Math 6510 Homework 1

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1

Let X be a topological space and suppose that x_0, x_1 are in the same path component. Since they are in the same component, there exists a (continuous) path $h : I \to X$ such that $h(0) = x_1, h(1) = x_0$. Let $\mathcal{P}$ be the set of homotopy classes of paths from x_0 to x_1 and define $F : \mathcal{P} \to \pi_1(X, x_0)$ by $F([f]) = [f \cdot h]$. This is a loop based at x_0 and as such $[f \cdot h] \in \pi_1(X, x_0)$. We need to show that this map is well-defined and bijective.

Well-Defined:
Suppose $f \simeq g$ or in other words $[f] = [g] \in \mathcal{P}$. If h_t is the homotopy between f, g and 1_t is the identity homotopy (i.e. $1_t(x) = x, \forall t$) then $h_t \cdot 1_t$ gives a homotopy between $f \cdot h$ and $g \cdot h$.

Injectivity:
Suppose that $[f \cdot h] = [g \cdot h]$ in $\pi_1(X, x_0)$. Then $[f \cdot h] \cdot [\bar{h}] = [f]$ and $[g \cdot h] \cdot [\bar{h}]$ so that $[f] = [g]$.

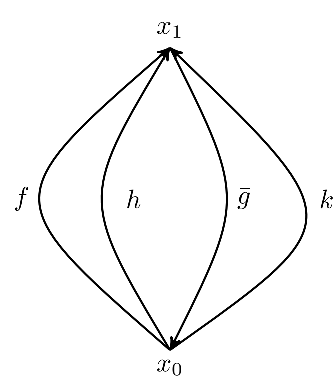
Surjectivity:
Let $[f] \in \pi_1(X, x_0)$. Note that $[f \cdot \bar{h}]$ is a path between x_0 and x_1 so that it makes sense to consider $F([f \cdot \bar{h}])$. It is clear that $F([f \cdot \bar{h}]) = [f]$.

2

Let X be path connected and suppose that h is the path between x_0 and x_1. Let $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$ be the change of basepoint homomorphism, $\beta_h([f]) = [h \cdot f \cdot \bar{h}]$. Suppose that $g \simeq h$. Then we want to show that $\beta_h([f]) = \beta_g([f]), \forall f \in \pi_1(X, x_0)$. Let H_t be the homotopy between g, h. Then $H_t \cdot 1_t \cdot \bar{H}_t$ is a homotopy between $[h \cdot f \cdot \bar{h}]$ and $[g \cdot f \cdot \bar{g}]$. Hence $\beta_h = \beta_g, \forall h \simeq g$.

3

Let X be a path connected space.
($\Rightarrow$) Suppose that $\pi_1(X, x_0)$ is Abelian for all $x_0 \in X$. From problem 1 we know that $\mathcal{P}(x_0, x_1) = \{[h] : h : I \to X, h(0) = x_0, h(1) = x_1\}$ is in bijective correspondance with $\pi_1(X, x_0)$ for any $x_1 \in X$. Since we already know that the change of basepoint homomorphism is a well-defined map, we can without the loss of generality choose a representative from each class in $\pi_1(X, x_0)$ that corresponds to a class in $\mathcal{P}(x_0, x_1)$. Now suppose that $[g], [h] \in \mathcal{P}$, so that $[h \cdot \bar{g}] \in \pi_1(X, x_0)$. To reiterate, the result of problem 1 states that every element of $\pi_1(X, x_0)$ is represented by such a loop, so this is equivalent to picking an arbitary loop in $\pi_1(X, x_0)$. We want to show that for arbitrary[1] $[f], [k] \in \mathcal{P}(x_0, x_1)$ that $\beta_{\bar{f}}([h \cdot \bar{g}]) = \beta_{\bar{k}}([h \cdot \bar{g}])$. The proof is more easily explained upon considering the following drawing:



[1] The result of problem two justifies looking at the homotopy classes in $\mathcal{P}$ as opposed to themselves

Now note the following:

$$\beta_{\bar{f}}([h \cdot \bar{g}]) = [\bar{f} \cdot h \cdot \bar{g} \cdot f] = [(\bar{f} \cdot h) \cdot (\bar{g} \cdot f)] = [(\bar{g} \cdot f) \cdot (\bar{f} \cdot h)] = [\bar{g} \cdot h] \tag{1}$$
$$= [\bar{g} \cdot k \cdot \bar{k} \cdot h] = [(\bar{g} \cdot k) \cdot (\bar{k} \cdot h)] = [\bar{k} \cdot h \cdot \bar{g} \cdot k] \tag{2}$$
$$= \beta_{\bar{k}}([h \cdot \bar{g}]) \tag{3}$$

The 3rd equalities of $(1), (2)$ follow from the fact that $\pi_1(X)$ is Abelian. This shows that the change of basepoint homomorphism is independent of path chosen.
($\Leftarrow$) Suppose that the basepoint-change homomorphisms β_h only depend on the endpoints of the path h. This means that if h, g are paths between $x_0, x_1 \in X$, then $\beta_h([f]) = \beta_g([f]), \forall f \in \pi_1(X, x_0)$. Now suppose that $x_1 = x_0$, for then h, g are simply loops in $\pi_1(X, x_0)$. Now let $f, g \in \pi_1(X, x_0)$ and note the following:

$$[g] = [g \cdot g \cdot \bar{g}] = \beta_g([g]) = \beta_f([g]) = [f \cdot g \cdot \bar{f}]$$

If we multiply on the left by $[f]$ we have $[g] \cdot [f] = [g \cdot f] = [f \cdot g \cdot \bar{f}] \cdot [f] = [f \cdot g \cdot \bar{f} \cdot f] = [f \cdot g]$ which shows that $\pi_1(X, x_0)$ is Abelian. Since X is path connected, this means that $\pi_1(X, x_0) \cong \pi_1(X, x_1), \forall x_1 \in X$.

Alternatively, note that we can show this rather quickly by using problem 1. Since the basepoint-change homomorphism only depends on endpoints, note that if h is a path from x_0 to x_1 and g is a path from x_1 to x_0, then $[h \cdot g] \in \pi_1(X, x_0)$. Since the change of basepoint homorphism is endpoint indpendent, $\beta_h = \beta_{\bar{g}}$. This gives an explicit homotopy between $h \cdot g$ and $g \cdot h$, so that from the bijection found in problem 1, $\pi_1(X)$ is necessarily Abelian.

