EE659: Assignment 5

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Question: 1

Let $f(\mathbf{x})$, where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathbb{R}^2$, be given by

$$f(\mathbf{x}) = 5x_1^2 + x_2^2 + 2x_1x_2 - 3x_1 - x_2.$$

(a) Express $f(\mathbf{x})$ in the form of $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - \mathbf{x}^\top\mathbf{b}$.

(b) Find the minimizer of f using the conjugate gradient algorithm. Use a starting point of $\mathbf{x}^{(0)} = (0,0)^\top$, where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$.

Proof. We begin by identifying the quadratic terms and linear terms in the function $f(\mathbf{x})$.

Quadratic terms:

- The coefficient of x_1^2 is 5, so $Q_{11} = 10$ (since we have a factor of $\frac{1}{2}$ in the quadratic form).
- The coefficient of x_2^2 is 1, so $Q_{22} = 2$.
- The coefficient of x_1x_2 is 2, so $Q_{12} = Q_{21} = 1$.

Thus, the matrix Q is:

$$Q = \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix}.$$

Linear terms:
The linear terms are $-3x_1$ and $-x_2$. Hence, the vector $\mathbf{b}$ is:

$$\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Therefore, we can express the function as:

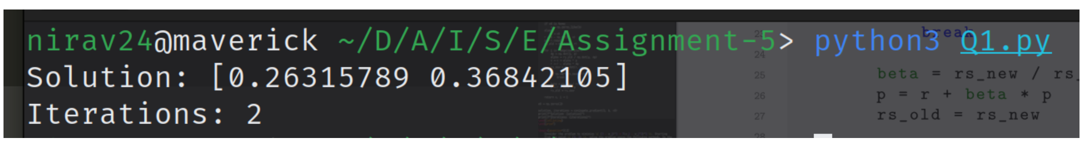
$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{x} - \mathbf{x}^\top \begin{bmatrix} 3 & 1 \end{bmatrix}.$$

For the second part, we resort to Python.

```
import numpy as np

Q = np.array([[10, 1], [1, 2]])
b = np.array([3, 1])

def conjugate_gradient(Q, b, x0=None, tol=1e-9, max_iter=1000):
```

```python
    if x0 is None:
        x = np.zeros_like(b)
    else:
        x = x0
    r = b - np.dot(Q, x)
    p = r.copy()
    rs_old = np.dot(r, r)

    for i in range(max_iter):
        Ap = np.dot(Q, p)
        alpha = rs_old / np.dot(p, Ap)
        x = x + alpha * p
        r = r - alpha * Ap
        rs_new = np.dot(r, r)

        if np.sqrt(rs_new) < tol:
            break

        beta = rs_new / rs_old
        p = r + beta * p
        rs_old = rs_new

    return x, i + 1

x0 = np.zeros(2)

solution, iterations = conjugate_gradient(Q, b, x0)
print(f"Solution: {solution}")
print(f"Iterations: {iterations}")
```

We get the results as shown below:



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Question: 2

Consider the problem to minimize $(3-x_1)^2 + 7(x_2 - x_1^2)^2$. Starting from the point $(0, 0)$, solve the problem using the following methods. Do the methods converge to the same point? If not, explain.

(a) The method of Fletcher and Reeves.

(b) The method of Davidon-Fletcher-Powell.

(a). We solve this numerically using Python.

```python
import numpy as np

def f(x):
    x1, x2 = x
    return (3 - x1)**2 + 7 * (x2 - x1**2)**2

def grad_f(x):
    x1, x2 = x
    df_dx1 = -2 * (3 - x1) - 14 * (x2 - x1**2) * 2 * x1
    df_dx2 = 14 * (x2 - x1**2)

    grad = np.array([df_dx1, df_dx2])
```

```python
    grad = np.clip(grad, -10, 10)

    return grad

def fletcher_reeves_method(x0, tol=1e-6, max_iter=1000):
    x = x0
    grad = grad_f(x)
    p = -grad
    iter_count = 0
    alpha = 0.01

    while np.linalg.norm(grad) > tol and iter_count < max_iter:
        x = x + alpha * p
        new_grad = grad_f(x)

        beta = np.dot(new_grad, new_grad) / np.dot(grad, grad)
        p = -new_grad + beta * p

        grad = new_grad
        iter_count += 1

        if np.any(np.isnan(grad)) or np.any(np.isinf(grad)):
            print("Gradient became NaN or Inf!")
            break

    return x, iter_count

x0 = np.array([0.0, 0.0])

solution_fletcher_reeves, iterations_fletcher_reeves = fletcher_reeves_method(x0)
print(f"Fletcher and Reeves solution: {solution_fletcher_reeves}")
print(f"Iterations: {iterations_fletcher_reeves}")
```

