Indefinite Integration Mantra

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Basic Rules

Assume that $f(x)$ and $g(x)$ are functions with anti-derivatives $\int f(x)\ \mathrm{d}x$ and $\int g(x)\ \mathrm{d}x$ respectively. Then the following hold :

1. $\int kf(x)\ \mathrm{d}x = k\int f(x)\ \mathrm{d}x$
2. $\int (f(x) \pm g(x))\ \mathrm{d}x = \int f(x)\ \mathrm{d}x \pm \int g(x)\ \mathrm{d}x$
3. If $\int f(x)\ \mathrm{d}x = F(x) + C$, then

$$\int f(ax+b)\ \mathrm{d}x = \frac{F(ax+b)}{a} + C$$

Integrals of Standard Functions

- $\int x^n\ \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad \text{if} \quad n \neq -1$
- $\int \frac{1}{x}\ \mathrm{d}x = \ln|x| + C$
- $\int \sin x\ \mathrm{d}x = -\cos x + C$
- $\int \cos x\ \mathrm{d}x = \sin x + C$
- $\int \tan x\ \mathrm{d}x = -\ln|\cos x| + C$
- $\int \sec x\ \mathrm{d}x = \begin{cases} \ln|\sec x + \tan x| + C \\ \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| \\ -\ln|\sec x - \tan x| + C \end{cases}$
- $\int \csc x\ \mathrm{d}x = \begin{cases} -\ln|\csc x + \cot x| + C \\ \ln\left|\tan\frac{x}{2}\right| \\ \ln|\csc x - \cot x| + C \end{cases}$
- $\int \cot x\ \mathrm{d}x = \ln|\sin x| + C$
- $\int \sec^2 x\ \mathrm{d}x = \tan x + C$
- $\int \csc^2 x\ \mathrm{d}x = -\cot x + C$
- $\int \sec x \tan x\ \mathrm{d}x = \sec x + C$
- $\int \csc x \cot x\ \mathrm{d}x = -\csc x + C$
- $\int e^x\ \mathrm{d}x = e^x + C$
- $\int a^x\ \mathrm{d}x = \frac{a^x}{\ln(a)} + C$
- $\int \frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x = \begin{cases} \sin^{-1} x + C \\ -\cos^{-1} x + C \end{cases}$
- $\int \frac{1}{1+x^2}\ \mathrm{d}x = \begin{cases} \tan^{-1} x + C \\ -\cot^{-1} x + C \end{cases}$
- $\int \frac{1}{x\sqrt{x^2-1}}\ \mathrm{d}x = \begin{cases} \sec^{-1} x + C \\ -\csc^{-1} x + C \end{cases}$
- $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
- $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
- $\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$
- $\int \sqrt{a^2-x^2}\ \mathrm{d}x = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int \sqrt{x^2+a^2}\ \mathrm{d}x = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2+a^2}\right| + C$
- $\int \sqrt{x^2-a^2}\ \mathrm{d}x = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2-a^2}\right| + C$
- $\int e^{ax}\cos bx\ \mathrm{d}x = \begin{cases} \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C \\ \frac{e^{ax}}{\sqrt{a^2+b^2}}\left(\cos bx - \tan^{-1}\frac{b}{a}\right) \end{cases}$
- $\int e^{ax}\sin bx\ \mathrm{d}x = \begin{cases} \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C \\ \frac{e^{ax}}{\sqrt{a^2+b^2}}\left(\sin bx - \tan^{-1}\frac{b}{a}\right) \end{cases}$

Standard Methods of Integration

Integration by Transformation

Simplify and convert the integrand into standard integrals.

Integration by Substitution

If $g(x)$ is a continuously differentiable function, then to evaluate integrals of the form

$$I = \int f(g(x))g'(x)\ \mathrm{d}x$$

we substitute $g(x) = t$ and $g'(x)\ \mathrm{d}x = dt$. This substitution reduces the integral to the form

$$\int f(t)\ \mathrm{d}t$$

After evaluating the integral, we substitute back the value of t.

