Math 6510 Homework 8

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§2.1 Problems

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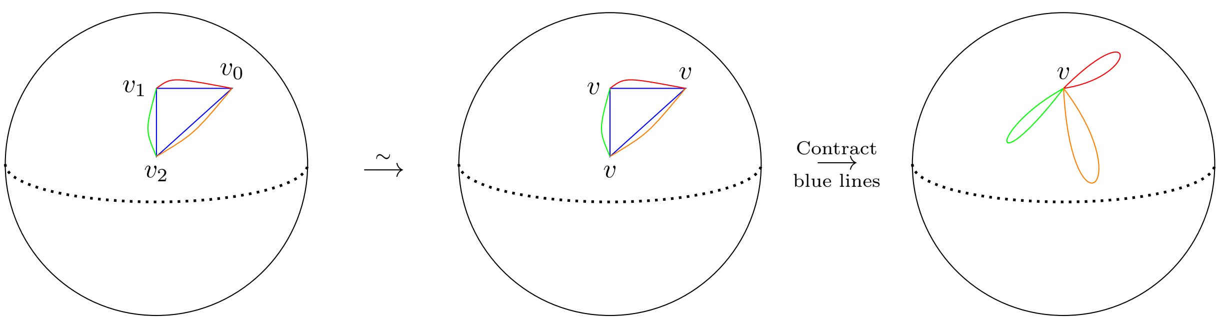
Problem. *a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $\mathbb{T}^2$ and A is a finite set of points in X*
b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown on Page 132. What are $X/A, X/B$?

a

Case 1: S^2
Let $|A| = n > 0$ since the case $n = 0$ is trivial. Now note that Since $S^2 \setminus \{x\} \cong \mathbb{R}^2$, if $|A| = 1$, then $H_n(X, A) \cong 0, \forall n$. Now suppose that $n > 1$ and let $A = \{a_0, \ldots, a_{n-1}\}$. Since the set A is totally disconnected and discrete as a subspace of S^2, there exist neighborhoods $\{N_i : a_i \in N_i\}$ such that N_i deformation retracts onto a_i. This is most easily seen from the fact that S^2 is a 2-manifold, so each of the neighborhood N_i is homeomorphic to a disk in $\mathbb{R}^2$, which is contractible. Hence we can apply Proposition 2.22, $H_n(X, A) \cong \tilde{H}_n(X/A)$. Via a homeomorphism, we can assume that the points in A are arranged in the form on a n-gon embedded in S^2. The easiest way to picture the space X/A is to place edges $[a_i, a_j], \forall i, j \in \{1, \ldots, n\}$, apply the quotient and then contract the edges [This is similar to example 0.8]. Now it is claimed that if we have regular n-gon $B := [a_0, a_1] \cup [a_1, a_2] \cup \ldots \cup [a_{n-1}, a_0]$ that we quotient by the relation $a_i \sim a_j, \forall i, j$, then $(B/\sim) \cong \vee_1^{n-1} S^1$. This is straightfoward, since if we label the edges $e_i := [a_i, a_{i+1}]$, then the quotient says that we have n edges that connect a single vertex to itself. Now let's consider how this looks on S^2. For the case $n = 3$, we have the following pictures:



Hence we are effectively generalizing the result of example 0.8 to show that $X/A \cong S^2 \bigvee (\vee_1^n S^1)$. From Corollary 2.25, which applies since $(S^2, \{v\})$ is a good pair from the aforementioned argument, this means that $\tilde{H}_n(X/A) \cong \tilde{H}_n(S^2) \oplus \underbrace{\tilde{H}_n(S^1) \oplus \cdots \oplus \tilde{H}_n(S^1)}_{n-1}$ so that we have have:

$$H_n(X, A) = \begin{cases} 0 & \text{if } n = 0, n > 2 \\ \mathbb{Z}^{n-1} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \end{cases} \tag{1}$$

Case 2: $\mathbb{T}^2$
The logic is precisely the same as in Case 1. Since $\mathbb{T}^2$ is a smooth 2-manifold, (X, A) is a good pair and Proposition

2.22 applies. We can again use a homeomorphism to arrange the points in A in an n-gon and effectively wedge $n-1$ copies of S^1 to the $\mathbb{T}^2$. Hence we have:

$$H_n(X, A) = \begin{cases} 0 & \text{if } n = 0, n > 2 \\ \mathbb{Z}^{n+1} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \end{cases} \tag{2}$$

b

In the first case, when we send A to a point, we effectively are taking the wedge sum of two tori. This can be seen more concretely as follows. We define $X = \mathbb{T}^2 \# \mathbb{T}^2$. For simplicity, we can picture this as two tori glued to the ends of a cylinder $S^1 \times I$ along some surface disk. By quotienting A, which is equal to slices of the cylinder, $S^1 \times \{k\}, \forall k \in I$, we send the cylinder to an interval I. Contracting this interval gives $\mathbb{T}^2 \vee \mathbb{T}^2$. The argument from part a) for $\mathbb{T}^2$ again applies so that if x_0 is the wedge point, $(\mathbb{T}^2, x_0)$ is a good pair. Hence using Proposition 2.22 and Corollary 2.25 we have,

$$H_n(X, A) \cong \tilde{H}_n(X/A) = \tilde{H}_n(\mathbb{T}^2 \vee \mathbb{T}^2) = \begin{cases} 0 & \text{if } n = 0, n > 2 \\ \mathbb{Z}^4 & \text{if } n = 1 \\ \mathbb{Z}^2 & \text{if } n = 2 \end{cases} \tag{3}$$

