FLUID MECHANICS

UNIT - IV Potential Flows.

* Stream Function and Velocity Potential in Fluid Dynamics. [Only for understanding]

→ The stream function and velocity potential are two powerful mathematical tools used to describe and analyse fluid flow, particularly in the context of aerodynamics.

→ They provide simplified methods for visualizing flow patterns and solving flow problems, especially for incompressible and irrotational flows.

* Stream Function :

→ The stream function, denoted by ψ, is a scalar function that represents the flow rate across a curve in a two-dimensional flow field.

→ It is defined in a way that automatically satisfies the continuity equation, which expresses the principle of conservation of mass in fluid dynamics.

* Significance :

→ In a two-dimensional flow, lines of constant ψ are streamlines, which represent the trajectories of fluid particles.

→ The difference in ψ between two streamlines represent the volumetric flow rate per unit depth between those two streamlines.

→ Therefore, the stream function provides a direct visual representation of flow patterns and satisfies continuity by construction.

* Mathematical Formulation :

→ The stream function is related to the velocity components (u, v) in a cartesian coordinate system as follows :

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

→ For axisymmetric flows in spherical coordinates (r, θ, ϕ), the streamline function ψ is defined as

$$\omega_\psi = \frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R} \quad \text{--- Cylindrical}$$

$$\omega_\psi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right] \quad \text{--- Spherical}$$

* Applications :

- Stream functions are used in various aerodynamics applications such as ;
- Visualizing flow patterns around airfoil and other aerodynamic bodies.
- Analyzing the flow in wind tunnels and other test facilities.
- Calculating the lift and drag forces on aerodynamic bodies.

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* Velocity Potential (ϕ) :

→ The Velocity Potential, denoted by ϕ, is a scalar function that can be defined for irrotational flows.

→ An irrotational flow is one where fluid particles do not rotate about their own axis.

→ This condition greatly simplifies the mathematical analysis of the flow.

* Significance ;

→ The gradient of the velocity potential directly gives the velocity vector.

→ If a velocity potential exists, it implies that the flow is irrotational.

* Mathematical Formulation : [IMP]

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

* Relationship between Stream Function ψ and Velocity Potential ϕ :

ψ = Constant in the equation of streamlines

ϕ = Constant in the equation of equipotential line.

* Equation of a streamline is $\psi(x, y) = \text{Constant}$

$$\therefore \quad d\psi = 0$$

$$\therefore \quad \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

We know $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

Putting this in the above

$$\therefore \quad v\,dx = u\,dy$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\psi = \text{constant}} = \frac{v}{u}$$ — Slope of Streamline.

* Equation of an equipotential line is $\phi(x, y) = \text{Constant}$

$$\therefore \quad \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

We know that $u = \frac{\partial \phi}{\partial x}$ and $v = \frac{\partial \phi}{\partial y}$

$$\therefore \quad u\,dx + v\,dy = 0$$

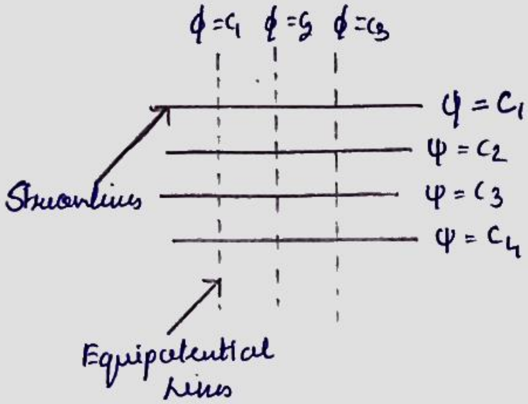
$$\therefore \quad \left(\frac{dy}{dx}\right)_{\phi = \text{constant}} = \frac{-u}{v}$$

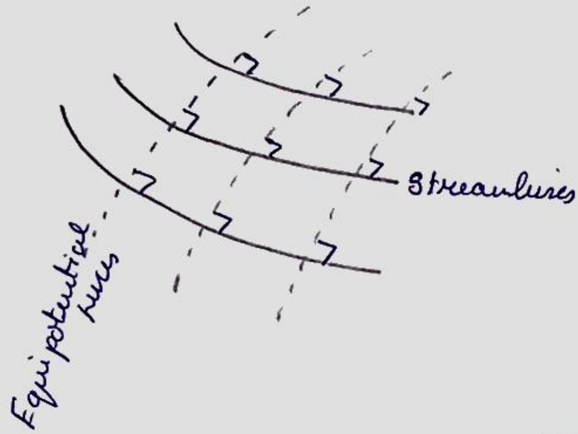
$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{\psi = \text{constant}}}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\psi = \text{constant}} \times \left(\frac{dy}{dx}\right)_{\phi = \text{constant}} = -1$$

→ Product of slope is -1.

→ Streamlines and Equipotential lines are Perpendicular to each other.





☆ Governing Equation for Potential Flows : [Understanding]

→ Potential flow is a simplified form of fluid flow, particularly useful in aerodynamics, that assumes that the flow is inviscid (frictionless), incompressible and irrotational.

