Math 6510 Homework 4

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Notational Note: In any presentation of a group G, I will explicitly set the relation equal to e as opposed to simply writing a presentation like $G \cong \langle a, b, c, d | abcd^{-1} \rangle$. This is a bit untraditional, but it will remove any possible confusion.

§0.0 Problems

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Problem. *Show that a homotopy equivalence $f : X \to Y$ induces a bijections between the sets of path-components of X and the set of path-components of Y and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Show the same for components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent in X.*

Since this is a problem from Chapter 0, we will show this without reference to $\pi_1(X, x_0)$ or $\pi_1(Y, y_0)$. Suppose $(A, x_0) \subset X$ is a pointed path-component of X and $(B, f(x_0)) \subset Y$ is the corresponding pointed, path-component of Y. Suppose that $f : X \to Y$ is a homotopy equivalence with $g : Y \to X$ as homotopy inverse. Then if $x_0, x_1 \in A$ and $\gamma : I \to A, \gamma(0) = x_0, \gamma(1) = x_1$ is a path in A, we can define the map $f \circ \gamma : I \to B$. This is continuous since f, γ are continuous and moreover, we have constructed a path in B from $f(x_0)$ to $f(x_1)$. On the other hand, if $\Gamma : [0, 1] \to Y$ is an arbitrary path in Y between $y_0, y_1 \in B$, then the path $g \circ \Gamma : [0, 1] \to X$ sends a path in B to a path in A. This implies that $g(B) \subset A$ so that every path in B corresponds to a path and A and vice-versa. Since all of the maps involved are continuous, this means that the restriction of f to path-components is bijective.

Since f is a homotopy equivalence, this means that $fg \simeq \mathbb{1}_Y, gf \simeq \mathbb{1}_X$. Let $f|_A = \tilde{f} : A \to f(A), g|_B = \tilde{g} : B \to g(B)$, so our goal is to show that $\tilde{f}\tilde{g} \simeq \mathbb{1}_Y, \tilde{g}\tilde{f} \simeq \mathbb{1}_X$. Firstly note that since $fg \simeq \mathbb{1}_Y$, $f\tilde{g} \simeq \mathbb{1}_B$ so that we can consider $\tilde{g} : B \to A$ and likewise $\tilde{f} : A \to B$. Now, as per Corollary 0.21, we want to show that the mapping cylinders $M_{\tilde{f}}, M_{\tilde{g}}$ deformation retract onto Y, X, respectively. But this is direct: the homotopies $g\tilde{f} \simeq \mathbb{1}_A, f\tilde{g} \simeq \mathbb{1}_B$ provide precisely the necessary homotopies.

To show this on components, simply note that if A, B are now components instead of path-components then the only continuous functions $t_A : A \to \{0, 1\}, t_B : B \to \{0, 1\}$ are constant. This means that $t_A \circ \tilde{g}, t_B \circ \tilde{f}$ are also constant, which implies that f, g are injective on components. Surjectivity follows immediately from the fact that $g(B) \subset A, f(A) \subset B$. The above mapping cylinder argument already accommodates the component case.

So why do this problem? According to Example 0.7, this result proves homotopy equivalence is effectively a component-wise feature, so that homotopy equivalence preserves the number of path-components and components.

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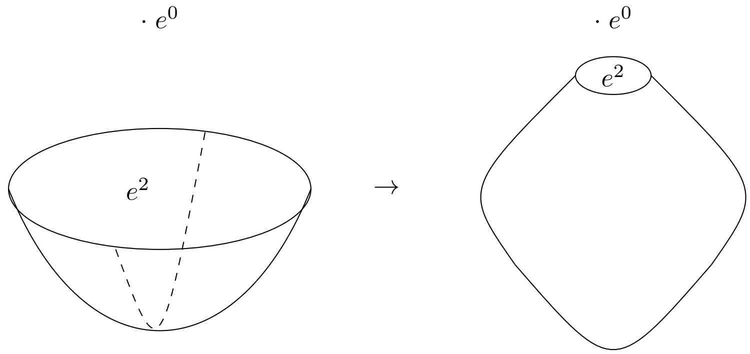
Problem. *Given $v, e, f \in \mathbb{Z}_+$ such that $v - e + f = 2$ construct a cell structure on S^2 having v 0-cells, e 1-cells and f 2-cells*

We can use a general construction for $f > 0$. We have three cases to consider: $f = 1, f = 2$ and $f \geq 3$.

Case 1. $f = 1$

For $v - e = 1$ we must have $f = 1$. However, we have a few subcases to consider:

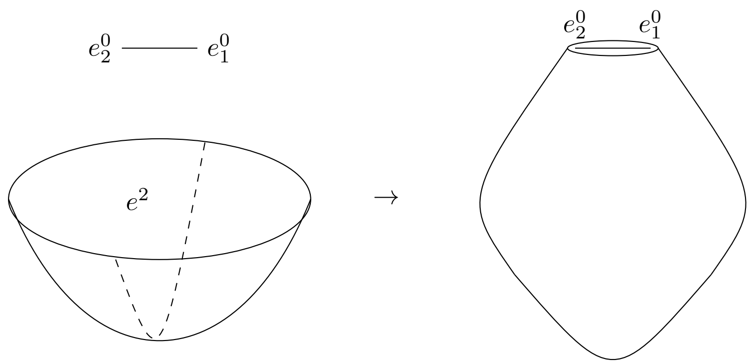
Subcase 1. *If $v = 1, e = 0$, then we have the trivial cell structure which glues the entire boundary of the 2-cell to the* *point (i.e. the attaching map φ is trivial/constant). Parts of the process are depicted below.*



