EE659 - Assignment 1

Q1) $W_1 = \text{Span}\{(1,1,0,-1), (0,1,3,1)\}$

$W_2 = \text{Span}\{(0,-1,-2,1), (1,2,2,-2)\}$

Notice that $1\cdot(0,-1,-2,1) + 1\cdot(1,2,2,-2) = (1,1,0,-1)$ - (1)

Suppose $\exists\, c, d \in \mathbb{R}$ such that $c\cdot(0,-1,-2,1) + d(1,2,2,-2) = (0,1,3,1)$

$\rightarrow$ By 1^{st} Coordinate $\rightarrow d = 0$

$\Rightarrow$ By 2^{nd} Coordinate $\rightarrow c = -1$

But by 3^{rd} Coordinate $\rightarrow$ Left Hand Side of the equation $= 2$

$\#$

Right Hand Side of the equation $= 3$

$\therefore (0,1,3,1) \notin \text{Span}\{(0,-1,-2,1), (1,2,2,-2)\}$ - (2)

$\Rightarrow (0,1,3,1), (0,-1,-2,1)$ & $(1,2,2,-2)$ are linearly independent & any vector in W_1+W_2 can be written using them as basis - (By ① & ②)

$$\boxed{\begin{array}{l}\therefore \text{Basis for } W_1+W_2 = \{(0,1,3,1), (0,-1,-2,1), (1,2,2,-2)\} \\ \text{Dimension of } W_1+W_2 = 3\end{array}}$$

Now if a vector $v \in W_1 \cap W_2$, then v can be written as $a\cdot(1,1,0,-1) + b(0,1,3,1) = c(0,-1,-2,1) + d(1,2,2,-2)$ for some $a,b,c,d \in \mathbb{R}$.

$$\Rightarrow b(0,1,3,1) = c(0,-1,-2,1) + d(1,2,2,-2) - a\left((0,-1,-2,1) + (1,2,2,-2)\right)$$

$$= (c-a)(0,-1,-2,1) + (d-a)(1,2,2,-2)$$

But since the 3 vectors are linearly independent,

$b = c-a = d-a = 0$

$\Rightarrow b = 0$ & $c = d = a$

$\rightarrow v = a(1,1,0,-1) \quad \forall\, a \in \mathbb{R}$

$$\boxed{\begin{array}{l}\therefore \text{Basis for } W_1 \cap W_2 = \{(1,1,0,-1)\} \\ \text{Dimension of } W_1 \cap W_2 = 1\end{array}}$$

Q2) Let $S = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 \mid x_1 = 3x_2,\ x_3 = x_4 = x_5\} = \text{Null}(T)$

Let T be the linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$

Now for some $x \in S$, $x = (3a, a, b, b, b)$ where $x_2 = a$ & $x_3 = b$

$= a(3,1,0,0,0) + b(0,0,1,1,1)$

Linearly Independent.

$\Rightarrow \dim(S) = 2$

But by the Fundamental Theorem for linear Maps, we have

$\dim(\mathbb{F}^5) = \dim(\text{Null}(T)) + \dim(\text{Im}(T))$

$\Rightarrow 5 = 2 + \dim(\text{Im}(T))$

$\Rightarrow \dim(\text{Im}(T)) = 3$

But since $T: \mathbb{F}^5 \to \mathbb{F}^2$, $\dim(\text{Im}(T)) \leq 2$. #.

$\therefore$ There cannot exist such a linear map with the mentioned properties.

Q3) a) True

Consider $x \in \ker(NM) \to NM(x) = 0 \Rightarrow M(x) \in \ker(N)$

Let $b = \{b_1, b_2, \dots, b_r\}$ be a basis for M.

We see that $\ker(M) \subseteq \ker(NM)$ so we can extend b to a basis for $\ker(NM)$.

Let $\{b_1, b_2, \dots, b_r, \dots b_n\}$ be a basis for $NM \Rightarrow M(b_i) \neq 0$ for $r+1 \leq i \leq n$.

Now we show that $\{N(b_{r+1}), N(b_{r+2}), \dots, N(b_n)\}$ is linearly independent.