* Derivation of Governing Equations :

1] Incompressible Continuity Equation :

For an incompressible flow, the density remains constant. The continuity equation, representing, conservation of mass, states that the divergence of the velocity fluid is zero.

$$\vec{\nabla} \cdot \vec{V} = 0 \quad --- \quad \vec{V} = \text{Velocity vector}.$$

2] Irrotational Flow and Velocity Potential:

Irrotational Flow means the fluid particles do not rotate about their own axes. Mathematically, the curl of the velocity field is zero:

$$\vec{\nabla} \times \vec{v} = 0$$

This condition allows the velocity field to be expressed as the gradient of a scalar function, called the velocity potential (ϕ):

$$\vec{v} = \nabla \phi$$

3] Laplace's Equation:

Substituting the velocity potential into the continuity equation yields Laplace's equation

$$\vec{\nabla}^2 \phi = 0$$

This is a second-order partial derivative that governs potential flow.

* Boundary Conditions:

1] Free-Stream Conditions: Far from any solid boundaries, the flow approaches a uniform free-stream velocity.

2] Solid Surface Conditions: On the surface of solid bodies, the velocity component normal to the surface is zero (no penetration). The tangential velocity is not constrained by the no-slip condition as it could be in viscous flows. This "slip" condition is a significant simplification arising from neglecting viscosity.

3] Free-Surface Conditions: If a free surface is present (e.g the interface between a liquid and a gas), the pressure is generally assumed to be constant (e.g. atmospheric pressure), and the Bernoulli's equation provides a relationship between the free surface elevation and the velocity.

* Derivation : [To heaven]

(i)
For an incompressible fluid , $\vec{\nabla} \cdot \vec{V} = 0$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad -(1)$$

For an irrotational flow, $\vec{\nabla} \times \vec{V} = 0$

$$\therefore u = \frac{\partial \phi}{\partial x} \text{ and } v = \frac{\partial \phi}{\partial y} \quad -(2)$$

Substituting eq (2) in (1) :

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\therefore \vec{\nabla}^2 \phi = 0 \quad --- \quad (*)$$

(ii)
Similarly , for an irrotational flow, $\vec{\nabla} \times \vec{V} = 0$

$$\therefore \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad -(3)$$

For an incompressible flow, we can define streamfunction ψ as :

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \quad -(4)$$

Substituting (4) in (3) :

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = 0$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\therefore \vec{\nabla}^2 \psi = 0 \quad --- \quad (*)$$

Numerical : [To Learn]

Q1: The velocity components of an incompressible, two dimensional flow are given by :

$$u = y^2 - x(1+x)$$

$$v = y(2x+1)$$

Find : (A) Does a streamline function exists ? If so, find it

(B) Does a velocity potential exists ? If so, find it.

Sol : For a velocity potential to exist, flow has to be irrotational :

$$\vec{\nabla} \times \vec{v} = 0$$

$$\therefore \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} y(2x+1) = 2y$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(y^2 - x - x^2) = 2y$$

$$\therefore 2y - 2y = 0$$

Thus, velocity field satisfies irrotational flow condition, hence velocity potential exists.

* To find velocity potential : $u = \frac{\partial \phi}{\partial x}$

$$\therefore y^2 - x - x^2 = \frac{\partial \phi}{\partial x}$$

Integrating on both sides w.r.t x

$$\therefore \phi(x,y) = xy^2 - \frac{x^2}{2} - \frac{x^3}{3} + f(y) + \text{Constant} \quad \text{— (1)}$$

* Now, also $v = \frac{\partial \phi}{\partial y} = 2xy + y$

Integrating on both sides w.r.t y

$$\therefore \phi(x,y) = \frac{2xy^2}{2} + \frac{y^2}{2} + f(x) + \text{Constant} \quad \text{— (2)}$$

* Comparing eq (1) and (2), we get:

$\therefore \quad f(x) = -\frac{x^2}{2} - \frac{x^3}{3} \qquad$ and $\quad f(y) = \frac{y^2}{2}$

$\therefore \; \phi(x, y)$ becomes:

$$\phi(x,y) = xy^2 + \frac{y^2}{2} - \frac{x^2}{2} - \frac{x^3}{3} + \text{Constant}$$

↳ Req. Ans.

Elementary Fluid Flows: [To Learn]

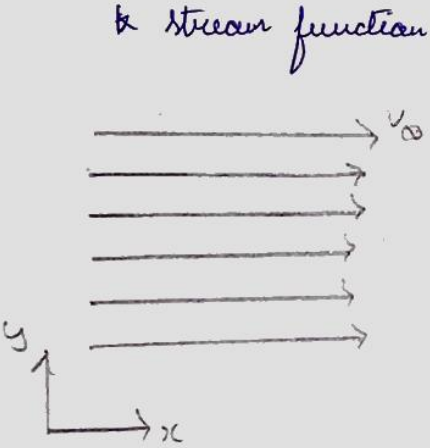
→ Elementary flows are fundamental flow patterns that, due to their simplified nature, are easily analysed and understood.

