Potential flows

POTENTIAL FLOWS - Steady, inviscid, incompressible and irrotational flow

Incompressible Flow: $\vec{\nabla} \cdot \vec{V} = 0$

Irrotational Flow: $\vec{\nabla} \times \vec{V} = 0$

Incompressible flow – The Stream Function

For a steady 2D flow, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

3D flow : $u\hat{i} + v\hat{j} + w\hat{k}$

2D : $u\hat{i} + v\hat{j}$

A clever variable transformation enables us to rewrite the continuity equation in terms of *one* dependent variable (ψ) instead of *two* dependent variables (u and v). We define the **stream function** ψ as

$$\boxed{u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

<u>Physical significance of stream function (ψ):</u>

- a single variable (ψ) replaces *two* variables (u and v)—once ψ is known, we can generate both u and v.
- we are guaranteed that the solution satisfies continuity
- Curves of constant ψ are **streamlines** of the flow

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Along a streamline: $\dfrac{dy}{dx} = \dfrac{v}{u}$

$$\Rightarrow \quad u\,dy = v\,dx$$

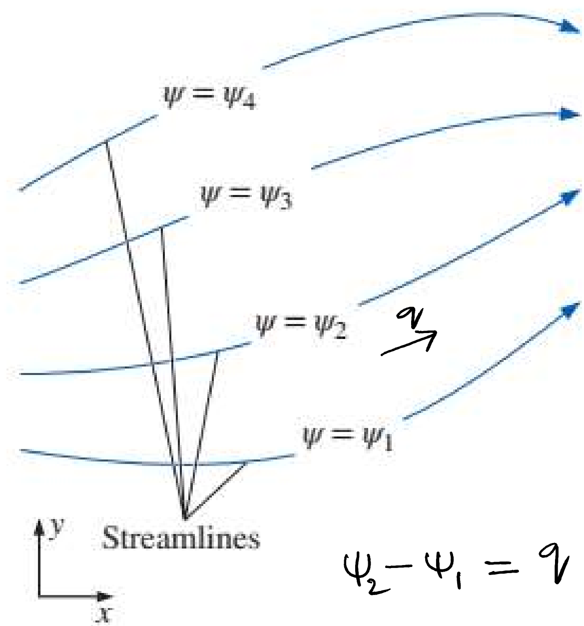
$$\Rightarrow \quad v\,dx - u\,dy = 0$$

$$\Rightarrow \quad -\frac{\partial \psi}{\partial x}dx - \frac{\partial \psi}{\partial y}dy = 0$$

$$\Rightarrow \quad \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

$$\Rightarrow \quad d\psi = 0$$

$$\Rightarrow \quad \boxed{\psi = \text{Constant}}$$



$$\psi_2 - \psi_1 = q$$

Another physically significant fact about the stream function:

The difference in the value of ψ from one streamline to another is equal to the volume flow rate per unit width between the two streamlines.

Irrotational flow – The velocity potential function

For an irrotational flow, $\vec{\nabla} \times \vec{V} = 0$,

$$\text{then } \vec{V} = \vec{\nabla}\phi. \qquad \vec{\nabla} \times \vec{\nabla}\phi = 0$$

- If the curl of a vector is zero, the vector can be expressed as the gradient of a scalar function ϕ, called the **potential function**.

- In an irrotational region of flow, the velocity vector can be expressed as the gradient of a scalar function called the velocity potential function.

$$\vec{V} = \vec{\nabla}\phi$$

$$u\hat{i} + v\hat{j} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}$$

$$\Rightarrow \quad \boxed{u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}}$$

for 2D flows, $\vec{\nabla} \times \vec{V} = 0$. $\Rightarrow$ $\begin{vmatrix} \hat{i} & -\hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0$

$$\Rightarrow \left(\underbrace{\cancel{\frac{\partial w}{\partial y}}}_{0} - \underbrace{\cancel{\frac{\partial v}{\partial z}}}_{0}\right)\hat{i} - \hat{j}\left(\underbrace{\cancel{\frac{\partial w}{\partial x}}}_{0} - \underbrace{\cancel{\frac{\partial u}{\partial z}}}_{0}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

