Research Methodology for Data Science

Colorization using Optimization

TU Delft

Overview

Lecturer

Klaus Hildebrandt

k.a.hildebrandt@tudelft.nl

EEMCS Building , Room 11.270

Formalities

Lectures:

- March 8, 15 and 22   8:45-10:30h

Assignment:

- Hand-in March 27 via Blackboard
  - Programming assignment in MatLab
  - Writing a short report
- Presentation on March 29, 8:00-11:00
  - Every group gets a 10 min. slot for presentation
  - All members must be present
  - Everyone needs to be able to present and explain everything
  - You only have to be present for your group presentation and 10min. in advance.

Topics

Problem modeling

- Using concepts from linear algebra and optimization

Problem solving

- Sparse matrix structures
- Algorithm for solving problems, in particular sparse linear systems

Hands-on

- Consider concrete applications
- Implement tools yourself using MatLab

Why Bother?

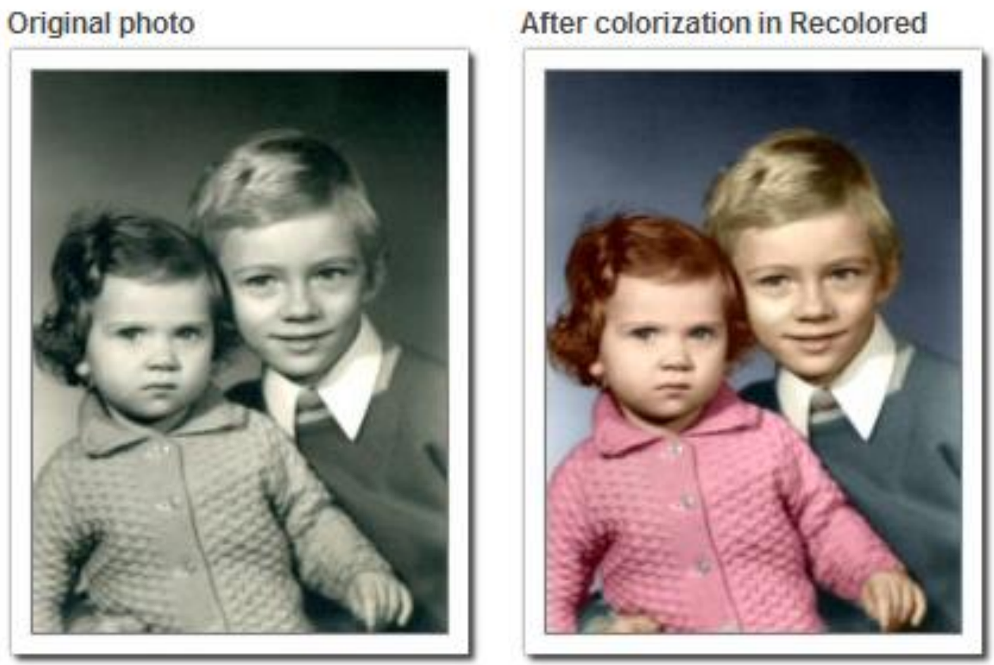
Powerful machinery

- Understand, implement and develop powerful algorithms
  - Many effective algorithms use optimization, sparse matrices …
- Use efficient tools for solving problems
  - Simplify implementation and testing
- Transfer concepts (ideas, algorithms) between fields

Image Colorization/Recolorization

Problem

- Add color to monochrome images or movies

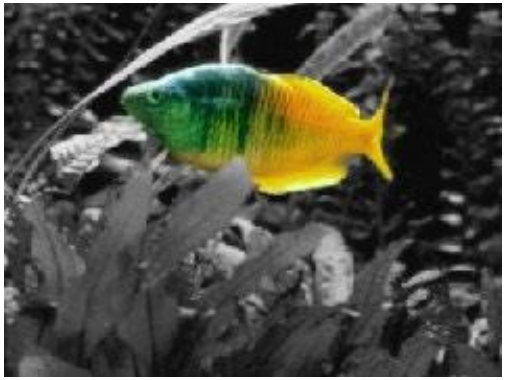


Recolored

Image Recolorization

Problem

- Change the colors in images or videos

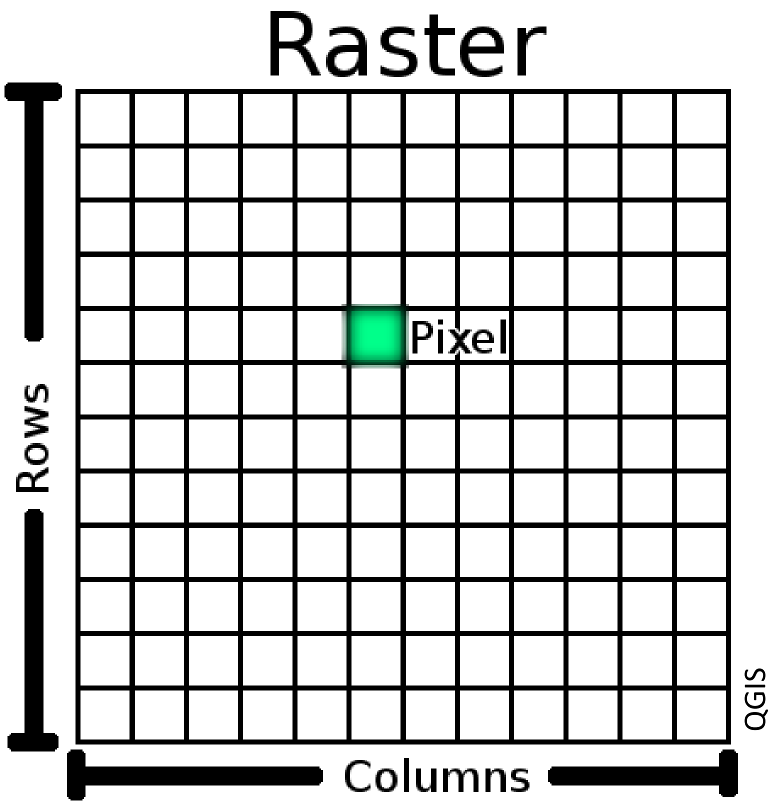


Weiss et al.