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(b). We solve this numerically using Python.

```python
import numpy as np

def f(x):
    x1, x2 = x
    return (3 - x1)**2 + 7 * (x2 - x1**2)**2

def grad_f(x):
    x1, x2 = x
    df_dx1 = -2 * (3 - x1) - 14 * (x2 - x1**2) * 2 * x1
    df_dx2 = 14 * (x2 - x1**2)
    return np.array([df_dx1, df_dx2])

def dfp_method(x0, tol=1e-6, max_iter=1000):
    x = x0
    grad = grad_f(x)
    H = np.eye(len(x))
    iter_count = 0

    while np.linalg.norm(grad) > tol and iter_count < max_iter:
        p = -np.dot(H, grad)  # search direction
        alpha = 0.01
        x_new = x + alpha * p
        grad_new = grad_f(x_new)

        s = x_new - x
        y = grad_new - grad

        H = H + np.outer(y, y) / np.dot(y, s) - np.dot(np.dot(H, np.outer(s, s)), H) / np.dot(s, np
            .dot(H, s))

```

```
        x = x_new
        grad = grad_new
        iter_count += 1

    return x, iter_count

x0 = np.array([0.0, 0.0])

solution_dfp, iterations_dfp = dfp_method(x0)
print(f"Davidon-Fletcher-Powell solution: {solution_dfp}")
print(f"Iterations: {iterations_dfp}")
```

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Question: 3

Consider the problem

$$\text{minimize} \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1 - 3x_4$$

subject to

$$2x_1 + x_2 + x_3 + 4x_4 = 7,$$
$$x_1 + x_2 + 3x_3 + x_4 = 6,$$
$$x_i \geqslant 0, \quad i = 1, 2, 3, 4$$

(a) Solve the problem using any optimization technique learned.

(b) Suppose the given initial guess is $\mathbf{x} = (2, 2, 1, 0)$, perform two iterations of the gradient projection method and comment on the directions obtained.

Proof. We will solve this problem using the method of Lagrange multipliers. The objective function is:

$$f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1 - 3x_4$$

The gradient of $f(x)$ is:

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4} \right)$$

We compute each partial derivative:

$$\frac{\partial f}{\partial x_1} = 2x_1 - 2, \quad \frac{\partial f}{\partial x_2} = 2x_2, \quad \frac{\partial f}{\partial x_3} = 2x_3, \quad \frac{\partial f}{\partial x_4} = 2x_4 - 3$$

Thus, the gradient is:

$$\nabla f(x) = (2x_1 - 2, 2x_2, 2x_3, 2x_4 - 3)$$

The constraints are:

$$2x_1 + x_2 + x_3 + 4x_4 = 7,$$
$$x_1 + x_2 + 3x_3 + x_4 = 6$$

We form the Lagrangian:

$$\mathcal{L}(x_1, x_2, x_3, x_4, \lambda_1, \lambda_2) = f(x_1, x_2, x_3, x_4) + \lambda_1(2x_1 + x_2 + x_3 + 4x_4 - 7) + \lambda_2(x_1 + x_2 + 3x_3 + x_4 - 6)$$

We need to solve the system:

$$
\begin{aligned}
2x_1 - 2 + 2\lambda_1 + \lambda_2 &= 0 \\
2x_2 + \lambda_1 + \lambda_2 &= 0 \\
2x_3 + \lambda_1 + 3\lambda_2 &= 0 \\
2x_4 - 3 + 4\lambda_1 + \lambda_2 &= 0 \\
2x_1 + x_2 + x_3 + 4x_4 &= 7 \\
x_1 + x_2 + 3x_3 + x_4 &= 6
\end{aligned}
$$

By solving this system, we obtain the values of $x_1, x_2, x_3, x_4, \lambda_1, \lambda_2$.

