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# MA403: ASSIGNMENT 2

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**Question: 6.1**

Determine which of the following functions are of bounded variation on  $[0, 1]$

(a)  $f(x) = x^2 \sin\left(\frac{1}{x}\right)$  if  $x \neq 0$ ,  $f(0) = 0$

(b)  $f(x) = \sqrt{x} \sin\left(\frac{1}{x}\right)$  if  $x \neq 0$ ,  $f(0) = 0$

(a). We calculate the derivative of  $f$ , which comes out to be  $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$  for all  $x \in (0, 1)$ . Now note that

$$|f'(x)| = \left| 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right| \leq \left| 2x \sin\left(\frac{1}{x}\right) \right| + \left| \cos\left(\frac{1}{x}\right) \right| \leq 2x + 1 < 3$$

for all  $x \in (0, 1)$ . Hence,  $f'$  is bounded on  $(0, 1)$  and hence  $f$  is of bounded variation on  $[0, 1]$ . ■

(b). We choose a partition given by

$$P = \left\{ 0, \frac{1}{(n+1)\frac{\pi}{2}}, \frac{1}{n\frac{\pi}{2}}, \dots, \frac{1}{\frac{\pi}{2}}, 1 \right\}$$

for some odd  $n \in \mathbb{N}$ . Then we have

$$\begin{aligned} \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| &= \sum_{i=1}^n \left| \sqrt{x_i} \sin\left(\frac{1}{x_i}\right) - \sqrt{x_{i-1}} \sin\left(\frac{1}{x_{i-1}}\right) \right| \\ &= 2\sqrt{\frac{2}{\pi}} \sum_{\substack{i=1 \\ i \text{ odd}}}^{n+1} \sqrt{\frac{1}{i}} \\ &> \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n+1} \sqrt{\frac{1}{i}} \end{aligned}$$

Since the summation  $\sum_{i=1}^{n+1} \sqrt{\frac{1}{i}}$  diverges as  $n \rightarrow \infty$ , we conclude that  $f$  is not of bounded variation on  $[0, 1]$ . ■

**Question: 6.2 (a)**

A function  $f$ , defined on  $[a, b]$ , is said to satisfy a uniform Lipschitz condition of order  $\alpha > 0$  on  $[a, b]$  if there exists a constant  $M > 0$  such that  $|f(x) - f(y)| < M|x - y|^\alpha$  for all  $x, y \in [a, b]$ . If  $f$  is such a function, show that  $\alpha > 1$  implies  $f$  is constant on  $[a, b]$ , whereas  $\alpha = 1$  implies that  $f$  is of bounded variation on  $[a, b]$ .

*Proof.* **Case 1:**  $\alpha > 1$

Let  $y > x$  be two points in the interval  $[a, b]$ . For every  $n \geq 1$ , divide the interval  $(x, y)$  into  $n$  subintervals  $(x_i, x_{i+1})$  where  $x_i = x + i \frac{y-x}{n}$  for  $i = 0, 1, \dots, n$  such that  $x_0 = x$  and  $x_n = y$ . The length of each subinterval is given by:

$$x_{i+1} - x_i = \frac{y-x}{n}.$$

By the hypothesis, for every  $i$ , we have:

$$|f(x_i) - f(x_{i+1})| < M(x_{i+1} - x_i)^\alpha = M \left( \frac{y-x}{n} \right)^\alpha.$$

Using the triangle inequality, we can estimate the total change in  $f$  from  $x$  to  $y$ :

$$|f(x) - f(y)| \leq \sum_{i=1}^n |f(x_i) - f(x_{i-1})| < \sum_{i=1}^n M \left( \frac{y-x}{n} \right)^\alpha = nM \left( \frac{y-x}{n} \right)^\alpha.$$

This simplifies to:

$$|f(x) - f(y)| < M(y-x)^\alpha n^{1-\alpha}.$$

Since we have  $\alpha > 1$ , it follows that  $1 - \alpha < 0$ . Therefore, as  $n \rightarrow \infty$ ,  $n^{1-\alpha} \rightarrow 0$ . Consequently, we get:

$$|f(x) - f(y)| < M(y-x)^\alpha n^{1-\alpha} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, we conclude:

$$f(x) = f(y).$$

Since  $y > x$  were arbitrary, we can conclude that  $f$  is constant on  $[a, b]$ .

**Case 2:**  $\alpha = 1$ 

Consider any partition  $P = \{a = x_0, x_1, \dots, x_n = b\}$  of  $[a, b]$ . For every  $i \in \{1, 2, \dots, n\}$ , we have:

$$|f(x_i) - f(x_{i-1})| < M|x_i - x_{i-1}|$$

since  $\alpha = 1$ . So summing up over all  $i$ , we get:

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| < M \sum_{i=1}^n |x_i - x_{i-1}| = M(b-a)$$

This means that  $f$  is of bounded variation on  $[a, b]$ . ■

**Question: 6.11**

Prove that every absolutely continuous function on  $[a, b]$  is continuous and of bounded variation on  $[a, b]$ .

**Definition: Absolutely Continuous Function**

A function  $f$  defined on  $[a, b]$  is said to be absolutely continuous on  $[a, b]$  if for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for every finite disjoint collection of open intervals  $\{(a_i, b_i)\}$  in  $[a, b]$  with  $\sum_i (b_i - a_i) < \delta$ , we have  $\sum_i |f(b_i) - f(a_i)| < \varepsilon$ .

*Proof.* Let  $f$  be an absolutely continuous function on  $[a, b]$ . By definition, if we choose only 1 interval  $(a, b)$  in the collection, then we have  $|f(b) - f(a)| < \varepsilon$  for  $|b - a| < \delta$ . This implies that  $f$  is uniformly continuous, and hence continuous on  $[a, b]$  since for every  $\varepsilon > 0$ , we can choose  $\delta > 0$  such that  $|b - a| < \delta \implies |f(b) - f(a)| < \varepsilon$ .

Let  $\varepsilon = 1$  and  $P' = \{a = x_0, \dots, x_n = b\}$  be a partition of  $[a, b]$  with the property that  $x_i - x_{i-1} = \frac{\delta}{2}$  for all  $i \in \{1, 2, \dots, n-1\}$ , and  $x_n - x_{n-1} \leq \frac{\delta}{2}$  where  $\delta$  is as in the definition of absolute continuity. Then, we have:

$$n = \left\lfloor \frac{2(b-a)}{\delta} \right\rfloor + 1.$$

Now pick any partition  $P = \{t_0, \dots, t_s\}$  and let  $P^* = P \cup P'$  such that  $P^* = \{z_0 = a, \dots, z_n = b\}$ . Let  $P_i^*$  be the set of points in  $P^*$  contained in  $[x_{i-1}, x_i]$  for  $i \in \{1, \dots, n\}$ , i.e.,

$$P_i^* = \{z_{i_k} \in P^* : z_{i_k} \in [x_{i-1}, x_i]\}$$

Thus, the sum can be estimated as follows:

$$\sum_{i=1}^s |f(t_i) - f(t_{i-1})| \leq \sum_{i=1}^n \sum_k |f(z_{i_k}) - f(z_{i_{k-1}})| \leq \sum_{i=1}^n \varepsilon = \sum_{i=1}^n 1 = n$$

Hence,  $f$  is of bounded variation on  $[a, b]$ . ■