The Z-transforms

Definition of the Z-Transform

Let $\{x(n)\}$ be a Sequence defined for $n = 0, \pm 1, \pm 2, \ldots$

Then the two sided Z-transform of the sequence $x(n)$ is defined as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}.$$

where z is a complex variable in general.

If $\{x(n)\}$ is a casual sequence, ie, $x(n) = 0$ for $n < 0$, then the Z-transform reduces to one-sided Z-transform. and its definition is

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}.$$

Unit Sample Sequence:

The Unit Sample Sequence $\delta(n)$ is defined as the sequence with values

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

The unit step sequence $u(n)$ has values

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

and

$$\delta(n-k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

$$u(n-k) = \begin{cases} 1 & \text{for } (n-k) \geq 0 \\ 0 & \text{for } (n-k) < 0 \end{cases}$$

Definition:-

If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n = 0, 1, 2, 3, \ldots,$ to ∞, T being the sampling period, then Z-transform of $f(t)$ is defined as

$$Z[f(t)] = F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}.$$

Properties and Theorems of Z-Transform.

Notations: $Z[f(t)] = F(z)$ and

$Z[x(n)] = X(z)$.

Theorem: 1

The Z-transform is linear. That is

$$Z[af(t)+bg(t)] = aZ[f(t)] + bZ[g(t)]$$

Proof:-

$$Z[af(t)+bg(t)] = \sum_{n=0}^{\infty} [af(nT)+bg(nT)]z^{-n}$$

$$= a\sum_{n=0}^{\infty} f(nT)z^{-n} + b\sum_{n=0}^{\infty} g(nT)z^{-n}$$ (by definition)

$$= aF(z) + bG(z)$$

$$= aZ[f(t)] + bZ[g(t)]$$

(or)

$$Z[a\{x(n)\}+b\{y(n)\}] = \sum_{n=0}^{\infty} [ax(n)+by(n)]z^{-n}$$

$$= a\sum_{n=0}^{\infty} x(n)z^{-n} + b\sum_{n=0}^{\infty} y(n)z^{-n}$$ (by defn).

$$= aX(z) + bY(z)$$

$$= aZ\{x(n)\} + bZ\{y(n)\}$$

Theorem : 2

$$Z[\delta(n)] = \sum_{n=\infty}^{\infty} \delta(n) z^{-n}$$

$$= 1 \quad \text{(by the definition of unit sample sequence)}$$

Theorem : 3

$$Z[u(n)] = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} \quad \text{(by the definition of unit step sequence)}$$

$$= \sum_{n=0}^{\infty} 1/z^{n}$$

$$= 1 + 1/z + 1/z^2 + \cdots$$

$$= (1 - 1/z)^{-1} \quad \text{if } |1/z| < 1$$

$$= \left(\frac{z-1}{z}\right)^{-1} \quad \text{if } 1 < |z|$$

$$= \frac{z}{z-1} \quad \text{if } |z| > 1$$

[we know that $(1-x)^{-1} = 1 + x + x^2 + \cdots$].

Theorem: 4 If $Z\{f(t)] = F(z)$

then $Z[a^n f(t)] = F(z/a)$

<u>Proof :</u>

$$Z\left[a^n f(t)\right] = \sum_{n=0}^{\infty} a^n f(nT) \cdot z^{-n} \qquad \text{(by defn.)}$$

$$= \sum_{n=0}^{\infty} f(nT) \cdot \left(z/a\right)^{-n}$$

$$= F\left(z/a\right)$$

(or)

If $Z\{x(n)\} = X(z)$ then $Z\{a^n x(n)\} = X(z/a)$

<u>Proof :</u>

$$Z\left[a^n x(n)\right] = \sum_{n=0}^{\infty} a^n x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) \left(z/a\right)^{-n}$$

$$= X\left(z/a\right).$$

Note: Theorem 4: we can say "<u>Dampling rule</u>."

<u>Theorem: 5</u>

$$Z\left[a^n u(n)\right] = \frac{z}{z-a} \quad \text{if } |z| > a.$$

<u>Proof :</u>

$$Z\left[a^n u(n)\right] = \sum_{n=0}^{\infty} a^n u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \qquad \text{by defn. of (unit step seq.)}$$

$$= \sum_{n=0}^{\infty} \left(a/z\right)^n$$

$$= 1 + a/z + (a/z)^2 + \cdots$$

$$= (1 - a/z)^{-1} \quad \text{if } |a/z| < 1$$

$$= \left(\frac{z-a}{z}\right)^{-1} \quad \text{if } |a| < |z|$$

$$= \frac{z}{z-a} \quad \text{if } |z| > |a|.$$

Theorem : 6

$$Z[n f(t)] = -z \frac{dF(z)}{dz}$$

Proof :-

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad \text{(defn.)}$$

$$\frac{dF(z)}{dz} = \frac{d}{dz} \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\,(-n)\, z^{-n-1}$$

$$= -\frac{1}{z} \underbrace{\sum_{n=0}^{\infty} n f(nT) z^{-n}}$$

$$= \frac{-1}{z} Z[n f(t)]$$

$$\therefore -z \frac{dF(z)}{dz} = Z[n f(t)]$$

(ie). $$Z[n f(t)] = -z \frac{dF(z)}{dz}$$

$$= 1 + a/z + (a/z)^2 + \cdots$$

$$= (1 - a/z)^{-1} \quad \text{if } |a/z| < 1$$

$$= \left(\frac{z-a}{z}\right)^{-1} \quad \text{if } |a| < |z|$$

$$= \frac{z}{z-a} \quad \text{if } |z| > |a|.$$

Theorem : 6

$$Z[n f(t)] = -z \frac{dF(z)}{dz}$$

Proof :-

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad \text{(defn.)}$$

$$\frac{dF(z)}{dz} = \frac{d}{dz} \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\,(-n)\, z^{-n-1}$$

$$= -\frac{1}{z} \underbrace{\sum_{n=0}^{\infty} n f(nT) z^{-n}}$$

$$= \frac{-1}{z} Z[n f(t)]$$

$$\therefore -z \frac{dF(z)}{dz} = Z[n f(t)]$$

(ie). $Z[n f(t)] = -z \dfrac{dF(z)}{dz}$

Theorem: 7

$$Z\left[f(t+kT)\right] = Z\left[f((n+k)T)\right] =$$

$$z^{k}\left[F(z) - f(0.T) - \frac{f(1.T)}{z} - \frac{f(2.T)}{z^{2}} - \dots - \frac{f((k-1)T)}{z^{k-1}}\right]$$

Proof:- $$Z\left[f((k+n)T)\right] = \sum_{n=0}^{\infty} f((n+k)T)\, z^{-n} \quad \text{(by defn.)}$$

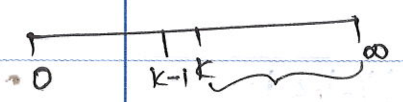
$$= \sum_{n=0}^{\infty} f((n+k)T) .\, z^{-n}\, z^{k} .\, z^{-k}$$

$$= z^{k} \sum_{n=0}^{\infty} f((n+k)T)\, z^{-(n+k)}$$

Put $n+k = m$.

$$= \; :$$

$$= z^{k} \sum_{m=k}^{\infty} f(mT)\, z^{-m}$$



$$= z^{k}\left[\underbrace{\sum_{m=0}^{\infty} f(mT) z^{-m}} - \sum_{m=0}^{k-1} f(mT) z^{-m}\right]$$

$$= z^{k}\left[F(z) - f(0) - \frac{f(T)}{z} - \frac{f(2T)}{z^{2}} - \dots - \frac{f((k-1)T)}{z^{k-1}}\right]$$

Note:-

If $f((n+k)T)$ is denoted by f_{n+k}, then

$$Z\left[f(t+kT)\right] = Z\left[f_{n+k}\right]$$

$$= z^{k}\left[F(z) - f_0 - \frac{f_1}{z} - \frac{f_2}{z^{2}} - \dots \frac{f_{k-1}}{z^{k-1}}\right]$$

Z- Transform of Standard Functions

Find the Z- transform of sequence $\{x(n)\}$ or $\{f_n\}$.
where $x(n)$ is given by

(1) $x(n) = k$	(6) $x(n) = \sin n\theta$	(11) $x(n) = n^2$
(2) $x(n) = (-1)^n$	(7) $x(n) = \cos n\theta$	
(3) $x(n) = a^n$	(8) $x(n) = r^n \sin n\theta$	
(4) $x(n) = n$	(9) $x(n) = r^n \cos n\theta$	
(5) $x(n) = na^n$	(10) $x(n) = n(n-1)$	

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

(1)
$$Z[k] = \sum_{n=0}^{\infty} k z^{-n} = k \sum_{n=0}^{\infty} z^{-n}$$

$$= k \sum_{n=0}^{\infty} 1/z^n$$

$$= k\left[1 + 1/z + 1/z^2 + \cdots\right]$$

$$= k\left[1 - 1/z\right]^{-1} \text{ if } |1/z| < 1$$

$$= k\left[\frac{z-1}{z}\right]^{-1} \text{ if } 1 < |z|$$

$$= \frac{kz}{z-1} \text{ if } |z| > 1.$$

Note:-

$$Z[1] = \frac{z}{z-1} \text{ if } |z| > 1$$

2) $Z\{(-1)^n\} = \sum_{n=0}^{\infty} (-1)^n z^{-n} = \sum_{n=0}^{\infty} \left(-1/z\right)^n$

$= 1 - 1/z + (1/z)^2 - (1/z)^3 + \cdots$

$= (1 + 1/z)^{-1}$ if $|1/z| < 1$

$= \left(\frac{z+1}{z}\right)^{-1}$ if $1 < |z|$.

