H9.

PROBLEM 1 *Long stable matches*

We showed in class that the stable matching algorithm ends in $O(n^2)$ time with a stable match. Find a set of preference inputs for 6 proposers and reviewers that requires at least 30 (in general, $n \cdot (n-1)$) proposals to be made before a stable matching is found no matter which order the proposals are made. Your answer should list the preferences of the proposers and the reviewers.

Solution:

Proposers:

$$P_1 : R_1 \succ R_2 \succ R_3 \succ R_4 \succ R_5 \succ R_6$$
$$P_2 : R_2 \succ R_3 \succ R_4 \succ R_5 \succ R_1 \succ R_6$$
$$P_3 : R_3 \succ R_4 \succ R_5 \succ R_1 \succ R_2 \succ R_6$$
$$P_4 : R_4 \succ R_5 \succ R_1 \succ R_2 \succ R_3 \succ R_6$$
$$P_5 : R_5 \succ R_1 \succ R_2 \succ R_3 \succ R_4 \succ R_6$$
$$P_6 : R_1 \succ R_2 \succ R_3 \succ R_4 \succ R_5 \succ R_6$$

Reviewers:

$$R_1 : P_2 \succ P_3 \succ P_4 \succ P_5 \succ P_6 \succ P_1$$
$$R_2 : P_3 \succ P_4 \succ P_5 \succ P_6 \succ P_1 \succ P_2$$
$$R_3 : P_4 \succ P_5 \succ P_6 \succ P_1 \succ P_2 \succ P_3$$
$$R_4 : P_5 \succ P_6 \succ P_1 \succ P_2 \succ P_3 \succ P_4$$
$$R_5 : P_6 \succ P_1 \succ P_2 \succ P_3 \succ P_4 \succ P_5$$
$$R_6 : P_1 \succ P_2 \succ P_3 \succ P_4 \succ P_5 \succ P_6$$

Note: The reviewers and proposers marked in red is one of the stable matching found using the algorithm.

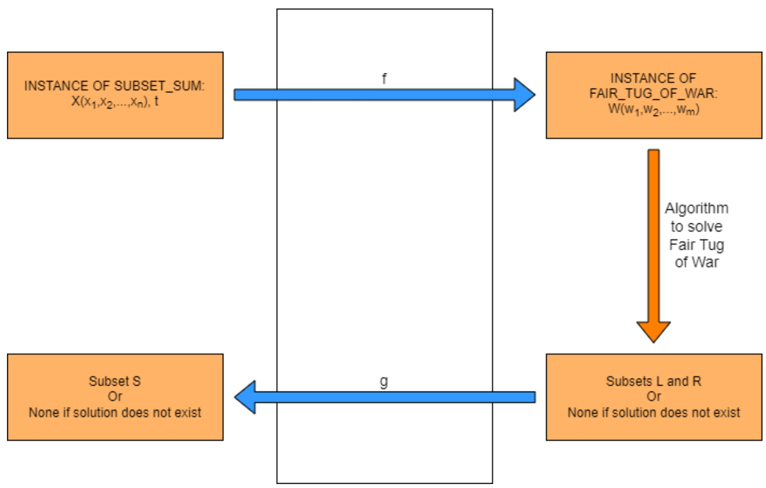
PROBLEM 2 *Fair Tug of War*

Recall the Tug of War problem from DP homework. Consider the closely related Fair Tug of War problem (FTW): given the non-negative weights of n people, $W = (w_1, w_2, \ldots, w_n)$, divide the people into two teams, L, R, such that every person in W is either in L or in R, and such that $\sum_{\ell \in L} w_\ell = \sum_{r \in R} w_r$. In other words, the two teams include everyone, and both teams' weights sum to exactly the same value. Note, the team sizes are not necessarily evenly split as was required in the Tug of War problem.

The subset sum problem is a famous NP-complete problem: given a list of non-negative numbers $L = (x_1, x_2, \ldots, x_n)$ and a target t, find a subset $S \subseteq L$ of numbers that sum to t, i.e., $\sum_{s \in S} s = t$.