→ They serve as the building blocks for understanding more complex flows encountered in fluid mechanics.

Types of Elementary Flow:

1] Uniform Flow: This is the simplest type of flow where the flow moves with a constant velocity in a single direction. Imagine a straight river with consistent current - this exemplifies uniform flow.

* Velocity Potential (ϕ): $\phi = U_x$ for flow in the x-direction
* Stream function (Ψ): $\Psi = U_y$ for flow in the x-direction.



Now, $\quad \phi = U_x = \frac{\partial \phi}{\partial x} \qquad \frac{\partial \phi}{\partial y} = v$

$\therefore \; V_\infty = \frac{\partial \phi}{\partial x} \qquad \therefore \frac{\partial \phi}{\partial y} = 0$

$\therefore \; \phi = V_\infty x + f(y) \qquad \therefore \; \phi = f(x)$

On Comparison, $f(y) = 0$

$\therefore \; \phi = V_\infty x \qquad \therefore \; x = \frac{\phi}{V_\infty} = \text{Constant}$

For ψ : $\frac{\partial \psi}{\partial y} = u$ $\frac{\partial \psi}{\partial x} = -v$

$\therefore \frac{\partial \psi}{\partial y} = V_\infty$ $\therefore \frac{\partial \psi}{\partial x} = 0$

$\therefore \partial \psi = V_\infty \, \partial y$ $\therefore \psi = f(y)$

$\therefore \psi = V_\infty y + f(x)$

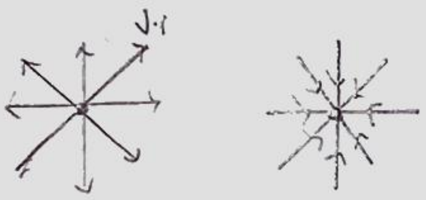
On Comparison, $f(x) = 0$

$$\therefore \boxed{\begin{aligned} \psi &= V_\infty y \\ \therefore\; y &= \frac{\psi}{V_\infty} = \text{Constant} \end{aligned}}$$

2] Source / Sink Flow : [To learn]

→ A source flow represents a point from which fluid emanates (spreads out) radially outwards in all directions, like water bubbling up from a spring.

→ A sink flow, conversely, is a point where fluid converges radially inward from all directions, akin to water draining down a plughole.



Source Sink

V_r – Radial Velocity

V_θ – Tangential Velocity.

* Source Flow :

$V_r \propto \frac{1}{r}$, $V_\theta = 0$

$\therefore \underbrace{V_r = \frac{C}{r}}_{(1)}$, $V_\theta = 0$ [$\because C$ – Constant]

To find C, consider volume flow rate at any distance r.

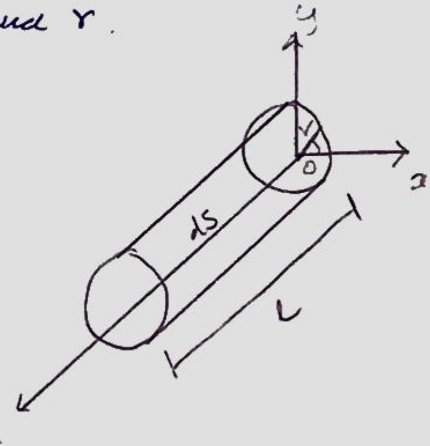
Volume Flow Rate q = Area × Velocity

$$q = 2\pi r l \times V_r$$

$$\therefore \frac{q}{l} = 2\pi r V_r$$

$$\therefore \lambda = 2\pi r V_r$$

$$\therefore V_r = \frac{\lambda}{2\pi r}$$



λ = Volume Flow Rate at any radius r per unit length.

λ = Source's Strength

* Putting the value of V_r in eq (1)

$$\therefore \frac{\lambda}{2\pi r} = \frac{C}{r}$$

$$\therefore C = \frac{\lambda}{2\pi}$$

* Now, For Velocity Potential : $\frac{\partial \phi}{\partial r} = V_r$ and $\frac{1}{r}\frac{\partial \phi}{\partial \theta} = V_\theta$

$$\therefore \frac{\partial \phi}{\partial r} = \frac{\lambda}{2\pi r} \qquad \therefore \frac{\partial \phi}{\partial \theta} = 0$$

$$\therefore \phi = \frac{\lambda}{2\pi} \ln r + f(\theta) \qquad \therefore \phi = f(r)$$

Comparing both, $f(\theta) = 0$

$$\boxed{\therefore \phi = \frac{\lambda}{2\pi} \ln r}$$

* Equipotential lines corresponds to lines of

$$\ln r = \text{Constant}$$

$$\therefore r = C_1$$

* For Streamline Functions, $\frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_r$ $\therefore \frac{\partial\psi}{\partial r} = -V_\theta$

$\therefore \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\lambda}{2\pi r}$ $\therefore \frac{\partial\psi}{\partial r} = 0$

$\therefore \psi = \frac{\lambda}{2\pi}\theta + f(r)$ $\therefore \psi = f(\theta)$

Comparing both the equations we get, $f(r) = 0$

$$\boxed{\therefore \psi = \frac{\lambda}{2\pi}\theta}$$

Streamlines corresponds to the lines of θ = constant

* For Sink : $-\lambda \rightarrow$ Sink's strength

$\therefore \psi = -\frac{\lambda}{2\pi}\theta$ and $\phi = -\frac{\lambda}{2\pi}\ln r$

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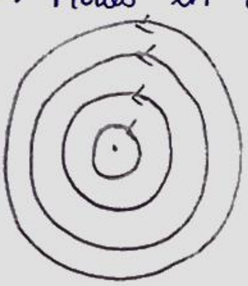
* Free Vortex Flow : [To learn]

→ Flows in closed circular streamlines is called a vortex flow.



