🚀 The Power of Exponentials and Roots

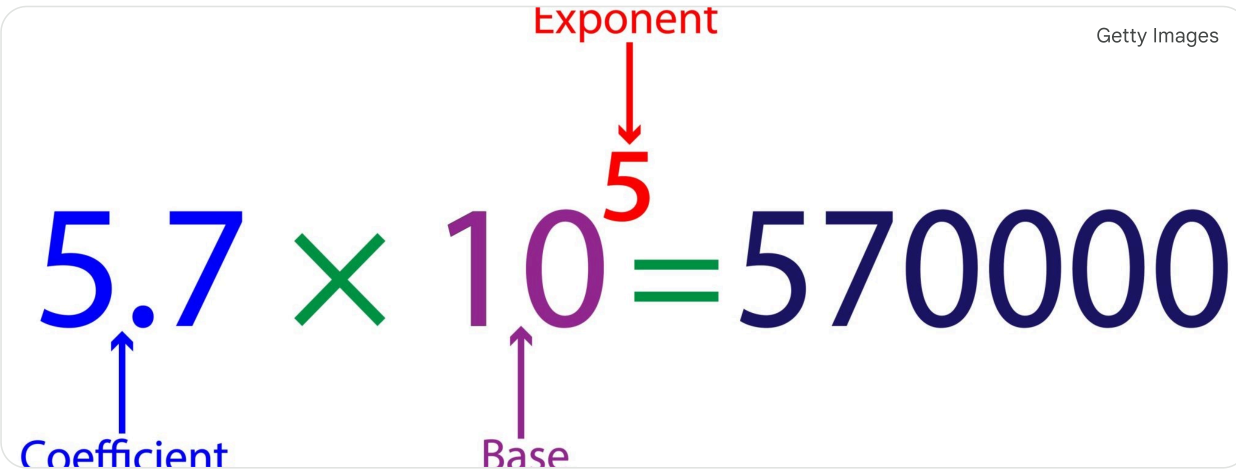
Welcome to the exciting world of **exponentials** (also called powers) and **roots**! This is a special way of handling **repeated multiplication**, which makes solving complex math problems much easier and faster.

Part 1: Exponentials (Powers)

An exponential expression tells you to multiply a number (the **base**) by itself a certain number of times (the **exponent**).

In the expression $\mathbf{a^m}$:

- $\mathbf{a}$ is the **Base** (the number being multiplied).
- $\mathbf{m}$ is the **Exponent** or **Power** (tells you *how many times* to multiply the base by itself).



For example:

- 2^3 (read as "2 to the power of 3" or "2 cubed") means:

$$2^3 = 2 \times 2 \times 2 = \mathbf{8}$$

- 5^4 (read as "5 to the power of 4") means:

$$5^4 = 5 \times 5 \times 5 \times 5 = \mathbf{625}$$

The Six Key Rules of Exponentials

These rules, sometimes called the Laws of Exponents, are powerful shortcuts!

1. Product of Powers Rule (Multiplying Exponentials)

When you multiply two exponential expressions that share the **same base**, you **add** the exponents.

$$\mathbf{a^m \times a^n = a^{m+n}}$$

- Example: $3^5 \times 3^2 = 3^{(5+2)} = \mathbf{3^7}$ *(You had five 3's and two 3's, giving you seven 3's multiplied together.)*

2. Quotient of Powers Rule (Dividing Exponentials)

When you divide two exponential expressions that share the **same base**, you **subtract** the exponents.

$$\mathbf{\frac{a^m}{a^n} = a^{m-n}}$$

- Example: $\frac{4^8}{4^3} = 4^{(8-3)} = \mathbf{4^5}$ *(The three 4's on the bottom cancel three of the 4's on the top, leaving $8 - 3 = 5$ total 4's.)*

3. Power of a Power Rule

When you raise an exponential expression to another power, you **multiply** the exponents.

$$\mathbf{(a^m)^n = a^{m \times n}}$$

- Example: $(10^4)^3 = 10^{(4\times3)} = \mathbf{10^{12}}$ *(You have 10^4 repeated 3 times: $10^4 \times 10^4 \times 10^4 = 10^{4+4+4} = 10^{12}$)*

4. Power of a Product Rule

When a product of different bases is raised to a power, you apply the power to **each base** inside the parentheses.

