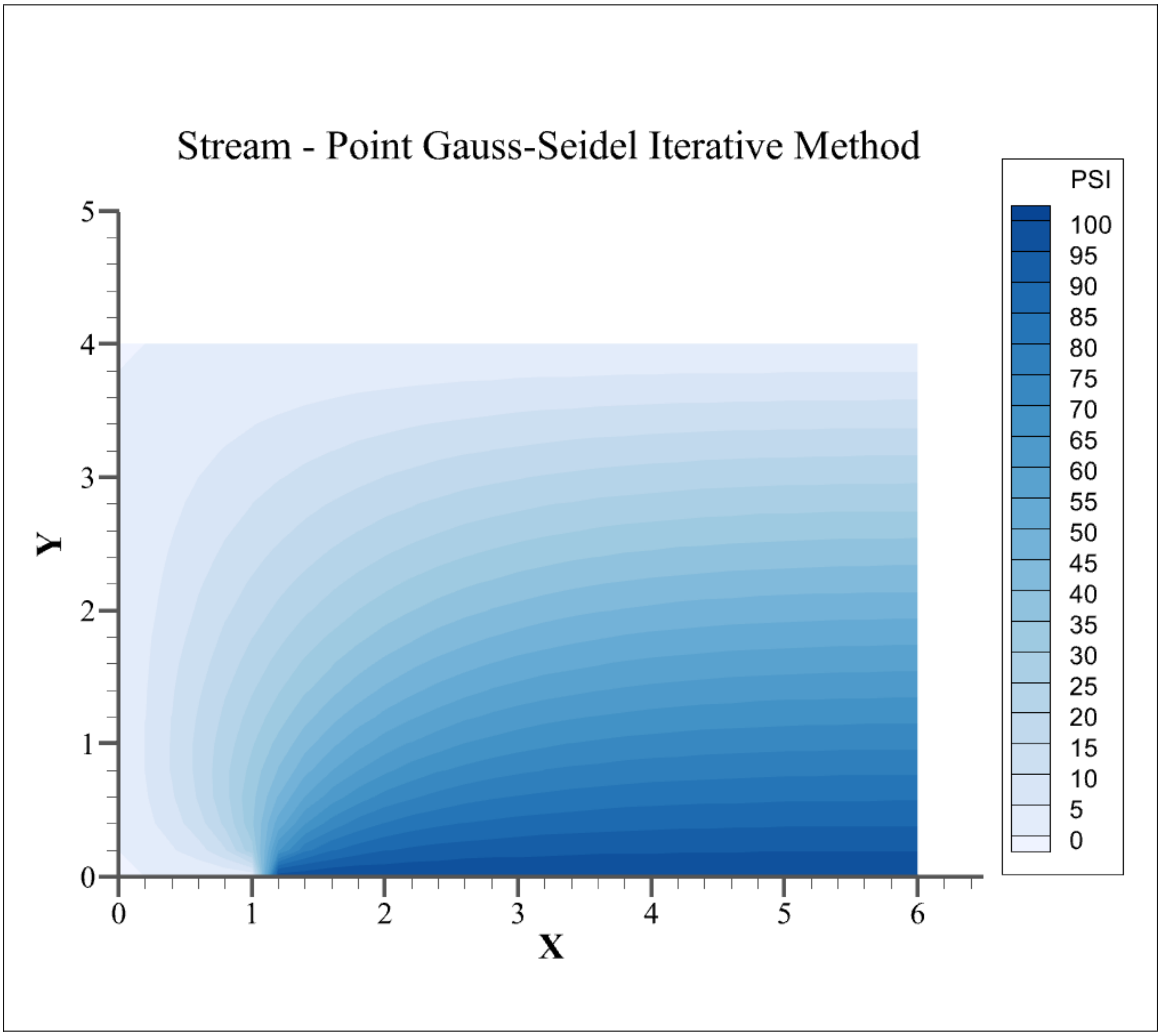


Fig 1.2: Stream contour using Point Gauss-Seidel Iterative method

Number of Iterations: 965

Time taken to converge to $\epsilon < 10^{-6}$: 2,913 msec

1.3. Point Successive Over Relaxation (PSOR) Iterative Method

Discretized equation:

$$\psi_{i,j}^{k+1} = (1-\omega)\psi_{i,j}^{k} + \frac{\omega}{2(1+\beta^2)}\left[\beta^2\,\psi_{i,j-1}^{k+1} + \psi_{i-1,j}^{k+1} + \psi_{i+1,j}^{k} + \beta^2\,\psi_{i,j+1}^{k}\right]$$

$\omega = 1.85$ taken

Contour:

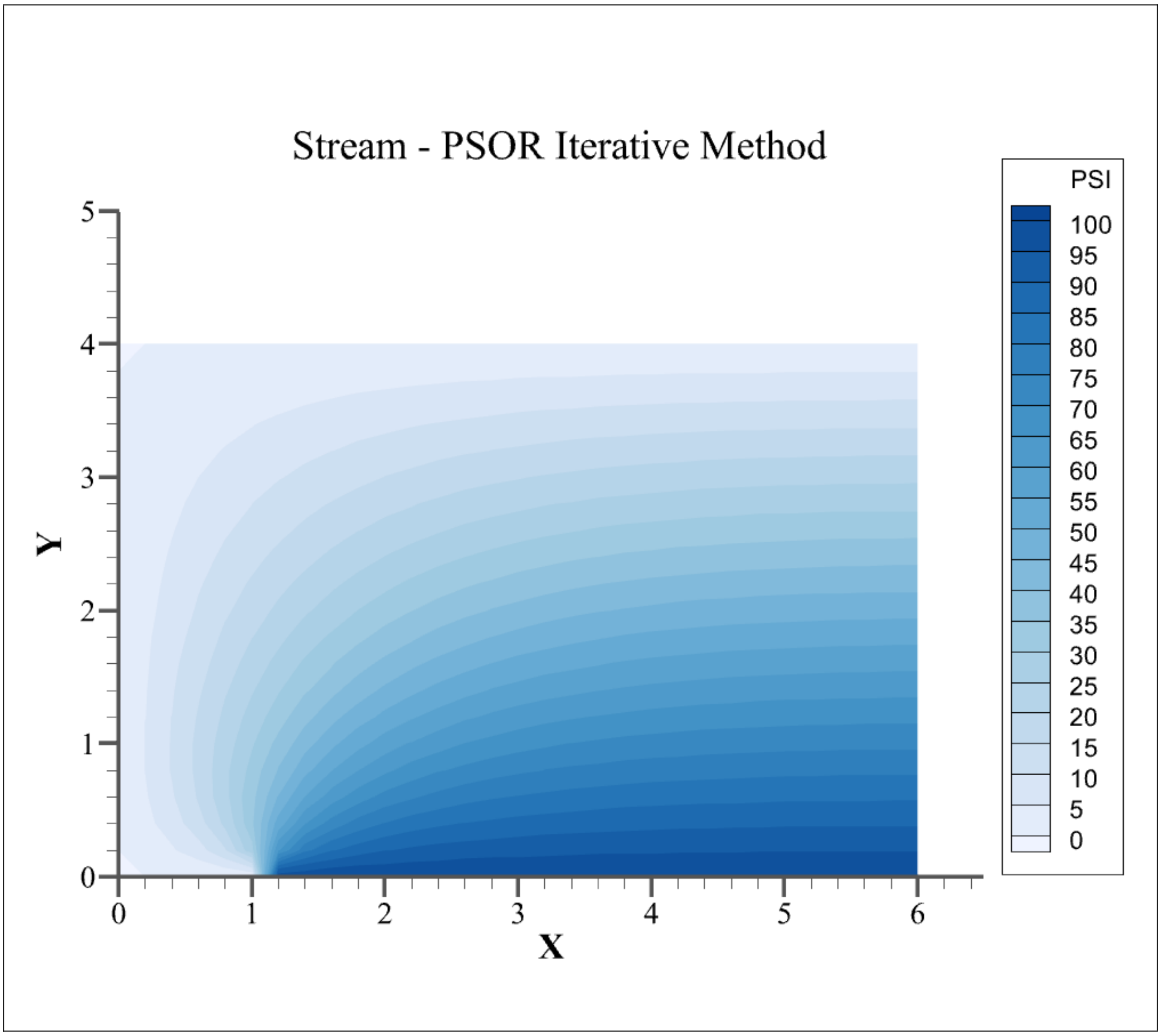


Fig 1.3: Stream contour using Point Successive Over Relaxation method

Number of Iterations: 81

Time taken to converge to $\epsilon < 10^{-6}$: 335 msec

1.4. Line Gauss-Seidel Iterative Method

Discretized equation: *x-sweep*

$$\psi_{i-1,j}^{k+1} - 2(1+\beta^2)\psi_{i,j}^{k+1} + \psi_{i+1,j}^{k+1} = -\beta^2(\psi_{i,j-1}^{k+1} + \psi_{i,j+1}^{k})$$

Contour:

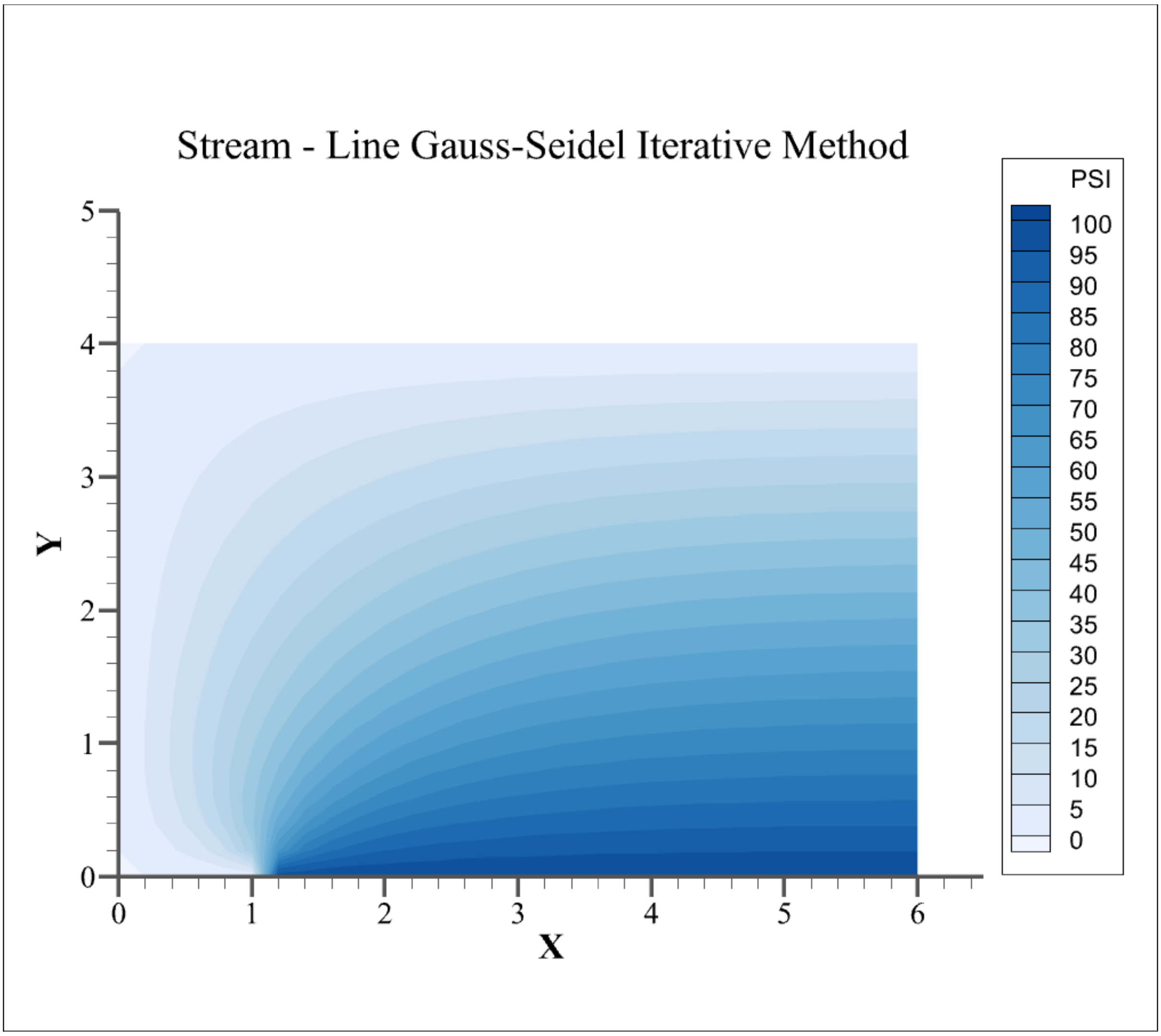


Fig 1.4: Stream contour using Line Gauss-Seidel Iterative method

Number of Iterations: 550

Time taken to converge to $\epsilon < 10^{-6}$: 2,932 msec

1.5. Alternating Direction Implicit (ADI) Method

Discretized equation:

x-sweep

$$\psi_{i-1,j}^{k+\frac{1}{2}} - 2(1+\beta^2)\psi_{i,j}^{k+\frac{1}{2}} + \psi_{i+1,j}^{k+\frac{1}{2}} = -\beta^2(\psi_{i,j-1}^{k+\frac{1}{2}} + \psi_{i,j+1}^{k})$$

y-sweep

$$\beta^2\psi_{i-1,j}^{k+1} - 2(1+\beta^2)\psi_{i,j}^{k+1} + \beta^2\psi_{i+1,j}^{k+1} = -(\psi_{i,j-1}^{k+1} + \psi_{i,j+1}^{k+\frac{1}{2}})$$

Contour:

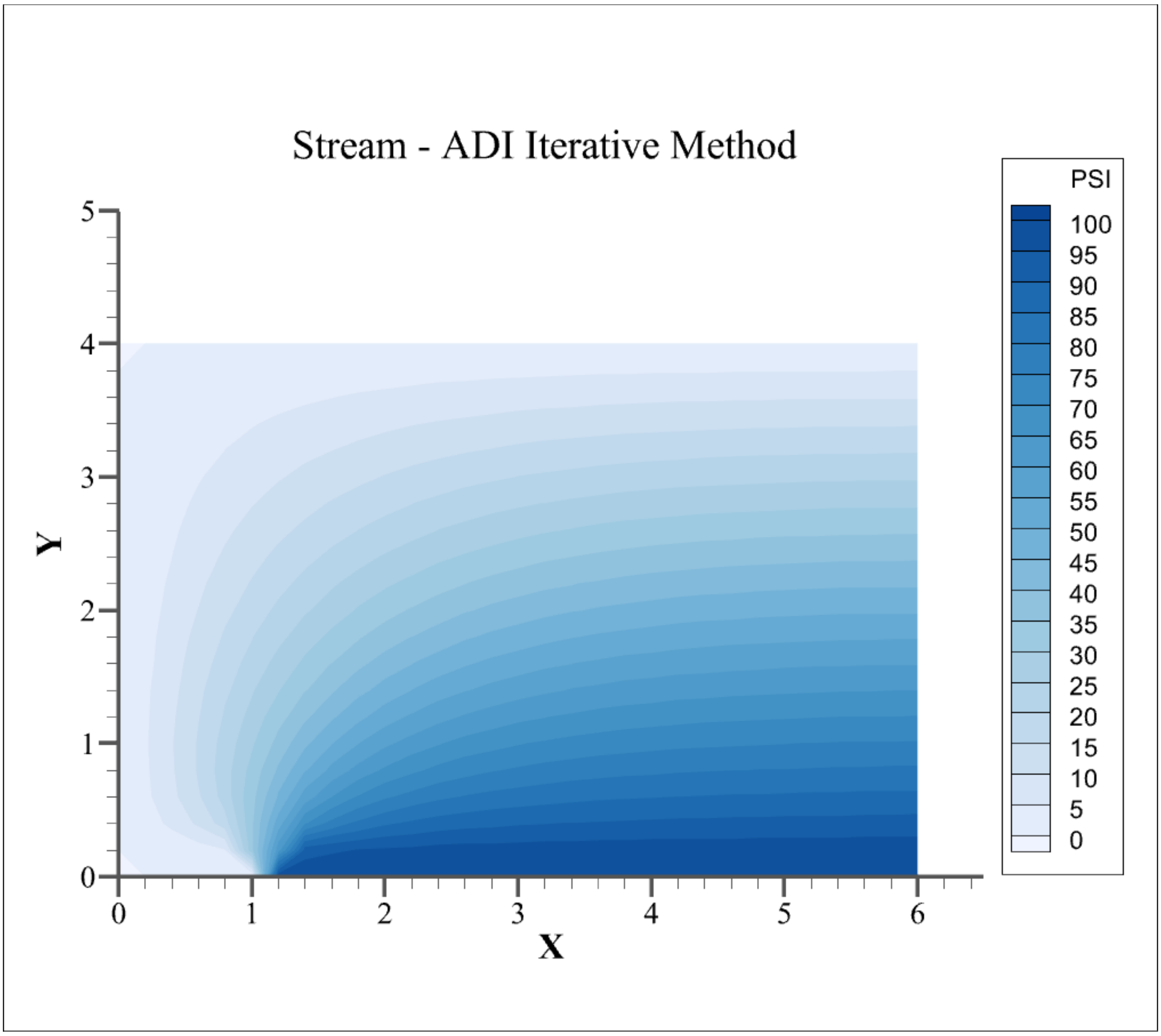


Fig 1.5: Stream Contour using Alternating Direction Implicit (ADI) method

Number of Iterations: 274

Time taken to converge to $\epsilon < 10^{-6}$: 2,261 msec

Plotting ω vs no. of iterations in PSOR method

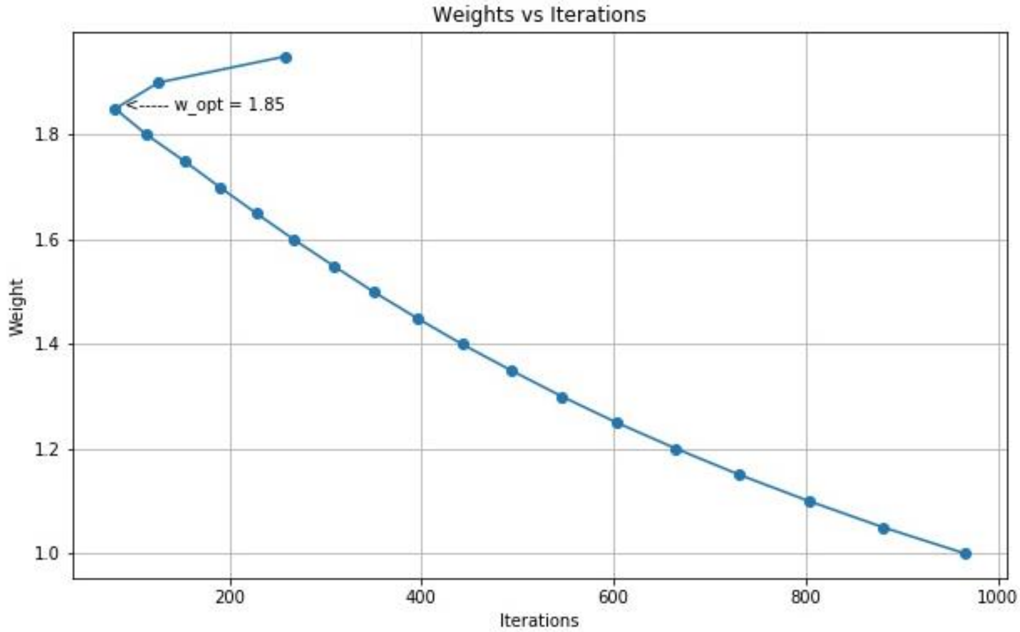


Fig 1.6: Weight vs Iterations for PSOR Method

$$\omega_{opt} = \frac{2\left(1-\sqrt{1-a}\right)}{a}, \quad where: a = \frac{\cos\left(\frac{\pi}{M-1}\right) + \beta^2 \cos\left(\frac{\pi}{N-1}\right)}{1+\beta^2}$$

$$\omega_{opt} = 1.828\ (from\ formula)$$

$$\omega_{opt} = 1.850\ (from\ graph)$$

Tabulating time taken for each Iterative Method

Table 1.1: Number of Iterations, Time taken and Time per iteration for each Iterative method

Method	Iterations	Time (ms)	Time/iter (ms)
Jacobian	1808	5134.99	2.84
Point Gauss-Seidel	965	2913.48	3.02
PSOR	81	335.16	4.14
Line Gauss-Seidel	550	2931.84	5.33
ADI	274	2260.92	8.25

