MA403: Assignment 2

Nirav Bhattad (23B3307)

Question: 6.1

Determine which of the following functions are of bounded variation on $[0, 1]$

(a) $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ if $x \neq 0$, $f(0) = 0$

(b) $f(x) = \sqrt{x} \sin\left(\frac{1}{x}\right)$ if $x \neq 0$, $f(0) = 0$

(a). We calculate the derivative of f, which comes out to be $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ for all $x \in (0, 1)$. Now note that

$$\left|f'(x)\right| = \left|2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right| \leqslant \left|2x \sin\left(\frac{1}{x}\right)\right| + \left|\cos\left(\frac{1}{x}\right)\right| \leqslant 2x + 1 < 3$$

for all $x \in (0, 1)$. Hence, f' is bounded on $(0, 1)$ and hence f is of bounded variation on $[0, 1]$. ■

(b). We choose a partition given by

$$P = \left\{0, \frac{1}{(n+1)\frac{\pi}{2}}, \frac{1}{n\frac{\pi}{2}}, \dots, \frac{1}{\frac{\pi}{2}}, 1\right\}$$

for some odd $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
\sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| &= \sum_{i=1}^{n} \left| \sqrt{x_i} \sin\left(\frac{1}{x_i}\right) - \sqrt{x_{i-1}} \sin\left(\frac{1}{x_{i-1}}\right) \right| \\
&= 2\sqrt{\frac{2}{\pi}} \sum_{\substack{i=1 \\ i \text{ odd}}}^{n+1} \sqrt{\frac{1}{i}} \\
&> \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n+1} \sqrt{\frac{1}{i}}
\end{aligned}$$

Since the summation $\sum_{i=1}^{n+1} \sqrt{\frac{1}{i}}$ diverges as $n \to \infty$, we conclude that f is not of bounded variation on $[0, 1]$. ■

Question: 6.2 (a)

A function f, defined on $[a, b]$, is said to satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a, b]$ if there exists a constant $M > 0$ such that $|f(x) - f(y)| < M|x - y|^\alpha$ for all $x, y \in [a, b]$. If f is such a function, show that $\alpha > 1$ implies f is constant on $[a, b]$, whereas $\alpha = 1$ implies that f is of bounded variation on $[a, b]$.

Proof. **Case 1:** $\alpha > 1$

Let $y > x$ be two points in the interval $[a, b]$. For every $n \geqslant 1$, divide the interval (x, y) into n subintervals (x_i, x_{i+1}) where $x_i = x + i\dfrac{y-x}{n}$ for $i = 0, 1, \ldots, n$ such that $x_0 = x$ and $x_n = y$. The length of each subinterval is given by:

$$x_{i+1} - x_i = \frac{y-x}{n}.$$

By the hypothesis, for every i, we have:

$$|f(x_i) - f(x_{i+1})| < M(x_{i+1} - x_i)^\alpha = M\left(\frac{y-x}{n}\right)^\alpha.$$

Using the triangle inequality, we can estimate the total change in f from x to y:

$$|f(x) - f(y)| \leqslant \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \sum_{i=1}^{n} M\left(\frac{y-x}{n}\right)^\alpha = nM\left(\frac{y-x}{n}\right)^\alpha.$$

This simplifies to:

$$|f(x) - f(y)| < M(y-x)^\alpha n^{1-\alpha}.$$

Since we have $\alpha > 1$, it follows that $1 - \alpha < 0$. Therefore, as $n \to \infty$, $n^{1-\alpha} \to 0$. Consequently, we get:

$$|f(x) - f(y)| < M(y-x)^\alpha n^{1-\alpha} \to 0 \quad \text{as } n \to \infty.$$

Thus, we conclude:

$$f(x) = f(y).$$

Since $y > x$ were arbitrary, we can conclude that f is constant on $[a, b]$.

Case 2: $\alpha = 1$

Consider any partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$. For every $i \in \{1, 2, \ldots, n\}$, we have:

$$|f(x_i) - f(x_{i-1})| < M|x_i - x_{i-1}|$$

since $\alpha = 1$. So summing up over all i, we get:

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < M\sum_{i=1}^{n} |x_i - x_{i-1}| = M\,(b-a)$$

This means that f is of bounded variation on $[a, b]$.

■

Question: 6.11

Prove that every absolutely continuous function on $[a, b]$ is continuous and of bounded variation on $[a, b]$.

Definition: Absolutely Continuous Function

A function f defined on $[a, b]$ is said to be absolutely continuous on $[a, b]$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite disjoint collection of open intervals $\{(a_i, b_i)\}$ in $[a, b]$ with $\sum_i (b_i - a_i) < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \varepsilon$.

Proof. Let f be an absolutely continuous function on $[a, b]$. By definition, if we choose only 1 interval (a, b) in the collection, then we have $|f(b) - f(a)| < \varepsilon$ for $|b - a| < \delta$. This implies that f is uniformly continuous, and hence continuous on $[a, b]$ since for every $\varepsilon > 0$, we can choose $\delta > 0$ such that $|b - a| < \delta \implies |f(b) - f(a)| < \varepsilon$.

Left $\varepsilon = 1$ and $P' = \{a = x_0, \ldots, x_n = b\}$ be a partition of $[a, b]$ with the property that $x_i - x_{i-1} = \frac{\delta}{2}$ for all $i \in \{1, 2, \ldots, n-1\}$, and $x_n - x_{n-1} \leqslant \frac{\delta}{2}$ where δ is as in the definition of absolute continuity.

Now pick any partition $P = \{t_0, \ldots, t_s\}$ and let $P^\star = P \bigcup P'$ such that $P^\star = \{z_0 = a, \ldots, z_n = b\}$. Let $P_i^\star$ be the set of points in $P^\star$ contained in $[x_{i-1}, x_i]$ for $i \in \{1, \ldots, n\}$, i.e.,

$$P_i^\star = \left\{z_{i_k} \in P^\star : z_{i_k} \in [x_{i-1}, x_i]\right\}$$

Thus, the sum can be estimated as follows:

$$\sum_{i=1}^{s} |f(t_i) - f(t_{i-1})| \leqslant \sum_{i=1}^{n} \sum_{k} |f(z_{i_k}) - f(z_{i_{k-1}})| \leqslant \sum_{i=1}^{n} \varepsilon = \sum_{i=1}^{n} 1 = n$$

Hence, f is of bounded variation on $[a, b]$. ■