$$\mathbf{(a \times b)^n = a^n \times b^n}$$

- Example: $(2 \times 3)^5 = \mathbf{2^5} \times \mathbf{3^5}$

5. Zero Exponent Rule

Any non-zero base raised to the power of **zero** is always $\mathbf{1}$.

$$\mathbf{a^0 = 1}, \text{where } \mathbf{a \neq 0}$$

- Example: $100^0 = \mathbf{1}$

6. Negative Exponent Rule

A negative exponent means you take the **reciprocal** (flip the fraction) of the base raised to the positive exponent.

$$\mathbf{a^{-n} = \frac{1}{a^n}}$$

- Example: $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ *(This moves the term to the opposite part of the fraction, changing the sign of the exponent.)*

Part 1.5: Why the Rules Work (The Logic)

Here's the simple logic behind each rule, by writing out the multiplication they represent:

Rule	Explanation (The "Why")	Example Breakdown
1. Product Rule: $a^m \times a^n = a^{m+n}$	The exponent counts the factors. When you multiply two groups of the same base, you are just combining the total count of factors. m factors of a plus n factors of a equals $m+n$ factors of a.	$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 = \mathbf{2^5}$
2. Quotient Rule: $\frac{a^m}{a^n} = a^{m-n}$	This is based on **cancellation**. Every factor of a in the denominator cancels out one factor of a in the numerator. You subtract the number of factors being cancelled (n) from the original number of factors (m).	$\frac{5^4}{5^2} = \frac{5\times5\times5\times5}{5\times5}$. Cancel two 5s from top/bottom. Left with $5 \times 5 = \mathbf{5^2}$
3. Power of a Power: $(a^m)^n = a^{m \times n}$	The outer exponent (n) tells you to repeat the inner expression (a^m) n times. Since a^m means m copies of a, repeating that n times gives you n groups of m copies, which means you have $m \times n$ total copies of a.	$(3^2)^3 = 3^2 \times 3^2 \times 3^2 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = \mathbf{3^6}$

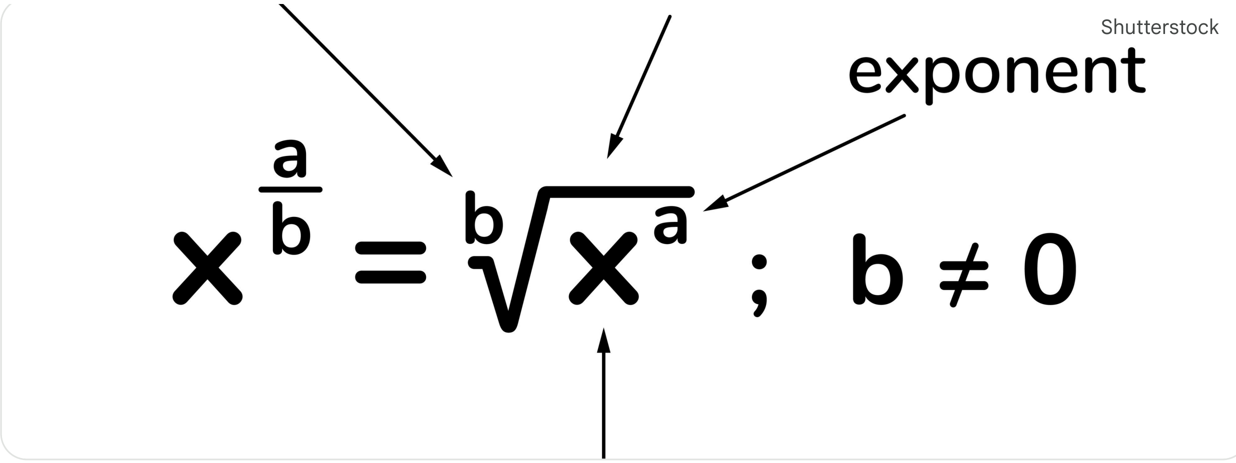
4. Power of a Product: $(a \times b)^n = a^n \times b^n$	The exponent applies to everything inside the parentheses. Because the order of multiplication doesn't matter (the commutative property), you can rearrange the factors so all the a's are together and all the b's are together.	$(2 \times 4)^3 = (2 \times 4) \times (2 \times 4) \times (2 \times 4)$. Rearrange: $(2 \times 2 \times 2) \times (4 \times 4 \times 4) = \mathbf{2^3} \times \mathbf{4^3}$
5. Zero Exponent: $a^0 = 1$	This rule is needed to make the Quotient Rule work for *all* situations. Since any number divided by itself equals 1, and the Quotient Rule says $\frac{a^m}{a^m} = a^{m-m} = a^0$, it must be true that $\mathbf{a^0 = 1}$.	$\frac{7^3}{7^3} = 1$ (by definition of division). Using the rule: $7^{3-3} = 7^0$. Since $7^0 = 1$.
6. Negative Exponent: $a^{-n} = \frac{1}{a^n}$	This rule keeps the pattern of **division** going. If $2^3 = 8$ and $2^2 = 4$ (divide by 2), and $2^1 = 2$ (divide by 2), and $2^0 = 1$ (divide by 2), the next step must be to divide by 2 again: $2^{-1} = 1 \div 2 = \frac{1}{2}$.	2^{-3}. Start at $2^0 = 1$. Divide by 2 three times: $1 \div 2 \div 2 \div 2 = \frac{1}{2 \times 2 \times 2} = \frac{1}{\mathbf{2^3}}$