$V_r = 0$ and $V_\theta \propto \frac{1}{r}$ C = Constant

$\therefore V_\theta = \frac{C}{r}$ — (1)

* Circulation : $\Gamma = -\oint \vec{v}\cdot\vec{dl}$

" It is defined as the negative line integral of velocity along the closed curve".

$\therefore \Gamma = -\oint \vec{V} \cdot \vec{dl}$

$\therefore \Gamma = -V_\theta \oint \vec{dl}$

$\therefore \Gamma = -V_\theta (2\pi r)$

$$\boxed{\therefore V_\theta = \frac{-\Gamma}{2\pi r}}$$

Comparing it with eq (1):

$\therefore V_\infty = \frac{-\Gamma}{2\pi r} = \frac{C}{r}$

$\therefore C = -\frac{\Gamma}{2\pi}$

$\rightarrow \Gamma$ is called the <u>Vortex's Strength</u>.

* <u>For Velocity Potential</u>: $\frac{\partial \phi}{\partial r} = V_r$ and $\frac{1}{r}\frac{\partial \phi}{\partial \theta} = V_\theta$

$\therefore \frac{\partial \phi}{\partial r} = 0$

$\therefore \partial\phi = f(\theta)$

$\therefore \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{-\Gamma}{2\pi r}$

$\therefore \partial\phi = \frac{-\Gamma}{2\pi}\partial\theta$

$\therefore \phi = \frac{-\Gamma}{2\pi}\theta + f(r)$

On comparing, $f(r) = 0$

$$\therefore \boxed{\phi = -\frac{\Gamma}{2\pi}\theta}.$$

θ = Constant on the equipotential lines.

* For Streamline Function: $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \phi}$ $\quad -V_\theta = \frac{\partial \psi}{\partial r}$

$\therefore 0 = \frac{\partial \psi}{\partial \phi}$

$\therefore \partial\psi = 0$

$\therefore \psi = f(r)$

$\frac{-\Gamma}{2\pi r} = \frac{\partial \psi}{\partial r}$

$\therefore \partial\psi = \frac{-\Gamma}{2\pi r}\partial r$

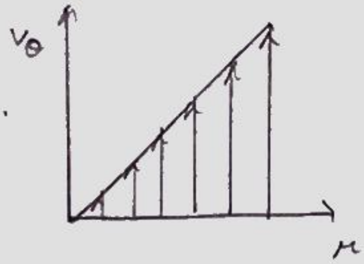
$\therefore \psi = \frac{-\Gamma}{2\pi}\ln r + f(\theta)$

On Comparing, $f(\theta) = 0$

$$\boxed{\therefore \psi = \frac{-\Gamma}{2\pi}\ln r}$$

r = Constant on the streamlines

Forced Vortex Flow: [To learn]



$V_r = 0 \quad V_\theta \propto r$

$\therefore v_\theta = \Omega r \qquad \Omega$ = Angular Velocity in flow field.

$\rightarrow \vec{\nabla} \times \bar{v} = \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]$

$= \frac{1}{r}\frac{\partial}{\partial r}(r^2 \Omega)$

$= \frac{1}{r} 2r\Omega$

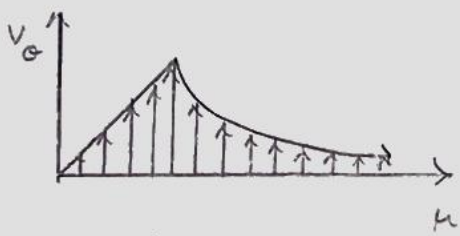
$0 \neq 2\Omega$

→ Rotational vortex.

→ Not a potential flow

→ Does not satisfied the irrotational flow condition.

Real Vortex: [To learn]



Rankine Vortex

→ All real vortices will be a combination of both forced vortex and a free vortex.

→ Example: Whirlpools, Tornados, cyclones.

Combination of Elementary Flows: [To learn]

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

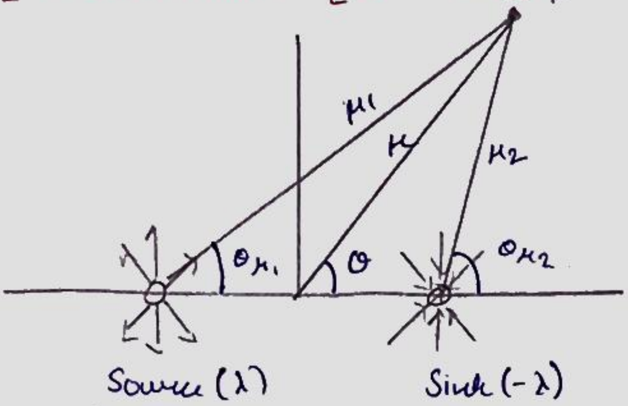
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

— 2nd Order Linear Equations.