On the other hand, when we send B to a point, we send one of the copies of $\mathbb{T}^2$ in X to a copy of the "earring" in Example 0.8. Since this earring is homotopic to $S^2 \vee S^1$, we have $X/B \cong \mathbb{T}^2 \# (S^2 \vee S^1)$. Now it is claimed that $X/B \cong \mathbb{T}^2 \vee S^1$. There are two ways of doing this, one requiring the manifold structure i.e. to define the connected sum) and the other using the intuition from Example 0.8. The intuition from Example 0.8 is clear when one compare the pictures in Example 0.8 to the picture on page 132.

The other way is a bit more rigorous. Recall that for two n-manifolds A, B, the connected sum $A \# B$ is defined by first considering an adapted charts $\varphi_A : U_A \to \mathbb{R}^n, \varphi_A(U_A) \approx B(0, R), \varphi_B : U_B \to \mathbb{R}^n, \varphi_B(U_B) \approx B(0, R + \epsilon)$, where $B(0, R) \subset \mathbb{R}^n$ the ball of radius R centered at the origin and U_A, U_B are open sets in A, B. Then we quotient $A \sqcup B$ by the relation $a \in A \sim \varphi_B^{-1} \circ \varphi_A(a) \in B$ to get $A \# B$. Now in the case we have, $\mathbb{T}^2 \# S^2$ first let the atlas $\{S^2 - \{N\}, S^2 - \{S\}\}$, where $\{N\}, \{S\}$ are the North and South Poles respectively, be used for S^2. Then if $U_A \subset \mathbb{T}^2$ is diffeomorphic to an open ball, the quotient of the connected sum sets U_A equal to $S^2 - \{N\}$ (or $S^2 - \{S\}$). This is equivalent to simply gluing on the point $\{N\}$ to $\mathbb{T}^2$, which will still give a $\mathbb{T}^2$. Now if the wedge point of the S^1 in $S^2 \vee S^1$ is $\{N\}$, then this shows that $\mathbb{T}^2 \# (S^2 \vee S^1) \cong \mathbb{T}^2 \vee S^1$.

Hence all the argument of part a) apply again and via Proposition 2.22 and Corollary 2.25 we have,

$$H_n(X, A) \cong \tilde{H}_n(X/A) = \tilde{H}_n(\mathbb{T}^2 \vee S^1) = \begin{cases} 0 & \text{if } n = 0, n > 2 \\ \mathbb{Z}^3 & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \end{cases} \tag{4}$$

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Problem. *Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis*

First let's show that it's Free Abelian, using exact sequences. We have the following exact sequence in homology,

$$\cdots \to H_n(\mathbb{Q}) \to H_n(\mathbb{R}) \to H_n(\mathbb{R}, \mathbb{Q}) \to H_{n-1}(\mathbb{Q}) \to \cdots$$

Since $\mathbb{Q}$ is a totally disconnected set, every point $q \in \mathbb{Q}$ is a path-component. Hence via Proposition 2.6 we have,

$$H_n(\mathbb{Q}) = \begin{cases} \oplus_{\alpha \in \mathbb{Q}} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathbb{R}$ is contactible, $\tilde{H}_n(\mathbb{R}) = 0, \forall n$, so that the long-exact sequence has truncated to the short-exact sequences,

$$\begin{aligned} &0 \to H_n(\mathbb{R}, \mathbb{Q}) \to H_{n-1}(\mathbb{Q}) \to 0 && \text{if } n > 1 \\ &0 \to H_1(\mathbb{R}, \mathbb{Q}) \to H_0(\mathbb{Q}) \to H_0(\mathbb{R}) \to 0 \end{aligned}$$

Using the prior results, this second sequence becomes:

$$0 \to H_1(\mathbb{R}, \mathbb{Q}) \to \oplus_{\alpha \in \mathbb{Q}} \mathbb{Z} \to \mathbb{Z} \to 0$$

From exactness, this implies that the map $H_1(\mathbb{R}, \mathbb{Q}) \to \oplus_{\alpha \in \mathbb{Q}} \mathbb{Z}$ is injective and since subgroups of free groups are free, $H_1(\mathbb{R}, \mathbb{Q})$ must be Free Abelian.

In order to find a basis for $H_1(\mathbb{R}, \mathbb{Q})$, we need to trace through the definitions in singular homology. Let's compute what cycles and boundaries look like in terms of the image of Δ^1.

