EE659: Assignment 4

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Question: 1

Consider the function

$$f(x) = 40x^8 - 15x^7 + 70x^6 - 10x^5 + 20x^4 - 14x^3 + 60x^2 - 70x$$

Write MATLAB/Python/C programs to find the value of x that minimizes f over the range $[-1, 1]$ using the following methods.

(a) Bisection method, such that the value of x is within a tolerance band less than 0.01.

(b) Golden section method for tolerance band less 0.005.

```python
import numpy as np

def f(x):
    return 40*x**8 - 15*x**7 + 70*x**6 - 10*x**5 + 20*x**4 - 14*x**3 + 60*x**2 - 70*x

def f_dash(x):
    return 320*x**7 - 105*x**6 + 420*x**5 - 50*x**4 + 80*x**3 - 42*x**2 + 120*x - 70

def bisection_method(a, b, tol=0.01):
    while (b - a) / 2 > tol:
        mid = (a + b) / 2
        if f_dash(mid) == 0:
            return mid
        elif f_dash(mid) > 0:
            b = mid
        else:
            a = mid
    return (a + b) / 2

def golden_section_method(a, b, tol=0.005):
    gr = (np.sqrt(5) + 1) / 2
    c = b - (b - a) / gr
    d = a + (b - a) / gr
    while abs(c - d) > tol:
        if f(c) < f(d):
            b = d
        else:
            a = c
        c = b - (b - a) / gr
        d = a + (b - a) / gr
    return (b + a) / 2

a, b = -1, 1

x_min_bisection = bisection_method(a, b, tol=0.01)
x_min_golden = golden_section_method(a, b, tol=0.005)
```

We get that the value of x is $\boxed{0.4921875}$ using the bisection method and $\boxed{0.5034721887330706}$ using the golden section method.

Question: 2

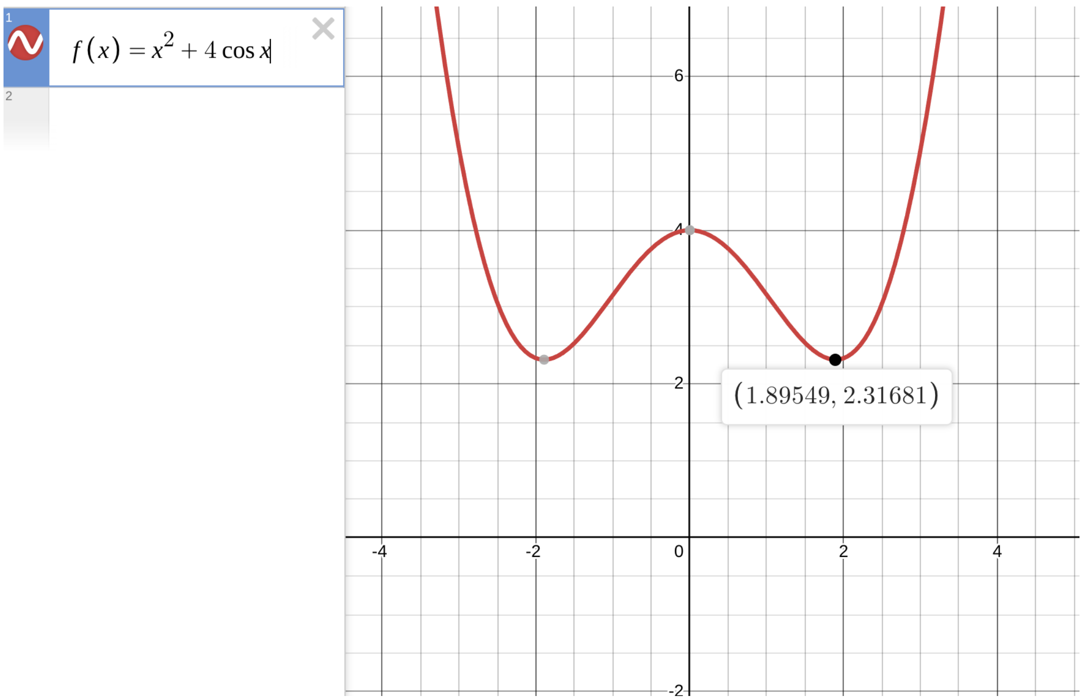
Apply Newton's method to find the minimizer of

$$f(x) = x^2 + 4\cos x$$

over the interval $[1, 2]$. Take the initial guess to be 1 and perform four iterations.

