EE659 - Assignment 3

Q1) a) $L(x_1, x_2, x_3, \lambda_1, \lambda_2) = (x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3) + \lambda_1(x_1 + 2x_2 - 3)$
$+ \lambda_2(4x_1 + 5x_3 - 6)$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 + 2x_2 + 4 + \lambda_1 + 4\lambda_2 = 0 \quad -(1)$$
$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_1 + 6x_2 + 5 + 2\lambda_1 = 0 \quad -(2)$$
$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 6 + 5\lambda_2 = 0 \Rightarrow \lambda_2 = -6/5 \quad -(3)$$
$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + 2x_2 - 3 = 0 \quad -(4)$$
$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 4x_1 + 5x_3 - 6 = 0 \quad -(5)$$

Solving equation (1), (2), (3), (4) & (5), we get $(x_1, x_2, x_3, \lambda_1, \lambda_2)$
$$= \left(\frac{16}{5}, \frac{-1}{10}, \frac{-34}{50}, 0, \frac{-6}{5}\right)$$

Basically after $\lambda_2 = -6/5$, we have 4 equations (linear) in 4 variables, combining equation (1) & (2) while eliminating λ_1, we get 3 equations in x_1, x_2, x_3 which can be easily solved using Cramer's Rule

minimum value $= 1377/100 = \underline{13.77}$ $\quad$ $\underline{(x_1, x_2, x_3) = \left(\frac{16}{5}, \frac{-1}{10}, \frac{34}{25}\right)}$

b) $L(x_1, x_2, \lambda) = 4x_1 + x_2^2 + \lambda(9 - x_1^2 - x_2^2)$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4 - 2\lambda x_1 = 0 \longrightarrow \lambda = \frac{2}{x_1} \quad (x_1 \neq 0)$$
$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 2\lambda x_2 = 0 \Rightarrow x_2(1 - \lambda) = 0$$
$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1^2 + x_2^2 - 9 = 0$$

$\longrightarrow$ ① $\lambda = 1 \Rightarrow x_1 = 2$, $x_2 = \pm\sqrt{5}$

$\longrightarrow$ ② $x_2 = 0 \longrightarrow x_1 = \pm 3$

$(2, \sqrt{5}) \longrightarrow$ objective $= 13$

$(2, -\sqrt{5}) \longrightarrow$ objective $= 13$

$(3, 0) \longrightarrow$ objective $= 12$

$(-3, 0) \longrightarrow$ objective $= -12$

$\therefore$ maximum value $=$ 13 at $(x_1, x_2) = (2, \sqrt{5})$ or $(2, -\sqrt{5})$

c) $L(x,y,\lambda) = xy + \lambda(x^2+4y^2-1)$

$\frac{\partial L}{\partial x} = 0 \Rightarrow 2\lambda x + y = 0$ —(1)

$\frac{\partial L}{\partial y} = 0 \Rightarrow 8\lambda y + x = 0$ —(2)

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^2+4y^2-1=0$ —(3)

$\frac{x}{y} = \frac{-1}{2\lambda}$ from equation (1)

$\frac{x}{y} = -8\lambda$ from equation (2)

$\therefore \frac{-1}{2\lambda} = -8\lambda$

$\Rightarrow \lambda = \pm\frac{1}{4}$

$\Rightarrow (x,y) = \left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{4}\right)$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right) \longrightarrow \frac{1}{4}$

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right) \longrightarrow \frac{-1}{4}$

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right) \longrightarrow -\frac{1}{4}$

$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right) \longrightarrow \frac{1}{4}$

$\therefore$ Maximum value $= 1/4$, attained at $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right)$ or $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right)$

Q2) a) $f(x_1,x_2,x_3) = x_1^2 + 3x_2^2 + x_3$ $\qquad g(x) = x_1^2+x_2^2+x_3^2-16=0$

$L(x_1,x_2,x_3,\lambda) = x_1^2+3x_2^2+x_3+\lambda(x_1^2+x_2^2+x_3^2-16)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1(\lambda+1)=0$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2(\lambda+3)=0$

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2\lambda x_3 + 1 = 0$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1^2+x_2^2+x_3^2-16=0$

$\lambda = -1 \longrightarrow x_2 = 0,\ x_3 = 1/2,\ x_1 = \pm 3\sqrt{7}/2$

$\lambda = -3 \longrightarrow x_1 = 0,\ x_3 = 1/6,\ x_2 = \pm 5\sqrt{23}/6$

$\lambda \neq -1,-3 \Rightarrow x_1 = x_2 = 0 \Rightarrow x_3 = \pm 4$

Local Extremizers $\longrightarrow$ $(0,0,-4) \xrightarrow{f} -4$

$(0,0,4) \xrightarrow{f} 4$

$\left(0, \frac{-5\sqrt{23}}{6}, \frac{1}{6}\right) \xrightarrow{f} \frac{577}{12}$

$\left(0, \frac{5\sqrt{23}}{6}, \frac{1}{6}\right) \xrightarrow{f} \frac{577}{12}$