Cycles: If $\varphi \in C_1(\mathbb{R}, \mathbb{Q})$ is a cycle, then $\partial_1 \varphi \in \mathbb{Q}$. Now note that $\varphi_1 \varphi \in \mathbb{Q} \iff \mathrm{Im}(\varphi) = [a, b]$, possibly $a = b$, $b - a \in \mathbb{Q}$.

Boundaries: On the other hand suppose that $0 \neq \varphi \in C_1(\mathbb{R}, \mathbb{Q})$ such that $\varphi = \partial_2 \Lambda, \Lambda \in C_2(\mathbb{R}, \mathbb{Q})$. This means that $\partial\Lambda \neq 0$ which can only happen if $\partial\Lambda : \partial\Delta^2 \to \mathbb{R}$ satisfies $\mathrm{Im}(\partial\Lambda) = [a, b]$ such that $b - a \in \mathbb{Q}$ and if $a = b$ then $a, b \notin \mathbb{Q}$. Note that $\mathrm{Im}(\partial\Lambda)$ must be an interval $[a, b]$ since $\partial\Delta^2$ is connected and compact.

From this, we see that $\mathrm{Im}\,\partial_2$ is the same as $\ker \partial_1$, except on maps $\sigma : \Delta^1 \to \mathbb{R}$ with $\sigma(\Delta^1)$ equal to a point. As such this means that $H_1(\mathbb{R}, \mathbb{Q}) = \ker \partial_1 / \ker \partial_2 \cong \langle \sigma : \Delta^1 \to \mathbb{R} \,|\, \sigma(\Delta^1) \in \mathbb{Q} \rangle \cong \mathbb{Q}$. Hence any basis for $\mathbb{Q}$ (as a group under addition) will be a basis of $H_1(\mathbb{R}, \mathbb{Q})$ by simply making the identification of the singular simplex that has image $q \in \mathbb{Q}$ with that of $\mathbb{Q}$. For example, we can take the countable basis $B = \{\sigma : \Delta^1 \to \mathbb{R} : \sigma(\Delta^1) = \{\frac{1}{n}\}$ for some $n \in \mathbb{N}$ or $\sigma(\Delta^1) = \{0\}\}$.

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Problem. *Compute the homology groups of the subspace $I \times I$ consisting of the four boundary edges plus all points in the interior whose first coordinate is rational*

First it is claimed that this space X is path-connected. Let $x, y \in X$ be such that $x = (p, q), y = (p', q'), p, q, p', q' \in I$. If $p = p'$ then the path is simply $\varphi(t) = (p, q' + \frac{1-t}{q-q'}(q - q')$. If $p \neq p'$, then a path is simply,

$$\varphi(t) = \begin{cases} (p, (1-3t)q) & t \in \left[0, \frac{1}{3}\right] \\ (p' + (3t-1)(p - p'), 0) & t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ (p', (3t-2)q') & t \in \left[\frac{2}{3}, 1\right] \end{cases}$$

$\therefore$ Proposition 2.7 gives $H_0(X) \cong \mathbb{Z}$.

Secondly, notice that since X is a subspace of a 2-dimensional manifold, all continuous maps $\sigma : \Delta^n \to X$ to X have the same boundary as the image as a map $\sigma : \Delta^{n-1} \to X$ for $n \geq 2$. Hence $H_n(X) \cong 0, \forall n \geq 2$.

Hence we are left with the computation of $H_1(X)$. First, let's consider two exact sequences, one for the pair (X, I^2) and one for the triple $(\partial I^2, X, I^2)$:

$$\cdots \to H_n(X) \to H_n(I^2) \to H_n(I^2, X) \to H_{n-1}(X) \to \cdots$$

$$\cdots \to H_n(X, \partial I^2) \to H_n(I^2, \partial I^2) \to H_n(I^2, X) \to H_{n-1}(X, \partial I^2) \to \cdots$$

Now we want to reduce these to easier-to-deal-with short exact sequences. Firstly, since I^2 is contractible $\tilde{H}_n(I^2) \cong 0, \forall n$ and $H_n(I^2, \partial I^2) \cong H_n(I^2/\partial I^2) \cong H_n(S^2)$ as $I^2/\partial I^2 \cong S^2$. Now it is claimed that $(X, \partial I^2)$ is a good pair. The set $A = \partial I^2 \cup \{(x, y) : x, y > 0, \min_{(x', y') \in \partial I^2} \|(x, y) - (x', y')\|_{\text{Taxi}} < \epsilon\}$, where $\|\cdot\|_{\text{Taxi}}$ is the Taxicab norm (or L^0 norm) on $\mathbb{R}^2$, is open in the subspace topology on I^2, since it is the intersection of I^2 with a taxicab-metered annulus around the the point $(\frac{1}{2}, \frac{1}{2})$. Hence $A' = A \cap X$ is open in X and is a neighborhood of ∂I^2. Explicitly we have $A' = \{(x, y) \in X : (x, y) \in \partial I^2$ or $x \in \mathbb{Q}, y < \epsilon$ or $x \in \mathbb{Q}, y \in (1 - \epsilon, 1)\}$. Now the following map $\varphi_q : A' \times I \to A'$:

$$\varphi_q(x, y, t) = \begin{cases} (x, (1-t)y, t) & \text{if } x = q \\ (x, y, t) & \text{if } x \neq q \end{cases}$$

This map is certainly continuous, since it is a simple linear retraction along one of the vertical lines in X and the identity everywhere else. Now let $f : \mathbb{N} \to \mathbb{Q}$ be the standard bijection between $\mathbb{N}, \mathbb{Q}$ which exists via a diagonalization argument.