Show that our Fair Tug of War problem is NP-complete.

Solution: For this, we show that SUBSETSUM $\leq_p$ FAIRTUGOFWAR.



Reduction from SUBSETSUM **to** FAIRTUGOFWAR**:**
Let $\bar{X} = \sum_{i=i}^{n} x_i$.

1. if $\bar{X} = t$, then SUBSETSUM(X, t) reduces to FAIRTUGOFWAR(X).
2. else, let $x' = |\bar{X} - 2t|$. Let $W = X \cup \{x'\}$, then SUBSETSUM(X, t) reduces to FAIRTUGOFWAR(W).

Proof: Given SUBSETSUM(X, t), 3 cases occur:

1. $\mathbf{t} = \frac{\bar{\mathbf{X}}}{\mathbf{2}}$:
Then the problem becomes the FAIRTUGOFWAR(X) directly as subsets $S \subseteq X$ and $X - S$ has $\sum_{s \in S} s = \sum_{s \in (X-S)} s = \frac{\bar{X}}{2}$.

2. $\mathbf{t} < \frac{\bar{\mathbf{X}}}{\mathbf{2}}$ (Thus $\bar{X} - 2t > 0$):
Thus, $x' = \bar{X} - 2t$. Let $S \subseteq X$ be a solution of SUBSETSUM(X, t).
Then $\sum_{s \in S} s = t$ and $\sum_{s \in (X-S)} s = \bar{X} - t$.
Let $S' = S \cup x'$, then $\sum_{s \in S'} s = t + \bar{X} - 2t = \bar{X} - t = \sum_{s \in (X-S)} s$.

Thus the problem reduces to dividing $W = X \cup x'$ into two subsets with equal sum($= \bar{X} - t$), which is FairTugOfWar(W).

3. $\mathbf{t} > \frac{\bar{\mathbf{X}}}{\mathbf{2}}$ (Thus $\bar{X} - 2t < 0$):
Thus, $x' = 2t - \bar{X}$. Let $S \subseteq X$ be a solution of SubsetSum(X, t).
Then $\sum_{s \in S} s = t$ and $\sum_{s \in (X-S)} s = \bar{X} - t$.
Let $S' = (X - S) \cup x'$, then $\sum_{s \in S'} s = \bar{X} - t + 2t - \bar{X} = t = \sum_{s \in S} s$.
Thus the problem again reduces to dividing $W = X \cup x'$ into two subsets with equal sum($= t$), which is FairTugOfWar(W).

Solution of SubsetSum$(X, t) = S$ **from solution of** FairTugOfWar$(W) = (L, R)$**:**

1. If solution of FairTugOfWar(W) does not exist, then solution of SubsetSum(X, t) also does not exist.
2. If $t = \frac{\bar{X}}{2}$, then S can either be L or R.
3. If $t < \frac{\bar{X}}{2}$, let L be the subset containing element x'. Then $S = L - x'$.
4. If $t > \frac{\bar{X}}{2}$, let L be the subset containing element x'. Then $S = R$.

Proof:

1. We were able to convert an instance of SubsetSum(X, t) to FairTugOfWar(W).
Thus, if SubsetSum(X, t) has a solution, if and only if FairTugOfWar(W) also has a solution.
2. If $t = \frac{\bar{X}}{2}$, then $\sum_{s \in L} s = \sum_{s \in R} s = \frac{\bar{X}}{2} = t$. Thus $S = L$ or $S = R$.
3. If $t < \frac{\bar{X}}{2}$, $\sum_{s \in L} s = X - t$.
Then, $\sum_{s \in L - x'} s = X - t - (X - 2t) = t$. Thus $S = L - x'$.
4. If $t > \frac{\bar{X}}{2}$, $\sum_{s \in R} s = t$ and since R does not contain x', so $S = R$.

Both Reduction and Solution Conversion can be done in $\Theta(n)$ time.

