



The fuzzball proposal for black holes

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ABSTRACT

The fuzzball proposal states that associated with a black hole of entropy S , there are $\exp S$ horizon-free non-singular solutions that asymptotically look like the black hole but generically differ from the black hole up to the horizon scale. These solutions, the fuzzballs, are considered to be the black hole microstates, while the original black hole represents the average description of the system. The purpose of this report is to review current evidence for the fuzzball proposal, emphasizing the use of AdS/CFT methods in developing and testing the proposal. In particular, we discuss the status of the proposal for 2 and 3 charge black holes in the D1–D5 system, presenting new derivations and streamlining the discussion of their properties. Results to date support the fuzzball proposal, but further progress is likely to require going beyond the supergravity approximation and sharpening the definition of a “stringy fuzzball”. We outline how the fuzzball proposal could resolve longstanding issues in black hole physics, such as Hawking radiation and information loss. Our emphasis throughout is on connecting different developments and identifying open problems and directions for future research.

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1. Introduction

The physics of black holes has been at the center stage of theoretical physics for over thirty years, as black holes raise puzzles that directly challenge many cherished fundamental physical properties, such as as unitarity and locality. Defining questions in this area have been

- Why does a black hole have entropy proportional to its horizon area?
- Is there information loss because of black holes?
- How does one resolve spacetime singularities, such as those inside black holes or in Big Bang cosmologies?

The fact that black holes appear to have entropy is puzzling on many counts. Typically, in a quantum system, the correspondence principle relates the quantum states to the classical phase space and the entropy to the volume of this phase space in Planck units. Black holes however are uniquely fixed in terms of the conserved charges they carry (black holes have “no hair” in classical general relativity) so the classical phase space is zero dimensional.¹ One could argue that perhaps the states counted by the Bekenstein–Hawking formula are purely quantum mechanical with no classical limit, but even in this case, one is faced with the puzzle that the entropy is proportional to the area of the horizon, rather than the volume enclosed in it, as one might have anticipated based on the fact that the entropy is an extensive quantity. This has been taken as a hint of a fundamental property of quantum gravitational theories, namely that they are *holographic* [1]: any $(d + 1)$ -dimensional gravitational theory should have a description in terms of d -dimensional quantum field theory without gravity with one degree of freedom per Planck area. We will discuss holography and its realization in the AdS/CFT correspondence extensively in this report.

Classically, black holes are completely black, but semi-classically they thermally radiate [2]. This fact considerably strengthens the case for taking seriously the analogy between the black hole laws and thermodynamics, and searching for an underlying statistical explanation of this thermodynamic behavior. More importantly, the thermal nature of the radiation has led to one of the biggest recent conundrums in theoretical physics: the information loss paradox [3]. Matter in a pure state may be thrown into black hole but only thermal radiation comes out. So it would appear as if a pure state has evolved into a mixed state, thus violating unitarity. This issue has been debated vigorously over the years, and has been taken as an indication that well accepted physical principles, such as unitarity or locality, may have to be abandoned, see [4] for reviews. As we will review later, recent progress based on the AdS/CFT correspondence implies a unitary evolution and the holographic nature of the correspondence also implies that spacetime locality is only approximate.

Black holes have a curvature singularity hidden behind their horizon. Near these singularities, Einstein gravity breaks down, and a fundamental question is how (and when) the quantum theory of gravity resolves these singularities and what

¹ Strictly speaking such uniqueness theorems have only been proven for very special systems, and in general black holes do carry hair. However the volume of the corresponding classical phase space is still far too small to account for black hole entropy upon quantization.

effect the resolution has. When considering black holes with macroscopic horizons, one might anticipate that semi-classical computations, such as those implying Hawking radiation, would be applicable. This conclusion has also been challenged in the literature, see [5–8] for a (small) sample of works in this direction. Here recent developments also offer a new perspective: non-singular spacetimes would generically differ from the black hole background up to the horizon scale, rather than only in the neighborhood of the singularity.

Over the last 15 years great progress has been achieved in string theory in addressing black hole issues, as we now briefly review. Firstly, for a class of supersymmetric black holes the black hole entropy was understood using D-branes. The basic idea is simple: supersymmetric states (generically) exist for all values of the parameters of the underlying theory. Changing the gravitational strength, one can interpolate between the description of the system as a black hole and the description in terms of bound states of D-branes. Thus one can compute the degeneracy in the D-brane description, and thanks to supersymmetry, extrapolate this result to the black hole regime. Starting from [9] such computations were done for a class of black holes and exact agreement was found with the Bekenstein-Hawking entropy formula. In more recent times, the agreement was extended to subleading orders, where on the gravitational side, one takes into account the effect of higher derivative terms, and on the D-brane side, one computes subleading terms in the large charge limit of the degeneracy formulas, see [10] for a review. These developments show that the gravitational entropy indeed has a statistical origin, but they do not directly address any of the other black hole issues, since the computations involve in an essential way an extrapolation from weak to strong coupling.

Further progress was achieved with the advent of the AdS/CFT correspondence. The asymptotically flat black holes under consideration have a near-horizon region that contains an AdS factor, so AdS/CFT is applicable. The black hole microstates can then be understood as certain supersymmetric states of the dual CFT. Since the AdS/CFT duality is (conjectured to be) an exact equivalence and the boundary theory is unitary, the dynamics of the black hole microstates is unitary. Although this, in principle, shows that there is no information loss, it still does not explain what is the gravitational nature of the black hole microstates, nor does it show where Hawking's original argument goes wrong.

The AdS/CFT correspondence however implies more: for every stable state of the CFT, there should exist a corresponding regular asymptotically AdS geometry that encodes in its asymptotics the vevs of gauge invariant operators in that state. Thus, for every CFT state that one counts to account for the black hole entropy, there should exist a corresponding asymptotically AdS geometry. These solutions will generically be stringy in the interior although some will be well described in the (super)gravity limit.

Now since these solutions have the same behavior near the AdS boundary as the near-horizon limit of the original asymptotically flat black hole, one can attach the asymptotically flat region as in the original black hole. We thus find that associated with the original black hole there is an exponential number of solutions that look like the black hole up to the horizon scale, but differ from it in the interior; the interior region is replaced by the asymptotically AdS solutions just discussed and there is one such solution per microstate. The ones that have a good supergravity description everywhere should not have horizons, since for those the semiclassical arguments that relate horizons to entropy should be applicable, and each of these solutions should correspond to a pure state. This is the fuzzball proposal for black holes, formulated in works of Mathur and collaborators in [11–14].

The purpose of this report is to review and make a critical appraisal of the fuzzball proposal. Other reviews on this subject include [15–18]. In the next section we introduce the fuzzball proposal, discuss its relation with the AdS/CFT correspondence, explaining more fully the argument in the preceding paragraph, and sketch why this proposal would resolve black hole puzzles. In Section 3 we explain how holography works. In particular, we discuss how the vevs of gauge invariant operators are encoded in supergravity solutions. Then in Section 4 we discuss the best understood and simplest example, namely the two charge D1–D5 system, where one can explicitly find all fuzzball geometries (visible in supergravity) and test the general arguments.

To make further progress with the fuzzball proposal, one would like to understand black holes which have macroscopic horizons, such as the 3-charge D1–D5–P system. Section 5 contains a discussion of what is known about the 3-charge system, in particular the candidate fuzzball geometries which have been constructed. Much of the work in the current literature has been focused on constructing explicit examples of fuzzball geometries, with a view to reproducing the entropy of the black hole. Many open questions remain as to how the fuzzball proposal would address black hole puzzles, and we discuss these issues in Section 6. Our emphasis throughout is on connecting the different developments and emphasizing open problems and directions for further research, rather than exhaustively reviewing the literature.

2. Generalities

2.1. What is the fuzzball proposal?

Consider a black hole solution with associated gravitational entropy S . According to the fuzzball proposal associated with this black hole, there are $\exp S$ horizon-free non-singular solutions that asymptotically look like the black hole but generically differ from the black hole up to the horizon scale. These solutions, the fuzzballs, are considered to be the black hole microstates, while the original black hole represents the average description of the system.

To complete this definition we should specify which class of theories we consider and what precisely we mean by “solutions”. We will work within the framework of string theory; black holes are then solutions of the corresponding low

energy effective action. Lower dimensional black holes, such as for example the 4d Schwarzschild black hole, are viewed as resulting from a corresponding 10d solution upon compactification. Black hole solutions typically involve only a very small number of the low energy fields, e.g. only the metric for the Schwarzschild black hole, or the metric plus a gauge field for the Reissner–Nordström solution.

The corresponding fuzzball solutions would, however, in general be solutions of the full theory, not just its low energy approximation. Only a subset of fuzzballs would solve the low energy field equations and, in general, these solutions would involve all low energy fields, not just the fields participating in the black hole solution. In fact, as it will become clear later on, it is crucial that the fuzzball solutions involve many other fields. This explains in part why people have not stumbled upon such an exponential number of regular solutions that resemble black holes.

Fuzzballs that involve string scale physics would, in general, only have a sigma model description or, if string field theory were adequately developed, they would be non-singular solutions of the string field equations. Some of these string solutions, however, may have an extrapolation to low energies, i.e. there would exist a corresponding supergravity solution, but it would contain small regions of high curvature. Furthermore, there would also be cases where the differences between fuzzball supergravity solutions are comparable to the corrections coming from the leading higher derivative corrections to the string theory effective action. In such cases, these solutions will not be reliably distinguishable within supergravity.

To properly define the dynamics of a system on a non-compact spacetime, we have to specify boundary conditions for all fields. We will loosely refer to the boundary conditions as non-normalizable modes and we will refer to the leading part of the asymptotics which is affected by dynamics as normalizable modes. All fuzzball solutions would share the same non-normalizable modes with the original black hole spacetime, but they would differ in the normalizable modes. The precise notion of normalizable and non-normalizable modes depends on the asymptotics under consideration, and will be clear in the specific examples of interest later.

2.2. Fuzzballs and black hole puzzles

The fuzzball proposal has the potential of resolving all black hole related puzzles, although more work is required in order to demonstrate this with sufficient precision.

Firstly, the black hole entropy becomes of a standard statistical origin: there is a corresponding solution for every black hole microstate. In other words, the entropy is related to the volume of the classical phase space. The fact that the entropy grows like an area will be seen to be a direct consequence of holography, at least in the examples where AdS/CFT is applicable.

Since the geometries have no horizons, there is no information loss either. Matter coming from infinity will escape back to infinity at late times. A typical fuzzball geometry is expected to look like the black hole asymptotically, but it would differ from it up to the horizon scale (although this has not been demonstrated to date for solutions with macroscopic non-extremal horizons). One might anticipate that this difference in the “inner horizon region” is responsible for obtaining a different answer than in Hawking’s original computation.

Boundary conditions and regularity in the interior are expected to fix the fuzzball solutions, and so the resolution of the black hole singularity is already built into this proposal. Notice also that the “size” of the fuzzball is determined dynamically from these requirements. We will discuss these issues in more detail in Section 6, after reviewing the current results and literature on the fuzzball proposal.

2.3. AdS/CFT and the fuzzball proposal

The black holes whose entropy we best understand microscopically have a near-horizon region that contains an AdS factor. For these black holes one can use the AdS/CFT correspondence, and for this reason the general discussion in the previous section can be made much more precise.

Let us consider for example the 3-charge D1–D5–P system, which was the first to be understood quantitatively. The near-horizon region is $\text{AdS}_3 \times S^3 \times X_4$, with X_4 either T^4 or $K3$ (more properly, the near-horizon region is $BTZ \times S^3 \times X_4$ [19]). The entropy of this system was originally computed in [9], by finding the degeneracy of the D1–D5–P bound states at weak coupling, and then extrapolating the result to the black hole phase. It was later realized that this computation is part of the AdS/CFT duality: what one counts is the degeneracy of certain supersymmetric states of the dual CFT.

Using gravity/gauge theory duality, however, one can say more. Given a state in (a deformation of) the CFT, the duality implies that there is a corresponding asymptotically AdS spacetime with non-trivial matter fields capturing the parameters of deformation and the vevs of gauge invariant operators in the given state. The detailed correspondence will be discussed in the next section, but for the current argument one only needs the existence of such a correspondence. Now consider one of the supersymmetric states counted in accounting for the entropy of, say, the Strominger–Vafa black hole. Associated with this state, there should exist a regular asymptotically AdS solution. We thus arrive at the conclusion that there should exist $\exp S$ regular solutions which asymptotically look like the near-horizon region of the original black hole. These solutions share the same non-normalizable modes as the near-horizon limit of the original black hole solution, since we are considering states and not deformations of the CFT, but differ in their normalizable modes, which capture the non-trivial

vevs in the state under consideration. One can now attach back² the asymptotically flat region to arrive at the conclusion that there should exist $\exp S$ regular solutions that look like the original black hole up to the horizon scale, but differ in the interior; the interior has been replaced by the asymptotically AdS solution corresponding to each state. These are the fuzzball solutions. The place where each solution starts to differ from the black hole is controlled by the vev of the lowest dimension operator in this state and solutions corresponding to different states are distinguished by the vevs of higher dimension operators [20–22].

What this argument emphatically does *not* imply is that the solutions would be supergravity solutions, and indeed, the majority of fuzzball solutions will not be, although some will be well described by supergravity. For states where operators dual to supergravity fields acquire large vevs, the solution will differ appreciably from the black hole solution already at the supergravity level (but still look like the black hole at the asymptotically flat infinity). For such solutions with small curvatures everywhere, one would anticipate that standard treatments that associate entropy with horizons would be valid. Since each of these solutions is meant to correspond to a pure state, the corresponding geometry should therefore be horizonless. Much of the current fuzzball literature has focused on finding such supergravity solutions.

On the other hand, there would be many cases/states where none of the operators dual to supergravity fields acquires a vev, or the vev is of string scale: the corresponding solutions will then agree with the original solution up to the string scale. One would not expect to find fuzzball solutions representing these states in supergravity, and indeed we will see this behavior exemplified in the 2-charge system in Section 4.

There will also be cases where a large fraction of the microstates of the original black hole have large vevs of operators dual to supergravity fields (chiral primaries), but these vevs differ from each other very little. Such states should not be distinguishable in supergravity. One might find supergravity fuzzball solutions corresponding to these states, which cannot be reliably distinguished, as the differences between them are of the same order as the corrections due to leading higher derivative terms. Then the relevant fuzzball solutions have an extrapolation to the supergravity regime, but one cannot really trust the distinctions between similar solutions. Again we will see this behavior occurring in the 2-charge system, and on general grounds, this must persist to other black holes with macroscopic horizons. There is also the possibility that states sharing the same vevs of chiral primaries and differing in the vevs of string states are best described in the low energy regime by solutions with “hair” capturing the vevs of the chiral primaries, which have either singularities or a horizon, with area smaller than the horizon area of the original black hole. The latter indicate that these states cannot be distinguished in supergravity. For instance, one could relate the black ring solutions to corresponding black holes with the same charges in this way.

Clearly, the low energy approximation will not suffice to describe such cases, and to make progress, one will need to work with backgrounds of the full string theory. In particular, to even define the properties of a generic fuzzball, one will need to address the question of the definition of entropy outside the geometric regime: when does a string theory background have entropy, and how is this defined given the worldsheet theory?

To date, one has relied on the geometric definition of entropy, with the entropy being associated with horizons. In the supergravity limit, one uses the Bekenstein-Hawking entropy of the horizon, with the generalization by Wald [23] being used when working perturbatively with higher derivative corrections. This geometric definition breaks down when dealing with string scale solutions, be they fuzzballs or black holes, and needs to be replaced.

Note that this issue can also not be circumvented in the case of so-called small black holes which do not have horizons in supergravity. In recent work (reviewed in [10]) such black holes have been treated by evaluating the leading corrections to supergravity on (singular) supergravity solutions, assuming that the corrected solution has a horizon, and then computing the Wald entropy. This approach is a priori unjustified, as the neglected higher corrections are not small on a string scale solution, and it works unreasonably well at reproducing the entropy of the dual CFT microstates. Such small black holes should properly be described by a background of the full string theory, in which the entropy would need to be defined from the worldsheet theory as above.

Given the state of current technology, most work on the fuzzball proposal has so far been in the context of supergravity solutions, as will be reviewed in Sections 4 and 5. The limitations of the supergravity approximation will however be a recurrent theme throughout, and we will return to discuss the need to go beyond supergravity in the final sections. Next, however, we will review the evidence for the fuzzball proposal obtained from finding and analyzing fuzzball supergravity solutions.

One should note here that the fuzzball proposal has been developed most in the context of asymptotically flat black holes in four and five dimensions, for which the near horizon region contains AdS_2 or AdS_3 factors. It is fuzzball solutions for these black holes which will be discussed in the following sections. Clearly the proposal is more generally applicable, and one could hope to make comparable progress at finding fuzzball solutions with other supersymmetric black hole systems. In particular, asymptotically AdS_5 black holes are a natural system to explore, as these fall into the best understood AdS/CFT duality, that with $\mathcal{N} = 4$ SYM in four dimensions.

Whilst one could carry out a detailed parallel fuzzball discussion for such black holes, we will not explore this case here for several reasons. First of all, it is natural to understand first, black holes which are asymptotically flat, and thus closer to

² As emphasized the fuzzball solutions and near-horizon region of the black hole have the same leading behavior near the AdS boundary (since they have the same non-normalizable modes) so the gluing is the same in both cases.

astrophysical black holes. Secondly, it turns out that the AdS_5 case is technically harder than the cases we discuss. The LLM bubbling solutions [24] describe all $1/2$ BPS states of the system (visible in supergravity) but the corresponding “black hole” does not have a macroscopic horizon in supergravity [25], so is not a good test case for the fuzzball proposal.

In the asymptotically AdS_5 case, it seems that one needs to break the supersymmetry to $1/16$ to obtain a black hole with macroscopic horizon area in supergravity, see for example [26]. With so little supersymmetry, even the counting of the black hole microstates is rather more subtle, as the degeneracy depends on the coupling, and the black hole entropy has not yet been reproduced from field theory. Moreover, finding explicit fuzzball solutions with so little supersymmetry is likely to be hard. Thus, whilst the fuzzball proposal should be applicable for this system, and indeed many other interesting black hole systems, we will focus on the better understood case of asymptotically flat black holes.

One should note that in using AdS/CFT arguments to support the fuzzball proposal there may be certain additional subtleties that we have not yet mentioned. Firstly, one should be slightly more careful in relating geometries to states: the fuzzball solutions for the $D1$ – $D5$ system should be in correspondence with states in the Higgs branch of a $(1 + 1)$ -dimensional theory. Due to the strong infrared fluctuations in $1 + 1$ dimensions, one should consider wavefunctions that spread over the whole of Higgs branch, rather than continuous moduli spaces of the quantum states. So more properly one should view the fuzzball solutions as dual to wavefunctions on the Higgs branch. These wavefunctions, however, may be localized around specific regions in the large N limit and indeed this issue does not seem to play a key role in any of the subsequent discussions.

Secondly, in AdS/CFT one can have multiple saddle points of the bulk action, with the same boundary conditions, the most well known example being thermal AdS and the Schwarzschild black hole. In this case, the onshell (renormalized) action determines the thermodynamically preferred solution, and there is a Hawking–Page transition between the two phases at a critical temperature. One might wonder whether such multiple saddle points could complicate the relationship between a given CFT microstate and a corresponding asymptotically AdS string background. In the current context, however, where we specify the state, this information determines the vevs of chiral primaries, so the boundary conditions include both the source and vev part of the solution. As we will review in the next section, this data determines a point in the phase space of the gravitational theory, and thus there should be a unique regular solution (if such a solution exists), see also [27] and references therein, for the corresponding discussion in the mathematics literature on hyperbolic manifolds.

Nonetheless one should bear in mind both caveats as an issue to be addressed in future when sharpening the definition of the fuzzball proposal in the string regime.

3. Holographic methods

In this section, we will summarize the status of holographic methods, with the emphasis being on summarizing results rather on derivations. The aim is to provide a handbook of holographic formulae and associated prescriptions that can be readily used, without having to delve into their derivation.

The basic principles of holography were laid out in the original papers [28–30]. In particular, the duality maps the spectrum of string theory on asymptotically $\text{AdS} \times X$, where X is a compact space, to the spectrum of gauge invariant operators of the dual QFT, and the string theory partition function, which is a function of boundary conditions posed on the conformal boundary of the spacetime, to the generating functional of correlation functions of the dual QFT, with the fields parameterizing the boundary conditions mapped to sources of the dual operators.

The duality relation in full generality is still very difficult to probe, so different approximations have been developed over the years, e.g. the low energy limit, the limit of long operators, the plane wave limit etc. In this report we focus on the low energy limit, where string theory is well approximated by supergravity. This limit typically corresponds to a strong coupling limit of the boundary theory. We will further consider the leading saddle point approximation of the bulk path integral, where the (logarithm of the) bulk partition function becomes equal to the on-shell supergravity action. In other words, we suppress supergravity loops. This typically corresponds to the large N limit in the dual theory. Within these approximations, the gravity/gauge theory duality equates the supergravity on-shell action to the generating functional of connected QFT correlators at strong coupling and large N .

Let us discuss first the case where the bulk solution is exactly $\text{AdS}_{d+1} \times X_q$, e.g. $\text{AdS}_5 \times S^5$ or $\text{AdS}_3 \times S^3 \times X_4$. In such cases, the dual theory is a d -dimensional conformal field theory (CFT_d) and there is a one to one correspondence between the supergravity KK spectrum and primary operators of the dual CFT. One can use the duality to compute correlation functions of primary operators at strong coupling and large N . Conformal field theories have vanishing 1-point functions, so the first non-trivial computation is that of a 2-point function and the latter can be computed holographically by solving the linearized fluctuation equation around $\text{AdS}_p \times X_q$ with prescribed boundary conditions at the conformal boundary of AdS_p . Higher n -point functions can be obtained by solving the $(n - 1)$ -th order fluctuation equations.

Solutions that are asymptotically $\text{AdS}_{d+1} \times X_q$ describe either deformations of the CFT_d or the CFT in a non-trivial state. In such cases the most elementary question is: what is the deformation parameter and/or the state? The state of the CFT is uniquely specified if one knows the expectation values of all gauge invariant operators in that state. Within the supergravity approximation we only have access to primary operators: one can reliably describe only deformations of the original CFT by primary operators and one can only compute the vevs of primary operators. The latter gives partial information about the state, but this information should be enough to specify the state within the approximations used (strong coupling, large N).

The parameters of deformation and the vevs of primary operators can be extracted by means of *algebraic manipulations* only from the asymptotic expansion of the supergravity solution. The solutions we discuss in this report describe states rather than deformations, so we will not discuss any further the case of deformations. In the next two subsections, we will discuss the issues involved in obtaining the vevs (1-point functions) of primary operators and describe how these are resolved leading to general formulae for the 1-point functions. Afterwards we focus on the case of interest, namely asymptotically $\text{AdS}_3 \times S^3$ solutions and present explicit formulae for the vevs of operators up to dimension 2. Higher point functions can be obtained by solving the fluctuation equations around the original solution [31–34] but this will not be reviewed here.

3.1. Holographic renormalization

The first issue one needs to address when attempting to carry out holographic computations, is that the on-shell action diverges, essentially due to the infinite volume of spacetime. This issue is dealt with by the formalism of holographic renormalization [35–38,31,32,39,34]; for a review see [33], and amounts to adding local boundary covariant counterterms to cancel the infinities. Actually the local boundary counterterms are required, irrespectively of the issue of finiteness, by the more fundamental requirement of the appropriate variational problem being well posed [40]. As is well known, the conformal boundary of asymptotically AdS spacetimes have a well-defined conformal class of metrics rather than an induced metric. This means that the appropriate variational problem involves keeping fixed a conformal class, and not an induced metric as in the usual Dirichlet problem for gravity in a spacetime with a boundary. The new variational problem requires the addition of further boundary terms, on top of the Gibbons–Hawking term, which turn out to be precisely the boundary counterterms, see [40] for the details and a discussion of the subtleties related to conformal anomalies.

The subject of holographic renormalization is extensively discussed in the literature, so we will only highlight a few points and introduce the notation to be used in later sections. The points that should be emphasized are:

- To obtain renormalized correlators, the main object of interest is the radial canonical momentum, rather than the on-shell action.
- The source and renormalized 1-point function are a conjugate pair, and can be considered as coordinates in the phase space of the gravitational theory.

Let us briefly discuss these points. Firstly, the asymptotic form of bulk fields, specialized to the $D = 3$ case of interest, is

$$\begin{aligned} ds_3^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2} \left(g_{(0)uv} + z^2 \left(g_{(2)uv} + \log(z^2) h_{(2)uv} + (\log(z^2))^2 \tilde{h}_{(2)uv} \right) + \dots \right) dx^u dx^v \\ \psi^1 &= z(\log(z^2) \psi_{(0)}^1(x) + \tilde{\psi}_{(0)}^1(x) + \dots); \\ \psi^k &= z^{2-k} \psi_{(0)}^k(x) + \dots + z^k \psi_{(2k-2)}^k(x) + \dots, \quad k \neq 1. \end{aligned} \quad (3.1)$$

In these expressions $(g_{(0)uv}, \psi_{(0)}^1(x), \psi_{(0)}^k(x))$ are sources for the stress energy tensor and scalar operators of dimension one and k respectively; as usual one must treat operators of dimension $\Delta = d/2$ separately, where d is the dimension of the boundary. Note that the 2-dimensional boundary coordinates are labeled by (u, v) . We will also use the notation $[\psi]_n$ to denote the coefficient of the z^n term in the asymptotic expansion of the field ψ .

Correlation functions can be computed, using the basic holographic dictionary that relates the on-shell gravitational action to the generating functional of correlators. The first variation can be done in all generality [37] yielding a relation between the 1-point function (in the presence of sources, so higher point functions can be obtained by further functional differentiation w.r.t. sources) and non-linear combinations of the asymptotic coefficients in (3.1). The underlying structure of the correlators is best exhibited in the radial Hamiltonian formalism, which is a Hamiltonian formulation with the radius playing the role of time. The Hamilton–Jacobi theory, introduced in this context in [41], then relates the variation of the on-shell action w.r.t. boundary conditions, thus the holographic 1-point functions, to radial canonical momenta. It follows that that one can bypass the on-shell action and directly compute renormalized correlators using radial canonical momenta π [39,34].

A fundamental property of asymptotically (locally) AdS spacetimes is that dilatations are part of their asymptotic symmetries. This implies that all covariant quantities can be decomposed into a sum of terms, each of which has definite scaling. For example, the radial canonical momentum π^k of scalar ψ^k has the expansion

$$\pi^k = \pi_{(2-k)}^k + \pi_{(1-k)}^k + \dots + \pi_{(k)}^k + \tilde{\pi}_{(k)}^k \log z^2 + \dots, \quad (3.2)$$

where the various terms have the dilatation weight indicated by the subscript,³ i.e. π_n^k has scaling weight n . These coefficients are in one to one correspondence with the asymptotic coefficients in (3.1) with the exact relation being in general non-linear. The advantage of working with dilatation eigenvalues, rather than with asymptotic coefficients is that the former are manifestly covariant while the latter in general are not: the asymptotic expansion (3.1) singles out one coordinate so it is not covariant. Holographic 1-point functions can be expressed most compactly in terms of eigenfunctions of the

³ Note that that the coefficient $\pi_{(k)}^k$ has an anomalous scaling transformation, $\delta \pi_{(k)}^k = -k \pi_{(k)}^k - 2 \tilde{\pi}_{(k)}^k$, due to conformal anomalies, see [39].

dilatation operator, and this explains the non-linearities found in explicit computations of 1-point functions. In particular, if one considers the case of a single scalar Ψ^k in an asymptotically (locally) AdS spacetime, the 1-point function is simply given by

$$\langle O^k \rangle = \pi_{(k)}^k. \quad (3.3)$$

Thus, we indeed see that source and the renormalized 1-point are conjugate variables, and one may consider them as coordinates in the classical phase space of the theory.

3.2. Kaluza–Klein holography

The discussion in the previous section involved a $(d + 1)$ dimensional supergravity theory which admits an AdS_{d+1} solution. However, the string theory backgrounds of interest which contain an AdS factor typically also involve a compact space, for example, $\text{AdS}_5 \times S^5$, $\text{AdS}_3 \times S^3 \times X_4$ etc. On general grounds, one expects that there is an effective $(d + 1)$ dimensional description. Thus provided one can obtain such a description, one can use holographic renormalization to obtain the 1-point functions of gauge invariant operators. The method of Kaluza–Klein holography provides an explicit algorithm for constructing the corresponding $(d + 1)$ dimensional action and extracting the vevs.

Note that generically the spheres appearing in these solutions have a radius which is of the same order as the AdS radius, so the higher KK modes are not suppressed relative to the zero modes and one cannot ignore them. In some cases it is nevertheless possible to only keep a subset of modes, because the equations of motion admit a solution with all modes, except the ones kept, set equal to zero, i.e. there exists a “consistent truncation”. The existence of such a truncation signifies the existence of a subset of operators of the dual theory that are closed under OPEs. The resulting theory is a $(d + 1)$ -dimensional gauged supergravity and such gauged supergravity theories have been the starting point for many investigations in AdS/CFT.

However, starting from a lower dimensional gauged supergravity is unsatisfactory on many grounds. Firstly, gauged supergravity captures the physics of only a very small subset of operators, typically that of the stress tensor supermultiplet. An infinite number of other operators, namely the ones dual to KK modes, are in principle accessible within the low energy limit, but one excludes them a priori. Secondly, the higher dimensional solutions are more fundamental as reduction of regular solutions may result in singular solutions. Thirdly, even in the case where a consistent truncation to a lower dimensional supergravity is possible, it is often very difficult, or indeed unknown, how to express known interesting higher dimensional solutions in a non-linear reduction ansatz that produces a corresponding solution of the gauged supergravity. Finally, a key conceptual question for holography is how the compact part of the geometry is encoded in QFT data and answering this question requires keeping all modes.

For these reasons, we will keep all KK modes in the reduction (so there is also never an issue of consistency). Naively, this results in an intractable problem of an infinite number of fields all coupled together. Recall however that the 1-point functions are extracted from the asymptotic expansion near the AdS boundary, and the fall off of the fields near the AdS boundary is fixed by their mass. This implies that to compute the holographic 1-point function of any given operator only a finite number of fields and a finite number of interactions are relevant. The method of Kaluza–Klein holography systematically constructs the lower dimensional action in a way that only the fields and interactions needed are kept at each step.

The steps involved in this construction are the following. The starting point is a D dimensional action that admits an $\text{AdS}_{d+1} \times X^q$ solution. We further assume that the harmonic analysis on the compact X^q is known, i.e. the set of spherical harmonics is known. This clearly is the case for $X^q = S^q$. In the first step, we consider fluctuations around this solution and expand the fluctuations in the harmonics of the compact space. Let ϕ denote collectively all fields, ϕ_o be the $\text{AdS}_{d+1} \times X^q$ solution and $\delta\phi$ the fluctuation, then schematically,

$$\begin{aligned} \phi(x, y) &= \phi_o(x, y) + \delta\phi(x, y) \\ \delta\phi(x, y) &= \sum_I \psi^I(x) Y^I(y) \end{aligned} \quad (3.4)$$

where x is a coordinate in the $(d + 1)$ non-compact directions, y is a coordinate in the compact directions and Y^I denotes collectively all spherical harmonics (scalar, vector, tensor and their covariant derivatives). Precise formulae for the case of interest will be presented in the next subsection.