$= \frac{z}{z+1}$ if $|z| > 1$

Note: $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ if $|x| < 1$.

③ $Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a/z)^n$

$= 1 + a/z + (a/z)^2 + \cdots$

$= (1 - a/z)^{-1}$ if $|a/z| < 1$

$= \left(\frac{z-a}{z}\right)^{-1}$ if $|a| < |z|$

$= \frac{z}{z-a}$ if $|z| > |a|$.

④ $Z[n] = \sum_{n=0}^{\infty} n z^{-n} = \sum_{n=0}^{\infty} n/z^n$

$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \cdots$

$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \cdots\right]$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right)^{-2} \quad \text{if} \quad |1/z| < 1$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right)^{-2} \quad \text{if} \quad 1 < |z|.$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right]^2 \quad \text{if} \quad |z| > 1$$

$$= \frac{1}{z}\left[\frac{z^2}{(z-1)^2}\right] \quad \text{if} \quad |z| > 1$$

$$= \frac{z}{(z-1)^2} \quad \text{if} \quad |z| > 1$$

Note:- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

if $|x| < 1$.

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(5) $$Z[na^n] = \sum_{n=0}^{\infty} n a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} n (a/z)^n$$

$$\Rightarrow a/z + 2(a/z)^2 + 3(a/z)^3 + \cdots$$

$$= \frac{a}{z}\left[1 + 2(a/z) + 3(a/z)^2 + \cdots\right]$$

$$= \frac{a}{z}\left[(1 - a/z)^{-2}\right] \quad \text{if} \quad |a/z| < 1$$

$$= \frac{a}{z}\left[\left(\frac{z-a}{z}\right)^{-2}\right] \quad \text{if} \quad |a|<|z|$$

$$= \frac{a}{z}\left[\left(\frac{z}{z-a}\right)^{2}\right] \quad \text{if} \quad |z|>|a|.$$

$$= \frac{a}{z}\left(\frac{z^2}{(z-a)^2}\right) \quad \text{if} \quad |z|>|a|$$

$$= \frac{az}{(z-a)^2} \quad \text{if} \quad |z|>|a|.$$

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6 & 7 $\Rightarrow$.

we know that $Z[a^n] = \frac{z}{z-a}$ if $|z|>|a|$.

Taking $a = e^{i\theta}$

$$Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}}$$

$$z[(e^{in\theta})] = \frac{z}{z-e^{i\theta}}.$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z-\cos\theta)-i\sin\theta} \times \frac{z-\cos\theta+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$= \frac{z[z-\cos\theta+i\sin\theta]}{(z-\cos\theta)^2-(i\sin\theta)^2}$$

$$= \frac{z(z - \cos\theta + i\sin\theta)}{z^2 + \cos^2\theta - 2z\cos\theta + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + i z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating real and imaginary parts, we get

$$Z[\cos n\theta] = \frac{z[z-\cos\theta]}{z^2 - 2z\cos\theta + 1} \quad \text{if} \quad |z| > 1$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \quad \text{if} \quad |z| > 1.$$

Note:-

$$|z| > |e^{i\theta}|$$

$$|z| > |\cos\theta + i\sin\theta|$$

$$|z| > \sqrt{\cos^2\theta + \sin^2\theta}$$

$$|z| > 1.$$

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(8) & (9) $\Rightarrow$:--

$$\underbrace{x(n) = r^n \sin n\theta \quad \& \quad x(n) = r^n \cos n\theta.}$$

Exercise:

Note:

$$Z[a^n] = \frac{z}{z-a} \quad \text{if} \quad |z| > |a|$$

Put $a = re^{i\theta}$

$$Z[r^n \cos n\theta] = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} \quad \text{if } |z| > |r|$$

Answer:

$$Z[r^n \sin n\theta] = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2} \quad \text{if } |z| > |r|$$

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(10) $Z[n^2] = Z[n \cdot n] =$

WKT $Z[n f(t)] = -z \frac{d}{dz} F(z)$

$$= -z \frac{d}{dz}\left[Z[f(t)]\right] = -z \frac{d}{dz}\left[Z[n]\right]$$

$$= -z \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) \quad u/v$$

$$= -z \left[\frac{(z-1)^2(1) - z(2(z-1)(1))}{(z-1)^4}\right] \quad (2z-2)$$

$$= -z \left[\frac{z^2 + 1 - \cancel{2z} - 2z^2 + \cancel{2z}}{(z-1)^4}\right]$$

$$= -z \left[\frac{-z^2 + 1}{(z-1)^4}\right]$$

$$= +z \left[\frac{z^2 - 1}{(z-1)^4}\right]$$

$$= z\left(\frac{(z+1)\cancel{(z-1)}}{(z-1)^{\cancel{4}}}\right) = \frac{z(z+1)}{(z-1)^3}.$$

(11)

$$Z[n(n-1)] = Z(n^2-n)$$
$$= Z(n^2) - Z(n)$$
$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$
$$= \frac{z(z+1) - z(z-1)}{(z-1)^3}$$
$$= \frac{\cancel{z^2}+z-\cancel{z^2}+z}{(z-1)^3} = \frac{2z}{(z-1)^3}.$$

Find the Z-transform of :

(1) $\{1/n\}$

(2) $\left\{\cos\frac{n\pi}{2}\right\}$

(3) $\left\{\frac{1}{n(n+1)}\right\}, n \geq 1$

(4) $u(n-1)$

(5) $3^n \delta(n-1)$

(6) $\cos\frac{n\pi}{2}\, u(n)$

(7) $\delta(n-k)$

(8) $ab^n \ (a \neq 0, b \neq 0)$

(9) $x(n) = \begin{cases} n, & n \geq 0 \\ 0 & n < 0 \end{cases}$

(10) $\frac{(n+1)(n+2)}{2}$

(11) $x(n) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{otherwise} \end{cases}$

(12) $x(n) = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n \leq 0 \end{cases}$

(13) $x(n) = \begin{cases} \frac{a^n}{n!} & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

① $Z\left\{ 1/n \right\} = \sum_{n=1}^{\infty} 1/n \, z^{-n}$

$$= \sum_{n=1}^{\infty} \frac{1}{nz^n} = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \dots + \text{to } \infty$$

$$= \frac{1}{z} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots$$

$$= -\log\left(1 - 1/z\right) \quad \text{if } |1/z| < 1$$

$$= \log\left(1 - 1/z\right)^{-1} \quad \text{if } 1 < |z|$$

$$= \log\left(\frac{z-1}{z}\right)^{-1} \quad \text{if } |z| > 1$$

$$\log\left(\frac{z}{z-1}\right) \quad \text{if } |z| > 1$$

Note: $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad |x| < 1.$

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② $Z\left\{ \cos\frac{n\pi}{2} \right\} = \sum_{n=0}^{\infty} \cos\frac{n\pi}{2} \cdot z^{-n}$

$$= 1 - \frac{1}{z^2} + \frac{1}{z^4} - \dots \text{to } \infty$$

$$= \left(1 + 1/z^2\right)^{-1} = \frac{z^2}{z^2+1} \quad \text{if } |z| > 1.$$

Note:

$n=0$, $\cos 0 = 1$ $\qquad$ $n=3$, $\cos\frac{3\pi}{2} = 0$.

$n=1$ $\cos\pi/2 = 0$ $\qquad$ $n=4$, $\cos\frac{4\pi}{2} = 1$

$n=2$ $\cos\pi$ -1 $\qquad$ ⋮

3) Let $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$.

$$\frac{1}{n\cancel{(n+1)}} = \frac{A(n+1)+Bn}{n\cancel{(n+1)}}$$

Put $n=-1$; $1 = -B$ $\therefore \boxed{B=-1}$ | Put $n=0$, $1 = A$ $\therefore \boxed{A=1}$

$$\frac{1}{n(n+1)} = \frac{+1}{n} - \frac{1}{n+1}.$$

$$Z\left\{\frac{1}{n(n+1)}\right\} = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} - \sum_{n=1}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \log\left(\frac{z}{z-1}\right) - \left[\sum_{n=1}^{\infty} \frac{1}{(n+1)z^n}\right]$$

$$= \log\left(\frac{z}{z-1}\right) - \left[\frac{1}{2z^1} + \frac{1}{3z^2} + \frac{1}{4z^3} + \cdots\right]$$

$$= \log\left(\frac{z}{z-1}\right) - \left[\frac{1}{2}\left(\frac{1}{z}\right)^1 + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= \log\left(\frac{z}{z-1}\right) - \frac{z}{z}\left[1-1+\frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right) + \cdots\right]$$

$$= \log\left(\frac{z}{z-1}\right) - z\left[-\frac{1}{z}+\frac{1}{z}+\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\underbrace{\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right)}$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left[-\log\left(\frac{z-1}{z}\right)\right]$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left(\log\left(\frac{z-1}{z}\right)^{-1}\right).$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left(\log\left(\frac{z}{z-1}\right)\right].$$

$$= (1-z)\log\left(\frac{z}{z-1}\right) + 1.$$

4) $Z\left[u(n-1)\right] = \sum_{n=1}^{\infty} u(n-1) \cdot z^{-n}$

we know that

$$\left[u(n-k) = \begin{cases} 1 & \text{for } (n-k) \geq 0 \\ 0 & \text{for } (n-k) < 0 \end{cases} \right]$$

$$\Rightarrow \sum_{n=1}^{\infty} z^{-n} = \sum_{n=1}^{\infty} \frac{1}{z^n}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left(1 - 1/z\right)^{-1} \quad \text{if } |1/z| < 1.$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right)^{-1} \quad \text{if } 1 < |z|$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right] \quad \text{if } |z| > 1.$$

$$= \frac{1}{z-1} \quad \text{if } |z| > 1.$$

we know that

$$\left[\delta(n-k) = \begin{cases} 1 & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}\right]$$

5) $Z[3^n \delta(n-1)] = \sum_{n=0}^{\infty} 3^n \delta(n-1) z^{-n} =$

here $k=1$

$= 3^1 z^{-1} = 3/z$.