Subcase 2. *Now suppose that $v \geq 2$. If $v = 2$, we can consider with the two 0-cells connected by the one 1-cell we have. Since we can line up all the v 0-cells and connect them with the $v - 1$ 1-cells (as below), the case of $v \geq 2$ will reduce to the $v = 2$ case.*

$$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5$$

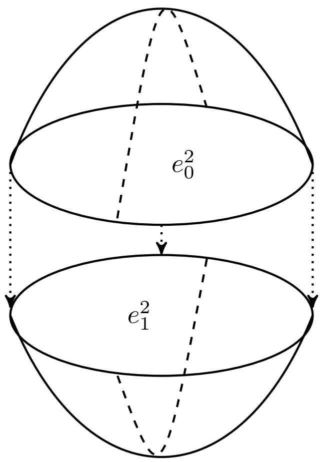
Without the loss of generality, we can assume that our 1-skeleton X^1 is a copy of $I = [0, 1]$. If θ is the angle form on the D^2 that we are going to glue, then the attaching map $\varphi : \partial D^2 \to X^1$ is defined by $\varphi(\theta) = |\sin\theta|$. This effectively glues each half of the circle to I, as the following picture illustrates:



Note that any further case can be reduced to this case, since a 1-skeleton with $v - e = 1$ can never be homeomorphic to a circle as $\chi(S^1) = \chi([0, 1]^2) = 4 - 4 = 0$.

Case 2. $f = 2$

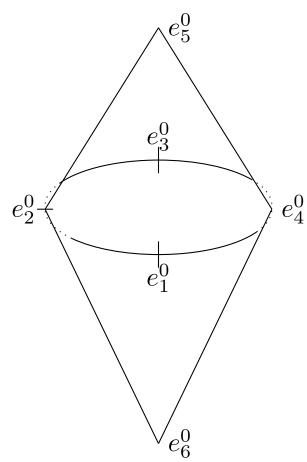
If $v - e = 0$, then $f = 2$, so that intuition dictates that this will be the construction of S^2 that glues two hemispheres to the equator circle. Since $v - e = 0$, we can construct the equator by simply ordering the v 0-cells in a ring and joining them with an 1-cell. One could be a bit more precise and place an 0-cell at $e^{\frac{2\pi k i}{v}} \in \mathbb{C}$ for $1 \leq k \leq v$. Then the e 1-cells are constructed to join $e^{\frac{2\pi k i}{v}}$ with $e^{\frac{2\pi (k+1) i}{v}}$. Given this 1-skeleton X^1 of a cyclic graph with v nodes, e, edges, we can simply glue the 2 2-cells, e^2_1, e^2_2 by the attaching maps that glue $\partial D^2 \cong S^1$ to $X^1 \cong S^1$. Visually we have the following:



Since $v - e = 0$, the 1-skeleton can be a circle, making the above gluing more rigorous (i.e. glue the boundaries to a center circle).

Case 3. $f \geq 3$

This process can be depicts in a fashion so that the CW complex 'looks' similar to that $f = 2$. Suppose $f = k$, so that $v - e = 2 - k < 0$. This means that there are $k - 2$ more edges than vertices. We will start with the 1-skeleton of a suspension of a circle. This is depicted below for the case $k = 4$:



In this case we can glue 4 faces (2-cells) along the cycles $[e^0_0, e^0_2, e^0_4], [e^0_0, e^0_4, e^0_2], [e^5_0, e^0_2, e^0_4], [e^5_0, e^0_4, e^0_2]$ (note the orientation change). Similarly, we can construct a homotopy S^1 by starting with a 1-skeleton of k vertices $\{e^0_1, \ldots, e^0_k\}$ and edges $\{e^1_1, \ldots, e^1_k\}$. We can expand this skeleton by suspending the circle via the additions of two vertices e^0_{k+1}, e^0_{k+2} and k edges $\{e^1_{k+1}, \ldots, e^1_{2k}\}$ such that at least one edge connects the homotopy S^1 as the center of the suspension to e^0_{k+1} and e^0_{k+2}. Note that the signature of these cycles is rather important, as we saw in the $k = 4$ case. Finally, the attaching maps for 2-cells are (non-unique) subsets of the set of all k-cycles in the symmetric group on $k + 2$ letters, S_{k+2}. The lack of uniqueness comes from the choice of map between the homotopy S^1 and the e^0_{k+1}, e^0_{k+2}; for example, one could have attached the suspension 0-cells to e^0_1 and e^0_4 instead.