The decomposition (3.4) is not unique because there are coordinate transformations,

$$X^{M'} = X^M - \xi^M(x, y) \quad (3.5)$$

where $X^M = \{x, y\}$ that transform the fluctuations ψ^I to each over or the background solution ϕ_o . In the supergravity literature on KK reduction one often imposes a gauge condition, most notably the de Donder gauge condition, to eliminate this issue. We instead construct gauge invariant combinations that have the property that in the de Donder gauge they coincide with gauge fixed variables. This is done by working out perturbatively in the number of fluctuations how each ψ^I transforms under (3.5) and then one constructs combinations $\hat{\psi}^I$ that transform as tensors, i.e. scalar combinations are

invariant under (3.5), $(d + 1)$ -dimensional vector combinations $A_\mu(x)$, transform as vectors, the metric $g_{\mu\nu}$ transforms as metric etc. Details of this procedure can be found in [42].

The aim is now to derive the equations that the gauge invariant modes $\hat{\psi}^I$ satisfy by substituting (3.4) in the D dimensional equations and working perturbatively in the number of fields. To linear order one obtains the spectrum, to quadratic order the cubic interactions etc. Note however that not all cubic (or higher) interactions are relevant for the computation of the 1-point function of any given operator. Only the ones that could modify the asymptotic coefficients that determine the vev need to be retained.

Expanding perturbatively in fluctuations one finds

$$\mathcal{L}_I \hat{\psi}^I = \mathcal{L}_{I\mathcal{J}\mathcal{K}} \hat{\psi}^{\mathcal{J}} \hat{\psi}^{\mathcal{K}} + \mathcal{L}_{I\mathcal{J}\mathcal{K}\mathcal{L}} \hat{\psi}^{\mathcal{J}} \hat{\psi}^{\mathcal{K}} \hat{\psi}^{\mathcal{L}} + \dots, \quad (3.6)$$

where $\mathcal{L}_{I\mathcal{J}\dots\mathcal{J}_n}$ is an appropriate differential operator. $\mathcal{L}_{I\mathcal{J}\dots\mathcal{J}_n}$ involves higher derivative terms and the set of field equations cannot generically be integrated into an action. However, one can always define $(d + 1)$ -dimensional fields ψ^I by a non-linear Kaluza–Klein reduction map of the fields $\hat{\psi}^I$:

$$\psi^I = \hat{\psi}^I + \mathcal{K}_{\mathcal{J}\mathcal{K}}^I \hat{\psi}^{\mathcal{J}} \hat{\psi}^{\mathcal{K}} + \dots, \quad (3.7)$$

where $\mathcal{K}_{\mathcal{J}\mathcal{K}}^I$ contains appropriate derivatives. The reduction map is such that the fields ψ^I do satisfy field equations which can be integrated into an action. Given this $(d + 1)$ -dimensional action, it is then straightforward to obtain the one point functions of operators in terms of the asymptotics of the fields ψ^I , using the well-developed techniques of holographic renormalization.

This results in the following general formula for an operator of dimension k

$$\langle O_k^I \rangle = \pi_{(k)}^I + \sum_{\mathcal{J}\mathcal{K}} a_{\mathcal{J}\mathcal{K}}^I \pi_{(k_1)}^{\mathcal{J}} \pi_{(k-k_1)}^{\mathcal{K}} + \dots \quad (3.8)$$

where $a_{\mathcal{J}\mathcal{K}}^I$ are numerical constants and the ellipses indicate higher powers of canonical momenta. The non-linear terms are related to extremal correlators, i.e. these terms are possible when the theory contains operators with dimension k_i , such that $\sum k_i = k$, and the numerical constants $a_{\mathcal{J}\mathcal{K}}^I$ are related to the extremal 3-point functions at the conformal point. Note that these formulae for the vevs apply to any solution of the same action. Given any solution, one can evaluate them to find the QFT data encoded by this solution.

To summarize, the radial canonical momenta are related (in general non-linearly) to the coefficients in the asymptotic expansion of the $(d + 1)$ -dimensional fields Ψ . These fields, in turn, are related to the D -dimensional coefficients ψ via the non-linear Kaluza–Klein map (3.7), which relates Ψ to the gauge invariant version $\hat{\psi}$ of ψ , which itself is a non-linearly related to ψ 's. One can now combine all these maps to produce a final formula for the vevs which is of the schematic form

$$\langle O_k^I(\vec{x}) \rangle = [\psi^I(\vec{x})]_k + \sum_{\mathcal{J}\mathcal{K}} b_{\mathcal{J}\mathcal{K}}^I [\psi^{\mathcal{J}}(\vec{x})]_{k_1} [\psi^{\mathcal{K}}(\vec{x})]_{(k-k_1)} + \dots \quad (3.9)$$

where $b_{\mathcal{J}\mathcal{K}}^I$ are numerical coefficients, \vec{x} are d -dimensional (boundary) coordinates, z is the Fefferman–Graham radial coordinates and the coefficients $[\psi^I]_k$ are asymptotic coefficients in

$$\delta\phi(z, \vec{x}, y) = \sum_{l,m} [\psi^I(\vec{x})]_m z^m Y^I(y). \quad (3.10)$$

Thus, if we are interested in extracting the vevs of gauge invariant operators from a given solution that is asymptotically $\text{AdS}_{d+1} \times X^q$, the procedure is to write it as the deviation from $\text{AdS}_{d+1} \times X^q$ in the form (3.10), extract the coefficients $[\psi^I(\vec{x})]_m$ and insert those in (3.9). This procedure was carried out for asymptotically $\text{AdS}_5 \times S^5$ solutions describing the Coulomb branch of $\mathcal{N} = 4$ SYM in [42,43] and for the LLM solutions in [44], resulting in strong tests of gravity/gauge theory duality away from the conformal point and for asymptotically $\text{AdS}_3 \times S^3$ solutions relevant for the fuzzball program in [21, 22]. In the next section, we present the detailed results for the case of interest.

3.3. Asymptotically $\text{AdS}_3 \times S^3$ solutions

In what follows, we will be interested in black hole and fuzzball solutions whose decoupling regions are asymptotic to $\text{AdS}_3 \times S^3 \times X_4$, where X_4 is T^4 or $K3$. Therefore one can use AdS/CFT methods to extract holographic data from these geometries and, in particular, the asymptotics of the six-dimensional solutions near the $\text{AdS}_3 \times S^3$ boundary encode the vevs of chiral primary operators in the dual field theory.

In the solutions of interest, only zero modes of the compact space X_4 are excited, so it is convenient to first compactify the solution over X_4 . Solutions of type IIB supergravity compactified on $K3$ give rise to solutions of $d = 6$, $N = 4b$ supergravity coupled to 21 tensor multiplets, constructed by Romans in [45]. Corresponding fuzzball solutions of type IIB on T^4 can also be expressed as solutions of $d = 6$, $N = 4b$ coupled to 5 tensor multiplets. These theories admit an $\text{AdS}_3 \times S^3$ solution, so

the program of Kaluza–Klein holography can be applied to obtain an effective three dimensional description of all relevant KK modes.

The bosonic field content of the $d = 6N = 4b$ theory with n_t tensor multiplets, is the graviton g_{MN} , 5 self-dual and n_t anti-self dual tensor fields and an $O(5, n_t)$ matrix of scalars \mathcal{M} which can be written in terms of a vielbein $\mathcal{M}^{-1} = V^T V$. Following the notation of [46] the bosonic field equations may be written as

$$\begin{aligned} R_{MN} &= 2P_M^{nr} P_N^{nr} + H_{MPQ}^n H_N^{PQ} + H_{MPQ}^r H_N^{PQ}, \\ \nabla^M P_M^{nr} &= Q^{Mnm} P_M^{nr} + Q^{Mrs} P_M^{ns} + \frac{\sqrt{2}}{3} H^{nMNP} H_{MNP}^r, \end{aligned} \quad (3.11)$$

along with Hodge duality conditions on the 3-forms

$$*_6 H_3^n = H_3^n, \quad *_6 H_3^r = -H_3^r. \quad (3.12)$$

In these equations (m, n) are $SO(5)$ vector indices running from 1 to 5, whilst (r, s) are $SO(n_t)$ vector indices running from 6 to $(5 + n_t)$. The 3-form field strengths are given by

$$H^n = G^A V_A^n; \quad H^r = G^A V_A^r, \quad (3.13)$$

where $A \equiv \{n, r\} = 1, \dots, (5 + n_t)$; $G^A = db^A$ are closed and the vielbein on the coset space $SO(5, n_t)/(SO(5) \times SO(n_t))$ satisfies

$$V^T \eta V = \eta, \quad V = \begin{pmatrix} V_A^n \\ V_A^r \end{pmatrix}, \quad \eta = \begin{pmatrix} I_5 & 0 \\ 0 & -I_{21} \end{pmatrix}. \quad (3.14)$$

The associated connection is

$$dV V^{-1} = \begin{pmatrix} Q^{mn} & \sqrt{2} P^{ms} \\ \sqrt{2} P^{rm} & Q^{rs} \end{pmatrix}, \quad (3.15)$$

where Q^{mn} and Q^{rs} are antisymmetric, and the off-diagonal block matrices P^{ms} and P^{rm} are transposed to each other. Q^{mn} and Q^{rs} are composite $SO(5)$ and $SO(n_t)$ connections which are solvable in terms of $5n_t$ physical scalars ϕ^{mr} via the Cartan–Maurer equation.

The six-dimensional field equation (3.11) admit an $AdS_3 \times S^3$ solution, such that

$$\begin{aligned} ds_6^2 &= \sqrt{Q_1 Q_5} \left(\frac{1}{z^2} (-dt^2 + dy^2 + dz^2) + d\Omega_3^2 \right); \\ G^5 &= H^5 \equiv G^{o5} = \sqrt{Q_1 Q_5} \left(\frac{dz}{z^3} \wedge dt \wedge dy + d\Omega_3 \right), \end{aligned} \quad (3.16)$$

with the vielbein being diagonal, and all other three forms (both self-dual and anti-self dual) vanishing. In what follows it is convenient to absorb the curvature radius $\sqrt{Q_1 Q_5}$ into an overall prefactor in the action, and work with the unit radius $AdS_3 \times S^3$. Perturbations of six-dimensional supergravity fields relative to an $AdS_3 \times S^3$ background may be expressed as

$$\begin{aligned} g_{MN} &= g_{MN}^o + h_{MN}; \quad G^A = G^{oA} + g^A; \\ V_A^n &= \delta_A^n + \phi^{nr} \delta_A^r + \frac{1}{2} \phi^{nr} \phi^{mr} \delta_A^m; \\ V_A^r &= \delta_A^r + \phi^{nr} \delta_A^n + \frac{1}{2} \phi^{nr} \phi^{ns} \delta_A^s. \end{aligned} \quad (3.17)$$

These fluctuations can then be expanded in a basis of spherical harmonics as follows:

$$\begin{aligned} h_{\mu\nu} &= \sum h_{\mu\nu}^I(x) Y^I(y), \\ h_{\mu a} &= \sum (h_{\mu}^{I\nu}(x) Y_a^{I\nu}(y) + h_{(s)\mu}^I(x) D_a Y^I(y)), \\ h_{(ab)} &= \sum (\rho^{Ia}(x) Y_{(ab)}^{Ia}(y) + \rho_{(v)}^{Ib}(x) D_a Y_b^{Iv}(y) + \rho_{(s)}^I(x) D_{(a} D_{b)} Y^I(y)), \\ h_a^a &= \sum \pi^I(x) Y^I(y), \\ g_{\mu\nu\rho}^A &= \sum 3D_{[\mu} b_{\nu\rho]}^{(A)I}(x) Y^I(y), \\ g_{\mu\nu a}^A &= \sum (b_{\mu\nu}^{(A)I}(x) D_a Y^I(y) + 2D_{[\mu} Z_{\nu]}^{(A)Iv}(x) Y_a^{Iv}(y)); \\ g_{\mu ab}^A &= \sum (D_\mu U^{(A)I}(x) \epsilon_{abc} D^c Y^I(y) + 2Z_\mu^{(A)Iv} D_{[b} Y_{a]}^{Iv}(y)); \\ g_{abc}^A &= \sum (-\epsilon_{abc} \Lambda^I U^{(A)I}(x) Y^I(y)); \\ \phi^{mr} &= \sum \phi^{(mr)I}(x) Y^I(y). \end{aligned} \quad (3.18)$$

Here (μ, ν) are AdS indices and (a, b) are S^3 indices, with x denoting AdS coordinates and y denoting sphere coordinates. Λ^I is defined in (C.1). The subscript (ab) denotes symmetrization of indices a and b with the trace removed. Relevant properties of the spherical harmonics are reviewed in Appendix C. We will often use a notation where we replace the index I by the degree of the harmonic k or by a pair of indices (k, I) where k is the degree of the harmonic and I now parameterizes their degeneracy, and similarly for I_v, I_t .

Imposing the de Donder gauge condition $D^A h_{aM} = 0$ on the metric fluctuations removes the fields with subscripts (s, ν) . In deriving the spectrum and computing correlation functions, this is therefore a convenient choice. The de Donder gauge choice is, however, not always a convenient choice for the asymptotic expansion of solutions; indeed the natural coordinate choice in our applications takes us outside de Donder gauge. As discussed in [42] this issue is straightforwardly dealt with by working with gauge invariant combinations of the fluctuations.

The linearized spectrum of the fluctuations was derived in [46], with the cubic interactions obtained in [47]. Let us briefly review the linearized spectrum derived in [46], focusing on fields dual to chiral primaries. Consider first the scalars. It is useful to introduce the following combinations which diagonalize the linearized equations of motion:

$$\begin{aligned} s_I^{(r)k} &= \frac{1}{4(k+1)} (\phi_I^{(5r)k} + 2(k+2)U_I^{(r)k}), \\ \sigma_I^k &= \frac{1}{12(k+1)} (6(k+2)\hat{U}_I^{(5)k} - \hat{\pi}_I^k). \end{aligned} \quad (3.19)$$

The fields $s^{(r)k}$ and σ^k correspond to scalar chiral primaries, with the masses of the scalar fields being

$$m_{s^{(r)k}}^2 = m_{\sigma^k}^2 = k(k-2). \quad (3.20)$$

The index r spans $6 \cdots 5 + n_t$ with $n_t = 5, 21$ respectively for T^4 and $K3$. Note also that $k \geq 1$ for $s^{(r)k}$; $k \geq 2$ for σ^k . The hats $(\hat{U}_I^{(5)k}, \hat{\pi}_I^k)$ denote the following. As discussed in [42], the equations of motion for the gauge invariant fields are precisely the same as those in de Donder gauge, provided one replaces all fields with the corresponding gauge invariant field. The hat thus denotes the appropriate gauge invariant field, which reduces to the de Donder gauge field when one sets to zero all fields with subscripts (s, ν) . For our purposes we will need these gauge invariant quantities only to leading order in the fluctuations, with the appropriate combinations being

$$\begin{aligned} \hat{\pi}_2^I &= \pi_2^I + \Lambda^2 \rho_{2(s)}^I; \\ \hat{U}_2^{(5)I} &= U_2^{(5)I} - \frac{1}{2} \rho_{2(s)}^I; \\ \hat{h}_{\mu\nu}^0 &= h_{\mu\nu}^0 - \sum_{\alpha, \pm} h_\mu^{1\pm\alpha} h_\nu^{1\pm\alpha}. \end{aligned} \quad (3.21)$$

Next consider the vector fields. It is useful to introduce the following combinations which diagonalize the equations of motion:

$$h_{\mu I_v}^\pm = \frac{1}{2} (C_{\mu I_v}^\pm - A_{\mu I_v}^\pm), \quad Z_{\mu I_v}^{(5)\pm} = \pm \frac{1}{4} (C_{\mu I_v}^\pm + A_{\mu I_v}^\pm). \quad (3.22)$$

For general k the equations of motion are Proca–Chern–Simons equations which couple (A_μ^\pm, C_μ^\pm) via a first order constraint [46]. The three dynamical fields at each degree k have masses $(k-1, k+1, k+3)$, corresponding to dual operators of dimensions $(k, k+2, k+4)$ respectively; the operators of dimension k are vector chiral primaries. The lowest dimension operators are the R symmetry currents, which couple to the $k = 1 A_\mu^{\pm\alpha}$ bulk fields. The latter satisfy the Chern–Simons equation

$$F_{\mu\nu}(A^{\pm\alpha}) = 0, \quad (3.23)$$

where $F_{\mu\nu}(A^{\pm\alpha})$ is the curvature of the connection, and the index $\alpha = 1, 2, 3$ is an $SU(2)$ adjoint index. We will here only discuss the vevs of these vector chiral primaries.

Finally there is a tower of KK gravitons with $m^2 = k(k+2)$, but only the massless graviton, dual to the stress energy tensor, will play a role here. Note that it is the combination $\hat{H}_{\mu\nu} = \hat{h}_{\mu\nu}^0 + \pi^0 g_{\mu\nu}^0$ which satisfies the Einstein equation; moreover one needs the appropriate gauge covariant combination $\hat{h}_{\mu\nu}^0$ given in (3.21).

3.4. Vevs of gauge invariant operators

Let us now give the expressions for the vevs of gauge invariant operators up to dimension two. These are expressed in terms of coefficients in the asymptotic expansions of the fields near the conformal boundary at $z = 0$.

Let us denote by $(\mathcal{O}_{S_I^{(r)k}}, \mathcal{O}_{\Sigma_I^k})$ the chiral primary operators dual to the fields $(s_I^{(r)k}, \sigma_I^k)$ respectively. The vevs of the scalar operators with dimension two or less can then be expressed in terms of the coefficients in the asymptotic expansion as

$$\begin{aligned} \langle \mathcal{O}_{S_i^{(r)1}} \rangle &= \frac{2N}{\pi} \sqrt{2} [s_i^{(r)1}]_1; & \langle \mathcal{O}_{S_i^{(r)2}} \rangle &= \frac{2N}{\pi} \sqrt{6} [s_i^{(r)2}]_2; \\ \langle \mathcal{O}_{\Sigma_i^2} \rangle &= \frac{N}{\pi} \left(2\sqrt{2} [\sigma_i^2]_2 - \frac{1}{3} \sqrt{2} a_{ij} \sum_r [s_i^{(r)1}]_1 [s_j^{(r)1}]_1 \right). \end{aligned} \quad (3.24)$$

Here $[\psi]_n$ denotes the coefficient of the z^n term in the asymptotic expansion of the field ψ . The coefficient a_{ij} refers to the triple overlap between spherical harmonics, defined in (C.5). Note that dimension one scalar spherical harmonics have degeneracy four, and are thus labeled by $i = 1, \dots, 4$. In deriving these vevs, one needs to follow the steps outlined in the previous section, in particular making use of the cubic coupling and Kaluza–Klein reduction formulae obtained in [47].

Now consider the stress energy tensor and the R symmetry currents. The three dimensional metric, which is given by $g_{\mu\nu}^o + \hat{H}_{\mu\nu}$, and the Chern–Simons gauge fields admit the following asymptotic expansions:

$$\begin{aligned} ds_3^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2} \left(g_{(0)\bar{\mu}\bar{\nu}} + z^2 \left(g_{(2)\bar{\mu}\bar{\nu}} + \log(z^2) h_{(2)\bar{\mu}\bar{\nu}} + (\log(z^2))^2 \tilde{h}_{(2)\bar{\mu}\bar{\nu}} \right) + \dots \right) dx^{\bar{\mu}} dx^{\bar{\nu}}; \\ A^{\pm\alpha} &= \mathcal{A}^{\pm\alpha} + z^2 A_{(2)}^{\pm\alpha} + \dots \end{aligned} \quad (3.25)$$

The vevs of the R symmetry currents $J_u^{\pm\alpha}$ are then given in terms of terms in the asymptotic expansion of $A_{\bar{\mu}}^{\pm\alpha}$ as

$$\langle J_{\bar{\mu}}^{\pm\alpha} \rangle = \frac{N}{4\pi} \left(g_{(0)\bar{\mu}\bar{\nu}} \pm \epsilon_{\bar{\mu}\bar{\nu}} \right) \mathcal{A}^{\pm\alpha\bar{\nu}}, \quad (3.26)$$

where $\epsilon_{\bar{\mu}\bar{\nu}}$ is the constant antisymmetric tensor such that $\epsilon_{01} = 1$. The vev of the stress energy tensor $T_{\bar{\mu}\bar{\nu}}$ is given by

$$\langle T_{\bar{\mu}\bar{\nu}} \rangle = \frac{N}{2\pi} \left(g_{(2)\bar{\mu}\bar{\nu}} + \frac{1}{2} R g_{(0)\bar{\mu}\bar{\nu}} + 8 \sum_r [\tilde{s}_i^{(r)1}]_1^2 g_{(0)\bar{\mu}\bar{\nu}} + \frac{1}{4} (\mathcal{A}_{(\bar{\mu}}^{+\alpha} \mathcal{A}_{\bar{\nu})}^{+\alpha} + \mathcal{A}_{(\bar{\mu}}^{-\alpha} \mathcal{A}_{\bar{\nu})}^{-\alpha}) \right) \quad (3.27)$$

where parentheses denote the symmetrized traceless combination of indices.

This summarizes the expressions for the vevs of chiral primaries with dimension two or less which were derived in [21]. Note that these operators correspond to supergravity fields, which are at the bottom of each Kaluza–Klein tower. The supergravity solution of course also captures the vevs of operators dual to the other fields in each tower. Expressions for these vevs involving all contributing non-linear terms were not explicitly derived in [21]: in general the vev of a dimension p operator will include contributions from terms involving up to p supergravity fields. Computing these in turn requires the field equations (along with gauge invariant combinations, KK reduction maps etc) up to p th order in the fluctuations.

Note that using only the linear term in a vev generically gives qualitatively the wrong answer. For example, when supersymmetry is unbroken, only operators which are the bottom components of supermultiplets can acquire vevs. Evaluating only the linearized expressions for the vevs of operators which are not bottom components of a supermultiplet in a supersymmetric background however usually gives a non-zero answer, which cannot be correct. In [48] linearized expressions for holographic vevs of operators were evaluated in the decoupling $\text{AdS}_3 \times S^3 \times X_4$ limit of the 2-charge supersymmetric black ring background. Non-zero vevs were found for the operators $\mathcal{O}_{\rho^{lt}}$ dual to the scalars ρ^{lt} given in (3.18). However, these operators (for dimension $\Delta \geq 3$) sit in the middle component of the spin 2 supermultiplet whose lowest component is a vector chiral primary (as can be inferred from Table 3 and Figure 1 of [46]) and cannot acquire a vev in a supersymmetric state. Thus the linearized expressions indeed give qualitatively wrong answers in this case.

Given an asymptotically AdS supergravity solution, one can extract the vevs, and the deformation parameters, by purely algebraic manipulations. This information characterizes the state of the dual theory and is the natural information to extract first. Higher point functions can be extracted by extending the holographic one-point functions to include sources, and then solving fluctuation equations in the geometry to sufficient order in the sources.

4. Two charge system

4.1. Naive geometry and D-branes

The 2-charge supersymmetric D1–D5 supergravity solution is

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{h_1 h_5}} (-dt^2 + dz^2) + \sqrt{h_1 h_5} dx^m dx^m + \sqrt{\frac{h_1}{h_5}} ds^2(X_4); \\ C &= (h_1^{-1} - 1) dt \wedge dz + *_4 dh_5, \quad e^{-2\Phi} = \frac{h_5}{h_1}. \end{aligned} \quad (4.1)$$

Here $ds^2(X_4)$ denotes the metric on T^4 or $K3$ respectively and the functions (h_1, h_5) are harmonic on R^4 , which has coordinates x^m . The notation $*_4$ denotes the Poincaré dual in the flat R^4 metric. Our conventions for the supergravity field

equations are given in [Appendix A](#). The solution describes D5-branes wrapping X_4 and intersecting D1-branes over the string directions (t, z) , with z being a periodic coordinate⁴ with radius R_z . Consider now single-centered solutions, such that $h_i = (1 + Q_i/r^2)$; the charges Q_i are related to the integral charges N_i as

$$Q_1 = \frac{(\alpha')^3 N_1}{V}; \quad Q_5 = \alpha' N_5, \quad (4.2)$$

where we have set the string coupling $g = 1$ and the volume v of X_4 is $(2\pi)^4 V$. Reducing over X_4 and z leads to the five dimensional metric

$$ds^2 = \lambda^{-2/3} dt^2 + \lambda^{1/3} (dr^2 + r^2 d\Omega_3^2) \quad (4.3)$$

where

$$\lambda = h_1 h_5 = \left(1 + \frac{Q_1}{r^2}\right) \left(1 + \frac{Q_5}{r^2}\right). \quad (4.4)$$

This is an asymptotically flat spacetime which has a naked singularity at $r = 0$: the Ricci scalar is given by

$$R = -\frac{1}{6} \left(\frac{2\partial_r^2 \lambda}{\lambda^{4/3}} + \frac{6\partial_r \lambda}{r\lambda^{4/3}} - \frac{(\partial_r \lambda)^2}{\lambda^{7/3}} \right), \quad (4.5)$$

and thus for $\lambda \sim r^{-2p}$ near $r = 0$

$$R = \frac{4}{3} p(p+1) r^{2p/3-2}. \quad (4.6)$$

In the 2-charge case, $p = 2$ and the curvature diverges at $r = 0$. As we shall see when we discuss the 3-charge case, the corresponding 5d metric is of exactly the same form but in that case λ contains the product of 3-harmonic functions. Thus for this case $p = 3$ and the curvature is finite at $r = 0$: there is an extremal horizon rather than a naked singularity at $r = 0$.

Consider now the decoupling limit:

$$\alpha' \rightarrow 0, \quad \rho = \frac{r}{\alpha'} \text{ fixed}, \quad R_z, \frac{v}{\alpha'^2} \text{ fixed}. \quad (4.7)$$

In this limit the constant in the harmonic functions drops out and one focuses on the region near $r = 0$. After rescaling the coordinates as

$$\rho \rightarrow \rho \frac{R_z}{\sqrt{Q_1 Q_5}}, \quad z \rightarrow \frac{z}{R_z}, \quad t \rightarrow \frac{t}{R_z} \quad (4.8)$$

the metric becomes

$$ds^2 = \sqrt{Q_1 Q_5} \left(\rho^2 (-dt^2 + dz^2) + \frac{d\rho^2}{\rho^2} + d\Omega_3^2 \right) + \sqrt{\frac{Q_1}{Q_5}} ds^2(X_4), \quad (4.9)$$

where $d\Omega_3^2$ is the metric on a unit 3-sphere and z has period 2π (and as usual we have set $\alpha' = 1$ after the limit was taken). The six-dimensional metric is thus the product of a locally AdS_3 spacetime and a 3-sphere, with AdS and sphere radii both equal to $(Q_1 Q_5)^{1/4}$. Had the z coordinate been non-compact, the non-compact part would have been AdS_3 in Poincaré coordinates and the apparent horizon at $\rho = 0$ would have merely been a coordinate horizon, which can be removed by introducing global coordinates for AdS_3 . Since z is compact, however, the three dimensional non-compact spacetime is instead the zero mass BTZ black hole. The spacetime (4.9) does not have a curvature singularity, but there are geodesics that terminate at $r = 0$. Extremal BTZ black holes have a horizon radius proportional to their mass and a singularity at $r = 0$. In the massless limit, the position of the horizon coincides with the singularity, so we end up with a naked singularity. In the 3-charge case one instead obtains a massive extremal BTZ black hole with horizon radius proportional to the new charge. The area of the horizon is proportional to the horizon radius and thus vanishes for the 2-charge system. Note that reducing the decoupled region over X^4 and z leads to a spacetime with a curvature singularity, explaining the singularity structure of (4.3), but this is due to the fact that we reduce over a circle of vanishing proper length.

Thus the 2-charge black hole does not have a macroscopic horizon within supergravity. As we will momentarily review, however, the entropy of the 2-charge system is $S \sim \sqrt{N_1 N_5}$ and one would expect that the D1–D5 solution with higher derivative corrections taken into account should have a horizon which accounts for this entropy. In this section we will see that supergravity fuzzball solutions for this case can be explicitly constructed, and matched with CFT microstates. However, the average fuzzball in this system is necessarily string scale, and thus the system can only provide an indicative toy example for the proposal as a whole.

⁴ Note we called z the Fefferman–Graham radial coordinate in the previous section. It should be clear from the context whether z is a Fefferman–Graham coordinate or the compact S^1 coordinate that the D1 and D5 brane wraps. We hope this will not cause any confusion.

4.2. CFT microstates

The dual CFT describing the low energy physics of the D1–D5 system is believed to be a deformation of the $\mathcal{N} = (4, 4)$ supersymmetric sigma model with target space $S^N(X_4)$, where $N = N_1 N_5$ and the compact space X_4 is either T^4 or $K3$, see [49] for a review. Most of the discussion below will be for the case of T^4 , although the results extend simply to $K3$. The orbifold point is roughly the analogue of the free field limit of $\mathcal{N} = 4$ SYM in the context of $\text{AdS}_5/\text{CFT}_4$ duality.

The Hilbert space of the orbifold theory decomposes into twisted sectors, which are labeled by the conjugacy classes of the symmetric group $S(N)$, the latter consisting of cyclic groups of various lengths. The various conjugacy classes that occur and their multiplicity are subject to the constraint

$$\sum_i n_i m_i = N, \quad (4.10)$$

where n_i is the length of the cycle (or the twist) and m_i is the multiplicity of the cycle. As emphasized in [50] there is a direct correspondence between the combinatorial description of the conjugacy classes and a second quantized string theory, the long and short string picture advocated in [51,52].

The point is that one can view the supersymmetric sigma model on $S^N(X_4)$ as the transversal fluctuations of a string with target space $R \times S^1 \times S^N(X_4)$, in lightcone gauge, with the string winding once around S^1 . However, the partition function of this single string decomposes into several distinct topological sectors, corresponding to the number of ways the string can be disentangled into separate strings that wind one or more times around $X_4 \times S^1$. The factorization of the conjugacy classes into a product of irreducible cyclic permutations n_i determines the decomposition into several strings of winding number n_i . Whilst the string picture is intuitive and useful for qualitative features, one will need the explicit orbifold CFT description for computations and it is therefore the latter we will use.

The chiral primaries and R ground states can be precisely described at the orbifold point. In particular the R ground states are associated with the cohomology of the internal manifold, T^4 or $K3$, and therefore the corresponding chiral primaries in the NS sector obtained from spectral flow are also labeled by the cohomology. For our discussions, only the states associated with the even cohomology will be relevant; let the NS sector chiral primaries be labeled as $\mathcal{O}_n^{(p,q)}$ where n is the twist and (p, q) labels the associated cohomology class. The degeneracy of the operators associated with the $(1, 1)$ cohomology is $h^{1,1}$ of the compact manifold.

The complete set of chiral primaries associated with the even cohomology is then built from products of the form

$$\prod_l (\mathcal{O}_{n_l}^{(p_l, q_l)})^{m_l}, \quad \sum_l n_l m_l = N, \quad (4.11)$$

where symmetrization over the N copies of the CFT is implicit. Spectral flow maps these chiral primaries in the NS sector to R ground states, where

$$\begin{aligned} h^R &= h^{NS} - j_3^{NS} + \frac{c}{24}; \\ j_3^R &= j_3^{NS} - \frac{c}{12}, \end{aligned} \quad (4.12)$$

where c is the central charge of the CFT. Each of the operators in (4.11) is mapped by spectral flow to a (ground state) operator of definite R -charge

$$\begin{aligned} \prod_{l=1} (\mathcal{O}_{n_l}^{(p_l, q_l)})^{m_l} &\rightarrow \prod_{l=1} (\mathcal{O}_{n_l}^{R(p_l, q_l)})^{m_l}, \\ j_3^R &= \frac{1}{2} \sum_l (p_l - 1) m_l, \quad \bar{j}_3^R = \frac{1}{2} \sum_l (q_l - 1) m_l. \end{aligned} \quad (4.13)$$

Note that Ramond operators which are obtained from spectral flow of primaries associated with the $(1, 1)$ cohomology have zero R charge. Explicit free field representations of these operators are reviewed in [49] but these will not be needed in what follows.