Plotting log(error) vs log(iterations) for all the methods

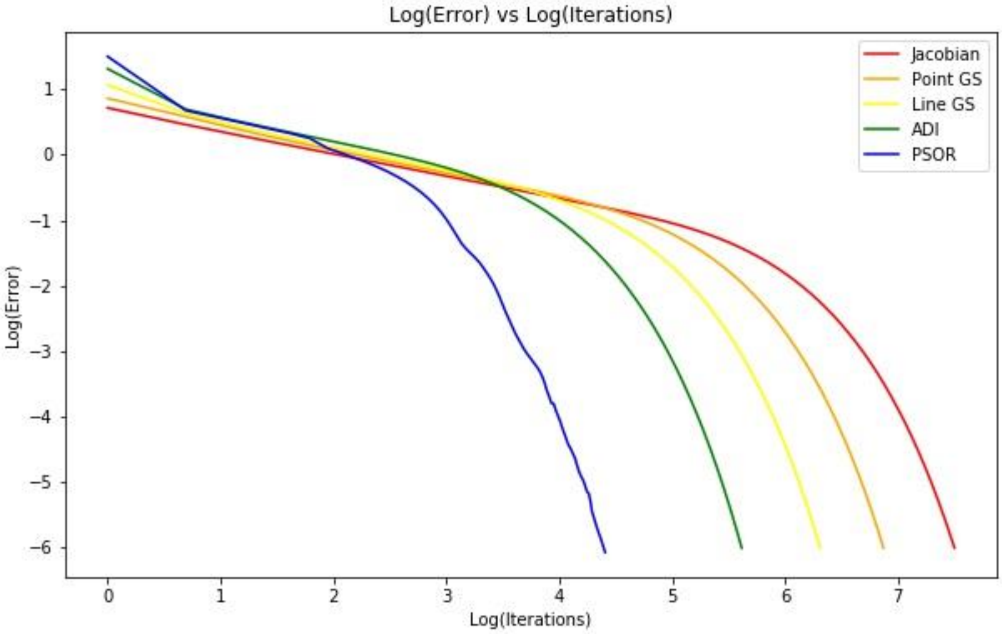
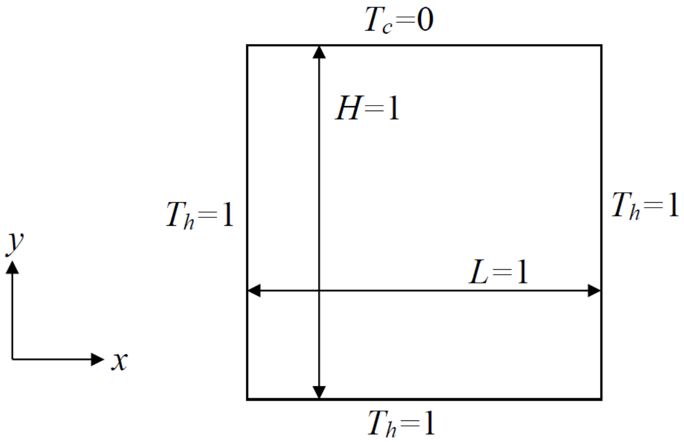


Fig 1.7: Log (error) vs Log (iterations) for all the methods

2. **Second Problem: (Temperature)**



Differential Equation: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

Discretized equation:

$$\beta^2 T_{i,j-1} + T_{i-1,j} - 2(1+\beta^2)T_{i,j} + T_{i+1,j} + \beta^2 T_{i,j+1} = 0$$

where, $\beta = \frac{\Delta x}{\Delta y}$

- Coding on python using *Jupyter Notebook*, and using the results to plot the contours in *Techplot*.

- Further plotting the number of iterations to converge upto $\epsilon = 10^{-6}$ for each iterative method on a graph using python.

- Tabulating the time taken for convergence through each iteration using python.

Taking the following as the inputs for the code:

$X = 1$

$Y = 1$

$\Delta x = 0.05$

$\Delta y = 0.05$

The Exact Solution from the equation given.

Equation: $T(x,y) = T_c + (T_h - T_c)\left[1 - 2\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n\pi}\frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}\sin\left(\frac{n\pi x}{L}\right)\right)\right]$

In the code, we consider $n = 1 \rightarrow 100$ as the values beyond that would give a negligible addition to the final value.

Contour:

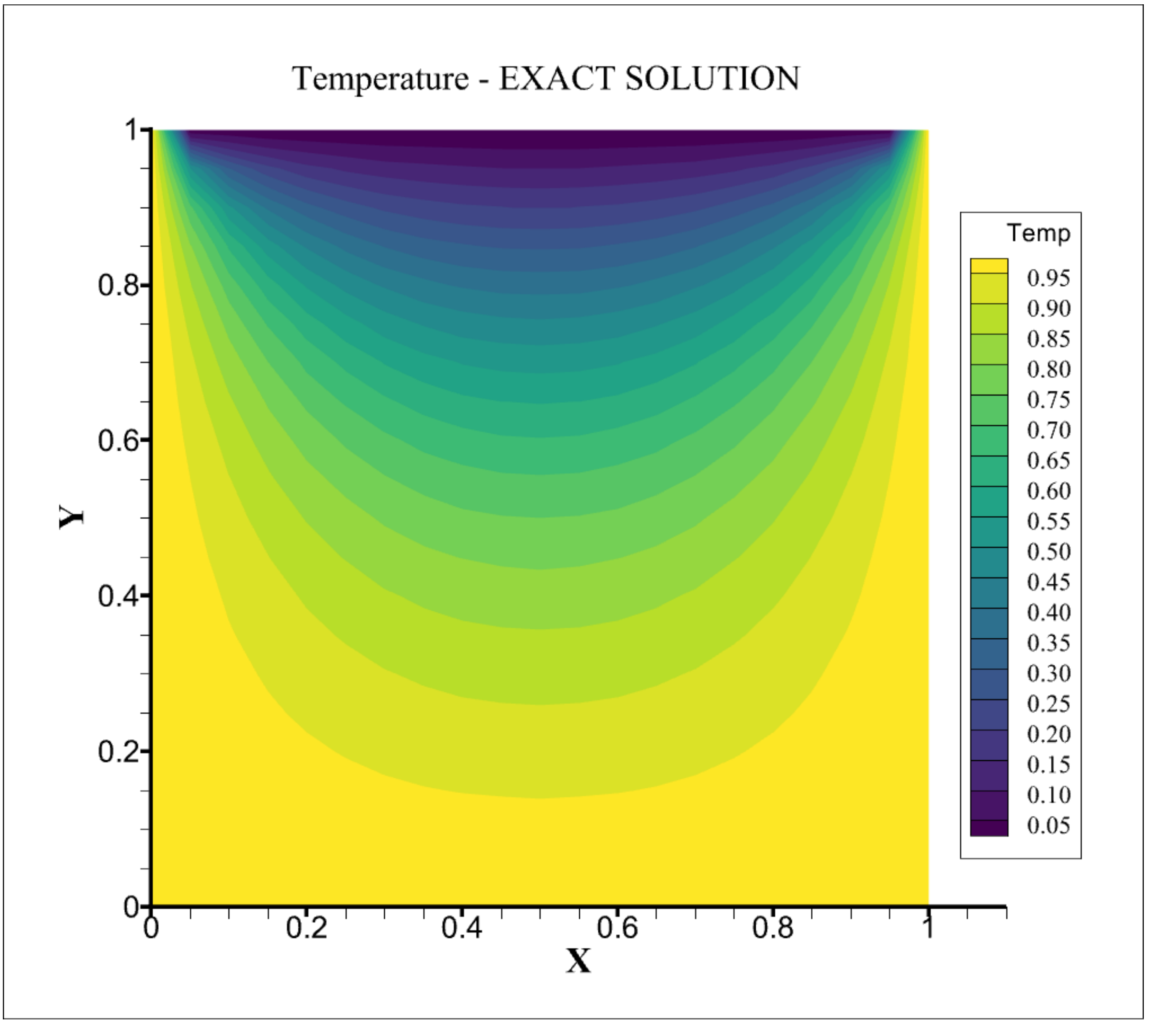


Fig 2.0: Temperature contour using formula for Exact Solution

2.1. Jacobi Iterative Method

Discretized equation:

$$T_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}\left[\beta^2\, T_{i,j-1}^{k} + T_{i-1,j}^{k} + T_{i+1,j}^{k} + \beta^2\, T_{i,j+1}^{k}\right]$$

