Derivatives Cheat Sheet

Nirav Bhattad

Fundamentals

By Definition, the derivative of function $f(x)$ at $x = a$ is given by : $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ if it exists and is finite.

Alternatively, we can also define $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ provided the limit exists and is finite.

- Right Hand Derivative (**RHD**) of f' at $x = a$ denoted by $f'(a^+)$ is defined by : $f'(a^+) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ provided the limit exists and is finite.
- Left Hand Derivative (**LHD**) of f' at $x = a$ denoted by $f'(a^-)$ is defined by : $f'(a^-) = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}$ provided the limit exists and is finite.
- If $f'(a)$ exists, then $f(x)$ is derivable at $x = a \implies f(x)$ is continuous at $x = a$.
- If $f(x)$ is derivable at $x = a$, then $f(x)$ is continuous at $x = a$. **But the converse is not true.** If $f(x)$ is continuous at $x = a$, then it need not be derivable at $x = a$.
- Derivability over an Interval $\to$ $f(x)$ is said to be derivable over an interval if it is derivable at each and every point of the interval.
- If 2 functions $f(x)$ and $g(x)$ are derivable at $x = a$, then sum, product, difference, composition* of the 2 functions will also be derivable at $x = a$ and if $g(a) \neq 0$, then the function $\frac{f(x)}{g(x)}$ will also be derivable at $x = a$.

Theorems of Derivatives

If u and v are derivable functions of x, then

- $\frac{\mathrm{d}(u \pm v)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \pm \frac{\mathrm{d}v}{\mathrm{d}x}$
- $\frac{\mathrm{d}}{\mathrm{d}x}(Ku) = K\frac{\mathrm{d}u}{\mathrm{d}x}$
- **Product Rule** $\to$ $\frac{\mathrm{d}}{\mathrm{d}x}(u \cdot v) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
- **Quotient Rule** $\to$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$
- **Chain Rule** If $y = f(u)$ and $u = g(x)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

Derivatives of Standard Functions

- $\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = nx^{n-1} \to x \in \mathbb{R}^+, n \in \mathbb{R}$
- $\frac{\mathrm{d}(a^x)}{\mathrm{d}x} = a^x \cdot \ln(a) \longrightarrow \frac{\mathrm{d}(e^x)}{\mathrm{d}x} = e^x$
- $\frac{\mathrm{d}(\ln(x))}{\mathrm{d}x} = \frac{1}{x}$
- $\frac{\mathrm{d}(\log_a(x))}{\mathrm{d}x} = \frac{1}{x\ln(a)}$
- $\frac{\mathrm{d}(\sin x)}{\mathrm{d}x} = \cos x$
- $\frac{\mathrm{d}(\cos x)}{\mathrm{d}x} = -\sin x$
- $\frac{\mathrm{d}(\tan x)}{\mathrm{d}x} = \sec^2 x$
- $\frac{\mathrm{d}(\sec x)}{\mathrm{d}x} = \sec x \cdot \tan x$
- $\frac{\mathrm{d}(\csc x)}{\mathrm{d}x} = -\csc x \cdot \cot x$
- $\frac{\mathrm{d}(\cot x)}{\mathrm{d}x} = -\csc^2 x$
- $\frac{\mathrm{d}(\text{Constant})}{\mathrm{d}x} = 0$
- $\frac{\mathrm{d}(\sin^{-1} x)}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}} \to |x| < 1$
- $\frac{\mathrm{d}(\cos^{-1} x)}{\mathrm{d}x} = \frac{-1}{\sqrt{1-x^2}} \to |x| < 1$
- $\frac{\mathrm{d}(\tan^{-1} x)}{\mathrm{d}x} = \frac{1}{1+x^2} \to x \in \mathbb{R}$
- $\frac{\mathrm{d}(\sec^{-1} x)}{\mathrm{d}x} = \frac{1}{|x|\sqrt{x^2-1}} \to |x| > 1$
- $\frac{\mathrm{d}(\csc^{-1} x)}{\mathrm{d}x} = \frac{-1}{|x|\sqrt{x^2-1}} \to |x| > 1$
- $\frac{\mathrm{d}(\cot^{-1} x)}{\mathrm{d}x} = \frac{-1}{1+x^2} \to x \in \mathbb{R}$
- For 2 functions $f(x)$ and $g(x)$,
  $[f(x)^{g(x)}]' = f(x)^{g(x)}\left[f'(x)\frac{g(x)}{f(x)} + g'(x) \cdot \ln f(x)\right]$