$\left(\frac{-3\sqrt{7}}{2}, 0, \frac{1}{2}\right) \xrightarrow{f} \frac{65}{4}$

$\left(\frac{3\sqrt{7}}{2}, 0, \frac{1}{2}\right) \xrightarrow{f} \frac{65}{4}$

b) Convert to polar coordinates - $x = r\cos\theta$, $y = r\sin\theta$

New problem is to find the local extremizers of r^2 subjected to

$r^2(8\cos^2\theta + 6\sin\theta\cos\theta + 6\sin^2\theta) = 200$

$$\Rightarrow r^2 = \frac{200}{8\cos^2\theta + 3\sin 2\theta + 6\sin^2\theta}$$

$$\Rightarrow r^2 = \frac{200}{6 + 2\cos^2\theta + 3\sin 2\theta}$$

$$\Rightarrow r^2 = \frac{200}{7 + \cos 2\theta + 3\sin 2\theta} \leq \frac{200}{7 - \sqrt{10}}$$

and

$$\Rightarrow r^2 = \frac{200}{7 + \cos 2\theta + 3\sin 2\theta} \geq \frac{200}{7 + \sqrt{10}}$$

Here we have used the well known inequality - $\sqrt{a^2+b^2} \leq a\cos\theta + b\sin\theta \leq \sqrt{a^2+b^2}$

with equality at $\cos\theta = \frac{\pm a}{\sqrt{a^2+b^2}}$, $\sin\theta = \frac{\pm b}{\sqrt{a^2+b^2}}$

minima is at $\cos 2\theta = \frac{1}{\sqrt{1^2+3^2}} \longrightarrow \cos 2\theta = \frac{1}{\sqrt{10}}$

$$\Rightarrow 2\theta = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

$$\Rightarrow \theta = \frac{1}{2}\cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \;\&\; \pi - \frac{1}{2}\cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

maxima is at $\cos 2\theta = \frac{-1}{\sqrt{1^2+3^2}} \longrightarrow \cos 2\theta = \frac{-1}{\sqrt{10}}$

$$\Rightarrow 2\theta = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right)$$

$$\Rightarrow \theta = \frac{1}{2}\cos^{-1}\left(\frac{-1}{\sqrt{10}}\right) \;\&\; \pi - \frac{1}{2}\cos^{-1}\left(\frac{-1}{\sqrt{10}}\right)$$

$\therefore$ Extrema at $(r,\theta) = \left(\frac{200}{7 \pm \sqrt{10}}, \frac{1}{2}\cos^{-1}\left(\frac{\mp 1}{\sqrt{10}}\right)\right) \;\&\; \left(\frac{200}{7 \pm \sqrt{10}}, \pi - \frac{1}{2}\cos^{-1}\left(\frac{\pm 1}{\sqrt{10}}\right)\right)$

local extrema $\longrightarrow$ $$\left(\frac{-10+\sqrt{10}}{117}\sqrt{3120+663\sqrt{10}},\ \sqrt{\frac{31200+6630\sqrt{10}}{1521}}\right)$$

$$\left(\frac{10-\sqrt{10}}{117}\sqrt{3120+663\sqrt{10}},\ -\sqrt{\frac{31200+6630\sqrt{10}}{1521}}\right)$$

$$\left(\frac{10+\sqrt{10}}{117}\sqrt{3120-663\sqrt{10}},\ \sqrt{\frac{31200-6630\sqrt{10}}{1521}}\right)$$

$$\left(-\frac{10+\sqrt{10}}{117}\sqrt{3120-663\sqrt{10}},\ -\sqrt{\frac{31200-6630\sqrt{10}}{1521}}\right)$$

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Q3) Using first order necessary conditions, we must have

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 4x + y - 6 = 0 \quad -(1)$$
$$\frac{\partial f}{\partial y} = 0 \Rightarrow x + 2y + z - 7 = 0 \quad -(2)$$
$$\frac{\partial f}{\partial z} = 0 \Rightarrow y + 2z - 8 = 0 \quad -(3)$$

From (1) & (2), we get $x + 2(6-4x) + z - 7 = 0$
$$\Rightarrow z = 7x - 5 \quad -(4)$$

Substituting (1) & (4) in (3), we get $(6-4x) + 2(7x-5) - 8 = 0$
$$\Rightarrow x = \frac{6}{5}$$
$$\Rightarrow y = \frac{6}{5}$$
$$\Rightarrow z = \frac{17}{5}$$
$$\Rightarrow (x,y,z) = \left(\frac{6}{5}, \frac{6}{5}, \frac{17}{5}\right) \longleftarrow \text{Critical Point}$$

Hessian Matrix $\longrightarrow \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Calculating eigenvalues $\longrightarrow (4-\lambda)(2-\lambda)^2 - (4-\lambda) - (2-\lambda) = 0$
$$\lambda^3 - 8\lambda^2 + 18\lambda - 10 = 0$$
Check graphically that all roots ≥ 0
or use intermediate value theorem to see that there is a root in $(0,1)$, $(2,3)$ & $(4,5)$ and so it has all eigenvalues non-negative & hence is semi-definite

Otherwise $\longrightarrow \det(4) = 4 > 1$
$$\det\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} = 7 > 0$$
$$\det(H) = 16 - 4 - 2 = 10 > 0$$
$\therefore$ H is positive ~~semi~~ definite, indicating that the critical point is local minimum

Since $f(x,y,z)$ is a quadratic function with positive definite hessian matrix, $f(x,y,z)$ is strictly convex and so the critical point is the global minimum.