Part 2: Roots (Radicals)

An **n-th root** is the **inverse** (opposite) of an exponential. When you find the n-th root of a number x, you are looking for a number a that, when multiplied by itself n times, equals x.

We write roots using the **radical symbol** $\sqrt{}$:

$$\sqrt[\mathbf{n}]{\mathbf{x}}$$



- The little number $\mathbf{n}$ is the **index** (it tells you how many times to multiply the number).
- The number $\mathbf{x}$ underneath is the **radicand**.

Types of Roots

1. **The Square Root (Index $n = 2$):** The index 2 is usually hidden.
   - $\sqrt{25} = \mathbf{5}$, because $5^2 = 5 \times 5 = 25$.
2. **The Cube Root (Index $n = 3$):**
   - $\sqrt[3]{8} = \mathbf{2}$, because $2^3 = 2 \times 2 \times 2 = 8$.
3. **The General N-th Root:**
   - $\sqrt[4]{16} = \mathbf{2}$, because $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

Fractional Exponents: The Connection

Roots can be rewritten as exponentials using **fractional exponents**. This is the most powerful rule linking the two concepts!

$$\sqrt[\mathbf{n}]{\mathbf{a}} = \mathbf{a}^{\mathbf{1/n}}$$

- The root's index (n) becomes the **denominator** (bottom number) of the fraction.
- **General Form:**

$$\sqrt[\mathbf{n}]{\mathbf{a}^{\mathbf{m}}} = \mathbf{a}^{\mathbf{m/n}}$$

Examples:

- $\sqrt{9} = 9^{1/2}$
- $\sqrt[3]{64^2} = 64^{2/3}$ (We can read this as "The cube root of 64, squared.")
  - Solution: $\sqrt[3]{64} = 4$, and then $4^2 = \mathbf{16}$.

Part 3: Homework Exercises (50 Problems!)

A. Simplify using the Exponential Rules (20 exercises)

Simplify the following expressions to a single base and exponent (e.g., 4^9).

1. $6^3 \times 6^5$
2. $10^9 \times 10^1$
3. $y^7 \times y^2$
4. $2^4 \times 2^{-1}$
5. $(-3)^2 \times (-3)^3$
6. $\frac{8^7}{8^4}$
7. $\frac{x^{10}}{x^3}$
8. $\frac{5^2}{5^5}$
9. $\frac{11^0}{11^2}$
10. $\frac{9^4}{9^4}$
11. $(2^3)^5$
12. $(4^{-2})^3$
13. $(a^6)^0$
14. $((-7)^3)^2$
15. $(10^{1/2})^4$
16. $(3 \times 5)^3$
17. $(4a)^2$
18. $(x^2y^3)^4$
19. 9^0
20. 7^{-3}

B. Evaluate (Find the actual number answer) (15 exercises)

Calculate the numerical value for each expression.

21. 4^3
22. 1^9
23. 10^5
24. $(-2)^4$
25. $\frac{6^5}{6^3}$
26. $\frac{3^2}{3^4}$
27. 5^{-2}
28. $2^3 \times 2^1$

29. $\sqrt{49}$
30. $\sqrt[3]{27}$
31. $\sqrt[4]{81}$
32. $\sqrt{100} + \sqrt[3]{1}$
33. $4^{1/2}$
34. $8^{1/3}$
35. $25^{3/2}$

C. Write as a Fractional Exponent or Radical (15 exercises)

Translate between radical notation and fractional exponent notation.