- If ψ_1 is a solution
- If ψ_2 is a solution

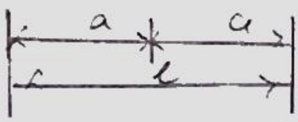
then $\underbrace{\psi_1 + \psi_2}_{\text{Physically possible potential flow}}$ is also a solution of this equation

✯ Doublet Flow: [To learn]



$$\psi_{source} = \frac{\lambda}{2\pi}\theta_1$$

$$\psi_{sink} = \frac{-\lambda}{2\pi}\theta_2$$



$l \to 0 \quad \lambda \to \infty$

$\therefore\ l\lambda = \text{Constant}$

At a point P in the flow,

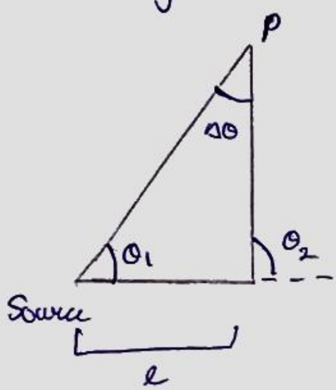
$$\psi = \psi_{source} + \psi_{sink}$$

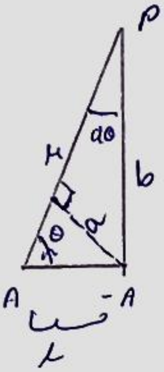
$$= \frac{\lambda}{2\pi}\theta_1 - \frac{\lambda}{2\pi}\theta_2$$

$$= \frac{\lambda}{2\pi}(\theta_1 - \theta_2)$$

$$= \frac{-\lambda}{2\pi}\Delta\theta$$

* Limiting case for a Doublet:





→ Let the distance l approach zero while the absolute magnitude of the strengths of the source and sink increases in such a fashion that the product ~~lA~~ $l\lambda$ remains constant.

→ In the limit, $l \to 0$ while ~~lA~~ $l\lambda$ remains constant, we obtain a special flow pattern defined as doublet.

→ The strength of doublet is defined by K and is defined as

$K = l\lambda$.

* Streamfunction of a doublet is:

$$\psi = \lim_{\substack{l \to 0 \\ K = l\lambda = \text{constant}}} \left(-\frac{\lambda}{2\pi} d\theta \right) \quad — (1)$$

From the diagram, $a = l\sin\theta$

$$\therefore \cos d\theta = \frac{r - l\cos\theta}{b}$$

$$\therefore b = \frac{r - l\cos\theta}{\cos d\theta}$$

Now, as $l \to 0$, $d\theta \to 0$, $\therefore \cos d\theta \approx 1$, $\tan d\theta \approx d\theta$

$$\therefore b = r - l\cos\theta$$

$$\therefore \tan d\theta = \frac{a}{b}$$

$$\therefore d\theta = \frac{a}{b} = \frac{l\sin\theta}{r - l\cos\theta} \quad — (2)$$

* Substituting (2) in (1), we get

$$\psi = \lim_{\substack{l \to 0 \\ K = l\lambda}} \left[-\frac{\lambda}{2\pi} \frac{l\sin\theta}{r - l\cos\theta} \right]$$

$$= \lim_{l \to 0} \left[-\frac{K}{2\pi} \frac{\sin\theta}{r - l\cos\theta} \right]$$

$$\boxed{\psi = \cancel{\lim} \; \frac{-K}{2\pi} \frac{\sin\theta}{r}}$$

$$\phi = \phi_{source} + \phi_{sink}$$

$$= \frac{\lambda}{2\pi} \ln r_1 \; \cancel{+} - \frac{\lambda}{2\pi} \ln r_2$$

$$= \frac{\lambda}{2\pi} \ln\left(\frac{r_1}{r_2}\right) = -\frac{\lambda}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

* From the diagram $r_1 = r$ and $r_2 = b = r - l\cos\theta$

$$\therefore \frac{r_2}{r_1} = \frac{r - l\cos\theta}{r} = 1 - \frac{l}{r}\cos\theta$$

$$\therefore \phi = \lim_{\substack{l \to 0 \\ K = l\lambda = \text{Constant}}} \left[\frac{-\lambda}{2\pi} \ln\left(\frac{r_2}{r_1}\right) \right] = \lim_{l \to 0} \left[\frac{-\lambda}{2\pi} \ln\left(1 - \frac{l}{r}\cos\theta\right) \right]$$

* Using Series Expansion for $\ln(1-x)$

$$\therefore \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

$$\therefore \phi = \lim_{l \to 0} \left[\frac{-\lambda}{2\pi} \left(-\frac{l}{r}\cos\theta - \frac{l^2}{2r^2}\cos^2\theta - \frac{l^3}{3r^3}\cos^3\theta - \dots \right) \right]$$

$$\boxed{\phi = \frac{K}{2\pi} \frac{\cos\theta}{r}}$$ --- Velocity Potential for a Doublet Flow.

