MA403: Assignment 1

Nirav Bhattad (23B3307)

Question

Test for convergence (p and q denote fixed real numbers):

(a) $\displaystyle\sum_{n=1}^{\infty} n^3 e^{-n}$

(b) $\displaystyle\sum_{n=1}^{\infty} p^n n^p \quad (p > 0)$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p - n^q} \quad (0 < q < p)$

(d) $\displaystyle\sum_{n=1}^{\infty} n^{-1-\frac{1}{n}}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{p^n - q^n} \quad (0 < q < p)$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \log(1 + \frac{1}{n})}$

(a). We can use the ratio test to test for convergence of the series $\displaystyle\sum_{n=1}^{\infty} n^3 e^{-n}$.

$$\begin{aligned}
\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \limsup_{n\to\infty} \left|\frac{(n+1)^3 e^{-(n+1)}}{n^3 e^{-n}}\right| \\
&= \limsup_{n\to\infty} \left|\frac{(n+1)^3}{n^3} \cdot e^{-1}\right| \\
&= \limsup_{n\to\infty} \left|\frac{n^3 + 3n^2 + 3n + 1}{n^3} \cdot e^{-1}\right| \\
&= \limsup_{n\to\infty} \left|\left(1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}\right) \cdot e^{-1}\right| \\
&= e^{-1} \\
&< 1
\end{aligned}$$

Since the limit is less than 1, the series $\displaystyle\sum_{n=1}^{\infty} n^3 e^{-n}$ converges. ∎

(b). We can use the ratio test to test for convergence of the series $\sum_{n=1}^{\infty} p^n n^p$.

$$\begin{aligned}
\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \limsup_{n\to\infty}\left|\frac{p^{n+1}(n+1)^p}{p^n n^p}\right| \\
&= \limsup_{n\to\infty}\left|\frac{p(n+1)^p}{n^p}\right| \\
&= \limsup_{n\to\infty}\left|p\left(1+\frac{1}{n}\right)^p\right| \\
&= \limsup_{n\to\infty} p \quad \left(\text{since } n\to\infty\text{, and } p>0\text{, so } \left(1+\frac{1}{n}\right)^p \to 1\right) \\
&= p
\end{aligned}$$

If $p < 1$, then the sequence is absolutely convergent by the Ratio Test and hence the series $\sum_{n=1}^{\infty} p^n n^p$ converges. If $p > 1$, then the series is divergent by the Ratio Test. If $p = 1$, then the series is given by $\sum_{n=1}^{\infty} n$, which is divergent.

Hence, the series $\sum_{n=1}^{\infty} p^n n^p$ converges if $0 < p < 1$, and diverges otherwise.

■

Remark

$\left(1+\frac{1}{n}\right)^p \to 1$ as $n \to \infty$ because $p > 0$. This can be seen as follows:

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^p &= \lim_{n\to\infty}\left(1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\cdots+\frac{p(p-1)\cdots(p-n+1)}{n^n}+\cdots\right) \\
&= 1+0+0+\cdots+0+\cdots \\
&= 1
\end{aligned}$$

Each successive term in the summation is of the form $\frac{p(p-1)\cdots(p-k+1)}{n^k}$, which tends to 0 as $n\to\infty$, as it is smaller than the previous term $\frac{p(p-1)\cdots(p-k+2)}{n^{k-1}}$. This can be seen as follows:

$$\begin{aligned}
\frac{p(p-1)\cdots(p-k+1)}{n^k} &= \frac{p}{n}\cdot\frac{p-1}{n}\cdots\frac{p-k+1}{n} \\
&< \frac{p}{n}\cdot\frac{p-1}{n}\cdots\frac{p-k+2}{n} \\
&= \frac{p(p-1)\cdots(p-k+2)}{n^{k-1}}
\end{aligned}$$

Since the limit of the 1st term is 0, the limit of the each successive term is also 0 and hence the limit of the entire summation is 1.

(c). We can use the limit comparison test to test for convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p - n^q}$.

Let $a_n = \dfrac{1}{n^p - n^q}$ and $b_n = \dfrac{1}{n^p}$. Then, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^p}{n^p - n^q} \\ &= \lim_{n\to\infty} \frac{1}{1 - n^{q-p}} \\ &= 1 \end{aligned}$$

where the last limit is equal to 1 because $p > q > 0$, and so $n^{q-p} \to 0$ as $n \to \infty$.