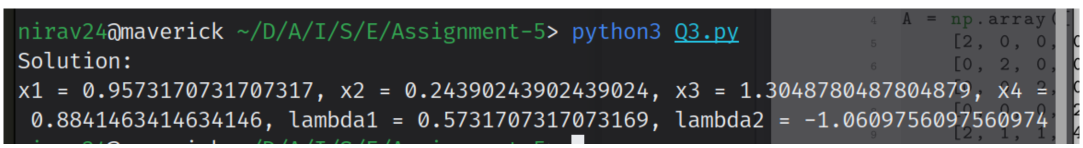
```python
import numpy as np
from scipy.linalg import solve

A = np.array([
    [2, 0, 0, 0, 2, 1],
    [0, 2, 0, 0, 1, 1],
    [0, 0, 2, 0, 1, 3],
    [0, 0, 0, 2, 4, 1],
    [2, 1, 1, 4, 0, 0],
    [1, 1, 3, 1, 0, 0]
])

b = np.array([2, 0, 0, 3, 7, 6])

solution = solve(A, b)

x1, x2, x3, x4, lambda1, lambda2 = solution
print(f"x1 = {x1}, x2 = {x2}, x3 = {x3}, x4 = {x4}, lambda1 = {lambda1}, lambda2 = {lambda2}")
```



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Question: 4

Consider the problem:

$$\text{minimize} \quad f(x) = x^{\frac{4}{3}}$$

Note that 0 is the global minimizer of f.

(a) Write down the algorithm for Newton's method applied to this problem.

(b) Show that as long as the starting point is not 0, the algorithm in part (a) does not converge to 0, no matter how close to 0 we start.

We are given the function $f(x) = x^{\frac{4}{3}}$ and need to apply Newton's method for minimization. The general update rule for Newton's method is:

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

First, we compute the first and second derivatives of the function $f(x)$:

$$f'(x) = \frac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}}$$

$$f''(x) = \frac{d}{dx}\left(\frac{4}{3}x^{\frac{1}{3}}\right) = \frac{4}{9}x^{-\frac{2}{3}}$$

Now, applying the Newton's method update rule:

$$x_{k+1} = x_k - \frac{\frac{4}{3}x_k^{\frac{1}{3}}}{\frac{4}{9}x_k^{-\frac{2}{3}}}$$

Simplifying the expression:

$$x_{k+1} = x_k - 3x_k$$

$$x_{k+1} = -2x_k$$

Thus, the algorithm for Newton's method applied to this problem is:

$$x_{k+1} = -2x_k$$

We now analyze the convergence behavior of the algorithm. The update rule derived earlier is:

$$x_{k+1} = -2x_k$$

This means that at each iteration, the value of x_k is multiplied by -2. Therefore, the sequence $\{x_k\}$ evolves as follows:

$$x_0, \quad x_1 = -2x_0, \quad x_2 = -2x_1 = 4x_0, \quad x_3 = -2x_2 = -8x_0, \quad \ldots$$

So, the sequence alternates between multiplying the previous value by -2. As we can observe, no matter how small $|x_0|$ is (as long as $x_0 \neq 0$), the values of x_k will either grow or shrink but will never approach zero. Specifically, the magnitude of x_k grows exponentially (in absolute value) because each iteration multiplies the previous value by 2.

Therefore, as long as the starting point $x_0 \neq 0$, the sequence will not converge to 0; rather, it will diverge. Thus, the algorithm does **not** converge to 0, regardless of how close the starting point is to zero, as long as $x_0 \neq 0$.

Question: 5

Solve the problem to minimize

$$2x_1 + 3x_2^2 + e^{2x_1^2+x_2^2}$$

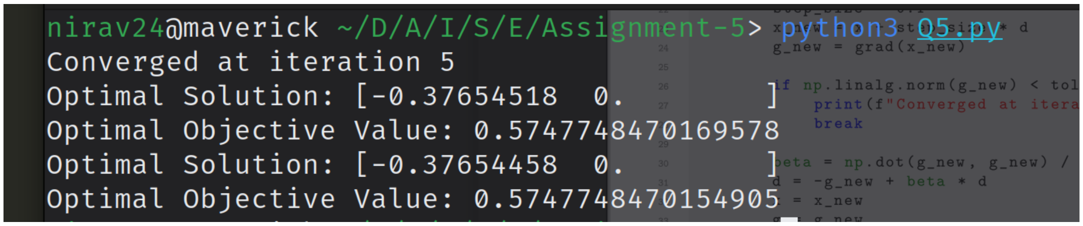
starting with the point $(1, 0)$, and using both the Fletcher and Reeves conjugate gradient method and the Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton method.