5

We will prove the following implications in order to arrive at $(a) \iff (b) \iff (c)$. $(a) \implies (b)$:
Suppose that every map $S^1 \to X$ is homotopic to a constant map, with image a point. Suppose $f : S^1 \to X$ and let the homotopy be $H : S^1 x I \to X$ so that $H|_{S^1 \times \{0\}} = f, H|_{S^1 \times \{1\}} = x_0$ for some $x_0 \in X$. Now parametrize D^2 by a radial coordinate $r \in [0, 1]$ and an angular coordinate $\phi \in [0, 2\pi)$ (which will serve as a well-defined angle form on S^1) and define a map $F : D^2 \to X, F(r, \phi) = H(\phi, 1 - r)$. This map is continuous since H is continuous and defined on all of D^2. Finally, note that $F(1, \phi) = f(\phi)$, so that F restricts to f on ∂D^2.
$(b) \implies (a)$:
Suppose that every map $S^1 \to X$ extends to a map $D^2 \to X$. Let $f : S^1 \to X$ be a continuous map and $\tilde{f} : D^2 \to X$ it's extension. Since D^2 is a contractible, there exists map $\varphi : D^2 \to \{x_0\}$ and $\psi : \{x_0\} \to D^2$ such that $\varphi\psi \simeq \mathbb{1}_{x_0}$ and $\psi\varphi \simeq \mathbb{1}_{D^2}$. As such, composing $\tilde{f}$ with ψ gives a homotopy H between $\tilde{f}$ and a constant map $g : D^2 \to X, g(D^2) = \tilde{f}(x_0)$. For instance, if $D^2 \subset \mathbb{R}^2$ and we choose $x_0 = 0$, then this homotopy is simply $H(x, t) = \tilde{f}(tx)$. Since $\tilde{f}|_{\partial D^2} = f$, this gives a homotopy between f and a constant map. In the case of $D^2 \subset \mathbb{R}^2$, this homotopy is explicitly given by simply restricting $H : D^2 \times I \to X$ to $\partial D^2 \times I$ or in other words $H(x, t) = \tilde{f}(tx), |x| = 1$.
$(b) \implies (c)$:
Note that every map $S^1 \to X$ is a loop and that if a map $f : S^1 \to X$ extends to a map $\tilde{f} : D^2 \to X$, then there is a homotopy between $f, \tilde{f}$. Using the argument of the previous implication, $(b) \implies (a)$, there is a homotopy between $\tilde{f}$ and a constant map. Since the binary relation "is homotopic to" is transitive, this implies that f is homotopic to a constant map. Since all of the above homotopies will preserve basepoint, this is a loop homotopy. Hence $[f]$ is trivial and $\pi_1(X, x_0) = 0, \forall x_0$ since the base of the loop and the choice of loop were arbitrary.
$(c) \implies (a)$:
Suppose that $\pi_1(X, x_0) = 0, \forall x_0 \in X$. The definition of a loop in a space X based at x_0 is a continuous map $\varphi : I \to X$ such that $\varphi(0) = \varphi(1) = x_0$. This is equivalent to a map $\tilde{\varphi} : S^1 \to X$ since we can compose φ with the quotient map $p : I \to S^1$ that maps $p(0) = p(1)$. Hence the loop space of X, denoted $[S^1, X]$ is the set of homotopy classes of all maps $S^1 \to X$. Since $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$ this implies that $[S^1, X]$ is necessarily trivial. One could form this category theoretically by consider the category $\mathcal{H}$ of homotopic loops in X. Within this category, we see that the above argument implies that $[S^1, X] \cong \{[h] \in \pi_1(X, x_0), \text{ for some } x_0 \in X\}$ under the "loop morphism" $[f] \in [S^1, X] \mapsto [f, f(\theta')]$. Note that this isomorphism, given the hypotheses of this problem is only valid because of the triviality conditions in $(a), (c)$. In the next problem, we will see that the same result is true, without such a triviality condition, if X is path-connected by using the inverse of the above loop morphism, a forgetful-like functor.

6

Let (X, x_0) be a path-connected, pointed space, define $[S^1, X] = \{[f] : f : S^1 \to X, f \text{ cts}\}$ and define the map $\phi : \pi_1(X, x_0) \to [S^1, X]$ by $[f, x_0] \mapsto [f]$ (i.e. forget the basepoint x_0). The idea is that $[S^1, X]$ is the set of equivalence classes of maps $S^1 \to X$ under a homotopy of *maps* whereas $\pi_1(X, x_0)$ is the set of equivalence classes of maps $S^1 \to X$ under homotopies of *loops* (i.e. homotopic relative to a distinguished point). As the map ϕ is the forgetful functor (inclusion) from the category of pointed topological spaces, $\overset{\circ}{\mathsf{Top}}$ to the category of topological spaces Top, the result of this problem depicts that for path-connected spaces, the spaces of map homotopies and loop homotopies are in one-to-one correspondance. Now we have to show two things:

- ϕ is surjective
- ϕ is injective on conjugacy classes, i.e. $\phi([f]) = \phi([g]) \iff \exists h \in \pi_1(X, x_0) \ni [hf\bar{h}] = [g]$

