```
---------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.891
Model:                            OLS   Adj. R-squared:                  0.890
Method:                 Least Squares   F-statistic:                     607.6
Date:                Tue, 23 Jan 2024   Prob (F-statistic):           2.04e-37
Time:                        22:20:19   Log-Likelihood:                -102.30
No. Observations:                  76   AIC:                             208.6
Df Residuals:                      74   BIC:                             213.3
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          3.2028      0.221     14.499      0.000       2.763       3.643
x              9.1104      0.370     24.650      0.000       8.374       9.847
==============================================================================
Omnibus:                        5.816   Durbin-Watson:                   1.896
Prob(Omnibus):                  0.055   Jarque-Bera (JB):                2.579
Skew:                          -0.112   Prob(JB):                        0.275
Kurtosis:                       2.126   Cond. No.                         4.42
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
---------------------------------------------------
```

We have already encountered some of the generated numbers like R2, F-statistic, etc. But the adjacent block of out contains many others … what are they, why are they important, and how to interpret them?

This is the output generated by the Notebook **SLR-using-OLX.ipynb**

Run it on the data that you have created for Exercise E1

AIC (Akaike Information Criteria) and BIC (Bayes Information ...)

- Both these are estimators of **prediction error**. They help in model selection.
- These numbers are used to compare across models. **A lower number indicates a better model**
- A difference in AIC or BIC value of 2, between models being compared, is considered significant. The model with a lower AIC or BIC value is designated as the better model, and becomes a candidate for selection.

- **Omnibus statistic:** This is a numeric value calculated from the skewness and kurtosis of the residuals (the difference between the predicted and actual values). A low value suggests the residuals are closer to a normal distribution, while a high value indicates deviation from normality.
- **Omnibus p-value:** This value represents the probability of observing the calculated Omnibus statistic, assuming the null hypothesis of normally distributed residuals is true. A low p-value (typically < 0.05) suggests there is significant evidence to reject the null hypothesis, implying the residuals are not normally distributed.

Jarque-Bera Test

- In the context of Ordinary Least Squares (OLS) regression, the Jarque-Bera test is used to **check the normality of the residuals.**
  - **Residuals**, the difference between the predicted and actual values in your model, play a crucial role in OLS analysis.
  - Their normality is one of the key assumptions for the validity of statistical inferences drawn from the model.
- A **low statistic** : Low value of the Jarque-Bera statistic (< 2) along with high p-value (ie. > 0.05) indicate that the residuals follow Normal Distribution.
- A **high statistic** : High value of the Jarque-Bera statistic (> 6) often accompanied with low p-values (ie. < 0.05) indicate that the residuals DO NOT follow Normal Distribution.

Durbin-Watson Test

- In the context of Ordinary Least Squares (OLS) regression, the Durbin-Watson (DW) test is a diagnostic tool used to check for **autocorrelation** in the residuals (errors) of the model. Autocorrelation occurs when there's a dependence between subsequent errors, meaning the error term at one point in time influences the error term at another point.
- Its value always falls between 0 and 4, with specific interpretations:
    - **2.0:** Indicates no autocorrelation (ideal scenario).
    - **0 to less than 2.0:** Suggests positive autocorrelation (errors tend to cluster together, either positive or negative).
    - **More than 2.0 to 4:** Suggests negative autocorrelation (errors tend to alternate between positive and negative).
- This test is used as a first check, and not a definitive test.

Condition Number

The Condition Number in Ordinary Least Squares (OLS) refers to a **measure of how sensitive the estimated coefficients are to small changes in the data**. It’s not directly related to any specific variable or error term, but rather evaluates the overall stability and robustness of the model’s solution. Its calculation is based on eigenvalues

- A **low Condition Number** indicates that the coefficients react minimally to small changes in the data (stable, robust model).
- A **high Condition Number** signifies that even slight data variations can significantly alter the coefficients (sensitive, potentially unstable model).

Why is it important?

- A high Condition Number suggests the model might be fitting noise or capturing spurious relationships due to its sensitivity to slight data changes. This makes the estimated coefficients less reliable and conclusions less trustworthy.
- In extreme cases, a very high Condition Number can lead to numerical issues during calculations, rendering the model estimation altogether unstable.

Interpretation:

There’s no single threshold for a “good” or “bad” Condition Number. However, in general:

- **Values below 10** are considered acceptable, indicating a relatively stable model.
- **Values above 30** raise concerns about sensitivity and potential instability.
- **Values above 100** are a strong indicator of an unreliable model requiring further investigation or improvement.

Output	Interpretation	Specific Limits/Values (if applicable)
R^2	Proportion of variance in the dependent variable explained by the model.	Range: [0, 1], higher values are desirable.
Adjusted R^2	R^2 adjusted for the number of predictors; a measure of model fit.	Like R^2, but adjusted for model complexity.
F-statistic	Tests the overall significance of the regression model.	Critical values based on significance level (e.g., 0.05).
AIC (Akaike's IC)	A measure of model goodness-of-fit, balancing complexity and fit.	Lower values are better; used for model comparison.
BIC (Bayesian IC)	Similar to AIC but penalizes model complexity more heavily.	Lower values are better; stricter penalty for complexity.
Log Likelihood	A measure of how well the model explains the observed data.	Higher values indicate better model fit.
Omnibus	Refers to a specific statistic and its associated p-value that test the normality of the residuals. It is a combination of multiple tests like Jarques-Bera test, Shapiro-Wilkes test and Kolmongoriv-Smirnov test.	For the residuals to have Normal Distribution, the Omnibus statistic should have low value and p-value should be > 0.05
Durbin-Watson test	Tests for autocorrelation in the residuals; values around 2 suggest no autocorrelation.	Range: [0, 4], close to 2 indicates no significant autocorrelation.
Jarque-Bera test	Tests for normality of residuals based on skewness and kurtosis. Test statistic value closer to zero implies residuals are normally distributed NULL Hypothesis: The residuals are Normally Distributed	Critical values based on significance level (e.g., 0.05). If p-value < 0.05, then NULL Hypothesis is rejected implying the residuals are NOT normally distributed. If residuals are normally distributed, p-value > 0.05.
Condition Number	Measures sensitivity to changes in input variables; high values indicate multicollinearity.	No strict limits; values above 30 may indicate multicollinearity.
Skew	A measure of the asymmetry of the residuals distribution.	Range: (-∞, ∞); 0 for a perfectly symmetric distribution.
Kurtosis	A measure of the "tailedness" of the residuals distribution.	Range: (-∞, ∞); 3 for a normal distribution (excess kurtosis).

MULTIPLE LINEAR REGRESSION.