The degeneracy of the Ramond ground states can be computed by counting the partitions of N into pairs of integers; for large N this results in a total degeneracy d such that

$$\log(d) \approx 2\pi \sqrt{\frac{C}{6} N}, \quad (4.14)$$

where $C = 12$ for T^4 and $C = 24$ for $K3$. (In counting the former one needs to include the odd dimensional cohomology.) Note that the degeneracies are clearly those of type IIB strings on T^4 and heterotic strings on T^4 respectively, with left moving excitation level N . It is also useful to note here that the distribution of ground states is highly localized around those with zero R charge: the degeneracy of ground states with zero R charge, d_0 , is given by [22], see also [53],

$$d_0 \approx d/N \quad (4.15)$$

for large N . Thus, as already mentioned, the 2-charge black hole has an entropy of order \sqrt{N} . By adding momentum one obtains a black hole with a regular horizon in supergravity; the corresponding microstates are left moving excited and right moving ground states, whose degeneracy for high excitation level can be computed via the Cardy formula. We will postpone detailed discussions of these states until Section 5.2.

The AdS/CFT dictionary relates the supergravity spectrum about the $\text{AdS}_3 \times S^3 \times X_4$ background to these chiral primary operators. In particular, in what follows we will make use of the relationship between (scalar) supergravity fields and chiral primaries. This map was established recently [54] by matching three point functions computed from supergravity with those computed in the orbifold CFT [55,56], assuming non-renormalization of the correlators. Such non-renormalization is supported by the matching of the correlators computed from the worldsheet theory with those computed in the orbifold CFT [57–59]. Labeling the orbifold operators by their twist and (j_3, \bar{j}_3) charges (m, \bar{m}) respectively, the correspondence is

$$\left(\begin{matrix} \mathcal{O}_{(n-1)m\bar{m}}^S \\ \mathcal{O}_{(n-1)m\bar{m}}^\Sigma \end{matrix} \right) \leftrightarrow \mathcal{M} \left(\begin{matrix} \mathcal{O}_{nm\bar{m}}^{0,0} \\ \mathcal{O}_{(n-2)m\bar{m}}^{2,2} \end{matrix} \right), \quad (4.16)$$

and

$$\mathcal{O}_{nm\bar{m}}^{(r)1,1} \leftrightarrow \mathcal{O}_{nm\bar{m}}^{S(r)}; \quad (4.17)$$

with the matrix \mathcal{M} being

$$\mathcal{M} = \frac{1}{\sqrt{2\Delta}} \begin{pmatrix} (\Delta+1)^{1/2} & -(\Delta-1)^{1/2} \\ (\Delta-1)^{1/2} & (\Delta+1)^{1/2} \end{pmatrix} \quad (4.18)$$

for $\Delta \geq 2$. Here $\mathcal{O}_{\Delta m\bar{m}}^\Phi$ denotes the operator of dimension Δ and R charges (m, \bar{m}) coupling to the bulk supergravity field Φ of corresponding mass and $SO(4)$ charges.

The formulae for the holographic vevs allow us to extract the expectation values of these operators, along with those of the stress energy tensor and R currents, from a given asymptotically $\text{AdS}_3 \times S^3 \times X_4$ supergravity solution. Given a proposed correspondence between the supergravity solution and a field theory state, one can use this data to test the correspondence. More generally n -point functions extracted from the bulk can also be compared to n -point functions in a given state, but there is no reason to anticipate that such correlation functions are non-renormalized.

4.3. Fuzzball solutions

The key observation in constructing fuzzball solutions for the 2-charge system is that D1–D5 is related by dualities to a fundamental string carrying momentum, and supergravity solutions for the latter have been known for more than a decade: the relevant solutions are general chiral null models. For the heterotic string, the general such model takes the form

$$\begin{aligned} ds^2 &= H^{-1}(-du dv + (K - 2\alpha' H^{-1} N^{(c)} N^{(c)}) dv^2 + 2A_I dx^I dv) + dx_I dx^I \\ B_{uv}^{(2)} &= \frac{1}{2}(H^{-1} - 1), \quad B_{vI}^{(2)} = H^{-1} A_I, \\ \Phi &= -\frac{1}{2} \ln H, \quad V_v^{(c)} = H^{-1} N^{(c)}, \end{aligned} \quad (4.19)$$

where $I = 1, \dots, 8$ labels the transverse directions and $V_m^{(c)}$ are Abelian gauge fields, with $((c) = 1, \dots, 16)$ labeling the elements of the Cartan of the gauge group. The equations of motion for the heterotic string are given in Appendix A; here the defining functions satisfy

$$\square H(x, v) = \square K(x, v) = \square A_I(x, v) = (\partial_I A^I(x, v) - \partial_v H(x, v)) = \square N^{(c)} = 0. \quad (4.20)$$

For the solution to correspond to a charged heterotic string with generic wave profile, one takes the following solutions

$$\begin{aligned} H &= 1 + \frac{Q}{|x - F(v)|^6}, \quad A_I = -\frac{Q \dot{F}_I(v)}{|x - F(v)|^6}, \quad N^{(c)} = \frac{q^{(c)}(v)}{|x - F(v)|^6}, \\ K &= \frac{Q^2 \dot{F}(v)^2 + 2\alpha' q^{(c)}(v) q^{(c)}(v)}{Q|x - F(v)|^6}, \end{aligned} \quad (4.21)$$

where $F^I(v)$ is an arbitrary null curve in R^8 ; $q^{(c)}(v)$ is an arbitrary charge wave, and $\dot{F}_I(v)$ denotes $\partial_v F_I(v)$. Such solutions were first discussed in [60,61], although the above has a more generic charge wave, lying in $U(1)^{16}$ rather than $U(1)$. Chiral null model solutions of type IIB are obtained by setting $N^{(c)} = 0$. More general solutions in which the fundamental strings carry condensates of fermion bilinears were considered in [62]; such solutions are necessary to account for all D1–D5 microstates, but for brevity we will not discuss them in detail here.

The F1-P solutions described by such chiral null models can be dualized to give corresponding solutions for the D1–D5 system as follows [11]. Compactify four of the transverse directions on a torus, such that x^i with $i = 1, \dots, 4$ are coordinates

on R^4 and x^ρ with $\rho = 5, \dots, 8$ are coordinates on T^4 . Then let $v = (t - z)$ and $u = (t + z)$ with the coordinate z being periodic with length $L_z \equiv 2\pi R_z$, and smear all harmonic functions over both this circle and over the T^4 , so that they satisfy

$$\square_{R^4} H(x) = \square_{R^4} K(x) = \square_{R^4} A_I(x) = \square_{R^4} N^{(c)} = 0, \quad \partial_t A^i = 0. \quad (4.22)$$

Thus the harmonic functions appropriate for describing strings with only bosonic condensates are [63]

$$\begin{aligned} H &= 1 + \frac{Q}{L_z} \int_0^{L_z} \frac{dv}{|x - F(v)|^2}; & A_I &= -\frac{Q}{L_z} \int_0^{L_z} \frac{dv \dot{F}_I(v)}{|x - F(v)|^2}; \\ N_{(c)} &= -\frac{Q}{L_z} \int_0^{L_z} \frac{dv q_c(v)}{|x - F(v)|^2}; & K &= \frac{Q}{L_z} \int_0^{L_z} \frac{dv (\dot{F}_i(v)^2 + \dot{F}_\rho(v)^2)}{|x - F(v)|^2}. \end{aligned} \quad (4.23)$$

Here $|x - F(v)|^2$ denotes $\sum_i (x_i - F_i(v))^2$.

Next, one follows an appropriate chain of dualities to obtain a D1–D5 solution. For D1–D5 on T^4 one starts with type IIB F1–P solutions on T^4 whilst for D1–D5 on K3 one starts with heterotic F1–P solutions and uses heterotic/IIA duality. The computational details of this duality chain are given in [22]. The final result in both cases can be written as

$$\begin{aligned} ds^2 &= \frac{f_1^{1/2}}{\tilde{f}_1 \tilde{f}_5} [-(dt - A_i dx^i)^2 + (dz - B_i dx^i)^2] + f_1^{1/2} f_5^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} ds^2(X_4), \\ e^{2\Phi} &= \frac{f_1^2}{f_5 \tilde{f}_1}, & B_{tz}^{(2)} &= \frac{\mathcal{A}}{f_5 \tilde{f}_1}, & B_{\tilde{\mu}i}^{(2)} &= \frac{\mathcal{A} \mathcal{B}_i^{\tilde{\mu}}}{f_5 \tilde{f}_1}, \\ B_{ij}^{(2)} &= \lambda_{ij} + \frac{2\mathcal{A} \mathcal{A}_{[i} B_{j]}}{f_5 \tilde{f}_1}, & B_{\rho\sigma}^{(2)} &= f_5^{-1} k^\gamma \omega_{\rho\sigma}^\gamma, & C^{(0)} &= -f_1^{-1} \mathcal{A}, \\ C_{tz}^{(2)} &= 1 - \tilde{f}_1^{-1}, & C_{\tilde{\mu}i}^{(2)} &= -\tilde{f}_1^{-1} \mathcal{B}_i^{\tilde{\mu}}, & C_{ij}^{(2)} &= c_{ij} - 2\tilde{f}_1^{-1} A_{[i} B_{j]}, \\ C_{tzij}^{(4)} &= \lambda_{ij} + \frac{\mathcal{A}}{f_5 \tilde{f}_1} (c_{ij} + 2A_{[i} B_{j]}), & C_{\tilde{\mu}ijk}^{(4)} &= \frac{3\mathcal{A}}{f_5 \tilde{f}_1} \mathcal{B}_{[i}^{\tilde{\mu}} c_{jk]}, \\ C_{tz\rho\sigma}^{(4)} &= f_5^{-1} k^\gamma \omega_{\rho\sigma}^\gamma, & C_{ij\rho\sigma}^{(4)} &= (\lambda_{ij}^\gamma + f_5^{-1} k^\gamma c_{ij}) \omega_{\rho\sigma}^\gamma, & C_{\rho\sigma\tau\pi}^{(4)} &= f_5^{-1} \mathcal{A} \epsilon_{\rho\sigma\tau\pi}, \end{aligned} \quad (4.24)$$

where we introduce a basis of self-dual and anti-self-dual 2-forms $\omega^\gamma \equiv (\omega^{\alpha+}, \omega^{\alpha-})$ with $\gamma = 1, \dots, b^2$ on the compact manifold X_4 . For both T^4 and K3 the self-dual forms are labeled by $\alpha_+ = 1, 2, 3$, whilst the anti-self-dual forms are labeled by $\alpha_- = 1, 2, 3$ for T^4 and $\alpha_- = 1, \dots, 19$ for K3. The intersections of these forms are defined by

$$d_\gamma \delta = \frac{1}{(2\pi)^4 V} \int_{X_4} \omega_2^\gamma \wedge \omega_2^\delta. \quad (4.25)$$

The solutions are expressed in terms of the following combinations of harmonic functions ($H, K, A_i, \mathcal{A}, \mathcal{A}^{\alpha-}$)

$$\begin{aligned} f_5 &= H; & \tilde{f}_1 &= 1 + K - H^{-1}(\mathcal{A}^2 + \mathcal{A}^{\alpha-} \mathcal{A}^{\alpha-}); & f_1 &= \tilde{f}_1 + H^{-1} \mathcal{A}^2; \\ k^\gamma &= (0_3, \sqrt{2} \mathcal{A}^{\alpha-}); & dB &= -*_4 d\mathcal{A}; & dc &= -*_4 df_5; \\ d\lambda^\gamma &= *_4 dk^\gamma; & d\lambda &= *_4 d\mathcal{A}; & \mathcal{B}_i^{\tilde{\mu}} &= (-B_i, A_i), \end{aligned} \quad (4.26)$$

where $\tilde{\mu} = (t, z)$ and the Hodge dual $*_4$ is defined over (flat) R^4 , with the Hodge dual in the Ricci flat metric on the compact manifold being denoted by $\epsilon_{\rho\sigma\tau\pi}$. The constant term in $C_{tz}^{(2)}$ is chosen so that the potential vanishes at asymptotically flat infinity.

We are interested in solutions for which the defining harmonic functions are given by

$$\begin{aligned} H &= 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|x - F(v)|^2}; & A_i &= -\frac{Q_5}{L} \int_0^L \frac{dv \dot{F}_i(v)}{|x - F(v)|^2}, \\ \mathcal{A} &= -\frac{Q_5}{L} \int_0^L \frac{dv \dot{\mathcal{F}}(v)}{|x - F(v)|^2}; & \mathcal{A}^{\alpha-} &= -\frac{Q_5}{L} \int_0^L \frac{dv \dot{\mathcal{F}}^{\alpha-}(v)}{|x - F(v)|^2}, \\ K &= \frac{Q_5}{L} \int_0^L \frac{dv (\dot{F}(v)^2 + \dot{\mathcal{F}}(v)^2 + \dot{\mathcal{F}}^{\alpha-}(v)^2)}{|x - F(v)|^2}. \end{aligned} \quad (4.27)$$

In these expressions Q_5 is the 5-brane charge and L is the length of the defining curve in the D1–D5 system, given by

$$L = 2\pi Q_5 / R_z, \quad (4.28)$$

where R_z is the radius of the z circle. Note that Q_5 has dimensions of length squared and is related to the integral charge via

$$Q_5 = \alpha' N_5 \quad (4.29)$$

(where g_s has been set to one). The D1-brane charge Q_1 is given by

$$Q_1 = \frac{Q_5}{L} \int_0^L dv (\dot{F}(v)^2 + \dot{\mathcal{F}}(v)^2 + \dot{\mathcal{F}}^{\alpha-}(v)^2), \quad (4.30)$$

and the corresponding integral charge is given by

$$Q_1 = \frac{N_1(\alpha')^3}{V}, \quad (4.31)$$

where $(2\pi)^4 V$ is the volume of the compact manifold.

Solutions with no internal excitations, that is $(\mathcal{F}(v), \mathcal{F}^{\alpha-}(v)) = 0$, were obtained in [11]. These Lunin-Mathur solutions involve only the graviton, dilaton and RR 2-form:

$$\begin{aligned} ds^2 &= \frac{1}{(f_1 f_5)^{1/2}} [-(dt - A_i dx^i)^2 + (dz - B_i dx^i)^2] + (f_1 f_5)^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} ds^2(X_4), \\ e^{2\phi} &= \frac{f_1}{f_5}, \\ C_{tz}^{(2)} &= 1 - f_1^{-1}, \quad C_{\tilde{\mu}i}^{(2)} = -f_1^{-1} \mathcal{B}_i^{\tilde{\mu}}, \quad C_{ij}^{(2)} = c_{ij} - 2f_1^{-1} A_{[i} B_{j]}, \end{aligned} \quad (4.32)$$

and are defined in terms of harmonic functions sourced on the curve $F^i(v)$ in R^4 :

$$f_5 = 1 + \frac{1}{L} \int_0^L \frac{Q_5 dv}{|x - F|^2}; \quad f_1 = 1 + \frac{1}{L} \int_0^L \frac{Q_5 (\partial_v F)^2 dv}{|x - F|^2}; \quad A = \frac{1}{L} \int_0^L \frac{Q_5 \partial_v F}{|x - F|^2}, \quad (4.33)$$

with $dB = -*_4 dA$. Note that corresponding 2-charge solutions carrying KK monopole charge and momentum which are related by dualities to these solutions were found in [64].

The mapping of the parameters from the original F1-P systems to the D1–D5 systems was discussed in [11]. The fact that the solutions take exactly the same form, regardless of whether the compact manifold is T^4 or $K3$, is unsurprising given that only zero modes of the compact manifold are excited. The solutions defined in terms of the harmonic functions (4.27) describe the complete set of two-charge fuzzballs for the D1–D5 system on $K3$. In the case of T^4 , these describe fuzzballs with only bosonic excitations; the most general solution would include fermionic excitations and thus more general harmonic functions of the type discussed in [62].

Given a generic fuzzball solution, one would first like to check whether the geometry is indeed smooth and horizon-free. There are no horizons, since the defining functions are positive definite. For the fuzzballs with no internal excitations, the question of singularity was discussed in [65], the conclusion being that the solutions are non-singular unless the defining curve $F^i(v)$ is non-generic and self-intersects. A generic fuzzball solution with internal excitations is also non-singular provided that the defining curve $F^i(v)$ does not self-intersect and $\tilde{F}_i(v)$ only has isolated zeroes [22].

One can illustrate that the fuzzballs are non-singular as follows [65]. The fuzzballs are defined in terms of harmonic functions, and potential singularities of the solutions are at the sources of these harmonic functions. Consider for simplicity, the case of fuzzballs with no internal excitations, for which the defining harmonic functions are sourced on a closed curve $F^i(v)$ in R^4 . In the neighborhood of a specific point on the curve, the harmonic function picks up contributions from a line of sources, and thus effectively behaves like a three-dimensional harmonic function. That is, parameterizing the R^4 as

$$ds_4^2 = dw^2 + dr^2 + r^2 d\Omega_2^2, \quad (4.34)$$

the direction w is aligned with the curve. The defining harmonic functions behave as

$$\int_{-\infty}^{\infty} \frac{dw}{w^2 + r^2} = \frac{\pi}{r}, \quad (4.35)$$

so that $f_a = Q_a/r$, whilst the defining one form behaves as

$$w = a \frac{dz}{r}. \quad (4.36)$$

Then the metric behaves as

$$\begin{aligned} ds^2 &= -\frac{2a}{\sqrt{Q_1 Q_5}} dw dt + \sqrt{Q_1 Q_5} \frac{dw^2}{r} \left(1 - \frac{a^2}{Q_1 Q_5}\right) \\ &\quad + \frac{r}{\sqrt{Q_1 Q_5}} (dz + a \cos \theta d\phi)^2 + \frac{\sqrt{Q_1 Q_5}}{r} (dr^2 + r^2 d\Omega_2^2) + \frac{\sqrt{Q_1}}{\sqrt{Q_5}} ds^2(X_4). \end{aligned} \quad (4.37)$$

When $a = \sqrt{Q_1 Q_5}$, which is exactly the condition that follows from the explicit forms of the full harmonic functions, this metric is the product of $R^{1,1} \times X_4 \times TN$ with TN Taub-Nut and thus the metric is non-singular at the curve location. We

should contrast this behavior with that of the naive solution (4.9). In the naive geometry, the circle S^1 parameterized by z is non-contractible but has vanishing proper length in the interior leading to singular behavior. Instead, in the fuzzball geometries the z circle is contractible and part of a Taub-NUT geometry, so the vanishing of its radius in the interior is akin to the behavior of polar coordinates near the origin. A generalization of this analysis shows that generic fuzzballs defined by non-intersecting curves are also non-singular [65,22].

Note that if there are no transverse excitations, $F^i(v) = 0$, the solution collapses to the naive singular solution. When there are only internal excitations, the corresponding microstates in the CFT are such that no supergravity operators acquire vevs. Thus fuzzballs corresponding to such states are not visible in the supergravity approximation at all.

Fuzzballs for which the defining curve is a circle in R^4 have been extensively used in the literature; these solutions are a special case of the Lunin-Mathur solutions (4.32). One chooses $F^i(v)$ to be a (multiwound) circle [66,67,63],

$$F^1 = \mu_n \cos \frac{2\pi n v}{L}, \quad F^2 = \mu_n \sin \frac{2\pi n v}{L}, \quad F^3 = F^4 = 0, \quad (4.38)$$

with the internal excitations being zero. In this case one can integrate the expressions for the harmonic functions to obtain the solution in the form

$$\begin{aligned} ds^2 &= f_1^{-1/2} f_5^{-1/2} \left(- \left(dt - \frac{\mu_n \sqrt{Q_1 Q_5}}{r^2 + \mu_n^2 \cos^2 \theta} \sin^2 \theta d\phi \right)^2 + \left(dz - \frac{\mu_n \sqrt{Q_1 Q_5}}{r^2 + \mu_n^2 \cos^2 \theta} \cos^2 \theta d\psi \right)^2 \right) \\ &\quad + f_1^{1/2} f_5^{1/2} \left((r^2 + \mu_n^2 \cos^2 \theta) \left(\frac{dr^2}{r^2 + \mu_n^2} + d\theta^2 \right) + r^2 \cos^2 \theta d\psi^2 + (r^2 + \mu_n^2) \sin^2 \theta d\phi^2 \right) \\ &\quad + f_1^{1/2} f_5^{-1/2} ds^2(X_4); \\ e^{2\phi} &= f_1 f_5^{-1}, \\ f_{1,5} &= 1 + \frac{Q_{1,5}}{r^2 + \mu_n^2 \cos^2 \theta}, \end{aligned} \quad (4.39)$$

whilst the RR 2-form field is as in (4.24) with

$$A = \mu_n \frac{\sqrt{Q_1 Q_5}}{(r^2 + \mu_n^2 \cos^2 \theta)} \sin^2 \theta d\phi; \quad B = -\mu_n \frac{\sqrt{Q_1 Q_5}}{(r^2 + \mu_n^2 \cos^2 \theta)} \cos^2 \theta d\psi. \quad (4.40)$$

The parameters (n, μ_n) labeling the curve are related to the charges via

$$n\mu_n = \frac{L}{2\pi} \sqrt{\frac{Q_1}{Q_5}} = \frac{\sqrt{Q_1 Q_5}}{R_z} \equiv \mu. \quad (4.41)$$

These solutions have been used for illustrative purposes, since the defining functions are in analytic form and the geometries preserve a $U(1)^2$ symmetry of the transverse R^4 . Furthermore, the fluctuation equations are separable, due to the symmetry, and thus two point functions can be obtained analytically.

One needs to bear in mind, however, that these solutions are not representative of generic fuzzballs. In this case, the defining curves are multi-wound, and therefore self-intersect, violating the condition for non singularity given above. Indeed except for the case of $n = 1$ the solutions have orbifold singularities at the curve location.

From the asymptotics one can see that the fuzzball solutions have the same mass and D1-brane, D5-brane charges as the naive solution; the latter are given in (4.29) and (4.31) whilst the ADM mass is

$$M = \frac{R_z V}{(\alpha')^4} (Q_1 + Q_5). \quad (4.42)$$

The two charge fuzzball geometries have the correct charges to describe Ramond ground states in the CFT. At the same time, to make quantitative progress with the fuzzball program, one would like to understand the detailed correspondence between these geometries and microstates. Holographic methods provide a powerful tool to address this question by extracting field theory data from a given geometry.

The first step is thus to extract the decoupling asymptotically AdS regions of the fuzzball geometries; this decoupling region is obtained by dropping the constant from (f_1, f_5) . Note that the very existence of such a region is a test that a given geometry corresponds to a CFT ground state. If by contrast, a geometry with the correct D1 and D5 charges does not have such a throat region, then it should not be interpreted as a 2-charge black hole microstate; it fails the first test.

It may be useful to comment here that *all* CFT quantities should be interpreted in the asymptotically AdS region of the geometry, be they conserved charges or scattering amplitudes. CFT data reconstructs the inner AdS region of the asymptotically flat geometry, but there is no known holographic description which reconstructs the full geometry. Indeed, if there were, an immediate corollary would be a holographic description of flat spacetime!

Given the asymptotically $AdS_3 \times S^3 \times X_4$ type IIB solutions, one can compactify on X_4 to obtain asymptotically $AdS_3 \times S^3$ solutions of six dimensional $\mathcal{N} = 4b$ supergravity. (Note that whilst a general T^4 compactification would give a solution of $\mathcal{N} = 8$ supergravity in six dimensions the solutions of interest here solve the equations of motion of the truncated theory.) These six-dimensional solutions can be analyzed in detail using holographic technology.

4.4. Map between geometries and microstates

The key question one would like to address is the correspondence between a given fuzzball solution and a Ramond ground state in the CFT. Motivated by the duality relation of the fuzzball solutions to the fundamental string system, a precise proposal was made in [20–22] for the correspondence between geometries and ground states; see also [68].

The fuzzball geometries are determined by giving a curve $F^I \equiv (F^I(v), \mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$ in an N dimensional space, where $N = 8$ when $X_4 = T^4$ and $N = 24$ when $X_4 = K3$. The corresponding state is now determined via the following steps.

1. Fourier expand the curve

$$F^I(v) = \sum_{n>0} \frac{1}{\sqrt{n}} (\alpha_n^I e^{-in\sigma^+} + (\alpha_n^I)^* e^{in\sigma^+}), \quad (4.43)$$

and consider the coherent state

$$|F^I\rangle = \prod_{n,I} |\alpha_n^I\rangle, \quad (4.44)$$

where $|\alpha_n^I\rangle$ is a standard coherent state, i.e. it satisfies

$$\hat{a}_n^I |\alpha_n^I\rangle = \alpha_n^I |\alpha_n^I\rangle. \quad (4.45)$$

At this stage \hat{a}_n^I are just auxiliary harmonic oscillators.

2. Expressing the coherent states in terms of Fock states, $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_k \frac{\alpha^k}{k!} (\hat{a}^\dagger)^k |0\rangle$, we identify in $|F^I\rangle$ the Fock states that satisfy the constraint

$$\prod (\hat{a}_{-n_l}^I)^{m_l} |0\rangle, \quad \sum n_l m_l = N_1 N_5, \quad (n_l > 0). \quad (4.46)$$

3. We retain only these terms from $|F^I\rangle$ and map the harmonic oscillators to CFT R operators via the dictionary

$$\begin{aligned} \frac{1}{\sqrt{2}} (\hat{a}_n^1 \pm i \hat{a}_n^2) &\leftrightarrow \hat{\mathcal{O}}_n^{R(\pm 1+1), (\pm 1+1)}, \\ \frac{1}{\sqrt{2}} (\hat{a}_n^3 \pm i \hat{a}_n^4) &\leftrightarrow \hat{\mathcal{O}}_n^{R(\pm 1+1), (\mp 1+1)}, \\ \hat{a}_n^{\tilde{\alpha}} &\leftrightarrow \hat{\mathcal{O}}_{\tilde{\alpha}n}^{R(1,1)} \end{aligned} \quad (4.47)$$

where $F^{\tilde{\alpha}} = (\mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$. This state is proposed to correspond via AdS/CFT to the fuzzball geometry, i.e. the vevs of gauge invariant operators in this state should agree with the one extract from the asymptotics of the solution.

The motivation for this map comes from the duality between the fuzzball solutions and the supergravity solutions describing a fundamental string carrying momentum. The FP solutions (4.19) solve the supergravity equations coupled to a macroscopic classical string source with the profile given by the curve F^I . On general grounds, one expects that the classical string source is produced by a coherent state of string oscillators. For the case at hand, these are BPS solutions and the auxiliary oscillator \hat{a}_n^I are identified with the left moving string oscillators. We now want to apply the duality to map the system to the D1–D5 frame. The FP states satisfying (4.46) are precisely the states that map across to D1–D5 states, and this is the reason we retained only this part of the coherent state. The one to one correspondence between string oscillators and R operators of the symmetric orbifold CFT in (4.47) goes back to [69]. Note however that while the stringy oscillators satisfy a Heisenberg algebra, the operator algebra of R ground states is more complicated.

4.5. Tests of correspondence

In this section, we will describe the methods used to probe the proposed correspondence between geometries and black hole microstates.

4.5.1. Holographic one point functions

Holographic methodology can be used to systematically test the correspondence. In particular, one can extract the holographic one point functions for chiral primaries from the geometry, and compare them to the vevs in the proposed dual state. Such computations were carried out in [20–22]. One first needs to reduce the asymptotically $\text{AdS}_3 \times S^3 \times X_4$ type IIB solutions on X_4 to obtain asymptotically $\text{AdS}_3 \times S^3$ solutions of six dimensional $\mathcal{N} = 4b$ supergravity (3.11), and then one can apply the holographic formulae given in (3.24), (3.26) and (3.27) to extract the vevs from the spacetime asymptotics.

Let us briefly sketch the steps involved. The compactification is straightforward although the explicit reduction formulae are rather complicated, particularly for $X_4 = K3$; details may be found in [21,22]. The resulting six-dimensional metric and three form fields are given by

$$ds^2 = \frac{1}{\sqrt{f_5 \tilde{f}_1}} [-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + \sqrt{f_5 \tilde{f}_1} dx_i dx^i, \quad (4.48)$$

$$G_{tyi}^A = \partial_i \left(\frac{m^A}{f_5 \tilde{f}_1} \right), \quad G_{\mu ij}^A = -2\partial_{[i} \left(\frac{m^A}{f_5 \tilde{f}_1} \mathcal{B}_{j]}^{\mu} \right),$$

$$G_{ijk}^A = \epsilon_{ijkl} \partial^l m^A + 6\partial_{[i} \left(\frac{m^A}{f_5 \tilde{f}_1} A_j B_{k]} \right),$$

with

$$m^n = \left(0_4, \frac{1}{4}(f_5 + F_1) \right), \quad (4.49)$$

$$\begin{aligned} m^r &= \frac{1}{4} \left((f_5 - F_1), -2A_\alpha, -\sqrt{2}N^{(c)}, 2A_5 \right) \\ &\equiv \frac{1}{4} \left((f_5 - F_1), -2\mathcal{A}^{\alpha-}, 2\mathcal{A} \right). \end{aligned}$$

Here $n = 1 \dots 5$ and $r = 6 \dots (n_t + 5)$ with the index $\alpha = 6, 7, 8$ and $F_1 = (1 + K)$. The corresponding $SO(5, n_t)$ matrix of scalar fields may be found in [21,22].

Before giving the vevs, let us remark that in solutions with only transverse excitations there are only two non-zero tensors (G^5, G^6). Such solutions involve only an $SO(1, 1)$ truncation of the $\mathcal{N} = 4b$ fields. Moreover, if one sets $f_1 = f_5$ then $G^6 = 0$, and the resulting solution solves the equations of minimal supergravity in six dimensions (i.e. with no additional tensors coupled). Looking ahead to the next section, we will see that almost all known fuzzball solutions for three charge black holes involve only the $SO(1, 1)$ truncation of the $\mathcal{N} = 4b$ fields; there are no analogues of the fuzzballs with internal excitations. It would be interesting to explore whether one can generate more general three charge solutions using the $SO(5, n_t)$ symmetry of the $\mathcal{N} = 4b$ theory.

The decoupling asymptotically $AdS_3 \times S^3$ region of the geometry is obtained by dropping the constant terms in the functions $(f_1, \tilde{f}_1, f_5, F_1)$. Next one can expand the defining functions for large radius and extract the appropriate fields (3.18) characterizing the perturbation with respect to $AdS_3 \times S^3$. Substituting these perturbations in the formulae (3.24), (3.26) and (3.27) gives the holographic vevs.