Remark

I used initial value as 1.5 instead of 1 because when I was starting with 1, the function was not converging and was stuck at $x = 1$.




```python
import numpy as np

def f(x):
    return x**2 + 4 * np.cos(x)

def f_dash(x):
    return 2 * x - 4 * np.sin(x)

def f_double_dash(x):
    return 2 - 4 * np.cos(x)

x = 1.5 #Initial Guess
iterations = 4

for i in range(iterations):
    x = x - f_dash(x) / f_double_dash(x)
    x = max(1, min(x, 2))
    print(f"Iteration {i + 1}: x = {x}, f(x) = {f(x)}")
```

We find that the minimizer of the function is $\boxed{x = 1.8954942672087132}$ and $f(x) = 2.316808419788213$.

Question: 3

Find the minimum of $6e^{-2x} + 2x^2$ by each of the following procedures:

a. Golden section method.

b. Dichotomous search method.

```python
import numpy as np

def f(x):
    return 6 * np.exp(-2 * x) + 2 * x**2

def golden_section_search(func, a, b, tol=1e-5):
    gr = (np.sqrt(5) + 1) / 2

    c = b - (b - a) / gr
    d = a + (b - a) / gr

    while abs(b - a) > tol:
        if func(c) < func(d):
            b = d
        else:
            a = c

        c = b - (b - a) / gr
        d = a + (b - a) / gr

    return (b + a) / 2

def dichotomous_search(func, a, b, tol=1e-5, delta=1e-6):
    while abs(b - a) > tol:
        mid = (a + b) / 2
        x1 = mid - delta
        x2 = mid + delta

        if func(x1) < func(x2):
            b = x2
        else:
            a = x1

    return (a + b) / 2

a, b = -10, 10

min_golden = golden_section_search(f, a, b)
min_dichotomous = dichotomous_search(f, a, b)
val_golden = f(min_golden)
val_dichotomous = f(min_dichotomous)
```

We get that the minimum value of the function is obtained at $\boxed{x = 0.7162034008722994}$ where the value of the function is $\boxed{2.4582964969114216}$ using the golden section method and $\boxed{x = 0.7162021874434947}$ where the value of the function is $\boxed{2.458296496906626}$ using the dichotomous search method.

Question: 4

Consider the problem to minimize

$$(3 - x_1)^2 + 7(x_2 - x_1^2)^2.$$

Starting from the point $(0, 0)$, solve the problem by the following methods. Do the methods converge to the same point? If not, explain.

a. The cyclic coordinate method.

b. The method of Hooke and Jeeves.