FTSOC, assume otherwise $\Rightarrow \sum_{i=r+1}^{n} \gamma_i N(b_i) = 0$. Since N is linear, it is equal to $\sum_{i=r+1}^{n} N(\gamma_i b_i) = 0 \Rightarrow \sum_{i=r+1}^{n} \gamma_i b_i \in \ker(M)$. But since $\{b_1, b_2, \dots, b_n\}$ is an independent set, $\gamma_i = 0\ \forall i \in \{r+1, \dots n\}$. This implies $\dim(\ker(N)) \geq n-r$

$\Rightarrow \underbrace{\dim(\ker(M))}_{=r} + \underbrace{\dim(\ker(N))}_{\geq n-r} \geq \underbrace{}_{n=}\dim(\ker(NM))$. Hence Proved!

b) $U = \mathbb{R}^4$, $V = \mathbb{R}^3$, $W = \mathbb{R}^2$

Basis u_i Basis v_i Basis w_i

$M(u_1) = M(u_2) = v_1$, $M(u_3) = M(u_4) = v_2$

$N(v_1) = N(v_2) = w_1$, $N(v_3) = w_2$

$\dim(\text{Im}(NM)) = 1$

$\dim(\text{Im}(N)) = 2$

$\dim(\text{Im}(M)) = 2$

Q3) c) $Tv = \lambda v$

3.1

$T^{-1}Tv = T^{-1}(\lambda v)$

$\Rightarrow v = \lambda(T^{-1}v)$

$\Rightarrow T^{-1}v = \frac{1}{\lambda}v$

d) Let $v_1, v_2, \dots, v_p$ be a basis of $Null(T)$, extend it to a basis of V $v_1, v_2, \dots, v_p, u_1, u_2, \dots, u_q$. Thus $\dim(Null(T)) = p$ & $\dim(V) = p+q$.

Denote $U = Span\{u_1, u_2, \dots, u_q\}$. Clearly $Span\{v_1, v_2, \dots, v_p\} \oplus Span\{u_1, \dots u_q\} = V = Null(T) \oplus U$. The restriction $T|_U$ is injective, because $U \cap Null(T) = \emptyset$

Hence $Tu_1, Tu_2, \dots Tu_p$ is a basis of $Range(T|_U)$.

Notice $Range(T) = T(V) = T(Null(T) \oplus U) = 0 + T(U) = Range(T|_U)$

$\Rightarrow \dim(Range(T)) = \dim(Range(T|_U)) = q$.

Hence, $\dim(V) = p+q = \dim(Null(T)) + \dim(Range(T))$

Q4) To prove V is a subspace $\longrightarrow$ ① $0 \in V$

② $\alpha v_1 + \underset{(1-\alpha)}{\cancel{\beta}} v_2 \in V \quad \forall v_1, v_2 \in V$ & $\alpha \in \mathbb{R}$

Since $x_0 \in C \rightarrow x_0 - x_0 = 0 \in V$

$v_1 \in V \rightarrow v_1 + x_0 \in C$

$v_2 \in V \rightarrow v_2 + x_0 \in C$

$\Rightarrow \alpha(v_1 + x_0) + (1-\alpha)(v_2 + x_0) \in C$

$\Rightarrow \alpha v_1 + (1-\alpha)v_2 + x_0 \in C$

$\Rightarrow \alpha v_1 + (1-\alpha)v_2 \in C$

Hence Proved!

Q5) a) Let x_1, λ_1 be such that $Ax_1 = \lambda_1 x_1$

$\boxed{\text{Let } x_2, \lambda_2 \ (\lambda_1 \neq \lambda_2) \text{ be such that } Ax_2 = \lambda_2}$

$A(\alpha_1 x_1 + \alpha_2 x_2) = \underline{\alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_2 x_2} \ (\neq \lambda'(\alpha_1 x_1 + \alpha_2 x_2))$

Not a cone.