  If $f(x) = \text{Constant}$, then
  $[f(x)^{g(x)}]' = f(x)^{g(x)}[g'(x) \cdot \ln f(x)]$

Some Chad Differentiation Tricks

- Differentiation of Implicit Functions : To find the derivative of $f(x, y) = 0$, we may use the following formula :
  $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\text{Partial Derivative of } f(x,y) \text{ w.r.t } x}{\text{Partial Derivative of } f(x,y) \text{ w.r.t } y}$$
- Differentiation of Functions in Parametric Form : If $x = f(t)$ and $y = g(t)$,
  $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$
- Differentiation using Logarithms :
  - $y = [f_1(x)]^{f_2(x)}$
    $\frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = (f_2{}'(x)) \ln f_1(x) + f_2(x)\frac{f_1{}'(x)}{f_1(x)}$
  - $y = \frac{\prod f_i(x)}{\prod g_i(x)}$
    $\frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \left[\sum \frac{f_i{}'(x)}{f_i(x)}\right] - \left[\sum \frac{g_i{}'(x)}{g_i(x)}\right]$
- Inverse Functions : If $y = f(x)$ and $x = g(y)$ are inverse functions, then
  $$g'(y) = \frac{1}{f'(x)}$$
  $$g''(y) = -\frac{f''(x)}{(f'(x))^3}$$
- Even and Odd Functions : If $f(x)$ is an even function, then $f'(x)$ will be an odd function, and conversely, if $f(x)$ is an odd function, then $f'(x)$ will be an even function.
- Leibnitz Theorem (Derivative for product of 2 functions) : For 2 functions u and v, n^{th} derivative of the product function $u \cdot v$ is given by
  $$y_n = \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r$$

Types of Discontinuities

- Removable Type of Discontinuities :
  - Missing Point Discontinuities
  - Isolated Point Discontinuities
- Non-Removable Type of Discontinuities
  - Finite Type
  - Infinite Type
  - Oscillatory Type

(Dis)continuity of Composite Functions

If $g(x)$ is defined as $g(x) = f(f(x))$, then discontinuities of $g(x)$ will be the union of the set of the discontinuities of $f(x)$ and $f(f(x))$.

Similarly if $h(x) = f(f(f(x)))$, the discontinuities of $h(x)$ will be union of the set of discontinuities of $g(x)$ and $f(f(f(x)))$.

Intermediate Value Theorem (IMVT)

The Intermediate Value Theorem states that, if $f(x)$ is continuous om $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point inside the interval.

Corollary : For a continuous function $f(x)$, if there exist $a, b \in \mathbb{R}$ such that $f(a) \cdot f(b) \leq 0$, i.e., they have opposite signs, then it is assured by IMVT that there exists a root in the interval $[a, b]$.

Tangents and Normals on the curve

The curve is given by $f(x, y) = 0$. We define $\frac{\mathrm{d}y}{\mathrm{d}x}$ as an expression we get after differentiating the equation of the curve w.r.t x.