The vevs of the stress energy tensor and of the R symmetry currents are

$$\langle T_{\mu\nu} \rangle = 0; \quad \langle J^{\pm\alpha} \rangle = \pm \frac{N}{2\pi} a^{\alpha\pm} (dz \pm dt), \quad (4.50)$$

where $a^{\alpha\pm}$ is a constant defined below in (4.54). The vanishing of the stress energy tensor is as anticipated, since these solutions should be dual to R vacua; however, the cancelation is very non-trivial. The vevs of the low lying chiral primaries were also computed, and expressed in terms of the harmonics of the curves defining the solutions:

$$f_{kl}^5 = \frac{1}{L(k+1)} \int_0^L dv (C_{i_1 \dots i_k}^l F^{i_1} \dots F^{i_k}), \quad (4.51)$$

$$f_{kl}^1 = \frac{Q_5}{L(k+1)Q_1} \int_0^L dv (\dot{F}^2 + \dot{\mathcal{F}}^2 + (\dot{\mathcal{F}}^{\alpha-})^2) C_{i_1 \dots i_k}^l F^{i_1} \dots F^{i_k},$$

$$(A_{kl})_i = -\frac{1}{L(k+1)} \int_0^L dv \dot{F}_i C_{i_1 \dots i_k}^l F^{i_1} \dots F^{i_k},$$

$$(\mathcal{A}_{kl}) = -\frac{\sqrt{Q_5}}{\sqrt{Q_1}L(k+1)} \int_0^L dv \dot{\mathcal{F}} C_{i_1 \dots i_k}^l F^{i_1} \dots F^{i_k},$$

$$\mathcal{A}_{kl}^{\alpha-} = -\frac{\sqrt{Q_5}}{\sqrt{Q_1}L(k+1)} \int_0^L dv \dot{\mathcal{F}}^{\alpha-} C_{i_1 \dots i_k}^l F^{i_1} \dots F^{i_k}.$$

Here the $C_{i_1 \dots i_k}^l$ are orthogonal symmetric traceless rank k tensors on \mathbb{R}^4 which are in one-to-one correspondence with the (normalized) spherical harmonics $Y_k^l(\theta_3)$ of degree k on S^3 . Fixing the center of mass of the whole system implies that

$$(f_{1i}^1 + f_{1i}^5) = 0. \quad (4.52)$$

The leading term in the asymptotic expansion of the transverse gauge field A_i can be written in terms of degree one vector harmonics as

$$A = \frac{Q_5}{r^2} (A_{ij})_i Y_1^j dY_1^i \equiv \frac{\sqrt{Q_1 Q_5}}{r^2} (a^{\alpha-} Y_1^{\alpha-} + a^{\alpha+} Y_1^{\alpha+}), \quad (4.53)$$

where $(Y_1^{\alpha-}, Y_1^{\alpha+})$ with $\alpha = 1, 2, 3$ form a basis for the $k = 1$ vector harmonics and we have defined

$$a^{\alpha\pm} = \frac{\sqrt{Q_5}}{\sqrt{Q_1}} \sum_{i>j} e_{\alpha ij}^{\pm} (A_{ij})_i, \quad (4.54)$$

where the spherical harmonic triple overlap $e_{\alpha ij}^{\pm}$ is defined in (C.6).

Note that the vevs of the R symmetry currents increase with the radius of the curve in R^4 . For example, the vevs for geometries based on the circular curves (4.38) are

$$\langle J^{\pm 3} \rangle = \pm \frac{N}{2\pi n} (dz \pm dt), \quad (4.55)$$

where the symmetry implies that only the $J^{\pm 3}$ components acquire vevs. In this case the R symmetry charges are proportional to the radius of the curve. Typical Ramond states have small R charges, and necessarily correspond to geometries sourced by curves of small radii.

The vevs of the scalar operators dual to the fields $(s_l^{(6)k}, \sigma_l^k)$ are given in terms of the transverse fluctuations:

$$\begin{aligned} \langle \mathcal{O}_{S_l^{(6)1}} \rangle &= \frac{N}{4\pi} (-4\sqrt{2}f_{1i}^5); \\ \langle \mathcal{O}_{S_l^{(6)2}} \rangle &= \frac{N}{4\pi} (\sqrt{6}(f_{2l}^1 - f_{2l}^5)); \\ \langle \mathcal{O}_{S_l^2} \rangle &= \frac{N}{4\pi} \sqrt{2}(-(f_{2l}^1 + f_{2l}^5) + 8a^{\alpha-}a^{\beta+}f_{l\alpha\beta}). \end{aligned} \quad (4.56)$$

The internal excitations of the fuzzball solutions are captured by the vevs of operators dual to the fields $s_l^{(r)k}$ with $r > 6$:

$$\begin{aligned} \langle \mathcal{O}_{S_l^{(5+n_l)1}} \rangle &= -\frac{N}{\pi} \sqrt{2}(\mathcal{A}_{1i}); & \langle \mathcal{O}_{S_l^{(6+\alpha_-)1}} \rangle &= \frac{N}{\pi} \sqrt{2}\mathcal{A}_{1i}^{\alpha_-}; \\ \langle \mathcal{O}_{S_l^{(5+n_l)2}} \rangle &= -\frac{N}{2\pi} \sqrt{6}(\mathcal{A}_{2l}); & \langle \mathcal{O}_{S_l^{(6+\alpha_-)2}} \rangle &= \frac{N}{2\pi} \sqrt{6}\mathcal{A}_{2l}^{\alpha_-}. \end{aligned} \quad (4.57)$$

Here $n_l = 5, 21$ for T^4 and $K3$ respectively, with $\alpha_- = 1, \dots, b^{2-}$ with $b^{2-} = 3, 19$ respectively. Thus each curve $(\mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$ induces corresponding vevs of operators associated with the middle cohomology of M^4 . Note the sign difference for the vevs of operators which are related to the distinguished harmonic function $\mathcal{F}(v)$.

One would like to reproduce these vevs from the field theory, using the proposed correspondence between geometries and Ramond ground states. Substantial progress in this direction was obtained in [20–22] where the detailed structure of these vevs was reproduced from the corresponding field theory calculations. The point is that one point functions in a given state may be related to three point functions at the conformal point. Consider a general state such that $|\Psi\rangle = O_\psi(0)|0\rangle$. Then the vev of an operator O_k of dimension k in this state is given by

$$\langle \Psi | O_k(\lambda^{-1}) | \Psi \rangle = \langle 0 | (O_\psi(\infty))^\dagger O_k(\lambda^{-1}) O_\psi(0) | 0 \rangle, \quad (4.58)$$

where λ is a mass scale. For scalar operators the 3-point function is uniquely determined by conformal invariance and the above computation yields

$$\langle \Psi | O_k(\lambda^{-1}) | \Psi \rangle = \lambda^k C_{\psi k \psi} \quad (4.59)$$

where $C_{\psi k \psi}$ is the fusion coefficient. Similarly, the expectation value of a symmetry current measures the charge of the state

$$\langle \Psi | j(\lambda^{-1}) | \Psi \rangle = \langle 0 | (O_\psi(\infty))^\dagger j(\lambda^{-1}) O_\psi(0) | 0 \rangle = q\lambda \langle \Psi | \Psi \rangle \quad (4.60)$$

where q is the charge of the operator O_ψ under j .

To reproduce the vevs, one needs to consider states $|\Psi\rangle$ which are ground states in the Ramond sector, or equivalently chiral primaries in the NS sector. The stress energy tensor and the R current expectation values are immediately reproduced from the proposed correspondence between curves and superpositions of Ramond ground states. Although the relevant three point functions of scalar chiral primaries have not been computed in full generality, the structure of the vevs given above is reproduced using selection rules for the three point functions, along with the large N expansion.

Selection rules are responsible for determining which scalar operators acquire a vev in a given superposition of Ramond ground states. For example, one can see from (4.57) and (4.51) that primaries associated with the $(1, 1)$ cohomology of X_4 acquire a vev only if both internal and transverse fluctuations of the fuzzball are non-zero. Moreover, the internal and transverse curves must share common Fourier modes for the dimension one operators to acquire vevs; further selection rules on the Fourier modes are needed for dimension two operators to acquire vevs. This structure can be reproduced in the field theory, using selection rules for the three point functions. A basic property of such three point functions is that they are only

non-zero when the total number of operators $\mathcal{O}_{\tilde{\alpha}}^{(1,1)}$ with a given index $\tilde{\alpha}$ in the correlation function is even. This implies that the $(1, 1)$ chiral primaries only acquire vevs when the fuzzball has internal excitations. When there are no internal internal fluctuations, there are no $\mathcal{O}_{\tilde{\alpha}}^{(1,1)}$ operators in the superposition defining the state, so total number of operators $\mathcal{O}_{\tilde{\alpha}}^{(1,1)}$ with a given index $\tilde{\alpha}$ in the three point function is never even. By contrast when there are internal excitations there are $\mathcal{O}_{\tilde{\alpha}}^{(1,1)}$ operators in the superposition defining the state, and thus $(1, 1)$ chiral primaries generically acquire vevs.

One can also easily see why the operator only acquires a vev if there are transverse excitations as well. All Ramond ground states associated with the middle cohomology have zero R charge, with the corresponding chiral primaries in the NS sector having the same charge $j_3^{NS} = \frac{1}{2}N$. Thus, a superposition involving only $\mathcal{O}^{(1,1)}$ operators has a definite R charge, and a charged operator cannot acquire a vev. Including transverse excitations means that the superposition of Ramond ground states contains charged operators, associated with the other cohomology, does not have definite R charge and therefore a charged operator can acquire a vev. Note that this point also explains why superpositions involving only $\mathcal{O}^{(1,1)}$ operators do not have dual fuzzball descriptions within supergravity: no operator which is dual to a supergravity mode can acquire a vev.

In fact one can go even further in matching field theory one point functions to the holographic vevs, using the fact that the most disconnected component of any correlator dominates at large N . In practice this implies that for many calculations the operators involved in the superposition behave like free harmonic oscillators – the deviation of the algebra from the free algebra is subleading in N – and thus even the numerical values of the vevs can be reproduced by harmonic oscillator algebra! This point was illustrated in detail in [21,22] using specific examples, such as elliptic curves. Given the recently refined dictionary relating supergravity fields to chiral primaries [54], and the proposed non-renormalization of the chiral ring, one could hope to push these calculations even further.

Another issue which is nicely demonstrated by the calculations in [21,22] is the indistinguishability of many solutions with supergravity: typically the vevs extracted from a solution defined by any given curve are small, comparable with the next order corrections to supergravity. Thus even fuzzballs which are non-singular in supergravity generally suffer from indistinguishability.

To summarize: the detailed holographic data extracted from these two charge geometries supports their interpretation as superpositions of Ramond ground states. Moreover, the data reinforces the general expectation that supergravity does not suffice to describe all the fuzzballs. Some cannot be seen at all in the supergravity approximation, whilst others cannot be reliably distinguished.

As we shall see later there is as yet no complete understanding of the space of fuzzball solutions for the 3-charge system: candidate solutions within supergravity seem to be characterized by a (constrained) set of curves along with additional parameters and the correspondence with states in the CFT is unclear. Extracting vevs from such geometries may thus provide a useful indicator as to the correspondence with black hole microstates.

4.5.2. Flight time and mass gap

Apart from matching the one-point functions, one could also test the correspondence by matching higher point functions. To make precision tests is rather hard in general. Firstly, to compute even a two-point function, one needs to solve a partial differential equation on the asymptotically $\text{AdS}_3 \times S^3$ background, which for a general fuzzball solution, is not even separable due to lack of symmetry. Secondly, n -point functions in a given geometry are related to $(n + 2)$ -point functions at the conformal point, and there is no reason to expect that these are non-renormalized even for chiral primaries. Thus the correlation functions computed in the orbifold CFT will not necessarily match those in the geometry.

Bearing in mind these caveats, estimates of the two point functions in symmetric geometries can give indicative information about the correspondence with CFT states, see [11,12,70,15,16]. For example, in the geometric approximation for the two point function of a massless scalar one considers null geodesics propagating inwards and then returning back to the boundary. The (finite) affine length of such a geodesic defines a characteristic scale or inverse mass gap. To give an example: one can consider the geometries obtained from a circular curve in R^4 given in (4.39). The decoupled region in the geometry has a metric:

$$ds^2 = \sqrt{Q_1 Q_5} \left(-(r^2 + \mu_n^2) d\hat{t}^2 + r^2 d\hat{z}^2 + \frac{dr^2}{(r^2 + \mu_n^2)} \right) + \sqrt{Q_1 Q_5} (d\theta^2 + \sin^2 \theta (d\phi + \mu_n d\hat{t})^2 + \cos^2 \theta (d\psi - \mu_n d\hat{z})^2) + \sqrt{Q_1/Q_5} ds^2(X_4), \quad (4.61)$$

where $\mu_n = \mu/n = \sqrt{Q_1 Q_5}/(nR_2)$ and \hat{z} is periodic with periodicity $2\pi\mu^{-1}$. Null radial geodesics such that $\theta = 0$ and (\hat{z}, ψ) are constant satisfy

$$\left(\frac{dr}{d\hat{t}} \right)^2 = (r^2 + \mu_n^2)^2, \quad (4.62)$$

and thus the total affine length of a geodesic propagating inwards from radial infinity and back out again is

$$\hat{T} = \int_0^\infty \frac{dr}{(r^2 + \mu_n^2)} = \frac{\pi}{2\mu_n}. \quad (4.63)$$

The corresponding energy scale μ_n characterizes the mass gap of the spectrum in this sector of the CFT and since μ^{-1} is the scale of \hat{z} , the dimensionless energy scale is $1/n$. (This is the same energy scale that is obtained from the holographic computation of the R charges, see (4.55).) This fits with the conjectured correspondence between these geometries and dual R charge eigenstates, $(\hat{\mathcal{O}}_n^{R(p,q)})^{N/n}$: excitations in the twist n sector are of the scale $1/n$. In the effective string picture, the excitation energy is also related to the time taken by excitations to propagate around the circle wound by the string.

In such a simple example, one does not acquire new information from this computation. In more complicated fuzzball geometries, however, such as 3-charge bubbling solutions to be discussed later, this estimation of the mass gap may be a useful indicator of the dual black hole microstate. At the same time, in geometries which break rotational symmetry and in which there are many parameters, the flight time is sensitive to the specific geodesic under consideration and has to be interpreted with care.

4.5.3. Geometric quantization

The fuzzballs are characterized by a curve in an 8 (T^4) or 24 ($K3$) dimensional space. One would like to enumerate them and compare them with the number of dual microstates, but naively one has an infinite number of solutions, since any classical curve determines a solution. The map between fuzzball solutions and microstates effectively discretizes the space of fuzzball solutions. From our earlier discussion it is clear that the classical curves are due to the states being approximately coherent; the coefficients in the Fourier expansion of the curve are eigenvalues of quantum harmonic oscillators.

Alternatively, note that the space of curves is the classical phase space of these gravitational solutions. The number of the corresponding quantum states should be obtained by appropriate quantization. These states should be in correspondence with the states in the dual CFT. One can thus consider quantizing this moduli space, and comparing the latter to the Ramond ground states in the CFT. For fuzzballs with only transverse excitations, namely (4.32), such that the solutions are determined by a smooth closed non-intersecting curve in four dimensions, the geometric quantization was carried out in [71]. It was found that the Fourier coefficients of the curve satisfy chiral boson commutation relations, with the value of the effective Planck constant agreeing with that obtained from dualising the fundamental string.

The basic idea of this quantization procedure is as follows. Consider any classical dynamical system with phase space coordinates satisfying the standard Poisson brackets

$$\{q^I, p^I\} = \delta^{II}. \quad (4.64)$$

Now restrict to a subspace M of the full phase space, parameterized by some coordinates x^A . The induced Poisson brackets can be extracted from pullback of the symplectic structure onto this subspace. The symplectic structure on the phase space is given by

$$\Omega = dp^I \wedge dq^I, \quad (4.65)$$

and the pullback Ω_M of this symplectic structure onto a subspace is given by

$$\begin{aligned} \Omega_M &= \omega_{AB}(x) dx^A \wedge dx^B, \\ \omega_{AB} &= \left(\frac{\partial p^I}{\partial x^A} \frac{\partial q^I}{\partial x^B} - \frac{\partial p^I}{\partial x^B} \frac{\partial q^I}{\partial x^A} \right). \end{aligned} \quad (4.66)$$

Then the induced Poisson brackets are given by the inverse of ω ,

$$\{x^A, x^B\} = \frac{1}{2} \omega^{AB}. \quad (4.67)$$

Once one has extracted the Poisson brackets from the symplectic form, one can quantize in the standard way to find the commutation relations.

In the case of interest, one considers the symplectic form of supergravity, restricted to the fuzzball solutions of interest, and extracts the appropriate Poisson brackets. Starting from the relevant part of the IIB supergravity action (in Einstein frame)

$$S_{IIB} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2} (\partial\Phi)^2 - \frac{1}{2} e^\Phi F_3^2 \right), \quad (4.68)$$

the symplectic form is

$$\Omega = \frac{1}{2\kappa_{10}^2} \int d\Sigma_m J^m \quad (4.69)$$

where the integration is over a Cauchy surface Σ and the symplectic current J^m is given by [72,73,71]

$$J^m = -\delta\Gamma_{ab}^m \wedge \delta(\sqrt{-g}g^{ab}) + \delta\Gamma_{ab}^b \wedge \delta(\sqrt{-g}g^{am}) - \delta(\sqrt{-g}e^{-\Phi}F^{ab}) \wedge \delta C_{ab} - \delta(\sqrt{-g}\partial^m\Phi) \wedge \delta\Phi. \quad (4.70)$$

Restricting to the fuzzball solutions (4.32) characterized by a curve $F_i(v)$ in R^4 , the pullback Ω_M of the symplectic form can be explicitly evaluated to give

$$\Omega_M = \frac{1}{2\alpha} \int \delta(\partial_v F_i(v)) \wedge \delta F_i(v) dv \quad (4.71)$$

which implies the Poisson bracket

$$\{F_i(v), \partial_v F_j(v')\} = \alpha \delta_{ij} \delta(v - v'). \quad (4.72)$$

The parameter α is found by explicit computation of the symplectic form to be given by

$$\alpha = \pi \frac{g_s^2}{R_z^2 V}, \quad (4.73)$$

in agreement with the value obtained by duality from the fundamental string. Thus, as previously stated, the defining curve of the fuzzball solution satisfies the standard Poisson bracket relations of chiral bosons. One can next promote the Poisson bracket to a quantum commutation relation, and introduce harmonic oscillators $(\hat{a}_n^i, (\hat{a}_n^i)^\dagger)$ for the curve. To count the number of black hole microstates obtained from these geometries, one must then restrict to curves satisfying the relation constraining the total D1-brane charge:

$$Q_1 = \frac{Q_5}{L} \int_0^L |\partial_v F^i|^2 dv. \quad (4.74)$$

As expected this reduces to counting the degeneracy of states at level $N_1 N_5$ in a system of four chiral bosons, namely counting the degeneracy of states satisfying

$$N_1 N_5 = \sum_{n=1}^{\infty} n \langle (\hat{a}_n^i)^\dagger \hat{a}_n^i \rangle, \quad (4.75)$$

which gives an entropy

$$S \approx 2\pi \sqrt{\frac{2}{3} N_1 N_5}. \quad (4.76)$$

It would be interesting to see if the quantization of general fuzzball geometries with internal excitations gives an analogous answer, corresponding to eight or twenty-four bosons for T^4 and $K3$ respectively. Given that fuzzballs with only internal excitations are not visible in supergravity, it seems possible that the symplectic form in the general case would have a more complicated structure than in (4.71), but the calculation has not been carried out.

Whilst the geometric quantization reproduces the expected entropy of the 2-charge microstates, one must clearly view this calculation with some caution. The 2-charge system does not have a horizon in supergravity, so the average fuzzball geometry must have a scale which is sub-string scale, and thus higher derivative corrections to supergravity are non-negligible for these geometries. It is not clear that quantizing the extrapolation of fuzzball geometries to the supergravity regime will in general give the correct counting, even though it does in this case.

Note that in this case, the number of microstates accounted for by the fuzzball solutions was also estimated using (probe) supertubes [74] in the dual D0-F1 system. In [75,76], the quantum states of the supertube were counted by directly quantizing the linearized Born-Infeld action near a round tube. This gives an entropy $S = 2\pi \sqrt{2 N_{D0} N_{F1}}$, where N_{D0} is the D0-brane charge and N_{F1} is the fundamental string charge. Backreacting a supertube with given profile in R^4 onto the supergravity fields, and then dualizing, should give the D1–D5 fuzzball solutions characterized by the same profile in R^4 . Thus the entropy obtained from quantizing the Born-Infeld action indicates the number of 2-charge fuzzball solutions, with the entropy matching the expected value. This approach is less direct than that discussed above, and is hard to generalize to the 3-charge system, where there is no such duality to a supertube description.

4.6. Open questions

Even given the detailed analysis already made in this system, there remain many interesting open questions and lessons for black holes with macroscopic horizons. Firstly, the recent demonstration that the chiral ring is non-renormalized, along with the matching between orbifold CFT operators and supergravity fields [54], would allow the vevs to be computed systematically at large N , and for the proposed correspondence to be tested to even higher precision using the holographic methods.

Furthermore, there is a detailed map from smooth geometries to black hole microstates, but not vice versa. Put differently, it is not understood directly in the D1–D5 system how a curve emerges. It is clearly important to understand this better, not least because this might lead to clues as to the defining data of 3-charge D1–D5–P fuzzball solutions. Note that a given R charge eigenstate does not in general appear to have a smooth supergravity description, whereas the smooth geometries

correspond to large superpositions of R charge eigenstates with the specific superposition determined by the classical curve. Thus the natural geometric basis does not coincide with the natural field theory basis, and the geometric basis does not sample well states with very small R charge.

Given the matching between geometries and microstates, one could consider coarse-graining over the fuzzball geometries, to understand how black hole properties emerge. Of course in this specific system the resulting black hole does not have a regular horizon in supergravity, so the coarse-graining can at best provide an indication of what happens for macroscopic black holes, but nonetheless it may be informative to explore this issue.

In general, the fact that many of the fuzzball geometries have very small parameters and are not well described using only supergravity could be interpreted as an indication that they will develop a horizon when higher derivative corrections are taken into account. For fuzzballs characterized by a generic curve this however seems unlikely: there is no singularity in the supersymmetric supergravity solution, so the higher derivative corrections are bounded.

Results in this section indicate that a more complete understanding of this system rests on understanding fuzzballs in the stringy regime. This is not unexpected, as the black hole does not have a macroscopic horizon, and thus one would clearly like to develop the fuzzball proposal for black holes with macroscopic horizons. As we shall see, however, even for large black holes, one is unlikely to evade the need to go beyond supergravity. Thus this 2-charge system may be useful as the simplest test case for both sharpening the definition of fuzzballs outside the supergravity regime and for developing calculational techniques to handle stringy fuzzballs.

5. Three and four charge systems

In the previous section, we discussed fuzzball solutions for the two charge system, and showed how AdS/CFT methodology can be used to make detailed tests of the correspondence between such geometries and CFT states. At the same time, the two charge black hole does not have a macroscopic horizon, so one would like to make progress with a system which does have a horizon in supergravity.

As a first step one would like to develop the fuzzball picture for asymptotically flat, (near) supersymmetric, black holes in four and five dimensions, which have finite area horizons in supergravity. This class includes static five-dimensional black holes with three charges, the first black hole whose entropy was explained microscopically in string theory by Strominger and Vafa [9], along with the rotating BMPV generalization [77]. In four dimensions black holes with a macroscopic horizon have four charges; for reviews of black hole solutions see [78].

There is a variety of reasons for first developing the picture for supersymmetric black holes. Perhaps most important is that in constructing candidate fuzzball solutions for supersymmetric black holes, one is not forced to solve the (highly non-linear) supergravity equations. One instead first solves the first order supersymmetry equations and then enforces, where necessary, additional restrictions arising from the equations of motion. Moreover, one can make use of the systematic classifications of supersymmetric solutions of supergravity that have been derived in recent years; in fact, almost all candidate three charge solutions derive from the classification of minimal supergravity in five dimensions [79]. Once supersymmetry is broken, one loses these powerful tools, and, as we will later review, very few fuzzball solutions are known.

Detailed microscopic explanations of the entropy are also only currently possible for supersymmetric black holes. Following the steps of Strominger and Vafa, one relates states in a D-brane system at weak coupling to a black hole at strong coupling, but the degeneracy of states in the weakly coupling theory can only be compared to the entropy of the black hole at strong coupling, if supersymmetry implies non-renormalization. This is the case for (the leading order term in) the entropy of supersymmetric black holes with large charges.

Without detailed knowledge of the properties of black hole microstates, it is hard to probe whether candidate fuzzball geometries do indeed correspond to such microstates. Even when one has a description in terms of D-branes at weak coupling, one cannot generically make a precise map between all data in this theory and that extracted from the fuzzball geometry.

One can, however, develop a precision map in cases where one can use AdS/CFT technology. Thus to push the fuzzball proposal further, it is natural to consider black holes which admit an AdS_3 factor in their near horizon regions, and relate the fuzzball geometries to states in the dual two dimensional conformal field theories. AdS/CFT technology is rather less developed for five and four dimensional black holes with AdS_2 near horizon regions, and thus we will not focus on such black holes here, although candidate fuzzball geometries have been constructed in these cases.

Before moving on to reviewing relevant properties of the black holes, one should note that whilst supersymmetric solutions can have finite horizon area, they do not have non-zero temperature. Thus one does not expect to address how temperature and Hawking radiation emerges in such a system. We will return to this issue in Section 6.

5.1. Black hole geometry and D-branes

In this section, we will review relevant properties of 3-charge black hole solutions. For what follows it is convenient to discuss solutions both of M theory compactified on a Calabi-Yau manifold, and of type IIB compactified on X_4 which is either T^4 or $K3$. Our conventions for the supergravity field equations are given in [Appendix A](#).

5.1.1. D1–D5 solutions

Let us begin with the type IIB solutions. Supersymmetric rotating (BMPV) strings wrapping a circle and carrying D1 charge Q_1 , D5 charge Q_5 along with momentum charge Q_p along the circle have a metric of the form

$$\begin{aligned} ds^2 = & \frac{1}{\sqrt{h_1 h_5}} \left(-(dt^2 - dz^2) + \frac{Q_p}{r^2} (dz - dt)^2 \right) + \sqrt{h_1 h_5} (dr^2 + r^2 (d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\phi^2)) \\ & + \frac{2\sqrt{Q_1 Q_5 Q_p}}{r^2 \sqrt{h_1 h_5}} (L_1 - L_2) (dt - dz) (\cos^2 \theta d\psi - \sin^2 \theta d\phi) + \sqrt{\frac{h_1}{h_5}} ds^2(X_4), \end{aligned} \quad (5.1)$$

with $h_i = 1 + Q_i/r^2$ and where $ds^2(X_4)$ denotes the metric on T^4 or $K3$ respectively. The corresponding RR 2-form potential and dilaton are given by

$$e^{-2\Phi} = \frac{h_5}{h_1}; \quad (5.2)$$

$$C = (h_1^{-1} - 1) dt \wedge dz - Q_5 \cos^2 \theta d\psi \wedge d\phi + \frac{\sqrt{Q_1 Q_5 Q_p}}{(r^2 + Q_1)} (L_2 - L_1) (dt - dz) (\cos^2 \theta d\psi - \sin^2 \theta d\phi).$$

The angular momentum charge is proportional to $(L_1 - L_2)$ and will be given below.

This solution follows from the supersymmetric limit of general non-extremal 3-charge black hole (actually black string) solutions of type IIB, with rotation constructed in [80]. The metric for these solutions is

$$\begin{aligned} ds^2 = & \frac{1}{\sqrt{H_1 H_5}} (-f(dt^2 - dz^2) + m(s_p dz - c_p dt)^2 + m(a_1 \cos^2 \theta d\psi + a_2 \sin^2 \theta d\phi)^2) \\ & + \sqrt{H_1 H_5} \left(\frac{r^2 dr^2}{(r^2 + a_1^2)(r^2 + a_2^2) - mr^2} + d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\phi^2 \right) \\ & + \frac{2m}{\sqrt{H_1 H_5}} (\cos^2 \theta ((a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p) dt + (a_2 s_1 s_5 c_p - a_1 c_1 c_5 s_p) dz) d\psi) \\ & + \frac{2m}{\sqrt{H_1 H_5}} (\sin^2 \theta ((a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p) dt + (a_1 s_1 s_5 c_p - a_2 c_1 c_5 s_p) dz) d\phi) \\ & + (a_2^2 - a_1^2) \frac{H_1 + H_5 - f}{\sqrt{H_1 H_5}} (\sin^4 \theta d\phi^2 - \cos^4 \theta d\psi^2) + \sqrt{\frac{H_1}{H_5}} ds^2(X_4), \end{aligned} \quad (5.3)$$

where the solution is parameterized by $(\delta_1, \delta_5, \delta_p)$ and we use the abbreviations $s_i = \sinh \delta_i$, $c_i = \cosh \delta_i$. The functions (H_1, H_5, f) are given by

$$H_i = f + m \sinh^2 \delta_i, \quad f = r^2 + a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta. \quad (5.4)$$

The RR 2-form potential and the dilaton are given by

$$\begin{aligned} C = & \frac{m \cos^2 \theta}{H_1} ((a_2 c_1 s_5 c_p - a_1 s_1 c_5 s_p) dt + (a_1 s_1 c_5 c_p - a_2 c_1 s_5 s_p) dz) \wedge d\psi \\ & + \frac{m \sin^2 \theta}{H_1} ((a_1 c_1 s_5 c_p - a_2 s_1 c_5 s_p) dt + (a_2 s_1 c_5 c_p - a_1 c_1 s_5 s_p) dz) \wedge d\phi \\ & - \frac{ms_1 c_1}{H_1} dt \wedge dz - \frac{ms_5 c_5}{H_1} (r^2 + a_2^2 + ms_1^2) \cos^2 \theta d\psi \wedge d\phi; \\ e^{-2\Phi} = & \frac{H_5}{H_1}. \end{aligned} \quad (5.5)$$

Thus the solution is labeled by six parameters (m, δ_i, a_1, a_2) along with the coupling constants and the two moduli (R_z, V) . The former is the radius of the z circle, and the latter is related to the volume v of X_4 by $v = (2\pi)^4 V$. The solution is asymptotically flat, approaching $R^{4,1} \times S^1 \times X_4$ as $r \rightarrow \infty$; here S^1 is the z circle. The horizons are located at

$$(r^2 + a_1^2)(r^2 + a_2^2) = mr^2, \quad (5.6)$$

and the topology of the horizon is $S^1 \times S^3 \times X_4$ where again the S^1 direction is associated with the z circle. This solution is hence a black string, and compactifying to five dimensions over $S^1 \times X_4$ produces a five-dimensional three charge rotating black hole.

The conserved charges and angular momenta are given by

$$\begin{aligned}
 Q_1 &= ms_1 c_1 = \frac{g\alpha'^3 N_1}{V}; \\
 Q_5 &= ms_5 c_5 = g\alpha' N_5; \\
 Q_p &= ms_p c_p = \frac{g^2(\alpha')^4}{VR_z^2} N_p, \\
 J_\psi &= -\frac{\pi m}{4G_5} (a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p) \\
 J_\phi &= -\frac{\pi m}{4G_5} (a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p),
 \end{aligned} \tag{5.7}$$

where (N_1, N_5, N_p) are the integral quantized charges and

$$\frac{\pi}{4G_5} = \frac{R_z V}{g^2(\alpha')^4}. \tag{5.8}$$

The mass is given by

$$M = \frac{\pi m}{4G_5} \left(s_1^2 + s_5^2 + s_p^2 + \frac{3}{2} \right). \tag{5.9}$$

Regular extremal black holes which saturate a BPS bound are obtained in the limit $m \rightarrow 0$ and $\delta_i \rightarrow \infty$ with $Q_i = ms_i^2$, $L_1 = a_1/\sqrt{m_1}$ and $L_2 = a_2/\sqrt{m_2}$ held fixed. The solution in this limit becomes (5.1).

