Twisted super Yangians of type AIII

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Overview of twisted Yangians

- Mathematical Physics
  - quantum integrable models with boundaries [Cherednik'84] [Sklyanin'88] ...
- Constructions and Representation Theory
  - types AI and AII (R-matrix presentation)[Olshanski'92] [Molev'97]
  - type AIII (R-matrix presentation) (reflection algebras) [Molev-Ragoucy'01]
  - all types ($\mathcal{J}$-presentation) [Mackay'02]
  - types BCD (R-matrix presentation) [Guay-Regelskis'14] [Guay-Regelskis-Wendlandt'16]
  - degenerations of twisted q-Yangians cf. [Drinfeld'86] [Gautam-Toledano-Laredo'10] [Guay-Ma'12]
  - finite $\mathcal{W}$-algebras of types BCD [Ragoucy'00] [Brown'07]
- Geometry
  - partial flag varieties of type B/C cf. [Bao-Kujawa-Li-Wang'14]
  - (conjecturally) equivariant homology on the Steinberg varieties of type B/C cf. [Ginzburg-Vasserot'93] [Nakajima'99] [Varagnolo'00]
    A new Drinfeld presentation is probably required.

Set $\varkappa = m + n$. The super Yangians $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ is

- a unital associative superalgebra ($\mathbb{Z}_2$-graded algebra) [Nazarov'91].
- Parity sequence: $\mathfrak{s} = (s_1, s_2, \cdots, s_\varkappa)$, where $s_i = \pm 1$ and $\#\{i|s_i = 1\} = m$; define $|i| \in \mathbb{Z}_2$ by the rule $s_i = (-1)^{|i|}$;
- Generators: $t_{ij}^{(r)}$ of parity $|i| + |j|$, $1 \leqslant i, j \leqslant \varkappa$ and $r \geqslant 1$;
- Defining relations:

$$\mathcal{R}_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)\mathcal{R}_{12}(u - v),$$

where $\mathcal{R}(u) = u\mathbb{I} - \mathcal{P}$ ($\mathcal{P}$ is the super flip operator) and

$$T(u) = (t_{ij}(u))_{i,j=1}^{\varkappa}, \qquad t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r}.$$

Let $T'(u) = (t'_{ij}(u)) = T^{-1}(u)$.

Twisted super Yangians

- Fix $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_\varkappa)$ where $\varepsilon_i = \pm 1$. Set $G = \mathrm{diag}(\boldsymbol{\varepsilon})$.
- Twisted super Yangian $\mathcal{B}_{\mathfrak{s},\boldsymbol{\varepsilon}}$, the subalgebra of $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ generated by the coefficients of entries of

$$B(u) = (b_{ij}(u)) = T(u)GT^{-1}(-u),$$

where $b_{ij}(u) \in \delta_{ij}\varepsilon_i + \mathcal{B}_{\mathfrak{s},\boldsymbol{\varepsilon}}[[u^{-1}]]u^{-1}$.
- Coideal subalgebra: $\Delta(b_{ij}(u)) \in \mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n}) \otimes \mathcal{B}_{\mathfrak{s},\boldsymbol{\varepsilon}}$.
- Unitary condition: $B(u)B(-u) = I$.
- Reflection equation:

$$R(u-v)B_1(u)R(u+v)B_2(v) = B_2(v)R(u+v)B_1(u)R(u-v).$$

- Deformations of twisted current superalgebras.

Known results and goal

For nonsuper case.

- [Molev-Ragoucy'01] Highest weight representation theory and classification of f.d. irreducible representations;
- [Chen-Guay-Ma'14] Schur-Weyl dual to dAHA of type B/C.

For super case.

- [Ragoucy-Satta'07] Nested algebraic Bethe ansatz.

Goal

For *arbitrary* $\mathfrak{s}$ and ε, we would like to

1. establish highest weight representation theory;
2. classify f.d. irreducible representations;
3. obtain Schur-Weyl duality with dAHA of type B/C.

Highest weight representations

A representation L of $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ is called highest weight if there exists a nonzero vector $\xi \in L$ such that L is generated by ξ and ξ satisfies

$$t_{ij}(u)\xi = 0, \qquad 1 \leqslant i < j \leqslant \varkappa,$$
$$t_{ii}(u)\xi = \lambda_i(u)\xi, \qquad 1 \leqslant i \leqslant \varkappa,$$

where $\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. The vector ξ is called a highest weight vector of L and the tuple $\boldsymbol{\lambda}(u) = (\lambda_i(u))_{1\leqslant i\leqslant\varkappa}$ is the highest weight of L.