Cedeño et al.

Images

Rasterized images



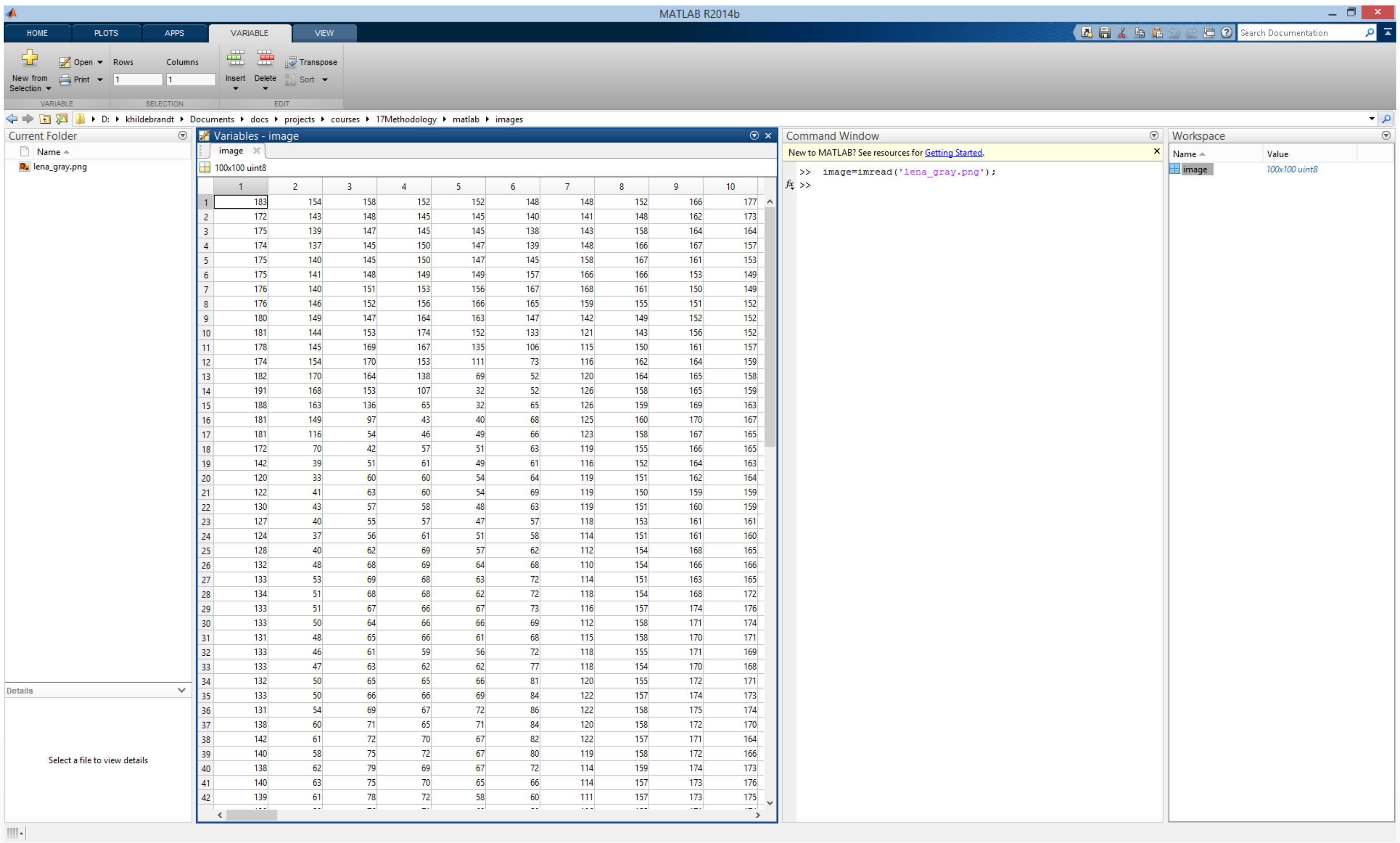
Pixels - Grayscale

Grayscale images

- Every pixel stores one number describing its luminance (light intensity)
- Typically luminance is represented by a
  - number in [0,1] or a
  - 8-bit integer (0 is black, 255 is white)

Images in MatLab

To load an image use: imread('myimage.jpg')

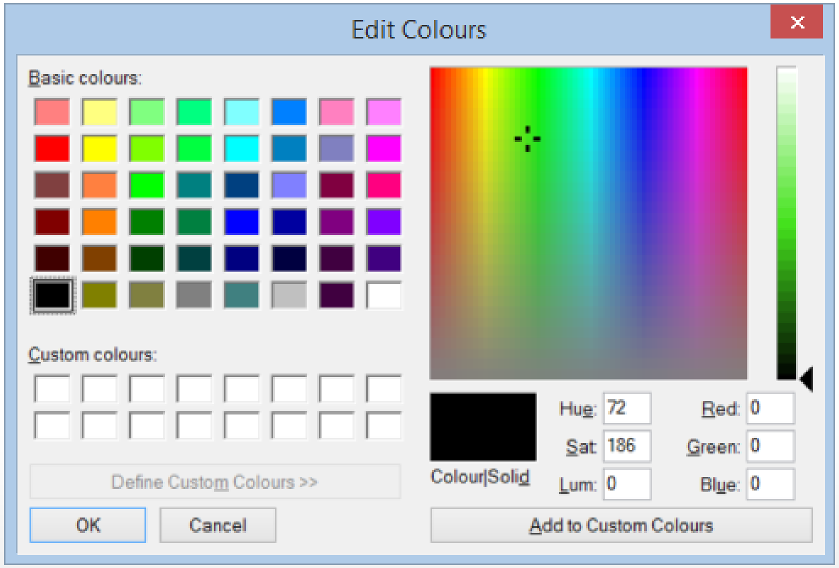


Pixels - Colored

There are various color models

Red-Green-Blue (RGB):