The decoupling limit of these solutions is obtained by focusing on the region $r^2 \ll Q_1, Q_5$; then H_1 and H_5 may be approximated by $H_1 \approx Q_1$ and $H_5 \approx Q_5$. One also focuses on near extremal solutions for which $Q_1, Q_5 \gg m, a_1^2, a_2^2$ so that $m \sinh \delta_1 \sinh \delta_5 \approx m \cosh \delta_1 \cosh \delta_5 \approx \sqrt{Q_1 Q_5}$. In this limit the metric can be written as

$$\begin{aligned}
 ds^2 &= -l^2 \left(\rho^2 - M_3 + \frac{J_3^2}{4\rho^2} \right) d\tau^2 + \left(\rho^2 - M_3 + \frac{J_3^2}{4\rho^2} \right)^{-1} l^2 d\rho^2 + l^2 \rho^2 \left(d\varphi - \frac{J_3}{2\rho^2} d\tau \right)^2 \\
 &\quad + l^2 d\theta^2 + \sin^2 \theta \left(d\phi + \frac{R_z}{l} (a_1 c_p - a_2 s_p) d\varphi + \frac{R_z}{l} (a_2 c_p - a_1 s_p) d\tau \right)^2 \\
 &\quad + l^2 \cos^2 \theta \left(d\psi + \frac{R_z}{l} (a_2 c_p - a_1 s_p) d\varphi + \frac{R_z}{l} (a_1 c_p - a_2 s_p) d\tau \right)^2 + \frac{\sqrt{Q_1}}{\sqrt{Q_5}} ds^2(X_4),
 \end{aligned} \tag{5.10}$$

with new coordinates $\varphi = z/lR_z$ and $\tau = t/lR_z$ with $l^2 = \sqrt{Q_1 Q_5}$ and

$$\rho^2 = \frac{R_z^2}{l^2} (r^2 + (m - a_1^2 - a_2^2) s_p^2 + 2a_1 a_2 s_p c_p). \tag{5.11}$$

The non-trivial six-dimensional metric is a twisted fibration of S^3 over the BTZ black hole, with the mass and angular momentum parameters (M_3, J_3) of the BTZ metric being

$$\begin{aligned}
 M_3 &= \frac{R_z^2}{l^2} ((m - a_1^2 - a_2^2) \cosh 2\delta_p + 2a_1 a_2 \sinh 2\delta_p); \\
 J_3 &= \frac{R_z^2}{l^2} ((m - a_1^2 - a_2^2) \sinh 2\delta_p + 2a_1 a_2 \cosh 2\delta_p).
 \end{aligned} \tag{5.12}$$

Restricting to the extremal limit in which $M_3 = J_3$ gives

$$\begin{aligned}
 ds^2 &= -l^2 \left(\rho - \frac{M_3}{2\rho} \right)^2 d\tau^2 + \left(\rho - \frac{M_3}{2\rho} \right)^{-2} l^2 d\rho^2 + l^2 \rho^2 \left(d\varphi - \frac{M_3}{2\rho^2} d\tau \right)^2 \\
 &\quad + l^2 d\theta^2 + \sin^2 \theta \left(d\phi + \frac{R_z}{l} \sqrt{Q_p} (L_1 - L_2) (d\varphi - d\tau) \right)^2 \\
 &\quad + l^2 \cos^2 \theta \left(d\psi + \frac{R_z}{l} \sqrt{Q_p} (L_1 - L_2) (d\tau - d\varphi) \right)^2 + \frac{\sqrt{Q_1}}{\sqrt{Q_5}} ds^2(X_4),
 \end{aligned} \tag{5.13}$$

which is a twisted fibration of S^3 over the extremal BTZ black hole.

The entropy of black hole solutions is given by

$$S = \frac{2\pi \rho_+}{4G_3} = 2\pi \left(\sqrt{NN_p - \frac{1}{4}(J_\phi - J_\psi)^2} + \sqrt{N\bar{N}_p - \frac{1}{4}(J_\phi + J_\psi)^2} \right) \quad (5.14)$$

where the effective three dimensional Newton constant is

$$\frac{1}{4G_3} = N = N_1 N_5 \quad (5.15)$$

and ρ_+ is the location of the outer event horizon. The Hawking temperature of the black hole is given by

$$T_H = \frac{(\rho_+^2 - \rho_-^2)}{2\pi \rho_+}, \quad (5.16)$$

where ρ_- is the location of the inner event horizon of the black hole.

Given the asymptotically $\text{AdS}_3 \times S^3$ region, one can obtain the corresponding data in the dual conformal field theory. In particular, making use of the formulae for the holographic vevs previously derived, one can extract the non-zero components of the stress energy tensor and R symmetry currents. Recall that the vevs are

$$\begin{aligned} \langle T_{ij} \rangle &= \frac{N}{2\pi} \left(g_{(2)ij} + \frac{1}{4} \mathcal{A}_{(i}^{+\alpha} \mathcal{A}_{j)}^{+\alpha} + \frac{1}{4} \mathcal{A}_{(i}^{-\alpha} \mathcal{A}_{j)}^{-\alpha} \right); \\ \langle J^{i\pm\alpha} \rangle &= \frac{N}{8\pi} (g_{(0)ij} \pm \epsilon_{ij}) \mathcal{A}^{\pm\alpha j}, \end{aligned} \quad (5.17)$$

where $g_{(n)ij}$ and $\mathcal{A}_{ij}^{\pm\alpha}$ are terms in the asymptotic expansion of the three dimensional metric and gauge fields respectively. The relevant three dimensional metric is the BTZ metric in the first line of (5.10) which in Fefferman–Graham form is

$$\begin{aligned} ds^2 &= \frac{l^2}{z^2} (dz^2 + dx^i dx^j (g_{(0)ij} + z^2 g_{(2)ij} + \dots)); \\ &= \frac{l^2}{z^2} \left(dz^2 + (-d\tau^2 + d\varphi^2) + \frac{1}{2} M_3 z^2 (d\tau^2 + d\varphi^2) + J_3 z^2 d\tau d\varphi + \dots \right), \end{aligned} \quad (5.18)$$

whilst the three dimensional gauge fields are, respectively,

$$\begin{aligned} A^{+3} &= \mathcal{A}^{+3} + \dots = \frac{R_z}{l} e^{\delta_p} (a_1 - a_2) (d\varphi - d\tau); \\ A^{-3} &= \mathcal{A}^{-3} + \dots = \frac{R_z}{l} e^{-\delta_p} (a_2 + a_1) (d\varphi + d\tau). \end{aligned} \quad (5.19)$$

Putting these values into the holographic formulae gives

$$\begin{aligned} \langle T \rangle &= \frac{1}{2\pi} (N_p (dx^+)^2 + \bar{N}_p (dx^-)^2); \\ \langle J^{+3} \rangle &= \frac{\sqrt{N\bar{N}_p}}{2\pi} (L_1 - L_2) dx^+ \equiv \frac{1}{2\pi} j^+ dx^+; \\ \langle J^{-3} \rangle &= -\frac{\sqrt{N\bar{N}_p}}{2\pi} (L_1 + L_2) dx^- \equiv \frac{1}{2\pi} j^- dx^-, \end{aligned} \quad (5.20)$$

where we define

$$N_p = \frac{VR_z^2}{4g^2(\alpha')^4} m e^{2\delta_p}; \quad \bar{N}_p = \frac{VR_z^2}{4g^2(\alpha')^4} m e^{-2\delta_p}, \quad (5.21)$$

and introduce light cone coordinates $dx^\pm = l(d\tau \pm d\varphi)$, which have periodicity 2π . Define $h = \int \langle T_{++} \rangle$ and $\bar{h} = \int \langle T_{--} \rangle$, and similarly $j^+ = \int \langle J^{+3} \rangle dx^+$, $j^- = \int \langle J^{-3} \rangle dx^-$. Then for the product of global AdS_3 with S^3 , $j^+ = j^- = L_1 = L_2 = 0$ and $h = \bar{h} = N_p = \bar{N}_p = -N/4$. For an extremal BTZ black hole, such that $M_3 = J_3$ one obtains $\bar{N}_p = 0$ and

$$h = N_p; \quad j^+ = \sqrt{N\bar{N}_p} (L_1 - L_2) \equiv J_\phi; \quad \bar{h} = j^- = 0, \quad (5.22)$$

where J_ϕ is the angular momentum with respect to asymptotically flat infinity given in (5.7).

One can reduce the black string solutions over the z circle and X_4 , to produce a three charge rotating black hole solution of five-dimensional supergravity, as was indeed done in the original BMPV paper [77]. For the supersymmetric case this leads to a five dimensional metric of the form

$$ds^2 = \lambda^{-2/3} (dt + k)^2 + \lambda^{1/3} (dr^2 + r^2 d\Omega_3^2) \quad (5.23)$$

where

$$\lambda = h_1 h_5 h_p = \left(1 + \frac{Q_1}{r^2}\right) \left(1 + \frac{Q_5}{r^2}\right) \left(1 + \frac{Q_p}{r^2}\right); \quad (5.24)$$

$$k = \frac{\sqrt{Q_1 Q_5 Q_p}}{r^2} (L_2 - L_1) (\cos^2 \theta d\psi - \sin^2 \theta d\phi).$$

Setting $k = 0$ gives the 3-charge Strominger–Vafa black hole [9], with the metric being of the same form as the 2-charge metric (4.3). In this case there is no curvature singularity at $r = 0$, see (4.5) and (4.6), but instead, a regular horizon. The same is true for the rotating BMPV 3-charge black hole. The near horizon region of the five-dimensional solutions is $\text{AdS}_2 \times S^3$ which is less amenable to detailed holographic analysis⁵ than the six-dimensional $\text{AdS}_3 \times S^3$ near horizon region. For this reason, we will focus on fuzzball geometries for the black string solutions.

5.1.2. M theory solutions

It is also useful to review briefly certain three charge solutions of M theory compactified on a Calabi-Yau. For simplicity let us first give the supersymmetric eleven-dimensional solution for a toroidal compactification describing orthogonally intersecting M2-branes:

$$ds^2 = - \left(\frac{1}{H_1 H_2 H_3} \right)^{2/3} (dt + k)^2 + (H_1 H_2 H_3)^{1/3} dx^m dx^m$$

$$+ \left(\frac{H_2 H_3}{H_1^2} \right)^{1/3} (dy_1^2 + dy_2^2) + \left(\frac{H_1 H_3}{H_2^2} \right)^{1/3} (dy_3^2 + dy_4^2) + \left(\frac{H_1 H_2}{H_3^2} \right)^{1/3} (dy_5^2 + dy_6^2), \quad (5.25)$$

where x^m are coordinates on R^4 and the four form is

$$F = F^1 \wedge dy_1 \wedge dy_2 + F^2 \wedge dy_3 \wedge dy_4 + F^3 \wedge dy_5 \wedge dy_6 \quad (5.26)$$

where the two forms can be written as

$$F^a = \frac{1}{2} d(H_a^{-1} (dt + k)), \quad (5.27)$$

with $a = 1, 2, 3$. The solutions are defined by three harmonic functions (H_1, H_2, H_3) on R^4 along with the one form k on R^4 :

$$H_a = \left(1 + \frac{Q_a}{r^2}\right); \quad (5.28)$$

$$k = \sqrt{Q_1 Q_2 Q_3} \frac{\omega}{r^2} (\cos^2 \theta d\psi - \sin^2 \theta d\phi),$$

where the metric on R^4 is

$$ds^2 = dr^2 + r^2 d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\phi^2 \quad (5.29)$$

and dk is self-dual in this metric, namely $dk - *_4 dk = 0$.

The charges Q_a are related to the integral M2-brane charges via $Q_a = l_p^2 V_a / (V_1 V_2 V_3)$ where $(2\pi)^2 V_a$ is the volume of each T^2 and l_p is the Planck length, related to the eleven-dimensional Newton constant G_{11} via

$$\frac{1}{16\pi G_{11}} = \frac{1}{(2\pi)^8 l_p^9}. \quad (5.30)$$

This metric describes an extremal 3-charge rotating black hole, with the decoupling region geometry being $\text{AdS}_2 \times S^3 \times T^6$. The 3 charges are the Q_a , with the angular momenta being

$$J_\psi = -J_\phi = \frac{\pi}{4G_5} \omega \sqrt{Q_1 Q_2 Q_3} = \omega \sqrt{N_1 N_2 N_3}, \quad (5.31)$$

and the entropy is given by

$$S = 2\pi \sqrt{N_1 N_2 N_3 (1 - \omega^2)}. \quad (5.32)$$

Clearly, this is the same formula as the extremal BPS limit of (5.14). Indeed, the type IIB on T^4 and M theory on T^6 solutions are related by dualities; first reduce the latter on the M theory circle y_6 to obtain a $(D2_{y_1 y_2} \perp D2_{y_3 y_4} \perp F1_{y_5})$ solution of

⁵ Note though that in this case the AdS_2 factor originates from a reduction of the extremal BTZ over the compact boundary direction, so one would anticipate the dual CFT to be a chiral CFT [81] and thus that there is better control over the duality.

type IIA and then T dualise on (y_3, y_4, y_5) to obtain a $(D4_{y_1 y_2 y_3 y_4 y_5} \perp D1_{y_5} \perp P_{y_5})$ solution of type IIB. Here $Dp_{y_1 \dots y_p}$ denotes the spatial directions wrapped by the Dp -brane.

Analogous three charge rotating five-dimensional black hole solutions for Calabi-Yau compactifications are well-known; the entropy of such wrapped M-brane configurations was first discussed in [82]. Moreover by replacing R^4 by a Taub-NUT space and compactifying on the NUT circle, one can obtain an extremal four-charge solution in four dimensions. However, for us the key feature of all of these solutions is that they give a near horizon geometry with an AdS_2 factor, rather than AdS_3 .

Candidate fuzzball solutions have been constructed for these black holes; indeed as we shall see almost all supersymmetric fuzzball solutions are constructed from a set of defining data, which can be used to build fuzzballs for both type IIB and M theory black holes. Detailed holographic analysis of type IIA and M theory systems is however more subtle than in the IIB solutions which have AdS_3 decoupling regions. One can easily build other three charge M theory solutions which do have AdS_3 near horizon regions, for example, by intersecting $(M2 \perp M5 \perp P)$ along a string. However the dual conformal field theory is far less understood in such cases, so again it is more natural to explore first the D1–D5–P system of type IIB.

5.2. CFT microstates

To find the microstates of the (non-extremal) 3-charge D1–D5–P black hole, we need to consider states in the D1–D5 CFT with $h = N_p$ and $\bar{h} = \bar{N}_p$. The asymptotic number of distinct states of this CFT can be obtained immediately by Cardy's formula [83]

$$d = \Omega \bar{\Omega}; \quad (5.33)$$

$$\Omega = \exp(2\pi \sqrt{c N_p / 6}); \quad \bar{\Omega} = \exp(2\pi \sqrt{c \bar{N}_p / 6}),$$

where $c = 6N_1 N_5$ is the central charge. Clearly this formula exactly reproduces the Bekenstein-Hawking entropy of the non-rotating black hole, which is somewhat surprising, as the states are far from supersymmetric when $\bar{N}_p \gg 1$.

Cardy's formula gives the entropy of a 3-charge black hole in a canonical ensemble, in which the R-charges are not fixed. However, just as for the 2-charge black hole, the majority of states being counted have zero R charge. Thus the difference between the density of states with zero R charge d_0 and the total density of states d is subleading for large N . The density of (supersymmetric) states with fixed and large j^+ and $h = N_p$ was computed by [77], in the case where $X_4 = K3$. This gives

$$S = 2\pi \sqrt{NN_p - (j^+)^2}, \quad (5.34)$$

exactly reproducing the Bekenstein-Hawking entropy (5.14).

Much subsequent work has been devoted to refining the computation of the black hole entropy; in particular, in recent times, substantial effort has gone into reproducing subleading terms in the entropy on both sides of the correspondence, see the review [10]. In the bulk, this involves evaluating higher derivative corrections on the leading order solution, whilst in the CFT one needs subleading terms in the asymptotic expansion of the degeneracy of states. In computing the latter, one can use an appropriate supersymmetric index, the elliptic genus in the case of $K3$ and a topological partition function in the case of T^4 .

Whilst the index is clearly a useful tool in the counting of BPS states, it is important to derive more properties of the microstates being counted, in order to deduce the corresponding properties of fuzzball solutions. There is surprisingly little literature on the relevant CFT states; for example, no correlation functions for non-primary operators have been explicitly computed even in the orbifold theory.

At the same time, one can already infer information about typical black hole microstates from the long/short string picture introduced in [52], which more formally corresponds to the decomposition of the Hilbert space of the orbifold theory into twisted sectors. The supersymmetric 3-charge microstates have left moving excitation level N_p and no right moving excitations. In the sector of twist n_i the momentum is quantized in units of $1/n_i$, although the total momentum is necessarily integral. This immediately implies [52] that the entropy is dominated by the highest twist (twist N) sector, or equivalently by the longest strings, as the momentum N_p can be partitioned maximally in this sector. Thus to establish properties of a typical microstate of a Strominger–Vafa black hole, it may suffice to explore states in this sector.

5.3. Survey of fuzzball solutions

The key step in constructing fuzzball solutions for the two charge D1–D5 system was to dualize known supergravity solutions for a fundamental string carrying an arbitrary profile wave. The three and four charge systems cannot, however, be dualized to any analogous system, and thus there is no systematic construction of fuzzball solutions corresponding to black hole microstates. Instead families and isolated examples of horizon-free, non-singular solutions with the correct conserved charges have been constructed using standard techniques for finding supergravity solutions. In particular, the two principle construction techniques are:

1. Supersymmetric classification techniques: In recent years there has been considerable progress in classifying supersymmetric solutions of supergravity theories. In the context of the fuzzball proposal, the classification of solutions

of minimal (ungauged) supergravity in five dimensions [79], and its extension to minimal supergravity coupled to Abelian vector multiplets, has proved particularly useful. As we will review below, every supersymmetric solution is built from a set of defining data, a four-dimensional hyper-Kähler space along with certain functions and forms on this space. Specific choices of this defining data reproduce the previously known supersymmetric rotating black hole and black ring solutions, but more general choices of defining data lead to horizon-free non-singular fuzzball solutions. Note that the classifications of six dimensional supergravity solutions found in [84,85] have also proved useful.

The five-dimensional fuzzball or, as they are often called, *bubbling*, solutions can be embedded into eleven-dimensional supergravity compactified on a Calabi-Yau or torus, in which case they should be related to microstates of M-brane black holes. By taking the hyper-Kähler space to be asymptotically locally flat (Taub-NUT), and reducing on the Taub-NUT circle one also obtains candidate geometries for four-dimensional four-charge black holes. Moreover, using a chain of dualities, the same defining data generates a D1–D5–P solution of type IIB on T^4 or $K3$, and thus gives candidate fuzzball geometries for the microstates of this system.

Below we will discuss in detail which black hole microstates are likely to be captured by such solutions. Let us remark here, however, that it is clear a priori that solutions of minimal supergravity coupled to Abelian vector multiplets will not be sufficient to capture all fuzzball solutions for a given black hole. In the two charge case, we saw that a finite fraction of the black hole entropy was associated with fuzzballs which also had internal excitations, along the compact part of the geometry. Such fuzzballs with internal excitations excite all type II supergravity fields, and when compactified along the compact manifold, give rise to generic solutions of *extended* rather than minimal supergravities in five or six dimensions.

In the three charge system we would also expect to find fuzzball solutions with internal excitations, as solutions of extended supergravity theories in five or six dimensions. However, there is to date no complete classification of solutions of the relevant extended supergravity theories, so finding fuzzballs with internal excitations is an open problem. We should emphasize here that such fuzzballs are likely to have qualitatively different properties to those without internal excitations (as they did in the two charge system) and thus one will need to know these properties before successfully coarse-graining over geometries.

2. Horizon-less limits of known black hole solutions: The second construction technique is to begin with known supergravity solutions describing rotating charged black objects, and then take careful limits of the parameters in the solutions, to obtain horizon-free non-singular solutions. Note that this technique is applicable to non-supersymmetric solutions, representing microstates of non-extremal black holes, and has been used to generate the few known fuzzball solutions for non-extremal black holes.

We will discuss below which black hole microstates are likely to be captured by such techniques, but let us already mention the main limitation of the technique. The most general non-extremal black hole and black ring solutions in five dimensions are labeled by a discrete number of parameters, and admit three Killing vectors (time, plus two rotational symmetries). The fuzzball geometries obtained by careful limits of the parameters are thus also highly symmetric, and are labeled by only a few parameters in addition to their conserved charges. Thus one does not obtain families of solutions, parameterized by arbitrary functions, as in the two charge case; one obtains only a discrete (small) number of fuzzball geometries. Moreover, the high degree of symmetry often allows one to identify uniquely the dual microstate, and it is found to be atypical.

There is by now a substantial literature on fuzzball solutions for the three and four charge cases. The aim of this section will not be to give a comprehensive discussion of every known solution, but instead to give an overview of the known solutions, emphasizing the connections between them. A comprehensive review of the known three charge fuzzball geometries was given in the review [18].

Solutions for three charge black holes and black rings have been discussed in [86–89,90,90–94]. In particular bubbling solutions for black holes and black rings, to be described below, were developed in [95–104]. Four charge solutions in four dimensions have been discussed in [105,106]. Non-supersymmetric solutions were found and their properties explored in [107–109].

5.3.1. Supersymmetric solutions of M theory

Let us begin with supersymmetric solutions of eleven-dimensional supergravity compactified on T^6 , which were obtained in [90] using the classification of solutions of minimal ungauged supergravity coupled to Abelian vector multiplets in five dimensions. The eleven-dimensional solutions of interest have a metric of the form

$$ds^2 = - \left(\frac{1}{Z_1 Z_2 Z_3} \right)^{2/3} (dt + k)^2 + (Z_1 Z_2 Z_3)^{1/3} h_{mn} dx^m dx^n + \left(\frac{Z_2 Z_3}{Z_1^2} \right)^{1/3} (dy_1^2 + dy_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2} \right)^{1/3} (dy_3^2 + dy_4^2) + \left(\frac{Z_1 Z_2}{Z_3^2} \right)^{1/3} (dy_5^2 + dy_6^2), \quad (5.35)$$

with the four form being

$$F = F^1 \wedge dy_1 \wedge dy_2 + F^2 \wedge dy_3 \wedge dy_4 + F^3 \wedge dy_5 \wedge dy_6. \quad (5.36)$$

Thus the solutions are defined by three functions (Z_1, Z_2, Z_3) , one form k and three two-forms (F^1, F^2, F^3) . The two forms can be written as

$$F^a = \theta^a - \frac{1}{2}d(Z_a^{-1}(dt + k)), \quad (5.37)$$

with $a = 1, 2, 3$. Then supersymmetric solutions are such that h_{mn} is hyper-Kähler, and the forms θ^a are self-dual and closed on the hyper-Kähler manifold H_4 , namely

$$\theta^a = *_4 \theta^a; \quad d\theta^a = 0, \quad (5.38)$$

with the Hodge dual taken in the metric h_{mn} . As we will see later in this section, smooth fuzzball solutions will require choosing pseudo hyper-Kähler base spaces, in which the signature changes in certain regions of the space. The functions Z_a and the one form k then satisfy

$$\begin{aligned} \square Z_a &= 2|\epsilon^{abc}|(\theta_b \cdot \theta_c); \\ dk + *_4 dk &= 2 \sum_a \theta^a Z_a, \end{aligned} \quad (5.39)$$

where $|\epsilon^{abc}|$ is the absolute value of the totally antisymmetric tensor; \square is the Laplacian on H_4 and for 2-forms on H_4 , $(\alpha \cdot \beta) \equiv \frac{1}{2}\alpha^{mn}\beta_{mn}$. Note that setting $Z_1 = Z_2 = Z_3$ and reducing on the torus, gives a solution of minimal ungauged supergravity in five dimensions; the effective five-dimensional solution is thus within the framework of that general classification [79].

The Killing spinors of these solutions are given by

$$\epsilon = (Z_1 Z_2 Z_3)^{-1/6} \epsilon_0, \quad (5.40)$$

where the 32-dimensional Majorana spinor satisfies the following projection conditions

$$\Gamma^{056} \epsilon_0 = \Gamma^{078} \epsilon_0 = \Gamma^{09(10)} \epsilon_0 = -\epsilon_0, \quad (5.41)$$

and furthermore the spinor ϵ_0 is covariantly constant on the hyper-Kähler space. Note that since $\Gamma^{0123456789(10)} = 1$ the projection conditions also imply that $\Gamma^{1234} \epsilon_0 = \epsilon_0$. The proof is reviewed in [Appendix B](#).

Reducing on the torus to five dimensions for general (Z_1, Z_2, Z_3) , gives solutions of $\mathcal{N} = 8$ ungauged supergravity in five dimensions, which involve only three scalars, three gauge fields and the metric. Thus the solutions involve only a subset of the $\mathcal{N} = 8$ fields; more general fuzzball solutions involving excitations along the T^6 would need to involve additional $\mathcal{N} = 8$ fields and have not yet been constructed in the three charge case.

Replacing $|\epsilon_{abc}|$ by the intersection form C_{abc} of a Calabi-Yau gives a solution of $\mathcal{N} = 2$ ungauged supergravity, corresponding to eleven-dimensional supergravity compactified on a Calabi-Yau. Further reducing the five-dimensional solutions along a compact isometry in the hyper-Kähler manifold results in a four-dimensional solution, corresponding to reduction of eleven-dimensional supergravity solutions on the product of a Calabi-Yau and a circle.

Solutions of this type, built from the defining data of a 4-dimensional hyper-Kähler space, along with harmonic forms and functions on this space, account for almost all known bubbling solutions for M theory and type IIA two, three and four charge black holes. The main class of exceptions are the two charge fuzzballs with internal excitations [22], given in (4.24), which switch on many additional fields in the compactified theory. More recently in [104] coordinate transformations have also been used to generate from known bubbling solutions new smooth three charge solutions, which involve internal excitations.

In what follows it is useful to work directly with the eleven-dimensional solution, rather than its dimensional reduction to five or four dimensions. In the next section, we will see how these solutions are dualized to the type IIB frame. Moreover, when analyzing regularity conditions, it is regularity in the uplifted solution which is relevant, not that in the dimensionally reduced solution. There are many known examples where the dimensionally reduced solution is singular, where the uplifted solution is not. Conditions for the absence of closed timelike curves can differ, and where they do again it is the conditions in the uplifted solution which are relevant.

5.3.2. Bubbling solutions of type IIB

The 11d solutions can be related to solutions of type IIB on T^4 by a simple chain of dualities. First reduce on the y_6 circle to obtain an F1-D2-D2 solution of type IIA, then T-dualize on (y_3, y_4, y_5) to obtain a D1-D5-P type IIB solution on T^4 .

The solutions in the type IIB frame can then be written in the form:

$$\begin{aligned} ds^2 &= -\frac{1}{\sqrt{Z_1 Z_2 Z_3}}(dt + k)^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}(dz + \mathcal{A}_3)^2 + \sqrt{Z_1 Z_2} dx_4^2 + \sqrt{\frac{Z_2}{Z_1}} dz_4^2, \\ e^{2\Phi} &= \frac{Z_2}{Z_1}; \quad F^{(3)} = (Z_1^{-2} Z_2 Z_3)^{-2/3} *_5 F_1 + F_2 \wedge (dz + \mathcal{A}_3), \end{aligned} \quad (5.42)$$

where dz_4^2 is the metric on T^4 or $K3$ and the dual $*_5$ is defined in the five-dimensional metric

$$ds_5^2 = -(Z_1 Z_2 Z_3)^{-2/3} (dt + k)^2 + (Z_1 Z_2 Z_3)^{1/3} dx_4^2. \quad (5.43)$$

They are therefore defined in terms of three functions Z_a with $a = 1, 2, 3$ and four one forms (ω, \mathcal{A}_a) with $\mathcal{F}_a = d\mathcal{A}_a$, along with the metric on the four-dimensional hyper-Kähler space. Introducing three harmonic self-dual one forms η_a on the base space the connections \mathcal{A}_a are

$$\mathcal{A}_a = -Z_a^{-1} (dt + k) + \eta_a. \quad (5.44)$$

The form k satisfies

$$dk + *dk = \sum_a Z_a d\eta_a, \quad (5.45)$$

whilst the functions Z_a satisfy

$$\square Z_a = \frac{1}{2} |\epsilon_{abc}| (d\eta_b)_{ij} (d\eta_c)^{ij}, \quad (5.46)$$

where \square is the Laplacian on the hyper-Kähler space. Clearly the defining data is the same as for the M theory and type IIA solutions.

The Killing spinors in this case can be written as

$$\epsilon = (Z_1 Z_2)^{-1/8} (Z_3)^{-1/4} \epsilon_0, \quad (5.47)$$

where the spinor ϵ_0 is covariantly constant on the hyper-Kähler space and satisfies the projection conditions

$$\epsilon_0 = \Gamma^{01} \epsilon_0 = \Gamma^{016789} \epsilon_0 = -i\epsilon_0^*. \quad (5.48)$$

These projection conditions along with the Majorana-Weyl condition imply that the covariantly constant (complex) spinor $\epsilon_0 = \epsilon_0^1 + i\epsilon_0^2$ is such that $\epsilon_0^1 = -\epsilon_0^2 \equiv \eta_0$ and

$$\Gamma^{01} \eta_0 = \Gamma^{016789} \eta_0 = \Gamma^{2345} \eta_0 = \eta_0. \quad (5.49)$$

Again we include for convenience the derivation of these spinors in [Appendix B](#).

5.3.3. Types of supersymmetric solutions

We have seen that the supersymmetric solutions of both type IIB and M theory are specified by a set of data, the metric h_{mn} on the hyper-Kähler space, along with three functions Z_a and one-forms (k, η_a) , satisfying (5.45) and (5.46). We will now review various black hole and fuzzball solutions characterized by different choices of the forms and functions. In view of what will follow, we will focus on solutions in the type IIB frame.

a. Explicit integration for Gibbons–Hawking base space

The most general defining data can be given in terms of three harmonic functions h_a and three harmonic one forms η_a on the hyper-Kähler base space. An additional one form k_- on the hyper-Kähler space, whose field strength is anti-self-dual, appears as an integration constant.