Theorem [Zhang'94]

- Every f.d. irreducible representation L of super Yangian $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ is a highest weight representation.
- Moreover, L contains a unique (up to proportionality) highest weight vector.
- Classification of f.d. irreducible reps for standard root system.

Highest weight representations

A representation V of $\mathcal{B}_{\mathfrak{s},\varepsilon}$ is called highest weight if there exists a nonzero vector $\eta \in V$ such that V is generated by η and η satisfies

$$b_{ij}(u)\eta = 0, \qquad 1 \leqslant i < j \leqslant \varkappa,$$
$$b_{ii}(u)\eta = \mu_i(u)\eta, \qquad 1 \leqslant i \leqslant \varkappa,$$

where $\mu_i(u) \in \varepsilon_i + u^{-1}\mathbb{C}[[u^{-1}]]$. The vector η is called a highest weight vector of V and the tuple $\boldsymbol{\mu}(u) = (\mu_i(u))_{1\leqslant i\leqslant\varkappa}$ is the highest weight of V.

Proposition

- Every f.d. irreducible representation V of the twisted super Yangian $\mathcal{B}_{\mathfrak{s},\varepsilon}$ is a highest weight representation.
- Moreover, V contains a unique (up to proportionality) highest weight vector.

Verma modules

- Give a weight $\boldsymbol{\mu}(u) = (\mu_i(u))_{1\leqslant i\leqslant \varkappa}$, $\mu_i(u) \in \varepsilon_i + u^{-1}\mathbb{C}[[u^{-1}]]$.
- Verma module $M(\boldsymbol{\mu}(u)) = \mathcal{B}_{\mathfrak{s},\boldsymbol{\varepsilon}}/\langle b_{ij}(u), b_{kk}(u) - \mu_k(u)\rangle$, for $1 \leqslant i < j \leqslant \varkappa$ and $1 \leqslant k \leqslant \varkappa$.
- $\rho_i = s_i + s_{i+1} + \cdots + s_{\varkappa}$.

Theorem

The Verma module $M(\boldsymbol{\mu}(u))$ is nontrivial if and only if

$$\mu_{\varkappa}(u)\mu_{\varkappa}(-u) = 1,$$
$$\tilde{\mu}_i(u)\tilde{\mu}_i(-u+\rho_{i+1}) = \tilde{\mu}_{i+1}(u)\tilde{\mu}_{i+1}(-u+\rho_{i+1}),$$

where $1 \leqslant i < \varkappa$ and

$$\tilde{\mu}_i(u) = (2u - \rho_{i+1})\mu_i(u) + \sum_{a=i+1}^{\varkappa} s_a\mu_a(u).$$

Tensor product

- $V(\boldsymbol{\mu}(u))$, the irreducible quotient of Verma module $M(\boldsymbol{\mu}(u))$ with highest weight vector η.
- $L(\boldsymbol{\lambda}(u))$, the irreducible quotient of Verma module over $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$ with highest weight vector ξ.
- Set

$$\tilde{b}_{ii}(u) = (2u - \rho_{i+1}) b_{ii}(u) + \sum_{a=i+1}^{\varkappa} s_a b_{aa}(u).$$

Then

- $L(\boldsymbol{\lambda}(u)) \otimes V(\boldsymbol{\mu}(u))$ is a $\mathcal{B}_{\mathfrak{s},\boldsymbol{\varepsilon}}$-module;
- $\Delta(\tilde{b}_{ii}(u)) \approx t_{ii}(u) t'_{ii}(-u) \otimes \tilde{b}_{ii}(u)$ on highest weight vector $\xi \otimes \eta$;
- $\tilde{b}_{ii}(u)(\xi \otimes \eta) = \lambda_i(u) \lambda'_i(-u) \tilde{\mu}_i(u)(\xi \otimes \eta)$, $1 \leqslant i \leqslant \varkappa$.

Sufficient conditions

Let $\varpi_i = s_i\varepsilon_i + s_{i+1}\varepsilon_{i+1} + \cdots + s_{\varkappa}\varepsilon_{\varkappa}$.