Then $\Phi := *_{i\in\mathbb{N}}\varphi_{f(i)}$, where $*$ is the product operation of the fundamental group, defines a deformation retraction[1] of A' onto ∂I^2 so that ∂I^2 has a neighborhood in X that deformation retracts onto it.
$\therefore (X, \partial I^2)$ is a good pair so $\tilde{H}_n(X/\partial I^2) \cong H_n(X, \partial I^2)$ via Proposition 2.22.

Now note that $X/\partial I^2 \cong \vee_{q\in\mathbb{Q}} S^1$.. To see this, first consider X as disjoint set of lines $L_q = \{(q,y) : y \in I\}$, indexed by $q \in \mathbb{Q}$ that are connected via the lines $A = \{(x,0) : x \in I\}, B = \{(x,1) : x \in I\}$. When we quotient out ∂I^2, we set $(q,0) \sim (q',0)$ and $(q,0) \sim (q,1), \forall q,q' \in \mathbb{Q}$ so that we get the wedge sum of circles from gluing the endpoints of all the L_q together. Hence:

$$\tilde{H}_n(X/\partial I^2) = \begin{cases} \oplus_{q\in\mathbb{Q}}\, \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

These considerations reduce us to the following short exact sequences:

$$0 \to H_2(I^2, X) \to H_1(X) \to 0$$
$$0 \to H_2(S^2) \to H_2(I^2, X) \to H_1(X/\partial I^2) \to 0$$

The first sequence gives $H_2(I^2, X) \cong H_1(X)$ while the second short exact sequence, upon inserting $H_2(S^2) \cong \mathbb{Z}, H_1(X/\partial I^2) \cong \oplus_{q\in\mathbb{Q}}\, \mathbb{Z}$ gives:

$$0 \to \mathbb{Z} \to H_2(I^2, X) \to \oplus_{q\in\mathbb{Q}}\, \mathbb{Z} \to 0$$

This implies that $\oplus_{q\in\mathbb{Q}}\, \mathbb{Z} \cong H_2(I^2, X)/\,\mathbb{Z}$ so we simply have $H_1(X) \cong H_2(I^2, X) \cong \oplus_{q\in\mathbb{Q}}\, \mathbb{Z}$.

In sum:

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \oplus_{q\in\mathbb{Q}}\, \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \tag{5}$$

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Problem. *Show that $\tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX), \forall n$, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified*

We know that $X \overset{\iota_1}{\hookrightarrow} CX \overset{\iota_2}{\hookrightarrow} SX$ under the (geometrically natural) identifications $\iota_1(x) = (x,0), \iota_2[x,a] = [x, \frac{1}{2}(a+1)]$, where we define $CX := X \times I/\{(x,1) \sim (x',1) : \forall x, x' \in X\}, SX = X \times I/\{(x,1) \sim (x',1), (x,0) \sim (x',0) : \forall x, x' \in X\}$. From the un-numbered example right before example 2.23, we know that $\tilde{H}_n(X \cup CA) \cong H_n(X, A)$ for a subspace $A \subset A$, where the gluing is the identity map, i.e. $a \in X \sim (a,0) \in CA$. In our case if we let $X = CX, A = X \times \{0\}$, then this gives $H_n(CX, X) \cong \tilde{H}_n(CX \cup C(X \times \{0\})) \cong \tilde{H}_n(CX \cup CX) \cong \tilde{H}_n(SX)$. Now since (CX, X) is a good pair (proven in the aforementioned example), we have the long exact sequence,

$$\cdots \to \tilde{H}_n(X) \to \tilde{H}_n(CX) \to \tilde{H}_n(CX/X) \to H_{n-1}(X) \to \cdots$$

Since CX is contactible, $\tilde{H}_n(CX) = 0$ so that we have the short exact sequence $0 \to \tilde{H}_n(CX/X) \to \tilde{H}_{n-1}(X) \to 0$, so that $\tilde{H}_n(CX, X) \cong \tilde{H}_{n-1}(X)$. Finally from Proposition 2.22, we have $\tilde{H}_n(CX/X) \cong H_n(CX, X) \cong \tilde{H}_n(SX)$, so that for all $n \geq 1$ we have $\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$.