☆ Summary: [☆☆☆☆☆ IMP]

Type of Flow	Velocity	ϕ	ψ
Uniform Flow in x-direction	$u = V_\infty$	$V_\infty x$	$V_\infty y$
Source	$V_r = \frac{\lambda}{2\pi r}$	$\frac{\lambda}{2\pi} \ln r$	$\frac{\lambda}{2\pi} \theta$
Vortex	$V_\theta = -\frac{\Gamma}{2\pi r}$	$-\frac{\Gamma}{2\pi} \theta$	$\frac{\Gamma}{2\pi} \ln r$
Doublet	$V_r = -\frac{K}{2\pi} \frac{\cos\theta}{r^2}$ $V_\theta = \frac{-K}{2\pi} \frac{\sin\theta}{r^2}$	$\frac{K}{2\pi} \frac{\cos\theta}{r}$	$-\frac{K}{2\pi} \frac{\sin\theta}{r}$

☆ Non-lifting Flow over a cylinder: [Understand Only]

→ When a fluid flows past a circular cylinder, the resulting flow field can be complex and depends on factors such as the fluid's viscosity and the speed of the flow.

Non-lifting Flow: This refers to a flow pattern where the cylinder does not experience a lift force, only drag. This typically occurs when the flow is symmetric around the cylinder.

Inviscid Flow: In an idealized scenario where the fluid is assumed to have no viscosity (inviscid), potential flow theory can be used to describe the flow field. This theory predicts a symmetrical pressure difference around the cylinder, resulting in zero drag.

D'Alembert's Paradox: This theoretical prediction of zero drag in inviscid flow contradicts the real-world observations, where drag is always present. This discrepancy is known as D'Alembert's Paradox.

* Viscous Flow:

→ In reality, all fluids have viscosity which significantly affects the fluid pattern.

→ Viscosity introduces friction between the fluid and the circular cylinder surface, creating a boundary layer.

- Boundary Layer: The boundary layer is a thin region near the cylinder surface where the fluid velocity changes from zero at the surface (due to no-slip condition) to the free stream velocity away from the surface.
- Separation: As the flow moves around the cylinder, the pressure increases on the downstream side. This adverse pressure gradient can cause the boundary layer to separate from the cylinder surface.

- Wake : Seperation leads to the formation of a wake behind the cylinder, a region of low pressure and recirculating flow

* Drag :

The seperation and wake formation are the primary reasons for drag on a circular cylinder in a viscous flow. The drag is composed of two components :

- Pressure Drag : This arises from the pressure difference between the front and rear of the cylinder due to wake.
- Friction Drag : This results from the viscous shear stresses acting on the cylinder within the boundary layer.

* Effect of Reynolds Number :

The effect of flow pattern and the drag experienced by the cylinder are strongly influenced by the Reynolds Number (Re), a dimensional quantity that represents the ratio of inertial forces to viscous forces in the flow.

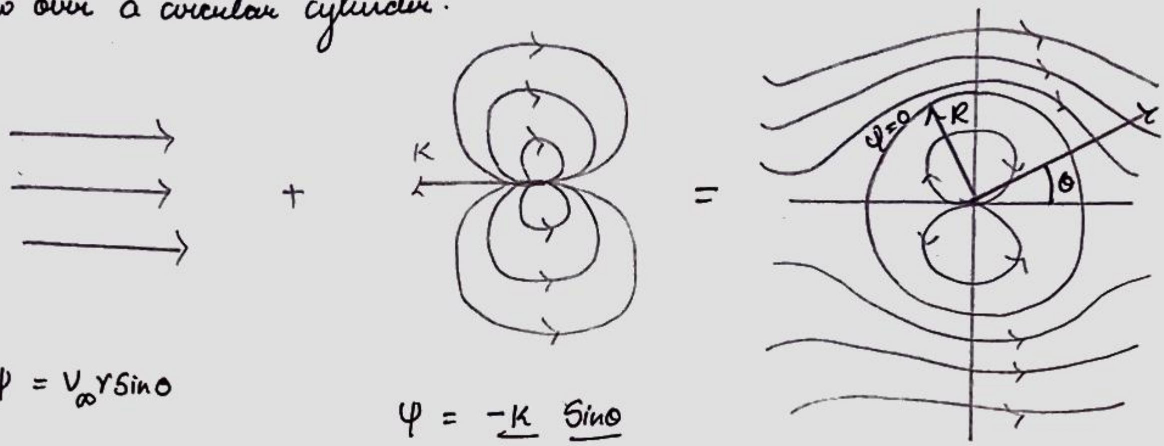
- Low Re (Creeping Flow) : At very low Re ($Re \ll 1$), viscous flow dominates, and the flow is smooth and attached to the cylinder. Drag is primarily due to friction.
- Moderate Re : As Re increases, the boundary layer becomes thinner, and eventually seperation occurs, leading to wake formation and increased pressure drag.
- High Re : At high Re, the flow in the wake ~~for~~ becomes turbulent, and the drag coefficient decreases, but remains significant due to large pressure drag component involved.