Q4) $f(x,y) = (y - x^2)^2 - x^2$

(a)

(1) $\frac{\partial f}{\partial x} = 2(y-x^2)(-2x) - 2x = (-2x)(2y - 2x^2 + 1) = 0$

(2) $\frac{\partial f}{\partial y} = 2(y-x^2) = 0$

To find the critical points of $f(x,y)$

Substituting (2) in (1), we get $(-2x) = 0 \Rightarrow x = 0$

$\Rightarrow y = 0$ (By (2))

$\therefore$ Critical Point $= (0,0)$

$\frac{\partial^2 f}{\partial x^2} \longrightarrow -2$

$\frac{\partial^2 f}{\partial y^2} \longrightarrow 2$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 0$

Hessian Matrix $= \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad$ $\det(H) = -4 < 0$

$\therefore$ (0,0) is a saddle point as $\det(H) < 0$ & the function does not have a local maxima or minima at this point

(b) Substituting Boundary Values of y

for $y = 1 \longrightarrow f(x,1) = 1 - 3x^2 + x^4 = f_1(x)$

for $y = -1 \longrightarrow f(x,-1) = 1 + x^2 + x^4 = f_2(x)$

$\frac{\partial f_1}{\partial x} = 0 \longrightarrow -6x + 4x^3 = 0$

$\rightarrow 2x^3 - 3x = 0$

$\Rightarrow x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$ $\qquad$ $f(0,1) = 1$

$f\left(\pm\sqrt{\frac{3}{2}}, 1\right) = 1 - \frac{9}{2} + \frac{9}{4} = -\frac{5}{4}$

$\frac{\partial f_2}{\partial x} = 0 \longrightarrow 2x + 4x^3 = 0$

$\Rightarrow 2x^3 + x = 0$

$\Rightarrow x(2x^2 + 1) = 0$

$\Rightarrow x = 0$ $\qquad$ $f(0,-1) = 1$

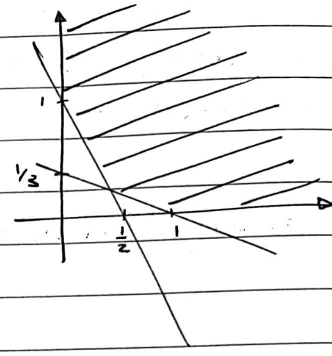
$\therefore$ Global maxima at (0,1) & (0,−1) with value 1.

Global minima at $\left(\pm\sqrt{\frac{3}{2}}, 1\right)$ with value $-5/4$.

Q5) (a) $L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = x_1^2 + 9x_2^2 - \lambda_1(2x_1 + x_2 - 1) - \lambda_2(x_1 + 3x_2 - 1) - \lambda_3 x_1 - \lambda_4 x_4$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - 2\lambda_1 - \lambda_2 - \lambda_3 = 0$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 18x_2 - \lambda_1 - 3\lambda_2 - \lambda_4 = 0$



Casework from now :

$\lambda_3 \neq 0 \longrightarrow x_1 = 0$

min. $x_2 = 1$

$\therefore$ value of min. function $= 9$

Used Karush-Kuhn-Tucker Conditions multiple times (complementary slackness condition majorly)

$\lambda_4 \neq 0 \longrightarrow x_2 = 0$

min. $x_1 = 1$

$\therefore$ value of min. function $= 1$

$\lambda_3 = \lambda_4 = 0$ nowonwards

$\lambda_1 \neq 0 \longrightarrow 2x_1 + x_2 - 1 = 0 \Rightarrow 2x_1 + x_2 = 1$

$(2x_1 + x_2)^2 \leq (x_1^2 + 9x_2^2)(4 + \frac{1}{9})$ (Cauchy-Schwartz Inequality)

$\Rightarrow (x_1^2 + 9x_2^2) \geq \frac{9}{37}$

$\hookrightarrow$ equality at $\frac{x_1}{2} = \frac{3x_2}{1/3} \Rightarrow x_1 = 18x_2$ & $2x_1 + x_2 = 1$

Intersection does not lie in the feasible region

$2x_1 + x_2 = 1$

Closest point on the line $\longrightarrow (2/5, 1/5)$

value of the function $= 13/25$

$\lambda_2 \neq 0 \longrightarrow x_1 + 3x_2 - 1 = 0 \Rightarrow x_1 + 3x_2 = 1$

$(x_1^2 + 9x_2^2)(1+1) \geq (x_1 + 3x_2)^2 = 1$ (Cauchy-Schwartz Inequality)

$\Rightarrow x_1^2 + 9x_2^2 \geq 1/2$

$\hookrightarrow$ equality at $\frac{x_1}{1} = \frac{3x_2}{1} \Rightarrow x_1 = 3x_2$ & $x_1 + 3x_2 = 1$

$\Rightarrow (x_1, x_2) = (1/2, 1/6)$

$\in$ feasible region

Min. value of the function $= 1/2$ at $(1/2, 1/6)$

$\lambda_3 = \lambda_4 = 0 \longrightarrow 2x_1 + x_2 = 1$

$x_1 + 3x_2 = 1 \longrightarrow (x_1, x_2) = (2/5, 1/5)$

$\longrightarrow$ value of function $= 13/25$

Having exhausted all cases, we safely claim that the minimum value is $1/2$ & is attained at $(1/2, 1/6)$

(b) We see that the constraint equation is redundant because of the square root in the objective function.