6) $Z\left[\cos\frac{n\pi}{2} u(n)\right] = \sum_{n=0}^{\infty} u(n) \cos\frac{n\pi}{2} z^{-n}$

$= \sum_{n=0}^{\infty} \cos\frac{n\pi}{2} z^{-n}$

[Refer 2nd problem in Page No: 15]

7) $Z[\delta(n-k)] = \sum_{n=0}^{\infty} \delta(n-k) z^{-n}$

$= \frac{1}{z^k}$, if k is a positive integer.

8) $Z[ab^n] = \sum_{n=0}^{\infty} ab^n z^{-n}$

$= \sum_{n=0}^{\infty} a\left(\frac{b}{z}\right)^n = a \sum_{n=0}^{\infty} (b/z)^n$

$= a\left[1 + \frac{b}{z} + \left(\frac{b}{z}\right)^2 + \dots \text{to } \infty\right]$

$= a\left[1 - b/z\right]^{-1}$ if $|b/z| < 1$

$= a\left[\frac{z-b}{z}\right]^{-1}$ if $|b| < |z|$

$$= a\left[\frac{z}{z-b}\right] \quad \text{if} \quad |z|>|b|.$$

$$= \frac{az}{z-b} \quad \text{if} \quad |z|>|b|.$$

9) $$Z[x(n)] = \sum_{n=0}^{\infty} n \cdot z^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$$

Refer page NO: 9
problem NO: 4

Note: Answer: $\frac{z}{(z-1)^2}$ if $|z|>1$

10) $$Z\left[\frac{(n+1)(n+2)}{2}\right] = \frac{1}{2} Z\left[n^2+3n+2\right]$$

$Z(2) = 2Z(1)$

$$= \frac{1}{2}\left[Z[n^2] + 3Z(n) + Z(2)\right]$$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}\right] \quad \text{if } |z|>1.$$

11) $$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$= z^{-k} = \frac{1}{z^k}.$$

12) $$Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{0} z^{-n}$$

$$= \sum_{n=-\infty}^{0} z^{-n}$$

$$= 1 + z + z^2 + \dots$$

$$= (1-z)^{-1} \text{ if } |z| < 1$$

$$= \frac{1}{1-z} \text{ if } |z| < 1$$

13) $$Z[x(n)] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \dots$$

$$= 1 + \frac{(a/z)}{1!} + \frac{(a/z)^2}{2!} + \frac{(a/z)^3}{3!} + \dots$$

$$= e^{a/z}$$

Note: $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots$

Find the Z-transform of f(t) where f(t) is given by

① t | ④ $\sin\omega t$ (HW)
② e^{-at} | ⑤ $\cos\omega t$
③ e^{at} (HW) | ⑥ t^k

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}.$$

① $$Z[t] = \sum_{n=0}^{\infty} (nT) z^{-n}$$

$$= T \sum_{n=0}^{\infty} n z^{-n}$$

$$= T \underbrace{\sum_{n=0}^{\infty} n \cdot \frac{1}{z^n}}$$

$$= \frac{Tz}{(z-1)^2}$$ [Refer page No: 9 problem no: 4]

② $$Z[e^{-at}] = \sum_{n=0}^{\infty} e^{-anT} \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{-aT})^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^{-aT}}{z}\right)^n$$

$$= 1 + \left(\frac{e^{-aT}}{z}\right)^1 + \left(\frac{e^{-aT}}{z}\right)^2 + \cdots$$

$$= \left(1 + \frac{e^{-aT}}{z} + \left(\frac{e^{-aT}}{z}\right)^2 + \cdots\right)$$

$$= \left(1 - \frac{e^{-aT}}{z}\right)^{-1} \text{ if } \left|\frac{e^{-aT}}{z}\right| < 1$$

$$= \left(\frac{z - e^{-aT}}{z}\right)^{-1} \text{ if } |e^{-aT}| < |z|$$

$$= \left(\frac{z}{z-e^{-aT}}\right) \text{ if } |z| > |e^{-aT})$$

③ III^{ly} $Z[e^{at}] = \dfrac{z}{z-e^{aT}}$ if $|z| > |e^{aT}|$. (Exercise).

⑤ $Z[\cos\omega t] = \displaystyle\sum_{n=0}^{\infty} \cos n\omega T \, z^{-n}$

$$= Z[\cos n\theta] \text{ where } \theta = \omega T$$

$$= \frac{z[z - \cos\omega T]}{z^2 - 2z\cos\omega T + 1}$$ [Refer page No: 11 & 12]

(or)

$$Z[\cos\omega t] = Z\left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$$

$$= \frac{1}{2}\left[Z(e^{i\omega t}) + Z[e^{-i\omega t})\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{i\omega T}} + \frac{z}{z - e^{-i\omega T}}\right]$$

$$= \frac{1}{2}\left[\frac{z(z-e^{-i\omega T}) + z\,(z-e^{i\omega T})}{(z-e^{i\omega T})(z-e^{-i\omega T})}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-i\omega T} + z^2 - ze^{i\omega T}}{z^2 - ze^{-i\omega T} - ze^{i\omega T} + 1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z\left[e^{i\omega T} + e^{-i\omega T}\right]}{z^2 - z\left[e^{i\omega T} + e^{-i\omega T}\right] + 1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z\,[2\cos\omega T)}{z^2 - z(2\cos\omega T) + 1}\right]$$

$$\Rightarrow \frac{2}{2}\left[\frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1}\right]$$

$$= \frac{z\,[z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$$

(4) $Z[\sin\omega t] = \left.\dfrac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}\right\} \rightarrow$ Exercise.

6) $Z[t^k]$

$$Z[t^k] = -Tz \frac{d}{dz} Z[t^{k-1}] \quad \text{(To prove)}$$

Pf:

$$Z[t^{k-1}] = \sum_{n=0}^{\infty} (nT)^{k-1} z^{-n}$$

$$\frac{d}{dz} Z[t^{k-1}] = \sum_{n=0}^{\infty} \frac{d}{dz} (nT)^{k-1} z^{-n}$$

$$= \sum (nT)^{k-1} (-n) z^{-(n+1)}$$

$$= -(Tz)^{-1} \sum_{n=0}^{\infty} (nT)^k z^{-n}$$

$$\therefore -Tz \frac{d}{dz} Z[t^{k-1}] = Z[t^k]$$

$$\therefore Z[t^k] = -Tz \frac{d}{dz} Z[t^{k-1}] \quad k = 1, 2, 3, \ldots$$

When $t = 1$ $\quad Z[t] = -Tz \frac{d}{dz} Z[1] = -Tz \frac{d}{dz} \left(\frac{z}{z-1}\right)$

$$= \frac{Tz}{(z-1)^2}$$

When $t = 2$ $\quad Z[t^2] = -Tz \frac{d}{dz} Z[t]$

$$= -Tz \frac{d}{dz} \left(\frac{Tz}{(z-1)^2}\right)$$

$$= \frac{T^2 z(z+1)}{(z-1)^3}$$

Shifting Theorem: 1

(i) If $Z[f(t)] = F(z)$, then $Z[e^{-at} f(t)] = F[ze^{aT}]$

(ii) If $Z[f(t)] = F(z)$, then $Z[e^{at} f(t)] = F[ze^{-aT}]$

Proof: (i) $F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$ (defn.)

$$Z[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (ze^{aT})^{-n}$$

$$= Z[f(t)]_{z \to ze^{aT}}$$

$$= F[ze^{aT}]$$

ie, $Z[e^{-at} f(t)] = Z[f(t)]_{z \to ze^{aT}}$

$$= F(z) \text{ where } z \to ze^{aT}.$$

Note:

Similarly, prove (ii)