→ SLR deals with only one independent variable, and takes the form:

$$y = a \cdot x + b \quad \text{or} \quad y = \beta_0 + \beta_1 x.$$

→ MLR deals with more than one independent variable, and takes the form:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots\cdots + \beta_k x_k$$

here $[x_1, x_2, x_3, \ldots x_k]$ are the independent variables, also known as "features"

A data Set for MLR will look as shown below

y	x1	x2	x3	x4
0.038117	0	0	0	0
0.896468	0.005556	3.09E-05	1.71E-07	9.53E-10
0.159546	0.011111	0.000123	1.37E-06	1.52E-08
0.863764	0.016667	0.000278	4.63E-06	7.72E-08
1.106349	0.022222	0.000494	1.1E-05	2.44E-07
1.010169	0.027778	0.000772	2.14E-05	5.95E-07
0.278498	0.033333	0.001111	3.7E-05	1.23E-06
1.114231	0.038889	0.001512	5.88E-05	2.29E-06
1.029804	0.044444	0.001975	8.78E-05	3.9E-06
0.37387	0.05	0.0025	0.000125	6.25E-06
0.971634	0.055556	0.003086	0.000171	9.53E-06
0.975377	0.061111	0.003735	0.000228	1.39E-05
1.079774	0.066667	0.004444	0.000296	1.98E-05
1.24279	0.072222	0.005216	0.000377	2.72E-05
0.644699	0.077778	0.006049	0.000471	3.66E-05
0.656177	0.083333	0.006944	0.000579	4.82E-05

features.

TRAIN DATA.

In MLR, the goal is to express 'y' as a linear combination of $x_1, x_2, \ldots$

$$\therefore \quad y_1 = \beta_0 + \beta_1 x_{11} + \beta_2 x_{21} + \ldots + \beta_k x_{k1} + e_1$$

$$y_2 = \beta_0 + \beta_1 x_{12} + \beta_2 x_{22} + \ldots + \beta_k x_{k2} + e_2$$

$$\vdots$$

$$y_m = \beta_0 + \beta_1 x_{1m} + \ldots\ldots + \beta_k x_{km} + e_m$$

'k' features

What is MLR?

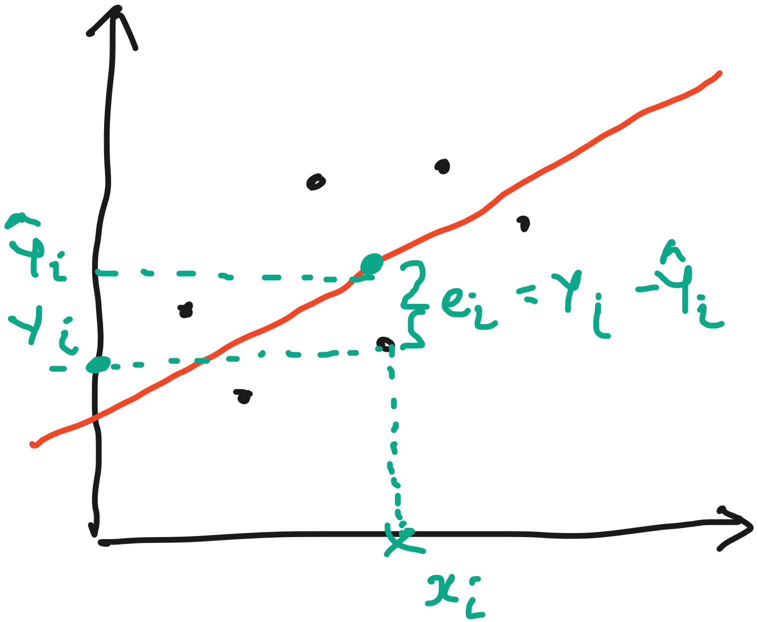
Using the values of all x_{ij} and the corresponding values of y_j, find out the most appropriate $\beta_0, \beta_1, \ldots \beta_k$.

How do we go about it?

The model that we create, ie. the Values of β_0, β_1, etc. that we identify Should be such that

$$\sum_{i=1}^{m} e_i^2$$ should be minimized

(minimize the sum of square of errors, SSE)

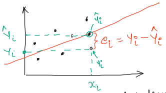


Note:

$$y_i = \beta_0 + \sum_{j=1}^{m} \beta_j x_{ij} + e_i$$

$$\hat{y}_i = \beta_0 + \sum_{j=1}^{m} \beta_j x_{ij}$$

MLR GRADIENT DESCENT



Matrices used in the derivations...

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}_{m\times 1} = \begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} & \cdots & x_{K1} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 1 & x_{1m} & x_{2m} & x_{3m} & \cdots & x_{Km} \end{bmatrix}_{m\times l} \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_K \end{bmatrix}_{l\times 1} + \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}_{m\times 1}$$

m = Number of observations (records)
K = Number of features (independent variables)
$l = K + 1$

→ $Y = X.\beta + E$ and $\underline{\hat{Y} = X.\beta}$ ... the regression 'line'

$E = Y - X.\beta$

$E^T.E = (Y - X\beta)^T (Y - X\beta)$ .... Sum of squares of errors

$J = \frac{1}{2m}(E^T.E)$ ... Cost function.

Goal: minimize J (ie: make $\frac{dJ}{d\beta} = 0$)

$$J = \frac{1}{2m}(Y - X\beta)^T(Y - X\beta)$$
$$= \frac{1}{2m}[(Y^T - \beta^T X^T)(Y - X\beta)]$$
$$= \frac{1}{2m}\left[Y^T.Y - Y^T(X\beta) - (\beta^T X^T)Y + (\beta^T X^T).(X\beta)\right]$$

$$\frac{dJ}{d\beta} = \frac{1}{2m}\left[\underbrace{\frac{d}{d\beta}(Y^T.Y)}_{A} - \underbrace{\frac{d}{d\beta}\{Y^T.(X\beta)\}}_{B} - \underbrace{\frac{d}{d\beta}\{(\beta^T X^T).Y\}}_{C} + \underbrace{\frac{d}{d\beta}\{(\beta^T X^T).(X\beta)\}}_{D}\right]$$

$A = \frac{d}{d\beta}(Y^T.Y) = 0$ ... Y is a vector of constants.

$B = \frac{d}{d\beta}\{Y^T.(X\beta)\} = Y^T.\frac{d}{d\beta}(X\beta) + \frac{dY^T}{d\beta}.(X\beta)$

$= Y^T.X + 0 = Y^T X$ ... Row vector $(1\times l)$

$C = \frac{d}{d\beta}\{(\beta^T X^T).Y\} = \beta^T X^T \frac{dY}{d\beta} + \frac{d(\beta^T X^T)}{d\beta}.Y$

$= 0 + X^T Y$ ... Column Vector $(l\times 1)$

$D = \frac{d}{d\beta}\{(\beta^T X^T).(X\beta)\} = (\beta^T X^T)\frac{d}{d\beta}(X\beta) + \frac{d}{d\beta}(\beta^T X^T).(X\beta)$

$= \underbrace{(\beta^T X^T X)}_{1\times l} + \underbrace{(X^T X\beta)}_{l\times 1}$

$$\frac{dJ}{d\beta} = \frac{1}{2m}\left[(\beta^T X^T X) + (X^T X\beta) - Y^T X - X^T Y\right]$$

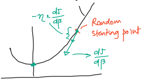
These are all 'vectors' and the results in the pairs shown below are equal in values. Hence re-arranging and simplifying ...

$$= \frac{1}{2m}\left[\underbrace{2.X^T X\beta}_{(l\times 1)} - \underbrace{2X^T Y}_{(l\times 1)}\right]$$

$$= X^T(X\beta - Y) \quad \ldots$$

$$\frac{dJ}{d\beta} = X^T(\hat{Y} - Y) \quad \ldots \text{Since } \hat{Y} = X\beta$$