36. $\sqrt[3]{5^2}$ (Write as a fractional exponent)
37. $\sqrt{a^7}$ (Write as a fractional exponent)
38. $\sqrt[5]{12}$ (Write as a fractional exponent)
39. $\sqrt[4]{x}$ (Write as a fractional exponent)
40. $y^{1/5}$ (Write as a radical)
41. $10^{1/2}$ (Write as a radical)
42. $7^{3/4}$ (Write as a radical)
43. $z^{5/3}$ (Write as a radical)
44. $\sqrt[5]{-32}$ (Evaluate)
45. $x^{1/2} \times x^{1/2}$ (Simplify the exponent)
46. $\frac{y^2}{y^{1/2}}$ (Simplify the exponent)
47. $\sqrt{\frac{1}{9}}$ (Evaluate)
48. $\sqrt[3]{64}$ (Evaluate)
49. $\sqrt{121}$ (Evaluate)
50. $(2^2)^{1/2}$ (Simplify and evaluate using the Power of a Power rule)

Part 4: Calculating Roots (The Bisection Method)

How do calculators find square roots and cube roots to so many decimal places? They use a strategy called the **Bisection Method**, which is just a super-precise, systematic way of **Guess and Check**.

The idea is that you find an interval (two numbers) where the answer must lie. Then, you test the number exactly in the middle of that interval to cut the search area in half! You repeat this process forever to get infinite decimal accuracy.

Example 1: Finding the Square Root of 5 ($\sqrt{5}$)

Our goal is to find a number x such that $x^2 = 5$.

Step 1: Find the Whole Number Bounds We check the whole numbers around 5:

- $2^2 = 4$
- $3^2 = 9$ Since 5 is between 4 and 9, the answer $\sqrt{5}$ must be between 2 and 3.
- **Interval:** $[2, 3]$

Step 2: Find the First Decimal Place (Tenths) We check numbers between 2 and 3:

- Try the midpoint 2.5: $(2.5)^2 = 6.25$ (Too high, so the answer is less than 2.5)
- Try 2.2: $(2.2)^2 = 4.84$ (Too low)
- Try 2.3: $(2.3)^2 = 5.29$ (Too high) Since 5 is between 4.84 and 5.29, the answer must be between 2.2 and 2.3.
- **New Interval:** $[2.2, 2.3]$

Step 3: Find the Second Decimal Place (Hundredths) We check numbers between 2.2 and 2.3:

- Try the midpoint 2.25: $(2.25)^2 = 5.0625$ (Too high)
- We know the answer is between 2.2 and 2.25. Let's test 2.23 and 2.24.
- $2.23^2 = 4.9729$ (Too low)
- $2.24^2 = 5.0176$ (Too high) The answer must be between 2.23 and 2.24.
- **New Interval:** $[2.23, 2.24]$

We now know that $\boldsymbol{\sqrt{5} \approx 2.23...}$. To get more decimals, you just keep going!

Example 2: Finding the Cube Root of 10 ($\sqrt[3]{10}$)

The process is exactly the same, but we **cube** (multiply by itself three times) our guesses instead of squaring them.

Step 1: Find the Whole Number Bounds

- $2^3 = 8$
- $3^3 = 27$ Since 10 is between 8 and 27, the answer $\sqrt[3]{10}$ must be between 2 and 3.
- **Interval:** $[2, 3]$

Step 2: Find the First Decimal Place (Tenths)

- Try 2.1: $(2.1)^3 = 9.261$ (Too low)
- Try 2.2: $(2.2)^3 = 10.648$ (Too high) The answer is between 2.1 and 2.2.
- **New Interval:** $[2.1, 2.2]$

Step 3: Find the Second Decimal Place (Hundredths)

- Try the midpoint 2.15: $(2.15)^3 = 9.938375$ (Too low, but very close!)
- Try 2.16: $(2.16)^3 = 10.077696$ (Too high) The answer is between 2.15 and 2.16.

- **New Interval:** $[2.15, 2.16]$

We now know that $\sqrt[3]{\mathbf{10}} \approx \mathbf{2.15...}$

This bisection method can be repeated infinitely to get infinite precision!