For a 2D flow, $\bar{V} = u\hat{i} + v\hat{j}$ $\quad w = 0$

(x,y)

For a 2D flow $\nabla \times \bar{V} = 0 \Rightarrow \boxed{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0}$

Relationship between Stream function ψ and Velocity potential ϕ

$\psi = \text{Constant}$ is the equation of a streamline

$\phi = \text{Constant}$ is the equation of an equipotential line

Equation of a streamline is $\psi(x, y) = \text{Constant}$

$$\Rightarrow d\psi = 0$$

$$\Rightarrow \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

We know $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$

$$\Rightarrow v\,dx = u\,dy$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\psi = \text{Constant}} = \frac{v}{u} \leftarrow \text{Slope of a streamline}$$

Equation of an equipotential line is $\phi(x,y) = \text{constant}$

$$\Rightarrow \quad d\phi = 0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

we know that $u = \dfrac{\partial \phi}{\partial x}$, $v = \dfrac{\partial \phi}{\partial y}$

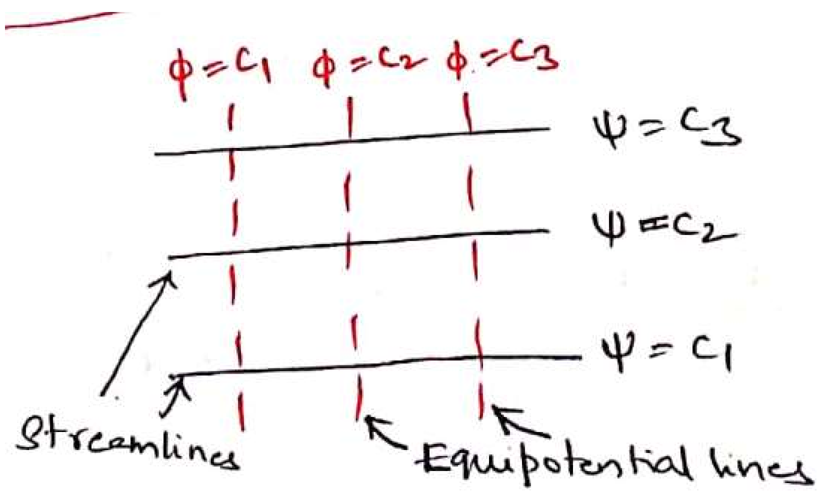
$$\Rightarrow \quad u\,dx + v\,dy = 0$$

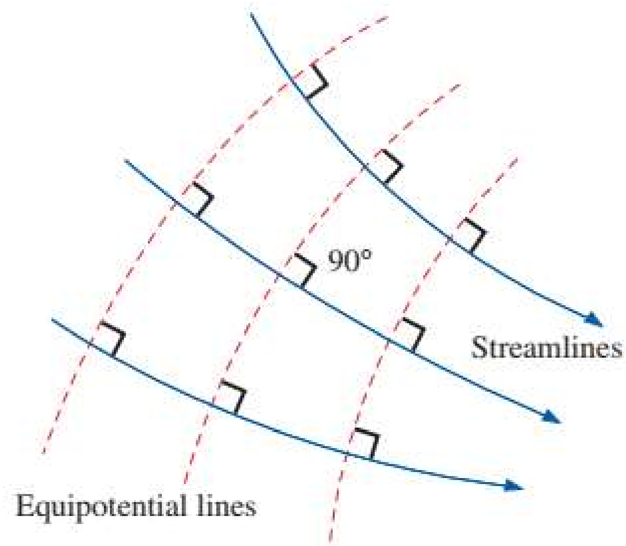
$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\phi = \text{constant}} = -\frac{u}{v} = -\frac{1}{\left(\frac{dy}{dx}\right)_{\psi = \text{constant}}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\phi = \text{const}} \left(\frac{dy}{dx}\right)_{\psi = \text{const}} = -1$$

Product of slopes is -1

Streamlines and equipotential lines are perpendicular to each other.
(ψ = Const) (ϕ = Constant)





Governing equations for Potential flows

For an incompressible flow, $\bar{\nabla}\cdot\bar{V}=0$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{— (1)}$$