That is, the scalar functions Z_a can be formally expressed as [\[90,92\]](#)

$$Z_a = h_a + \frac{1}{2} |\epsilon_{abc}| \square^{-1} ((d\eta_b)_{ij} (d\eta_c)^{ij}), \quad (5.50)$$

whilst the three forms \mathcal{A}_a are given by

$$\mathcal{A}_a = Z_a^{-1} (dt + k) + \eta_a. \quad (5.51)$$

For the asymptotically flat limit of the metric to be manifest, one can use a (constant) gauge transformation on \mathcal{A}_3 and write it as

$$\mathcal{A}_3 = (Z_3^{-1} - 1)(dt + k) + (k + \eta_3). \quad (5.52)$$

Since the form k satisfies

$$dk + *dk = - \sum_a Z_a d\eta_a \quad (5.53)$$

the solution implicitly includes an integration constant. There is always the freedom to add to k any form k_- which satisfies

$$dk_- + *_4 dk_- = 0. \quad (5.54)$$

Explicit integrated solutions are not known in general. In the case of hyper-Kähler base spaces with a $U(1)$ isometry which is preserved, the equations can however be integrated [112]. Writing the base space in the Gibbons–Hawking form [110, 111], the metric is

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V dx^i dx^i \quad (5.55)$$

where x^i with $i = 2, 3, 4$ are coordinates on R^3 and the connection A satisfies $*_3 dA = dV$. Then the solution may be written in terms of seven harmonic functions (K_a, L_a, M) on R^3 as

$$\begin{aligned} Z_a &= V^{-1} K_b K_c + L_a; \\ \eta_a &= V^{-1} K_a (d\psi + A) - *_3 dK_a, \\ k &= k_\psi (d\psi + A) + \hat{k}_i dx^i; \\ k_\psi &= V^{-2} K_a K_b K_c + \frac{1}{2} V^{-1} \sum_a K_a L_a + M; \\ *_3 d\hat{k} &= V dM - M dV + \frac{1}{2} \sum_a (K_a dL_a - L_a dK_a). \end{aligned} \quad (5.56)$$

Here $*_3$ denotes the Hodge dual on R^3 . Note that the harmonic functions (K_a, M) define (η_a, k_-) respectively. This explicit integration is central to most of the fuzzball solutions which have been constructed, with the strategy being to choose harmonic data on R^3 such that regularity conditions are satisfied.

Finding explicit solutions for the case in which the hyper-Kähler base space does not have a $U(1)$ isometry is an open problem. In the 2-charge system fuzzballs in which the symmetry of the transverse R^4 is completely broken form a less singular and more representative basis. One might expect the same to be true in the 3-charge system, and thus one would like to solve explicitly in the non-symmetric case.

b. Black holes and black rings

Supersymmetric three charge rotating black hole and black ring geometries can be realized as specific solutions within this framework. In fact the defining data for the supersymmetric rotating black holes was already given in (5.28): three harmonic functions sourced at the origin of an R^4 base space, along with one anti-self dual form (the integration constant k_-).

Let us now consider the supersymmetric black rings found and analyzed in [114, 112, 90, 115, 113]. The defining functions can be written as

$$\begin{aligned} Z_a &= 1 + \frac{Q_a}{\Sigma} - \frac{1}{2} |\epsilon_{abc}| \frac{q_b q_c R^2 \cos 2\theta}{\Sigma^2}; \\ \Sigma &= (r^2 + R^2 \cos^2 \theta); \\ \mathcal{A}_a &= Z_a^{-1} (dt + k) + \frac{q_a R^2}{\Sigma} (\sin^2 \theta d\varphi - \cos^2 \theta d\phi); \\ k_\phi &= -\frac{r^2 \cos^2 \theta}{2\Sigma^2} \left(\sum_a q_a Q_a - q_1 q_2 q_3 \left(1 + \frac{2R^2 \cos^2 2\theta}{\Sigma} \right) \right); \\ k_\varphi &= -\frac{\sum_a q_a R^2 \sin^2 \theta}{\Sigma} + \left(1 + \frac{R^2}{r^2} \right) \tan^2 \theta k_\phi, \end{aligned} \quad (5.57)$$

with $a = 1, 2, 3$ and the base space being R^4 with the metric

$$dx_4^2 = \Sigma \left(\frac{dr^2}{(r^2 + R^2)} + d\theta^2 \right) + (r^2 + R^2) \sin^2 \theta d\varphi^2 + r^2 \cos^2 \theta d\phi^2. \quad (5.58)$$

The defining harmonic functions and harmonic forms in this case are simply

$$\begin{aligned} h_a &= \left(1 + \frac{Q_a}{\Sigma} \right); \\ \eta_a &= \frac{q_a R^2}{\Sigma} (\sin^2 \theta d\varphi - \cos^2 \theta d\phi). \end{aligned} \quad (5.59)$$

In the form k is included an integration constant (5.54)

$$k_- = -\frac{\sum_a q_a R^2}{2\Sigma} (\sin^2 \theta d\varphi + \cos^2 \theta d\phi). \quad (5.60)$$

In contrast to the black hole solution, the harmonic functions in the black ring are sourced on a circle in R^4 , located at $r = 0$, $\theta = \pi/2$. Since the solution preserves $U(1)^2$ isometries of the hyper-Kähler base (R^4), it can also be rewritten as a solution of type (a) on a Gibbons–Hawking base space. The metric on R^4 given in (5.58) can be rewritten in Gibbons–Hawking form via the coordinate transformations

$$r \cos \theta = 2\sqrt{\rho} \cos\left(\frac{1}{2}\theta_3\right); \quad r^2 \sin^2 \theta + R^2 = 4\rho; \quad \phi = \frac{1}{2}(\psi + \phi_3); \quad \varphi = \frac{1}{2}(\psi - \phi_3), \quad (5.61)$$

to give the Gibbons–Hawking metric

$$dx_4^2 = V^{-1}(d\psi + \cos \theta_3 d\phi_3)^2 + V dx^i dx_i, \quad (5.62)$$

where the R^3 metric is written as

$$dx_i dx^i = d\rho^2 + \rho^2(d\theta_3^2 + \sin^2 \theta_3 d\phi_3^2) \quad (5.63)$$

and the Gibbons–Hawking potential is $V = 1/\rho$. Thus the black ring solutions written in this coordinate system are such that the harmonic functions are sourced at $\rho = \frac{1}{4}R^2$ and $\theta_3 = \pi$. The source circle thus wraps the fibre, and is located at a point on R^3 . Letting the Cartesian coordinates be $x^1 = \rho \cos \theta_3$, $x^2 = \rho \sin \theta_3 \cos \phi_3$ and $x^3 = \rho \sin \theta_3 \sin \phi_3$, then the source is located at $\vec{x}_0 \equiv (-\frac{1}{4}R^2, 0, 0)$. Noting that

$$(r^2 + R^2 \cos^2 \theta) = 4 \left(\rho^2 + \frac{1}{2}R^2 \rho \cos \theta_3 + \frac{R^4}{16} \right)^{1/2} = 4|x - x_0|, \quad (5.64)$$

the defining harmonic functions in R^3 are given by

$$\begin{aligned} h &= \frac{1}{|x - x_0|}; & L_a &= 1 + \frac{1}{4}(Q_a - q_b q_c)h; \\ K_a &= -\frac{1}{2}q_a h; & M &= \frac{1}{4} \sum_a q_a \left(1 - \frac{1}{4}R^2 h \right). \end{aligned} \quad (5.65)$$

Multi-centered supersymmetric black rings, and black saturns, may then be immediately constructed by choosing instead defining harmonic functions in R^3 which are multi-centered [112,90,113]. Note, however, that each point source in R^3 corresponds to a circle of sources in R^4 , except when the source is located at the origin of R^3 , in which case the radius of the wrapped fiber is zero, and the harmonic function is sourced at the origin of R^4 . The latter choice of harmonic function gives as above a supersymmetric black hole.

Let us consider the solutions in the type IIB frame, where they are understood in the context of the D1–D5–P system. The single supersymmetric black ring is characterized by seven parameters, (Q_a, q_a, R) along with the moduli (R_z, V) , where R_z is the radius of the z circle and $v = (2\pi)^4 V$ is the volume of X_4 . The integral charges N_a of the black ring are given by [114, 90,115]

$$Q_1 = \frac{g(\alpha')^3}{V} N_1; \quad Q_2 = g\alpha' N_5; \quad Q_3 = \frac{g^2(\alpha')^4}{VR_z^2} N_p, \quad (5.66)$$

where (N_1, N_5, N_p) are the D1-brane, D5-brane and momentum charges respectively. The dimensionless (non-conserved) dipole charges n_a are given by

$$q_1 = \frac{g\alpha'}{R_z} n_1; \quad q_2 = \frac{g(\alpha')^3}{VR_z} n_2; \quad q_3 = R_z n_3, \quad (5.67)$$

and the angular momenta at asymptotically flat infinity are

$$J_\phi = \frac{1}{2} \sum_a n_a N_a - \frac{1}{2} n_1 n_2 n_3; \quad J_\psi = J_\phi + \frac{R_z V}{(\alpha')^4 g^2} (q_1 + q_2 + q_3) R^2. \quad (5.68)$$

The entropy of the black ring can be written as

$$S = 2\pi \sqrt{n_1 n_2 n_3 \delta - \gamma^2} \quad (5.69)$$

where

$$\begin{aligned} \gamma &= \frac{1}{2}(n_3 N_3 - n_1 N_1 - n_2 N_2 + n_1 n_2 n_3); \\ \delta &= \frac{N_1 N_2}{n_3} - n_1 N_1 - n_2 N_2 + n_1 n_2 n_3 - \frac{q_3 R^2}{C}, \\ C &= \frac{(\alpha')^4}{R_z V}. \end{aligned} \quad (5.70)$$

Given that the supersymmetric black ring has the same charges as the three charge black hole, it should be interpreted as a specific mixed state in the D1–D5 CFT, with momentum N_p , characterized by the parameters (n_a, R) . Explicitly identifying this mixed state is an open problem. Presumably candidate 3-charge fuzzball geometries with the same angular momentum as the black ring should correspond to microstates of both the 3-charge black hole and the 3-charge black ring. However, identifying which geometries would be relevant for the black ring is likely to be rather difficult, unless the corresponding CFT mixed state can be identified.

In the context of the fuzzball proposal, one might wonder whether there exist horizon-less non-singular limits of the black ring solutions. These would have D1–D5–P charges and could correspond to microstates of both the black hole and of the black ring. The case $\delta = 0 = \gamma$ is known to give a solution without a horizon, but with an orbifold singularity [116]. Clearly, other restrictions on the parameters also give solutions with zero entropy, such as for example $\gamma = \delta = n_1 n_2 n_3$, and it is possible that specific restrictions could give rise to regular horizon-free geometries. This issue has not yet been systematically investigated.

c. Lunin–Mathur solutions

It is useful to show explicit how the two-charge Lunin–Mathur fuzzball solutions (4.32) are contained within this solution set. Let us first restrict the defining data to three harmonic functions h_a and a single one form $\eta^3 \equiv \eta$ on the hyper-Kähler space, taken now to be R^4 . In these solutions the remaining three one-forms \mathcal{A}_a are defined in terms of (h_a, η) as

$$\begin{aligned}\mathcal{A}_1 &= h_1^{-1}(dt + k); & \mathcal{A}_2 &= h_2^{-1}(dt + k); \\ \mathcal{A}_3 &= (h_3^{-1} - 1)(dt + k) + (k - \eta) \equiv (h_3^{-1} - 1)(dt + k) + b.\end{aligned}\quad (5.71)$$

Substituting these expressions into (5.42) the three form can be written as

$$F^{(3)} = d(h_2^{-1}(dt + k) \wedge (dz + b)) + *_4 dh_1. \quad (5.72)$$

This is the same form as in the Lunin–Mathur solutions, as one would expect, since the gravitational wave does not couple to the RR field strengths. However, the form k satisfies the relation

$$dk + *_4 dk = h_3 d\eta, \quad (5.73)$$

and therefore dk *cannot* be harmonic except when h_3 is constant. On setting $\eta = 0$ and choosing the harmonic functions to be of the standard (single-centered) form

$$h_a = \left(1 + \frac{Q_a}{r^2}\right) \quad (5.74)$$

one clearly recovers the three charge static extremal black string. Choosing η to be non-zero, for the same choice of harmonic functions, reproduces the three charge rotating extremal BMPV black string.

On setting $h_3 = 1$ one recovers the Lunin–Mathur geometries as follows. Let $d\eta = dA + *_4 dA \equiv dA + dB$, where the definition of B is that $dB = *_4 dA$, so that $\eta = A + B$. Then (5.73) is solved by letting $k = A$ which implies that in (5.72) $b = B$. Choosing the explicit forms of the harmonic functions to be those given in (4.33) one indeed obtains the solutions given in (4.32).

It is important to note however that the two charge fuzzball solutions with internal excitations cannot be obtained within this framework, as they involve many additional fields in type IIB, and hence in the compactified theory.

d. Bubbling geometries

We now turn our attention to the main class of fuzzball geometries which have been constructed, the so-called bubbling solutions [95–104,106]. Given the detailed discussion of these backgrounds in other works, for example the review [18], we will simply highlight their main features here.

The basic idea of the bubbling geometries is to use a non-trivial “ambipolar” $U(1)$ invariant hyper-Kähler base space, whose signature changes in different regions. That is, the potential for the Gibbons–Hawking space is

$$V = \sum_i \frac{q_i}{|x - x_i|}, \quad \sum_i q_i = 1. \quad (5.75)$$

The constraint on the total charge is necessary for the space to be asymptotically flat. In regions where V changes sign the signature of the base space changes from $(+, +, +, +)$ to $(-, -, -, -)$. Denote the 2-dimensional surfaces where V changes sign by Σ_α ; then $V(\Sigma_\alpha) = 0$. For the complete metric signature to remain unchanged this means that

$$Z_a V \geq 0 \quad (5.76)$$

everywhere, which in turn requires that the Z_a change sign at Σ_α , also. Taking the remaining seven harmonic functions (L_a, K_a, M) defined in (5.56) to be sourced at the same locations x_i , so that

$$L_a = 1 + \sum_i \frac{l_a^i}{|x - x_i|}; \quad K_a = \sum_i \frac{k_a^i}{|x - x_i|}; \quad M = m + \sum_i \frac{m^i}{|x - x_i|}, \quad (5.77)$$

one then chooses the parameters (l_a^i, k_a^i, m^i) so that (Z_a, η_a, k) are finite at the locations of the harmonic function sources, x_i . In particular this implies that

$$l_a^i = -\frac{k_b^i k_c^i}{q_i}; \quad m^i = \frac{k_1^i k_2^i k_3^i}{2(q_i)^2}; \quad m = -\frac{1}{2} \sum_{i,a} k_a^i. \quad (5.78)$$

These values are chosen so as to cancel the poles in the defining functions.

Note that the functions Z_a for large $|x|$ are expanded as

$$Z_a = 1 + \frac{1}{|x|} \left(\sum_i l_a^i + \sum_{i,j} k_b^i k_c^j \right) + \dots \equiv 1 - \frac{1}{|x|} \sum_i \frac{\tilde{k}_b^i \tilde{k}_c^i}{q_i} + \dots, \quad (5.79)$$

where

$$\tilde{k}_a^i = k_a^i - q_i \sum_j k_a^j; \quad \sum_i \tilde{k}_a^i = 0. \quad (5.80)$$

One can show that the supergravity solutions are invariant under

$$K_a \rightarrow K_a + c_a V, \quad (5.81)$$

for any constant c_a , see [95,97], and thus physical quantities such as mass which are expressed in terms of Z_a must depend on the invariant quantities \tilde{k}_a^i rather than k_a^i .

For the supergravity solutions to be regular requires:

1. Absence of singularities, in particular (a) at the locations of the sources in the harmonic functions and (b) where the base space signature changes.
2. Absence of closed timelike curves and Dirac-Misner strings.

Conditions (5.76) and (5.78) are sufficient to ensure that the solution is non-singular, see the detailed analysis of [97]. Note that although the defining functions in the solution are built from harmonic functions which have sources at the Gibbons–Hawking centers, there are no delta function sources in the defining functions [95,97]. The solution is therefore regular at the locations of the sources. One can also show that the solution is regular where the base space signature changes, namely where $V \rightarrow 0$. Consider the five-dimensional part of the metric

$$ds_5^2 = -(Z_1 Z_2 Z_3)^{-2/3} (dt + k)^2 + (Z_1 Z_2 Z_3)^{1/3} (V^{-1} (d\psi + A)^2 + V dx^i dx^i) \quad (5.82)$$

in a neighborhood of a hypersurface Σ_a where $V(\Sigma_a) = 0$. Using the explicit forms of the functions one can show that in the neighborhood of a point on Σ_a

$$ds_5^2 \approx -2dt d\varphi + dx^i dx^i. \quad (5.83)$$

Here $d\varphi = d\psi + A_{\Sigma_a}$ with A_{Σ_a} the Gibbons–Hawking gauge field and (t, x^i) have been appropriately rescaled. The metric is regular at Σ_a , with the Killing vector ∂_t becoming null on this surface. A detailed analysis of the regularity of the solutions on these hypersurfaces can be found in [97].

Removing closed timelike curves restricts the parameters further; for example, in the eleven-dimensional geometries one has to ensure that

$$Z_1 Z_2 Z_3 V^2 - k_\psi^2 V \geq 0 \quad (5.84)$$

globally. Recall that the k_ψ was given in (5.56). Solving the constraints this equation makes on the parameters (k_a^i, q_i, \vec{x}_i) in general is rather difficult. However, one can derive constraints which remove Dirac-Misner strings from \hat{k} , defined in (5.56); these so-called *bubble equations* are

$$\sum_{j \neq i} \Pi_{ij}^1 \Pi_{ij}^2 \Pi_{ij}^3 \frac{q_i q_j}{r_{ij}} = -2 \left(m q_i + \frac{1}{2} \sum_a k_a^i \right); \quad (5.85)$$

$$\Pi_{ij}^a = \left(\frac{k_a^i}{q_j} - \frac{k_a^j}{q_i} \right); \quad r_{ij} = |x_i - x_j|.$$

In specific examples, one finds that satisfying these equations is sufficient to guarantee the global absence of CTCs, but this is not always the case and identifying systematically the conditions required to remove closed timelike curves remains an open problem.

The three conserved charges with respect to asymptotically flat infinity can be expressed in terms of the defining data $(\tilde{k}_a^i, q_i, \tilde{x}_i)$ as

$$Q_a = -2\mathcal{N}|\epsilon_{abc}|\sum_{i=1}^N \frac{\tilde{k}_i^b \tilde{k}_i^c}{q_i}, \quad (5.86)$$

where the prefactor \mathcal{N}_a depends on whether one is considering the type IIB or M theory case. In the former case, the Q_a are the conserved D1, D5 and momentum charges, and the normalization is

$$\mathcal{N}_{IIB} = \frac{R_z V}{(\alpha')^4 g^2}. \quad (5.87)$$

In the M theory case, the Q_a are the conserved membrane charges, and the appropriate normalization is

$$\mathcal{N}_M = \frac{\pi}{4G_5} = \frac{\pi V_6}{4G_{11}}, \quad (5.88)$$

with G_d the d -dimensional Newton constant and V_6 the volume of the six torus (or of the Calabi-Yau). The solutions also have conserved angular momenta with respect to asymptotically flat infinity. The generic solution breaks the $SO(4)$ rotational invariance of the transverse R^4 at infinity to $U(1)$, and therefore there are non-zero angular momenta (J_ψ, \vec{J}) , where \vec{J} defines a three dimensional vector in the R^3 of the Gibbons–Hawking metric (5.55). Then one finds that

$$J_\psi = \frac{4}{3}\mathcal{N}|\epsilon_{abc}|\sum_{i=1}^N \frac{\tilde{k}_i^a \tilde{k}_i^b \tilde{k}_i^c}{q_i^2},$$

$$\vec{J} = 8\mathcal{N}\vec{D}, \quad \vec{D} \equiv \sum_{i=1}^N \vec{D}_i, \quad \vec{D}_i \equiv \sum_a \tilde{k}_i^a \vec{x}_i.$$

J_ψ and J_{ϕ_3} correspond to the left and right moving components J_+ and J_- of the CFT R symmetry currents, respectively. More generally \vec{J} should correspond to the right moving $SU(2)$ CFT R current J_-^a . Thus all the bubbling solutions should correspond to R charge eigenstates in the left moving (excited) sector, but need not be R charge eigenstates in the right moving (ground state) sector. Note however that the angular momenta given above are with respect to asymptotically flat infinity, and do not automatically coincide precisely with the vevs of the R symmetry currents, extracted from the decoupling region in the geometry; one needs to extract the vevs of the R symmetry currents using the holographic formulae.

The main goal of the literature has been to find data sets $(\tilde{k}_a^i, q_i, \tilde{x}_i)$ such that the supergravity solution is regular and has the same charges as a black hole or black ring with macroscopic horizon area. For example, recall that a supersymmetric BMPV black hole with integral D1–D5–P charges (N_1, N_5, N_p) and angular momentum $J_+ = j_+$ has entropy

$$S = 2\pi\sqrt{N_1 N_5 N_p - j_+^2}, \quad (5.89)$$

so for a black hole with macroscopic horizon area, one needs $j_+^2 \ll N_1 N_5 N_p$. The regular solutions first constructed had $j_+^2 \sim N_1 N_5 N_p$, and thus could not correspond to microstates of a macroscopic BMPV black hole. Later, however, it was observed that $j_+^2 = kN_1 N_5 N_p$ (with the fraction k being smaller than one) could be achieved in scaling solutions [99,101,102], namely solutions in which all the centers are at

$$\vec{x}_i = \mu \vec{y}_i, \quad (5.90)$$

with $\mu \rightarrow 0$ and \vec{y}_i finite. These so-called deep microstate solutions were explored numerically and analytically in [99,101,102].

Recalling the behavior in the 2-charge system it is perhaps unsurprising that the bubbling points need to be clustered at the origin. In the 2-charge geometries, the radius of the curve in R^4 determining the solution was related to the angular momenta, with solutions of small angular momenta deriving from curves of small radii, see the discussion around (4.55). Most of the Ramond ground states have small R charge, and thus the typical fuzzball should be characterized by a curve of small radius. Here we see analogous behavior in the 3-charge system: the defining harmonic functions are sourced on curves in the four-dimensional base space, which must have small radii for the fuzzball to have typical R charges.

It also seems natural that one needs to cluster the centers. In the 2-charge system it seems unlikely that solutions characterized by disconnected curves describe bound states of D1 branes and D5 branes. They can have the same charges and angular momenta as typical black hole microstates, but are most likely related to Coulomb branch physics. Here in the 3-charge system one sees that a scaling solution in which the centers cluster is needed to even obtain a solution with typical charges.

Note that candidate fuzzball geometries for black rings can be found by taking a single center at the origin with $q_1 = 1$, and clustering the remaining points at some distance ρ from the origin. The scale ρ determines the radius of the corresponding black ring. In Section 5.4 we will consider the correspondence between such geometries and D1–D5–P microstates.

5.3.4. Limits of black hole solutions and spectral flow

The second technique used to find candidate fuzzball geometries involves starting with known non-extremal charged rotating black hole solutions, and then taking careful limits of the parameters to obtain solutions with no horizons or singularities. More generally, one could start with any rotating solution of Einstein equations, apply boosts and dualities to obtain solutions carrying the required charges, and then restrict the parameters to obtain horizon-free non-singular solutions. Thus, as previously mentioned, one could also look for fuzzball limits of black ring solutions. A systematic exploration of the possibilities has not yet been carried out.

The principal advantage of this technique is that it does not rely on supersymmetry, and thus allows one to find fuzzball solutions for non-extremal black holes. The main drawback, however, is that the known black hole and black ring solutions are highly symmetric and characterized by only a small number of parameters, which are further restricted by demanding regularity and absence of horizons. Thus, the resulting fuzzball solutions typically have only a few parameters in addition to the required conserved charges, and correspond to rather atypical black hole microstates.

Another related solution generating technique is “spectral flow”: one begins with a given near horizon geometry and makes a coordinate transformation, which preserves the AdS asymptotics. In particular, one can generate certain 3-charge D1–D5–P geometries from 2-charge D1–D5 geometries in this way [88,89,70,91]. Gluing back the asymptotically flat region then gives a fuzzball geometry of the 3-charge D1–D5–P black hole. One should note that these solution generating techniques are often called spectral flow transformations, although it is unclear whether all such named transformations indeed correspond to spectral flow in the CFT.

Before the development of the bubbling solutions using the classification of supergravity solutions, most candidate fuzzball solutions for the 3-charge and 4-charge systems were found by techniques of this kind. Supersymmetric fuzzballs for the D1–D5–P system were found in [88,89,70,91], whilst supersymmetric fuzzballs for the D1–D5–KK system were found in [92] and for the 3-charge supersymmetric black ring in [93]. In [94] time dependent solutions carrying 3 charges were found and in [109] non-supersymmetric fuzzball solutions of the D1–D5–KK system were found and analyzed.

To illustrate these techniques, let us focus on the example of the non-extremal D1–D5–P solutions found in [107] and analysed further in [108]. General non-extremal 3-charge black hole solutions with rotation were given in Eq. (5.3). Smooth geometries with no horizons can be obtained by demanding that the singularity where $g_{rr}^{-1} = 0$ be nothing but a coordinate singularity, analogous to that of polar coordinates at the origin of R^2 . There are four conditions on the parameters needed to ensure regularity (assuming the momentum charge is non-zero):

$$\begin{aligned} m &= a_1^2 + a_2^2 - a_1 a_2 \frac{c_1^2 c_5^2 c_p^2 + s_1^2 s_5^2 s_p^2}{s_1 c_1 s_5 c_5 s_p c_p}; \\ \frac{j+j^{-1}}{s+s^{-1}} &= (l-n), \quad \frac{j-j^{-1}}{s-s^{-1}} = (l+n), \quad (l, n \in \mathbb{Z}); \\ R_z &= \frac{m s_1 c_1 s_5 c_5 \sqrt{s_1 c_1 s_5 c_5 s_p c_p}}{\sqrt{a_1 a_2 (c_1^2 c_5^2 c_p^2 - s_1^2 s_5^2 s_p^2)}}, \end{aligned} \quad (5.91)$$

where R_z is the radius of the z direction and the parameters (j, s) are given by

$$j \equiv \left(\frac{a_2}{a_1} \right)^{1/2}, \quad s \equiv \left(\frac{s_1 s_5 s_p}{c_1 c_5 c_p} \right)^{1/2} \leq 1. \quad (5.92)$$

The conserved charges and angular momenta are given by

$$\begin{aligned} Q_1 &= m s_1 c_1 = \frac{g \alpha'^3 N_1}{V}; & Q_5 &= m s_5 c_5 = g \alpha' N_5; \\ Q_p &= m s_p c_p = \frac{g^2 (\alpha')^4}{V R_z^2} N_p, & N_p &= N_1 N_5 l n; \\ J_\psi &= -\frac{\pi m}{4 G_5} (a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p) = -N_1 N_5 l \\ J_\phi &= -\frac{\pi m}{4 G_5} (a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p) = N_1 N_5 n, \end{aligned} \quad (5.93)$$

where (N_1, N_5, N_p) are the integral charges and

$$\frac{\pi}{4 G_5} = \frac{R_z V}{g^2 (\alpha')^4}. \quad (5.94)$$

The first expressions for (Q_p, J_ψ, J_ϕ) hold generally, with the restrictions to the solutions of interest being given in terms of the integers (l, n) . The mass is given by

$$M = \frac{\pi m}{4 G_5} \left(s_1^2 + s_5^2 + s_p^2 + \frac{3}{2} \right). \quad (5.95)$$

Restricting the parameters to ensure regularity, and focussing on near extremal configurations, this can be rewritten as

$$M = \frac{\pi}{4G_5} (Q_1 + Q_5 + Q_p) + \frac{1}{2R_z} N_1 N_5 (l^2 + n^2 - 1) \equiv M_{BPS} + \Delta M. \quad (5.96)$$

Thus one can see that if one fixes the three charges (Q_1, Q_5, Q_p) , along with the moduli, then the solution is characterized by only one other (integral) parameter. This parameter controls both the non-extremality and the angular momenta, so the solution gives precisely one microstate of a non-extremal D1–D5–P system with fixed non-extremality.

It is useful to rewrite the six-dimensional metric as a fibration over a four-dimensional base space, in order to facilitate comparison with the general forms of supersymmetric solutions. Of course, in the non-supersymmetric case, the base space does not have any special character, but the hyper-Kähler structure is recovered in the supersymmetric limit. The six-dimensional part of the metric can thus be written as

$$\begin{aligned} ds^2 = & \frac{1}{\sqrt{H_1 H_5}} \left(-(f - m)(d\tilde{t} - (f - m)^{-1} m c_1 c_5 (a_1 \cos^2 \theta d\psi + a_2 \sin^2 \theta d\phi))^2 \right. \\ & + f(d\tilde{z} + f^{-1} m s_1 s_5 (a_2 \cos^2 \theta d\psi + a_1 \sin^2 \theta d\phi))^2 \left. + \sqrt{H_1 H_5} \left(\frac{r^2 dr^2}{(r^2 + a_1^2)(r^2 + a_2^2) - m r^2} + d\theta^2 \right. \right. \\ & + (f(f - m))^{-1} [f(f - m) + f a_2^2 \sin^2 \theta - (f - m) a_1^2 \sin^2 \theta d\phi^2 + 2 m a_1 a_2 \sin^2 \theta \cos^2 \theta d\psi d\phi \\ & \left. \left. + (f(f - m) + f a_1^2 \cos^2 \theta - (f - m) a_2^2 \cos^2 \theta) d\psi^2 \right] \right), \end{aligned} \quad (5.97)$$

where $\tilde{t} = t_{c_p} - z_{s_p}$ and $\tilde{z} = z_{c_p} - t_{s_p}$. In the supersymmetric limit this six-dimensional metric can indeed be rewritten in terms of an ambipolar hyper-Kähler base and harmonic functions on this base [91].

The decoupling region of this geometry is obtained by replacing $\tilde{H}_1 = Q_1$ and $\tilde{H}_5 = Q_5$ and also approximating $m s_1 s_5 = m c_1 c_5 = \sqrt{Q_1 Q_5}$. This gives

$$\begin{aligned} ds^2 = & \frac{1}{\sqrt{Q_1 Q_5}} \left(-(f - m)(d\tilde{t} - (f - m)^{-1} \sqrt{Q_1 Q_5} (a_1 \cos^2 \theta d\psi + a_2 \sin^2 \theta d\phi))^2 \right. \\ & + f(d\tilde{z} + f^{-1} \sqrt{Q_1 Q_5} (a_2 \cos^2 \theta d\psi + a_1 \sin^2 \theta d\phi))^2 \left. + \sqrt{Q_1 Q_5} \left(\frac{r^2 dr^2}{(r^2 + a_1^2)(r^2 + a_2^2) - m r^2} + d\theta^2 \right. \right. \\ & + (f(f - m))^{-1} [f(f - m) + f a_2^2 \sin^2 \theta - (f - m) a_1^2 \sin^2 \theta d\phi^2 + 2 m a_1 a_2 \sin^2 \theta \cos^2 \theta d\psi d\phi \\ & \left. \left. + (f(f - m) + f a_1^2 \cos^2 \theta - (f - m) a_2^2 \cos^2 \theta) d\psi^2 \right] \right). \end{aligned} \quad (5.98)$$

This metric in turn can be rewritten as the twisted fibration of S^3 over BTZ as in (5.10). The regularity conditions of (5.91) then translate to $J_3 = 0$ and $M_3 = -1$, in which case the BTZ part of the metric becomes global AdS_3 . Imposing both the regularity conditions and setting $m \rightarrow 0$ restricts the fibration further, as we shall see below.