* Rotation and Magnus Effect :

If the cylinder is rotating, the flow pattern becomes asymmetric, and a lift force is generated perpendicular to the flow direction. This phenomenon is known as the Magnus effect.

★ Derivation : [To learn]

→ The combination of a uniform flow and a doublet produces the flow over a circular cylinder.



$\psi = V_\infty r \sin\theta$

Uniform Flow

$\psi = \frac{-K}{2\pi} \frac{\sin\theta}{r}$

Doublet

$\psi = V_\infty r \sin\theta - \frac{K}{2\pi} \frac{\sin\theta}{r}$

Flow over a cylinder

→ $\psi = V_\infty r \sin\theta - \frac{K}{2\pi} \frac{\sin\theta}{r}$

$$\psi = V_\infty r \sin\theta \left[1 - \frac{K}{2\pi V_\infty r^2}\right]$$

Let $R^2 = \frac{K}{2\pi V_\infty}$

$$\boxed{\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right)}$$ --- It is the stream function for a uniform flow ~~over~~ - doublet combination.

* When $r = R \Rightarrow \psi = 0$

$r = R$ is the equation of circle.

Therefore, $\psi = 0$ is the stream function for the flow over a circle of radius R.

* Velocity Field:

- Radial Velocity (V_r):

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}\left[V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right)\right]$$

$$= \frac{1}{r}\left(V_\infty r \cancel{\sin\theta} \cos\theta\right)\left(1 - \frac{R^2}{r^2}\right)$$

$$\boxed{V_r = \left(1 - \frac{R^2}{r^2}\right) V_\infty \cos\theta}$$

- Tangential Velocity (V_θ):

$$V_\theta = -\frac{\partial \psi}{\partial r} = -\frac{\partial}{\partial r}\left[V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right)\right]$$

$$= -\left[V_\infty r \sin\theta \frac{2R^2}{r^3} + \left(1 - \frac{R^2}{r^2}\right) V_\infty \sin\theta\right]$$

$$\boxed{V_\theta = -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin\theta}$$

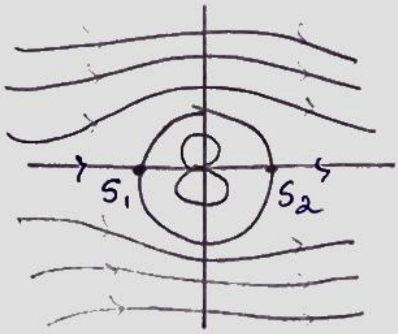
* To Locate the Stagnation Points:

[Stagnation Points are the points in the flow where velocity is zero].

$\therefore V_r = 0 \Rightarrow \left(1 - \frac{R^2}{r^2}\right) V_\infty \cos\theta = 0$

$\therefore V_\theta = 0 \Rightarrow -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin\theta = 0$

* Simultaneously solving these two equations for r and θ, we find that there are two stagnation points, located at $(r, \theta) = (R, 0)$ and (R, π)



→ Note that the $\psi = 0$ streamline, since it goes through the stagnation points, is dividing the streamline.

→ That is, all the flow inside $\psi = 0$ (inside the circle) comes from the doublet and the flow outside $\psi = 0$ (outside the circle) comes from the uniform flow.

→ Therefore, we replace the flow inside the circle by a solid body, and the external flow will not know the difference.

→ Consequently the inviscid, irrotational, incompressible flow over a circular cylinder R can be synthesized by adding a uniform flow with velocity V_∞ and a doublet of strength K, where R is related to V_∞ and K through

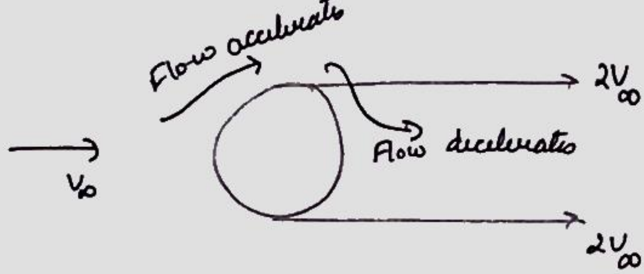
$$R = \sqrt{\frac{K}{2\pi V_\infty}}$$

☆ Velocity distribution over a cylinder: [To learn]

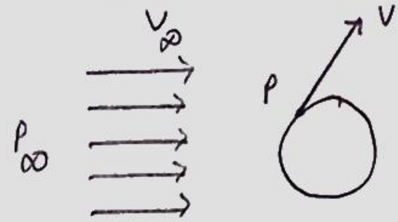
$$V_r = \left(1 - \frac{R^2}{r^2}\right) V_\infty \cos\theta \quad \xrightarrow[r = R]{\text{Substituting}} \quad V_r = 0$$

$$V_\theta = -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin\theta \qquad\qquad V_\theta = -2V_\infty \sin\theta$$