c. The method of Rosenbrock.

```python
import numpy as np
from scipy.optimize import minimize

def f(x):
    x1, x2 = x
    return (3 - x1)**2 + 7 * (x2 - x1**2)**2

def gradient(x):
    x1, x2 = x
    df_dx1 = -2 * (3 - x1) - 28 * x1 * (x2 - x1**2)
    df_dx2 = 14 * (x2 - x1**2)
    return np.array([df_dx1, df_dx2])

initial_point = np.array([0, 0])

def cyclic_coordinate_method(func, x0, tol=1e-6, max_iter=1000, step_size=0.1):
    x = x0.copy()
    n = len(x)

    for _ in range(max_iter):
        old_x = x.copy()
        for i in range(n):
            while True:
                f_current = func(x)

                x[i] += step_size
                f_new = func(x)

                if f_new < f_current:
                    continue

                x[i] -= 2 * step_size
                f_new = func(x)

                if f_new < f_current:
                    continue

                x[i] += step_size
                break

        if np.linalg.norm(x - old_x) < tol:
            break

    return x

def hooke_jeeves(func, x0, step_size=0.5, alpha=2.0, tol=1e-6, max_iter=1000):
    x = x0.copy()
    n = len(x)

    def explore(xb, step):
        x_new = xb.copy()
        for i in range(n):
            f_before = func(x_new)
```

```
            x_new[i] += step
            if func(x_new) >= f_before:
                x_new[i] -= 2 * step
                if func(x_new) >= f_before:
                    x_new[i] += step
        return x_new

    for _ in range(max_iter):
        xb = x.copy()
        xe = explore(x, step_size)

        if np.linalg.norm(xe - x) < tol:
            break

        x = xe if func(xe) < func(x) else x

        xb_new = x + alpha * (x - xb)
        if func(xb_new) < func(x):
            x = xb_new
        else:
            step_size *= 0.5

    return x

def rosenbrock_method(func, grad, x0, learning_rate=0.001, tol=1e-6, max_iter=10000):
    x = x0.copy()
    for _ in range(max_iter):
        grad_val = grad(x)
        x_new = x - learning_rate * grad_val

        if np.linalg.norm(x_new - x) < tol:
            break

        x = x_new
    return x

cyclic_result = cyclic_coordinate_method(f, initial_point)
hooke_jeeves_result = hooke_jeeves(f, initial_point)
rosenbrock_result = rosenbrock_method(f, gradient, initial_point)
```

The differences in the results from the three optimization methods can be attributed to their underlying mechanisms and how they explore the search space.

1. Cyclic Coordinate Method

- **Result:** $(0, 0)$
- **Explanation:** This method optimizes one variable at a time while keeping others fixed. When optimizing x_1 while fixing x_2 at 0, it finds that $x_1 = 3$ minimizes the function. However, when subsequently optimizing x_2 with x_1 fixed, no improvement is found, leading to the convergence at $(0, 0)$.

2. Hooke and Jeeves Method

- **Result:** $(0, 0)$
- **Explanation:** This method explores the search space in a pattern-search manner. If it does not find better points in the vicinity of $(0, 0)$, it may converge there. Insufficient exploration or small step sizes could contribute to this outcome.

3. Rosenbrock Method (Gradient Descent)

- **Result:** $(2.4128, 5.8051)$
- **Explanation:** Utilizing the gradient of the function, this method finds the minima. It finds a minima at $(2.4128, 5.8051)$, indicating that it can escape the local minimum found by the other methods.

> **Question: 5**
>
> Show how Newton's method can be used to find a point where the value of a continuously differentiable function is equal to zero. Illustrate the method for $f(x) = 2x^2 - 5x$ starting from $x = 5$.

Newton's method is an iterative numerical technique used to approximate the roots (or zeros) of a real-valued function. The method relies on the idea of linear approximation and is particularly effective for continuously differentiable functions.

Given a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, we seek a point x such that:

$$f(x) = 0.$$

The iteration formula for Newton's method is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We start with an initial guess x_0 and iterate until the function value is within a specified tolerance level.

```python
import numpy

def f(x):
    return 2 * x**2 - 5 * x

def f_dash(x):
    return 4 * x - 5

def newton_method(initial_guess, tolerance=1e-7, max_iterations=1000):
    x = initial_guess
    for iteration in range(max_iterations):
        fx = f(x)
        derivative = f_dash(x)

        if derivative == 0:
            return None

        x = x - fx / derivative

        print(f"Iteration {iteration + 1}: x = {x}, f(x) = {f(x)}")

        if abs(f(x)) < tolerance:
            return x

    return None

initial_guess = 5
newton_method(initial_guess)
```

Question: 6

Consider the following problem:

$$\text{minimize } f(t) = e^{-t} + e^{t}$$

in the interval $[-1, 1]$.