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Problem. *Enumerate all subcomplexes of S^∞, with the cell structure on S^∞ that has S^n as its n-skeleton*

From the note on page 7, we know that we can inductively define S^n a complex consisting of S^{n-1}, e^n_1, e^n_2, with the n-cells attached via gluing their boundaries to S^{n-1}. This means that we can describe S^n as the union of two copies of each k-cell, $k \leq n$. In other words, $S^n = e_1^0 \cup e_2^0 \cup e_1^1 \cup e_2^1 \cup \ldots \cup e_1^n \cup e_2^n$. If A is any closed subcomplex of S^n containing the k-cell e_1^k, then A must also contain the boundary of the cell e_1^k in order to be closed. The boundary of this cell[1] is simply S^{k-1}, so that any closed subspace containing a k-cell must contain *all* l-cells in the complex for $l < k$. This intuitively means that the only subcomplexes are hemispheres of the form $e_\alpha^k \cup e_1^{k-1} \cup e_2^{k-1} \cdots$. Hence, we can formally write the set of all subcomplexes of $S^n, \mathcal{S}_n$ as:

$$\mathcal{S}_n := \{e_1^0, e_2^0, e_\alpha^k \cup e_1^{k-1} \cup e_2^{k-1} \cdots \cup e_1^0 \cup e_2^0,\ e_1^k \cup e_2^k \cup e_1^{k-1} \cup e_2^{k-1} \cdots \cup e_1^0 \cup e_2^0 | \alpha \in \{1,2\}, 1 \leq k \leq n\} \tag{1}$$

Hence, in the direct limit topology, we can simply define the set of all subcomplexes of S^∞, $\mathcal{S}_\infty$ as,

$$\mathcal{S}_\infty = \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{S}_n \tag{2}$$

Note that this satisfies the direct limit topology, in that $\mathcal{S}_\infty \cap \mathcal{S}_n, \mathcal{S}_\infty \cap S^n \neq \emptyset$, $\forall n \in \mathbb{N} \cup \{0\}$.

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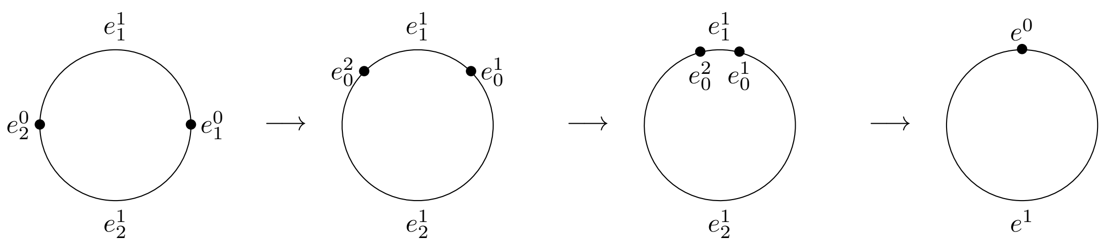
Problem. *Show that S^∞ is contractible*

We will proceed by induction and use some analytic arguments to show contractibility. Using the construction where the k-skeleton of S^n is simply S^k, we will first show show that S^{n-1} as the $n-1$ skeleton of S^n contracts to a point *in* S^n.

Claim. *S^{n-1} contracts to a point in S^n*

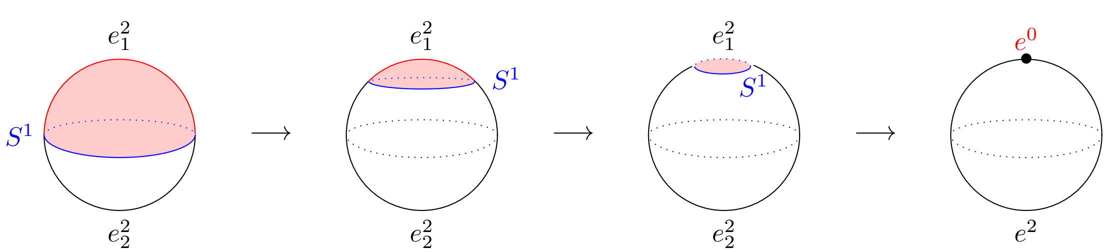
Proof. We will proceed by induction. The base cases (the first two will be shown for illustration of the technique used in the inductive step) include showing that $S^0 \subset S^1$ contracts to a point and $S^1 \subset S^2$ contracts to a point. In the case of $S^0 \subset S^1$, we have $S_0 = \{e^{i\pi}, 1\} \subset S^1$, so that we can contract S^0 along the arc $A = \{e^{i\theta} \in S^1 : \theta \in (0, \pi)\}$. This process actually deforms the equatorial CW structure of $S^1_{eq} = S^0 \cup e_1^1 \cup e_2^1$ to the simple CW structure of $S^1_{sim} = e^0 \cup e^1$. Visually this process is:



Now note that while S^1 *itself* is not contractible, when embedded in S^2 as the equator, it *is* contractible. After all, it is simply a loop in S^2 based at some point on the equator, so as $\pi_1(S^2, x_0) = 0$, it is null-homotopic. However, this doesn't show contractibility; this comes directly from the CW complex structure. The idea is more easily seen in pictures; however, before giving the pictures, let us give the explicit homotopy $H : S^2 \times I \to S^2$ between S^1 and a pole on S^2 (S^2 is thought of as embedded in $\mathbb{R}^3$ and given the equatorial CW structure):

$$H(x, y, z, t) = \{(x, y, t) : x^2 + y^2 = 1 - t^2\} \tag{3}$$

Note that $H(x, y, z, 0)$ is simply the equatorial S^1 while $H(x, y, z, 1)$ is simply a point. Visually we have:



Note that the 1-cell e_1^2, colored in red, and it's boundary S^1, colored in blue, form a closed hemisphere (a subcomplex, based on the previous problem) that contracts to a point.