We switch to polar coordinates, in which the problem becomes:

Find the minimum & maximum values for

$$f(r,\theta) = r^2 \sin\theta\cos\theta + \sqrt{9-r^2}$$

$$\frac{\partial f}{\partial r} = 0 \Rightarrow 2r\sin\theta\cos\theta + \frac{(-2r)}{2\sqrt{9-r^2}} = 0$$

$$\frac{\partial f}{\partial \theta} = 0 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 0$$

$$\Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 0$$

$$r = 0 \quad \text{or} \quad \theta = \frac{\pi}{4} + n\pi/2 \quad n \in \mathbb{Z}$$

$$\theta = \pi/4 \longrightarrow 2r \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \frac{r}{\sqrt{9-r^2}} = 0$$

$$\Rightarrow r = \frac{r}{\sqrt{9-r^2}}$$

$$\Rightarrow \sqrt{9-r^2} = 1$$

$$\Rightarrow r = 2\sqrt{2}$$

$$f(r,\theta) = (2\sqrt{2})^2 \cdot 1/\sqrt{2} \cdot 1/\sqrt{2} + \sqrt{9-8}$$

$$= 4 + 1$$

$= 5 \longleftarrow$ Maximum value of $f(r,\theta)$ = Maximum value of $f(x,y)$

$$\theta = 3\pi/4 \longrightarrow 2r \cdot \frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) - \frac{r}{\sqrt{9-r^2}} = 0$$

$$\Rightarrow r + \frac{r}{\sqrt{9-r^2}} = 0$$

$$\Rightarrow r = 0$$

Considering points on the boundary with $\theta = \frac{3\pi}{4}$, we have

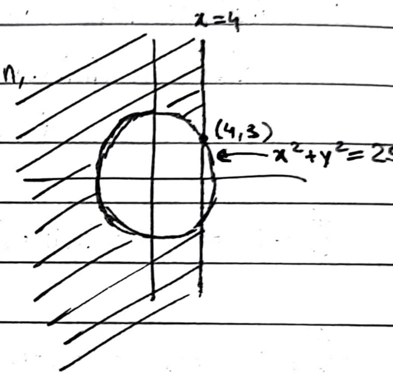
$$f(r,\theta) = 3^2\left(\frac{1}{\sqrt2}\right)\left(-\frac{1}{\sqrt2}\right) - \sqrt{9-9}$$
$$= -9/2 - 0$$
$$= -4.5$$

minimum value

We will get the same values when $\theta = 5\pi/4$ & $\theta = 7\pi/4$.

$\therefore$ maxima $= 5$, minima $= -4.5$

Q6) Since $\nabla f(x,y) \neq 0$ anywhere in the feasible region, the point optimizing the function must lie on the boundary.



$x = 4 \longrightarrow 5x^2 + 6y^2 = 80 + 6y^2 \geq 80 + 6(9)$
$= 134$

since the only points on $x=4$ in the feasible region have $y \leq -3$ or $y \geq 3$.

attained at $(4,3)$ or $(4,-3)$

$x^2+y^2 = 25 \longrightarrow$ We use Lagrange Multipliers

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y) = (5x^2+6y^2) + \lambda(x^2+y^2-25)$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 10x = 2\lambda x$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow 12y = 2\lambda y$$
$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^2+y^2-25 = 0$$

If $x = 0 \longrightarrow y = \pm 5$

value of the function $= 6(\pm5)^2 = 150$

If $y = 0 \longrightarrow x = \pm 5$

$\longrightarrow x = 5$ does not lie in the feasible region

value of the function $= 5(5)^2 = 125$

If $x \neq 0$ & $y \neq 0 \longrightarrow \lambda = \frac{10x}{2x} = \frac{12y}{2y} \Rightarrow 5 = 6$ # Contradiction

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$\therefore$ The minimum value of the function is 125 attained at $(-5,0)$.