① Find $Z[e^{-at}]$

Soln: we know that, $Z[e^{-at} f(t)] = [Z(f(t))]_{z \to ze^{aT}}$

$$Z[e^{-at} \cdot 1] = Z[1]_{z \to ze^{aT}}$$

$$= \left(\frac{z}{z-1}\right)_{z \to ze^{aT}} \qquad \left(\because Z(1) = \frac{z}{z-1}\right)$$

$$= \frac{ze^{aT}}{ze^{aT}-1}$$

$$\therefore Z[e^{-at}] = \frac{ze^{aT}}{ze^{aT}-1}$$

② Find $[e^{-at} t]$

Soln:

we know that, $Z[e^{-at} f(t)] = Z(f(t))_{z \to ze^{aT}}$

$$Z[e^{-at} t] = [Z(t)]_{z \to ze^{aT}}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to ze^{aT}} \qquad \left(\because Z(t) = \frac{Tz}{(z-1)^2}\right)$$

$$= \frac{Tze^{aT}}{(ze^{aT}-1)^2}$$

Find

(3) $Z[e^{-at}\sin bt]$

we know that $Z[e^{-at}f(t)] = Z[f(t)]_{z \to ze^{aT}}$

$$Z[e^{-at}\sin bt] = [Z[\sin bt]]_{z \to ze^{aT}}$$

$$= \left.\frac{z\sin bT}{z^2 - 2z\cos bT + 1}\right|_{z \to ze^{aT}}$$

$$= \frac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$$

Find

(4) $Z[e^{3t}\cos t]$.

$$Z[e^{3t}\cos t] = Z(\cos t)\big|_{z \to ze^{-3T}}$$

$$= \left.\frac{z(z - \cos T)}{z^2 - 2z\cos T + 1}\right|_{z \to ze^{-3T}}$$

$$= \frac{ze^{-3T}(ze^{-3T} - \cos T)}{z^2e^{-6T} - 2ze^{-3T}\cos T + 1}$$

(5) Find $Z(e^{-t}t^2)$

Soln:- $Z[e^{-t}t^2) = Z(t^2)_{z \to ze^T}$

$$= \left. \frac{T^2 z(z+1)}{(z-1)^3} \right|_{z \to ze^T}$$

$$= \frac{T^2 \cdot ze^T(ze^T+1)}{(ze^T-1)^3}$$

If $Z\{f(t)] = F(z)$ then $Z[a^n f(t)) = F(z/a)$

If $Z\{x(n)] = X(z)$ then $Z[a^n x(n)) = X(z/a)$

(1) Find $Z[a^n n]$

soln: we know that, $Z[a^n x(n)) = X(z/a)$

$$Z[a^n n] = [Z(n)]_{z \to z/a} \qquad \left[\because Z(n) = \frac{z}{(z-1)^2}\right]$$

$$= \left. \frac{z}{(z-1)^2} \right|_{z \to z/a}$$

$$= \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{az}{(z-a)^2}.$$

(2) Find $Z\left[\frac{a^n}{n!}\right]$

Note: In page NO. 20, problem NO. 13. The same problem can be solved also by formula

soln:- we know that, $Z[a^n x(n)] = X(z/a)$

$$Z\left[a^n \frac{1}{n!}\right] = \left[Z\left[\frac{1}{n!}\right]\right]_{z \to z/a}$$

$$= e^{1/z}\Big|_{z \to z/a} \qquad \because Z\left[\frac{1}{n!}\right] = e^{1/z}$$

$$= e^{\frac{1}{z/a}} = e^{a/z}.$$

(3) Find $Z\left[\frac{a^n}{n}\right]$

soln:- we know that, $Z[a^n x(n)] = X(z/a)$

$$Z\left[a^n \cdot \frac{1}{n}\right] = \left[Z\left[\frac{1}{n}\right]\right]_{z \to z/a}$$

$$= \log \frac{z}{z-1}\Big|_{z \to z/a} \qquad \because Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$$

$$= \log\left(\frac{z/a}{z/a - 1}\right)$$

$$= \log\left(\frac{z/a}{\frac{z-a}{a}}\right) = \log\left[\frac{z}{z-a}\right]$$

(4) Find $Z[a^n t]$

Soln:- we know that, $Z[a^n f(t)] = F(z/a)$

$$Z(a^n t) = (Z(t))_{z \to z/a}$$

$$= \left.\frac{Tz}{(z-1)^2}\right|_{z \to z/a}$$

$$= \frac{\frac{Tz}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{\frac{Tz}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{Tz}{a} \times \frac{a^2}{(z-a)^2}$$

$$= \frac{Taz}{(z-a)^2}$$

(5) Find $Z[a^n e^{at}]$

Soln:- we know that, $Z[a^n f(t)] = F(z/a)$

$$Z[a^n e^{at}] = [Z(e^{at})]_{z \to z/a}$$

$$= \left.\frac{z}{z-e^{aT}}\right|_{z \to z/a}$$

$$= \frac{\frac{z}{a}}{\frac{z}{a}-e^{aT}} = \frac{\frac{z}{a}}{\frac{z-ae^{aT}}{a}} = \frac{z}{z-ae^{aT}}$$

Shifting Theorem II.

If $Z[f(t)] = F(z)$, then $Z[f(t+T)] = z[F(z) - f(0)]$.

Proof :-

$$Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f((n+1)T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f((n+1)T) . z^{-n} . z^{1} . z^{-1}$$

$$= z \sum_{n=0}^{\infty} f((n+1)T) . z^{-(n+1)}$$

Put $n+1 = k$.

$$= z \sum_{k=1}^{\infty} f(kT) z^{-k}$$

$$= z \left[\underbrace{\sum_{k=1}^{\infty} f(kT) z^{-k} + f(0)} - f(0) \right]$$

$$= z \left[\sum_{k=0}^{\infty} f(kT) z^{-k} - f(0) \right]$$

$$= z \left[F(z) - f(0) \right]$$

① Find $Z\left[e^{3(t+T)}\right]$

Soln: we know that, $Z[f(t+T)] = z[F(z) - f(0)]$

$$= z\left[Z[f(t)] - f(0)\right]$$

here $f(t) = e^{3t}$, $f(0) = e^{0} = 1$

$$Z[f(t)] = Z(e^{3t}) = \frac{z}{z - e^{3T}}$$

$$\therefore Z\left[e^{3(t+T)}\right] = z\left[F(z) - f(0)\right]$$

$$= z\left[\frac{z}{z - e^{3T}} - 1\right]$$

$$= z\left[\frac{z - z + e^{3T}}{z - e^{3T}}\right]$$

$$= \frac{z e^{3T}}{z - e^{3T}}$$

(or)

$$Z\left[e^{3(t+T)}\right] = Z\left[e^{3t+3T}\right]$$

$$= Z\left[e^{3T} e^{3t}\right]$$

$$= e^{3T} Z\left[e^{3t}\right]$$

$$= e^{3T}\left[\frac{z}{z - e^{3T}}\right] = \frac{z e^{3T}}{z - e^{3T}}.$$

(2) Find $Z[\sin(t+T)]$

we know that, $Z[f(t+T)] = z(F(z) - f(0))$

$$= z[Z[f(t)] - f(0)]$$

here $f(t) = \sin t$, $f(0) = 1$.

$$Z[f(t)] = Z(\sin t) = \frac{z \sin T}{z^2 - 2z\cos T + 1}$$

$$\therefore Z[\sin(t+T)] = z\left[\frac{z \sin T}{z^2 - 2z\cos T + 1} - 0\right]$$

$$= \frac{z^2 \sin T}{z^2 - 2z\cos T + 1}$$

(or)

$$Z[\sin(t+T)] = Z[\sin t \cos T + \cos t \sin T]$$

$$= \cos T\, Z[\sin t] + \sin T\, Z(\cos t)$$

$$= \cos T\left[\frac{z \sin T}{z^2 - 2z\cos T + 1}\right] + \sin T\left[\frac{z(z - \cos T)}{z^2 - 2z\cos T + 1}\right]$$

$$= \frac{z \sin T \cos T + z^2 \sin T - z \sin T \cos T}{z^2 - 2z\cos T + 1}$$

$$= \frac{z^2 \sin T}{z^2 - 2z\cos T + 1}$$

Initial value Theorem :-

If $Z[f(t)] = F(z)$, then $f(0) = \lim_{z \to \infty} F(z)$.