- 3 color values are stored
- Typically
  - each value is in [0,1] or
  - each value is a 8-bit integer

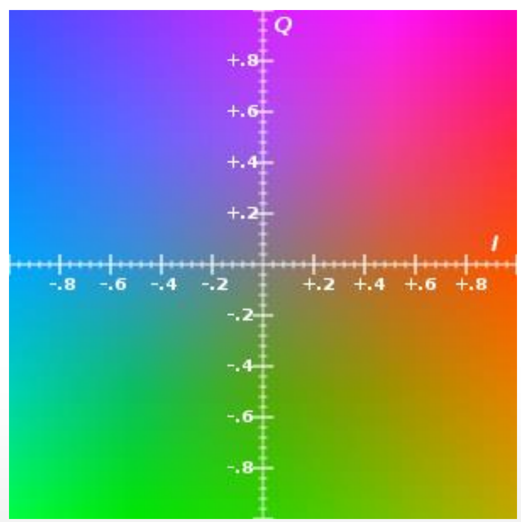


Paint color editor

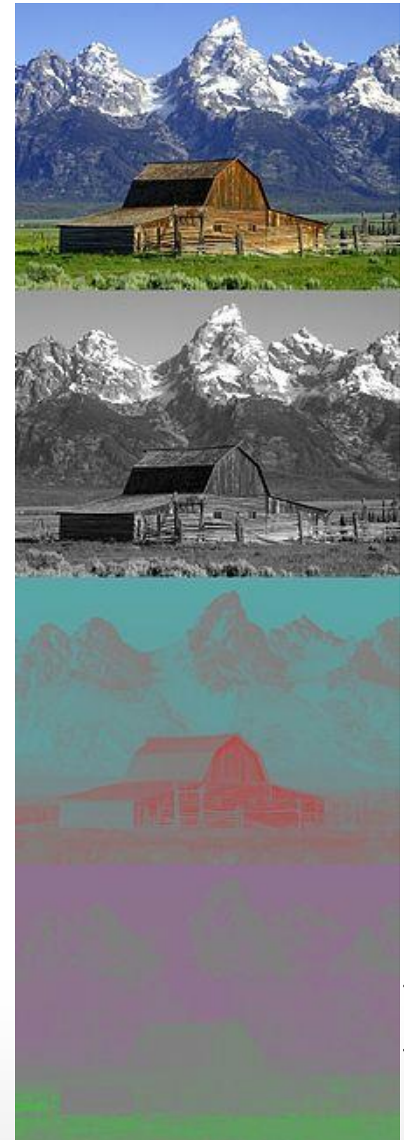
Pixels - Colored

Color space YIQ

- 3 color values are stored
- Y for luminance
- I,Q for chrominance (color)
- YIQ is the color space used by the (analog) NTSC color TV system



IQ for Y=0.5



YIQ

Y

I

Q

Wikipedia

Colors - RGB 2 YIQ

To map from RGB to YIQ

- For RGB values between 0 and 1 and
  Y in [0,1], I,Q in [-0.5,0.5]

$$\begin{bmatrix} Y \\ I \\ Q \end{bmatrix} = \begin{bmatrix} 0.299 & 0.587 & 0.114 \\ 0.596 & -0.274 & -0.322 \\ 0.211 & -0.523 & 0.312 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

In MatLab use

- rgb2ntsc, ntsc2rgb

Colorization using Optimization

Approach

- Given grayscale image and color annotations create colored image
- The scribbled colors are propagated to all pixels

Levin et al.

Propagation of Colors

Propagation

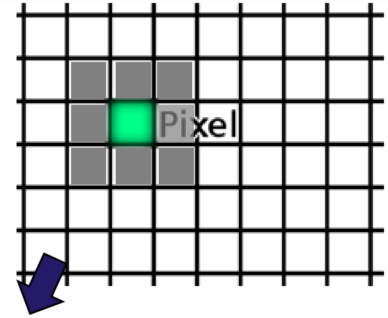
- Neighboring pixels with similar intensities should get similar colors

$$Y \rightarrow I, Q$$

Luminance channel

Color channels

Formulating the Optimization Problem



Notation

- Every pixel gets an index 1,2,…,n
- The Y, I, Q values of the i^{th} pixel are Y_i, I_i, Q_i
- We denote the set of neighbors of pixel i by $N(i)$
- We denote the set of pixels for which colors are given by C and by $\bar{I}_i$ and $\bar{Q}_i$ the given color values

Minimization problem (only for $\boldsymbol{I}$)

Analog for $\boldsymbol{Q}$

Find the color values I_i such that

$$E(I) = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j\in N(i)} w_{ij}(I_i - I_j)\right)^2$$

is minimal and $I_i = \bar{I}_i$ for all $i \in C$

Formulating the Optimization Problem

Generic form

Find minimizer of the objective among all candidates satisfying the constraints!