Q7) (a) $L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (x_1 - 3/2)^2 + (x_2 - t)^4 + \lambda_1 g_1(x) + \lambda_2 g_2(x) + \lambda_3 g_3(x) + \lambda_4 g_4(x)$

Primal Feasibility $\longrightarrow g_i(x) \leq 0$

True trivially (we consider $g_i(x)$ as negative of function in question)

$\mp x_1 \mp x_2 - 1 \leq 0$

Dual Feasibility $\longrightarrow \lambda_i \geq 0$

True by choice of λ_i

Complementary Slackness $\longrightarrow g_1(1,0) = 0$

$g_2(1,0) = 0$

$g_3(1,0) = -2 \Rightarrow \lambda_3 = 0$

$g_4(1,0) = -2 \Rightarrow \lambda_4 = 0$

Stationary $\longrightarrow \frac{\partial L}{\partial x_1} = 0 \Rightarrow 2(1 - 3/2) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$

$\Rightarrow -1 + \lambda_1 + \lambda_2 = 0$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4(x_2 - t)^3 + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0$

$\Rightarrow -4t^3 + \lambda_1 - \lambda_2 = 0$

$\Rightarrow -4t^3 + \lambda_1 - (1 - \lambda_1) = 0$

$\Rightarrow \lambda_1 = \frac{4t^3+1}{2} \geq 0$ (By dual feasibility condition)

$\Rightarrow 4t^3 + 1 \geq 0$

$\Rightarrow t^3 \geq \frac{-1}{4}$

$\Rightarrow \boxed{t \geq -\frac{1}{\sqrt[3]{4}}}$

$\lambda_1 + \lambda_2 = 1$ & $\lambda_2 \geq 0 \longrightarrow \lambda_1 \leq 1$

$\Rightarrow \frac{4t^3+1}{2} \leq 1$

$\Rightarrow 4t^3 \leq 1$

$\Rightarrow \boxed{t \leq \frac{1}{\sqrt[3]{4}}}$

$\therefore -\frac{1}{\sqrt[3]{4}} \leq t \leq \frac{1}{\sqrt[3]{4}}$

(b) $t = 1 \longrightarrow$

minimize $(x - 3/2)^2 + (y-1)^4$ subject to $|x| + |y| \leq 1$

$$\frac{\partial f}{\partial x} = 2(x - 3/2)$$
$$\frac{\partial f}{\partial y} = 4(y-1)^3$$

$\nabla f = 0$ at $x = 3/2, y = 1$ which does not lie in the feasible region.

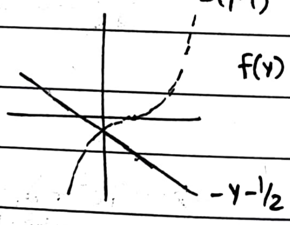
$\therefore$ The minima must be a boundary point.

Case I) $x \geq 0, y \geq 0$

Constraint simplifies to $x + y = 1$

$L(x, y, \lambda) = (x - 3/2)^2 + (y-1)^4 + \lambda(x+y-1) = 0$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2(x - 3/2) + \lambda = 0$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow 4(y-1)^3 + \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x + y - 1 = 0$$



$\Rightarrow (x - 3/2) = 2(y-1)^3$

$\Rightarrow (1 - y - 3/2) = 2(y-1)^3$

$\Rightarrow -y - 1/2 = 2(y-1)^3$

$\Rightarrow 2(y-1)^3 + y + 1/2 = 0$

Graphically we see that there can be only 1 root

at $y = 0$, $f(y) = -2 + 1/2 = -1.5$

at $y = 1$, $f(y) = 1 + 1/2 = 1.5$

$\therefore \exists$ a root between 0 & 1 $\longrightarrow$ Point of minima

Case II) $x \geq 0, y \leq 0$ $\longrightarrow$ Constraint becomes $x - y = 1$

We get the 3 Lagrange Equations as

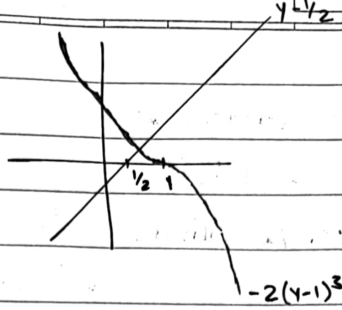
$$2(x - 3/2) + \lambda = 0$$
$$4(y-1)^3 - \lambda = 0$$
$$x - y - 1 = 0$$

$\Rightarrow (x-3/2) = -2(y-1)^3$

$\Rightarrow (1+y-3/2) = -2(y-1)^3$

$\Rightarrow y-1/2 = -2(y-1)^3$

$\Rightarrow 2(y-1)^3 + y - 1/2 = 0$



Only one root graphically

$f(1/2) = 2(-1/8) + 1/2 - 1/2 < 0$

$f(1) = 0 + 1 - 1/2 = 1/2 > 0$

$\therefore$ The root lies b/w $1/2$ & 1 (Contradiction since $0 \geq y$)

Case III) $x \leq 0,\ y \geq 0 \longrightarrow -x+y = 1$ Constraint

We get the 3 Lagrange Equations as

$$2(x-3/2) - \lambda = 0$$

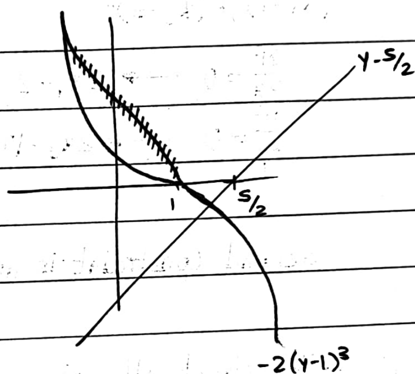
$$4(y-1)^3 + \lambda = 0$$

$$-x+y-1 = 0$$

$\Rightarrow (x-3/2) = -2(y-1)^3$

$\Rightarrow (y-1-3/2) = -2(y-1)^3$

$\Rightarrow 2(y-1)^3 + y - 5/2$



only 1 root graphically

$f(1) = 1 - 5/2 < 0$

$f(5/2) = 2 \times \frac{27}{8} + 0 > 0$

$\therefore$ The root satisfies $1 < y_* < 5/2$

$\Rightarrow 1 < x_* + 1 < 5/2$

$\Rightarrow 0 < x_* < 3/2$ # (Contradiction, we assumed $x \leq 0$)