Contour:

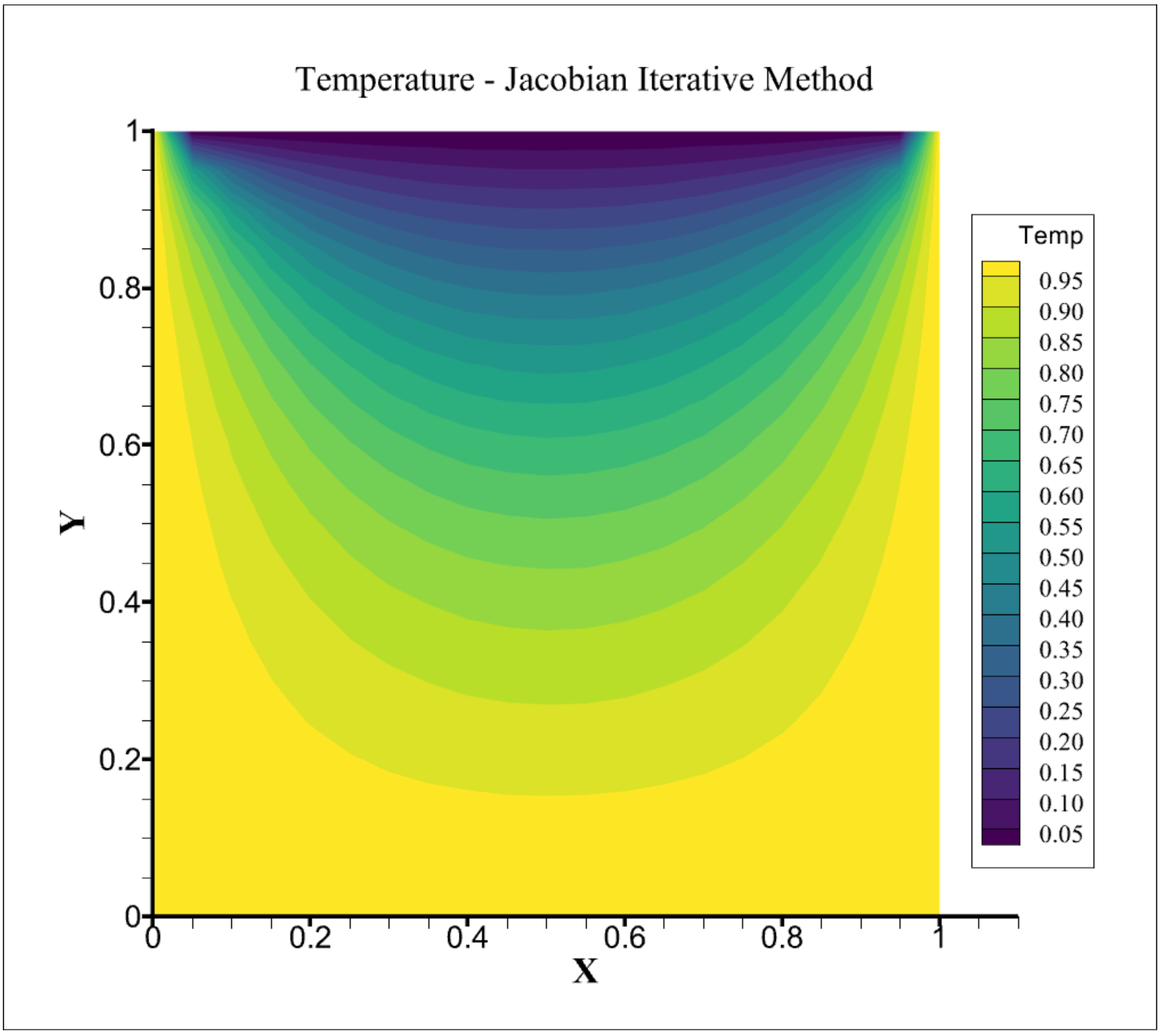


Fig 2.1: Temperature contour using Jacobian Iterative method

Number of Iterations: 724

Time taken to converge to $\epsilon < 10^{-6}$: 1,454 msec

2.2. Point Gauss-Seidel Iterative Method

Discretized equation:

$$T_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}\left[\beta^2\, T_{i,j-1}^{k+1} + T_{i-1,j}^{k+1} + T_{i+1,j}^{k} + \beta^2\, T_{i,j+1}^{k}\right]$$

Contour:

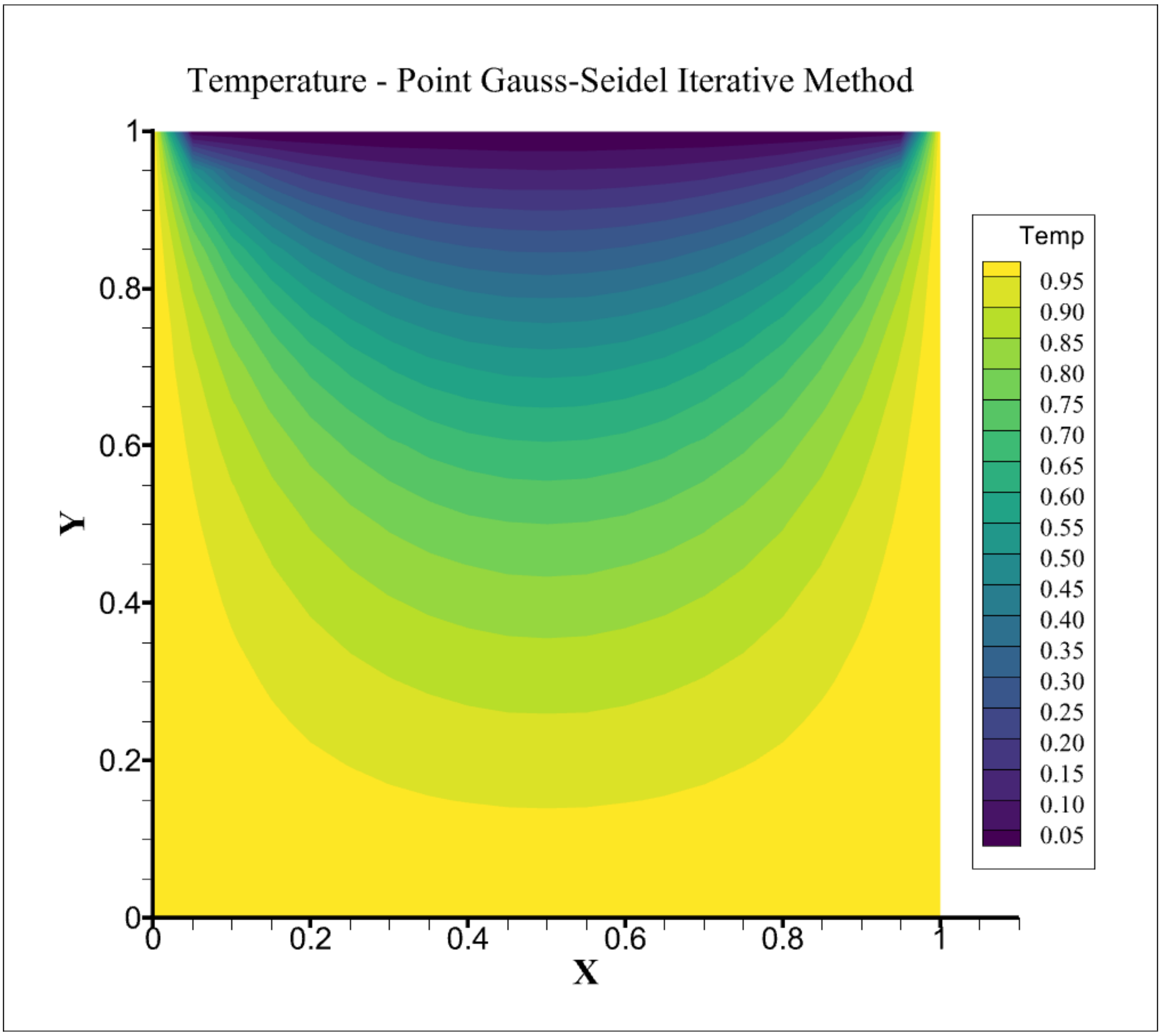


Fig 2.2: Temperature contour using Point Gauss-Seidel Iterative method

Number of Iterations: 389

Time taken to converge to $\epsilon < 10^{-6}$: 620 msec

2.3. Point Successive Over Relaxation (PSOR) Iterative Method

Discretized equation:

$$T_{i,j}^{k+1} = (1-\omega)T_{i,j}^{k} + \frac{\omega}{2(1+\beta^2)}\left[\beta^2\, T_{i,j-1}^{k+1} + T_{i-1,j}^{k+1} + T_{i+1,j}^{k} + \beta^2\, T_{i,j+1}^{k}\right]$$

$\omega = 1.85$ taken

Contour:

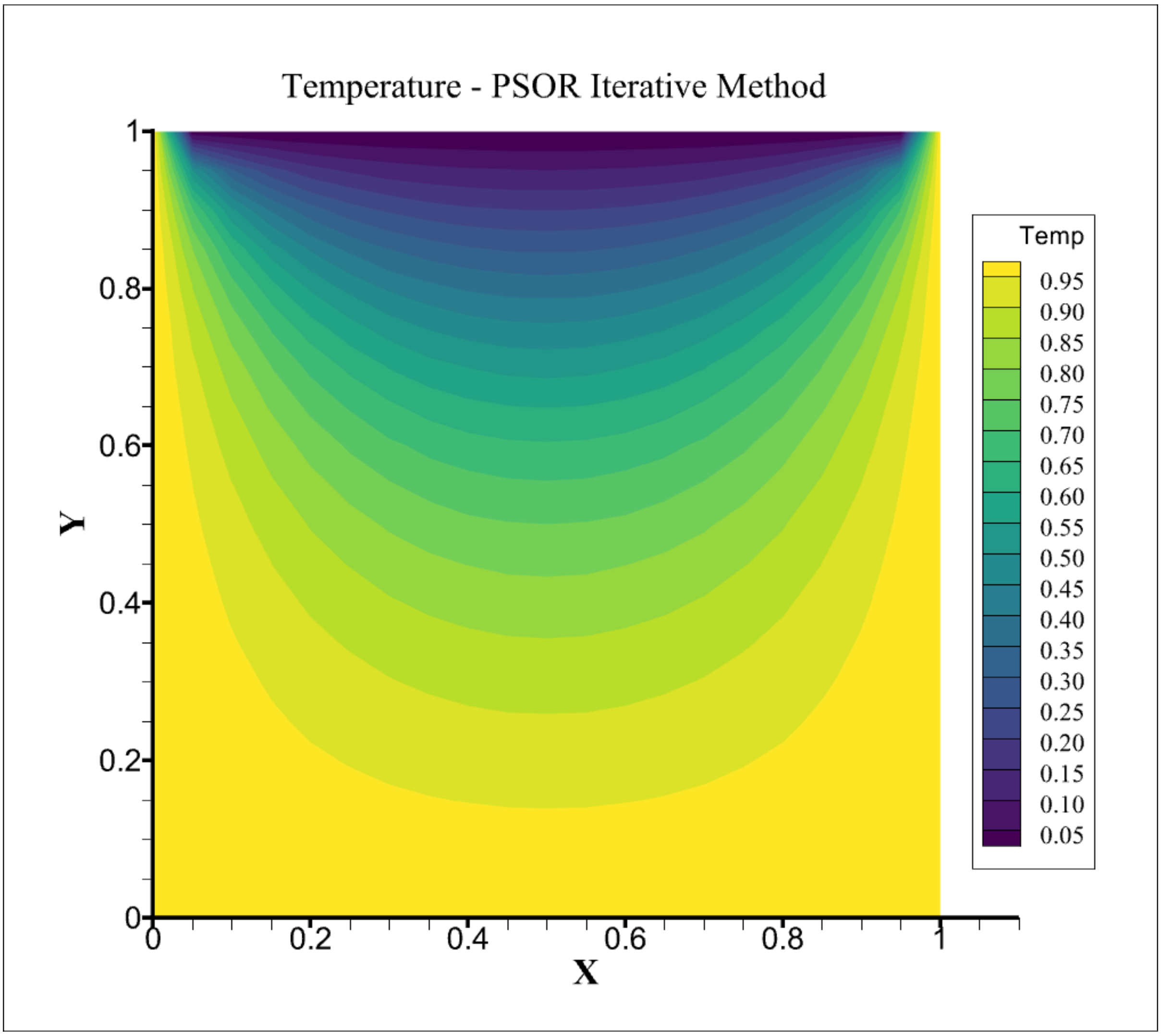


Fig 2.3: Temperature contour using Point Successive Over Relaxation method

Number of Iterations: 81

Time taken to converge to $\epsilon < 10^{-6}$: 252 msec

2.4. *Line Gauss-Seidel Iterative Method*

Discretized equation: *x-sweep*

$$T_{i-1,j}^{k+1} - 2(1+\beta^2)T_{i,j}^{k+1} + T_{i+1,j}^{k+1} = -\beta^2(T_{i,j-1}^{k+1} + T_{i,j+1}^{k})$$