We can now use this special case of $CX \cup CX$ as the base case for induction on the number of cones identified at their base. Our induction hypothesis is,

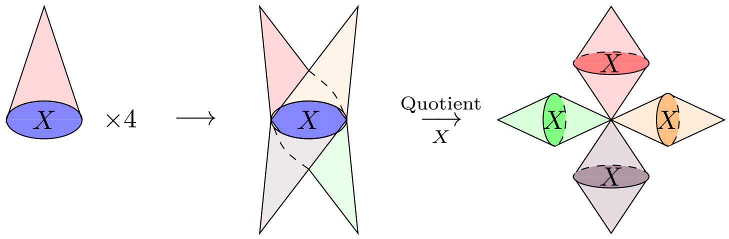
$$\tilde{H}_{k+1}(\cup_{i=1}^n CX) \cong \oplus_{i=1}^{n-1} \tilde{H}_{k+1}(SX) \cong \oplus_{i=1}^{n-1} \tilde{H}_k(X)$$

[1] For completeness we have the following explicit definition of Φ:

$$\Phi(x,y,t) = \begin{cases} (x, g(x,t)\cdot y, t) & \text{if } x \in \mathbb{Q} \text{ and } t \in [2^{-(f^{-1}(x)+1)}, 2^{-f^{-1}(x)}] \\ (x,y,t) & \text{if } y = 0 \text{ or } x \notin \mathbb{Q} \text{ or } t \notin [2^{-(f^{-1}(x)+1)}, 2^{f^{-1}(x)}] \end{cases}$$

where $g(x,t) = \frac{1}{2^{-(f^{-1}(x)+1)}}(2^{-(f^{-1}(x)+1)} - t)$

where the last isomorphism comes from the base case. This intuition for this comes from the fact that $CX/X \cong SX$ and if we have a disjoint union $\sqcup_1^n CX$ of cones and identify all of their bases, and then collapse the base to a point, we have a bunch of suspensions wedges at the point $[X \times \{0\}]$. Pictorially for $n = 4$ this is the following:



Now if we consider the fibration $X \hookrightarrow \cup_1^n CX \twoheadrightarrow \cup_1^n CX/X \times \{0\}$, we see that X can also be viewed (naturally) as a subspace of $\cup_1^n CX/X \times \{0\}$. Using the same steps as before, we see that,

$$\tilde{H}_{k+1}(\cup_1^{n+1} CX) \cong \tilde{H}_{k+1}\left((\cup_1^n CX) \cup C(X \times \{0\})\right) \cong \tilde{H}_{k+1}(\cup_1^n CX, X) \cong \tilde{H}_{k+1}(\cup_1^n CX/X) \cong \tilde{H}_{k+1}(\vee_1^n SX) \cong \oplus_{i=1}^n \tilde{H}_k(X)$$

where again the last isomorphism comes from the base case.

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Problem. *Show that the second barycentric subdivision of a Δ-complex is a simplicial complex. Namely, show that the first barycentric subdivision produces a Δ-complex with the property that each simplex has all its vertices distinct, then show that for a Δ-complex with this property, barycentric subdivision produces a simplicial complex*

From the definitions in the book, "A simplicial complex is a Δ-complex such that all the vertices for any given simplex are distinct and if two simplices have the same set of vertices, then they are the same simplex." Let us first show that barycentric subdivision leads to a Δ-complex such that each simplex has distinct vertices. Note that $B(X)$ for a Δ-complex X simply means "the barycentric subdivision of X."

Claim. *Given a Δ-complex X with k-simplices (i.e. k-skeleton) X^k, then $B(X^k)$ is comprised of simplices with distinct vertices.*

Base Case
In this case, we have a 1-simplex (i.e. an edge) homeomorphic to a compact interval $[a, b]$. Barycentric subdivision gives us two intervals $I_1\left[a, \frac{a+b}{2}\right], I_2\left[\frac{a+b}{2}, b\right]$. These clearly have distinct vertices (one has the vertex a, the other has b).

Induction Step
Our induction hypothesis is that if X is a Δ-complex then for all $m < n$, the subcomplex generated by all of the m-simplices in X forms a simplicial complex. For each n-simplex $X^n := [x_0, \ldots, x_n]$ with barycenter b in X, we have n subsimplices $X_i^n = [b, x_0, \ldots, \hat{x}_i, \ldots, x_n]$. Each of these subsimplices has a boundary

$$\partial X_i^n = \sum_{j<i}(-1)^j[b, x_0, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_n] + \sum_{j>i}(-1)^{j-1}[b, x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n]$$

The $n-1$ simplices that comprise this boundary satisfy the induction hypothesis, so that $[b, x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n] \neq [b, x_0, \ldots, \hat{x}_{i'}, \ldots, \hat{x}_{j'}, \ldots, x_n]$ unless $i = i', j = j'$. Note that all of the vertices in X^n are contained in $\partial X_i^n, \partial X_j^n$ as long as $i \neq j$. Since each X_i^n of the barycentric subdivision has a different vertex x_i removed, this shows that $X_i^n = X_j^n \iff i = j$ which means that all the subsimplices have distinct vertices. Hence $B(X^n)$ is made up n-simplices that all have distinct vertices.