[1] I suppose one should be careful and say the boundary of the disc D_α^k which gives rise to e_α^k

The inductive step follows the same procedure. Suppose that $S^{k-1} \subset S^k$ is contractible, $\forall k < n$. Since $S^n = S^{n-1} \cup e_1^n \cup e_2^n$, we can do precisely the same process as illustrated above, except that we contract S^{n-1} along e_1^n or e_2^n. Since $S^{n-1} \cup e_1^n \cong D^n$ (this is after the quotient of the attaching map), such a contraction exists. $\square$

Now let's consider the case of finite n, with $S^1 \subset S^2 \subset S^3 \subset \cdots \subset S^n$. Each of the above contractions is in fact a deformation retraction of the closure of the cell $e_1^k \subset S^k$ to a point; call this deformation retraction $r_t^k : S^k \to S^k$ and define local coordinates $(\vec{x}, \theta)$ on S^k such that $\vec{x}$ describes the coordinates on S^{k-1} whereas θ is the angle function for new coordinate on S^k. Then we can explicitly define the endpoints of the retraction:

$$r_0^k(x) = \mathbb{1}_{S^k_{eqn}} \qquad r_1^k(x) = \begin{cases} (\vec{0}, \frac{\pi}{2}) & \text{if } x \in e_1^k \cup S^{k-1} \\ (\vec{x}, 2\theta) & \text{if } x \in e_2^k \end{cases} \tag{4}$$

where $(\vec{0}, \frac{\pi}{2})$ is the "North Pole" of S^k and $\mathbb{1}_{S^k_{eqn}}$ is the identity relative to the equatorial manifold structure. In other words, we are considering S^k as a topological manifold, with coordinates induced by standard spherical coordinates on cells in the CW structure. Now the idea is to compose these retractions, defining a new functions $R_t^k : S^k \to S^k, R_t^k = r_{t_1}^1 * r_{t_2}^2 * \cdots * r_{t_k}^k$, where the binary relation $*$ is defined by,

$$f_t * g_t(x) = \begin{cases} f_{2t}(x) & \text{if } 0 \le t \le \frac{1}{2} \\ g_{1-2t}(x) & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{5}$$

and is defined iff $f_1 = g_0$. In our case, this condition is satisfied, since the $k+1^{\text{th}}$ retraction restracts the sphere S^k that comes out of retracting the copy of S^{k-1} on S^k, i.e. the image of $r_1^k(S^{k-1}) \cong S^k_{sim}$. Note that after applying this composition of k retractions, we are left with the CW complex $\{e^0, e^k\}$. This can be easily seen in the case of $k = 2$ by the pictures on the previous page, however, if one traces through all of the retractions and uses (4), the general case becomes clear. To complete the proof, we therefore need show that as $k \uparrow \infty$, that this CW structure on S^∞ intersects k-cells with an arbitrarily small volume, $\forall k$.

Now note that when we take the limiting composition,

$$R_t^\infty : S^\infty \to S^\infty, \lim_{k \uparrow \infty} r_t^1 * r_t^2 * \cdots r_t^k \tag{6}$$

the equation (5) tells us that the domain of n^{th} retraction in I will be a diadic interval $[\frac{1}{2^n}, \frac{1}{2^{n+1}}]$. Based on the spatial construction of the retractions (i.e. how they contract the k-cell e_1^k), this means know that for arbitrary $0 < \epsilon < 1$, there a dyadic number $\frac{1}{2^k} < \epsilon$ such that for $t_k \in (0, \frac{1}{2^p}), m(e_1^k \cap r_t^k(S^k) \cap S^\infty) < c(k,p), \forall t_k, k$ where m is the k-dimensional Lebesgue Measure and $c(k,p)$ is a bounded function, monotonically decreasing, positive function in k with limit 0. The reason that the volume decreases is because each k-cell has finite volume (i.e. giving the boundedness condition) and as it is contracted to a point, this volume goes to zero. From an analytic standpoint, this is because each k-cell is homeomorphic to an open k-ball and a standard result in analysis is that the volume of a k-ball, with regard to the k-dimensional Lebesgue measure, goes to 0 as $k \uparrow \infty$. Since S^∞ inherits the direct limit topology, this relation holds for arbitrarily large k. This means that arbitrarily few k-cells are in this contracted version of S^∞. In other words, the cell structure becomes $\{e_0\}$ in this limit.

$\therefore S^\infty$ is contractible.