We use python to solve the above problem. The code is as follows:

```
import numpy as np
from scipy.optimize import minimize

def objective(x):
    x1, x2 = x
    exp_term = np.exp(np.clip(2 * x1**2 + x2**2, -50, 50))
    return 2 * x1 + 3 * x2**2 + exp_term

def gradient(x):
    x1, x2 = x
    exp_term = np.exp(np.clip(2 * x1**2 + x2**2, -50, 50))
    grad_x1 = 2 + 4 * x1 * exp_term
    grad_x2 = 6 * x2 + 2 * x2 * exp_term
    return np.array([grad_x1, grad_x2])

def fletcher_reeves_cg(func, grad, x0, tol=1e-6, max_iter=100):
    x = x0
    g = grad(x)
    d = -g
```

```
    for i in range(max_iter):
        g = grad(x)
        step_size = 0.1
        x_new = x + step_size * d
        g_new = grad(x_new)

        if np.linalg.norm(g_new) < tol:
            print(f"Converged at iteration {i}")
            break

        beta = np.dot(g_new, g_new) / np.dot(g, g)
        d = -g_new + beta * d
        x = x_new
        g = g_new

    return x, func(x)

def bfgs(func, grad, x0, tol=1e-6, max_iter=50, epsilon=1e-8):
    x = x0
    n = len(x)

    H = np.eye(n)

    f = func(x)
    g = grad(x)

    for _ in range(max_iter):
        p = -np.dot(H, g)

        alpha = 1.0
        x_new = x + alpha * p
        f_new = func(x_new)
        g_new = grad(x_new)

        if np.linalg.norm(g_new) < tol:
            return x_new, f_new

        s = x_new - x
        y = g_new - g

        rho = np.dot(y, s)
        if abs(rho) < epsilon:
            print("Warning: Small curvature. Skipping Hessian update.")
            return x_new, f_new

        rho = 1.0 / rho

        H = np.dot(np.eye(n) - rho * np.outer(s, y), np.dot(H, np.eye(n) - rho * np.outer(y, s))) +
             rho * np.outer(s, s)

        x = x_new
        f = f_new
        g = g_new

    return x, f

x0 = np.array([0.0, 0.0])

x_opt_cg, f_opt_cg = fletcher_reeves_cg(objective, gradient, x0)
print(f"Optimal Solution: {x_opt_cg}")
print(f"Optimal Objective Value: {f_opt_cg}")

x_opt_bfgs, f_opt_bfgs = bfgs(objective, gradient, x0)
print(f"Optimal Solution: {x_opt_bfgs}")
print(f"Optimal Objective Value: {f_opt_bfgs}")
```

Output on running the above code:



Question: 6

Consider the following problem:

$$\text{Minimize} \quad (x_1 - 5)^2 + (x_2 - 3)^2$$

subject to

$$3x_1 + 2x_2 \leqslant 6,$$
$$-4x_1 + 2x_2 + 2 \leqslant 4$$

Formulate a suitable barrier problem with the initial parameter equal to 1. Use an unconstrained optimization technique starting with the point $(0, 0)$ to solve the barrier problem.

```python
import numpy as np
from scipy.optimize import minimize

def objective(x):
    x1, x2 = x
    return (x1 - 5)**2 + (x2 - 3)**2

def barrier(x, mu):
    x1, x2 = x
    g1 = 6 - (3 * x1 + 2 * x2)
    g2 = 2 - (-4 * x1 + 2 * x2)

    if g1 <= 0 or g2 <= 0:
        return np.inf

    return -mu * (np.log(g1) + np.log(g2))

def barrier_objective(x, mu):
    return objective(x) + barrier(x, mu)

def solve_barrier_problem(mu, x0):
    result = minimize(barrier_objective, x0, args=(mu), method='CG', options={'disp': True})
    return result.x, result.fun

mu = 1
x0 = np.array([0.0, 0.0])

solution, objective_value = solve_barrier_problem(mu, x0)

print("Optimal Solution: ", solution)
print("Optimal Objective Value: ", objective_value)
```

Question: 7

Consider the following problem:

$$\text{Minimize} \quad x_1^2 + 2x_2^2$$

subject to

$$2x_1 + 3x_2 - 6 \leqslant 0, \quad -x_2 + 1 \leqslant 0.$$

(a) Find the optimal solution to this problem.