PROBLEM 3 *Ministry of Museums (20pts Extra Credit)*

The ministry of museums wants to ensure that every town either has a museum or is next to a town that has a museum. Given a map of a country consisting of towns and roads between them, the *Museum* problem is to decide whether k museums can be placed in towns so as to satisfy the Ministry's goal.

Let language $L = \left\{ (G,k) \;\middle|\; \begin{array}{l} G \text{ is a graph, and } k \text{ museums can be placed} \\ \text{at verticies such that every vertex either has} \\ \text{a museum or is adjacent to one with a mu-} \\ \text{seum.} \end{array} \right\}$.

1. Argue that $L \in NP$. (Recall, this means showing that it is easy to verify an instance in polynomial time given a witness.)

2. Next show that L is NP-complete by showing a reduction from one of the NP-complete problems we discussed in class.

Solution:

1. Let S be the set of towns having museums. We can verify whether S is a solution of Museum$(G(V,E),k)$ using the following algorithm:

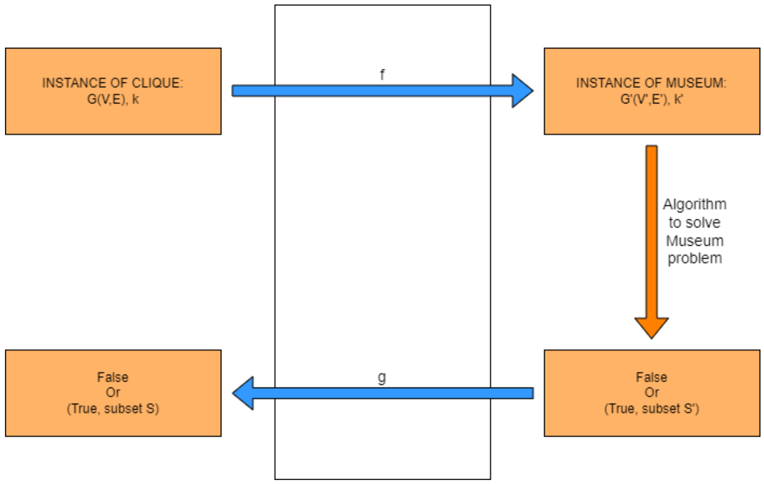
   VerifyMuseum$(G(V,E),k,S)$

   if $|S| \neq k$
       return *False*
   for all $v \in V$
       if $v \notin S$ **and** $\forall u \in adj(v)$, $u \notin S$
           return *False*
   return True

   VerifyMuseum$(G(V,E),k,S)$ checks if for all towns $v \in V$, either v has a museum or any of its neighbours have museum and returning *True* only if all of the towns satisfy this condition.

   The time complexity for VerifyMuseum is $\Theta(E)$ since each pair $(u,v) \in V$ connected by roads are checked atmost twice. Since we can verify whether S is a solution of Museum$(G(V,E),k)$ in a polynomial time, Museum $\in NP$.

2. For this, we show that $\textsc{Clique} \leq_p \textsc{Museum}$.



Reduction from $\textsc{Clique}(G(V,E),k)$ **to** $\textsc{Museum}(G'(V',E'),k')$**:**
Let $G^c(V,E^c)$ be a graph such that for every vertices $u, v \in V$, if $e(u,v) \notin E$ then $e(u,v) \in E^c$.
G has a k-Clique iff G^c has a solution to k-Museum.
$\textsc{Clique}(G(V,E),k)$ can be reduced to $\textsc{Museum}(G^c(V,E^c),k)$

Proof:
Let $S \subseteq V$ be a solution to $\textsc{Clique}(G(V,E),k)$. $|S| = k$.
Consider any vertex $v \in V - S$. Then there exists a vertex $u \in S$ such that $e(u,v) \notin E$.
Thus $e(u,v) \in E^c$.
It means that v is connected to atleast one vertex $u \in S$ in G^c. Since $|S| = k$, we have a subset of V in G^c such that $\forall v \in V$, either $v \in S$ or $\exists u \in adj(v)$ such that $u \in S$. Hence S is a solution of $\textsc{Museum}(G^c(V,E^c),k)$.