Our next claim is,

Claim. *Given a Δ-complex X such that each k-simplex has its vertices distinct, then $B(X)$ is a simplicial complex*

Let X^n be the n-skeleton of X. We again proceed by induction.

Base Case:
By hypothesis, X^1 comprises of 1-simplices X_i^1. Now suppose that X_i^1, X_j^1 are 1-simplices such that $\partial X_i^1 = \partial X_j^1 = \{a, b\}$ for some $a, b \in X^0$. Then under barycentric subdivsion, we introduce two points $c_k \in X_k^1, k \in \{i, j\}$ such that

$B(X_j^1) \cong [a, c_i] \cup [c_i, b], k \in \{i, j\}$. Since $c_i \neq c_j \neq a \neq b$ this implies that $B(X_i^1), \forall i$ has distinct vertices. Since this holds for arbitrary X_i^1, this means that $B(X^1)$ is comprised of simplices with different vertices
$\therefore B(X^1)$ is a simplicial complex

Inductive Step:
Now suppose that $B(X^m)$ is a simplicial complex $\forall m < n$ and suppose that $X_i^n = [x_0, \ldots, x_n], X_j^n = y_0, \ldots, y_n]$ are two n-simplices in X^n that have the same vertices. Again, barycentric subdivision introduces two barycenters b_i, b_j such that,

$$B(X_i^n) = \sum_k [b_i, x_0, \ldots, \hat{x}_k, \ldots, x_n]$$
$$B(X_j^n) = \sum_k [b_j, y_0, \ldots, \hat{y}_k, \ldots, y_n]$$

Using precisely the same argument as in the last claim, using $\partial B(X_i^n), \partial B(X_j^n)$ is an alternating sum of $n-1$ faces that are coned over b_i, b_j, respectively. From the induction hypothesis, each of these faces has distinct vertices, so that we again have:

$$[b_i, x_0, \ldots, \hat{x}_k, \ldots, x_n] = [b_i, x_0, \ldots, \hat{x}_{k'}, \ldots, x_n] \iff k = k'$$
$$[b_j, y_0, \ldots, \hat{y}_k, \ldots, y_n] = [b_j, y_0, \ldots, \hat{y}_{k'}, \ldots, y_n] \iff k = k'$$

This implies that $B(X_i^n), B(X_j^n)$ is made up of n-simplices with distinct vertices. Since this holds for arbitrary X_i^n, X_j^n this means that $B(X^n)$ is made up of n-simplices with distinct vertices.
$\therefore B(X)$ is a simplicial complex by induction
$\therefore$ Any Δ-complex can be transformed to a simplicial complex via two subsequent barycentric subdivisions[2]

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Problem. *Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$, if $X = [0,1]$ and A is the sequence $(1, \frac{1}{2}, \frac{1}{3}, \cdots)$ together with its limit* 0.

It is claimed that $H_1(X, A)$ is countable, whereas $\tilde{H}_1(X/A)$ is uncountable. Consider the following Δ-structure on $[0, 1]$: Let the points $\{1, \frac{1}{2}, \ldots, \frac{1}{n} \ldots\}$ serve as vertices and let the intervals $[\frac{1}{n} \frac{1}{n+1}]$ be edges. Then the induced Δ-structure on A is simply the vertices. This gives quotient chain groups of $\Delta^1(X)/\Delta^1(A) = \Delta^1(X) \cong \oplus_{n \in \mathbb{N}} \mathbb{Z}$ and $\Delta^0(X)/\Delta^0(A) = 0$. Hence our chain complex is simply,

$$\cdots \to 0 \xrightarrow{\partial} \oplus_{n \in \mathbb{N}} \mathbb{Z} \xrightarrow{\partial} 0$$

This implies that $H_1(X, A) \cong H_1^\Delta(X, A) \cong \oplus_{n \in \mathbb{N}} \mathbb{Z}$ which is countable as since the direct sum enforces a finiteness criteria.

On the other hand, the quotient X/A is simply the Hawaiian Earring from Example 1.25 for which there exists a surjective homomorphism $\pi_1(X/A) \twoheadrightarrow \prod_{i=1}^\infty \mathbb{Z}$ which is uncountable. The ideal answer would be to simply use the Hurewicz Theorem of Appendix 2.A, which says that $H_1(X/A) \cong \mathbf{Ab}(\pi_1(X/A))$ and argue that composing with the quotient map $q : G \to G/\{[g, h] : g, h \in H\}$ will preserve cardinality. However, it turns out that $\prod_{i=1}^\infty \mathbb{Z}$ (known as the *Baer-Specker Group*) is *not* free Abelian. Hence we will have to effectively retrace the steps of Example 1.25, but in terms of maps $\sigma : \Delta^1 \to X/A$. Since $\Delta^1 \cong I$, this is no different from the case of π_1, since we only have to ensure that for any singular chain $\sigma : \Delta^1 \to X/A$, $\sigma(0) = \sigma(1) = (0, 0)$.