If one focuses on this decoupled region of the geometry, it is easy to understand why the non-extremality is pinned to the angular momentum, and moreover what the corresponding CFT state is. Consider the geometry

$$\begin{aligned} ds^2 = & \sqrt{Q_1 Q_5} \left(-(r^2 + \gamma_1^2 \mu^2) dt^2 + r^2 dy^2 + \frac{dr^2}{r^2 + \gamma_1^2 \mu^2} \right) \\ & + \sqrt{Q_1 Q_5} (d\theta^2 + \cos^2 \theta (d\psi + j_\psi)^2 + \sin^2 \theta (d\phi + j_\phi)^2) + (Q_1/Q_5)^{1/2} dz^2 \end{aligned} \quad (5.99)$$

$$F = \sqrt{Q_1 Q_5} (dt \wedge dy \wedge dr + \cos \theta \sin \theta d\theta \wedge (d\phi + j_\phi) \wedge (d\psi + j_\psi)),$$

with constant dilaton $e^{-2\Phi} = Q_5/Q_1$ and $y \sim y + 2\pi\mu^{-1}$, $\mu = \sqrt{Q_1 Q_5}/R$. Let

$$j_\phi = j_+ dx^+ + j_- dx^-; \quad j_\psi = j_+ dx^+ - j_- dx^-, \quad (5.100)$$

where $dx^\pm = \mu(dt \pm dy)$. The geometry is regular if $\gamma_1^2 = 1$, and has permissible conical singularities if $\gamma_1 = 1/n$ with $n \in \mathbb{Z}$. One can immediately extract the holographic stress energy tensor and R symmetry currents using the formulae for the holographic vevs given in (5.20)

$$\langle T \rangle = \frac{N}{2\pi} ((j_+)^2 - \gamma_1^2)(dx^+)^2 + ((j_-)^2 - \gamma_1^2)(dx^-)^2; \quad \langle J^{\pm 3} \rangle = \frac{N}{2\pi} j_\pm dx^\pm, \quad (5.101)$$

with $N = N_1 N_5$. The vevs of scalar operators vanish. For the case of interest, one finds that $\gamma_1 = 1$ and

$$j_\pm = \frac{1}{2}(l \pm n), \quad (5.102)$$

and thus

$$h = N \left(\frac{1}{4} (l + n)^2 \right); \quad \bar{h} = N \left(\frac{1}{4} (l - n)^2 \right); \quad j_+ = \frac{1}{2} N (l + n); \quad j_- = \frac{1}{2} N (l - n). \quad (5.103)$$

Note that the momentum charge at asymptotically flat infinity is proportional to $(h - \bar{h})$ whilst the mass depends on $(h + \bar{h})$. The CFT state with these charges corresponds to the spectral flow of the NS-NS vacuum. One can see this by recalling that under a spectral flow in the left sector by $\alpha_L = (l + n)$ units and in the right sector by $\alpha_R = (l - n)$ units,

$$h \rightarrow h - \alpha_L j_+ + \frac{c}{24} \alpha_L^2; \quad j_+ \rightarrow j_+ + \frac{c}{12} \alpha_L, \quad (5.104)$$

with corresponding expressions for (\bar{h}, j_-) , where $c = 6N$. Letting $\alpha_R = 1$ and $\alpha_L = 2p + 1$ gives a BPS state in the RR sector which is in the right moving vacuum and in a left moving excited state. When (α_R, α_L) are odd one obtains a non-supersymmetric state in the RR sector which is a microstate of the non-extremal black hole.

Since these states are related by spectral flow to the identity, it is rather simple to compute properties of these states, for example, decay rates of the non-supersymmetric states. (We will defer discussion of this issue to Section 6). At the same time, they are atypical black hole microstates, and thus their properties cannot be assumed to be representative. To be more precise, consider the microstates of the BPS D1–D5–P (Strominger–Vafa) static black hole. Counting states with fixed (N_1, N_5, N_p) and either states with $(j_+ = j_- = 0)$ or *all* states gives the same leading contribution to the entropy, namely the famous result

$$S = 2\pi \sqrt{N_1 N_5 N_p}. \quad (5.105)$$

The point is that most states have $(j_+ = j_- = 0)$, so the difference between the total degeneracy of states d_t with fixed (N_1, N_5, N_p) and arbitrary R-charges, and the degeneracy of states d_0 with fixed (N_1, N_5, N_p) and $(j_+ = j_- = 0)$ is subleading. Fixing j_+ and j_- corresponds to working in the canonical ensemble, whilst allowing for all (j_+, j_-) corresponds to working in the grand canonical ensemble. Clearly the BPS state discussed above, which has $j_+ \gg 1$, does not contribute to the canonical ensemble for the Strominger–Vafa black hole, and it is a highly non-representative state in the grand canonical ensemble, which is peaked around states with zero R-charge. The BPS state above could be interpreted as a microstate of a rotating BMPV black hole with this value of j_+ . However, recalling the BMPV entropy formula, and noting that $j_+^2 = N_1 N_5 N_p$, one sees there are not enough microstates with this value of j_+ to give a black hole with macroscopic horizon area.

More generally, even if the state did not have atypical R-charges, since it is obtained by spectral flow of the identity only the stress energy tensor and R currents can acquire expectation values. By contrast, a typical microstate will be characterized by the vevs of operators within that state, and these vevs will determine the scale at which the geometry starts to differ from the naive black hole geometry.

Before moving to the correspondence between geometries and microstates we should comment that the known non-extremal fuzzball solutions have superradiant instabilities; see for example [117,118,108,119]. This is another indication that these solutions are not representative. As we will discuss in Section 6, one expects non-extremal fuzzballs to be unstable, with their decay rate matching the decay rate of the corresponding non-BPS microstates in the CFT. The radiation emitted by a typical fuzzball should be very similar to that of the non-extremal black hole. However, the known non-extremal fuzzball solutions decay much faster; there is no contradiction, as the decay rate matches that of the corresponding CFT states, see [120,121], but these atypical fuzzballs are not representative.

5.4. Correspondence between geometries and microstates

One of the most important outstanding issues is to derive the correspondence between the candidate fuzzball geometries and the black hole microstates. One would first like to match sufficient data to be sure that the geometries do indeed correspond to black hole microstates, rather than other states with the same charges. Then one would like to explore what fraction of the black hole microstates is captured by the known fuzzball geometries. No systematic analysis of the correspondence has yet been carried out, and a number of issues in matching geometries to microstates are rather puzzling.

The bubbling geometries [95,97–104] are viewed as the most promising candidates for duals to typical black hole microstates. First of all, they have the correct charges to correspond to three charge black holes, and in scaling solutions, one can obtain angular momenta comparable to that of a macroscopic BMPV black hole. By replacing the asymptotically flat base space by an asymptotically locally flat (Taub-NUT) space such that $V \rightarrow 1$ as $r \rightarrow \infty$ one can also obtain candidate geometries for four charge black holes in four dimensions, as were discussed in [106], again with the same charges as typical black hole microstates.

Secondly, the scaling solutions [99,101,102] have throat regions, such that there exists a decoupling region with the same asymptotics as in the black hole geometry. In the context of the type IIB solutions, this implies that bubbling solutions, like the corresponding black hole or black ring, admit an $\text{AdS}_3 \times S^3 \times X_4$ decoupling region. This is a necessary condition for the bubbling solution to correspond to a black hole microstate: the AdS/CFT dictionary implies that any microstate is dual to an asymptotically $\text{AdS}_3 \times S^3 \times X_4$ geometry, with the information encoded in the asymptotics determining the specific microstate.

Solutions of M theory compactified on a Calabi-Yau admit decoupling $\text{AdS}_2 \times S^3$ or $\text{AdS}_3 \times S^2$ regions, depending on the Taub-NUT charge. In the former case, there has been considerable progress on counting the relevant BPS states, via the counting of curves in the Calabi-Yau, and on understanding the quiver quantum mechanics of the D0-D2-D4-D6 branes which wrap these curves and generate the black hole. At the same time, it is hard to use detailed AdS/CFT technology to analyze these bubbling geometries, as one is forced to work within the rather poorly understood $\text{AdS}_2/\text{CFT}_1$ correspondence, and there is no precise dictionary developed between the decoupling AdS_2 regions and the dual theory. In the other case, where one does have an AdS_3 region in an M theory solution, one does not have a good description of the corresponding 2-dimensional CFT, given the limiting understanding of M-brane theories.

Thus, to make full use of AdS/CFT technology to identify specific bubbling solutions, it is natural to work instead with the type IIB solutions, which admit an $\text{AdS}_3 \times S^3 \times X_4$ decoupling region and correspond to states in the relatively well understood D1–D5 conformal field theory. A clear advantage of working in this system is that one already has an identification of the geometries dual to the microstates of the 2-charge black hole, namely the Ramond ground states.

Let us turn to the correspondence between bubbling geometries in the type IIB frame and D1–D5-P microstates in the conformal field theory. Recall first the correspondence between two charge geometries and Ramond ground states: the former are characterized by a curve whose Fourier coefficients determine the corresponding dual superposition of Ramond ground states. Restricting to ground states built from the universal cohomology of X_4 , the curve is a generic closed curve in \mathbb{R}^4 . In the special case where the curve is a circle, the corresponding dual is a specific R charge eigenstate Ramond ground state.

Note that this dictionary determines the dual superposition of Ramond ground states given the curve defining a smooth supergravity solution. As discussed in [21,22] the geometric dual of a generic R charge eigenstate is not known; most likely it is not well described in supergravity, but one can also not exclude that the dual consists of multi-center concentric circular curves. The latter is consistent with both the symmetry of the CFT state and the charges, and cannot be excluded by comparing vevs of operators, since these are just too small for reliable comparisons [21]. A priori, however, it seems rather unnatural that the geometric dual should be multi-center, as this generically signals Coulomb branch, rather than Higgs branch, physics.

Moving on to the three charge bubbling geometries, recall that the defining data for these geometries is the data at the centers of the Gibbons–Hawking metric, namely $(\tilde{k}_i^a, q_i, \vec{x}_i)$. Here q_i is the nut charge of the i th center; \vec{x}_i is its position in \mathbb{R}^3 , and \tilde{k}_i^a determines the associated contributions to the charges. Lifting this data into four-dimensional language, to make contact with the two charge geometries, the defining data is associated with a set of circles on the four-dimensional base space, with \vec{x}_i determining their radii and relative orientation. In particular, if the \vec{x}_i are parallel, the circles are concentric and $|\vec{x}_i|$ determines their radii. The latter solutions preserve a $U(1)^2$ isometry group of the base space, and should correspond to R charge eigenstates in the conformal field theory.

The basic question is therefore: how does the data $(\tilde{k}_i^a, q_i, \vec{x}_i)$ capture the dual D1–D5-P microstate? Consider first the case where the \vec{x}_i are parallel, so the corresponding microstate must be an R charge eigenstate. Then, as a first guess, one might think that the multi-center data corresponds to fractionation in the CFT. Recall that the Ramond ground states in the orbifold CFT language as

$$\prod_j \mathcal{O}_{Rn_j}^{a_j \bar{a}_j} |0\rangle, \quad \sum_j n_j = N, \quad (5.106)$$

where each twist n_j operator $\mathcal{O}_{Rn_j}^{a_j \bar{a}_j}$ is associated with the (a_j, \bar{a}_j) cohomology and corresponds on spectral flow to an NS chiral primary. The same operator can occur multiple times, with appropriate symmetrization. The left moving excited states with total momentum N_p are obtained by exciting these ground states, with the momentum being distributed between the different twist sectors. In the string picture, this corresponds to distributing the momentum between each group of multiwound strings. Note that in such a distribution, the momenta p_j given to each twist sector n_j is non-negative.

It would be rather natural if this fractionation were also present in the corresponding geometries. One might first try to identify each effective string with a Gibbons–Hawking center, but this naive identification does not seem to be correct. Recall that the total charges are given by

$$Q_a = -2\mathcal{N}|\epsilon_{abc}| \sum_i \frac{\tilde{k}_i^b \tilde{k}_i^c}{q_i} \equiv \sum_i (Q_a)_i, \quad (5.107)$$

with $\sum_i \tilde{k}_i^a = 0$ and $\sum_i q_i = 1$. In particular, the contributions to the three charges from centers with positive q_i cannot all be positive. At most, two out of the three $(Q_a)_i$ are positive, with the other negative, or only one of the three is non-zero and positive. This does not however agree with the CFT fractionation, where the total D1–D5-P charges are sums over only positive contributions.

Actually this disagreement is not really surprising: one should be rather careful about associating charge contributions to each center, as there are no sources there. Whilst the solution is built from harmonic functions sourced at the Gibbons–Hawking centers, all the defining functions appearing in the solution are finite at these centers. The distinguished hypersurfaces in the full solution are clearly those where the Killing vector which is timelike at infinity becomes null (5.83),

which happens when the base space signature changes. Thus it may be physically more natural to parameterize the solutions in terms of functions which have poles at these surfaces.

Indeed, this is explicitly demonstrated by the analysis of [91] for the specific 3-charge geometries obtained from limits of Cvetic–Youm black hole solutions. Whilst one can rewrite these solutions in terms of six harmonic functions on an ambipolar Gibbons–Hawking base space, the physical interpretation is more manifest in the original Cvetic–Youm coordinates, in which the defining functions are harmonic on R^4 with poles at the sources for the D-brane charges. An important open question is thus whether one can reparameterize the bubbling solutions in such a way that the connection with fractionation in the CFT is manifest, and moreover such that one can systematically solve the equations for absence of closed timelike curves. For recent progress along related lines see [104]: here coordinate transformations are used to relate smooth three charge solutions with particular hyper-Kähler bases to solutions in which several of the Gibbons–Hawking centers are replaced by two charge (Lunin–Mathur) geometries. Such techniques may help to understand which geometries correspond to bound states and how the space of smooth three-charge solutions is parameterized.

One should mention here that it is possible that the parameterization in terms of the ambipolar Gibbons–Hawking space may still be natural for the M theory geometries; perhaps the centers are related to the nodes in the quiver quantum mechanics as has been suggested in [101,102]. Nonetheless, a systematic solution of the regularity conditions in the M theory system is also still lacking.

Furthermore, without solving such conditions, it would be hard to carry out geometric quantization for these fuzzball geometries; one clearly needs to determine these constraints before quantization. Of course, even if one could carry out the geometric quantization, it would be hard to make further progress without determining which black hole microstates the bubbling geometries sample, i.e. without carrying out the matching.

Another related puzzle is the flight time calculation in the scaling bubbling geometries: this gives a finite answer for symmetric geometries, which can be interpreted in terms of the field theory mass gap, but for non-symmetric geometries the time can become infinite [101,102] although one would still anticipate an energy gap in the field theory. Thus either such geometries do not correspond to microstates, or the calculation needs to be interpreted more carefully.

6. Fuzzballs and black hole physics

The previous sections have focused on the explicit construction of fuzzball solutions in supergravity for near supersymmetric black holes, and the matching of these solutions with black hole microstates. Most of the literature on the fuzzball proposal has concentrated on this issue, since finding and matching candidate geometries for black holes with macroscopic horizons has turned out to be difficult, even in the supersymmetric case. Results to date can be regarded as evidence for the fuzzball proposal, but to make further detailed and quantitative progress on issues such as coarse-graining one would anticipate the need to match better the candidate bubbling geometries with CFT data and to understand fuzzball solutions in the stringy regime.

At the same time, one can already envisage how many key conceptual questions could be answered, and there are many interesting and suggestive results in the current literature. Therefore in this section we will consider how longstanding issues in black hole physics are addressed by the fuzzball proposal.

6.1. Quantizing fuzzball geometries

Suppose one finds fuzzball geometries in supergravity with the correct charges to correspond to microstates of a given black hole. Typically the geometries will be characterized by continuous functions, and will not be countable. Knowledge of the holographic map between these solutions and the CFT microstates should determine how the continuous functions are discretized, as in the case of 2-charge geometries. Alternatively, one could consider quantizing the geometries, as in the discussion in Section 4.5.3. This would allow one to estimate the number of states accounted for by the geometries, and, if it reproduced the black hole entropy, this could be interpreted as evidence that the geometries do describe black hole microstates.

At the same time, it is not clear why quantizing the fuzzball geometries *visible in supergravity* should reproduce the full black hole entropy in general. In the 2-charge case, quantizing the extrapolation of the fuzzball solutions to supergravity (for the subset of solutions that this was done) indeed reproduced the black hole entropy, even though the average solution was string scale, but one cannot assume that this agreement will persist in other, less supersymmetric systems. If geometric quantization of the fuzzball solutions found in supergravity does reproduce the black hole entropy, it would suggest that the extrapolation of the fuzzballs to supergravity is more representative than one may have anticipated. Moreover, one would then be able to address the important issue of coarse-graining, to demonstrate explicitly how black hole properties emerge.

6.2. Finite temperature

Much of the discussion of previous sections has been restricted to the case of supersymmetric (zero temperature) solutions, since one can then use powerful tools to find supergravity solutions. Clearly, restricting to zero temperature removes an important feature of black hole physics: extremal black holes do not Hawking radiate. So an important question

is how the fuzzball proposal works for a non-extremal black hole. Given the very few known candidate fuzzball geometries for non-extremal black holes, the issue of Hawking radiation has not been discussed in generality. However, in cases where AdS/CFT is applicable, one can understand how the non-extremal fuzzballs will radiate, and moreover use the specific known solutions as a testing ground.

First let us recall how Hawking radiation of asymptotically AdS_{d+1} black holes is computed in the dual conformal field theory. Consider radiation of a scalar field ϕ , which corresponds to the dual gauge invariant operator \mathcal{O}_ϕ of dimension Δ_ϕ . Let the emitted radiation have d -dimensional momentum \vec{p} . Now consider a state $|i\rangle$ in the boundary field theory which is dual to an asymptotically AdS_{d+1} spacetime. The rate of emission of the scalar field ϕ with momentum \vec{p} is computed from the decay rate of the corresponding field theory state $|i\rangle$. The differential decay rate $d\Gamma(i)$ is given by

$$d\Gamma(i) = \sum_f |M_{fi}|^2 d\Omega \quad (6.1)$$

where $|f\rangle$ are possible final states, $d\Omega$ is an appropriate phase space factor, and the matrix element M_{fi} is given by

$$\mathcal{N} \langle f | \mathcal{O}_\phi(\vec{p}) | i \rangle, \quad (6.2)$$

with \mathcal{N} a normalization constant (fixed by two point functions at the conformal point). In practice, one computes this by using the optical theorem to relate the sum of $|M_{fi}|^2$ over final states into an appropriate discontinuity of the analytically continued Euclidean two point function.

Let us now focus on the case of interest, a 2d CFT. To compute the decay rate of the BTZ black hole corresponding to a thermal state in a (grand) canonical ensemble, one must average over initial states weighted by the appropriate Boltzmann factors. The analytic continuation of the Euclidean thermal two point function for the scalar operator of dimension Δ_ϕ is given by

$$\Pi(x^+ x^-) \equiv \langle \mathcal{O}_\phi^\dagger(0, 0) \mathcal{O}_\phi(x^+, x^-) \rangle_{\text{thermal}} = \mathcal{C} \left[\frac{\pi T_+}{\sinh(\pi T_+ x^+)} \right]^{\Delta_\phi} \left[\frac{\pi T_-}{\sinh(\pi T_- x^-)} \right]^{\Delta_\phi}, \quad (6.3)$$

where \mathcal{C} is a normalization factor, $x^\pm = t \pm z$ are lightcone coordinates and T_\pm are the left and right moving temperatures respectively. Expressed in terms of the inverse temperature $\beta = T^{-1}$ conjugate to the energy ω and the chemical potential μ_z conjugate to the momentum p_z , $2T_\pm^{-1} = \beta \pm \mu_z$. In terms of the BTZ temperature T_H given in (5.16) and the inner and outer horizon locations ρ_\pm these quantities are

$$T_+^{-1} = \beta T_H^{-1} \left(1 + \frac{\rho_-}{\rho_+} \right); \quad T_-^{-1} = \beta T_H^{-1} \left(1 - \frac{\rho_-}{\rho_+} \right). \quad (6.4)$$

Then the emission rate Γ_e is proportional to the discontinuity in the two point function

$$\Gamma_e = \mathcal{F} \int dt dz e^{i\omega t - ip_z z} \Pi(t + i\epsilon, z) \quad (6.5)$$

where \mathcal{F} is the flux of the emitted particle. For example, in the case of $\Delta_\phi = 2$ and $p_z \rightarrow 0$ and inserting appropriate normalization factors one obtains

$$\Gamma_e = \frac{\pi^3 Q_1 Q_5 \omega}{(e^{\frac{\omega}{2T_+}} - 1)(e^{\frac{\omega}{2T_-}} - 1)} \quad (6.6)$$

in agreement with the corresponding Hawking radiation rate computed in the bulk. More details of this calculation may be found in [122]. Note that detailed balance implies that the emission rate $\Gamma_e = e^{-\omega/T_H} \Gamma$ where Γ is the absorption rate.

To compute instead the decay (or absorption) rate of a particular state $|i\rangle$ in an ensemble one needs to compute the appropriate discontinuity in the two point function in that state. For a typical member of a thermal ensemble, one would expect the variance from the thermal spectral density to be small. For an atypical member of the ensemble the decay rate can however differ substantially.

Now suppose one has a candidate asymptotically AdS_{d+1} fuzzball geometry dual to the specific state $|i\rangle$: how should the instability of the state to decay by the operator \mathcal{O}_ϕ be reflected in the bulk, which has no horizon so does not Hawking radiate? The answer is that modes of the bulk field ϕ with real momentum \vec{p} should be tachyonic, with the imaginary part of the frequency reflecting the decay rate.

This is precisely what has been found for the specific known examples of non-extremal fuzzball solutions, such as that discussed in the previous section. The instability of the bulk solutions precisely matches the computed decay rate of the corresponding CFT microstate [120,121]. However, the CFT microstate under consideration is a highly atypical member of the thermal ensemble, and thus the emitted radiation is not even approximately thermal: $\Gamma_e \approx \omega^2$, compared to (6.6). This rapid decay rate is responsible for the superradiant behavior in the bulk solution.

6.3. Path integral approach

Despite its computational and conceptual difficulties, the Euclidean path integral approach to gravity rather naturally accounts for black hole thermodynamics. The Euclidean path integral is

$$Z = \int D[g]D[\Phi]e^{-I_E(g,\Phi)}, \quad (6.7)$$

where g is the metric, Φ collectively denotes matter fields and I_E is the Euclidean action. Whilst making sense of the integration over geometries remains an open problem, it is interesting that in the saddle point approximation the onshell Euclidean action for a solution of the Einstein (supergravity) equations behaves as the free energy F , i.e.

$$I_E = \beta F = \beta(E - \mu_i Q_i) - S, \quad (6.8)$$

with β the inverse temperature, μ_i the conjugate potentials to conserved charges Q_i (angular momentum, electric charge etc) and S the Bekenstein–Hawking entropy. This identification is consistent with standard thermodynamic relations, for example

$$S = -\left(\frac{\partial F}{\partial T}\right)_{Q_i} = \left(\beta \frac{\partial I_E}{\partial \beta} - I_E\right). \quad (6.9)$$

Note that extremal black holes should be treated as a limiting case of non-extremal black holes, in which case their entropy is non-zero. If one works directly at extremality ($T = 0$) the Euclidean action is linear in β , and thus the entropy would seem to vanish.

The fuzzball solutions are, however, solutions of the gravitational field equations with the same asymptotics as the black hole, and are thus also saddle points of the path integral. Fuzzball solutions which are stationary in the Lorentzian correspond to topologically trivial Euclidean solutions, since the imaginary time Killing vector has no fixed point sets. There is a natural thermodynamic interpretation of the fuzzball solutions: each solution has the same mass E and conserved charges Q_i as the black hole, and quantization of the solutions should give a total degeneracy $\mathcal{D} = e^S$. Therefore the Euclidean action I_{f_a} associated with each fuzzball f_a should be

$$I_{f_a} = \beta F = \beta(E - \mu_i Q_i). \quad (6.10)$$

Clearly by construction one finds that

$$\int_{f_a} D[g_a]D[\Phi_a]e^{-I_{f_a}} = e^{-I_{f_a}+S} = e^{-I_{BH}}, \quad (6.11)$$

where the functional integral is over all fuzzball geometries f_a , with metric g_a and matter fields Φ_a , and I_{BH} is the action for the black hole of the same mass and charges.

Evidently one would like to derive these formulae explicitly in a specific example, by quantizing the set of fuzzball solutions. However, the argument above suggests that in the integration over geometries one should not include both black holes and fuzzballs simultaneously, as this is over counting. One should either include all topologically trivial configurations with the same charges (namely the fuzzballs) or one should include the topologically non-trivial configuration (namely the black hole). In cases where AdS/CFT is applicable, this fits exactly with field theory expectations: one either sums over pure states or in equilibrium configurations typical states may be represented by thermal states, but one does not simultaneously include both pure and mixed states when computing a partition function.

Hawking's proposed resolution of the information loss paradox [123] also makes use of the path integral over asymptotically AdS metrics. However, both topologically trivial and non-trivial configurations are included simultaneously in the path integral. The former are unitary and the latter are not, but it is argued that contributions to correlation functions from topologically non-trivial configurations fall off rapidly, and do not contribute at late times. Thus just as in the fuzzball proposal one effectively only includes the topologically trivial configurations in the path integral although of course the justification is rather different.

6.4. Distinguishing between a black hole and a fuzzball

Given that the black hole curvature is small at the horizon scale, it may initially seem surprising that the fuzzball solutions already start to differ from the black hole at this scale. One might have argued that deviations from the black hole spacetime which resolve its singularity should be localized near the curvature singularity. At the same time, standard results in black hole physics already indicate that the horizon itself is deeply connected with quantum gravity effects. Firstly, the existence of a closed trapped surface is an input into the singularity theorems relevant to black holes. As we review below, it is hard to evade these theorems within supergravity when a horizon is present. Secondly, Hawking radiation is associated with the existence of a horizon, and thus with the well known trans-Planckian and backreaction issues indicating the breakdown of semiclassical approximations.

One might worry that the small differences between the fuzzball geometry and that of the black hole at the horizon scale manifest as substantial differences in the measurements of an observer at infinity. In asymptotically AdS examples, however, we have discussed in detail how a black hole microstate is characterized by vevs and higher point functions; the deviation of these from the corresponding ensemble average is small for a typical microstate, and thus distinguishing between black hole and fuzzball requires a large sequence of measurements over a long timescale. Essentially the same point was made by Hawking in [123]: an observer at spatial infinity would need to make measurements over an infinite time to be certain that a black hole formed and then evaporated.

Inside the outer future event horizon of a non-extremal black hole the Killing vector ∂_t is spacelike, and the radius necessarily decreases along future directed timelike geodesics. In a corresponding fuzzball solution, the geometry at sub-horizon scales may be radically different, with for example, ∂_t remaining timelike, so cannot an observer at infinity detect this difference? Again this question needs to be phrased more precisely in terms of measurements accessible to the observer: scattering of particles and measurement of emitted radiation. The former corresponds to excitation and subsequent decay by radiation of the black hole with the latter corresponding to spontaneous radiation. A typical non-extremal fuzzball should emit radiation with only small deviations from the thermal spectrum of the black hole, which would take the observer a long time to measure. Discussions of the small scale of deviations of correlation functions in a typical state from that of the thermal state can be found in [124,125]; see also [126].

Note that atypical non-extremal fuzzballs can contain ergoregions, in which a Killing vector which is timelike at infinity becomes spacelike. Such ergoregions are responsible both for superradiance processes (mirroring excitation and decay of the black hole) and for spontaneous emission (corresponding to Hawking radiation). However these atypical fuzzballs decay much faster than the corresponding black hole, as we saw in Section 6.2. One can also understand Hawking radiation from static black holes in terms of spontaneous emission from an ergoregion bounded by the horizon. The key difference between the black hole and the atypical fuzzball in this respect is that in former case the ergoregion straddles the horizon, whereas for the fuzzball, there is no horizon to cloak it.

6.5. Gravitational collapse

Whilst eternal black holes represent a simplified system for explicit analysis, one would also like to apply the fuzzball proposal to astrophysical gravitational collapse. In astrophysical collapse, it is believed that given sufficiently dense and generic initial data the formation of first a closed trapped surface and then a (cloaked) singularity is inevitable. The fuzzball proposal in this context would be that supergravity solutions exist which describe the collapse of initial data, avoiding horizon and singularity formation. The late-time quasi-stationary solution should differ from the corresponding black hole spacetime only at sub-horizon scales.

To date, only stationary fuzzball solutions have been found, and it would be difficult to find the time dependent solutions needed to describe gravitational collapse. However, in cases where one can use AdS/CFT, the initial data corresponding to a pure state in the field theory should collapse to form a fuzzball. Recall that the fuzzball solutions generically involve all supergravity fields (and beyond) reflecting the fact that in a generic state all primary (and other) operators acquire a vev. If one specifies initial data only for a subset of fields then one is restricting to a subspace of the phase space, and this could be interpreted as tracing over the complementary part of the phase space, so this initial data should correspond to mixed states. Such initial data should collapse to form black holes. It would clearly be interesting to investigate this further, and understand how this connects to results from numerical simulations of gravitational collapse. In particular, it is known that even for spherically symmetric collapse horizon formation can be avoided for non-generic initial data [127]. For the non spherically symmetric collapse relevant to fuzzballs fewer results are known (see the review [128]) but perhaps the conditions on initial data needed to avoid horizon formation can be related to conditions on the dual theory being in a pure state.

It is also interesting to note here that many of the results of numerical relativity support the fuzzball proposal: by adding just one scalar field to the action, one can find numerical solutions of the Einstein equations which are horizonless, non-singular and asymptote to the black hole solution. In particular, in a recent paper [129] numerical solutions were found which could be interpreted as fuzzballs for the Schwarzschild black hole.

6.6. Singularity theorems

The fuzzball proposal states that each black hole microstate corresponds to a non-singular horizon-free geometry. In this section, we will consider how the proposal is reconciled with the general proofs of singularity theorems.

We have seen in explicit examples that fuzzball solutions for spherically symmetric black holes break the spherical symmetry, and involve many more of the supergravity fields than the black hole solution. This, at first sight, seems somewhat reminiscent of proposals for singularity resolution in general relativity suggested in the 1960s. The first singularity theorems showed that there would be singularities in symmetric solutions under certain reasonable conditions, but the results depended on the symmetry being exact. It was thus suggested by a number of authors (in particular, the Russian school of Lifshitz, Khalatnikov and co-workers [130]) that singularities were the result of symmetries and would not occur in more general solutions. Indeed this idea seemingly fits with the fuzzball proposal, as the non-singular solutions are less symmetric and involve more fields.

However, the idea that singularities were non-generic was soon disfavored, in part because the Russian school found that explicit families of non-symmetric solutions did contain singularities [131], but principally because of the Penrose–Hawking proofs [132] of singularity theorems without assumptions of symmetry. So how is the fuzzball proposal reconciled with the singularity theorems?

To understand this it is useful to recall the three main conditions required to prove the existence of a singularity, or to be more precise, geodesic incompleteness of the spacetime:

1. An energy condition;
2. Condition on global structure;
3. Gravity strong enough to trap a region.

The exact conditions depend on the theorem under consideration, see [133]; for example, the second condition can be the existence of a non-compact Cauchy surface or that the chronology condition holds throughout the spacetime. More importantly for our purposes, the third condition can be that the spatial cross-section of the universe is closed, or that there is a closed trapped surface in the spacetime. The proofs input these conditions into the Raychaudhuri equation to show that timelike or null geodesics converge within finite affine length, and thus that the spacetime is geodesically incomplete.

The fuzzball solutions which are non-singular in supergravity evade the singularity theorems via the third condition: the supergravity fields do satisfy the weak and/or strong energy condition, and there is an appropriate non-compact Cauchy surface, but there is no closed trapped surface or horizon in the spacetime!