Contour:

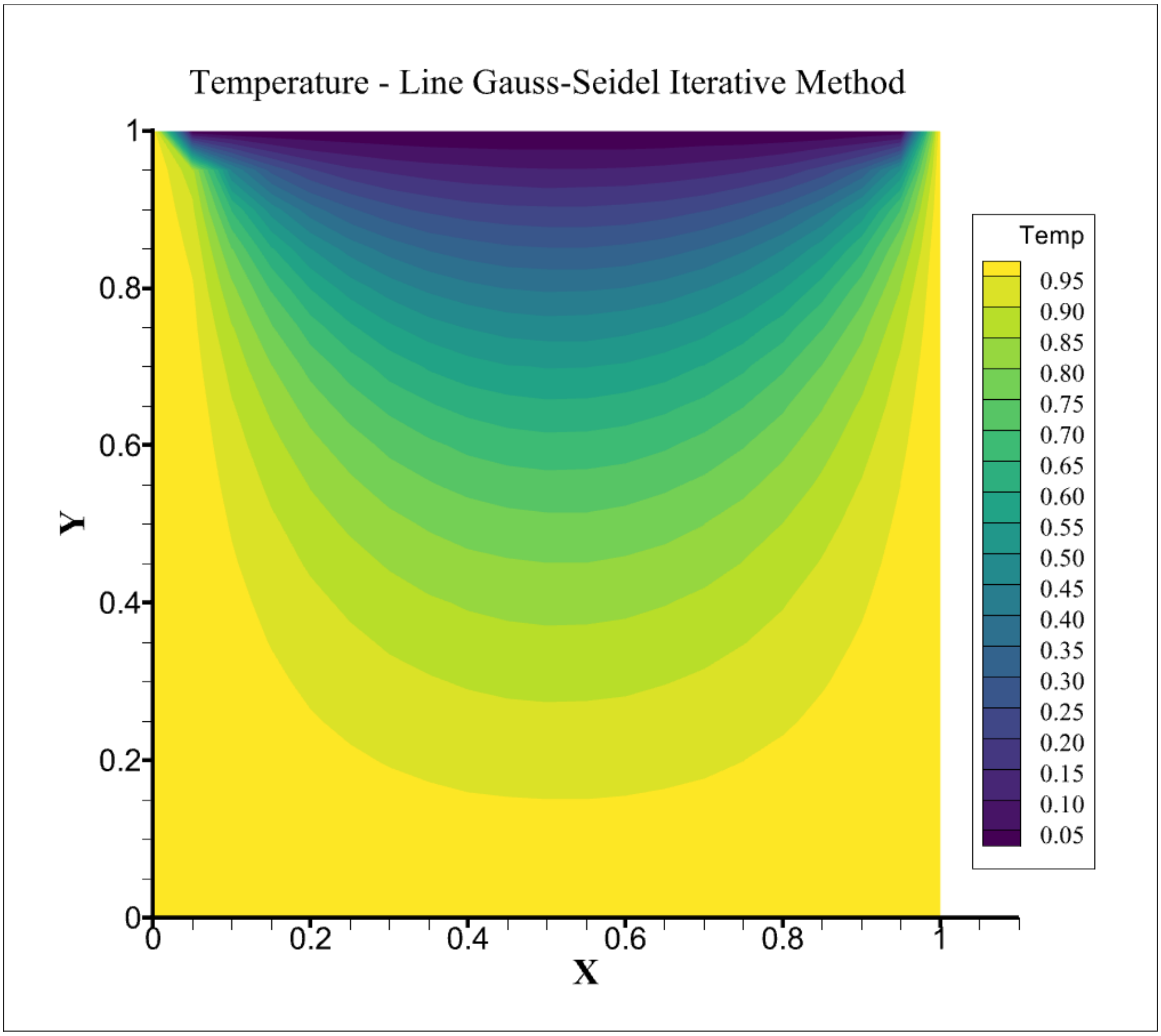


Fig 2.4: Temperature contour using Line Gauss-Seidel Iterative method

Number of Iterations: 203

Time taken to converge to $\epsilon < 10^{-6}$: 1,060 msec

2.5. Alternating Direction Implicit (ADI) Method

Discretized equation:

x-sweep

$$T_{i-1,j}^{k+\frac{1}{2}} - 2(1+\beta^2)T_{i,j}^{k+\frac{1}{2}} + T_{i+1,j}^{k+\frac{1}{2}} = -\beta^2(T_{i,j-1}^{k+\frac{1}{2}} + T_{i,j+1}^{k})$$

y-sweep

$$\beta^2 T_{i-1,j}^{k+1} - 2(1+\beta^2)T_{i,j}^{k+1} + \beta^2 T_{i+1,j}^{k+1} = -(T_{i,j-1}^{k+1} + T_{i,j+1}^{k+\frac{1}{2}})$$

Contour:

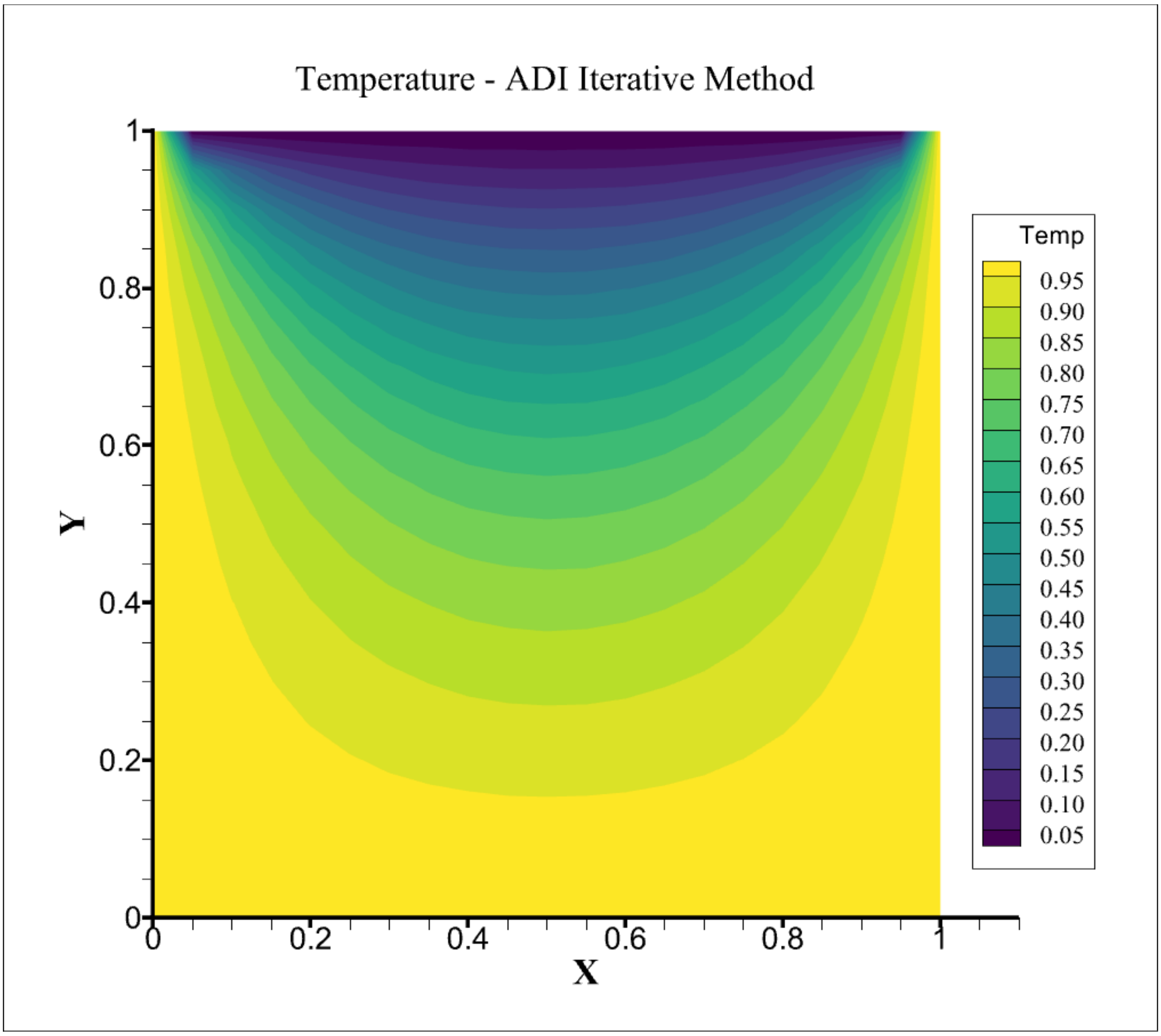


Fig 2.5: Temperature Contour using Alternating Direction Implicit (ADI) method

Number of Iterations: 110

Time taken to converge to $\epsilon < 10^{-6}$: 784 msec

Plotting ω vs no. of iterations in PSOR method

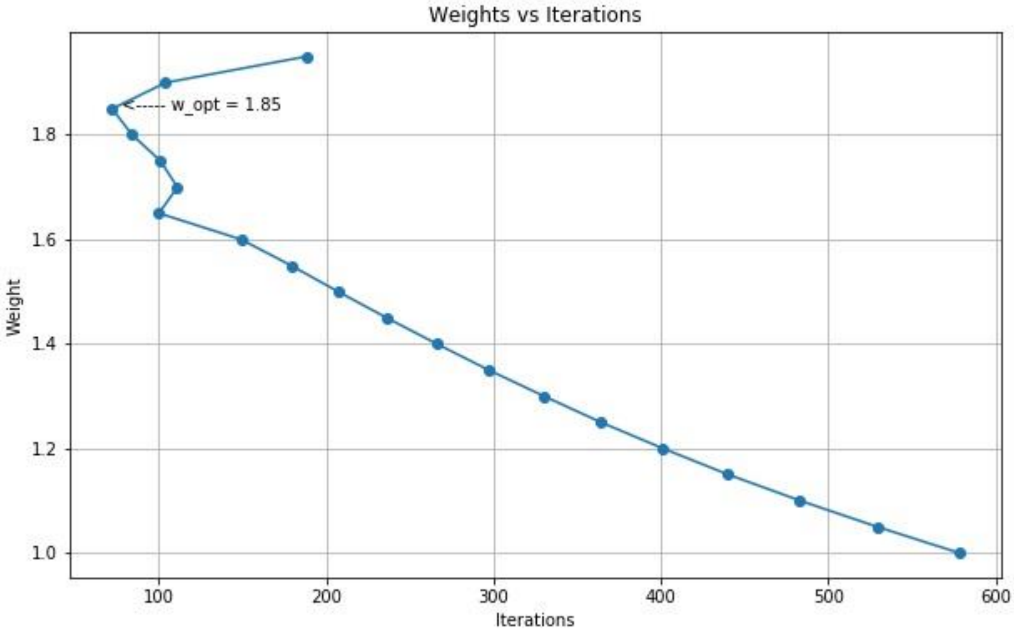


Fig 2.6: Weight vs Iterations for PSOR Method

$$\omega_{opt} = \frac{2\left(1-\sqrt{1-a}\right)}{a}, \quad where: a = \frac{\cos\left(\frac{\pi}{M-1}\right) + \beta^2 \cos\left(\frac{\pi}{N-1}\right)}{1+\beta^2}$$

$$\omega_{opt} = 1.800\ (from\ formula)$$

$$\omega_{opt} = 1.850\ (from\ graph)$$

Tabulating time taken for each Iterative Method

Table 2.1: Number of Iterations, Time taken and Time per iteration for each Iterative method

Method	Iterations	Time (ms)	Time/iter (ms)
Jacobian	724	1454.32	2.01
Point Gauss-Seidel	389	620.17	1.59
PSOR	81	252.52	3.12
Line Gauss-Seidel	203	1060.10	5.22
ADI	110	783.78	7.13

Plotting log(error) vs log(iterations) for all the methods

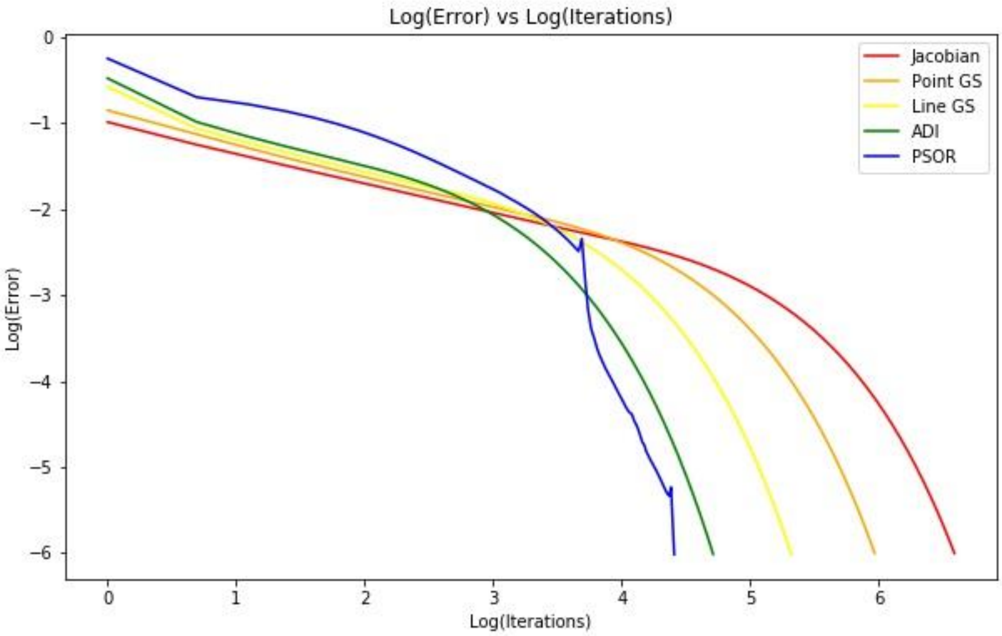


Fig 2.7: Log (error) vs Log (iterations) for all the methods