As in example 1.25, consider the retractions $r_n : X \to C_n$, where C_n is the n-th circle (i.e. the circle that corresponds to $[\frac{1}{n}, \frac{1}{n+1}]$ in X). Each retraction leads to a surjective induced map on homology, $(r_n)_* : \tilde{H}_1(X/A) \to \tilde{H}_1(C_n) \cong \mathbb{Z}$. This map is surjective simply because the singular simplex $\sigma : I \to X/A, \sigma(0) = \frac{1}{n}, \sigma(1) = \frac{1}{n+1}$ maps to the generator of $\tilde{H}_1(C_n)$ under $(r_n)_*$. Now just as in the example, we can consider the product of the maps $(r_n)_*$ which we denote $\rho : \tilde{H}_1(X/A) \to \prod_{i=1}^\infty \mathbb{Z}$. This is the direct product since *any* arbitrary sequence of integers $\{a_n\}_n \in \mathbb{N}$ corresponds to a unique loop in X/A (as depicted in the figure next to Example 1.25) that winds around each circle C_n, k_n times. This map is also surjective, since we can wind Δ^1 around any C_n an arbitrary number of times. Since $\prod_{i=1}^\infty \mathbb{Z}$ is uncountable, this implies that $\tilde{H}_1(X/A)$ is also uncountable.
$\therefore H_1(X, A) \not\cong \tilde{H}_1(X/A)$

[2]But you'd never want to do this!

27

Problem. *Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.*
a) Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
b) For the case of the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs — there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs. Note that a homotopy equivalence of pairs $(X, A) \to (Y, B)$ is also a homotopy equivalence for the pairs obtained by replacing A and B by their closures

a

This is pretty much a direct application of the 5-lemma. Let $g = f|_A$ be the homotopy equivalence between A and B. Using the exact sequences for the pairs $(X, A), (Y, B)$ we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "g_*,\cong"] & H_n(X) \arrow[r] \arrow[d, "f_*,\cong"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "\tilde{f}_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "g_*,\cong"] & H_{n-1}(X) \arrow[r] \arrow[d, "f_*,\cong"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

where we define $\tilde{f}$ by $\tilde{f} \circ q = f$ for the quotient map $q : C_n(X) \to C_n(X)/C_n(A)$. Since the sequences on the top and bottom are exact and all other maps outside of $\tilde{f}_*$ are isomorphisms of Abelian groups, the 5-lemma prescribes that $\tilde{f}_*$ is an isomorphism, i.e. $H_n(X, A) \cong H_n(Y, B)$.

b

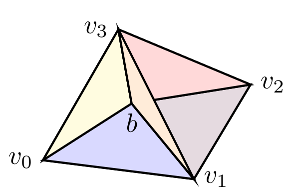
Firstly, note that the main difference between this question and part *a*) is that the homotopy equivalence $D^n - \{0\} \simeq S^{n-1}$ is *not* the restriction of the homotopy equivalence $D^n \simeq D^n$ via $\mathbb{1}_{D^n}$. The fact about closures in the problem statement follows directly from the homotopy extension property for $(\bar{A}, A)$, as it is clear that $\bar{A} \cup \{0\} \cup A \times I$ is a retract of $\bar{A} \times I$. If g is the homotopy inverse of f, then the same argument can be used for the pair $(\bar{B}, B)$.

Suppose that the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$ was a homotopy equivalence. From the fact in the problem statement, this implies that there is a homotopy equivalence between $(D^n, \mathrm{Cl}\left(S^{n-1}\right))$ and $(D^n, \mathrm{Cl}\left(D^n - \{0\}\right))$. But $\mathrm{Cl}(D^n - \{0\}) = D^n$ and $\mathrm{Cl}(S^{n-1}) = S^{n-1}$ (since S^{n-1} is complete as a metric space) which implies that there exists a homotopy equivalence between D^n and S^{n-1} which is a contradiction.

28

Problem. *Let X be the cone on the 1-skeleton of Δ^3, the union of all line segments joining points in the six edges of Δ^3 to the barycenter of Δ^3. Compute the local homology groups $H_n(X, X - \{x\}), \forall x \in X$. Define ∂X to be the subspace of points x such that $H_n(X, X - \{x\}) = 0, \forall n$, and compute the local homology groups $H_n(\partial X, \partial X - \{x\}) = 0, \forall n$. Use these calculations to determine which subsets $A \subset X$ have the property that $f(A) \subset A, \forall f : X \to X$, homeomorphism*

Let's start with the geometric picture:



Note that the outer faces are not colored; only the colored faces are part of X. Let's first write out the algebraic facts gleaned from the long-exact sequence before considering removing various points. The long exact sequence for (reduced) local homology is,

$$\cdots \to \tilde{H}_n(U - \{x\}) \to \tilde{H}_n(U) \to H_n(U, U - \{x\}) \xrightarrow{\partial} \tilde{H}_{n-1}(U - \{x\}) \to \cdots$$

Moreover, note that $\tilde{H}_n(U) \cong 0$, since U is a a neighborhood and every point in the above diagram has a neighborhood that is contractible. Hence we get isomorphisms $H_n(U, U - \{x\}) \cong \tilde{H}_{n-1}(U - \{x\}), \forall n$.