For an irrotational flow, we can define ϕ such that $\bar{V}=\bar{\nabla}\phi$

$$\Rightarrow u=\frac{\partial \phi}{\partial x} \text{ and } v=\frac{\partial \phi}{\partial y} \quad \text{— (2)}$$

Substituting (2) in (1) $\Rightarrow$ $\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)=0$

$$\Rightarrow \boxed{\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0} \quad \text{— (A)} \quad \Rightarrow \nabla^2\phi=0$$

Similarly, For an irrotational flow $\bar{\nabla} \times \bar{V} = 0$

$$\Rightarrow \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{— (3)}$$

For an incompressible flow, we can define streamfunction ψ such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \qquad \text{— (4)}$$

Substitute (4) in (3) $\Rightarrow$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0} \qquad \text{— (B)} \qquad \Rightarrow \nabla^2 \psi = 0$$

** A 2D irrotational and incompressible flow, both ϕ and ψ satisfies "Laplace Equation".

** Any solution of the Laplace Equations (A) and (B) represent a potential flow.

6.13 The velocity components of an incompressible, two-dimensional velocity field are given by the equations

$$u = y^2 - x(1 + x)$$
$$v = y(2x + 1) = 2xy + y$$

(a) Does a stream functions exists? If so, find it.
(b) Does a velocity potential exists? If so, find it.

Sol: For a stream function ψ to exist, the velocity field has to satisfy continuity equation

Continuity eqn : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 0 - 1 - 2x \qquad \frac{\partial v}{\partial y} = 2x + 1$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -1 - 2x + 2x + 1 = 0$

Velocity field satisfies continuity eqn

$\Rightarrow$ stream function exists

$u = \frac{\partial \psi}{\partial y} \qquad v = \frac{-\partial \psi}{\partial x}$

$\frac{\partial \psi}{\partial y} = y^2 - x - x^2$

$\psi(x,y) = \frac{y^3}{3} - xy - x^2 y + f(x) + \text{Const.}$ —— (1)

$\frac{\partial \psi}{\partial x} = -v = -2xy - y$

Integrate w.r.t x on both sides

$\psi(x,y) = -x^2 y - xy + f(y) + \text{Const.}$ —— (2)

By comparing (1) and (2) $\Rightarrow f(x) = 0$, $f(y) = \frac{y^3}{3}$

$$\psi(x,y) = -x^2 y - xy + \frac{y^3}{3} + \text{Constant}$$

6.13 The velocity components of an incompressible, two-dimensional velocity field are given by the equations

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(a) Does a stream functions exists? If so, find it.

✓(b) Does a velocity potential exists? If so, find it.

For a velocity potential to exist, flow has to be irrotational $\Rightarrow \nabla \times \bar{V} = 0 \Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$$\frac{\partial v}{\partial x} = 2y + 0 \qquad = 2y$$

$$\frac{\partial u}{\partial y} = 2y + 0 + 0 \quad = 2y$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2y - 2y = 0$$

$\Rightarrow$ Velocity field satisfies irrotational flow condition

$\Rightarrow$ Velocity potential exists

Velocity potential ϕ

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial \phi}{\partial x} = u = y^2 - x - x^2$$

Integrate w.r.t x on both

$$\phi(x, y) = xy^2 - \frac{x^2}{2} - \frac{x^3}{3} + f(y) + \text{const.} \quad \text{—(1)}$$

$$\frac{\partial \phi}{\partial y} = v = 2xy + y$$

Integrate w.r.t y on both

$$\phi(x, y) = xy^2 + \frac{y^2}{2} + f(x) + \text{const} \quad \text{—(2)}$$

Comparing (1) & (2)

$$f(x) = -\frac{x^2}{2} - \frac{x^3}{3} \qquad f(y) = \frac{y^2}{2}$$

$$\underline{\underline{\phi(x, y)}} = xy^2 + \frac{y^2}{2} - \frac{x^2}{2} - \frac{x^3}{3} + \text{const}$$

Summary

incompressible and irrotational flows

Stream function ψ $\quad u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}$ — From incompressible flow condition $\bar{\nabla} \cdot \bar{V} = 0$

Velocity potential ϕ $\rightarrow$ $u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$ — From irrotational flow condition $\bar{\nabla} \times \bar{V} = 0$

