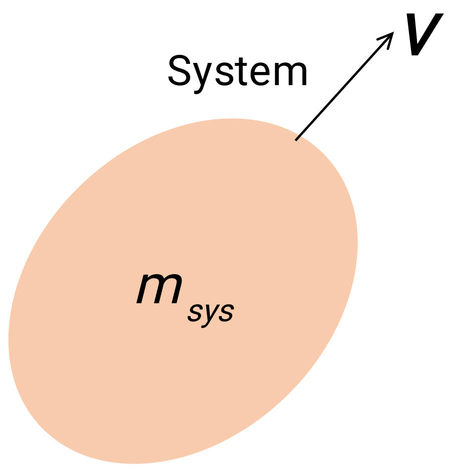
The Reynolds Transport Theorem

Conservation parameters: mass (m), momentum (mV) and energy (E)

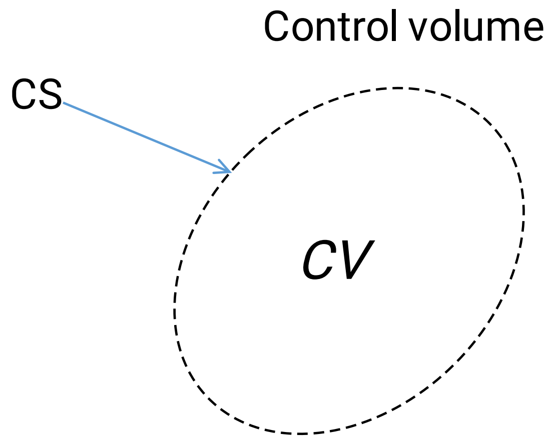
B - Any flow parameter
b - Amount of that parameter per unit mass = B/ m

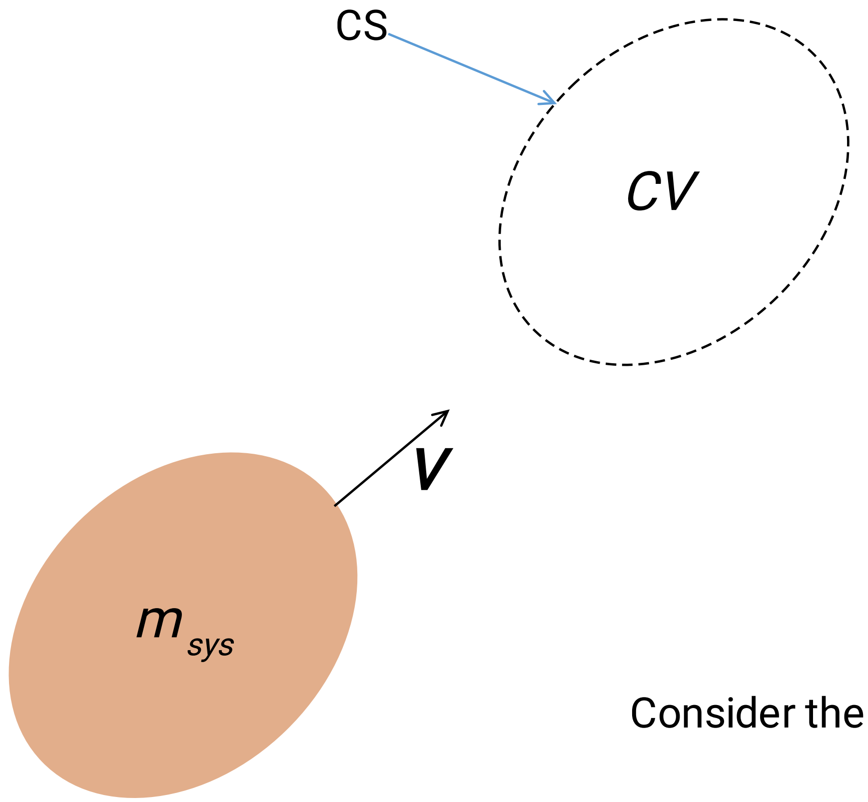
	B	b=B/m
Mass	m	1
Momentum	$m\vec{V}$	$\vec{V}$
Energy	E	e