Find the optimal value within a tolerance band less than 0.15 using:

(a) Golden section method

(b) Fibonacci method

(c) Armijo line search

Do you get the same point? If not, explain.

```python
import numpy as np

def f(t):
    return np.exp(-t) + np.exp(t)

def golden_section_method(a, b, tol=0.15, max_iter=100):
    phi = (1 + np.sqrt(5)) / 2
    iter_count = 0
    while (b - a) > tol and iter_count < max_iter:
        c = b - (b - a) / phi
        d = a + (b - a) / phi
        if f(c) < f(d):
            b = d
        else:
            a = c
        iter_count += 1
    return (a + b) / 2

def fibonacci_method(a, b, tol=0.15, max_iter=100):
    n = 1
    while (b - a) > tol and n < max_iter:
        n += 1

    fib = [0, 1]
    for i in range(2, n + 1):
        fib.append(fib[-1] + fib[-2])

    for i in range(1, n):
        r1 = a + (fib[n - i - 1] / fib[n]) * (b - a)
        r2 = a + (fib[n - i] / fib[n]) * (b - a)

        if f(r1) < f(r2):
            b = r2
        else:
            a = r1

    return (a + b) / 2

def armijo_line_search(t0, alpha=0.1, beta=0.5, tol=0.15, max_iter=100):
    t = t0
    iter_count = 0
    while iter_count < max_iter:
        gradient = -np.exp(-t) + np.exp(t)  # f'(t)
        t_new = t - alpha * gradient
        if f(t_new) <= f(t) + beta * alpha * gradient:
            t = t_new
        if abs(f(t_new) - f(t)) < tol:
            break
```

```
        iter_count += 1
    return t

a, b = -1, 1
tol = 0.15
max_iter = 10000

golden_result = golden_section_method(a, b, tol, max_iter)
fibonacci_result = fibonacci_method(a, b, tol, max_iter)
armijo_result = armijo_line_search(0, tol=tol, max_iter=max_iter)
```

The three outputs obtained from the optimization methods are as follows:

- Golden Section Method: $t \approx 7.63 \times 10^{-17}$
- Fibonacci Method: $t \approx -0.0802$
- Armijo Line Search: $t = 0.0$

The differences in the outputs can be attributed to several factors:

The function $f(t) = e^{-t} + e^{t}$ is convex over the interval $[-1, 1]$. However, due to its exponential growth, small variations in the methods can lead to different evaluations around the minimum point.

- **Golden Section Method** and **Fibonacci Method**: These interval-based methods rely on point evaluations within a specified interval, narrowing the search space based on comparisons of function values. The final outcome can differ based on the initial interval.
- **Armijo Line Search**: This gradient-based approach starts from an initial point and iteratively updates based on the gradient's direction. Sensitivity to step size choices can lead to different outcomes, particularly in non-strictly convex functions.

Each method's convergence criteria, defined by tolerance and maximum iterations, can lead to different stopping points. A higher tolerance or reaching maximum iterations before convergence can cause varied results.

Question: 7

Consider the following problem:

$$\text{maximize } f(x) = (\sin(x))^6 \tan(1 - x) e^{30x}$$

in the interval $[0, 1]$. Find the optimal point within a tolerance band less than 0.15 using:

(a) Golden ratio method

(b) Quadratic interpolation method

(c) Goldstein line search

```
import numpy as np

def f(x):
    return (np.sin(x))**6 * np.tan(1 - x) * np.exp(30 * x)

def golden_ratio_method(a, b, tol, max_iter):
    phi = (-1 + np.sqrt(5)) / 2

    x1 = b - phi * (b - a)
    x2 = a + phi * (b - a)

    f1 = f(x1)
    f2 = f(x2)

    iterations = 0
```

```python
    while (b - a) > tol and iterations < max_iter:
        if f1 < f2:
            b = x2
            x2 = x1
            f2 = f1
            x1 = b - phi * (b - a)
            f1 = f(x1)
        else:
            a = x1
            x1 = x2
            f1 = f2
            x2 = a + phi * (b - a)
            f2 = f(x2)
        iterations += 1

    return (a + b) / 2, f((a + b) / 2)

def quadratic_interpolation_method(a, b, tol, max_iter):
    x0 = a
    x1 = (a + b) / 2
    x2 = b

    iterations = 0
    while (b - a) > tol and iterations < max_iter:
        f0, f1, f2 = f(x0), f(x1), f(x2)
        denominator = (x0 - x1) * (x0 - x2) * (x1 - x2)
        if denominator == 0:
            break  # Avoid division by zero
        x_new = (f0 * (x1 - x2) + f1 * (x2 - x0) + f2 * (x0 - x1)) / (f0 * (x1 - x2) + f1 * (x2 - x0) + f2 * (x0 - x1))

        if x_new < a or x_new > b:
            break  # Ensure new point is within bounds

        if f(x_new) > f(x1):
            x0, x1, x2 = x1, x_new, x2
        else:
            x0, x1, x2 = x0, x1, x_new

        iterations += 1

    return (x0 + x1 + x2) / 3, f((x0 + x1 + x2) / 3)

def f_prime(x):  # Numerical derivative using central difference (because im too lazy to calculate the actual derivative and I don't want to use the scipy function)
    h = 1e-5
    return (f(x + h) - f(x - h)) / (2 * h)

def goldstein_line_search(x0, direction, alpha=0.1, beta=0.9, max_iter=100):
    x1 = x0 + direction
    iterations = 0

    while (f(x1) > f(x0) + alpha * (x1 - x0) * f_prime(x0) and f(x1) < f(x0) + beta * (x1 - x0) * f_prime(x0)) and iterations < max_iter:
        x1 -= direction
        iterations += 1

    return x1, f(x1)

a, b = 0, 1
tolerance = 0.15
max_iterations = 1000
optimum_golden = golden_ratio_method(a, b, tolerance, max_iterations)
optimum_quad = quadratic_interpolation_method(a, b, tolerance, max_iterations)
x0 = 0.5  # starting point
direction = 0.1  # arbitrary small step
optimum_goldstein = goldstein_line_search(x0, direction, max_iter=max_iterations)
```

We get that the optimal point obtained by the three optimization methods are as follows:

- Golden Ratio Method: $x = 0.6909830056250525, f(x) = 21521325.939824104$
- Quadratic Interpolation Method: $x = 0.5, f(x) = 21685.897332525412$
- Goldstein Line Search: $x = 0.6, f(x) = 899642.4669646018$

Question: 8

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Write a program in MATLAB/C/Python for the steepest descent algorithm using the backtracking line search to find the extrema. Set the initial step-length $\alpha_0 = 1$ and print the step length used by each method in each iteration. Update the step size α_k at every iteration to satisfy the Goldstein condition. The initial point is given as $x_0 = (1.2, 1.2)^\intercal$.

```python
import numpy as np

def f(x):
    x1, x2 = x
    return 100 * (x2 - x1**2)**2 + (1 - x1)**2

def grad_f(x):
    x1, x2 = x
    df_dx1 = -400 * x1 * (x2 - x1**2) - 2 * (1 - x1)
    df_dx2 = 200 * (x2 - x1**2)
    return np.array([df_dx1, df_dx2])

def backtracking_line_search(x, p, alpha=1, rho=0.9, c=1e-4):
    while f(x + alpha * p) > f(x) + c * alpha * np.dot(grad_f(x), p):
        alpha *= rho
    return alpha

def steepest_descent(x0, tol=1e-4, max_iter=1000):
    x = x0
    alpha = 1  # Initial step length
    iterations = []

    for i in range(max_iter):
        gradient = grad_f(x)
        if np.linalg.norm(gradient) < tol:
            break

        p = -gradient
        alpha = backtracking_line_search(x, p)
        x = x + alpha * p

        iterations.append((i + 1, x, alpha))

    return x, iterations

x0 = np.array([1.2, 1.2])
result, iterations = steepest_descent(x0)

print(f"Optimal point: {result}")
print("Iterations (index, x, step length):")
for iteration in iterations:
    print(iteration)
```