- Variables: I_i (with $i \notin C$)
- Objective:

$$E(I) = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j \in N(i)} w_{ij}(I_i - I_j)\right)^2$$

- Constraints: I_i=$\bar{I}_i$ for all $i \in C$

What Happens?

Optimization problem

- Variables: I_i (with $i \notin C$)
- Objective: $E(I) = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j \in N(i)} w_{ij}(I_i - I_j)\right)^2$
- Constraints: $I_i = \bar{I}_i$ for all $i \in C$

Minimize difference between color at a pixel and a weighted average of the neighbors!

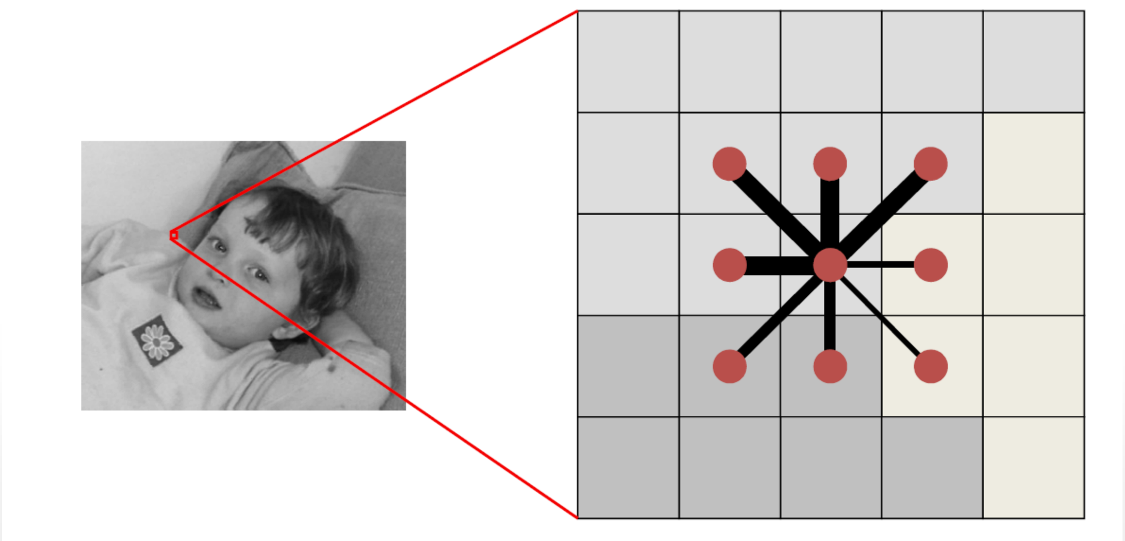
The Weights

The weights

Variance of luminance around i

Normalization of weights

- $\widetilde{w}_{ij} = e^{-(Y_i - Y_j)^2/\sigma_i}$
- $w_{ij} = \widetilde{w}_{ij} / (\sum_{j \in N(i)} \widetilde{w}_{ij})$
- Weights are positive and $\sum_{j \in N(i)} w_{ij} = 1$
- Effect: pixels with similar luminance value have stronger influence on each others color



Solve the Problem!

How can such a problem be solved?

Take a step back!

Polynomials

Quadratic polynomial on $\mathbb{R}$

Quadratic term

- General form: $f(x) = \frac{1}{2} a\, x^2 + b\, x + c$

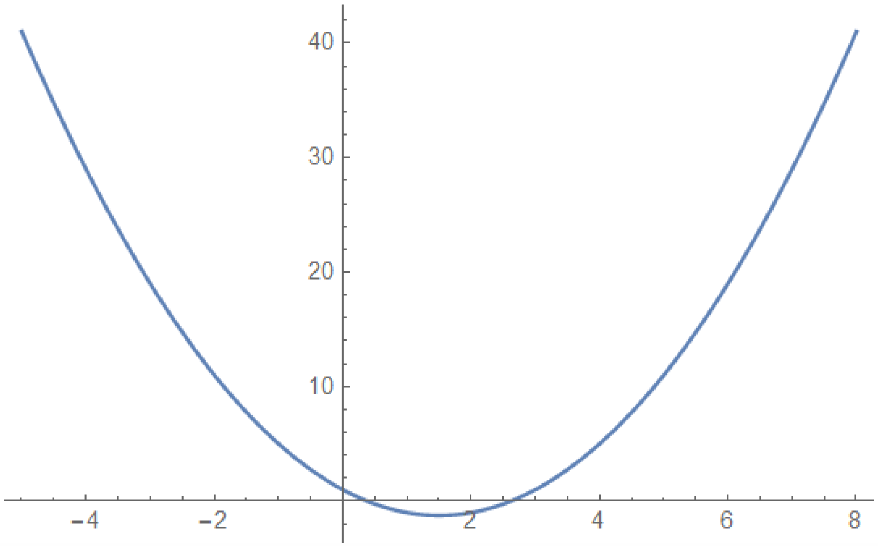
constant term

Linear term

- $a, b, c \in \mathbb{R}$
- Example:

```
In[4]:= Plot[x^2 - 3 x + 1, {x, -5, 8}]
```

Out[4]=



Euler Lagrange Equation

Critical points

- The derivative vanishes
- Example in $\mathbb{R}$

$$f(x) = \frac{1}{2} a\, x^2 + b\, x + c$$

$$\frac{d}{dx} f(x) = ax\ + b$$

- Critical point satisfies the linear equation

$$ax\ + b = 0$$

- Solution x is a minimum if $a > 0$
  and a maximum if $a < 0$

Quadratic Polynomial

In $\mathbb{R}^2$:

- General form:

$$f(x_1, x_2) = a_1 x_1^2 + a_2 x_1 x_2 + a_2 x_2^2 + b_1 x_1 + b_2 x_2 + c$$

Quadratic Polynomial

In $\mathbb{R}^2$:

- General form:

$$f(x_1, x_2) = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + b_1 x_1 + b_2 x_2 + c$$

$$= \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \left(\qquad\qquad \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \left(\qquad\qquad \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c$$

Quadratic Polynomial

In $\mathbb{R}^2$:

- General form:

$$f(x_1, x_2) = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + b_1 x_1 + b_2 x_2 + c$$

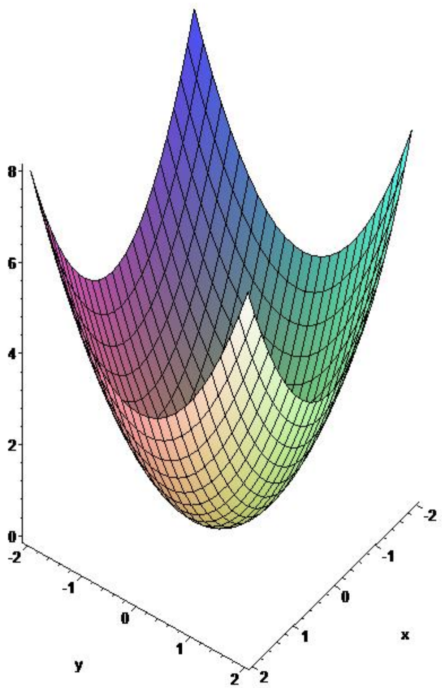
$$= \frac{1}{2}\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 2a_1 & a_2 \\ a_2 & 2a_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c$$

$$= \frac{1}{2} x^T A x + b^T x + c$$

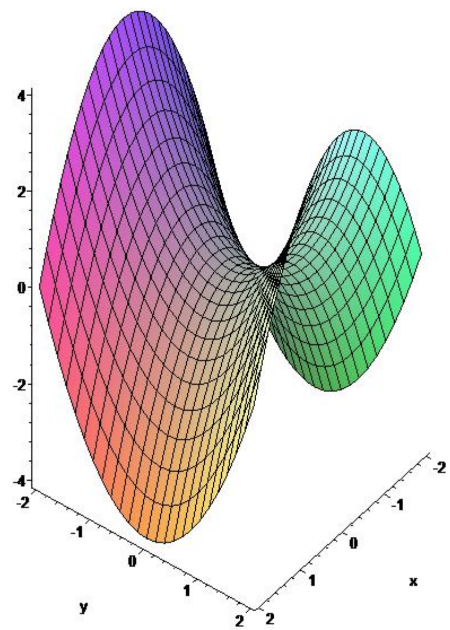
Optimization

Quadratic Polynomials

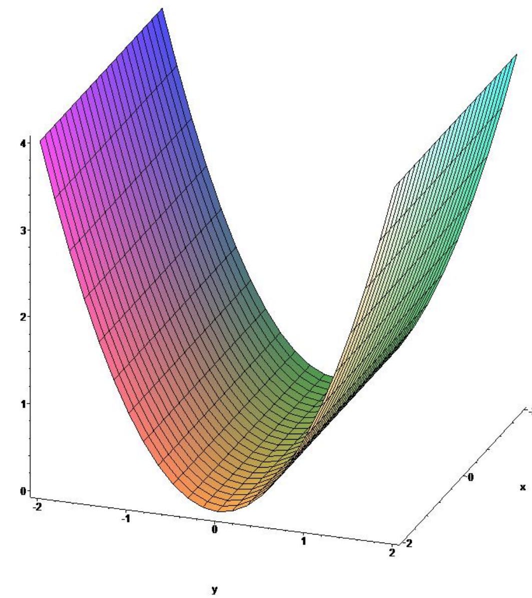
- Example: in $\mathbb{R}^2$



$\lambda_1 = 1$, $\lambda_2 = 1$



$\lambda_1 = 1$, $\lambda_2 = -1$



$\lambda_1 = 1$, $\lambda_2 = 0$

Quadratic Polynomial

In $\mathbb{R}^n$:

- General form:

$$f(x) = \frac{1}{2}x^T Ax + b^T x + c$$

- $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n, c \in \mathbb{R}$
- A is symmetric---- Remark: this is no restriction. If A is not symmetric, the symmetric matrix $\tilde{A} = \frac{1}{2}(A + A^T)$ yields the same function.

Euler Lagrange Equation

A critical points

- The derivative vanishes
- Example in $\mathbb{R}^n$

$$f(x) = \frac{1}{2} x^T A x + b^T\, x + c$$

$$\frac{d}{dx} f(x) = Ax\; + b$$

- The critical point satisfies the linear equation

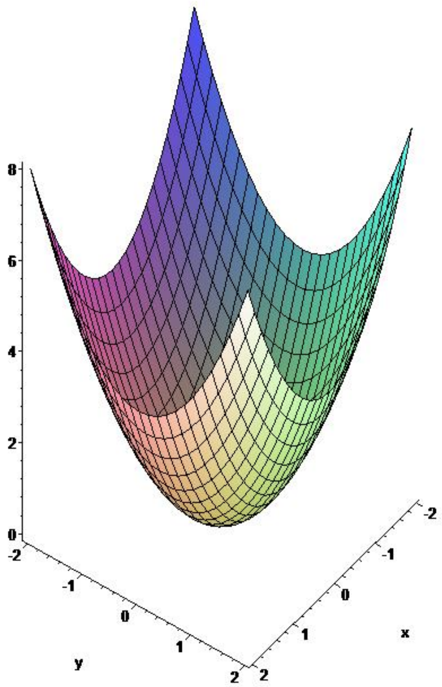
$$Ax\; + b = 0$$

- Solution x is a minimum if A is positive definite (only positive eigenvalues) and a maximum if A is negative definite (only negative eigenvalues)