Proof :-

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= f(0.T) + \frac{f(1.T)}{z} + \frac{f(2.T)}{z^2} + \cdots \text{ to } \infty$$

$$= f(0) + \frac{1}{z} f(T) + \frac{1}{z^2} f(2T) + \cdots \text{ to } \infty$$

Taking limit as $z \to \infty$

$$\lim_{z \to \infty} F(z) = \lim_{z \to \infty} \left[f(0) + \frac{1}{z} f(T) + \frac{1}{z^2} f(2T) + \cdots + \text{to } \infty \right]$$

$$= f(0) \qquad \because \lim_{n \to \infty} 1/n \to 0.$$

$$\therefore \lim_{z \to \infty} F(z) = f(0).$$

Final value theorem :

If $Z[f(t)] = F(z)$ then $\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$

Proof :

$$Z[f(t+T) - f(t)] = \sum_{n=0}^{\infty} [f(nT+T) - f(nT) z^{-n}$$

$$Z[f(t+T)] - Z(f(t)] = \sum_{n=0}^{\infty} [f((n+1)T) - f(nT)] z^{-n}$$

$$zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty} [f((n+1)T) - f(nT)] z^{-n}$$

$$(z-1)F(z) - zf(0) = \sum_{n=0}^{\infty} [f((n+1)T) - f(nT)] z^{-n}$$

Taking limit as $z \to 1$,

$$\lim_{z \to 1} [(z-1)F(z) - zf(0)] = \lim_{z \to 1} \sum_{n=0}^{\infty} [f((n+1)T) - f(nT)] z^{-n}$$

$$\lim_{z \to 1} (z-1)F(z) - f(0) = \sum_{n=0}^{\infty} f((n+1)T) - f(nT)$$

$$= (f(\cancel{T}) - f(0)) + [f(\cancel{2T}) - f(\cancel{T})] + f(3T) - f(\cancel{2T}) + \cdots + [f((n+1)T) - f(nT)] + \cdots + f(\infty)$$

$$= f(\infty) - f(0)$$

$$= \lim_{t \to \infty} f(t) - f(0)$$

$$\lim_{z \to 1} (z-1) F(z) - \cancel{f(0)} = \lim_{t \to \infty} f(t) - \cancel{f(0)}$$

$$\therefore \lim_{z \to 1} (z-1) F(z) = \lim_{t \to \infty} f(t).$$

Problems :-

① If $F(z) = \dfrac{5z}{(z-2)(z-3)}$ find $f(0)$ and $\lim\limits_{t \to \infty} f(t)$

Soln :- By initial value thm,

$$f(0) = \lim_{z \to \infty} F(z)$$

$$= \lim_{z \to \infty} \frac{5z}{(z-2)(z-3)}$$

$$= \lim_{z \to \infty} \frac{5z}{z^2 - 5z + 6} = \frac{\infty}{\infty}.$$

$$= \lim_{z \to \infty} \frac{5}{2z - 5} \quad \text{(by L' hospital's rule)}$$

$$= 0$$

By Final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)\, F(z)$$

$$= \lim_{z\to 1} (z-1) \frac{5z}{(z-2)(z-3)} = 0.$$

---

(2) If $F(z) = \dfrac{2z}{z-e^{-T}}$ find $\lim\limits_{t\to\infty} f(t)$ and $f(0)$.

Soln: By initial value thm,

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{2z}{z-e^{-T}}$$

$$= \lim_{z\to\infty} \frac{\cancel{z}}{\cancel{z}} \left[\frac{2}{1-\frac{e^{-T}}{z}} \right] \quad \left[\because \lim_{z\to\infty} \frac{e^{-T}}{z} \to 0 \right]$$

$$= 2$$

By Final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)\, F(z)$$

$$= \lim_{z\to 1} (z-1) \frac{2z}{z-e^{-T}} = 0.$$

---

(3) If $F(z) = \dfrac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$ find $f(0)$ also find $\lim\limits_{t \to \infty} f(t)$.

Soln:-

By initial value thm,

$$f(0) = \lim_{z \to \infty} F(z)$$

$$= \lim_{z \to \infty} \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1} = \frac{\infty}{\infty}$$

$z^2 - z\cos aT$

$$\Rightarrow \lim_{z \to \infty} \frac{2z - \cos aT}{2z - 2\cos aT} \quad \text{[by L-Hospital's rule)}$$

$$= \lim_{z \to \infty} \frac{\cancel{z}\left[2 - \frac{\cos aT}{z}\right]}{\cancel{z}\left[2 - \frac{\cos aT}{z}\right]} \qquad \lim_{z \to \infty} \frac{\cos aT}{z} \to 0$$

$$= \lim_{z \to \infty} \frac{2}{2} = 1.$$

By Final value thm,

$$\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$$

$$= \lim_{z \to 1} (z-1) \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$$

$$= 0.$$

Shift property.

$Z\{x(n-m)\} = z^{-m}X(z)$ where $x(n)$ is a causal sequence and m is a positive integer.

Proof:

$$Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m)z^{-n} \quad \text{(by defn)}$$

Put $n-m = k$

$n = m+k.$

$$\Rightarrow \sum_{k=-m}^{\infty} x(k)\, z^{-(m+k)}$$

$$= \sum_{k=0}^{\infty} x(k) z^{-m} \cdot z^{-k} \quad [\because \{x(n)\} \text{ is a causal seq.) ie, } x(n)=0 \text{ for } n<0.$$

$$= z^{-m} \sum_{k=0}^{\infty} x(k) z^{-n}$$

$$= z^{-m} X(z).$$

$$\therefore Z\{x(n-m)\} = z^{-m}X(z).$$

Note:

$$Z^{-}[z^{-m}X(z)] = \{x(n-m)\} = [Z^{-1}X(z)]_{n \to n-m}.$$

Inverse Z-Transform :

The inverse z-transform of $Z\{x(n)\} = X(z)$ is defined as $Z^{-1}[X(z)] = \{x(n)\}$.

Method : 1 Long Division Method :

Since Z-transform is defined by the series $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$, to find the inverse Z-transform $x(n) = Z^{-1}[X(z)]$ of $X(z)$, expand $X(z)$ in the proper power series and collect the coefficient of z^{-n} to get $x(n)$.

Find inverse Z-transform of

(1) $\dfrac{10z}{(z-1)(z-2)}$

(2) $\dfrac{(z+2)z}{z^2+2z+4}$

(3) $\dfrac{2z(z^2-1)}{(z^2+1)^2}$

(4) $\dfrac{z}{z^2+7z+10}$ (Exercise).

1) $X(z) = \dfrac{10z}{(z-1)(z-2)} = \dfrac{10z}{z^2-3z+2} = \dfrac{10z}{z^2\left[1-\frac{3z}{z^2}+\frac{2}{z^2}\right]}$

$= \dfrac{10z^{-1}}{1-3z^{-1}+2z^{-2}}$

$$
\begin{array}{r|l}
 & 10z^{-1} + 30z^{-2} + 70z^{-3} + 150z^{-4} + \cdots \\
\hline
1-3z^{-1}+2z^{-2} & 10z^{-1} \\
 & 10z^{-1} - 30z^{-2} + 20z^{-3} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 30z^{-2} - 20z^{-3} \\
 & 30z^{-2} - 90z^{-3} + 60z^{-4} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 70z^{-3} - 60z^{-4} \\
 & 70z^{-3} - 210z^{-4} + 140z^{-5} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 150z^{-4} - 140z^{-5}
\end{array}
$$

We know that, $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$10z^{-1} + 30z^{-2} + 70z^{-3} + 150z^{-4} + \cdots = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$

$\therefore$ $x(0) = 0$

$x(1) = 10$

$x(2) = 30$

$x(3) = 70$

$x(4) = 150$

$\vdots$

In general, $x(n) = 10(2^n - 1), \quad n = 0, 1, 2, \ldots$

(ii) $X(z) = \dfrac{z^2+2z}{z^2+2z+4} = \dfrac{\cancel{z^2}\left(1+\frac{2z}{z^2}\right)}{\cancel{z^2}\left(1+\frac{2z}{z^2}+\frac{4}{z^2}\right)} = \dfrac{1+2z^{-1}}{1+2z^{-1}+4z^{-2}}$

$$
\begin{array}{r|l}
 & 1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \cdots \\
\hline
1+2z^{-1}+4z^{-2} & 1+2z^{-1} \\
 & \underline{\overset{(-)}{1} + \overset{(-)}{2z^{-1}} + \overset{(-)}{4z^{-2}}} \\
 & \quad -4z^{-2} \\
 & \quad \underline{\overset{(+)}{-4z^{-2}} \overset{(+)}{-8z^{-3}} \overset{(+)}{-16z^{-4}}} \\
 & \qquad 8z^{-3} + 16z^{-4} \\
 & \qquad \underline{\overset{(-)}{8z^{-3}} + \overset{(-)}{16z^{-4}} + \overset{(-)}{32z^{-5}}} \\
 & \qquad\qquad -32z^{-5}
\end{array}
$$

WKT, $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \cdots = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$

$\therefore$ $x(0) = 1$, $x(1) = 0$, $x(2) = -4$, $x(3) = 8$,

$x(4) = 0$, $x(5) = -32$, ...

$\therefore$ The sequence is $1, 0, -4, 8, 0, -32 \ldots$

(iii) $$X(z) = \frac{2z(z^2-1)}{(z^2+1)^2} = \frac{2z^3-2z}{z^4+2z^2+1} = \frac{z^{\cancel{3}}\left[2-\frac{2z}{z^3}\right]}{z^4\left[1+\frac{2z^2}{z^4}+\frac{1}{z^4}\right]}$$

$$= \frac{z^{-1}\left[2-2z^{-2}\right]}{1+2z^{-2}+z^{-4}} = \frac{2z^{-1}-2z^{-3}}{1+2z^{-2}+z^{-4}}.$$

$$X(z) = 2z^{-1} - 6z^{-3} + 10z^{-5} - 14z^{-7} + \cdots \quad \text{(Exercise)}.$$

we know that,

$$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$2z^{-1} - 6z^{-3} + 10z^{-5} - 14z^{-7} + \cdots = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

$\therefore\ x(0) = 0,\quad x(1) = 2,\quad x(2) = 0,\quad x(3) = -6,\quad x(4) = 0,$

$x(5) = 10,\quad x(6) = 0,\quad x(7) = -14,\ \ldots\ldots$

In general, $x(n) = 2n \sin\frac{n\pi}{2}, \quad n = 0, 1, 2, \ldots$

Method 2 (Partial Fraction Method).

