Math 6510 Homework 7

Tarun Chitra

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§2.1 Problems

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Problem. *Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.*

Let $r : X \to A$ be a retraction from X to A, and let $i : A \hookrightarrow X$ be the inclusion map. Then $ri = \mathbb{1} : A \to A$, so $\mathbb{1}_A = \mathbb{1}_* = (ri)_* = r_* i_* : H_n(A) \to H_n(A)$. Since $\mathbb{1}$ is injective, $r_* i_*$ is injective. We know that if we have maps f, g such that gf is injective, then f must be injective. Thus since $r_* i_*$ is injective, $i_* : H_n(A) \to H_n(X)$ is injective.

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Problem. *Show that chain homotopy of chain maps is an equivalence relation.*

Let $f : C_n \to C'_n$ be a chain map. Define $P : C_n \to C'_{n+1}$ by $P(\sigma) = 0$ (i.e. P is the zero map). Then $P\partial + \partial P = 0 = f - f$, so f is chain homotopic to f. Now let $f, g : C_n \to C'_n$ be two chain maps with f chain homotopic to g, so there is a $P : C_n \to C'_{n+1}$ such that $f - g = P\partial + \partial P$. Then $g - f = -(f - g) = -(P\partial + \partial P) = (-P)\partial + \partial(-P)$, so $(-P) : C_n \to C'_{n+1}$ is such that $g - f = (-P)\partial + \partial(-P)$, so g is chain homotopic to f.

Finally suppose $f, g, h : C_n \to C'_{n+1}$ be such that f is chain homotopic to g and g is chain homotopic to h. Then there exist maps $P_1, P_2 : C_n \to C'_{n+1}$ such that

$$f - g = P_1\partial + \partial P_1 \tag{1}$$
$$g - h = P_2\partial + \partial P_2. \tag{2}$$

Adding equations (1) and (2), we get $f - h = (P_1 + P_2)\partial + \partial(P_1 + P_2)$, so $(P_1 + P_2) : C_n \to C'_{n+1}$ is such that $f - h = (P_1 + P_2)\partial + \partial(P_1 + P_2)$, so f is chain homotopic to h. Thus chain homotopy of chain maps is an equivalence relation.

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Problem. *Verify that $f \simeq g$ implies $f_* = g_*$ for induced homomorphisms of reduced homology groups.*

Theorem 2.10 says that if $f \simeq g$, then $f_* = g_* : H_n(X) \to H_n(Y)$. We know that for $n > 0$, $\tilde{H}_n \cong H_n$, so this means that for $n > 0$, $f_* = g_* : \tilde{H}_n(X) \to \tilde{H}_n(Y)$.

We look at the only remaining case, when $n = 0$. Assume $f \simeq g : X \to Y$, let $f_*, g_* : H_0(X) \to H_0(Y)$ be the induced homomorphisms $H_0(X)$ to $H_0(Y)$, and $\tilde{f}_*, \tilde{g}_* : \tilde{H}_0(X) \to \tilde{H}_0(Y)$ be the induced homomorphisms on $\tilde{H}_0(X)$ to $\tilde{H}_0(Y)$. Our goal is to show that $f \simeq g$ implies $\tilde{f}_* = \tilde{g}_*$.

When $n = 0$, $H_0 = \tilde{H}_0 \oplus \mathbb{Z}$, so $\tilde{H}_0 = H_0/\mathbb{Z}$. Then $\tilde{f}_*, \tilde{g}_* : H_0(X)/\mathbb{Z} \to H_0(Y)/\mathbb{Z}$, i.e. $\tilde{f}_*, \tilde{g}_*$ are just f_*, g_* acting on the quotient of $H_0(X)$ by $\mathbb{Z}$. There is a one-to-one correspondence between $H_0(X)/\mathbb{Z}$ and a subgroup $H'(X) \subseteq H_0(X)$, and between $H_0(Y)/\mathbb{Z}$ and a subgroup $H'(Y) \subseteq H_0(Y)$. Thus since $f \simeq g$, we have $f_* = g_* : H_0(X) \to H_0(Y)$, so

$$f_*|_{H'(X)} = g_*|_{H'(X)} : H'(X) \to H'(Y). \tag{3}$$

But since $H'(X) \cong \tilde{H}_0(X)$ and $H'(Y) \cong \tilde{H}_0(Y)$ and $\tilde{f}_*, \tilde{g}_*$ are just f_*, g_* acting on $\tilde{H}_0(X)$, equation (3) says that $\tilde{f}_* = \tilde{g}_* : \tilde{H}_0(X) \to \tilde{H}_0(Y)$.