Some Important Substitutions

If you see a radical which is annoying, use these substitutions which usually simplify the integrals :)

- $a^2 + x^2 \longrightarrow x = a\tan\theta$ or $x = a\cot\theta$
- $a^2 - x^2 \longrightarrow x = a\sin\theta$ or $x = a\cos\theta$
- $x^2 - a^2 \longrightarrow x = a\sec\theta$ or $x = a\csc\theta$
- $\sqrt{\frac{a-x}{a+x}} \longrightarrow x = a\cos 2\theta$
- $\sqrt{\frac{x}{a-x}} \longrightarrow x = a\sin^2\theta$
- $\sqrt{\frac{x}{a+x}} \longrightarrow x = a\tan^2\theta$
- $\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\longrightarrow x = \alpha\cos^2\theta + \beta\sin^2\theta$
- $\sqrt{\frac{x-\alpha}{x-\beta}} \longrightarrow x = \alpha\sec^2\theta - \beta\tan^2\theta$

Integration by Parts

If $u(x)$ and $v(x)$ are 2 functions, then

$$\int uv\ \mathrm{d}x = u\int v\ \mathrm{d}x - \int\left(\int v\ \mathrm{d}x\right)\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$$

A rule-of-thumb for the preference of the first function $u(x)$ is **LIATE** - Logarithmic, Inverse Trigonometric, Algebraic, Trigonometric, Exponential.

Integration by Partial Fractions

It is used to integrate functions of the form $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$

1. Convert $\dfrac{P(x)}{Q(x)}$ to a *proper* rational function, *i.e.* degree of $P(x) <$ degree of $Q(x)$. If degree of $P(x) \geq$ degree of $Q(x)$, convert the function to a *proper* function by long divsison.
2. Factorise $Q(x)$ into linear and quadratic factors (Possible by Fundamental Theorem of Algebra).
3. Do **Partial Fraction Decomposition** of the function.
    - If all the factors of $g(x)$ are linear and non repeating : $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{x-a_1} + \dfrac{A_2}{x-a_2} + \cdots + \dfrac{A_n}{x-a_n}$ Substitute x as the roots of the linear factors to find the coefficients (Short-Cut Method).
    - If $g(x)$ is expressible as the product of linear factors such that some of them are repeating $g(x) = (x-a_1)^k(x-a_2)\cdots(x-a_n)$

      Then $\dfrac{P(x)}{Q(x)}$ can be represented as
      $$\frac{A_1}{x-a_1} + \cdots + \frac{A_k}{(x-a_n)^k} + \frac{B_2}{x-a_2} + \cdots + \frac{B_n}{x-a_n}$$
    - If $g(x)$ has irreducible non-repeating quadratic factors of the form ax^2+bx+c, we assume the partial fraction of the type $\dfrac{Ax+B}{ax^2+bx+c}$ and find the coefficients by comparing coefficients.
    - If $g(x)$ has irreducible and repeating quadratic factors. For every sich factor of the form $(ax^2+bx+c)^k$, we consider the partial fraction
      $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

Reduction Formulae

- $I_n = \displaystyle\int \sin^n x \, \mathrm{d}x$

  $I_n = \dfrac{(n-1)I_{n-2} - \sin^{n-1} x \cos x}{n}$
- $I_n = \displaystyle\int \cos^n x \, \mathrm{d}x$

  $I_n = \dfrac{(n-1)I_{n-2} + \cos^{n-1} x \sin x}{n}$
- $I_n = \displaystyle\int \tan^n x \, \mathrm{d}x$

  $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$
- $I_n = \displaystyle\int \cot^n x \, \mathrm{d}x$

  $I_n = -\dfrac{\cot^{n-1} x}{n-1} - I_{n-2}$
- $I_n = \displaystyle\int \sec^n x \, \mathrm{d}x$

  $I_n = \dfrac{\sec^{n-2} x \tan x + (n-2)I_{n-2}}{n-1}$
- $I_n = \displaystyle\int \csc^n x \, \mathrm{d}x$

  $I_n = \dfrac{-\csc^{n-2} x \cot x + (n-2)I_{n-2}}{n-1}$

Some General Techniques Used to solve Integrals

- If there are terms under radical sign in the denominator, rationalise
- If the integrand consists of product of trigonometric functions, use product-to-sum identities to convert to standard integrals.
- While substituting, usually it is best to choose the inner part of a composite function, such as a quantity raised to a power.
- $\displaystyle\int \frac{\mathrm{d}x}{ax^2+bx+c}, \int \frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\, \mathrm{d}x$

  Complete the square and use the standard integrals
- $\displaystyle\int \frac{(px+q)\mathrm{d}x}{ax^2+bx+c}, \int \frac{(px+q)\mathrm{d}x}{\sqrt{ax^2+bx+c}}, \int (px+q)\sqrt{ax^2+bx+c}\, \mathrm{d}x$

  Write the linear term in the form of $A(ax^2+bx+c)' + B$ where A and B are constants, and then use the standard integrals.
- $\displaystyle\int \frac{(ax^2+bx+c)\,\mathrm{d}x}{px^2+qx+r}, \int \frac{(ax^2+bx+c)\,\mathrm{d}x}{\sqrt{px^2+qx+r}}, \int (ax^2+bx+c)\sqrt{px^2+qx+r}\, \mathrm{d}x$