The Gradient Descent process



① assume some value for β. eg $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$
② Using $\hat{Y} = X\beta$, evaluate $\hat{Y}$
③ Calculate $\frac{dJ}{d\beta}$ as per the above expression by assuming some value for η (eg: 0.05)
④ Calculate new values for β using the following expression

$$\beta_{new} \leftarrow \beta_{old} - \eta(\nabla J)$$

⑤ Repeat steps 2 to 4 till $[ABS(\beta_{new} - \beta_{old})]$ it reaches a threshold level (eg: 0.0001)

MULTIPLE LINEAR REGRESSION. (MLR)

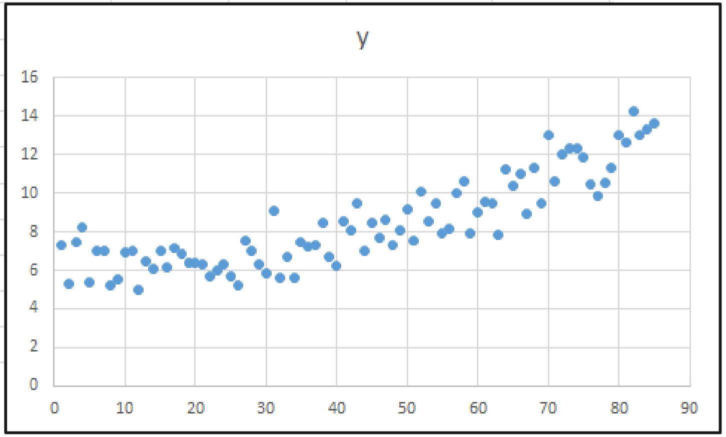
- In MLR, more than one independent variable x_i potentially determine the dependent variable 'y'

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \cdots\cdots + \beta_K x_K$$

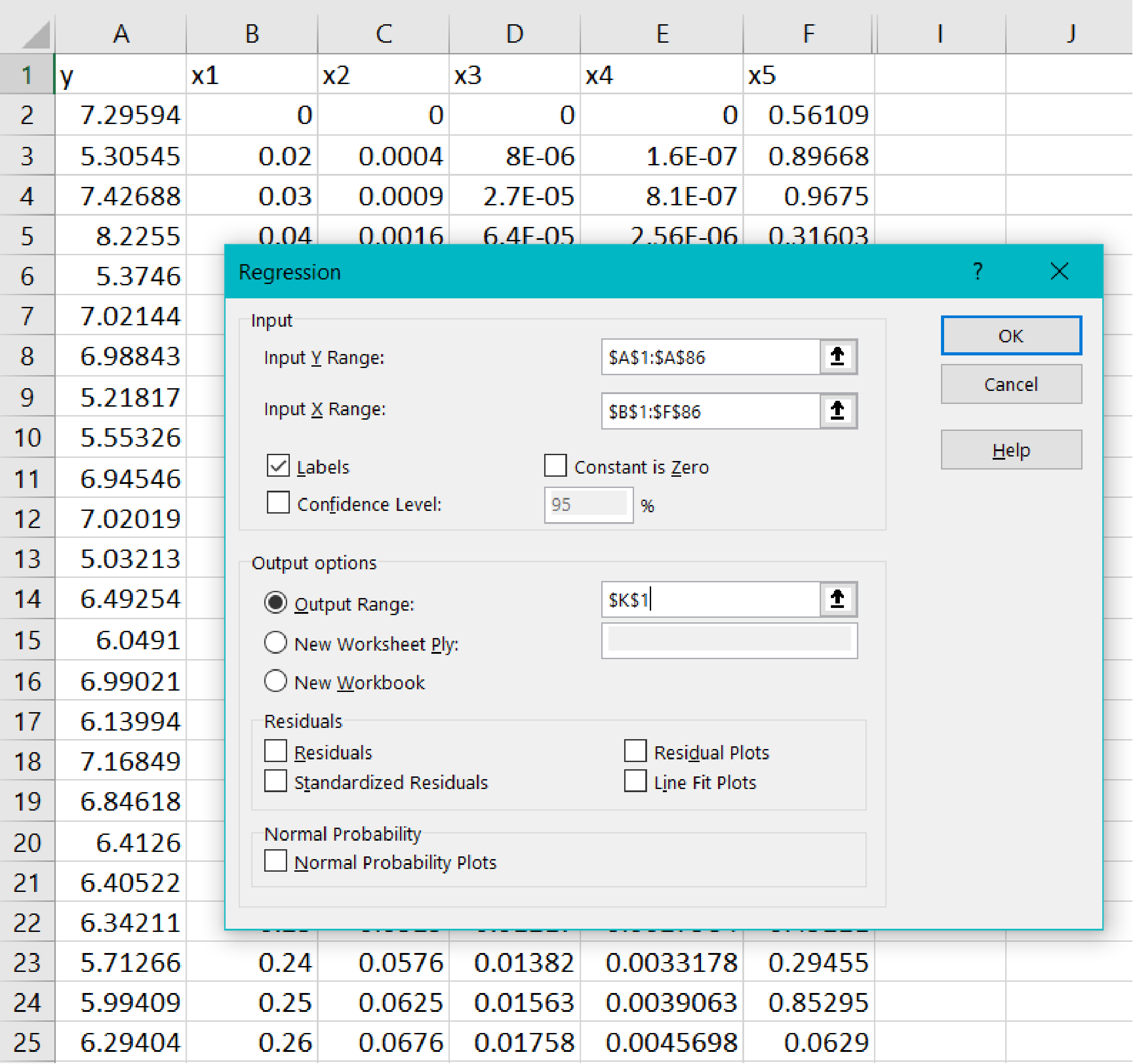
- MLR involves calculating the coefficients β_i using the given dataset (TRAIN DATA).
- As in the case of SLR, these values are obtained by minimizing the <u>SSE</u> (explained in a separate document)
- MLR involves identifying the most relevant predictors as explained subsequently.

MLR using the dataset **data-set-for-MLR.xlsx**

	A	B	C	D	E	F
1	y	x1	x2	x3	x4	x5
2	7.29594	0	0	0	0	0.56109
3	5.30545	0.02	0.0004	8E-06	1.6E-07	0.89668
4	7.42688	0.03	0.0009	2.7E-05	8.1E-07	0.9675
5	8.2255	0.04	0.0016	6.4E-05	2.56E-06	0.31603
6	5.3746	0.05	0.0025	0.00013	6.25E-06	0.74414
7	7.02144	0.06	0.0036	0.00022	1.296E-05	0.19197
8	6.98843	0.07	0.0049	0.00034	2.401E-05	0.86862
9	5.21817	0.08	0.0064	0.00051	4.096E-05	0.74443
10	5.55326	0.09	0.0081	0.00073	6.561E-05	0.801
11	6.94546	0.1	0.01	0.001	0.0001	0.05507
12	7.02019	0.11	0.0121	0.00133	0.0001464	0.61864
13	5.03213	0.12	0.0144	0.00173	0.0002074	0.80112
14	6.49254	0.13	0.0169	0.0022	0.0002856	0.78146
15	6.0491					
16	6.99021					
17	6.13994					
18	7.16849					
19	6.84618					
20	6.4126					
21	6.40522					
22	6.34211					
23	5.71266					
24	5.99409					
25	6.29404					



- The dataset in the file consists of the **train** dataset of 85 observations and the **test** dataset of 15 observations
- We will create an MLR model using the train dataset and subsequently validate the model using the test dataset
- We start by creating an MLR model using all the **x** variables (also known as ***features***)
- A scatter plot of **y** reveals that the observations are non-linear …
    - So, will **Linear Regression** be able to create a good and acceptable model??