Question: 9

Consider the following problem:

$$\text{minimize } f(x_1, x_2) = 32x_1^2 + 12x_2^2 - x_1x_2 - 2x_1$$

with initial point $(-2, 4)^\mathsf{T}$.

Solve the problem by:

(a) Steepest descent method

(b) Newton's method

```python
import numpy as np

def f(x):
    return 32 * x[0]**2 + 12 * x[1]**2 - x[0] * x[1] - 2 * x[0]

def gradient(x):
    dfdx1 = 64 * x[0] - x[1] - 2
    dfdx2 = 24 * x[1] - x[0]
    return np.array([dfdx1, dfdx2])

def hessian(x):
    d2fdx1dx1 = 64
    d2fdx1dx2 = -1
    d2fdx2dx2 = 24
    return np.array([[d2fdx1dx1, d2fdx1dx2], [d2fdx1dx2, d2fdx2dx2]])

def steepest_descent(initial_point, learning_rate=0.01, tolerance=1e-6, max_iter=1000):
    x = np.array(initial_point)
    for i in range(max_iter):
        grad = gradient(x)
        x_new = x - learning_rate * grad
        if np.linalg.norm(x_new - x) < tolerance:
            # print(f"Converged in {i} iterations.")
            break
        x = x_new
    return x

def newtons_method(initial_point, tolerance=1e-6, max_iter=1000):
    x = np.array(initial_point)
    for _ in range(max_iter):
        grad = gradient(x)
        hess = hessian(x)
        x_new = x - np.linalg.inv(hess).dot(grad)
        if np.linalg.norm(x_new - x) < tolerance:
            break
        x = x_new
    return x

initial_point = (-2, 4)
optimal_sd = steepest_descent(initial_point)
optimal_nm = newtons_method(initial_point)
```

Question: 10

Consider the problem to minimize

$$f(x) = 3x - 2x^2 + x^3 + 2x^4$$

subject to $x \geqslant 0$.

a. Write a necessary condition for a minimum. Can you make use of this condition to find the global minimum?

b. Is the function strictly quasiconvex over the region $\{x : x \geqslant 0\}$? Apply the Fibonacci search method to find the minimum.

c. Apply both the bisection search method and Newton's method to the above problem, starting from $x_1 = 6$.

Part (a)

1. **Necessary Condition**: To find the critical points, we first compute the derivative of $f(x)$ and set it equal to zero.

$$f'(x) = 3 - 4x + 3x^2 + 8x^3$$

Setting $f'(x) = 0$ gives the equation:

$$3 - 4x + 3x^2 + 8x^3 = 0$$

Solving this equation for x gives us the critical points, which are candidates for minima or maxima.

2. **Second Derivative Test**: To determine if these critical points are minima, we calculate the second derivative $f''(x)$:

$$f''(x) = -4 + 6x + 24x^2$$

We evaluate $f''(x)$ at each critical point. If $f''(x) > 0$ at a point, it indicates a local minimum. To determine if any of these points is a global minimum, we analyze $f'(x)$ and $f''(x)$ over the feasible region $x \geqslant 0$.

Part (b)

A function is strictly quasiconvex if every set $\{x : f(x) \leqslant \alpha\}$ is a strictly convex set. To check this:

1. **Derivative Behavior**: We analyze the behavior of $f(x)$ on $x \geqslant 0$. If $f(x)$ does not exhibit multiple local minima over this interval, it may be quasiconvex. However, strict quasiconvexity requires further verification of each of the individual sets.