- For a point on the curve
  - Equation of Tangent :
  $$(y - y_1) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{(x_1,y_1)} \cdot (x - x_1)$$
  - Equation of Normal :
  $$(y - y_1) = -\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)_{(x_1,y_1)} \cdot (x - x_1)$$
- For a point (a, b) not on the curve
  - Equation of Tangent :
  Solve the following 2 equations simultaneously -
  $$\frac{(y_1 - b)}{(x_1 - a)} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{(x_1,y_1)}$$
  $$f(x_1, y_1) = 0$$
  - Equation of Normal :
  Solve the following 2 equations simultaneously -
  $$\frac{(y_1 - b)}{(x_1 - a)} = -\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)_{(x_1,y_1)}$$
  $$f(x_1, y_1) = 0$$

Conditions for a line to be Tangent to a given Curve at a point

Slope of Line = Slope of tangent to the curve at the point of contact
$$-\frac{a}{b} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{(x_1,y_1)}$$

Angle Between Curves

$$m_1 = \left(\frac{\mathrm{d}f_1(x)}{\mathrm{d}x}\right)_{(x_1,y_1)} \text{ and } m_2 = \left(\frac{\mathrm{d}f_2(x)}{\mathrm{d}x}\right)_{(x_1,y_1)}$$

Acute angle between them is given by
$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Lengths in terms of derivatives

- Length of Tangent : $|y| \cdot \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}$
- Length of Normal : $|y| \cdot \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}y}\right)^2}$
- Length of Sub-Tangent : $\left|y \cdot \frac{\mathrm{d}x}{\mathrm{d}y}\right|$
- Length of Sub-Normal : $\left|y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\right|$

Monotonicity

A function which in a given interval is increasing or decreasing is called 'Monotonic' in that interval.

If $f'(x) \geq 0$ at a point $x = a$, then the function at this point is increasing (or precisely non-decreasing). If $f'(x) \leq 0$, the function $f(x)$ at this point is decreasing (or precisely non-increasing). Even if $f'(a)$ is not defined, $f(x)$ can still be increasing or decreasing.

Rolle's Theorem

Let $f(x)$ be a function subject to the following conditions :

1. $f(x)$ is a continuous function of x in the closed interval of $[a, b]$.
2. $f(x)$ is differentiable for every point in the interval (a, b).
3. $f(a) = f(b)$

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = 0$

Mean Value Theorems

Lagrange's Mean Value Theorem

If a function $f(x)$ is

1. continuous in the interval $[a, b]$.
2. differentiable in the interval (a, b).

Then there exists at least one point $x = c$ in the interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Cauchy's Mean Value Theorem

If functions $f(x)$ and $g(x)$ are both continuous in the interval $[a, b]$, differentiable in the interval (a, b), and $g'(x)$ is not zero in the interval (a, b), then there exists some point $x = c$ in (a, b) such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

Taylor's Theorem

Let $f : [a, x] \to \mathbb{R}$. If

1. $f^{(n-1)}$ exists and is continuous on $[a, x]$.
2. $f^{(n)}$ exists on (a, x).

Then there exists $c \in (a, x)$ such that
$$f(x) = f(a) + \frac{f^1(a)}{1!}(x-a)^1 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^n(c)}{n!}(x-a)^{(n)}$$

Maxima - Minima

Types

- Absolute Maxima
- Relative/Local Maxima
- Absolute Minima
- Relative/Local Minima

Necessary Condition

If $f(x)$ is a maximum or minimum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

Using Second Order Derivative

- $f(c)$ is a minimum value of the function $f(x)$ if $f'(c) = 0$ (If it exists) and $f''(c) > 0$ (Only if it exists).
- $f(c)$ is a maximum value of the function $f(x)$ if $f'(c) = 0$ (If it exists) and $f''(c) < 0$ (Only if it exists).

Point(s) of Inflection

The sign of the second order derivative determines the concavity of the curve.

- $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0 \implies ConcaveUpwards$
-
- $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0 \implies ConcaveDownwards$

At the point of Inflection, we find that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ flips sign.

Inflection Points can also occur if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ fails to exist.

Newton-Raphson's Method for Approximations

1. Devise a *good* function $f(x)$
   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
   Do this till $n \to \infty$ (For practical purposes, take approximate value and do it a few times to improve accuracy)
2. $f(x + \Delta x) = f(x) + \frac{\mathrm{d}y}{\mathrm{d}x}(\Delta x)$

Nirav Bhattad, M Prakash Institute, Personal Website