b) $(\alpha_1 x_1^T + \alpha_2 x_2^T) A (\alpha_1 x_1 + \alpha_2 x_2)$

3.2

$= \alpha_1^2 x_1^T A x_1 + \alpha_2^2 x_2^T A x_2 + \dots < 0$

May be more negative

Not a cone

c) $x^T(\alpha_1 A_1 + \alpha_2 A_2)x = x^T A_1 x \cdot \alpha_1 + x^T A_2 x \cdot \alpha_2 \leq 0$

Cone

d) $(\alpha A^T)(\alpha A) = \alpha^2 (A^T A) = \alpha^2 I \neq I$ Not a cone

Q6) $(A, B, C, D, E) = (x_1, x_2, x_3, x_4, x_5)$

minimize $- 20x_1 + 5x_2 + 5x_3 + 2x_4 + 7x_5$

subject to $\longrightarrow 0.4x_1 + 1.2x_2 + 0.6x_3 + 0.6x_4 + 12.2x_5 \geq 70$

$\longrightarrow 6x_1 + 10x_2 + 3x_3 + x_4 + 0x_5 \geq 50$

$\longrightarrow 0.4x_1 + 0.6x_2 + 0.4x_3 + 0.2x_2 + 2.6x_5 \geq 12$

$\longrightarrow x_i \geq 0 \;\; \forall i \in [5].$

Adding Slack Variables $\longrightarrow$ Matrix looks like

$$\left[\begin{array}{cccccccc|c} -0.4 & -1.2 & -0.6 & -0.6 & -12.2 & 1 & 0 & 0 & -70 \\ -6 & -10 & -3 & -1 & 0 & 0 & 1 & 0 & -50 \\ -0.4 & -0.6 & -0.4 & -0.2 & -2.6 & 0 & 0 & 1 & -12 \end{array}\right]$$

$-20 \quad -5 \quad -5 \quad -2 \quad -7$

4.1

$\longrightarrow \underline{\frac{50}{6}}, \; \frac{70}{0.4}, \; \frac{12}{0.4}$

Solving it (I solved in rough & don't have time to complete it),

we get $\boxed{(x_1, x_2, x_3, x_4, x_5) = \left(0, 5, 0, 0, \frac{320}{61}\right)}$

$\boxed{\& \text{ min. cost } = \frac{3765}{61}}$ ★

Q7) The convex polygon can be modelled as a linear programming problem with n conditions,

$$y \geq a_i x + b_i \quad \forall\, i \in [1,t]$$
$$y \leq a_i x + b_i \quad \forall\, i \in [t+1,n] \qquad i \in \mathbb{N}$$

So, to find the max. radius, we just need to solve the following problem

$$\max\ r \qquad (\overset{(x,y)}{} \in P)$$
$$\text{subject to} \quad r \leq \frac{|y - a_i x - b_i|}{\sqrt{a_i^2 + b_i^2}} \quad \forall\, i \in [n]$$

Q8) The convex hull $\operatorname{conv}\{x_1, x_2, \ldots, x_k\}$ is the smallest convex set containing the points $\{x_1, x_2, \ldots, x_k\}$. Since each x_i is on the boundary of C, and C is convex, any combination of $x_1, x_2, \ldots, x_k$ must also lie in C.