Case IV) $x \leq 0,\ y \leq 0 \longrightarrow -x-y = 1$ Constraint

We get the 3 Lagrange Equations as

$$2(x-3/2) - \lambda = 0$$

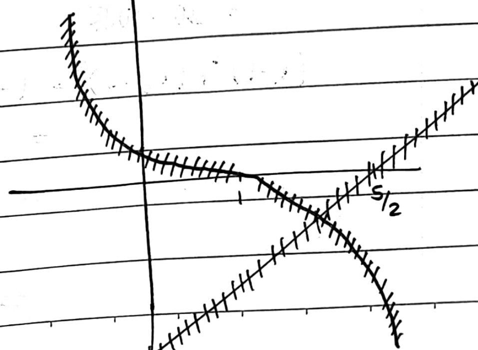
$$4(y-1)^3 - \lambda = 0$$

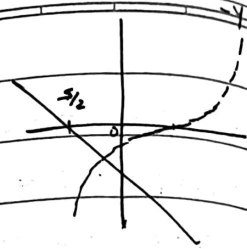
$$-x-y-1 = 0$$

$\Rightarrow (x-3/2) = 2(y-1)^3$

$\Rightarrow (-y-1-3/2) = 2(y-1)^3$

$\Rightarrow 2(y-1)^3 + y + 5/2 = 0$





$f(0) = 2(-1) + 5/2 > 0$

$f(-5/2) = 2(-7/2)^3 < 0$

$\therefore$ The root y_* satisfies $-5/2 < y_* < 0$

$\Rightarrow -5/2 < -x_* - 1 < 0$

$\Rightarrow -3/2 < -x_* < 1$

$\Rightarrow 3/2 > x_* > -1$

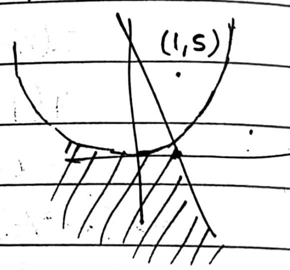
only 1 root graphically

Contradiction to assumption that $|x| + |y| = 1$.

Hence, the only active constraint is $g_1(x) = 1 - x - y$

Q8) a) $x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26$ subject to $x_2 - 5x_1^2 \le 0$, $10x_1 + x_2 \le 10$

$(x_1-1)^2 + (x_2-5)^2$



$\frac{\partial f}{\partial x} = 0 \longrightarrow 2x_1 - 2 = 0 \longrightarrow x_1 = 1$

$\frac{\partial f}{\partial y} = 0 \longrightarrow 2x_2 - 10 = 0 \longrightarrow x_2 = 5$

Second Constraint is not satisfied.

$\therefore$ We check all points on the boundary.

minima $\Rightarrow 10x_1 + 5x_1^2 = 10$

$\Rightarrow x_1^2 + 2x_1 - 2 = 0$

$\Rightarrow (x_1 + 1)^2 = 3$

$\Rightarrow x_1 = -1 \pm \sqrt{3}$

$x_1 = -1 + \sqrt{3}$

$x_2 = 10 - 10(x_1)$

$= 10 - 10(-1 + \sqrt{3})$

$= 20 - 10\sqrt{3}$

$(x_1-1)^2 + (x_2-5)^2 = (-1+\sqrt{3}-1)^2 + (20 - 10\sqrt{3} - 5)^2$

$= (2-\sqrt{3})^2 + (15 - 10\sqrt{3})^2$

$= 7 - 4\sqrt{3} + 25(3 - 2\sqrt{3})^2$

$= 7 - 4\sqrt{3} + 25(21 - 12\sqrt{3})$

$= 532 - 304\sqrt{3}$ at $(-1+\sqrt{3}, 20 - 10\sqrt{3})$

Date:

b) $x_1^2+x_2^2$ subject to $x_1 \geq 0$, $x_2 \geq 0$, $x_1+x_2 \geq 5$

$$L(x_1, x_2, \lambda) = x_1^2+x_2^2 + \lambda(5-x_1-x_2)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 5 = x_1 + x_2$$

$$\Rightarrow 2x_1 = 2x_2$$

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow x_1 = x_2 = 5/2$$

$f(5/2, 5/2) = \frac{25}{4} + \frac{25}{4} = \frac{25}{2}$ ← minima

no maxima exists as x_1 & x_2 are not upperbounded.