→ On the surface of the cylinder, $V = V_\theta = -2V_\infty \sin\theta$



★ Pressure Distribution over a cylinder : [To Learn]



Co-efficient of Pressure : $C_p = \dfrac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2}$

Applying Bernoulli's equation between 1 and 2 :

$$P_\infty + \frac{1}{2}\rho V_\infty^2 = P + \frac{1}{2}\rho V^2$$

$$\therefore \quad P - P_\infty = \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2$$

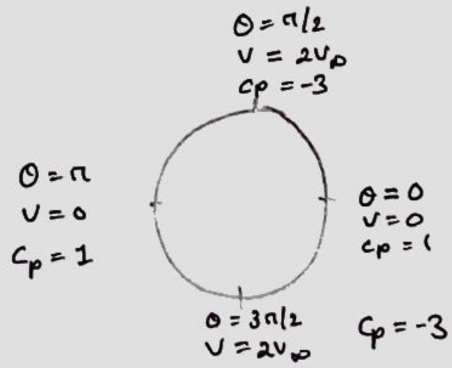
$$\therefore \quad P - P_\infty = \frac{1}{2}\rho V_\infty^2 \left(1 - \frac{V^2}{V_\infty^2}\right)$$

$$\therefore \quad \frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \left(\frac{V}{V_\infty}\right)^2$$

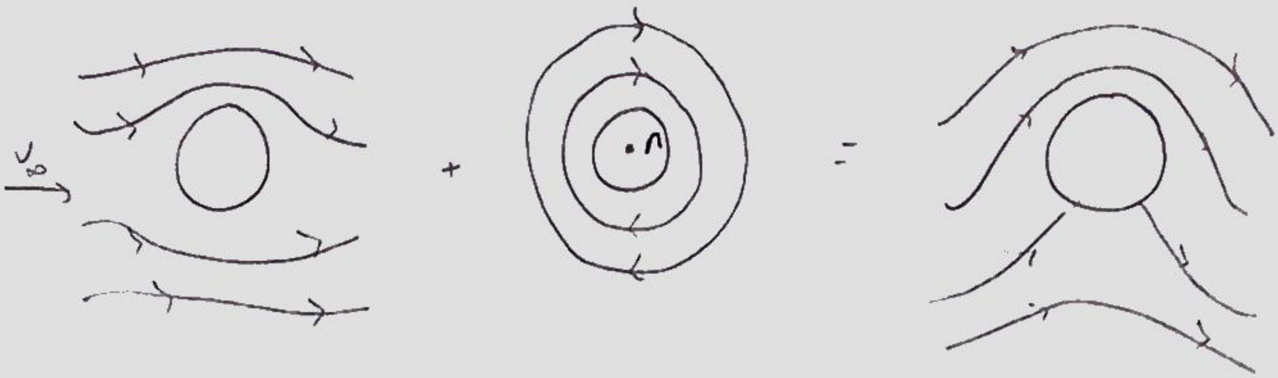
$$\therefore \quad C_p = 1 - \left(\frac{V}{V_\infty}\right)^2$$

$$\therefore \quad C_p = 1 - \left(\frac{-2V_\infty \sin\theta}{V_\infty}\right)^2$$ [$\because$ On the cylinder surface $V = -2V_\infty \sin\theta$]

$$\therefore \quad C_p = 1 - 4\sin^2\theta$$



☆ Lifting Flow over a circular cylinder - Flow over a rotating cylinder.



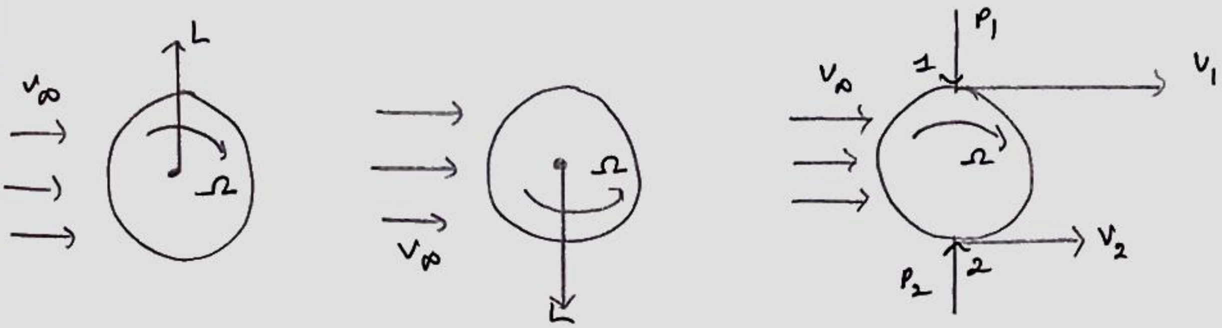
Non-Lifting Flow over a cylinder

Vortex of Strength Γ

Lifting Flow over a cylinder

→ A rotating cylinder or sphere moving in a fluid produces lift force.

↳ Magnus Effect OR Robin-Magnus Effect.



$V_1 > V_2$

$\therefore P_1 < P_2$