Probabilities and Statistics I - Finals: Formulae

Continuous Random Variables

Uniform distribution:

$$\mu = \frac{A+B}{2}; \quad \sigma^2 = \frac{(B-A)^2}{12}; \quad f(x;A,B) = \begin{cases} \frac{1}{B-A}, & x \in [A,B]; \\ 0, & \text{otherwise} \end{cases}$$

Normal and standard normal distribution:

$$n(x;\mu,\sigma) = n(z;0,1,) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}; \quad z = \frac{x-\mu}{\sigma}$$

Thm. 6.3: $X \sim b(x;n,p) \xrightarrow[np>5\ ;\ nq>5]{n\to\infty} X \sim n(z;0,1)$; with continuity correction: $\pm 0.5\mathbb{Z}$

Gamma and exponential ($\alpha = 1$) function and distribution: $\mu = \alpha\beta$, $\sigma^2 = \alpha\beta^2$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx = (\alpha-1)\Gamma(\alpha-1); \quad \Gamma(n) = (n-1)!$$

$$\text{For } \alpha > 0,\ \beta > 0 :: f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Functions of Random Variables: Let $Y_i = u_i(\vec{X})$, $X_i = w_i(\vec{Y})$; u,w are injective functions.

Discrete cases :: Thm. 7.1: $X \sim f(x) \iff g(y) = f[w(x)]$

Thm. 7.2: $X_1, X_2 \sim f(x_1,x_2) \iff g(y_1,y_2) = f[w_1(x_1,x_2), w_2(x_1,x_2)]$

$$J = w'(y) \quad ; \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Continuous cases :: Thm. 7.3: $X \sim f(x) \iff g(y) = f[w(x)]\|J\|$

Thm. 7.4: $X_1, X_2 \sim f(x_1,x_2) \iff g(y_1,y_2) = f[w_1(x_1,x_2), w_2(x_1,x_2)]\|J\|$

Moment-Generating Functions ; r^{th} moment: $\mu = \mu_r'$ and $\sigma^2 = \mu_2' - \mu^2$; $\mu_r' = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0}$

$$\mu_r' = \mathbb{E}[X^r] = \begin{cases} \sum_x x^r f(x) \\ \int_{-\infty}^\infty x^r f(x)\,dx \end{cases} \qquad M_X(t) = \mathbb{E}[e^{tX}] = \begin{cases} \sum_x e^{tx} f(x) \\ \int_{-\infty}^\infty e^{tx} f(x)\,dx \end{cases} = \mathcal{L}\{f_X(-x)\}(t)$$

Thm. 7.7: Uniqueness. Let X and Y be two random variables with moment-generating functions $M_X(t)$ and $M_Y(t)$, respectively. If $M_X(t) = M_Y(t)$ for all values of t, then X and Y have the same probability distribution.

Thm. 7.8: $M_{X+a}(t) = e^{at}M_X(t)$; **Thm. 7.9:** $M_{aX}(t) = M_X(at)$

Thm. 7.10: Independent-bases. If $X_1, X_2, \ldots, X_n$ are independent random variables with moment-generating functions $M_{X_1}(t), M_{X_2}(t), \ldots, M_{X_n}(t)$, respectively, and $Y = X_1 + X_2 + \cdots + X_n$, then $M_Y(t) = M_{X_1}(t)M_{X_2}(t)\ldots M_{X_n}(t)$.

Thm. 7.11: If $X_1, X_2, \ldots, X_n$ are independent random variables having normal distributions with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively, then the random variable $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ has a normal distribution with mean $\mu_Y = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$ and variance $\sigma_Y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n\sigma_n^2$.

Thm. 7.12: If $X_1, X_2, \ldots, X_n$ are mutually independent random variables that have, respectively, chi-squared distributions with $v_1, v_2, \ldots, v_n$ degrees of freedom, then the random variable $Y = X_1 + X_2 + \cdots + X_n$ has a chi-squared distribution with $v = v_1 + v_2 + \cdots + v_n$ degrees of freedom.

Col. 7.1-2: If $X_1, X_2, \ldots, X_n$ are independent random variables and X_i follows a normal distribution with mean μ_i and variance σ_i^2 for $i = 1, 2, \ldots, n$, then the random variable

$$Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i} \right)^2 \text{ has a chi-squared distribution with } v = \text{n degrees of freedom.}$$

Fundamental sampling: A **population** consists of the totality of the observations with which we are concerned. A **sample** is a subset of a population. Any function of a random variable constituting a random sample is called a **statistic**. The probability distribution of a statistic is called a **sampling distribution**. Let $X_1, X_2, \ldots, X_n$ be n independent random variables, each having the same probability distribution $f(x)$. Define $X_1, X_2, \ldots, X_n$ to be a **random sample** of size n from the population $f(x)$ and write its joint probability distribution as $f(x_1, x_2, \ldots, x_n) = f(x_1) f(x_2) \ldots f(x_n)$.

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \ ; \ \tilde{x} = \begin{cases} X[\frac{n+1}{2}] & \text{odd,} \\ \frac{1}{2}\left(X[\frac{n}{2}] + X[\frac{n}{2}+1]\right) & \text{even.} \end{cases} \ ; \ S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{n(n-1)} \left[n \sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i \right)^2 \right]$$

$$\text{Central Limit Theorem: } \bar{X} \sim N\left(\mu, \sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}\right) \text{ as } n \longrightarrow \infty$$

$$\text{Sampling Distribution of the Difference of Means: } \bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

Chi-squared statistic and distribution:

$$\chi^2 = V = \frac{(n-1)S^2}{\sigma^2}; \quad f(x; v) = \frac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2}, \ x > 0 \ ; \quad \chi_\alpha^2 : P(\chi^2 > \chi_\alpha^2) = \alpha$$

t-distribution statistic and distribution ($n \geq 30$):

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{Z}{\sqrt{V}/v}; \quad f(t; v) = \frac{\Gamma((v+1)/2)}{\sqrt{v\pi}\Gamma(v/2)} \left(1 + \frac{t^2}{v}\right)^{-(v+1)/2} \quad ; \quad t_\alpha : P(T > t_\alpha) = \alpha$$

F-distribution: $f_{1-\alpha}(v_1, v_2) = 1/f_\alpha(v_2, v_1)$

$$F = \frac{U/v_1}{V/v_2}; \quad f(F; v_1, v_2) = \frac{\sqrt{\left(\frac{v_1 F}{v_1 F + v_2}\right)^{v_1} \left(\frac{v_2}{v_1 F + v_2}\right)^{v_2}}}{F\, B(v_1/2, v_2/2)} \quad ; \quad F_\alpha : P(F > F_\alpha) = \alpha$$

One and two sample estimation

$$\text{Known } \sigma^2 : \mu \in \left[\bar{x} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{x} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right] \quad \text{Unknown } \sigma^2 : \mu \in \left[\bar{x} - t_{\alpha/2} \frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2} \frac{s}{\sqrt{n}}\right]$$

$$\text{C.I. for } \sigma^2 : \sigma^2 \in \left[\frac{(n-1)s^2}{\chi_{\alpha/2}^2}, \frac{(n-1)s^2}{\chi_{1-\alpha/2}^2}\right] \quad ; \quad \frac{\sigma_1^2}{\sigma_2^2} \in \left[\frac{s_1^2/s_2^2}{F_{\alpha/2}}, \frac{s_1^2/s_2^2}{F_{1-\alpha/2}}\right]$$