Find the inverse Z-transform of

(1) $\dfrac{z}{z^2+7z+10}$ (2) $\dfrac{z^2+z}{(z-1)(z^2+1)}$ (3) $\dfrac{kTz}{(z-1+kT)(z-1)}$

(4) $\dfrac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$ (5) $\dfrac{z}{(z-1)^2(z+1)}$

① $X(z) = \dfrac{z}{z^2+7z+10}$.

$$\therefore \frac{X(z)}{z} = \frac{1}{z^2+7z+10} = \frac{1}{(z+5)(z+2)}$$

Consider $\dfrac{1}{(z+5)(z+2)} = \dfrac{A}{z+5} + \dfrac{B}{z+2}$

$$\frac{1}{(z+5)(z+2)} = \frac{A(z+2)+B(z+5)}{(z+5)(z+2)}$$

$$1 = A(z+2) + B(z+5).$$

Solve, $A = -\frac{1}{3}$; $B = \frac{1}{3}$.

$$\therefore \frac{X(z)}{z} = \frac{\frac{-1}{3}}{z+5} + \frac{\frac{1}{3}}{z+2}$$

$$X(z) = \frac{-1}{3}\frac{z}{z+5} + \frac{1}{3}\frac{z}{z+2}$$

Taking inverse z- trans. on b.s;

$$z^{-1}[X(z)] = z^{-1}\left[\frac{-1}{3}\frac{z}{z+5} + \frac{1}{3}\frac{z}{z+2}\right]$$

$$= \frac{-1}{3} z^{-1}\left[\frac{z}{z+5}\right] + \frac{1}{3} z^{-1}\left[\frac{z}{z+2}\right]$$

$$x(n) = \frac{-1}{3}(-5)^n + \frac{1}{3}(-2)^n, \quad n=0,1,2,\ldots$$

WKT,

$$Z[a^n] = \frac{z}{z-a}.$$

---

4) Let $X(z) = \dfrac{z^2}{(z-1/2)(z-1/4)}$

$$\frac{X(z)}{z} = \frac{z}{(z-1/2)(z-1/4)}$$

Consider $\dfrac{z}{(z-1/2)(z-1/4)} = \dfrac{A}{z-1/2} + \dfrac{B}{z-1/4}.$

solve, $A=2$, $B=-1$.

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1/2} - \frac{1}{z-1/4}.$$

$$X(z) = \frac{2z}{z-1/2} - \frac{z}{z-1/4}$$

Taking inverse - z trans. on both sides, we get.

$$z^{-1}[X(z)] = z^{-1}\left[RHS \right]$$

$$x(n) = 2z^{-1}\left[\frac{z}{z-1/2}\right] - z^{-1}\left[\frac{z}{z-1/4}\right]$$

$$x(n) = 2(1/2)^n - (1/4)^n, \quad n=0,1,2,\ldots$$

$$\therefore x(n) = 2(1/2)^n - (1/4)^n, \quad n=0,1,2,\ldots$$

---

5) Let $X(z) = \dfrac{z}{(z-1)^2(z+1)}$

$$\frac{X(z)}{z} = \frac{1}{(z-1)^2(z+1)}$$

Consider $\dfrac{1}{(z+1)(z-1)^2} = \dfrac{A}{z+1} + \dfrac{B}{z-1} + \dfrac{C}{(z-1)^2}$

Solve, $A = 1/4$, $B = -1/4$, $C = 1/2$.

$$\therefore \frac{X(z)}{z} = \frac{1/4}{z+1} - \frac{1/4}{z-1} + \frac{1/2}{(z-1)^2}$$

$$X(z) = \frac{1}{4}\frac{z}{z+1} - \frac{1}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

Taking inverse Z-trans. on b.s, we get

$$z^{-1}[X(z)] = z^{-1}(RHS)$$

$$x(n) = \frac{1}{4} Z^{-1}\left(\frac{z}{z+1}\right) - \frac{1}{4} Z^{-1}\left(\frac{z}{z-1}\right) + \frac{1}{2} Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= \frac{1}{4}(-1)^n - \frac{1}{4} + \frac{1}{2} n, \quad n = 0, 1, 2, \ldots \quad \because Z(n) = \frac{z}{(z-1)^2}$$

Convolution of Sequences:

The Convolution of two sequences $\{x(n)\}$ and $\{y(n)\}$ is defined as

$$w(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

Note: If it is one sided (right sided) sequence, take $x(k) = 0,\ y(k) = 0$ for $k < 0$.

Convolution Theorem :-

If $w(n)$ is the Convolution of two sequences $x(n)$ and $y(n)$, then

$$Z[w(n)] = W(z) = Z[x(n)] \cdot Z[y(n)] = X(z) \cdot Y(z)$$

Problems :-

① Find the Z-transform of the Convolution of $x(n) = u(n)$ & $y(n) = a^n u(n)$.

Solution :-

we know that, $Z[x(n)] = Z(u(n)) = X(z) = \sum_{n=0}^{\infty} u(n) z^{-n}$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{z}{z-1} \quad \text{if } |z| > 1.$$

$$Z[y(n)] = Z[a^n u(n)] = Y(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \frac{z}{z-a} \quad \text{if } |z| > |a|.$$

∴ The Z-transform of the Convolution of $x(n)$ and $y(n)$ is

$$W(z) = \frac{z}{z-1} \cdot \frac{z}{z-a} = \frac{z^2}{(z-1)(z-a)} \quad \text{if } |z| > \max(|a|, 1).$$

2) Find the Z-transform of the Convolution of $x(n) = a^n u(n)$ and $y(n) = b^n u(n)$.

Solution :-

$$Z[x(n)] = Z[a^n u(n)] = X(z) = \frac{z}{z-a} \quad \text{if } |z| > |a|$$

$$Z[y(n)] = Z[b^n u(n)] = Y(z) = \frac{z}{z-b} \quad \text{if } |z| > |b|.$$

$W(z) =$ Z - transform of Convolution of $x(n)$ and $y(n)$.

$$= \frac{z^2}{(z-a)(z-b)} \quad \text{if } |z| > \max(|a|, |b|).$$

The Convolution of two Causal sequence $x(n)$ and $y(n)$, we define

$$\{x(n)\} * \{y(n)\} = \sum_{k=0}^{n} x(n-k)\,y(k).$$

Convolution theorem:

If $f(n)$ and $g(n)$ are two Causal sequences,

$$Z\left[\{f(n) * g(n)\}\right] = Z\{f(n)\} \cdot Z\{g(n)\} = F(z) \cdot G(z)$$

Note:

$$Z^{-1}\left[F(z) \cdot G(z)\right] = f(n) * g(n)$$

$$= \sum_{k=0}^{n} f(n-k)\,g(k)$$

$$= Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$$

Problems :-

1. Find the inverse Z-transform of $\frac{z^2}{(z-a)^2}$ using Convolution theorem.

Solution :-

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n.$$

$$= \sum_{k=0}^{n} a^{n-k} \cdot a^k = \sum_{k=0}^{n} a^n$$

$$= (n+1)\, a^n.$$

---

2. Find the inverse Z-transform of $X(z) = \frac{z^2}{(z-1/2)(z-1/4)}$ using Convolution theorem.

Solution :-

$$Z^{-1}\left[X(z)\right] = Z^{-1}\left[\frac{z^2}{(z-1/2)(z-1/4)}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1/2} \cdot \frac{z}{z-1/4}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1/2}\right] * Z^{-1}\left[\frac{z}{z-1/4}\right]$$

$$= (1/2)^n * (1/4)^n$$

$$= \sum_{k=0}^{n} (1/2)^{n-k} \cdot (1/4)^{k}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} (1/2)^{-k} (1/4)^{k}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} (1/2)^{-k} \left((1/2)^{2}\right)^{k}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} (1/2)^{-k} \left(\frac{1}{2}\right)^{2k}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} (1/2)^{k}$$

$$= \left(\frac{1}{2}\right)^{n} \left[1 + (1/2) + (1/2)^{2} + \dots + \left(\frac{1}{2}\right)^{n}\right]$$

$$= \left(\frac{1}{2}\right)^{n} \left[\frac{1 - (1/2)^{n+1}}{1 - 1/2}\right] \text{ by G.P.}$$

$$\left[a + ar + ar^{2} + \dots + ar^{n} = \frac{a(1 - r^{n+1})}{1 - r}, \text{ if } r < 1\right]$$

$$= \left(\frac{1}{2}\right)^{n} \left[\frac{1 - (1/2)^{n+1}}{1/2}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} \left(1 - (1/2)^{n+1}\right)$$

$$= (1/2)^{n-1} - (1/2)^{2n}.$$

3) Find the inverse Z-transform of $\frac{8z^2}{(2z-1)(4z-1)}$ by using Convolution theorem.