Proposition

Suppose the highest weight $\boldsymbol{\mu}(u)$ satisfies

$$\frac{\tilde{\mu}_i(u)}{\tilde{\mu}_{i+1}(u)} = \frac{(2\varepsilon_i u - \varepsilon_i\rho_{i+1} + \varpi_{i+1} + 2\gamma)\lambda_i(u)\lambda_{i+1}(-u+\rho_{i+1})}{(2\varepsilon_{i+1}u - \varepsilon_{i+1}\rho_{i+2} + \varpi_{i+2} + 2\gamma)\lambda_{i+1}(u)\lambda_i(-u+\rho_{i+1})}, \quad (1)$$

where $1 \leqslant i < \varkappa$, $\gamma \in \mathbb{C}$, and $\boldsymbol{\lambda}(u) = (\lambda_i(u))_{1\leqslant i\leqslant\varkappa}$ is a highest weight such that the $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$-module $L(\boldsymbol{\lambda}(u))$ is f.d., then $V(\boldsymbol{\mu}(u))$ is f.d.

Conjecture

If the irreducible $\mathcal{B}_{\mathfrak{s},\boldsymbol{\varepsilon}}$-module $V(\boldsymbol{\mu}(u))$ is finite-dimensional, then there exist $\gamma \in \mathbb{C}$ and a highest weight $\boldsymbol{\lambda}(u) = (\lambda_i(u))_{1\leqslant i\leqslant\varkappa}$ such that

1. the equations (1) hold, and
2. the $\mathrm{Y}(\mathfrak{gl}^{\mathfrak{s}}_{m|n})$-module $L(\boldsymbol{\lambda}(u))$ is finite-dimensional.

Classifications

Note that if $\varepsilon_i = \varepsilon_{i+1}$, then

$$\frac{2\varepsilon_i u - \varepsilon_i \rho_{i+1} + \varpi_{i+1} + 2\gamma}{2\varepsilon_{i+1} u - \varepsilon_{i+1} \rho_{i+2} + \varpi_{i+2} + 2\gamma} = 1.$$

We say that $\boldsymbol{\varepsilon}$ is simple if $\#\{i \,|\, \varepsilon_i \neq \varepsilon_{i+1}\} \leqslant 1$.

Theorem

The conjecture is true for the following cases,

1. if $n = 0, 1$ and $\boldsymbol{\varepsilon}$ is arbitrary;
2. if $\mathfrak{s}$ is the standard parity sequence and $\boldsymbol{\varepsilon}$ is simple (and other cases that can be obtained from this by odd reflections).

The case when $n = 0$ and $\boldsymbol{\varepsilon}$ is simple is due to [Molev-Ragoucy'01].

Schur-Weyl duality

- Let $\mathcal{H}^{l}_{\vartheta_1,\vartheta_2}$ be the degenerate affine Hecke algebra of type B/C.
- Let $p = \#\{i \,|\, \varepsilon_i = 1, 1 \leqslant i \leqslant \varkappa\}$.

The following are super generalizations of [Chen-Guay-Ma'14], cf. [Drinfeld'85] [Arakawa'98] [Nazarov'99] [L-Mukhin'20].

Theorem

If $\vartheta_1 = 1$ and $\vartheta_2 = \varpi_1$, then the following holds.

1. Exist a Drinfeld functor $\mathcal{D}^{\varepsilon}_{\mathfrak{s},\boldsymbol{\varepsilon}}$ from the category of $\mathcal{H}^{l}_{\vartheta_1,\vartheta_2}$-modules to the category of $\mathcal{B}_{\mathfrak{s},\boldsymbol{\varepsilon}}$-modules of level l.
2. If $\max\{p, \varkappa - p\} < l$, then the Drinfeld functor $\mathcal{D}^{\varepsilon}_{\mathfrak{s},\boldsymbol{\varepsilon}}$ is an equivalence of categories.
3. Drinfeld functor $\mathcal{D}^{\varepsilon}_{\mathfrak{s},\boldsymbol{\varepsilon}}$ maps simple $\mathcal{H}^{l}_{\vartheta_1,\vartheta_2}$-modules to either 0 or simple $\mathcal{B}_{\mathfrak{s},\boldsymbol{\varepsilon}}$-modules.

Thank you!