Invoking the Linear Regression functionality of Excel and selecting the variables …

y	x1	x2	x3	x4	x5
7.29594	0	0	0	0	0.56109
5.30545	0.02	0.0004	8E-06	1.6E-07	0.89668
7.42688	0.03	0.0009	2.7E-05	8.1E-07	0.9675
8.2255	0.04	0.0016	6.4E-05	2.56E-06	0.31603
5.3746	0.05	0.0025	0.00013	6.25E-06	0.74414
7.02144	0.06	0.0036	0.00022	1.296E-05	0.19197
6.98843	0.07	0.0049	0.00034	2.401E-05	0.86862
5.21817	0.08	0.0064	0.00051	4.096E-05	0.74443
5.55326	0.09	0.0081	0.00073	6.561E-05	0.801
6.94546	0.1	0.01	0.001	0.0001	0.05507
7.02019	0.11	0.0121	0.00133	0.0001464	0.61864
5.03213	0.12	0.0144	0.00173	0.0002074	0.80112
6.49254	0.13	0.0169	0.0022	0.0002856	0.78146
6.0491	0.15	0.0225	0.00338	0.0005063	0.68791
6.99021	0.16	0.0256	0.0041	0.0006554	0.59236
6.13994	0.17	0.0289	0.00491	0.0008352	0.20935
7.16849	0.18	0.0324	0.00583	0.0010498	0.96488
6.84618	0.19	0.0361	0.00686	0.0013032	0.48896
6.4126	0.21	0.0441	0.00926	0.0019448	0.50876
6.40522	0.22	0.0484	0.01065	0.0023426	0.71971
6.34211	0.23	0.0529	0.01217	0.0027984	0.49221
5.71266	0.24	0.0576	0.01382	0.0033178	0.29455
5.99409	0.25	0.0625	0.01563	0.0039063	0.85295
6.29404	0.26	0.0676	0.01758	0.0045698	0.0629

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.912936908
R Square	0.833453799
Adjusted R Square	0.8229129
Standard Error	0.991752189
Observations	85

– ok

ANOVA

	df	*SS*	*MS*	*F*	*gnificance F*
Regression	5	388.848	77.7697	79.0686	2.7E-29
Residual	79	77.7022	0.98357		
Total	84	466.551			

→ ok

	Coefficients	*andard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	6.908046001	0.59691	11.5731	1.1E-18	5.71993	8.09616
x1	-9.846555611	7.41707	-1.32755	0.18815	-24.6099	4.91676
x2	39.10941696	30.7379	1.27235	0.20698	-22.0728	100.292
x3	-42.57714279	46.9855	-0.90618	0.3676	-136.099	50.9451
x4	20.26808239	23.6657	0.85643	0.39435	-26.8374	67.3736
x5	0.10562362	0.41144	0.25672	0.79806	-0.71333	0.92457

None of these values are acceptable since their corresponding p-values are all MUCH GREATER than 0.05

So, we DISCARD x_5 – which has the highest p-value, and re-create the model.

	A	B	C	D	E	F
1	y	x1	x2	x3	x4	x5
2	7.29594	0	0	0	0	0.56109
3	5.30545	0.02	0.0004	8E-06	1.6E-07	0.89668
4	7.42688	0.03	0.0009	2.7E-05	8.1E-07	0.9675
5	8.2255	0.04	0.0016	6.4E-05	2.56E-06	0.31603
6	5.3746	0.05	0.0025	0.00013	6.25E-06	0.74414
7	7.02144	0.06	0.0036	0.00022	1.296E-05	0.19197
8	6.98843	0.07	0.0049	0.00034	2.401E-05	0.86862
9	5.21817	0.08	0.0064	0.00051	4.096E-05	0.74443
10	5.55326	0.09	0.0081	0.00073	6.561E-05	0.801
11	6.94546	0.1	0.01	0.001	0.0001	0.05507
12	7.02019	0.11	0.0121	0.00133	0.0001464	0.61864
13	5.03213	0.12	0.0144	0.00173	0.0002074	0.80112
14	6.49254	0.13	0.0169	0.0022	0.0002856	0.78146
15	6.0491	0.15	0.0225	0.00338	0.0005063	0.68791
16	6.99021	0.16	0.0256	0.0041	0.0006554	0.59236
17	6.13994	0.17	0.0289	0.00491	0.0008352	0.20935
18	7.16849	0.18	0.0324	0.00583	0.0010498	0.96488
19	6.84618	0.19	0.0361	0.00686	0.0013032	0.48896
20	6.4126	0.21	0.0441	0.00926	0.0019448	0.50876
21	6.40522	0.22	0.0484	0.01065	0.0023426	0.71971
22	6.34211	0.23	0.0529	0.01217	0.0027984	0.49221
23	5.71266	0.24	0.0576	0.01382	0.0033178	0.29455
24	5.99409	0.25	0.0625	0.01563	0.0039063	0.85295
25	6.29404	0.26	0.0676	0.01758	0.0045698	0.0629

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.912860812
R Square	0.833314862
Adjusted R Square	0.824980605
Standard Error	0.985945239
Observations	85

ANOVA

	df	*SS*	*MS*	*F*	*gnificance F*
Regression	4	388.783	97.1959	99.9867	2.6E-30
Residual	80	77.767	0.97209		
Total	84	466.551			

	Coefficients	*andard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	6.982707415	0.51821	13.4746	2.8E-22	5.95143	8.01398
x1	-9.95691922	7.36125	-1.35261	0.17999	-24.6063	4.69243
x2	39.18955261	30.5563	1.28254	0.20336	-21.6194	99.9986
x3	-42.50561677	46.7095	-0.91	0.36556	-135.461	50.4493
x4	20.19742663	23.5256	0.85853	0.39316	-26.62	67.0148

Discard x_4 and proceed ...

y	x1	x2	x3	x4	x5
7.29594	0	0	0	0	0.56109
5.30545	0.02	0.0004	8E-06	1.6E-07	0.89668
7.42688	0.03	0.0009	2.7E-05	8.1E-07	0.9675
8.2255	0.04	0.0016	6.4E-05	2.56E-06	0.31603
5.3746	0.05	0.0025	0.00013	6.25E-06	0.74414
7.02144	0.06	0.0036	0.00022	1.296E-05	0.19197
6.98843	0.07	0.0049	0.00034	2.401E-05	0.86862
5.21817	0.08	0.0064	0.00051	4.096E-05	0.74443
5.55326	0.09	0.0081	0.00073	6.561E-05	0.801
6.94546	0.1	0.01	0.001	0.0001	0.05507
7.02019	0.11	0.0121	0.00133	0.0001464	0.61864
5.03213	0.12	0.0144	0.00173	0.0002074	0.80112
6.49254	0.13	0.0169	0.0022	0.0002856	0.78146
6.0491	0.15	0.0225	0.00338	0.0005063	0.68791
6.99021	0.16	0.0256	0.0041	0.0006554	0.59236
6.13994	0.17	0.0289	0.00491	0.0008352	0.20935
7.16849	0.18	0.0324	0.00583	0.0010498	0.96488
6.84618	0.19	0.0361	0.00686	0.0013032	0.48896
6.4126	0.21	0.0441	0.00926	0.0019448	0.50876
6.40522	0.22	0.0484	0.01065	0.0023426	0.71971
6.34211	0.23	0.0529	0.01217	0.0027984	0.49221
5.71266	0.24	0.0576	0.01382	0.0033178	0.29455
5.99409	0.25	0.0625	0.01563	0.0039063	0.85295
6.29404	0.26	0.0676	0.01758	0.0045698	0.0629