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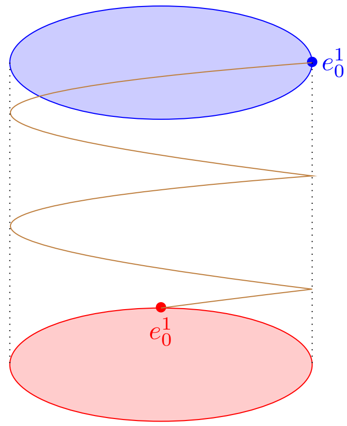
Problem. *a) Show that the mapping cylinder of every map $f : S^1 \to S^1$ is a CW complex*
b) Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts

a

Let $x_0 \in S^1$ be a basepoint and take two 0-cells e_0^i. One 0-cell corresponds to x_0, while the other corresponds to $f(x_0)$. Now construct two circles by attaching both ends of a 1-cell $e_1^i, i \in \{1, 2\}$ to a single 0-cells. Now let n be the degree of f [2] and recall that a degree n map has a winding number of n. This means that we can should connect e_0^1 to e_0^2 by a 1-cell that winds around the (soon-to-be) cylinder n times before arriving at e_0^2. The intuition behind this is from our problem on Dehn Twists; the degree represents the number of times we apply the twisting map to a geodesic on $S^1 \times I$.

[2] Recall that on Homework 2 we proved that every map $S^1 \to S^1$ is homotopic to degree n map

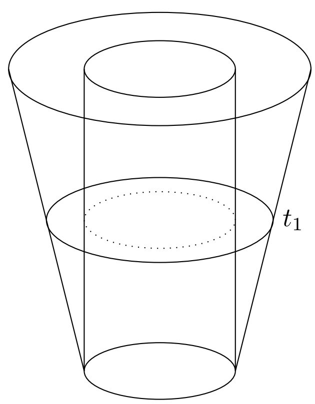
One might also say that this is the "natural path" between e_0^1, e_0^2 in the mapping cylinder C_f. For degree 2 map, this process might look like the following:



Finally, we can attach a 2-cell that attaches its boundary to the 1-skeleton (above) by first wrapping around the top circle, along the 1-cell between e_0^1 and e_0^2 and finally along the boundary of the bottom circle.

b

Both the annulus and the Möbius band deformation retract onto their central circles. Let the deformation retract of the annulus be f_t and the deformation retract of the Möbius band be g_t. Constructing CW structures for the individual mapping cylinders is straight forward, since we can choose k points $\{t_i\}_{i=1}^k \subset [0,1]$ and we can construct the annuli/twisted band at each t_i by simply looking using f_{t_i} or g_{t_i}. The case of the annulus is depicted in the illustration below:



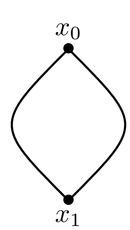
Now we can take the S^1 ends of the mapping cylinders C_{f_t}, C_{g_t} and glue them by associating $f_0(s) \sim g_0(2s)$. The doubling map will "add" the twist needed to smoothly retract onto the Mö strip. Again, the CW complex is pretty straightforward, since we are constructing another cylinder from two smaller cylinders.

§1.2 Problems

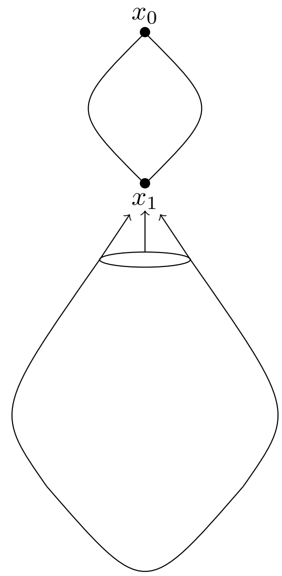
7

Problem. *Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X, x_0)$*

From Example 0.8 in Chapter 0, we see that $X \simeq S^1 \vee S^2$, so that $\pi_1(X, x_0) \cong \mathbb{Z} * 0 \cong \mathbb{Z}$. In terms of a cell structure, we can model X on $S^1 \vee S^2$. Consider the following 1-skeleton, built off of a 0-skeleton $\{x_0, x_1\}$



We can then attach a 2-cell to x_0 or x_1 as in the problem of how to put a cell structure on S^2 with $f = 1$ in order to get the space $S^2 \vee S^1$. Visually, this one skeleton is the following (the arrows represent the attaching map):



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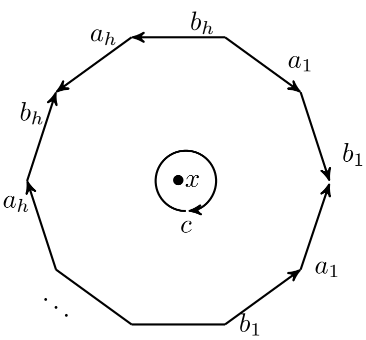
Problem. *In the surface M_g of genus g, let C be a circle that separates M_g into two compact subsurfaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto its boundary circle C, and hence M_g does not retract onto C. But show that M_g* does *retract onto the non-separating circle C in the figure on page 53.*

First let's show that M'_h doesn't retract onto the boundary of the disk removed for the connected sum. Suppose that a retraction $\exists\, r : M'_h \to C$. From Proposition 1.17, the induced map $\iota_* : \pi_1(C, x_0) \to \pi_1(M'_h, x_0)$ is injective for some $x_0 \in C$, where $\iota : C \hookrightarrow M'_h$ is the natural inclusion. Note that M'_h is homotopy equivalent to the punctured genus h surface[3] so that $\pi_1(M'_h, x_0) \cong \pi_1(M_h \setminus \{x\}, x'_0)$, where x'_0 is a point arbitrarily close to the puncture in $M_h \setminus \{x\}$. We can compute the fundamental group of $M_h \setminus \{x\}$ using the fundamental polygon,



