MA403: Assignment 2

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Question: 6.1

Determine which of the following functions are of bounded variation on $\left[0,1\right]$

(a)
$$f(x) = x^2 \sin\left(\frac{1}{x}\right)$$
 if $x \neq 0$, $f(0) = 0$

(b)
$$f(x) = \sqrt{x} \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0, f(0) = 0$$

(a). We calculate the derivative of f, which comes out to be $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ for all $x \in (0,1)$. Now note that

$$\left| f'(x) \right| = \left| 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right| \le \left| 2x \sin\left(\frac{1}{x}\right) \right| + \left| \cos\left(\frac{1}{x}\right) \right| \le 2x + 1 < 3$$

for all $x \in (0,1)$. Hence, f' is bounded on (0,1) and hence f is of bounded variation on [0,1].

(b). We choose a partition given by

$$P = \left\{0, \frac{1}{(n+1)\frac{\pi}{2}}, \frac{1}{n\frac{\pi}{2}}, \dots, \frac{1}{\frac{\pi}{2}}, 1\right\}$$

for some odd $n \in \mathbb{N}$. Then we have

$$\sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} \left| \sqrt{x_i} \sin\left(\frac{1}{x_i}\right) - \sqrt{x_{i-1}} \sin\left(\frac{1}{x_{i-1}}\right) \right|$$

$$= 2\sqrt{\frac{2}{\pi}} \sum_{\substack{i=1 \ i \text{ odd}}}^{n+1} \sqrt{\frac{1}{i}}$$

$$> \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n+1} \sqrt{\frac{1}{i}}$$

Since the summation $\sum_{i=1}^{n+1} \sqrt{\frac{1}{i}}$ diverges as $n \to \infty$, we conclude that f is not of bounded variation on [0,1].

Question: 6.2 (a)

A function f, defined on [a,b], is said to satisfy a uniform Lipschitz condition of order $\alpha>0$ on [a,b] if there exists a constant M>0 such that $|f(x)-f(y)|< M|x-y|^{\alpha}$ for all $x,y\in [a,b]$. If f is such a function, show that $\alpha>1$ implies f is constant on [a,b], whereas $\alpha=1$ implies that f is of bounded variation on [a,b].

Proof. Case 1: $\alpha > 1$

Let y > x be two points in the interval [a, b]. For every $n \ge 1$, divide the interval (x, y) into n subintervals (x_i, x_{i+1}) where $x_i = x + i \frac{y - x}{n}$ for i = 0, 1, ..., n such that $x_0 = x$ and $x_n = y$. The length of each subinterval is given by:

$$x_{i+1} - x_i = \frac{y - x}{n}.$$

By the hypothesis, for every i, we have:

$$|f(x_i) - f(x_{i+1})| < M(x_{i+1} - x_i)^{\alpha} = M\left(\frac{y - x}{n}\right)^{\alpha}.$$

Using the triangle inequality, we can estimate the total change in f from x to y:

$$|f(x) - f(y)| \le \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \sum_{i=1}^{n} M\left(\frac{y-x}{n}\right)^{\alpha} = nM\left(\frac{y-x}{n}\right)^{\alpha}.$$

This simplifies to:

$$|f(x) - f(y)| < M(y - x)^{\alpha} n^{1 - \alpha}.$$

Since we have $\alpha > 1$, it follows that $1 - \alpha < 0$. Therefore, as $n \to \infty$, $n^{1-\alpha} \to 0$. Consequently, we get:

$$|f(x) - f(y)| < M(y - x)^{\alpha} n^{1 - \alpha} \to 0 \text{ as } n \to \infty.$$

Thus, we conclude:

$$f(x) = f(y).$$

Since y > x were arbitrary, we can conclude that f is constant on [a, b].

Case 2: $\alpha = 1$

Consider any partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ of [a, b]. For every $i \in \{1, 2, \dots, n\}$, we have:

$$|f(x_i) - f(x_{i-1})| < M|x_i - x_{i-1}|$$

since $\alpha = 1$. So summing up over all i, we get:

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < M \sum_{i=1}^{n} |x_i - x_{i-1}| = M (b - a)$$

This means that f is of bounded variation on [a, b].

Question: 6.11

Prove that every absolutely continuous function on [a, b] is continuous and of bounded variation on [a, b].

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Definition: Absolutely Continuous Function

A function f defined on [a,b] is said to be absolutely continuous on [a,b] if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite disjoint collection of open intervals $\{(a_i,b_i)\}$ in [a,b] with $\sum_i (b_i-a_i) < \delta$, we have $\sum_i |f(b_i)-f(a_i)| < \varepsilon$.

Proof. Let f be an absolutely continuous function on [a,b]. By definition, if we choose only 1 interval (a,b) in the collection, then we have $|f(b)-f(a)|<\varepsilon$ for $|b-a|<\delta$. This implies that f is uniformly continuous, and hence continuous on [a,b] since for every $\varepsilon>0$, we can choose $\delta>0$ such that $|b-a|<\delta\Longrightarrow |f(b)-f(a)|<\varepsilon$.

Left $\varepsilon=1$ and $P'=\{a=x_0,\ldots,x_n=b\}$ be a partition of [a,b] with the property that $x_i-x_{i-1}=\frac{\delta}{2}$ for all $i\in\{1,2,\ldots,n-1\}$, and $x_n-x_{n-1}\leqslant\frac{\delta}{2}$ where δ is as in the definition of absolute continuity. Then, we have:

$$n = \left| \frac{2(b-a)}{\delta} \right| + 1.$$

Now pick any partition $P = \{t_0, \dots, t_s\}$ and let $P^* = P \cup P'$ such that $P^* = \{z_0 = a, \dots, z_n = b\}$. Let P_i^* be the set of points in P^* contained in $[x_{i-1}, x_i]$ for $i \in \{1, \dots, n\}$, i.e.,

$$P_i^* = \{ z_{i_k} \in P^* : z_{i_k} \in [x_{i-1}, x_i] \}$$

Thus, the sum can be estimated as follows:

$$\sum_{i=1}^{s} |f(t_i) - f(t_{i-1})| \leqslant \sum_{i=1}^{n} \sum_{k} |f(z_{i_k}) - f(z_{i_{k-1}})| \leqslant \sum_{i=1}^{n} \varepsilon = \sum_{i=1}^{n} 1 = n$$

Hence, f is of bounded variation on [a, b].