Of course, as we have emphasized throughout this report, a generic fuzzball solution should be described as a solution of the full string theory equations of motion, not just as a supergravity background. In such cases, the singularity theorems are simply not applicable. For the same reason, whilst the singularity theorems imply that any black object in supergravity has a singularity behind its horizon, stringy corrections are expected to resolve these singularities.

6.7. Emergence of horizons

The main objective of the fuzzball proposal is to show that black hole properties emerge upon coarse-graining over fuzzball geometries. In particular, one would like to understand the emergence of the defining property of a macroscopic black hole, its horizon. Given the incomplete understanding of fuzzball geometries for even supersymmetric macroscopic black holes, such a coarse-graining has not yet been carried out.

One can however use rather general arguments to infer the scale of the coarse-grained geometry. One first needs to estimate how much a typical fuzzball geometry differs from the naive black hole solution, but this is a question which is easy to answer, having set up the AdS/CFT dictionary. Near the conformal boundary, in the far UV of the field theory, all fuzzball solutions should look like the naive solution, since in the UV we see only the conformal behavior. The solutions should start differing at energies comparable to the scale set by the vev of the most relevant gauge invariant operator. Using AdS/CFT this translates into a certain radial distance scale.

Suppose one estimates the vev of the lowest dimension operator in a typical microstate of the black hole, and then translates this value into a characteristic radial scale r_c . Now compute the area of a spatial surface of constant radius r_c : the area of this so-called stretched horizon (in the naive metric) should approximately reproduce the black hole horizon area. A version of this argument was given in [11,12,134,135], although here we use the AdS/CFT dictionary and the vevs to determine more systematically the relevant radial scale.

It is straightforward to carry out this calculation for the 2-charge black hole. In this case the vev of the most relevant gauge invariant operator is expressed in terms of the curve characterizing the fuzzball solution. Thus the radial scale r_c is determined by the scale of a typical curve. Now recall that the defining curve $F^a(v)$ in k -dimensional space, with $a = 1, \dots, k$, behaves as

$$F^a(v) = \mu \sum_{n>0} (f_n^a \exp(2\pi i n v/L) + (f_n^a)^* \exp(-2\pi i n v/L)), \quad (6.12)$$

with $\mu = \sqrt{Q_1 Q_5}/R_z$, $L = 2\pi Q_5/R_z$ and

$$Q_1 = \frac{Q_5}{L} \int_0^L (\partial_v F^a)^2 dv \rightarrow 2 \sum_{a,n>0} n (f_n^a (f_n^a)^*) = 1. \quad (6.13)$$

A typical curve $F^a(v)$ satisfying this constraint will have a scale of the order of

$$r_c \sim \frac{\mu}{\sqrt{N}}. \quad (6.14)$$

Now recall that the naive 2-charge metric (in the string frame) in the decoupling region is

$$ds^2 = \frac{r^2}{\sqrt{Q_1 Q_5}} (-dt^2 + dz^2) + \sqrt{Q_1 Q_5} \left(\frac{dr^2}{r^2} + d\Omega_2^2 \right) + \frac{\sqrt{Q_1}}{\sqrt{Q_5}} ds^2(X_4), \quad (6.15)$$

with $e^{2\phi} = Q_1/Q_5$. Then the area of a spatial surface at constant r_c in the Einstein frame metric ($g_{\text{ein}} = e^{-\phi/2}g$) is \mathcal{A} where

$$\mathcal{A} = (2\pi)^6 \pi V R_z \sqrt{Q_1 Q_5} r_c \sim (2\pi)^6 \pi (\alpha')^4 \sqrt{N}. \quad (6.16)$$

Putting the area into the Bekenstein-Hawking entropy formula gives

$$S = \frac{2\mathcal{A}}{(2\pi)^6 (\alpha')^4} \sim 2\pi \sqrt{N}. \quad (6.17)$$

Hence the area of the stretched horizon reproduces the microscopic entropy of the 2-charge black hole. Of course one should be a little cautious of this computation, as the typical scale implied by (6.14) is

$$r_c \sim \frac{(\alpha')^2}{R_z \sqrt{V}}, \quad (6.18)$$

which is clearly substring scale for the compactification radii of interest, as indeed it must be for the 2-charge black hole which has no horizon in supergravity.

Whilst these general arguments can be used to infer the characteristic scale of a typical fuzzball geometry, explicitly demonstrating that a horizon emerges upon coarse-graining remains an open problem. One first needs to construct a representative basis of fuzzball geometries for a black hole with a macroscopic horizon before one can explore quantization and coarse-graining. The three charge system seems a promising test case for exploring such issues.

Note that if the fuzzball proposal is correct, then coarse-graining over horizonless backgrounds of trivial topology should result in the black hole background, whose topology is non-trivial. The black hole should have a global feature, a horizon, which none of the individual fuzzballs possess. On the Euclidean section this translates into periodicity in imaginary time, and non-trivial topology.

It may at first seem surprising that the black hole has a property which is not shared by any of the fuzzballs. Of course, given that this property is associated with entropy, this must be the case if the fuzzball proposal is to be correct. Moreover, the emergence of non-trivial topology under coarse-graining precisely mirrors the corresponding field theory behavior: thermal states in the field theory are characterized by periodicity in imaginary time, which is not a property of any state in the thermal ensemble.

6.8. Stringy fuzzballs

Throughout this report, we have emphasized that many of the fuzzballs for any given black hole will not be well-described by supergravity. To develop the fuzzball proposal, and indeed to understand more rigorously small black holes, one will most likely need to work with backgrounds of the full string theory. Clearly, there are many technical obstacles to overcome in this area: string theory in backgrounds with Ramond-Ramond fluxes is still rather poorly understood, even given the progress with the pure spinor formalism, and solving the worldsheet string theory in curved backgrounds is generically very hard.

At the same time, not all issues relating to stringy fuzzballs are likely to be intractable. Suppose one considers a worldsheet theory in an asymptotically $\text{AdS}_3 \times S^3 \times X_4$ fuzzball background, which is sufficiently strongly curved that the supergravity approximation does not hold. Just as in the case of $\text{AdS}_3 \times S^3 \times X_4$, the information that one extracts from the worldsheet theory is the (holographic) correlation functions.

Throughout this report, we have expressed the correspondence between fuzzballs and black hole microstates in terms of these correlation functions: the one-point functions are used to identify the specific microstate whilst the two-point functions characterize the decay rate of non-BPS fuzzballs etc. Even when a geometric description is accessible, the correspondence is not expressed in terms of the local geometry, but rather in terms of measurements made in the asymptotic region. Thus the absence of a geometric description does not necessitate a complete reformulation of the fuzzball proposal: the information one computes from the worldsheet theory is exactly the information needed to probe the fuzzball proposal.

In the supergravity regime, one could compute holographic one-point functions from a given supergravity solution from algebraic manipulations even when solving fluctuation equations to extract higher correlation functions is intractable. The same should be true for the worldsheet theory: given the string spectrum in the $\text{AdS}_3 \times S^3$ background one would expect that one can extract holographic one-point functions from the worldsheet theory in asymptotically $\text{AdS}_3 \times S^3$ backgrounds even if one cannot fully solve the worldsheet theory. Moreover, one would anticipate being able to determine whether a given worldsheet theory corresponds to a black hole (i.e. there is entropy) or to a fuzzball. One should be able to address in general terms how one point functions and entropy are encoded in the worldsheet theory, and this would be interesting in itself.

7. Conclusions

The fuzzball proposal is a promising idea, which has the potential to resolve longstanding black hole related puzzles. In the few years since this proposal was suggested, a body of evidence in support of the proposal has been found. In particular, families of fuzzball solutions visible within supergravity have been found for various black hole systems in

string theory, and holographic technology has been used to match their properties with those of black hole microstates. All computations, be they of holographic one point functions, scattering or geometric quantization, support the interpretation of these geometries as black hole microstates. The aim of this report was to collect, in one place, all current evidence, emphasizing interconnections and pointing out open problems and avenues for further research. We further sketched how key issues in black hole physics could be addressed by the fuzzball proposal.

A recurrent theme in this report was the use of the AdS/CFT correspondence to test, motivate and perhaps even explain why (a form) of the fuzzball proposal should hold, at least for black holes which admit AdS near horizon regions. In our view, precision holography would be needed if this interesting idea is to become a physical model that would either be falsified, or explain black holes at a quantitative level, and for this reason, we placed particular emphasis on this topic.

Whilst the fuzzball proposal has the potential to address black hole issues, substantial work and technical progress is still needed to demonstrate explicitly how the key issues are resolved. One would like to show in concrete examples how black hole properties, such as the horizon and Hawking radiation, emerge upon coarse-graining over fuzzballs. To address these issues, one will need to understand better the holographic map between solutions and microstates and how to coarse-grain geometries. Most likely one will also be forced to work with backgrounds of the full string theory, rather than just supergravity.

The fuzzball proposal is a concrete idea which could explain black hole physics, and is very natural from a holographic perspective. Given its potential and the increasing body of supporting evidence, it merits further development and investigation.

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Appendix A. Conventions for field equations and supersymmetry

The equations of motion for IIA supergravity are:

$$\begin{aligned} e^{-2\Phi} \left(R_{mn} + 2\nabla_m \nabla_n \Phi - \frac{1}{4} H_{mpq}^{(3)} H_n^{(3)pq} \right) - \frac{1}{2} F_{mp}^{(2)} F_n^{(2)p} - \frac{1}{2 \cdot 3!} F_{mpqr}^{(4)} F_n^{(4)pqr} + \frac{1}{4} G_{mn} \left(\frac{1}{2} (F^{(2)})^2 + \frac{1}{4!} (F^{(4)})^2 \right) &= 0, \quad (\text{A.1}) \\ 4\nabla^2 \Phi - 4(\nabla \Phi)^2 + R - \frac{1}{12} (H^{(3)})^2 &= 0, \\ dH^{(3)} = 0, \quad dF^{(2)} = 0, \quad \nabla_m F^{(2)mn} - \frac{1}{6} H_{pqr}^{(3)} F^{(4)npqr} &= 0, \\ \nabla_m (e^{-2\Phi} H^{(3)mnp}) - \frac{1}{2} F_{qr}^{(2)} F^{(4)qmp} - \frac{1}{2 \cdot (4!)^2} \epsilon^{npm_1 \dots m_4 n_1 \dots n_4} F_{m_1 \dots m_4}^{(4)} F_{n_1 \dots n_4}^{(4)} &= 0, \\ dF^{(4)} = H^{(3)} \wedge F^{(2)}, \quad \nabla_m F^{(4)mnpq} - \frac{1}{3! \cdot 4!} \epsilon^{npqm_1 \dots m_3 n_1 \dots n_4} H_{m_1 \dots m_3}^{(3)} F_{n_1 \dots n_4}^{(4)} &= 0. \end{aligned}$$

The corresponding equations for type IIB are:

$$\begin{aligned} e^{-2\Phi} \left(R_{mn} + 2\nabla_m \nabla_n \Phi - \frac{1}{4} H_{mpq}^{(3)} H_n^{(3)pq} \right) - \frac{1}{2} F_m^{(1)} F_n^{(1)} - \frac{1}{4} F_{mpq}^{(3)} F_n^{(3)pq} - \frac{1}{4 \cdot 4!} F_{mpqrs}^{(5)} F_n^{(5)pqrs} \\ + \frac{1}{4} G_{mn} \left((F^{(1)})^2 + \frac{1}{3!} (F^{(3)})^2 \right) &= 0, \\ 4\nabla^2 \Phi - 4(\nabla \Phi)^2 + R - \frac{1}{12} (H^{(3)})^2 &= 0, \\ dH^{(3)} = 0, \quad \nabla_m (e^{-2\Phi} H^{(3)mnp}) - F_m^{(1)} F^{(3)mnp} - \frac{1}{3!} F_{mqr}^{(3)} F^{(5)mqrnp} &= 0, \quad (\text{A.2}) \\ dF^{(1)} = 0, \quad \nabla_m F^{(1)m} + \frac{1}{6} H_{pqr}^{(3)} F^{(3)pqr} &= 0, \\ dF^{(3)} = H^{(3)} \wedge F^{(1)}, \quad \nabla_m F^{(3)mnp} + \frac{1}{6} H_{mqr}^{(3)} F^{(5)mqrnp} &= 0, \\ dF^{(5)} = d(*F^{(5)}) = H^{(3)} \wedge F^{(3)}, \end{aligned}$$

where the Hodge dual of a p -form ω_p in d dimensions is given by

$$(*\omega_p)_{i_1 \dots i_{d-p}} = \frac{1}{p!} \epsilon_{i_1 \dots i_{d-p} j_1 \dots j_p} \omega_p^{j_1 \dots j_p}, \quad (\text{A.3})$$

with $\epsilon_{01 \dots d-1} = \sqrt{-g}$. The RR field strengths are defined as

$$F^{(p+1)} = dC^{(p)} - H^{(3)} \wedge C^{(p-2)}. \quad (\text{A.4})$$

The equations of motion for the heterotic theory are:

$$\begin{aligned} 4\nabla^2 \Phi - 4(\nabla \Phi)^2 + R - \frac{1}{12} (H^{(3)})^2 - \alpha' (F^{(c)})^2 &= 0, \\ \nabla_m (e^{-2\Phi} H^{(3)mnr}) &= 0, \\ R^{mn} + 2\nabla^m \nabla^n \Phi - \frac{1}{4} H^{(3)mrs} H_{rs}^{(3)n} - 2\alpha' F^{(c)mr} F_r^{(c)n} &= 0, \\ \nabla_m (e^{-2\Phi} F^{(c)mn}) + \frac{1}{2} e^{-2\Phi} H^{(3)mrs} F_{rs}^{(c)} &= 0. \end{aligned}$$

$F_{mn}^{(c)}$ with $(c) = 1, \dots, 16$ are the field strengths of Abelian gauge fields $V_m^{(c)}$; we consider here only supergravity backgrounds with Abelian gauge fields. This restriction means that the gauge field part of the Chern–Simons form in H_3 ,

$$H^{(3)} = dB^{(2)} - 2\alpha' \omega_3(V) + \dots, \quad (\text{A.5})$$

does not play a role in the supergravity solutions, nor does the Lorentz Chern–Simons term denoted by the ellipses.

The action for eleven-dimensional supergravity is

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g_{11}} \left(R - \frac{1}{12} F^2 - \frac{1}{432} \epsilon^{M_1 \dots M_{11}} F_{M_1 \dots M_4} F_{M_5 \dots M_8} A_{M_9 \dots M_{11}} \right), \quad (\text{A.6})$$

whilst the supersymmetry variation of the gravitino is

$$\delta \Psi_M = \left(D_M + \frac{1}{144} (\Gamma_M^{NPQR} - 8\delta_M^N \Gamma^{PQR}) F_{NPQR} \right) \epsilon \quad (\text{A.7})$$

with ϵ a 32-component Majorana spinor.

Our conventions for the (truncated) type IIB action are

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_{10}} \left(e^{-2\Phi} (R_{10} + 4(\partial \Phi)^2) - \frac{1}{12} (F^{(3)})^2 + \dots \right), \quad (\text{A.8})$$

where $2\kappa_{10}^2 = (2\pi)^7 g_s^2 (\alpha')^4$; for the most part, we set $g_s = 1$ since it plays no role in our discussion. The part of the supersymmetry variations relevant here are then

$$\begin{aligned} \delta \lambda &= \Gamma^m \partial_m \Phi \epsilon + \frac{i}{12} e^\Phi \Gamma^{mnp} F_{mnp} \epsilon^*, \\ \delta \Psi_m &= D_m \epsilon - \frac{i}{48} e^\Phi F_{npq} \Gamma^{npq} \Gamma_m \epsilon^* \end{aligned} \quad (\text{A.9})$$

where $\epsilon = \epsilon_1 + i\epsilon_2$ is a complex Majorana–Weyl spinor.

Appendix B. Killing spinors

In this appendix, we will review the derivation of the Killing spinors of the bubbling solutions (5.35) and (5.42). The proof of the Killing spinors for the M theory solutions (5.35) runs as follows. Introduce a vielbein

$$e^{\hat{0}} = f^{-1/3} (dt + k); \quad e^{\hat{m}} = f^{1/6} \tilde{e}^{\hat{m}}; \quad e^{\hat{5}} = \left(\frac{Z_2 Z_3}{Z_1^2} \right)^{1/6} dy_1, \dots \quad (\text{B.1})$$

where $f = Z_1 Z_2 Z_3$, $\hat{m} = 1 \dots 4$ and $\tilde{e}^{\hat{m}}$ a vielbein for the hyper-Kähler metric h_{mn} . The ellipses denote corresponding expressions for the $e^{\hat{\mu}}$ with $\hat{\mu} = 5 \dots 10$. The spin connection is

$$\begin{aligned} \omega^{\hat{0}\hat{m}} &= \frac{1}{3} f^{-3/2} \tilde{e}^{m\hat{m}} \partial_m f (dt + k) + f^{-1/2} (dk)_{im} \tilde{e}^{m\hat{m}} dx^i; \\ \omega^{\hat{m}\hat{n}} &= \frac{1}{6} f^{-1} \partial_n f \tilde{e}^{n\hat{m}} \tilde{e}^{\hat{n}} + \tilde{\omega}^{\hat{m}\hat{n}} + \frac{1}{2} f^{-1} (dk)_{nm} \tilde{e}^{m\hat{m}} \tilde{e}^{n\hat{n}} (dt + k); \\ \omega^{\hat{5}\hat{m}} &= \frac{1}{6} \left(\frac{\partial_m (Z_2 Z_3)}{Z_2 Z_3 \sqrt{Z_1}} - 2 \frac{\partial_m Z_1}{\sqrt{Z_1^3}} \right) \tilde{e}^{m\hat{m}} dy_1, \end{aligned} \quad (\text{B.2})$$

with corresponding expressions for the other $\omega^{\hat{m}\hat{n}}$. Here $\tilde{\omega}$ is the spin connection for the hyper-Kähler space. Now the components of the Killing spinor equations along the T^6 directions reduce to

$$\left(\partial_{y_1} + \frac{1}{12} \omega_{y_1}^{\hat{5}\hat{m}} \Gamma_{\hat{5}\hat{m}} - \frac{1}{12} \Gamma_{\hat{5}\hat{m}} \tilde{\epsilon}^{\hat{m}\hat{n}} \left(\frac{\partial_m Z_2}{Z_2 \sqrt{Z_1}} \Gamma^{\hat{0}\hat{7}\hat{8}} + \frac{\partial_m Z_3}{Z_3 \sqrt{Z_1}} \Gamma^{\hat{0}\hat{9}(\hat{1}\hat{0})} - 2 \frac{\partial_m Z_1}{\sqrt{Z_1^3}} \Gamma^{\hat{0}\hat{5}\hat{6}} \right) \right) \epsilon \\ + \frac{1}{6Z_1 \sqrt{Z_2 Z_3}} \Gamma^{\hat{5}\hat{m}\hat{n}} \left(Z_2 f_{\hat{m}\hat{n}}^2 \Gamma^{\hat{7}\hat{8}} + Z_3 f_{\hat{m}\hat{n}}^3 \Gamma^{\hat{9}(\hat{1}\hat{0})} - 2Z_1 f_{\hat{m}\hat{n}}^1 \Gamma^{\hat{5}\hat{6}} \right) \epsilon = 0,$$

where the two forms f^a are defined as

$$f_{\hat{m}\hat{n}}^a \tilde{\epsilon}^{\hat{m}} \wedge \tilde{\epsilon}^{\hat{n}} = \theta^a - \frac{1}{2} Z_a^{-1} dk. \quad (\text{B.3})$$

The projection conditions

$$\Gamma^{056} \epsilon = \Gamma^{078} \epsilon = \Gamma^{09(10)} \epsilon = -\epsilon \quad (\text{B.4})$$

along with the duality of the two forms θ^a , are sufficient to ensure these equations are satisfied for spinors which are independent of the torus coordinates. The time component of the Killing spinor equation is also satisfied given the same conditions. The components along the hyper-Kähler space reduce, using the projections to

$$\left(\partial_m + \frac{1}{4} \tilde{\omega}_m^{\hat{m}\hat{n}} \Gamma_{\hat{m}\hat{n}} + \sum_{a=1}^3 \frac{\partial_m Z_a}{6Z_a} \right) \epsilon = 0, \quad (\text{B.5})$$

and are thus satisfied, provided that the explicit form of the spinor is

$$\epsilon = (Z_1 Z_2 Z_3)^{-1/6} \epsilon_0 \quad (\text{B.6})$$

with ϵ_0 covariantly constant on the base hyper-Kähler space. It is useful to extract the explicit form of these spinors on a Gibbons–Hawking space, namely a hyper-Kähler space with a $U(1)$ isometry. The metric on this space can be written in the form

$$ds_4^2 = V^{-1} (d\psi + A)^2 + V dx^i dx^i \quad (\text{B.7})$$

where x^i with $i = 2, 3, 4$ are coordinates on R^3 and the connection A satisfies $*_3 dA = dV$. Introducing the vierbein

$$\tilde{e}^1 = V^{-1/2} (d\psi + A); \quad \tilde{e}^i = V^{1/2} dx^i, \quad (\text{B.8})$$

the spin connection is

$$\tilde{\omega}^{1i} = -\frac{1}{2} V^{-2} \partial_i V (d\psi + A) + \frac{1}{2} V^{-1} (dA)_{ij} dy^j; \quad (\text{B.9}) \\ \tilde{\omega}^{ij} = \frac{1}{2} V^{-2} (dA)_{ij} (d\psi + A) + \frac{1}{2} V^{-1} \partial_j V dx^i.$$

Thus the Killing spinor equations become

$$\left(\partial_\psi - \frac{1}{4} V^{-2} \partial_i V (\Gamma^{1i} + \epsilon_{ijk} \Gamma^{jk}) \right) \epsilon = 0; \quad (\text{B.10}) \\ \left(\partial_i - \frac{1}{4} (V^{-2} A_i \partial_k V - V^{-1} \partial_j V \Gamma^{i1}) \right) (\Gamma^{1j} + \epsilon_{jkl} \Gamma^{kl}) \epsilon = 0.$$

These equations are solved for $\Gamma^{1234} \epsilon = \epsilon$ for constant ϵ_0 .

Next let us consider the type IIB bubbling solutions (5.42). To prove the spinors for the type IIB solution, introduce a vielbein such that

$$\hat{e}^0 = (Z_1 Z_2)^{-1/4} Z_3^{-1/2} (dt + k); \quad \hat{e}^1 = Z_3^{1/2} (Z_1 Z_2)^{-1/4} (dz + \mathcal{A}_3); \quad (\text{B.11}) \\ \hat{e}^{\hat{m}} = (Z_1 Z_2)^{1/4} \tilde{e}^{\hat{m}}; \quad \hat{e}^{\hat{a}} = (Z_2 / Z_1)^{1/4} dy^a,$$

where $\hat{m} = 2, 3, 4, 5, \hat{a} = 6, 7, 8, 9$ and $\tilde{e}^{\hat{m}}$ is a vierbein on the hyper-Kähler space. Note that the three form can conveniently be written in terms of this vielbein as

$$F^{(3)} = (Z_1 Z_2)^{1/4} Z_3^{-1/2} (d\eta_2 - Z_2^{-1} dk) \wedge \hat{e}^1 + Z_1^{1/2} Z_2^{-3/2} dZ_2 \wedge \hat{e}^0 \wedge \hat{e}^1 \\ + *_4 dZ_1 + Z_1^{5/4} Z_2^{-3/4} Z_3^{-1/2} \hat{e}^0 \wedge (d\eta_1 - Z_1^{-1} *_4 dk), \quad (\text{B.12})$$

where $*_4$ refers to the Hodge dual on the hyper-Kähler space.

The associated spin connection takes the form

$$\begin{aligned}
 \omega^{01} &= -\frac{1}{2}(Z_1 Z_2)^{-1/4} Z_3^{-3/2} \partial_{\hat{m}} Z_3 \tilde{e}^{\hat{m}}; \\
 \omega^{0\hat{m}} &= -\frac{1}{4}(Z_1 Z_2)^{1/4} \left(\frac{\partial^{\hat{m}} Z_1}{Z_1} + \frac{\partial^{\hat{m}} Z_2}{Z_2} \right) e^0 - (Z_1 Z_2)^{-1/2} \left(\frac{1}{2} Z_3^{-1/2} \partial^{\hat{m}} Z_3 (dz + \eta_3) - \frac{1}{2} Z_3^{-1/2} (dk)^{\hat{m}\hat{n}} \tilde{e}_{\hat{n}} \right); \\
 \omega^{1\hat{m}} &= -\frac{1}{4}(Z_1 Z_2)^{1/4} \left(\frac{\partial^{\hat{m}} Z_1}{Z_1} + \frac{\partial^{\hat{m}} Z_2}{Z_2} \right) e^1 \\
 &\quad + (Z_1 Z_2)^{-1/2} \left(\frac{1}{2} Z_3^{-1/2} \partial^{\hat{m}} Z_3 (dz + \eta_3) - \frac{1}{2} (Z_3^{1/2} (d\eta_3)^{\hat{m}\hat{n}} + Z_3^{-1/2} (dk)^{\hat{m}\hat{n}}) \tilde{e}_{\hat{n}} \right); \\
 \omega^{\hat{m}\hat{n}} &= \tilde{\omega}^{\hat{m}\hat{n}} + \frac{1}{4} \tilde{e}^{\hat{m}} \left(\frac{\partial^{\hat{n}} Z_1}{Z_1} + \frac{\partial^{\hat{n}} Z_2}{Z_2} \right) - \frac{1}{2} (d\eta_3)^{\hat{m}\hat{n}} (Z_3 (dz + \eta_3) + (dt + k)) - \frac{1}{2} (dk)^{\hat{m}\hat{n}} (dz + \eta_3); \\
 \omega^{\hat{a}\hat{m}} &= \frac{1}{4} (Z_1 Z_2)^{-1/4} \left(\frac{\partial^{\hat{m}} Z_2}{Z_2} - \frac{\partial^{\hat{m}} Z_1}{Z_1} \right) e^{\hat{a}}
 \end{aligned} \tag{B.13}$$

where $\partial_{\hat{m}} Y = \tilde{e}_{\hat{m}}^m \partial_m Y$. Noticing that

$$(dt + k) = Z_3^{1/2} (Z_1 Z_2)^{1/4} e^0; \quad (dz + \eta_3) = Z_3^{-1/2} (Z_1 Z_2)^{1/4} (e^0 - e^1), \tag{B.14}$$

all terms in the spin connection can be expressed in terms of the vielbein.

The spin connection together with the conveniently expressed three form allow us to check the type IIB supersymmetric equations (A.9). The dilatino equation reduces to

$$\begin{aligned}
 i \Gamma^{\hat{m}} \tilde{e}_{\hat{m}}^m \left(\frac{\partial_m Z_2}{Z_2} - \frac{\partial_m Z_1}{Z_1} \right) \epsilon^* + (Z_1 Z_2 Z_3)^{-1/2} (dk)_{\hat{m}\hat{n}} \Gamma^{\hat{m}\hat{n}} (\Gamma^0 + \Gamma^1) \epsilon \\
 + (Z_1 Z_2 Z_3)^{-1/2} \left(Z_2 (d\eta_2)_{\hat{m}\hat{n}} \Gamma^1 + Z_1 (d\eta_1)_{\hat{m}\hat{n}} \Gamma^0 \right) \Gamma^{\hat{m}\hat{n}} \epsilon + \Gamma^{\hat{m}01} \tilde{e}_{\hat{m}}^m \frac{\partial_m Z_2}{Z_2} \epsilon - \frac{\partial_m Z_1}{Z_1} \tilde{e}_{\hat{m}}^m \epsilon^{\hat{m}\hat{n}\hat{p}\hat{q}} \Gamma_{\hat{n}\hat{p}\hat{q}} \epsilon = 0.
 \end{aligned} \tag{B.15}$$

This equation is satisfied by the spinors (5.47), using the projection conditions

$$\epsilon = \Gamma^{01} \epsilon = \Gamma^{0126789} \epsilon = -i\epsilon, \tag{B.16}$$

along with the self-duality of the forms $d\eta_a$. Next consider the components of the gravitino equation along the compact directions, which reduces to:

$$\omega_a^{\hat{a}\hat{m}} \Gamma_{\hat{m}} \epsilon = -\frac{i}{24} F_{MNP}^{(3)} \Gamma^{MNP} \epsilon^*. \tag{B.17}$$

This is also satisfied using the projection conditions (5.47) along with the self-duality of the forms $d\eta_a$, as are the components of the gravitino equation along the string directions. Finally the gravitino equation along the hyper-Kähler space reduces to:

$$\left(\partial_m + \frac{1}{4} \tilde{\omega}_m^{\hat{m}\hat{n}} \Gamma_{\hat{m}\hat{n}} + \frac{\partial_m Z_3}{4Z_3} + \frac{\partial_m Z_1}{8Z_1} + \frac{\partial_m Z_2}{8Z_2} \right) \epsilon = 0. \tag{B.18}$$

This is satisfied, provided that the spinor is

$$\epsilon = (Z_1 Z_2)^{-1/8} Z_3^{-1/4} \epsilon_0, \tag{B.19}$$

where the spinor ϵ_0 is covariantly constant on the hyper-Kähler space.

Appendix C. Properties of spherical harmonics

Scalar, vector and tensor spherical harmonics on the unit radius S^3 satisfy the following equations

$$\begin{aligned}
 \square Y^I &= -\Lambda_k Y^I, \\
 \square Y_a^{Iv} &= (1 - \Lambda_k) Y_a^{Iv}, \quad D^a Y_a^{Iv} = 0, \\
 \square Y_{(ab)}^{I_t} &= (2 - \Lambda_k) Y_{(ab)}^{I_t}, \quad D^a Y_{k(ab)}^{I_t} = 0,
 \end{aligned} \tag{C.1}$$

where $\Lambda_k = k(k+2)$ and the tensor harmonic is traceless. It will often be useful to explicitly indicate the degree k of the harmonic; we will do this by an additional subscript k , e.g. degree k spherical harmonics will also be denoted by Y_k^I , etc.

\square denotes the d’Alambertian along the three sphere. The vector spherical harmonics are the direct sum of two irreducible representations of $SU(2)_L \times SU(2)_R$, which are characterized by

$$\epsilon_{abc} D^b Y^{clv\pm} = \pm(k+1) Y_a^{lv\pm} \equiv \lambda_k Y_a^{lv\pm}. \quad (C.2)$$

The degeneracy of the degree k representation is

$$d_{k,\epsilon} = (k+1)^2 - \epsilon, \quad (C.3)$$

where $\epsilon = 0, 1, 2$ respectively for scalar, vector and tensor harmonics. For degree one vector harmonics l_v is an adjoint index of $SU(2)$, and will be denoted by α . We use normalized spherical harmonics, such that

$$\int Y^I Y^J = \Omega_3 \delta^{IJ}; \quad \int Y^{aI} Y_a^{Jv} = \Omega_3 \delta^{IJv}; \quad \int Y^{(ab)I} Y_{(ab)}^{Jt} = \Omega_3 \delta^{IJt}, \quad (C.4)$$

where $\Omega_3 = 2\pi^2$ is the volume of a unit 3-sphere. We define the following triple integrals as

$$\int Y^I Y^J Y^K = \Omega_3 a_{IJK}; \quad (C.5)$$

$$\int (Y_1^{\alpha\pm})^a Y_1^j D_a Y_1^i = \Omega_3 e_{\alpha ij}^{\pm}. \quad (C.6)$$

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