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.912019254
R Square	0.83177912
Adjusted R Square	0.825548717
Standard Error	0.984343752
Observations	85

ANOVA

	df	*SS*	*MS*	*F*	*gnificance F*
Regression	3	388.067	129.356	133.503	2.9E-31
Residual	81	78.4835	0.96893		
Total	84	466.551			

	Coefficients	*andard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	6.717731373	0.4156	16.164	4.5E-27	5.89082	7.54464
x1	-4.499675935	3.70651	-1.21399	0.22828	-11.8745	2.87513
x2	14.05360633	8.73189	1.60946	0.11141	-3.32013	31.4273
x3	-2.717152907	5.81602	-0.46718	0.64162	-14.2892	8.8549

Discard x_3 and proceed ...

	A	B	C	D	E	F
1	y	x1	x2	x3	x4	x5
2	7.29594	0	0	0	0	0.56109
3	5.30545	0.02	0.0004	8E-06	1.6E-07	0.89668
4	7.42688	0.03	0.0009	2.7E-05	8.1E-07	0.9675
5	8.2255	0.04	0.0016	6.4E-05	2.56E-06	0.31603
6	5.3746	0.05	0.0025	0.00013	6.25E-06	0.74414
7	7.02144	0.06	0.0036	0.00022	1.296E-05	0.19197
8	6.98843	0.07	0.0049	0.00034	2.401E-05	0.86862
9	5.21817	0.08	0.0064	0.00051	4.096E-05	0.74443
10	5.55326	0.09	0.0081	0.00073	6.561E-05	0.801
11	6.94546	0.1	0.01	0.001	0.0001	0.05507
12	7.02019	0.11	0.0121	0.00133	0.0001464	0.61864
13	5.03213	0.12	0.0144	0.00173	0.0002074	0.80112
14	6.49254	0.13	0.0169	0.0022	0.0002856	0.78146
15	6.0491	0.15	0.0225	0.00338	0.0005063	0.68791
16	6.99021	0.16	0.0256	0.0041	0.0006554	0.59236
17	6.13994	0.17	0.0289	0.00491	0.0008352	0.20935
18	7.16849	0.18	0.0324	0.00583	0.0010498	0.96488
19	6.84618	0.19	0.0361	0.00686	0.0013032	0.48896
20	6.4126	0.21	0.0441	0.00926	0.0019448	0.50876
21	6.40522	0.22	0.0484	0.01065	0.0023426	0.71971
22	6.34211	0.23	0.0529	0.01217	0.0027984	0.49221
23	5.71266	0.24	0.0576	0.01382	0.0033178	0.29455
24	5.99409	0.25	0.0625	0.01563	0.0039063	0.85295
25	6.29404	0.26	0.0676	0.01758	0.0045698	0.0629

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.911770714
R Square	0.831325834
Adjusted R Square	0.82721183
Standard Error	0.979640446
Observations	85

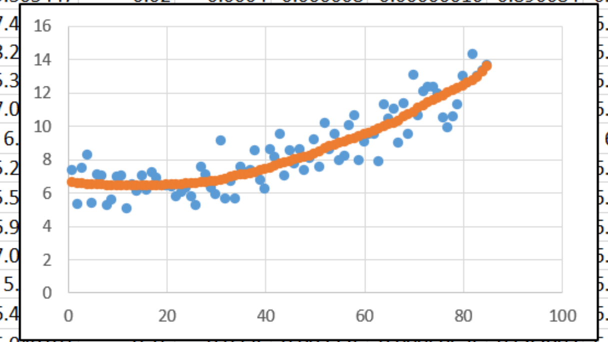
ANOVA

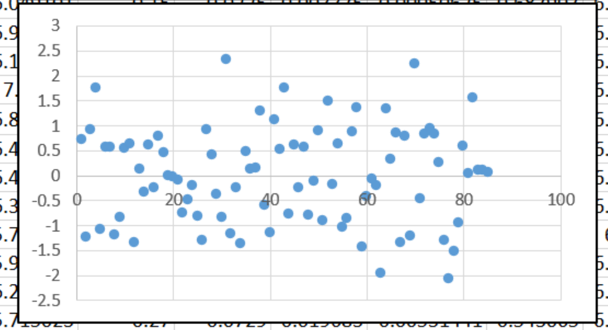
	df	*SS*	*MS*	*F*	*gnificance F*
Regression	2	387.856	193.928	202.072	2E-32
Residual	82	78.695	0.9597		
Total	84	466.551			

	Coefficients	*andard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	6.589411803	0.31041	21.2281	4.6E-35	5.97191	7.20692
x1	-2.914457293	1.48449	-1.96328	0.053	-5.86757	0.03866
x2	10.03277678	1.46738	6.8372	1.3E-09	7.11369	12.9519

The model seems ok now ... p-value of x_1 is very close to 0.05, so we choose to keep it ...

	A	B	C	D	E	F	G	H
1	y	x1	x2	x3	x4	x5	ycap	e
2	7.295939	0	0	0	0	0.561088	6.589412	0.706527
3	5.305447	0.02	0.0004	0.000008	0.00000016	0.896684	6.535136	-1.22969
4	7.4						.511008	0.915876
5	8.2						.488886	1.736618
6	5.3						.468771	-1.09418
7	7.0						.450662	0.570781
8	6.						6.43456	0.55387
9	5.2						.420465	-1.2023
10	5.5						.408376	-0.85512
11	6.9						.398294	0.547164
12	7.0						.390218	0.629977
13	5.						.384149	-1.35202
14	6.4						.380086	0.112449
15	6.0						.377981	-0.32888
16	6.9						.379938	0.610267
17	6.1						.383901	-0.24396
18	7.						.389871	0.778618
19	6.8						.397848	0.448328
20	6.4						.419821	-0.00722
21	6.4						.433818	-0.02859
22	6.3						.449821	-0.10771
23	5.7						6.46783	-0.75517
24	5.9						.487846	-0.49375
25	6.2						.509869	-0.21582
26	5.7						.533898	-0.81888
27	5.							





SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.911770714
R Square	0.831325834
Adjusted R Square	0.82721183
Standard Error	0.979640446
Observations	85

ANOVA

	df	*SS*	*MS*	*F*	*gnificance F*
Regression	2	387.8555	193.9278	202.0722	2.04E-32
Residual	82	78.69502	0.959695		
Total	84	466.5505			

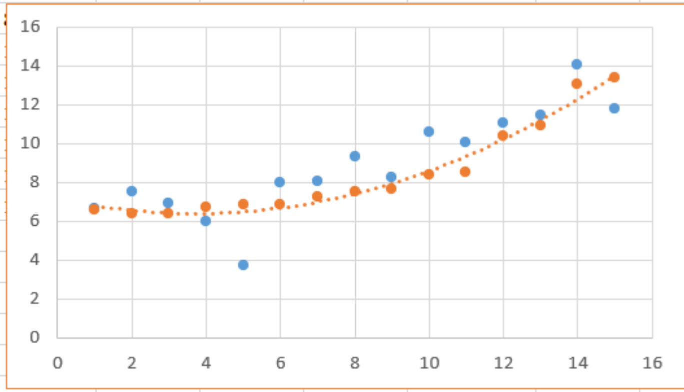
	Coefficients	*andard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	6.589411803	0.31041	21.22811	4.59E-35	5.971908	7.206916
x1	-2.914457293	1.484485	-1.96328	0.053004	-5.86757	0.038656
x2	10.03277678	1.467381	6.837202	1.32E-09	7.113689	12.95186

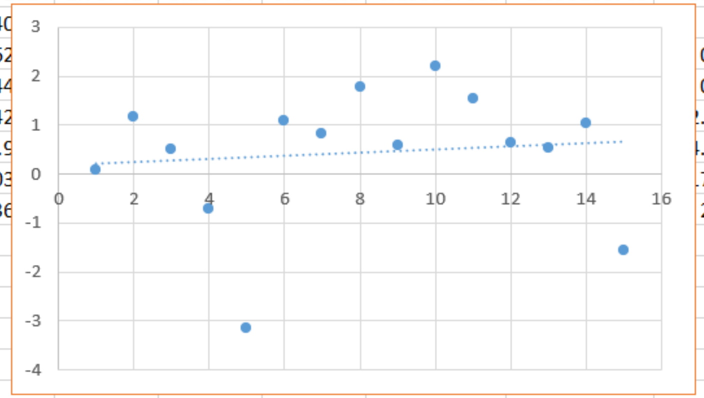
- Plot of y (blue dots) and y_cap (red dots) seems to indicate that the model has captured the non-linear nature of **y** (how? why?)
- Plot of residuals also indicate no visible trend
- Model indicators like R2 and F-statistic also seem Ok
- Overall - the model seems to be good and acceptable based its performance on the train data
- We now need to check its performance on the test data …

The illustrated method of iteratively eliminating the **least important features** features, based on their p-values, is known as **BACKWARD FEATURES ELIMINATION**

Model performance on **Test Data**

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	y	x1	x2	x3	x4	x5	**ycap**	**e**	e_sq	y-y_bar	y-y_bar_sq	y_cap-y_bar	y_cap-y_bar_sq				
2	6.65375	0.01	0.0001	0.000001	0.00000001	0.419067	6.561271	0.092479	0.008552	-2.24148	5.024246447	-2.333962333	5.447380171			Coefficients	
3	7.539954	0.14	0.0196	0.002744	0.00038416	0.209814	6.37803	1.161924	1.350067	-1.35528	1.836780485	-2.517202634	6.3363091		Intercept	6.589411803	
4	6.914127	0.2	0.04	0.008	0.0016	0.375459	6.407831	0.506296	0.256335	-1.98111	3.924780608	-2.487401425	6.18716585		x1	-2.914457293	
5	6.006237	0.33	0.1089	0.035937	0.01185921	0.752656	6.72021	-0.71397	0.509758	-2.889	8.346296524	-2.175022553	4.730723107		x2	10.03277678	
6	3.702046	0.36	0.1296	0.046656	0.01679616	0.393727	6.840455	-3.13841	9.849613	-5.19319	26.96919177	-2.054777793	4.222111778				
7	7.970599	0.37	0.1369	0.050653	0.01874161	0.416966	6.88455	1.08605	1.179504	-0.92463	0.854947091	-2.010683095	4.042846509		SSE	28.70223718	
8	8.067204	0.44	0.1936	0.085184	0.03748096	0.815868	7.249396	0.817808	0.66881	-0.82803	0.685631517	-1.645836662	2.708778319		MSE_test	1.913482479	
9	9.289438	0.48	0.2304	0.110592	0.05308416	0.327166	7.502024	1.787414	3.194847	0.394205	0.155397395	-1.393208769	1.941030673		MSE_train	0.9258238	
10													1.5746623				
11													0.28402637		y_bar	8.895232841	
12													0.12281082		SST	99.03386445	
13													.325148205		SSR	81.29945607	
14													.203988792				
15													7.16036458		R2	0.820925817	(SSR/SST)
16													20.0121095		R2	0.71017755	(1 - SSE/SST)





why would this be more correct?

- The y and ycap plots seem to indicate that the model, created using train data also performs reasonably well on the test data
- R2 value on test data seems Ok and close to the R2 value using train data
-