c) $L(x_1, x_2, \lambda_1, \lambda_2) = (x_1^2 + 6x_1x_2 - 4x_1 - 2x_2) + \lambda_1(x_1^2 + 2x_2 - 1) + \lambda_2(2x_1 - 2x_2 - 1)$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 + 6x_2 - 4 + 2\lambda_1 x_1 + 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 6x_1 - 2 + 2\lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1^2 + 2x_2 = 1$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 2x_1 - 2x_2 = 1$$

$\rightarrow$ ~~$x_1 = -1 \pm \sqrt{3}$~~

~~$x_2 = x_1 - \frac{1}{2} = -\frac{3}{2} \pm \sqrt{3}$~~

d) Convert to polar coordinates

$$r^2(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta) = 1$$

$$r^2(\cos^2\theta - \sin\theta) \leq 0 \Rightarrow \cos^2\theta \leq \sin\theta$$

$$r^2(1+\sin 2\theta) = 1$$

extremizer $\rightarrow \sin 2\theta = 1$ (−1 wont work because $r^2 \times 0 \neq 1$)

$$2\theta = \frac{\pi}{2} + 2n\pi$$

$$\theta = \frac{\pi}{4} + n\pi$$

at $\theta = \frac{\pi}{4} \longrightarrow \cos^2\theta = \frac{1}{2} \leq \frac{1}{\sqrt{2}} = \sin\theta$

at $\theta = \frac{\pi}{4} + \pi \longrightarrow \cos^2\theta = \frac{1}{2} \geq -\frac{1}{\sqrt{2}} = \sin\theta$

$\therefore$ extremizer at $(r, \theta) = (1/\sqrt{2}, \pi/4) = (1/2, 1/2)$ in cartesian

Assuming $x_1, x_2 \geq 0$, because if $x_1 \leq 0$ is permisible, we can choose x_1 very small, tending to 0 & negative & $x_2 \to 6^+$ & the overall sum would tend to $-\infty$

minimize

Q9) a) $6x_1 + \frac{96}{x_1} + \frac{4x_2}{x_1} + \frac{x_1}{x_2}$ subject to $x_1 + x_2 = 6$

$6x_1 + \frac{96}{x_1} \geq 2\sqrt{6 \cdot 96} = 48$ By AM-GM Inequality $\longrightarrow$ equality at $6x_1 = \frac{96}{x_1}$

$\Rightarrow x_1 = 4$

$\frac{4x_2}{x_1} + \frac{x_1}{x_2} \geq 2\sqrt{4 \cdot 1} = 4$ By AM-GM Inequality $\longrightarrow$ equality at $\frac{4x_2}{x_1} = \frac{x_1}{x_2}$

$\Rightarrow \frac{x_1}{x_2} = 2$

$\therefore 6x_1 + \frac{96}{x_1} + \frac{4x_2}{x_1} + \frac{x_1}{x_2} \geq 48 + 4 = 52$

$\Rightarrow x_2 = 2$

If we choose $(x_1, x_2) = (4,2)$ we see that it satisfies the constraint equation and at (x_1, x_2) we attain the global minima of the function.

ALITER: Represent x_2 in terms of x_1 using the constraint equation & it becomes one-variable optimization

Assuming $x_i \geq 0$ for the same reason as above

b) minimize $\frac{250}{x_1} + 1090x_1 - 7x_2 - 5x_3 + \frac{7x_2^2}{40x_1} + \frac{5x_3^2}{16x_1}$ subject to $x_1 + x_3 = 4$, $x_2 + x_3 = 12$

$\frac{7x_2^2}{40x_1} - 7x_2 = 7x_2\left(\frac{x_2}{40x_1} - 1\right) = 7x_2\left(\frac{x_2 - 40x_1}{40x_1}\right)$

$\frac{5x_3^2}{16x_1} - 5x_3 = 5x_3\left(\frac{x_3}{16x_1} - 1\right) = 5x_3\left(\frac{x_3 - 16x_1}{16x_1}\right)$

$4 - x_1 = 12 - x_2$

$\Rightarrow x_2 = 8 + x_1$

$\frac{250}{x_1} + 1000x_1 - 7x_2 - 5x_3 + \frac{7x_2^2}{40x_1} + 70x_1 + \frac{5x_3^2}{16x_1} + 20x_1$

$\frac{7x_2^2}{40x_1} + 70x_1 \geq 2\sqrt{\frac{7x_2^2}{40x_1} \cdot 70x_1} = 7x_2 \quad -(1)$

$\frac{5x_3^2}{16x_1} + 20x_1 \geq 2\sqrt{\frac{5x_3^2}{16x_1} \cdot 20x_1} = 5x_3 \quad -(2)$

$\Rightarrow \left(\frac{250}{x_1} + 1090x_1 - 7x_2 - 5x_3 + \frac{7x_2^2}{40x_1} + \frac{5x_3^2}{16x_1}\right) \geq \left(\frac{250}{x_1} + 1000x_1\right)$

AM-GM (see 1 & 2)