Soln:- Given $\frac{8z^2}{(2z-1)(4z-1)} = \frac{\cancel{8}z^2}{\cancel{2}(z-1/2)\cancel{4}(z-1/4)}$

$$= \frac{z^2}{(z-1/2)(z-1/4)}.$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n} \quad \left(\begin{matrix}\text{Refer}\\ \text{eg. 2}\end{matrix}\right)$$

4) Using Convolution theorem evaluate inverse Z-transform of $\frac{z^2}{(z-1)(z-3)}$.

Solution:-

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1} \cdot \frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (1)^n * 3^n$$

$$= \sum_{k=0}^{n} 1^{n-k}\, 3^{k}$$

$$= \sum_{k=0}^{n} 3^{k} = 1+3+3^{2}+\cdots+3^{n}$$

$$\left[a+ar+ar^{2}+\cdots+ar^{n} = \frac{a(r^{n+1}-1)}{r-1} \text{ if } |r|>1 \right]$$

$$= \frac{(3^{n+1}-1)}{2}$$

Exercise problems :-

1. Find $Z^{-1}\left[\dfrac{z^{2}}{(z-a)(z-b)}\right]$ Ans: $\dfrac{b^{n+1}-a^{n+1}}{b-a}$

2. Find $Z^{-1}\left[\dfrac{z^{2}}{(z-1/2)(z+1/4)}\right]$ Ans: $\dfrac{2}{3}\left(\dfrac{1}{2}\right)^{n}+\dfrac{1}{3}\left(-1/4\right)^{n}$

3. Find $Z^{-1}\left[\dfrac{z^{2}}{(z+a)^{2}}\right]$ Ans: $(n+1)(-a)^{n}$

Evaluation of Residues of $f(z)$:-

① Residue of $f(z)$ at its simple pole $z = z_0$ is given by

$$= \lim_{z \to z_0} (z - z_0) f(z).$$

② Residue of $f(z)$ at its pole $z = z_0$ of order n is given by

$$= \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \right]$$

Inverse Integral method :

The inverse Z transform of $X(z)$ is given by

$$x(n) = \frac{1}{2\pi i} \oint_C X(z) \cdot z^{n-1} dz$$

$= \Sigma R =$ Sum of the residues of $X(z) z^{n-1}$ at all isolated singularities of $X(z) z^{n-1}$.

C is a circle with centre at origin and Radius R (which is large enough to contain all the singularities)

1. Find the inverse Z-transform of $\frac{z}{(z-1)(z-2)}$.

<u>solution</u>:

Let $X(z) = \frac{z}{(z-1)(z-2)}$

$$X(z).z^{n-1} = \frac{z \cdot z^{n-1}}{(z-1)(z-2)} = \frac{z^n}{(z-1)(z-2)}$$

$z=1$ is a simple pole

$z=2$ is also a simple pole.

Residue of $X(z).z^{n-1}$ at $z=1 = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{z^n}{\cancel{(z-1)}(z-2)} = -1$

Residue of $X(z).z^{n-1}$ at $z=2 = \lim_{z \to 2} \cancel{(z-2)} \cdot \frac{z^n}{(z-1)\cancel{(z-2)}} = 2^n$

$\therefore$ $x(n) = \Sigma R$ where ΣR is the sum of the residues of $X(z).z^{n-1}$ at the isolated singularities.

ie, $x(n) = 2^n - 1$, $n = 0, 1, 2, \ldots$

(2) Find $Z^{-1}\left[\frac{z^2-3z}{(z+2)(z-5)}\right]$

Solution:-

Let $X(z) = \frac{z^2-3z}{(z+2)(z-5)}$

$$X(z).\, z^{n-1} = \frac{(z^2-3z)\, z^{n-1}}{(z+2)(z-5)}$$

$$= \frac{z^{n+1}-3z^n}{(z+2)(z-5)}$$

$z=-2$ is a simple pole.

$z=5$ is a " ".

Residue of $X(z).\, z^{n-1}$ at $z=-2$ =

$$\lim_{z\to -2} \cancel{(z+2)} \frac{z^{n+1}-3z^n}{\cancel{(z+2)}(z-5)} = \frac{(-2)^{n+1}-3(-2)^n}{(-7)}$$

$$\Rightarrow \frac{(-2)^n[-2-3]}{-7} = \frac{5}{7}(-2)^n$$

Residue of $X(z).\, z^{n-1}$ at $z=5$ is

$$\lim_{z\to 5} \cancel{(z-5)} \frac{z^{n+1}-3z^n}{(z+2)\cancel{(z-5)}} = \frac{5^{n+1}-35^n}{5+2}$$

$$= \frac{5^n[5-3]}{7} = \frac{2}{7}(5)^n$$

$\therefore\ x(n) = \sum R$

$$= \frac{5}{7}(-2)^n + \frac{2}{7}(5)^n, \quad n=0,1,2,\ldots$$

3) Find $Z^{-1}((X(z))$ where $X(z) = \dfrac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$.

soln:

Given $X(z) = \dfrac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$

$$X(z) \cdot z^{n-1} = \frac{(4z^2 - 2z) \cdot z^{n-1}}{z^3 - 5z^2 + 8z - 4}$$

$$= \frac{4z^{n+1} - 2z^n}{z^3 - 5z^2 + 8z - 4}$$

Poles are given by $z^3 - 5z^2 + 8z - 4 = 0$.

$$\begin{array}{c|cccc} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$z = 1$ is one of pole.

$z^2 - 4z + 4 = 0$.

$z = 2, 2$.

$\therefore$ $z = 1$ is a simple pole

$z = 2$ is a pole of order 2.

Residue of $X(z) \cdot z^{n-1}$ at $z = 1$ $= \lim_{z \to 1} \cancel{(z-1)} \left(\dfrac{4z^{n+1} - 2z^n}{\cancel{(z-1)}(z-2)^2} \right)$

$$= \frac{4(1)^{n+1} - 2(1)^n}{(1-2)^2} = 2$$

$\frac{vu' - uv'}{v^2}$

Residue of $X(z) \cdot z^{n-1}$ at $z=2$ of order 2 =

$$\frac{1}{(2-1)!} \lim_{z \to 2} \frac{d^1}{dz^1} \cancel{(z-2)^2} \frac{4z^{n+1} - 2z^n}{(z-1)\cancel{(z-2)^2}}$$

$$= \frac{1}{1} \lim_{z \to 2} \frac{d}{dz} \left(\frac{4z^{n+1} - 2z^n}{(z-1)} \right) \frac{u}{v}$$

$$= \lim_{z \to 2} \left[\frac{(z-1)\left(4(n+1)z^{n+\cancel{1}-\cancel{1}} - 2nz^{n-1}\right) - (4z^{n+1} - 2z^n)(1)}{(z-1)^2} \right]$$

$$= \left[\frac{(2-1)\left(4(n+1)2^n - 2n\,2^{n-1}\right) - 4\,2^{n+1} + 2\,2^n}{(2-1)^2} \right]$$

$(1)^2 = 1$

$$\Rightarrow \quad 4(n+1)2^n - 2n\,2^{n-1} - 4\,2^{n+1} + 2 \cdot 2^n$$

$$\Rightarrow \quad 4n2^n + 42^n - n2^n - 4(2^n \cdot 2) + 2 \cdot 2^n$$

$$= 4n2^n + 42^n - n2^n - 82^n + 22^n.$$

$$= n3 \cdot 2^n - 2 \cdot 2^n$$

$$= 2^n(3n - 2)$$

$\therefore \; x(n) = \sum R.$

$$x(n) = 2 + 2^n(3n-2), \quad n = 0, 1, 2, \ldots$$

(4) Find the inverse Z-transform of $\frac{z(z+1)}{(z-1)^3}$.

Soln:- Let $X(z) = \frac{z(z+1)}{(z-1)^3}$

$$X(z)\cdot z^{n-1} = \frac{(z^2+z)z^{n-1}}{(z-1)^3}$$

$$X(z)\cdot z^{n-1} = \frac{z^{n+1}+z^n}{(z-1)^3}$$

$z=1$ is a pole of order 3.