From this picture it is quite apparent that we have 5 cases to consider:

1. Interior point of X^2
2. Any of the vertices
3. The barycenter
4. Interior points of the edge $[v_i, v_j], i \neq j$
5. Interior points of the edge $[v_i, b]$

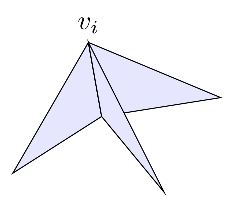
Interior Point of X^2

If x_1 is an interior point, then U is homeomorphic to an open set in $\mathbb{R}^2$ and as such $U - \{x_1\} \cong S^1$. Hence:

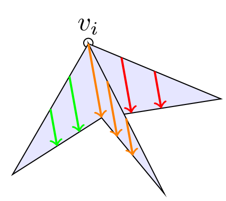
$$H_n(X, X - \{x_1\}) \cong \tilde{H}_{n-1}(S^1) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Any of the vertices

Let $x_2 = v_i, i \in \{0, \ldots, 3\}$. Any neighborhood of v_i will be of the form,



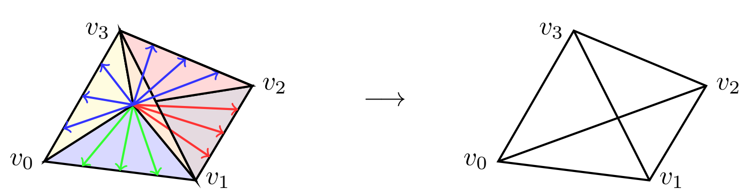
This is a star-shaped neighborhood of a point[3] and since $v_i \in \partial X$, we can retract the neighborhood $U - \{v_i\}$ via the following retraction:



Hence $\tilde{H}_n(U - \{x_2\}) = 0, \forall n$ so $H_n(U, U - \{x\}) = 0, \forall n$.

The Barycenter

Let $x_3 = b$. If we remove the barycenter, we can retract to the 1-skeleton of Δ^3 as follows:



Hence to compute $\tilde{H}_n(X - \{b\}) \cong \tilde{H}_n(X^1)$, we need the homology of X^1, the 1- skeleton of Δ^3. Note that $\Delta_0(X^1) \cong \mathbb{Z}^4, \Delta_1(X^1) \cong \mathbb{Z}^6, \Delta_n(X^1) \cong 0, \forall n > 1$. If we let $v_0, \ldots, v_3$ be the generators of Δ_0 and $e_1, \ldots, e_6$ be the generators of

[3] The Poincaré Lemma and the de Rham theorem give us the result almost immediately here

Δ_1, then we have,

$$\partial_1 e_1 = v_1 - v_0 \partial_1 e_2 = v_2 - v_1 \qquad \partial_1 e_3 = v_2 - v_0$$
$$\partial_1 e_4 = v_3 - v_0 \partial_1 e_5 = v_3 - v_1 \qquad \partial_1 e_6 = v_3 - v_2$$

A straight-forward computation shows that a basis for $\ker \partial_1$ is $\{e_1 - e_2 - e_3, e_1 - e_4 - e_5, e_2 - e_5 - e_6, e_3 - e_4 - e_6\}$ so $\tilde{H}_1(X^1) \cong \mathbb{Z}^4$. Hence we have:

$$H_n(U, U - \{x_3\}) = H_n(U, U - \{b\} \cong \tilde{H}_{n-1}(U - \{b\}) \cong \tilde{H}_{n-1}(X^1) = \begin{cases} \mathbb{Z}^4 & \text{if } n = 2 \\ 0 & \text{else} \end{cases} \tag{7}$$

Interior Points of an edge $[v_i, v_j], i \neq j$

Let $x_4 \in [v_i, v_j], x_4 \neq v_i, v_j$. Any neighborhood U of x_4 will be star-shaped and look similar to the case of a vertex v_i. Using the same retraction, we can show that $U - \{x_4\}$ is contractible so we again have $H_n(U, U - \{x\}) = 0, \forall n$.

Interior Points of the edge $[v_i, b]$

Let $x_5 \in [v_i, b], x_5 \neq x_i, b$. We will again have a star-shaped neighborhood. The only difference from this case and the case of the barycenter is that x_5 only meets 3 faces as opposed to the 6 that the barycenter meets. As such we can use the same retraction to retract those three faces. In this case, we get one of the barycenter tetrahedra, say $[b, v_0, v_1, v_2]$ (minus the base face) as well as the edges $[v_1, v_3], [v_2, v_3], [v_0, v_3], [b, v_3]$. There are two nested tetrahedra; however, the inner tetrahedra with face contributes nothing to homology. Hence we have the same homology as in the barycenter case:

$$H_n(U, U - \{x_5\}) \cong \tilde{H}_{n-1}(X^1) = \begin{cases} \mathbb{Z}^4 & \text{if } n = 2 \\ 0 & \text{else} \end{cases} \tag{8}$$