The arrow labeled c that is in the center of the polygon is a non-trivial generator that arises from attaching a punctured 2-cell to the 1-skeleton. This implies that the relation we mod out is actually $c\prod_i [a_i, b_i]$ based on the definition of the normal subgroup of Proposition 1.26. Hence we have,

$$\pi_1(M_h \setminus \{x\}, x'_0) \cong \left\langle a_1, b_1 \ldots, a_h, b_h, c \,|\, c\prod_i [a_i, b_i] \right\rangle \tag{7}$$

Now recall that for a group G, we define the abelianization morphism by $\mathbf{ab}(G) := G/\langle [a_i, a_j], \forall a_i, a_j \in H \rangle$, where H is the set of generators of G. If we factor $\pi_1(C, x_0) \to \pi_1(M'_h, x_0)$ through the quotient of the abelianization, then we get an injective map $\mathbf{ab}[C] \to \mathbf{ab}[\pi_1(M'_h, x_0)]$. However, when we abelianize $\pi_1(M'_h, x_0)$, we set

[3] Lee, *Introduction to Smooth Manifolds*, Problem 7-10

$[a_i, b_i] = [a_i, a_j] = [b_i, b_j] = [a_i, c] = [b_i, c] = e, \forall i, j$, so that the relation $c\prod_i [a_i, b_i] = e$ becomes $c = e$. In particular, this means that $\mathbf{ab}[C] \xrightarrow{0} \mathbf{ab}[\pi_1(M'_h, x_0)]$. This implies that the generator corresponding to the removed circle is a non-trivial element of the kernel of this map, contradicting the injectivity of this map.

$\therefore$ The retraction $r : M'_h \to C$ cannot exist.

Now we want to show that M_g retracts onto C'. Let $X^1 = \bigvee_{\alpha=1}^{2g} S^1$ be the 1-skeleton of M_g. From the result of Proposition A.5, we know that $\forall\, 0 < \epsilon < 1$, a neighborhood $N_\epsilon(X^1)$ retracts onto X^1. By properly scaling (which is a homeomorphism for open sets in $\mathbb{R}^2$) the 2-skeleton of M_g, we can retract M_g onto X^1. Since the circle C' is simply one of the circles in X^1, we can then retract X^1 onto C' by collapsing all of the other $2g-1$ circles. Note that this is *not* a deformation retract.

11

Problem. *The* ***mapping torus*** *T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.*

Let x_0 be a basepoint and suppose that $[a], [b]$ are the generators of $\pi_1(X, x_0)$. Under the quotient of the mapping torus, the loops $[a], [b] \in \pi_1(X, x_0)$ become equivalent to the loop $f_*([a]), f_*([b]) \in \pi_1(X, f(x_0))$. This means that $a^{-1}f_*([a]) = b^{-1}f_*([b]) = e$ in $\pi_1(T_f, x_0)$. Since $f_*([a]), f_*([b])$ are simply words in $[a], [b]$. As mentioned in the hint, we can construct the 1-skeleton X^1 of T_f to be $(S^1 \vee S^1) \vee S^1$. From the above discussion, the fundamental group of X^1 is,

$$\pi_1(X^1, x_0) \cong \langle a, b | a^{-1}f_*([a]) = b^{-1}f_*([b]) = e\rangle * \mathbb{Z} = \langle a, b, c | a^{-1}f_*([a]) = b^{-1}f_*([b]) = e\rangle$$

In the case that $[a] \overset{f_*}{\mapsto} [a]^k, [b] \overset{f_*}{\mapsto} [b]^l$, we have $\pi_1(X^1, x_0) \cong \mathbb{Z}_{k-1} * \mathbb{Z}_{l-1} * \mathbb{Z}$ from the result of problem $A1$.

Now the attaching maps will simply commute the generators a, b with the meridianal generator of the torus c, so that we end up with the presentation:

$$\pi_1(X^1, x_0) \cong \langle a, b, c | a^{-1}f_*([a]) = b^{-1}f_*([b]) = [a, c] = [b, c] = e\rangle \tag{8}$$

16

Problem. *Show that the fundamental group of the surface of infinite genus shown on page 53 is free on an infinite number of generators*

Recall that we can define $M_g \cong \overbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}^{g} := \#_g\, \mathbb{T}^2$. Since a connected sum $A\#B$ can be thought of as $(A \setminus D_1^n) \sqcup (B \setminus D_2^n)/\sim$, where the equivalence relation[4] is $x \in \partial D_1^n, y \in \partial D_2^n, x \sim y \iff x = y \in \mathbb{R}^n$. Now since M_g is constructed from 2-manifolds, we can use a modified version of Proposition 1.26. The inclusion $\#_{g-1}\, \mathbb{T}^2 \hookrightarrow \#_g\, \mathbb{T}^2$ induces a map of π_1 that adds the generators a_g, b_g and edits the commutation relation to $\prod_{i=1}^g [a_i, b_i]$. Formally, we can define a homomorphism $\Lambda_g : \pi_1(\#_g\, \mathbb{T}^2, x_0) \to \pi_1(\#_{g+1}\, \mathbb{T}^2, x_0)$ such that does the following:

$$\Lambda_g(a_i) = a_i \qquad \text{For } i \in \{1, \dots, g\} \tag{9}$$
$$\Lambda_g(b_i) = b_i \qquad \text{For } i \in \{1, \dots, g\} \tag{10}$$
$$\Lambda_g\Big(\prod_{i=1}^{g}[a_i, b_i]\Big) = a_{g+1}^{-1}b_{g+1}^{-1} \tag{11}$$