- The MSE values are differing, indicating some level of **overfitting** to the train data. However, the size of the test data is small, so the errors are possibly magnified.
- Usually, the technique of **cross-validation** is used - wherein multiple test data sets are used to evaluate the model. This results in an unbiased error estimate on test data.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.831
Model:                            OLS   Adj. R-squared:                  0.827
Method:                 Least Squares   F-statistic:                     202.1
Date:                Fri, 26 Jan 2024   Prob (F-statistic):           2.04e-32
Time:                        14:19:48   Log-Likelihood:                -117.33
No. Observations:                  85   AIC:                             240.7
Df Residuals:                      82   BIC:                             248.0
Df Model:                           2
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          6.5894      0.310     21.228      0.000       5.972       7.207
x1            -2.9145      1.484     -1.963      0.053      -5.868       0.039
x2            10.0328      1.467      6.837      0.000       7.114      12.952
==============================================================================
Omnibus:                        1.380   Durbin-Watson:                   2.318
Prob(Omnibus):                  0.502   Jarque-Bera (JB):                1.164
Skew:                           0.080   Prob(JB):                        0.559
Kurtosis:                       2.450   Cond. No.                         23.2
==============================================================================
```

Exercise:

1. Re-create and validate the MLR model yourself, using the steps outlined in this document
2. Use the statsmodels based OLS function to repeat **all** these steps, and analyze the additional metrics created (Omnibus, Durbin-Watson, Jarque-Bera, AIC, BIC, Condition Number, etc.) at each stage

(the dataset has been uploaded to Moodle)

What if the relationship between the dependent variable (y) and the independent variables (X) is not linear as in the case below? We have seen that the regression errors are not random and the other important regression parameters like R2 are also very low.

How do we remedy this situation?

	y	x1	ycap	ei
2	0.038116834	0	0.078554	-0.04044
3	0.896467788	0.005555556	0.083296	0.813172
4	0.159545792	0.011111111	0.088032	0.071514
5	0.863764416	0.016666667	0.092756	0.771008
6	1.106349076	0.022222222	0.097463	1.008886
7	1.010169458	0.027777778	0.102147	0.908022
8	0.278498289	0.033333333	0.106803	0.171695
9	1.114230685	0.038888889	0.111424	1.002807
10	1.029803908	0.044444444	0.116005	0.913799
11	0.373869889	0.05	0.12054	0.25333
12	0.971634374	0.055555556	0.125024	0.84661
13	0.975376766	0.061111111	0.129452	0.845925
14	1.079774246	0.066666667	0.133817	0.945957
15	1.242790434	0.072222222	0.138116	1.104675
16	0.644698738	0.077777778	0.142341	0.502358
17	0.656177067	0.083333333	0.146489	0.509688
18	1.09549189	0.088888889	0.150554	0.944938
19	1.115274736	0.094444444	0.154532	0.960743
20	1.512547878	0.1	0.158416	1.354131
21	0.639395626	0.105555556	0.162204	0.477192

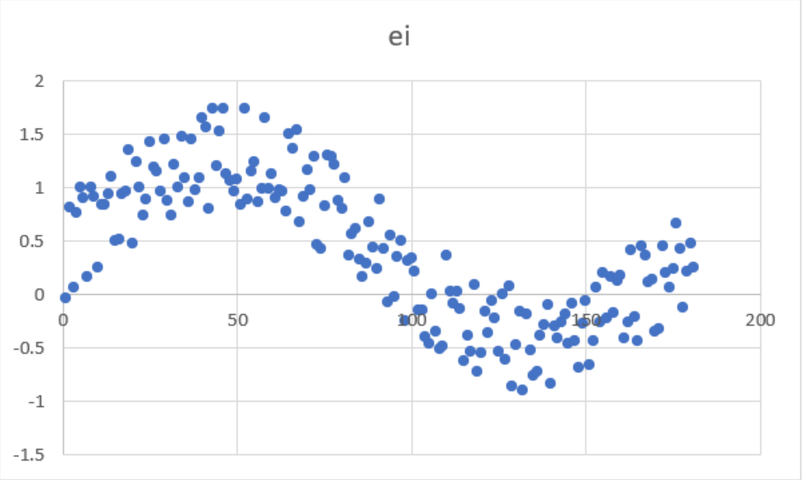
SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.713818524
R Square	0.509536885
Adjusted R Square	0.506796867
Standard Error	0.530607579
Observations	181

ANOVA

	df	*SS*
Regression	1	52.35633091
Residual	179	50.3964481
Total	180	102.752779

	Coefficients	*Standard Error*							
Intercept	1.417122114	0.078553776	18.04015	3.95E-42	1.262112	1.572133	1.262112	1.572133	0.547807
x1	-1.852833934	0.135870553	-13.6368	1.69E-29	-2.12095	-1.58472	-2.12095	-1.58472	12.74861



We can see that the error plot is not random, and it follows a pattern. This indicates that forcing a **line** to model this data results in incorrect results. We need to **introduce non-linear independent variables** in the system so that the Multiple Linear Regression method can 'use' this non-linearity to produce the desired non-linear y_cap.

So, we introduce additional columns x2, x3, x4 such that

x2 = x1 * x1
x3 = x1 * x1 * x1
x4 = x1 * x1 * x1 * x1

Note: The method is still Linear Regression. It is **Linear Regression of non-linear independent variables**.

The resulting regression method is known as **Polynomial Regression** - since polynomial terms are introduced as independent variable to handle non-linearity in y.

In general, introducing additional **x** variables to improve the performance of ML methods is known as **Feature Engineering**.

Hence, **Polynomial Regression** can be said to be an application of the **Feature Engineering** technique.

You are encouraged to create a dataset that is not good for being regressed by a line, but gets adequately represented by a Polynomial Regression.

After introducing the polynomial terms and carrying out MLR, the results are as follows:

- The first chart shows y and y_cap (blue and organge)
- The second chart shows the error scatter plot
- We observe that the R2 value is now quite good and all the p-values, except that for x4, are much less than 0.05. This indicates that x4 is not significant and needs to be dropped.

	A	B	C	D	E	F	G	H
1	y	x1	x2	x3	x4		ycap	ei
2	0.038116834	0	0	0	0		0.31406	-0.27594
3	0.896467788	0.005555556	3.09E-05	1.71E-07	9.53E-10		0.381398	0.51507
4	0.159545792	0.011111111	0.000123	1.37E-06	1.52E-08		0.446403	-0.28686
5	0.863764416	0.016666667	0.000278	4.63E-06	7.72E-08		0.509106	0.354658
6	1.106349076	0.022222222	0.000494	1.1E-05	2.44E-07		0.569538	0.536811
7	1.010169458	0.027777778	0.000772	2.14E-05	5.95E-07		0.627729	0.38244
8	0.278498289	0.033333333	0.001111	3.7E-05	1.23E-06		0.683711	-0.40521
9	1.114230685	0.038888889	0.001512	5.88E-05	2.29E-06		0.737514	0.376717
10	1.029803908	0.044444444	0.001975	8.78E-05	3.9E-06		0.789169	0.240635
11	0.373869889	0.05	0.0025	0.000125	6.25E-06		0.838706	-0.46484
12	0.971634374	0.055555556	0.003086	0.000171	9.53E-06		0.886155	0.085479
13	0.975376766	0.061111111	0.003735	0.000228	1.39E-05		0.931548	0.043829
14	1.079774246	0.066666667	0.004444	0.000296	1.98E-05		0.974913	0.104861
15	1.242790434	0.072222222	0.005216	0.000377	2.72E-05		1.016282	0.226508
16	0.644698738	0.077777778	0.006049	0.000471	3.66E-05		1.055685	-0.41099
17	0.656177067	0.083333333	0.006944	0.000579	4.82E-05		1.09315	-0.43697
18	1.09549189	0.088888889	0.007901	0.000702	6.24E-05		1.128709	-0.03322
19	1.115274736	0.094444444	0.00892	0.000842	7.96E-05		1.162391	-0.04712