Governing equations for potential flows:

$$\nabla^2 \phi = 0 \quad \text{and} \quad \nabla^2 \psi = 0 \quad \text{— Laplace Equations}$$

(or)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

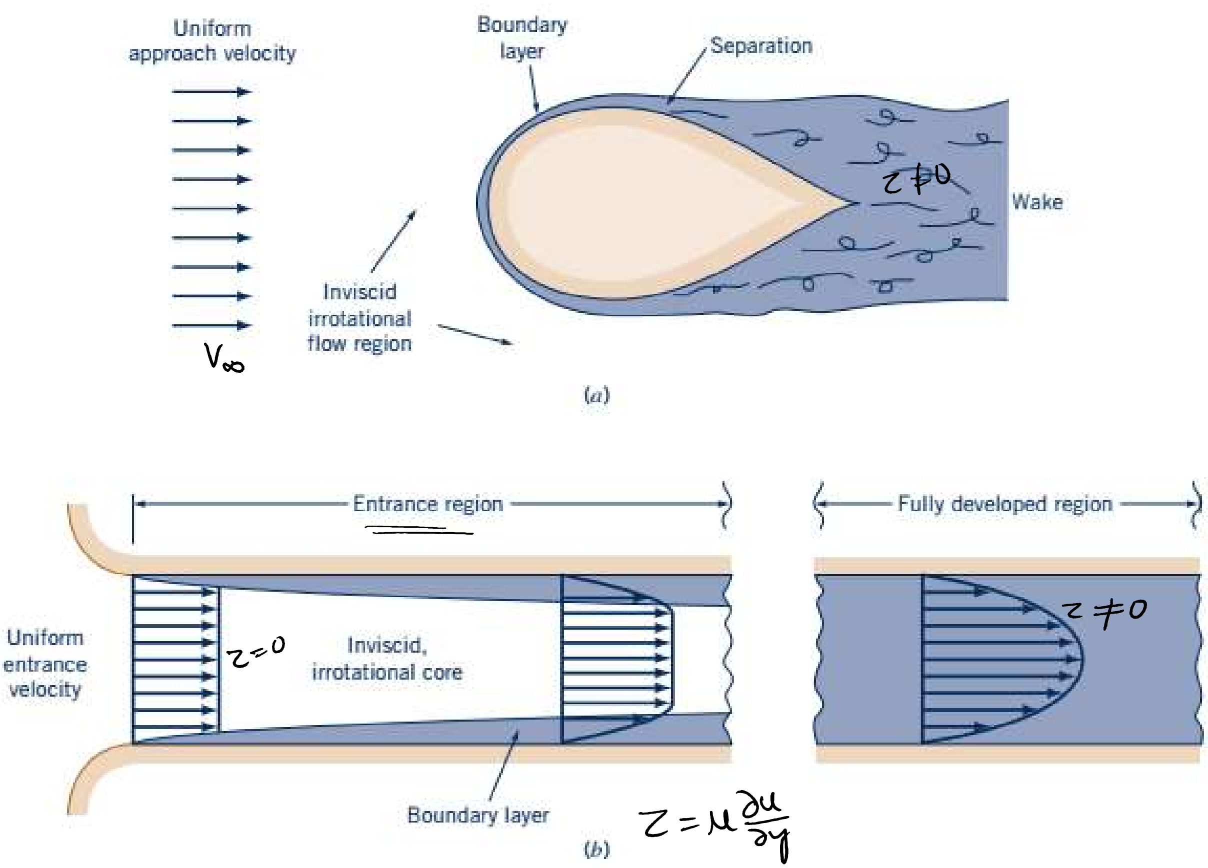
* These equations are linear PDE. By solving these equations with boundary conditions, we obtain ϕ and ψ.

* obtain velocities from ϕ $\quad u = \frac{\partial \phi}{\partial x}$ and $v = \frac{\partial \phi}{\partial y}$

* $p + \frac{\rho V^2}{2} = \text{constant}$

obtain pressure from Bernoulli's

Where are the potential flow solutions valid in real cases?



■ **Figure 6.14** **Various regions of flow: *(a)* around bodies; *(b)* through channels.**