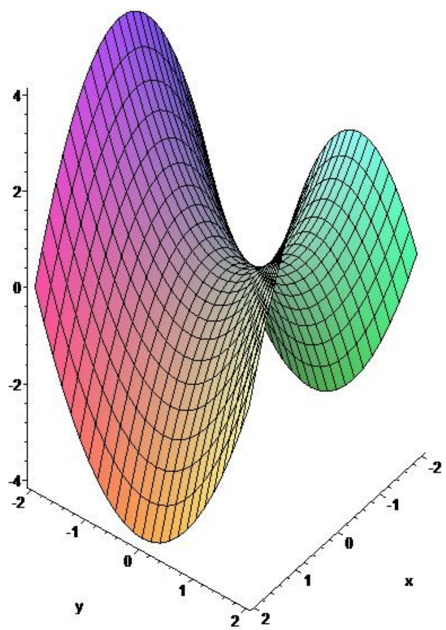
Optimization

Quadratic Polynomials

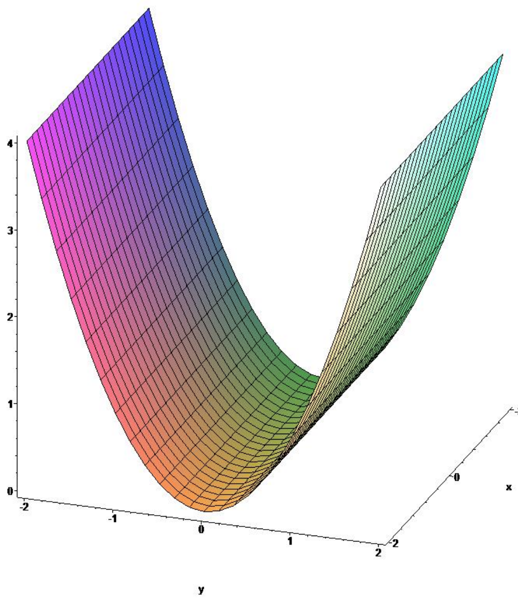
- Example: in $\mathbb{R}^2$



$\lambda_1 = 1, \lambda_2 = 1$



$\lambda_1 = 1, \lambda_2 = -1$



$\lambda_1 = 1, \lambda_2 = 0$

Examples of Quadratic Polynomials

Least squares problems

- Standard least squares problems minimize a convex quadratic polynomial
- 1-d example: $k, l \in \mathbb{R}$

$$f(x) = \frac{1}{2}(l\,x - k)^2 = \frac{1}{2}l^2 x^2 - k\,l\,x + \frac{1}{2}k^2$$

- Standard form of quadratic polynomial:
  - $a = l^2, b = -kl, c = k^2$

$$f(x) = \frac{1}{2}a\,x^2 + b\,x + c$$

Examples of Quadratic Polynomials

Least squares problems

- Standard least squares problems minimize a convex quadratic polynomial

- n-d example: $L \in \mathbb{R}^{m \times n}, k \in \mathbb{R}^m$

$$\|\boldsymbol{x}\|^2 = x^T x$$

$$f(x) = \frac{1}{2} \|L\, x - k\|^2$$

  - $m = n$

$$\left\| \left(\quad \right) \left(\quad \right) - \left(\quad \right) \right\|^2$$

Examples of Quadratic Polynomials

Least squares problems

- Standard least squares problems minimize a convex quadratic polynomial

- n-d example: $L \in \mathbb{R}^{m \times n}, k \in \mathbb{R}^m$

$$\|\boldsymbol{x}\|^2 = x^T x$$

$$f(x) = \frac{1}{2}\|L\,x - k\|^2$$

  - $n > m$

$$\left\|\left(\qquad\qquad\qquad\right)\left(\quad\right) - \left(\quad\right)\right\|^2$$

  - $n < m$

$$\left\|\left(\qquad\qquad\right)\left(\quad\right) - \left(\quad\right)\right\|^2$$

Examples of Quadratic Polynomials

Least squares problems

- Standard least squares problems minimize a convex quadratic polynomial

- n-d example: $L \in \mathbb{R}^{m \times n}, k \in \mathbb{R}^m$

$$\|\boldsymbol{x}\|^2 = x^T x$$

$$f(x) = \frac{1}{2}\|L\,x - k\|^2$$

- Standard form of quadratic polynomial:
  - $A = ?, b = ?, c = ?$

$$f(x) = \frac{1}{2}x^T A x + b^T\,x + c$$

Examples of Quadratic Polynomials

Least squares problems

- Standard least squares problems minimize a convex quadratic polynomial

- n-d example: $L \in \mathbb{R}^{m\times n}, k \in \mathbb{R}^m$

$\|\boldsymbol{x}\|^2 = x^T x$

$$f(x) = \frac{1}{2}\|L\,x - k\|^2 = \frac{1}{2}(L\,x - k)^T(L\,x - k)$$
$$= \frac{1}{2}x^T L^T Lx - k^T Lx + \frac{1}{2}k^T k$$

- Standard form of quadratic polynomial:
  - $A = ?, b = ?, c = ?$

$$f(x) = \frac{1}{2}x^T Ax + b^T\,x + c$$

Examples of Quadratic Polynomials

Least squares problems

- Standard least squares problems minimize a convex quadratic polynomial

- n-d example: $L \in \mathbb{R}^{m\times n}, k \in \mathbb{R}^m$

$\|\boldsymbol{x}\|^2 = x^T x$

$$f(x) = \frac{1}{2}\|L\,x - k\|^2 = \frac{1}{2}(L\,x - k)^T(L\,x - k)$$

$$= \frac{1}{2}x^T L^T L x - k^T L x + \frac{1}{2}k^T k$$

- Standard form of quadratic polynomial:
  - $A = L^T L, b = -L^T k, c = k^T k$

$$f(x) = \frac{1}{2}x^T A x + b^T\, x + c$$

- Solution: $L^T L\mathrm{x} = L^T k$

Back to Colorization

Objective:

$$E(I) = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j\in N(i)} w_{ij}(I_i - I_j)\right)^2$$

All terms are quadratic: $I_i I_j$ or I_i^2.