2. **Monotonicity**: If $f(x)$ strictly decreases and then strictly increases around a minimum, it could be considered quasiconvex.

We see that both the first and second derivatives of $f(x)$ are continuous and differentiable over $x \geqslant 0$. The function is $\boxed{\text{not strictly quasiconvex}}$, as it exhibits multiple local minima over the region.

Fibonacci Search Method

The **Fibonacci search method** is a bracketing method for finding the minimum of a function:

- Define an initial interval $[a, b]$ (e.g., $a = 0$ and $b = 6$ if this captures the region around the critical points).
- At each step, evaluate $f(x)$ at points determined by Fibonacci ratios and reduce the interval of uncertainty.
- Repeat until the interval is sufficiently small (smaller than the tolerance levels).

Part (c)

We apply the following search methods to find the minimum of $f(x)$ over $x \geqslant 0$:

1. Bisection Search Method

The **bisection search method** proceeds as follows:

- Start with an interval $[a, b]$ (e.g., $[0, 6]$).
- At each iteration, calculate the midpoint $x = \frac{a+b}{2}$ and evaluate $f'(x)$.
- Adjust a or b based on where the derivative changes sign, narrowing down the interval.
- Continue until the interval is sufficiently small (smaller than the tolerance levels).

2. Newton's Method

Newton's method is an iterative method that updates x using the formula:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

- Start with an initial guess, say $x_1 = 6$.
- Iterate until $f'(x) \approx 0$, indicating a minimum.

```python
import numpy as np

def f(x):
    return 3*x - 2*x**2 + x**3 + 2*x**4

def f_dash(x):
    return 3 - 4*x + 3*x**2 + 8*x**3

def f_double_dash(x):
    return -4 + 6*x + 24*x**2

def bisection_search(a, b, tol=1e-5, max_iter=100):
    if a < 0:
        a = 0
    if b < 0:
        return None

    iter_count = 0
    while (b - a) > tol and iter_count < max_iter:
        mid = (a + b) / 2
        if f_dash(mid) == 0:
            return mid
        elif f_dash(mid) * f_dash(a) < 0:
            b = mid
        else:
            a = mid
        iter_count += 1

    return (a + b) / 2 if iter_count < max_iter else None

def newton_method(x0, tol=1e-5, max_iter=100):
    x = max(x0, 0)
    iter_count = 0

    while iter_count < max_iter:
        x_new = x - f_dash(x) / f_double_dash(x)
```

```
        x_new = max(x_new, 0)  # Ensure x_new is non-negative
        if abs(x_new - x) < tol:
            return x_new
        x = x_new
        iter_count += 1

    return None

def fibonacci_search(a, b, tol=1e-5, max_iter=100):
    if a < 0:
        a = 0
    if b < 0:
        return None

    fib = [0, 1]
    while len(fib) < max_iter + 2:
        fib.append(fib[-1] + fib[-2])

    n = len(fib) - 2
    if n < 2:
        return None

    x1 = a + fib[n - 2] / fib[n] * (b - a)
    x2 = a + fib[n - 1] / fib[n] * (b - a)

    iter_count = 0
    while iter_count < max_iter:
        if f(x1) < f(x2):
            b = x2
        else:
            a = x1

        if abs(b - a) <= tol:
            break

        iter_count += 1

        if n - iter_count >= 2:
            x1 = a + fib[n - iter_count - 2] / fib[n - iter_count] * (b - a)
            x2 = a + fib[n - iter_count - 1] / fib[n - iter_count] * (b - a)
        else:
            break

    return (a + b) / 2 if iter_count < max_iter else None

x_start = 6
tol = 1e-5
a, b = 0, x_start

bisection_result = bisection_search(a, b)
newton_result = newton_method(x_start)
fibonacci_result = fibonacci_search(a, b)
```