Now $\frac{250}{x_1} + 1000x_1 \geq 2\sqrt{\frac{250}{x_1} \cdot 1000x_1}$

$= 2 \cdot 500$

$= 1000$

Equality at $\frac{250}{x_1} = 1000x_1$

$\Rightarrow x_1 = 1/2$

$\frac{7x_2^2}{40x_1} = 70x_1$

$\Rightarrow x_2^2 = 400x_1^2$

$\Rightarrow x_2 = 20x_1$

$\Rightarrow x_2 = 20(1/2) = 10$

$x_3 = 16x_1$

$\frac{4}{17}$ $\frac{64}{17} = x_3$

$\frac{140}{17} = 3 \quad 33 = 71s.$

$\frac{5x_3^2}{16x_1} = \cancel{\ } 20x_1$

$\frac{4}{13}$ $\frac{48}{13}$ $\frac{108}{13} = 27$

$\Rightarrow x_3^2 = 64x_1^2$

$\Rightarrow x_3 = 8x_1$

$\Rightarrow x_3 = 8(1/2) = 4.$

$\frac{7x_2^2}{40x_1} = K$

$\sqrt{40K} = 35$

$252 - \frac{80}{21} = \frac{172}{21} = x_2$

$\frac{5x_3^2}{16x_1} + 240x_1 \geq 2\sqrt{\frac{5x_3^2}{16x_1} \cdot 240x_1}$

$x_1 + x_3 = 4$

$125 = K$

$\frac{5x_3^2}{16x_1} = Kx_1$

$\Rightarrow 5x_3^2 = 16Kx_1^2$

$\Rightarrow x_3 = 4\sqrt{\frac{K}{5}} \cdot x_1$

$x_3 = 20x_1$

$4 \times 3x_1$

$172/4 = 43$ $K = 45$

$K = 80$ $\frac{4}{21} = x_1$

$x_3 = \frac{80}{21}$

$\frac{7x_2^2}{40x_1} + \cancel{\ } x_1 \geq 2\sqrt{\frac{7x_2^2}{40x_1} \cdot 200}$

$\frac{5x_3^2}{16x_1} + 80x_1 \geq 2\sqrt{\frac{5x_3^2}{16x_1} \cdot 80x_1} = 10x_3$

Q10) a) $L(x, y, z, \lambda) = 6x + 3y + 2z + \lambda(4x^2 + 2y^2 + z^2 - 70)$

$\frac{\partial L}{\partial x} = 0 \Rightarrow 6 + 8x\lambda = 0 \Rightarrow x = -\frac{3}{4\lambda}$

$\frac{\partial L}{\partial y} = 0 \Rightarrow 3 + 4y\lambda = 0 \Rightarrow y = -\frac{3}{4\lambda}$

$\frac{\partial L}{\partial z} = 0 \Rightarrow 2 + 2z\lambda = 0 \Rightarrow z = -\frac{1}{\lambda}$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 4x^2 + 2y^2 + z^2 = 70$

$4\left(\frac{3}{4\lambda}\right)^2 + 2\left(\frac{3}{4\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 70$

$\Rightarrow \frac{9}{4\lambda^2} + \frac{9}{8\lambda^2} + \frac{1}{\lambda^2} = 70$

$\Rightarrow 18 + 9 + 8 = 70(8\lambda^2)$

$\Rightarrow 35 = 70(8\lambda^2)$

$\Rightarrow \lambda^2 = \frac{1}{16}$

$\Rightarrow \lambda = \pm\frac{1}{4}$

$\lambda = -1/4 \longrightarrow (x, y, z) = (3, 3, 4) \xrightarrow{f} 36$ (Maxima)

$\lambda = +1/4 \longrightarrow (x, y, z) = (-3, -3, -4) \xrightarrow{f} -36$ (Minima)

$\therefore$ Maxima is at $(x, y, z) = (3, 3, 4)$ & value of objective function is 36

Minima is at $(x, y, z) = (-3, -3, -4)$ & value of objective function is -36

Assuming $x \geq 0$ & $y \geq 0$, since otherwise minima tends to 0 (take $x \to -\infty$ & $y \to \infty$ with $x^3 + y^3 = 16$)

b) $L(x, y, \lambda) = e^{xy} + \lambda(x^3 + y^3 - 16)$

$\frac{\partial L}{\partial x} = 0 \Rightarrow ye^{xy} + 3\lambda x^2 = 0 \Rightarrow ye^{xy} = -3\lambda x^2 \quad -(1)$

$\frac{\partial L}{\partial y} = 0 \Rightarrow xe^{xy} + 3\lambda y^2 = 0 \Rightarrow xe^{xy} = -3\lambda y^2 \quad -(2)$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^3 + y^3 = 16$

Dividing (1) by (2), we get $\frac{y}{x} = \frac{x^2}{y^2}$ (Assuming $x \neq 0$ & $y \neq 0$)

$\Rightarrow x^3 = y^3$

$\Rightarrow x^3 = y^3 = 8$

$\Rightarrow x = y = 2$

$f(2,2) = e^{2 \cdot 2} = e^4 \leftarrow$ maxima

$F(0, \sqrt[3]{16}) = e^0$

$F(\sqrt[3]{16}, 0) = e^0$

minima becaus $xy \geq 0$ always