Residue of $X(z)\cdot z^{n-1}$ at $z=1$ of order $3 =$

$$\frac{1}{2!}\lim_{z\to 1}\frac{d^{3-1}}{dz^{3-1}}\cancel{(z-1)^3}\frac{z^{n+1}+z^n}{\cancel{(z-1)^3}}$$

$$\Rightarrow \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^{n+1}+z^n\right)$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left(\frac{d}{dz}\left(z^{n+1}+z^n\right)\right)$$

$(n+1)\, z^{n+\cancel{1}-\cancel{1}}$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left((n+1)z^n + nz^{n-1}\right)$$

$$= \frac{1}{2}\lim_{z\to 1}\left[(n+1)(n)z^{n-1} + n(n-1)z^{n-2}\right]$$

$$= \frac{1}{2}\left[(n+1)n\,1^{(n-1)} + n(n-1)\,1^{(n-2)}\right]$$

$$= \frac{1}{2}\left[(n^2+n)+n(n-1)\right] = \frac{1}{2}\left[n^2+\cancel{n}+n^2-\cancel{n}\right]$$

$$= n^2$$

3. Find $Z^{-1}\left[\frac{z}{z^2+1}\right]$ by using Residue method.

$$X(z)\, z^{n-1} = \frac{z^n}{z^2+1}$$

Singular points are given by $z^2+1=0 \Rightarrow z = \pm i$

$+i, -i$ are simple poles.

$$\text{Res}\,(z=i) = \lim_{z\to i} \cancel{(z-i)} \frac{z^n}{(z+i)\cancel{(z-i)}} = \frac{i^n}{2i}$$

$$\text{Res}\,(z=-i) = \lim_{z\to -i} \cancel{(z+i)} \frac{z^n}{\cancel{(z+i)}(z-i)} = \frac{(-i)^n}{(-2i)}$$

$$\therefore\ Z^{-1}\left[\frac{z}{z^2+1}\right] = \text{Sum of the residues}$$

$$= \frac{i^n}{2i} + \frac{(-i)^n}{-2i}$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \qquad\qquad -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

$$i^n = \cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} \qquad\qquad (-i)^n = \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}$$

$$i^n - (-i)^n = 2i\sin\frac{n\pi}{2}$$

$$Z^{-1}\left[\frac{z}{z^2+1}\right] = \frac{i^n - (-i)^n}{2i} = \frac{\cancel{2i}\sin\frac{n\pi}{2}}{\cancel{2i}} = \sin\frac{n\pi}{2}.$$

Applications to Solving Finite Difference Equations.

$$Z[y(n)] = Y(z)$$

$$Z[y(n+1)] = zY(z) - zy(0)$$

$$Z[y(n+2)] = z^2Y(z) - z^2y(0) - zy(1)$$

$$\vdots$$

$$Z[y(n+k)] = z^k\left[Y(z) - y(0) - \frac{y(1)}{z} - \frac{y(2)}{z^2} - \cdots - \frac{y(k-1)}{z^{k-1}}\right]$$

1) Solve : $y_{n+2} - 4y_{n+1} + 4y_n = 0$ given $y_0 = 1$ and $y_1 = 0$.

Soln :-

Taking Z-transform on both sides of the difference equation, we get

$$Z[y_{n+2} - 4y_{n+1} + 4y_n] = Z[0]$$

$$Z[y_{n+2}] - 4Z[y_{n+1}] + 4Z[y_n] = 0.$$

$$z^2Y(z) - z^2y(0) - zy(1) - 4[zY(z) - zy(0)] + 4Y(z) = 0.$$

$$z^2Y(z) - z^2(1) - \cancel{z(0)} - 4[zY(z) - z(1)] + 4Y(z) = 0 \quad (\because y_0 = 1 \ \& \ y_1 = 0)$$

$$z^2Y(z) - z^2 - 4zY(z) + 4z + 4Y(z) = 0.$$

$$(z^2 - 4z + 4)Y(z) = z^2 - 4z$$

$$Y(z) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$Y(z)\cdot z^{n-1} = \frac{(z^2-4z)\,z^{n-1}}{(z-2)^2}$$

$$Y(z)\cdot z^{n-1} = \frac{z^{n+1}-4z^{n}}{(z-2)^2}$$

$z=2$ is a pole of order 2.

Residue of $Y(z)\cdot z^{n-1}$ at $z=2$ of order 2 $(n=1)$ $\Big\} = \frac{1}{1!}\lim \frac{d}{dz}\cancel{(z-2)^2}\frac{z^{n+1}-4z^{n}}{\cancel{(z-2)^2}}$

$$= \lim_{\to 2} (n+1)z^{n} - 4(nz^{n-1}).$$

$$\Rightarrow (n+1)2^{n} - 4n(2^{n-1}) \qquad 2^{2+n-1}$$

$$= n2^{n}+2^{n} - n(2\cdot 2^{n})$$

$$\Rightarrow 2^{n}(n+1-2n)$$

$$= 2^{n}(1-n).$$

$$\therefore\ y(n) = 2^{n}(1-n)\ ,\ n=0,1,2,\ldots$$

2) Solve : $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$.

Soln :-

$$y_{n+2} + 6y_{n+1} + 9y_n = 2^n .$$

Taking Z-trans. on b.s of the difference eqn,

$$Z[y_{n+2} + 6y_{n+1} + 9y_n] = Z(2^n)$$

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z(y_n) = z/z-2$$

$$[z^2Y(z) - z^2y(0) - zy(1)] + 6[zY(z) - y_0] + 9Y(z) = z/z-2$$

$$z^2Y(z) - \cancel{z^2(0)} - \cancel{z(0)} + 6zY(z) - \cancel{6(0)} + 9Y(z) = z/z-2$$

$$Y(z)[z^2 + 6z + 9] = z/z-2$$

$$Y(z) = \frac{z}{(z^2+6z+9)(z-2)}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

Solve, $A = \frac{1}{25}$ $B = -\frac{1}{25}$ $C = -\frac{1}{5}$

$$\therefore Y(z) = \frac{1/25}{z-2} - \frac{1/25}{z+3} - \frac{1/5}{(z+3)^2}$$

Taking inverse Z-transform on both sides, we get

$$Z^{-1}[Y(z)] = Z^{-1}[RHS]$$

$$y(n) = \frac{1}{25} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5}\left[\frac{z}{(z+3)^2}\right]$$

$$= \frac{1}{25}(2^n) - \frac{1}{25}(-3)^n - \frac{1}{5}\left[\frac{-3z}{-3(z+3)^2}\right]$$

$n(-3)^n$

$$= \frac{2^n}{25} - \frac{(-3)^n}{25} - \frac{1}{5} n(-3)^{n-1}.$$

$$\left[\because \; Z[a^n] = \frac{z}{z-a} \text{ and } Z[na^n] = \frac{az}{(z-a)^2}\right]$$

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3) Solve : $x(n+2) - 3x(n+1) + 2x(n) = 0$ given $x(0) = 0$, $x(1) = 1$.

Soln:

Given $x(n+2) - 3x(n+1) + 2x(n) = 0$.

Taking Z-trans. on both sides, we get

$$Z[x(n+2) - 3x(n+1) + 2x(n)] = Z(0)$$

$$Z[x(n+2)] - 3Z[x(n+1)] + 2Z(x(n)] = 0.$$

$$z^2X(z) - z^2\cancel{x(0)} - zx(1) - 3[zX(z) - \cancel{x(0)}] + 2X(z) = 0$$

(0) (0)

$$z^2X(z) - zx(1) - 3zX(z) + 2X(z) = 0.$$

$$[z^2 - 3z + 2]X(z) - z(1) = 0.$$

$$X(z) = \frac{z}{z^2 - 3z + 2}$$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{(z-2)} + \frac{B}{(z-1)}$$

Solve, $A = 1$, $B = -1$.

$$\therefore \frac{X(z)}{z} = \frac{1}{z-2} + \frac{-1}{z-1}$$

$$X(z) = \frac{z}{z-2} - \frac{1}{z-1}$$

Taking inverse z-transform on b.s, we get

$$x(n) = 2^n - 1, \quad n = 0, 1, 2, \ldots$$

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4 Solve: $y(n) - y(n-1) = u(n) + u(n-1)$.

we know that,

$$Z[x(n-m)] = z^{-m} X(z) \qquad Z(u(n)) = \frac{z}{z-1}$$

$$u(n-m) = z^{-m} \frac{z}{z-1}$$

Taking Z-trans. on both sides, we get

$$Z[y(n) - y(n-1)] = Z[u(n) + u(n-1)]$$

$$= Z[y(n)] - Z[y(n-1)] = Z(u(n)) + Z[u(n-1)]$$

$$Y(z) - z^{-1} Y(z) = \frac{z}{z-1} + z^{-1} \cdot \frac{z}{z-1}$$

$$Y(z)\left[1 - \frac{1}{z}\right] = \frac{z}{z-1} + \frac{1}{\cancel{z}} \cdot \frac{\cancel{z}}{z-1}$$

$$Y(z)\left[\frac{z-1}{z}\right] = \frac{z}{z-1} + \frac{1}{z-1}$$

$$Y(z) = \frac{z+1}{z-1} \times \frac{z}{z-1}$$

$$Y(z) = \frac{z^2+z}{(z-1)^2}$$

$$Y(z)\cdot z^{n-1} = \frac{z^2+z}{(z-1)^2} \cdot z^{n-1} =$$

$$Y(z)\cdot z^{n-1} = \frac{z^{n+1}+z^n}{(z-1)^2}$$

$z=1$ is pole of order 2.

Residue of $Y(z)\cdot z^{n-1}$ at $z=1$ of order 2 $\Big\} = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz} \cancel{(z-1)^2} \cdot \frac{z^{n+1}+z}{\cancel{(z-1)^2}}$

$$= \lim_{z\to 1} \frac{d}{dz}\left(z^{n+1}+z^n\right)$$

$$= \lim_{z\to 1} (n+1)\, z^{n+1-\cancel{1}} + n z^{n-1}$$

$$\Rightarrow n+1+n = 1+2n.$$

$\therefore x(n) = R$ where R is residue of $Y(z)\cdot z^{n-1}$

$\therefore x(n) = 1+2n, \quad n = 0, 1, 2, \ldots$

Exercise problems:

1. Solve: $y(n) - ay(n-1) = u(n)$
2. Solve: $y_{n+2} + y_n = 2$ given $y_0 = y_1 = 0$.
3. Solve: $y_{n+2} - 4y_n = 0$ using Z-transform.