It is clear that this is a homomorphism that has the desired properties. Note that in (11) we are effectively "adding" two generators as we increment the genus by 1. Now in the spirit of classifying spaces, we can denote this with a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(\#_{g-1}\, \mathbb{T}^2, x_0) \arrow[r, "\Lambda_{g-1}"] & \pi_1(\#_g\, \mathbb{T}^2, x_0) \arrow[r, "\Lambda_g"] & \pi_1(\#_{g-1}\, \mathbb{T}^2, x_0) \arrow[r] & \cdots \\
\cdots \arrow[r, hook] & (\#_{g-1}\, \mathbb{T}^2, x_0) \arrow[r, hook] \arrow[u, "\boldsymbol{\pi_1}"] & (\#_g\, \mathbb{T}^2, x_0) \arrow[r, hook] \arrow[u, "\boldsymbol{\pi_1}"] & (\#_{g+1}\, \mathbb{T}^2, x_0) \arrow[r, hook] \arrow[u, "\boldsymbol{\pi_1}"] & \cdots
\end{tikzcd}$$

[4]This is more formally defined by considering a manifold, removing an n-disk and using a chart centered around the origin to properly define $x = y$.

Using this diagram, we can let $g \uparrow \infty$ (as one would in the construction of BG or Gr_∞) and from the construction of Λ_g, we see that we continually add two generators while only having one relation. This means that $\lim_{g\uparrow\infty}\pi_1(\#_g \mathbb{T}^2, x_0)$ is infinitely-generated.

Additional Problems

A1

Problem. *Compute $\pi_1(X)$, where X is the quotient space of a torus $S^1 \times S^1$ obtained by identifying points on the circle $S^1 \times \{y_0\}$ that differ by rotations of the circle by $\frac{2\pi}{n}$ for y_0 a basepoint in the second S^1 factor and $n \geq 2$ a fixed integer.*

Claim: $\pi_1(X) \cong \langle a, b | a^n = [a, b] = e\rangle$

Recall that in Problem 1.1.17 from homework 2, we constructed infinitely many retractions of $S^1 \vee S^1 \to S^1$. Let $q : S^1 \to S^1/\mathbb{Z}_n$ be the quotient map of the hypothesis[5] and consider the sequence of maps,

$$S^1 \vee S^1 \twoheadrightarrow S^1 \xrightarrow{q} S^1/\mathbb{Z}_n \hookrightarrow (S^1/\mathbb{Z}_n) \vee S^1$$

The first map induces an injection $\pi_1(S^1, x_0) \to \pi_1(S^1 \vee S^1, \tilde{x}_0)$ so that the above sequence will give us a way to express $\pi_1\left((S^1/\mathbb{Z}_n) \vee S^1, x_0\right)$ in terms of the generators a, b of $\pi_1(S^1 \vee S^1, \tilde{x}_0)$. As such we only need to compute $\pi_1(S^1/\mathbb{Z}_n, x_0)$ in terms of the generator of $\pi_1(S^1, x_0)$. Now let $a : I \to S^1$ be a degree 1 loop in S^1; then under the quotient, a is mapped to a degree n loop. Since the homotopy class of a serves as the sole generator and identity of $\pi_1(S^1, x_0)$, this implies that the induced homomorphism $q_* : S^1 \to S^1/\mathbb{Z}^n$ sends the identity to the degree n loop in $S^1/\mathbb{Z}^n$. Since identities are preserved under homomorphism, this implies that $q_*([a]^n) = e$ in $\pi_1(S^1/\mathbb{Z}^n)$. Hence $\pi_1(S^1/\mathbb{Z}^n, x_0) \cong \langle a | a^n = e\rangle \cong \mathbb{Z}_n$. This means that the 1-skeleton of X, X^1 has a fundamental group $\pi_1(X^1, x_0) \cong \pi_1(S^1/\mathbb{Z}^n, y_0) * \pi_1(S^1, y_0) \cong \mathbb{Z}_n * \mathbb{Z}$. From proposition 1.26, attaching the 2-cells of the torus to this 1-skeleton[6] will simply mod out the commutator $[a, b]$ so that we have the desired claim.

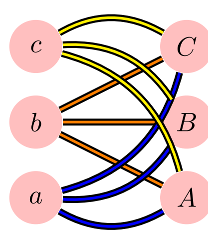
A2

Problem. *Let $K = K_{3,3}$ be the graph obtained by joining each one of 3 vertices a, b, c to each one of 3 other vertices A, B, C by an edge. Thus there are 9 edges in total. Let X be obtained from K by attaching a 2-cell along each loop formed by a cycle of 4 edges in K. Show that $\pi_1(X) = 0$ Generalize to the graph $K_{p,q}$ obtained by joining each of p points to each of q other points.*

$K_{3,3}$

Claim: $\pi_1(K_{3,3}) = 0$

By definition, $K_{3,3}$ is the complete, bipartitite graph on 6 nodes. Since the graph is bipartite, this implies that any cycle must contain an even number of edges. To see this suppose that A, B are the disjoint sets of 3 vertices. Then any path that leaves A must go to B (it would not be bipartite if $\exists$ an edge $A \to A$) and any path from B must go to A; from this it is clear that a cycle needs to contain an even number of edges. Now let $T \subset K_{3,3}$ be the minimal spanning tree[7] of $K_{3,3}$. In the case of $K_{3,3}$, the spanning tree is somewhat easily visualized; the drawing below will help give some intuition behind the CW structure that will be constructed.