20	1.512547878	0.1	0.01	0.001	0.0001		1.194225	0.318323
21	0.639395626	0.105555556	0.011142	0.001176	0.000124		1.224242	-0.58485
22	1.406626554	0.111111111	0.012346	0.001372	0.000152		1.25247	0.154157
23	1.172479286	0.116666667	0.013611	0.001588	0.000185		1.278939	-0.10646
24	0.909355558	0.122222222	0.014938	0.001826	0.000223		1.303679	-0.39432
25	1.067679037	0.127777778	0.016327	0.002086	0.000267		1.326718	-0.25904
26	1.603060114	0.133333333	0.017778	0.00237	0.000316		1.348086	0.254974

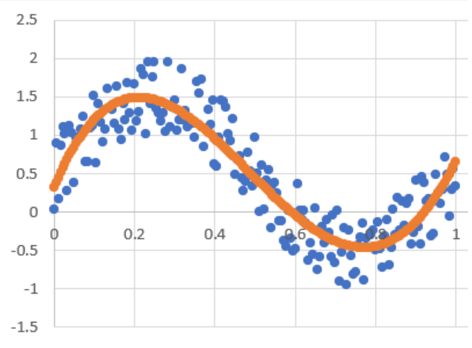
SUMMARY OUTPUT

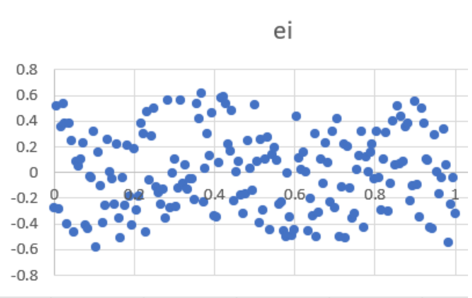
Regression Statistics	
Multiple R	0.913571
R Square	0.834612
Adjusted R	0.830853
Standard E	0.310737
Observatic	181

ANOVA

	df	*SS*	*MS*	*F*	*ignificance F*
Regressior	4	85.75865	21.43966	222.0403	1.26E-67
Residual	176	16.99412	0.096558		
Total	180	102.7528			

	Coefficients	*andard Erro*	*t Stat*	*P-value*	*Lower 95%*	*Uppe*
Intercept	0.31406	0.11176	2.810135	0.005513	0.093498	0.5
x1	12.33273	1.5577	7.917267	2.62E-13	9.258555	15.
x2	-38.302	6.359842	-6.02247	9.77E-09	-50.8533	-25
x3	30.31208	9.568272	3.167978	0.00181	11.42877	49.
x4	-4.00187	4.746218	-0.84317	0.400277	-13.3687	5.





After dropping x4 from the model, the results are as follows:

- All p-values are now much lower than the threshold 0.05

This technique of starting off with all features and then **dropping** non-significant features one at a time is known as **Backward Feature Selection / Engineering.**

Backward feature engineering is a feature selection technique that removes features one by one until the model performance reaches a peak, and it is used to optimize the performance of the machine learning model by only including the most affecting feature and removing the least affecting feature.

y	x1	x2	x3	x4		ycap	ei
0.038116834	0	0	0	0		0.369343	-0.33123
0.896467788	0.005555556	3.09E-05	1.71E-07	9.53E-10		0.430539	0.465929
0.159545792	0.011111111	0.000123	1.37E-06	1.52E-08		0.48971	-0.33016
0.863764416	0.016666667	0.000278	4.63E-06	7.72E-08		0.54688	0.316884
1.106349076	0.022222222	0.000494	1.1E-05	2.44E-07		0.602072	0.504278
1.010169458	0.027777778	0.000772	2.14E-05	5.95E-07		0.655307	0.354862
0.278498289	0.033333333	0.001111	3.7E-05	1.23E-06		0.706611	-0.42811
1.114230685	0.038888889	0.001512	5.88E-05	2.29E-06		0.756005	0.358226
1.029803908	0.044444444	0.001975	8.78E-05	3.9E-06		0.803512	0.226292
0.373869889	0.05	0.0025	0.000125	6.25E-06		0.849155	-0.47529
0.971634374	0.055555556	0.003086	0.000171	9.53E-06		0.892958	0.078676
0.975376766	0.061111111	0.003735	0.000228	1.39E-05		0.934943	0.040434
1.079774246	0.066666667	0.004444	0.000296	1.98E-05		0.975133	0.104641
1.242790434	0.072222222	0.005216	0.000377	2.72E-05		1.013552	0.229239
0.644698738	0.077777778	0.006049	0.000471	3.66E-05		1.050221	-0.40552
0.656177067	0.083333333	0.006944	0.000579	4.82E-05		1.085164	-0.42899
1.09549189	0.088888889	0.007901	0.000702	6.24E-05		1.118405	-0.02291
1.115274736	0.094444444	0.00892	0.000842	7.96E-05		1.149965	-0.03469
1.512547878	0.1	0.01	0.001	0.0001		1.179868	0.33268
0.639395626	0.105555556	0.011142	0.001176	0.000124		1.208137	-0.56874
1.406626554	0.111111111	0.012346	0.001372	0.000152		1.234795	0.171832
1.172479286	0.116666667	0.013611	0.001588	0.000185		1.259864	-0.08738
0.909355558	0.122222222	0.014938	0.001826	0.000223		1.283368	-0.37401
1.067679037	0.127777778	0.016327	0.002086	0.000267		1.30533	-0.23765
1.603060114	0.133333333	0.017778	0.00237	0.000316		1.325772	0.277288
1.367903685	0.138888889	0.01929	0.002679	0.000372		1.344718	0.023186

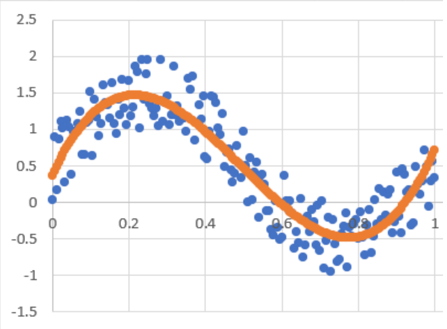
SUMMARY OUTPUT

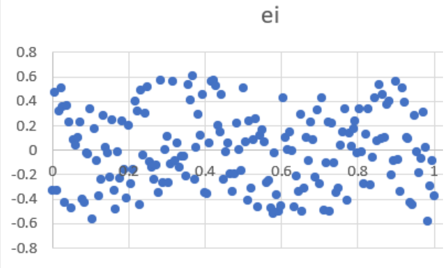
Regression Statistics	
Multiple R	0.913205
R Square	0.833943
Adjusted R	0.831129
Standard E	0.310483
Observatic	181

ANOVA

	df	*SS*	*MS*	*F*	*ignificance F*
Regressior	3	85.69001	28.56334	296.3007	9.58E-69
Residual	177	17.06277	0.0964		
Total	180	102.7528			

	Coefficients	*andard Err*	*t Stat*	*P-value*	*Lower 95%*	*Uppe*
Intercept	0.369343	0.090432	4.084204	6.69E-05	0.190879	0.54
x1	11.19878	0.785338	14.25982	3.25E-31	9.648946	12.
x2	-33.1662	1.827791	-18.1455	3.01E-42	-36.7732	-29
x3	22.30833	1.201303	18.57012	2.04E-43	19.93761	24.





Backward v/s Forward Feature Engineering

Forward Feature Engineering	Backward Feature Engineering
Starts with an empty feature set and iteratively adds one feature at a time based on their performance	Starts with a complete set of features and removes features one by one until the model performance reaches a peak
Goal is to identify the most accurate and informative features that contribute to the predictive power of the model	Goal is to identify the most accurate and relevant features that can be used in a model
Iteratively adds features to the model	Iteratively removes features from the model
Can be a more time-consuming process than backward feature engineering	Can be a more systematic approach than forward feature engineering
Can be useful when the number of features is relatively small	Can be useful when the number of features is relatively large
Can be prone to overfitting if too many features are added to the model	Can be prone to underfitting if too many features are removed from the model
Can be used in combination with backward feature engineering to optimize the feature selection process	Can be used in combination with forward feature engineering to optimize the feature selection process

In summary, forward feature engineering and backward feature engineering are two techniques used in machine learning for selecting relevant features to include in a model. Forward feature engineering starts with an empty feature set and iteratively adds one feature at a time based on their performance, while backward feature engineering starts with a complete set of features and removes features one by one until the model performance reaches a peak. Both techniques have their advantages and disadvantages and can be used in combination to optimize the feature selection process.

Exercise-1

- Try Backward Feature Elimination by adding polynomial and other relevant functions as base features to the data set in **non-linear-data-set-for-regression.csv**
- Try the Forward Feature Selection method for the same dataset
- Try the mixed approach (forward + backward) feature selection on the dataset.

Exercise-2

- Perform Linear Regression by adding appropriate features (polynomial / others) to the uploaded dataset **sine-segment-perturbed.csv.** What conclusions can you make?