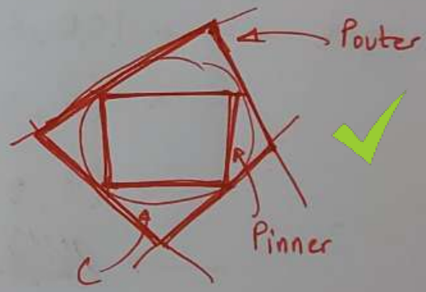
$$\Longrightarrow P_1 \subseteq C.$$

Now $P_{outer} = \{x \mid a_i^T(x - x_i) \leq 0,\ i = 1, 2, \ldots, k\}$

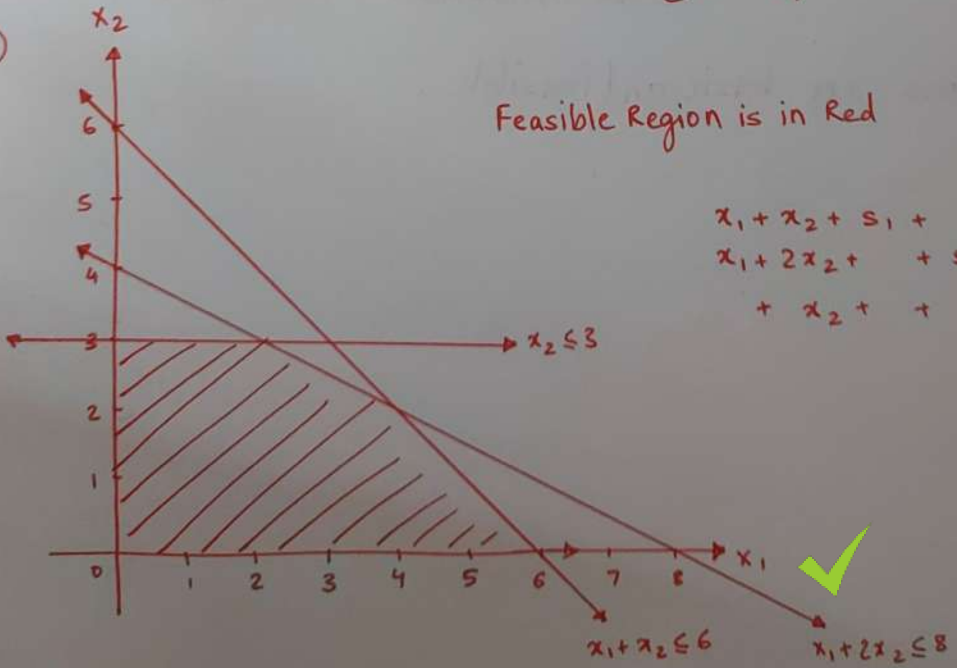
$$= \bigcap_{i=1}^{k} \{x \mid a_i^T(x - x_i) \leq 0\} \supseteq C$$

$$\Longrightarrow C \subseteq P_{outer}$$

Hence Proved.



Q9)



Feasible Region is in Red

$$x_1 + x_2 + s_1 + \quad + \quad = 6$$
$$x_1 + 2x_2 + \quad + s_2 + \quad = 8$$
$$+ x_2 + \quad + \quad + s_3 = 3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 6 \\ 1 & 2 & 0 & 1 & 0 & 8 \\ 0 & 1 & 0 & 0 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 6 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -1 & 5 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \longleftarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 6 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

solution $\longrightarrow (2, 3, 1, 0, 0)$

similarly, we get $\longrightarrow (0, 0, 6, 8, 3)$

$\longrightarrow (4, 2, 0, 0, 1)$

$\longrightarrow (0, 3, 3, 2, 0)$

$\longrightarrow (6, 0, 0, 2, 1)$

All these 5 solutions are basic and feasible. ✓

6.1

Q10) ($\Rightarrow$) If P has extreme point x^*

At x^*, some subset of the inequality constraints will become equalities

$\rightarrow$ let the number be $k \Rightarrow k \geq n$

These constraints can be written as $\underbrace{A}_{k \times n \text{ matrix}} x^* = b$

Since x^* is an extreme point, it must be feasible & should lie on the intersection of the hyperplanes defined by the k equality constraints. Therefore, A must have full rank $\Rightarrow$ Vectors out of the k equalities should span the n-dimensional space.

$\Rightarrow \exists$ a subset of n linearly independent vectors.

($\Leftarrow$) Assume that among the vectors $\{a_j\}$, $\exists$ subset of n-LI vectors.

Consider matrix A formed by those n-vectors.

We find point P as solution of $Ax = b$

$\uparrow$ RHS of the constraint equations

7.1

Since A is of full rank, the system has a unique solution (P) which is the extreme point.

Index of comments

3.1 Q3(c,d) those statement are true or false.

3.2 The justification for cone and convex cone have been used wronly

4.1 Q6. The formulation of LP is correct. However, the intermediate steps of solving the LP are required.

6.1 Q9. there are a total of 9 basic solutions. And 5 basic feasible solutions. You found basic feasible solutions only.

7.1 Q10. try to prove using mathematical arguments rather than giving more statements. Also, the arguments are not enough to prove the claim.