Surjectivity:
Let $[f] \in [S^1, X]$ and let θ be the standard angle form on S^1 and let $\theta' \in [0, 2\pi)$ be an arbitrary angle. Since X is path-connected, there is a path h between $f(\theta')$ and the basepoint x_0. Then the map $hf\bar{h}$ is a loop based at x_0 that corresponds to $[f]$ under ϕ. We still need to establish that the map is well-defined, since the class $[f] \in [S^1, X]$ is only defined up to a homotopy of maps. Suppose that $f, g \in [f]$, homotopic via a map $L : S^1 \times I \to X$ such that $L|_{S^1\times\{0\}} = f, L|_{S^1\times\{1\}} = g$. Let $h, \tilde{h}$ be paths between x_0 and $f(\theta'), g(\theta')$. Then we need to show that $hf\bar{h} \simeq h'g\bar{h}$ is a homotopy of loops. Since X is path-connected, $\exists$ a path h_t between x_0 and $H(\theta', t)$, $\forall t \in I$, so define the map $H : X \times I \to X$ by $H|_{X\times\{t\}} = h_t$. This is a continuous map (since the path h_t is a continuous map for all t) and as such defines a homotopy between h, h'. Then the map $H_t \cdot L_t \cdot \bar{H}_t$, where $\cdot$ is the group operation within π_1 defines a homotopy between $hf\bar{h}$ and $h'g\bar{h}$.

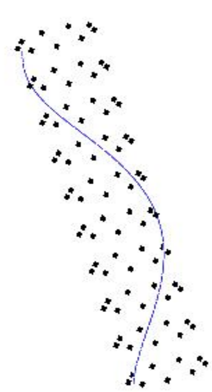
$\therefore \phi$ is surjective

Injectivity on Conjugacy Classes:
($\Rightarrow$) Suppose that $\phi([f]) = \phi([g])$. This means that as maps, $f \simeq g$ via a homotopy H. Since f, g are loops based at x_0, we need to ensure that H is a homotopy of *loops*. This means that if $f(\theta') = g(\hat{\theta}) = x_0$ then we need to construct a loop such that $H_t(\hat{\theta}) = x_0, \forall t$, where $H_t = H|_{X\times\{t\}}$. Since X is path-connected, let h be a path between x_0 and $f(\hat{\theta})$. Then $h \cdot f \cdot \bar{h}$ is a loop based at x_0 that is *loop* homotopic to g.
($\Leftarrow$) Now suppose that there exists an $[h] \in \pi_1(X, x_0)$ such that $hf\bar{h} \simeq g$. We want to show that $\phi([f]) = \phi([g])$ or in other words, $f \simeq g$ as a homotopy of maps. Now note that as a loop, $h : [0, 1] \to X, h(0) = h(1) = x_0$ is homotopic (as a map) to the constant map at x_0 via the homotopy $H : I \times I \to X, H(x, t) = h(tx)$. Hence $hf\bar{h}$ is homotopic as a map to f by using the aforementioned homotopy H and $\bar{H}$.

7

A picture of the map $f : S^1 \times I \to S^1 \times I, f(\theta, s) = (\theta + 2\pi s, s)$ for a fixed θ is the following:

Cylinder for Problem 7



We can construct an explicit homotopy to one of the boundary circles, namely the map $F : (S^1 \times I) \times I \to S^1 \times I$ defined by $F(\theta, s, t) = (\theta + 2\pi ts, s)$. This is certainly continuous and $F(\theta, s, 0) = \mathbb{1}_{S^1\times I}, F(\theta, s, 1) = f(\theta, s)$. Note that this homotopy only restricts to the identity, $\forall t \in I$ on $S^1 \times \{0\}$. Now we need to show that there doesn't exist a homotopy $H : (S^1 \times I) \times I \to S^1 \times I$ that restricts to the identity on $S^1 \times \{1\} \cup S^1 \times \{0\}$ for all $t \in I$. Suppose that such a homotopy existed. Now fix $\theta_0 \in [0, 2\pi)$ and consider the set $K_t := \{H_t(\theta_0, s) : s \in [0,1]\}$. Now note that $K_0 = \{f(\theta_0, s) : s \in [0,1]\} = \{(\theta_0 + 2\pi s, s) : s \in [0,1]\}$ which is simply the image of f. Let $p_1 : S^1 \times I \to S^1$ be the projection onto the first coordinate. Then the set $\{(p_1(k), 0) : k \in K_0\}$ is homeomorphic to $[\omega_1] \in \pi_1(S^1 \times \{0\}, x_0)$, the loop that wraps around S^1 once. This can be seen in the above image, if you simply project the helix down onto the base S^1. Now note that the set $K_1 = \{(\theta_0, s) : s \in [0,1]\}$ is simply a line on the cylinder. Projecting this down onto the base S^1 yields a constant loop, $[\omega_0] \in \pi_1(S^1 \times \{0\}, x_0)$. This implies that H is a homotopy between $[\omega_0]$ and $[\omega_1]$ a contradiction.

8

No, the Borsuk-Ulam Theorem does *not* hold for $\mathbb{T}^2 = S^1 \times S^1$. We can think of $S^1 \times S^1 \subset \mathbb{C}^2$, defined by $S^1 \times S^1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\} = \{(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \in \mathbb{C}^2 : \theta_1, \theta_2 \in [0,1)\}$. Now let $n \in \mathbb{N}$ be odd and consider the map $(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \mapsto (\theta_1^n, \theta_2^n)$. The inversions $z_1 \mapsto -z_1, z_2 \mapsto -z_2$ send $\theta_1 \mapsto \theta_1 + \frac{1}{2}, \theta_2 \mapsto \theta_2 + \frac{1}{2}$. If this map were to satisfy $f(x,y) = f(-x,-y)$, then we would have $\theta_1^n = (\theta_1 + \frac{1}{2})^n, \theta_2^n = (\theta_2 + \frac{1}{2})^n$. This means that we have to show that the polynomials,

$$(\theta_1 + \pi)^n - \theta_1^n = \sum_{i=0}^{n-1} \binom{n}{i} \theta_1^i \left(\frac{1}{2}\right)^{n-i} \tag{4}$$

$$(\theta_2 + \pi)^n - \theta_2^n = \sum_{i=0}^{n-1} \binom{n}{i} \theta_2^i \left(\frac{1}{2}\right)^{n-i} \tag{5}$$

don't have any roots in $[0,1)$. For $n = 3$ this is true, since we get a quadratic with only imaginary roots. However, note that it is not possible for a root of either polynomial to be in $[0,1)$, since we would have a finite sum of strictly positive numbers. Hence we have an infinite sequence of polynomials that violate the Borsuk-Ulam Theorem