This implies that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p - n^q}$ converges if and only if $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p}$ converges. Since the latter converges, whenever $p > 1$, the former also converges. Whenever $0 < p \leqslant 1$, the latter diverges, and so does the former. ■

Remark: p-series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is known as the p-series. It converges if $p > 1$ and diverges if $0 < p \leqslant 1$.

Proof. We can use the Cauchy condensation test to prove the convergence/divergence of the p-series. Let $a_n = \dfrac{1}{n^p}$. Then, we have

$$\begin{aligned} 2^n a_{2^n} &= 2^n \cdot \frac{1}{(2^n)^p} \\ &= 2^{n(1-p)} \end{aligned}$$

If $p = 1$, then $2^{n(1-p)} = 1$, which is a constant. So, the condensed series becomes $\displaystyle\sum_{n=1}^{\infty} 1$, which diverges. If $p > 1$, then $2^{n(1-p)} \to 0$ as $n \to \infty$, and so the condensed series becomes a geometric series, with sum given by $\dfrac{1}{1 - 2^{1-p}}$, which converges. If $0 < p < 1$, then $2^{n(1-p)} \to \infty$ as $n \to \infty$, and so the condensed series becomes a geometric series with common ratio greater than 1, which diverges.

This implies that the original series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leqslant 1$. ■

(d). We can use the limit comparison test to test for convergence of the series $\displaystyle\sum_{n=1}^{\infty} n^{-1-\frac{1}{n}}$.

Let $a_n = n^{-1-\frac{1}{n}}$ and $b_n = n^{-1}$. Then, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^{-1-\frac{1}{n}}}{n^{-1}} \\ &= \lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^{-1} \\ &= 1 \end{aligned}$$

This implies that $\sum_{n=1}^{\infty} n^{-1-\frac{1}{n}}$ converges if and only if $\sum_{n=1}^{\infty} n^{-1}$ converges. Since the latter diverges, the former also diverges. ■

Remark: $\lim n^{\frac{1}{n}} = 1$

We can compute the limit $\lim_{n\to\infty} n^{\frac{1}{n}}$ as follows:

For all $n \in \mathbb{N}$, $n^{\frac{1}{n}} > 1$, so we can write $n^{\frac{1}{n}} = 1 + h_n$, where $h_n = n^{\frac{1}{n}} - 1 > 0$. Then, we have

$$\begin{aligned} n &= (1+h_n)^n \\ &= 1 + nh_n + \frac{n(n-1)}{2}h_n^2 + \cdots + h_n^n \\ &> \frac{n(n-1)}{2}h_n^2 \\ \implies h_n &< \sqrt{\frac{2n}{n(n-1)}} = \sqrt{\frac{2}{n-1}} \end{aligned}$$

Taking limits on both sides, we get

$$\begin{aligned} \lim_{n\to\infty} h_n &\leqslant \lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0 \\ \implies \lim_{n\to\infty} n^{\frac{1}{n}} &= \lim_{n\to\infty} (1+h_n) = 1 \end{aligned}$$

(e). We can use the limit comparison test to test for convergence of the series $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n}$.

Let $a_n = \frac{1}{p^n - q^n}$ and $b_n = \frac{1}{p^n}$. Then, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{p^n}{p^n - q^n} \\ &= \lim_{n\to\infty} \frac{1}{1 - \left(\frac{q}{p}\right)^n} \\ &= 1 \end{aligned}$$

This implies that $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{p^n}$ converges. Since the latter converges, whenever $p > 1$, the former also converges. Whenever $0 < p \leqslant 1$, the latter diverges, and so does the former. ■

Remark

The series $\sum_{n=1}^{\infty} \frac{1}{p^n}$ is a geometric series, with sum given by $\frac{1}{p-1}$. It converges if $p > 1$ and diverges if $0 < p \leqslant 1$ because the common ratio must be less than 1 for convergence.

(f). General term of the series is given by $a_n = \dfrac{1}{n\log(1+\frac{1}{n})}$. We can use the limit comparison test to test for convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\log(1+\frac{1}{n})}$.

Let $b_n = 1$, then we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{1}{n\log(1+\frac{1}{n})} \\ &= 1\end{aligned}$$

This implies that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\log(1+\frac{1}{n})}$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} 1$ converges. Since the latter diverges, the former also diverges.

■

Remark

We see that $\displaystyle\lim_{n\to\infty} \frac{1}{n\log(1+\frac{1}{n})}$ tends to 1 as follows:

By definition of euler's constant e, we have that $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Taking logarithm on both sides, we get $\displaystyle\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) = 1$. Hence, $\displaystyle\lim_{n\to\infty} \frac{1}{n\log(1+\frac{1}{n})} = 1$.