Hence $E(I)$ is quadratic

$$E(I) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\, I_i I_j = I^T A I$$

But what are the a_{ij}?

Back to Colorization

Objective:

$$E(I) = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j\in N(i)} w_{ij}(I_i - I_j)\right)^2$$

- Consider matrix $L \in \mathbb{R}^{n\times n}$ with
  - $L_{ii} = \sum_{j\in N(i)} w_{ij}$ diagonal entries
  - $L_{ij} = -w_{ij}$ for $j \in N(i)$
  - $L_{ij} = 0$ other entries

Then $(L\,I)_i = \sum_{j\in N(i)} w_{ij}(I_i - I_j)$

$$E(I) = \frac{1}{2}\|L\,I\|^2 = \frac{1}{2} I^T L^T L I$$

Back to Colorization

Objective:

$$E(I) = \frac{1}{2}\|L\,I\|^2 = \frac{1}{2}I^T L^T L I$$

Hence we have to solve $L^T L I = 0$

Are we missing something?

Yes the constraints!

Constraints

Two variants:

- Hard constraints: constraints are exactly satisfied

$$I_i = \bar{I}_i \text{ for all } i \in C$$

- Soft constraints: constraints are not exactly satisfied, but only in least squares sense.
    - Add the term $a\sum_{i\in C}(I_i - \bar{I}_i)^2$ to the objective, where $a > 0$ is a weight.

Soft Constraints

Least squares term

$$a\frac{1}{2}\sum_{i\in S}(I_i - \bar{I}_i)^2$$

To compute the solution, set up the linear system

- Standard form: $S \in \mathbb{R}^{m\times n}$, $\bar{I} \in \mathbb{R}^m$, m number of constraints

$$E_C(I) = \frac{1}{2}\|S\,I - \bar{I}\|^2$$

  - $n > m$

$$\left\| \begin{pmatrix} \quad\quad\quad \end{pmatrix} \begin{pmatrix} \quad \end{pmatrix} - \begin{pmatrix} \quad \end{pmatrix} \right\|^2$$

What is S and what is $\bar{I}$?

Soft Constraints

Selector Matrix

S is a rectangular matrix that selects the pixels that are constrained

- Let $\{c_1, c_2, \ldots, c_m\}$ be the constrained pixels. Then the selector matrix S has the entries

$$S_{ic_i} = 1 \text{ and all other entries are zero}$$

- $\bar{I} \in \mathbb{R}^m$ lists the color values I of the constraint pixels

$$E_C(I) = \frac{1}{2} \|S\, I - \bar{I}\|^2$$

Soft Constraints

Resulting linear system?

- For $E_C(I)$ we have:

$$E_C(I) = \frac{1}{2}||S\,I - \bar{I}||^2 = \frac{1}{2}I^T S^T S I - \bar{I}^T S I + \bar{I}^T \bar{I}$$

- Putting all together: To solve the soft constraint problem we minimize the objective

$$E(I) + aE_C(I)$$

by solving the linear system

$$(L^T L + aS^T S)I = aS^T \bar{I}$$

Hard Constraints

Linear system for hard constraints

- Use Lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$
  - Artificial variables are discarded after solving
- The displacements are the solution of the system

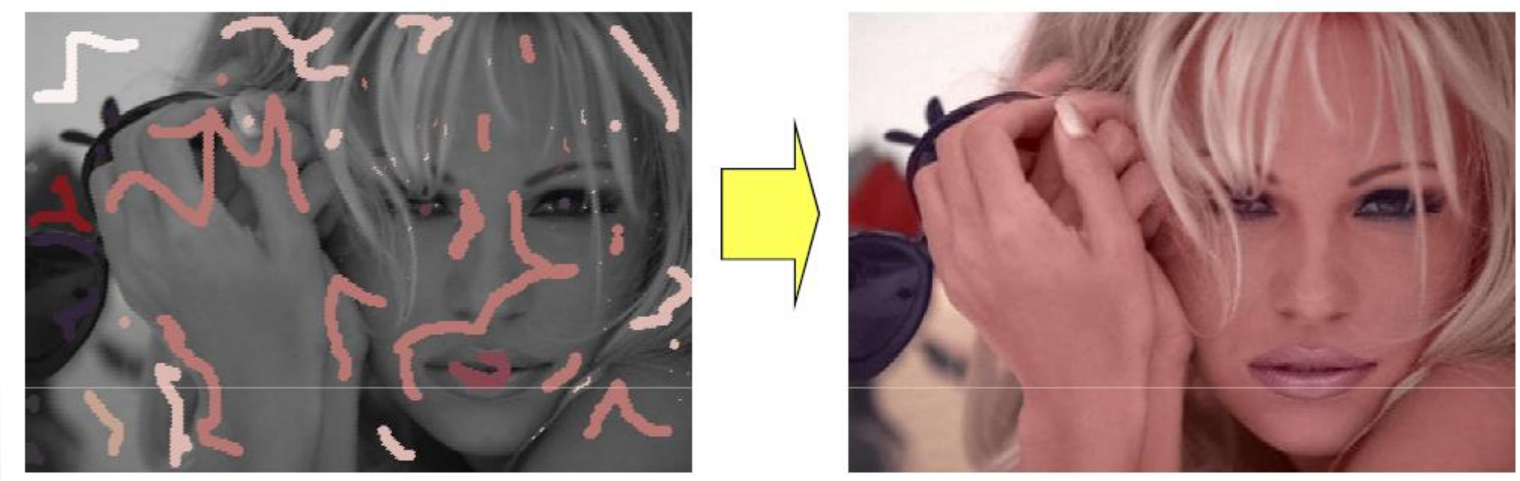
$$\begin{bmatrix} L^T L & S^T \\ S & 0 \end{bmatrix} \begin{bmatrix} I \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{I} \end{bmatrix}$$