Now for each 2-cell e_2^i attached to this graph, define $A_i := T \cup \varphi_i(e_2^i)$, where φ_i is the attaching map. By definition, this will give a closed subset (since the 2-cell is itself a closed disk before being attached) that is glued onto a spanning tree. Note that $T \cap \varphi_i(e_2^i) \neq$, since T passes through every vertex. Moreover, A_i is contractible since T, e_2^i are both contractible. From proposition $A4$, we know that $\exists$ an open, contractible neighborhood X_i of A_i. It is clear that $\{X_i\}$ is an open cover of $K_{3,3}$ and that each intersection $X_i \cap X_j$ is path-connected, since $T \in X_i \cap X_j$. Hence $\pi_1(K_{3,3}) = 0$, since we have found a contractible cover that satisfies van Kampen's Theorem.

[5]It is pretty clear that this is a $\mathbb{Z}_n$ action since applying the map $\theta \to \theta + \frac{2\pi}{n}$, n times gives the identity

[6]Don't forget to compose the attaching map of $S^1/\mathbb{Z}^n$ with the quotient map

[7]I will use the standard combinatorial theorem that $\forall$ graphs $G, \exists$ a minimal spanning tree

$K_{p,q}$

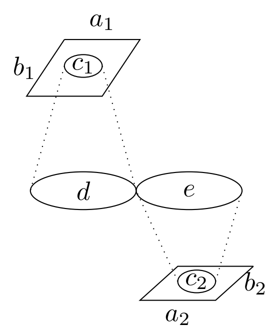
Claim: $\pi_1(K_{p,q}) = 0$

In fact the same construction from above works, as long as we can show that the set $\{X_i\}$ covers $K_{p,q}$. However, this is trivially true since every vertex is contained in at least one 4-cycle. Moreover, since X_i contains $T, \forall i$, this cover still satisfies the property $X_i \cap X_j \cap X_k$ is path-connected $\forall i, j, k$. Hence $\pi_1(K_{p,q}) = 0$ as well.

A3

Problem. *Consider a 2-torus $\mathbb{T}^2$ and let T' be obtained from $\mathbb{T}^2$ by deleting a small open disk. Furthermore, let X be obtained from a new copy of $\mathbb{T}^2$ by attaching two copies of T', identifying their boundary circles with longitudinal and meridian circles $S^1 \times \{y_0\}$ and $\{x_0\} \times S^1$ in $\mathbb{T}^2$. Find a presentation for $\pi_1(X)$ with 4 generators.*

As shown in problem 1.2.9 of this problem set, the fundamental group of T' is $\pi_1(T') \cong \langle a, b, c \,|\, c[a, b] = e\rangle$ (See the fundamental polygon drawn in that problem). Let X_1, X_2 be the two copies of T' that we are gluing with $\pi_1(X_i) = \langle a_i, b_i, c_i \,|\, c_i[a_i, b_i] = e\rangle$. Our goal is to formally apply van Kampen's Theorem, but before doing that let's first appeal to intuition. We are gluing the boundary circles represented by the generators c_i onto two generators d, f of $\pi_1(\mathbb{T}^2)$. As such, we have a 1-skeleton that looks like the following:



From this 1-skeleton, we can read off the desired relation by using the fact that $c_1 \sim d, c_2 \sim f$ under the quotient. This gives us a relation of $[a_1, b_1]df[a_2, b_2] = e$. Now we can express this relation solely in terms of the generating set $\{a_1, b_1, a_2, b_2\}$ as follows. Recall that for each copy of T', we have the relation $c_i[a_i, b_i] = e$ which means that $c_i = [b_i, a_i]$. Finally, recall that after applying Proposition 1.26 when we attach 2-cells, we add the relation $[d, f] = e$, so that we have,

$$[d, f] = dfd^{-1}f^{-1} = [b_1, a_1][b_2, a_2][a_1, b_1][a_2, b_2] = e$$

or in other words,

$$\pi_1(X) \cong \langle a_1, b_1, a_2, b_2 | [b_1, a_1][b_2, a_2][a_1, b_1][a_2, b_2] = e\rangle \tag{12}$$

While this may seem informal, we can apply van Kampen's Theorem to ensure that this is the proper fundamental group. Note that if we let X_3 be the central torus (i.e. corresponding to the loops d, e in the above figure), then the set $\{X_i\}$ is an open cover of X. All of the intersections $X_i \cap X_j = S^1$ as we see in the figure, while $X_1 \cap X_2 \cap X_3 = \emptyset$, which are all path-connected. The generators of the fundamental groups of these circles formally represent the transition from the above diagram to algebraic relations.