(b) Formulate a suitable function with an initial penalty parameter $\mu = 1$.

(c) Starting from the point $\begin{bmatrix} 2 & 4 \end{bmatrix}^\top$, solve the resulting problem by a suitable unconstrained optimization technique.

Proof. First, rewrite the constraints:

$$g_1(x_1, x_2) = 2x_1 + 3x_2 - 6 \leqslant 0, \quad g_2(x_1, x_2) = -x_2 + 1 \leqslant 0.$$

The Lagrange function for this problem is:

$$\mathcal{L}(x_1, x_2, \mu_1, \mu_2) = x_1^2 + 2x_2^2 + \mu_1(2x_1 + 3x_2 - 6) + \mu_2(-x_2 + 1).$$

The KKT conditions are:

$$\nabla_{x_1}\mathcal{L} = 0, \quad \nabla_{x_2}\mathcal{L} = 0,$$

along with the complementary slackness conditions:

$$\mu_1(2x_1 + 3x_2 - 6) = 0, \quad \mu_2(-x_2 + 1) = 0.$$

Solving these equations taking cases when μ_1 is or is not 0 and μ_2 is or is not 0, we get the optimal solution as $\begin{bmatrix} 0 & 1 \end{bmatrix}^\top$. Therefore, the optimal solution is:

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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Question: 8

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\|\mathbf{x}\|^2 \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $m < n$ and A is full row rank. Show that if $\mathbf{x}^{(0)} \in \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$, then the projected steepest descent algorithm converges to the solution in one step.

Proof. Let the objective function be:

$$f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2 = \frac{1}{2}\mathbf{x}^\top\mathbf{x}.$$

The gradient of $f(\mathbf{x})$ is (by derivative of quadratic form):

$$\nabla f(\mathbf{x}) = \mathbf{x}.$$

The projected steepest descent algorithm iteratively updates the solution using the gradient direction and projects the result onto the feasible set defined by $A\mathbf{x} = \mathbf{b}$.

The update rule for the projected steepest descent method is:

$$\mathbf{x}^{(k+1)} = \text{Proj}_{A\mathbf{x}=\mathbf{b}}\left(\mathbf{x}^{(k)} - \alpha_k \nabla f(\mathbf{x}^{(k)})\right),$$

where α_k is the step size at the k-th iteration and $\text{Proj}_{A\mathbf{x}=\mathbf{b}}$ denotes the projection onto the feasible set $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$.

Since $\mathbf{x}^{(0)}$ satisfies the constraint $A\mathbf{x}^{(0)} = \mathbf{b}$, the starting point is already in the feasible set. The gradient at this point is:

$$\nabla f(\mathbf{x}^{(0)}) = \mathbf{x}^{(0)}.$$

The update step is:

$$\mathbf{x}^{(1)} = \text{Proj}_{A\mathbf{x}=\mathbf{b}}\left(\mathbf{x}^{(0)} - \alpha_0 \mathbf{x}^{(0)}\right).$$

If we choose $\alpha_0 = 1$ (the value which optimizes the objective function), the update becomes:

$$\mathbf{x}^{(1)} = \text{Proj}_{A\mathbf{x}=\mathbf{b}}\left(\mathbf{x}^{(0)} - \mathbf{x}^{(0)}\right) = \text{Proj}_{A\mathbf{x}=\mathbf{b}}\left(\mathbf{0}\right).$$

Thus, the projection of the origin $\mathbf{0}$ onto the feasible set $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$ will give the point in the feasible set closest to the origin. This point is the optimal solution to the problem.

Since the feasible set is defined by $A\mathbf{x} = \mathbf{b}$, the projection of the origin $\mathbf{0}$ onto this set is the solution $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{b}$. Hence, the projected steepest descent method converges to the optimal solution in one step. ■