- Solve the system for I and λ. I describes the solution

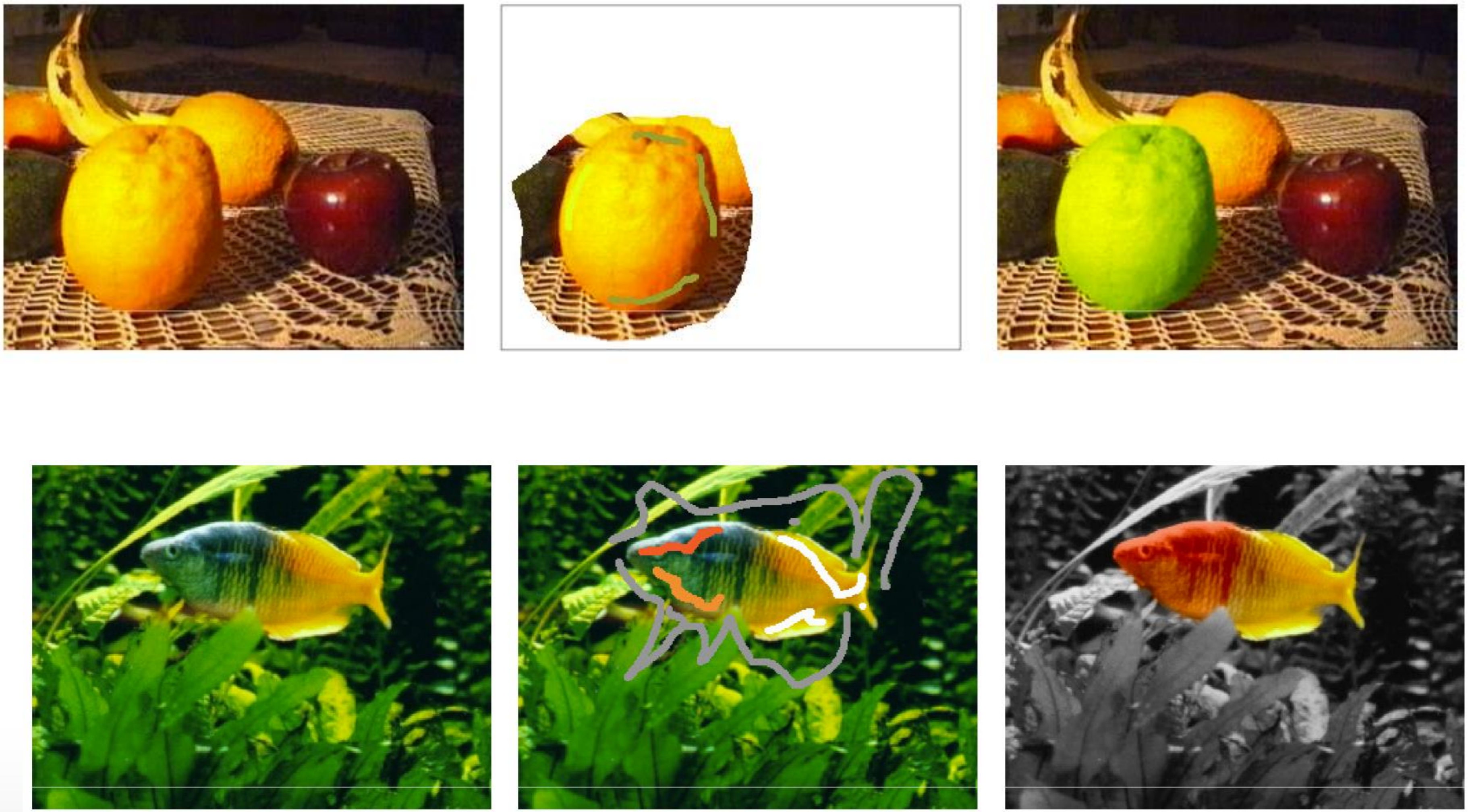
Results – Comparison to Original

Results

Results



Recoloring



Summary of Algorithm

Colorization

- Load image and scribbles
  - Convert to YIQ color space
- Construct matrices L,S and vectors $\bar{I}$ and $\bar{Q}$
- Construct linear systems for I and Q and solve them

$$(L^T L + aS^T S)I = aS^T \bar{I}$$

- Save resulting image
  - Convert resulting image to RGB color space

Literature

Research Paper on Colorization

- Colorization using optimization
  Anat Levin, Dani Lischinski, and Yair Weiss
  ACM SIGGRAPH 2004
  DOI=http://dx.doi.org/10.1145/1186562.1015780

Book Chapter on Quadratic Programs with Constraints

- Numerical Optimization
  Jorge Nocedal, S Wright
  Springer 2006
  Chapter 16: Quadratic Programming