  Substitute $ax^2+bx+c = \lambda(px^2+qx+r) + \mu(px^2+qx+r)' + \gamma$, and then use the standard integrals.
- $\displaystyle\int \frac{1}{a\cos^2 x + b\sin^2 x}\,\mathrm{d}x, \int \frac{1}{a+b\sin^2 x}\,\mathrm{d}x, \int \frac{1}{a+b\cos^2 x}\,\mathrm{d}x, \int \frac{1}{(a\sin x + b\cos x)^2}\,\mathrm{d}x, \int \frac{1}{a+b\sin^2 x + c\cos^2 x}\,\mathrm{d}x$

  Divide numerator and Denominator by $\cos^2 x$, replace $\sec^2 x$, if any, in denominator by $1+\tan^2 x$, and put $\tan x = t$ so that $\sec^2 x\, \mathrm{d}x = dt$
- $\displaystyle\int \frac{1}{a\sin x + b\cos x}\,\mathrm{d}x, \int \frac{1}{a+b\sin x}\,\mathrm{d}x, \int \frac{1}{a+b\cos x}\,\mathrm{d}x, \int \frac{1}{a\sin x + b\cos x + c}\,\mathrm{d}x$

  Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, and then substitute $\tan\dfrac{x}{2} = t$ and use standard integrals.
- $\displaystyle\int \frac{1}{a\sin x + b\cos x}\,\mathrm{d}x$

  can also be solved using the substitution $a = r\cos\theta$ and $b = r\sin\theta$.
- $\displaystyle\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c}\,\mathrm{d}x, \int \frac{p\cos x + q\sin x}{a\cos x + b\sin x}\,\mathrm{d}x$

  Express numerator as $\lambda\,(\text{Denominator}) + \mu(\text{Denominator})' + \gamma$
- $\displaystyle\int e^x(f(x) + f'(x))\,\mathrm{d}x = e^x f(x) + C$
- $\displaystyle\int \frac{1}{\text{Linear}\sqrt{\text{Linear}_1}}\,\mathrm{d}x, \int \frac{1}{\text{Quadratic}\sqrt{\text{Linear}_1}}\,\mathrm{d}x$

  Substitute Linear_1 as t and use standard integrals.
- $\displaystyle\int \frac{1}{\text{Linear}_1\sqrt{\text{Quadratic}}}\,\mathrm{d}x$

  Substitute Linear_1 as $\dfrac{1}{t}$ and use standard integrals.
- $\displaystyle\int \frac{1}{\text{Quadratic}\sqrt{\text{Quadratic}_1}}\,\mathrm{d}x$

  Substitute $\dfrac{\text{Quadratic}_1}{\text{Quadratic}} = t^2$ and use standard integrals.
- $\displaystyle\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}\,\mathrm{d}x$

  Substitute $x = \dfrac{1}{t}$, and then substitute the term inside the radical sign as u^2 and use standard integrals.

Commonly used Integrals

- $\displaystyle\int \sin^2(x) = \frac{x}{2} - \frac{\sin 2x}{4}$

  As a corollary $\displaystyle\int \cos^2(x) = \frac{x}{2} + \frac{\sin 2x}{4}$

- $\displaystyle\int \tan^2 x \, \mathrm{d}x = \tan x - x + C$

  Similarly $\displaystyle\int \cot^2 x \, \mathrm{d}x = -\cot x - x + C$

- $\displaystyle\int \ln x \, \mathrm{d}x = x \ln x - x + C$

- $\displaystyle\int \sin^{-1} x \, \mathrm{d}x = x \sin^{-1} x + \sqrt{1-x^2} + C$

- $\displaystyle\int \cos^{-1} x \, \mathrm{d}x = x \cos^{-1} x - \sqrt{1-x^2} + C$

- $\displaystyle\int \tan^{-1} x \, \mathrm{d}x = x \tan^{-1} x + \frac{\ln(1+x^2)}{2} + C$

- $\displaystyle\int \sec^{-1} x \, \mathrm{d}x = x \sec^{-1} x - \ln(x + \sqrt{x^2-1}) + C$

- $\displaystyle\int \csc^{-1} x \, \mathrm{d}x = x \csc^{-1} x + \ln(x + \sqrt{x^2-1}) + C$

- $\displaystyle\int \cot^{-1} x \, \mathrm{d}x = x \cot^{-1} x - \frac{\ln(1+x^2)}{2} + C$

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