9

We want to prove the Ham-Sandwich Theorem for $\mathbb{R}^3$: If A_1, A_2, A_3 are compact sets in $\mathbb{R}^3$, then $\exists$ a plane that *simultaneously* divides the three sets into subsets of equal measure. Without the loss of generality, assume that there exists a neighborhood U of $0 \in \mathbb{R}^3$ such that $A_i \cap U = \emptyset$. By transforming coordinates with an affine translation, we can always ensure that this happens. First, let's reduce the problem to considering points on the unit sphere, $S^2 \hookrightarrow \mathbb{R}^3$. For each point in $a_i \in A_i$ (for any i), there exists a unique line $\mathbf{x}_i$ connecting a_i and $0 \in \mathbb{R}^3$. A standard fact about 3-dimensional geometry is that it takes 3 points (or vectors) to determine a plane up to origin (i.e. to specify an *affine* plane that is homeomorphic to the 2-dimensional Affine Space A^2). Since we have three sets, a point $(a_1, a_2, a_3) \in (A_1, A_2, A_3)$ specifies (up to location of origin) a unique plane passing through the points a_1, a_2, a_3. Since each point corresponds to a line, we can map a_i to $\mathbf{x}_i \cap S^2$. More precisely, we construct a map $\Lambda : \mathbb{R}^3 \setminus \{0\} \to S^2, \Lambda(x) = [x]$, where the equivalence relation is $x \equiv y \iff x = \lambda y, \lambda > 0$. This is similar to the construction of $\mathbb{RP}^2$ except that we are looking at equivalence classes of rays. Note that this map is continuous (it is the retraction of $\mathbb{R}^3 \setminus \{0\}$ onto S^2) and one of the maps of a homotopy equivalence, so that in particular, $\Lambda(A_i) \subset S^2$ is compact. Thus we can specify a plane that intersects all three sets by specifying points $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \in \Lambda(A_1) \times \Lambda(A_2) \times \Lambda(A_3)$, up to translation. Since we are measure "equal division" of the sets A_1, A_2, A_3 by using the Lebesgue measure m on $\mathbb{R}^3$ (which is the unique, translational-invariant, complete Radon measure on $\mathbb{R}^3$), knowing a plane "up to translation" will not change our results. Another elementary fact from geometry[2] is that a plane in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into two disjoint subsets R_+ and R_-. Given a point $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$, denote the partition due to the plane containing (a_1, a_2, a_3) by $\{R_+(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3), R_-(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)\}$. Now suppose for a contradiction that there does not exist a plane that divides $A_i, \forall i$. Then this means that there is no point $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \in S^2$ such that $m(A_i \cap R_+(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)) = m(A_i \cap R_-(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3))$. For brevity, let $m_i^{\pm}(\tilde{a_1}, \tilde{a_2}, \tilde{a_3}) := m(A_i \cap R_+(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)) - m(A_i \cap R_-(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)), m^{\mp} = -m^{\pm}$. Now consider the function, $f : S^2 \times S^2 \times S^2 \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ defined by:

$$f(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = \left(\left(m_1^{\pm}(\tilde{a_1}, \tilde{a_2}, \tilde{a_3}), m_1^{\mp}(\tilde{a_1}, \tilde{a_2}, \tilde{a_3})\right), \left(m_2^{\pm}(\tilde{a_1}, \tilde{a_2}, \tilde{a_3}), m_2^{\mp}(\tilde{a_1}, \tilde{a_2}, \tilde{a_3})\right), \left(m_3^{\pm}(\tilde{a_1}, \tilde{a_2}, \tilde{a_3}), m_3^{\mp}(\tilde{a_1}, \tilde{a_2}, \tilde{a_3})\right)\right) \tag{6}$$

This map is continuous, since we are composing a restriction morphism with the Lebesgue measure yielding a measure that is absolutely continuous (and has compact support) with respect to m. The projection $p_i f$ of f onto its ith coordinate can be considered a continuous map $S^2 \to \mathbb{R}^2$ so that by the Borsuk-Ulam theorem, there exists $x \in S^2$ such that $f(x) = f(-x)$. This implies that $m_i^{\pm}(x) = m_i^{\pm}(-x), m_i^{\mp} = m_i^{\mp}(-x)$. The equations are redundant and we

[2] *Unlike* the Jordan Curve Theorem!

only need to consider, without the loss of generality, the $\pm$ case. Since each point $x \in S^2$ maps to the intersection of two open sets this gives $m_i^+(x) - m_i^-(x) = m_i^+(-x) - m_i^-(-x)$. As $x \neq -x \forall x \in S^2$, the σ-additivity of the Lebesgue measure gives $m^+(x \cup (-x)) = m^+(x) - m^+(-x) = m^-(x) - m^-(-x) = m^-(x \cup (-x))$. This implies that $\exists$ a point where $m^+ = m^-$, contradicting our assumption that no such point exists. Tracing back through the reductions and reformulation of this problem as a map on the sphere, this means that there must exist a plane that divides each A_i.