Consider a



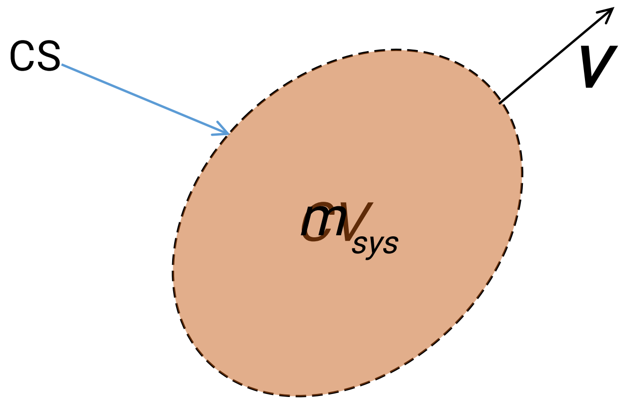
and





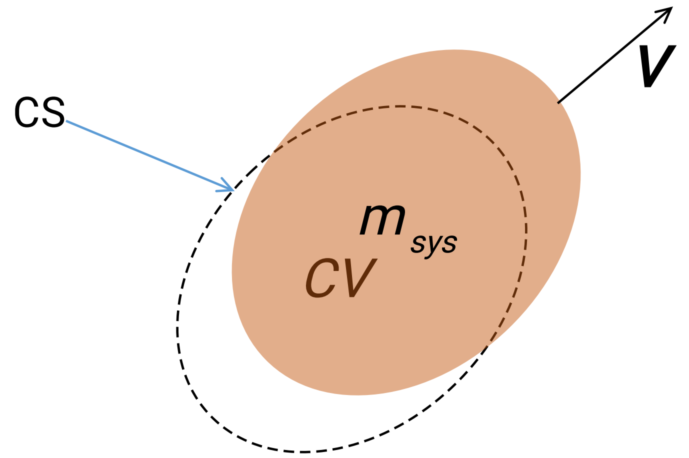
Consider the case of System flowing through the Control Volume

At a time instant t



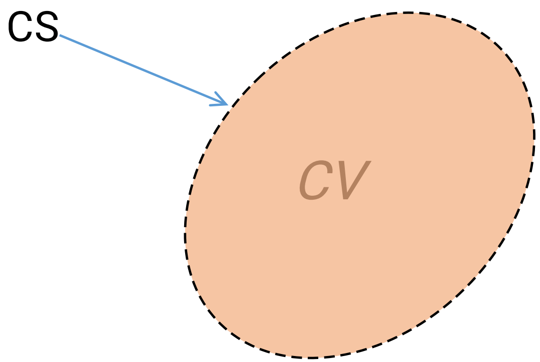
the system occupies the control volume

At a time instant t+δt



the system moves out of CV

At time t



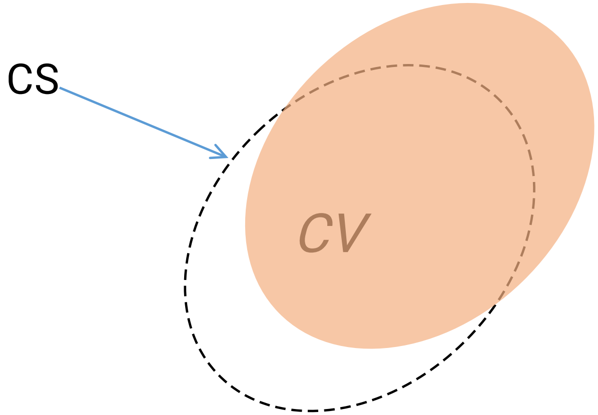
At time t, the system occupies the control volume

Sys = CV

At time t,

$$B_{sys}(t) = B_{CV}(t)$$

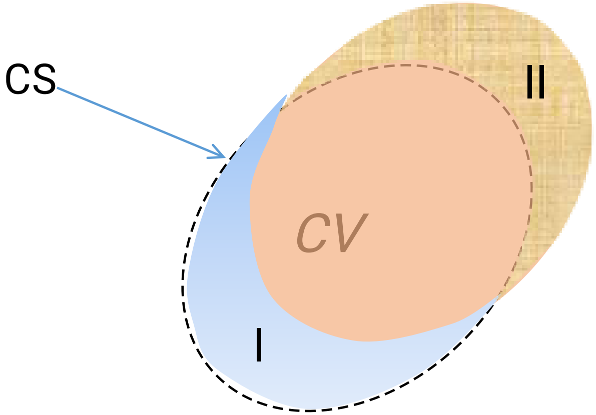
At time $t+\delta t$



At time $t+\delta t$, the system moving out of CV

Sys =

At time $t+\delta t$



At time $t+\delta t$, the system moving out of CV

Sys = CV − I + II

At time $t+\delta t$,

$$B_{sys}(t+\delta t) = B_{CV}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t)$$

Change in the amount of B in the system in the time interval δt is:

$$\frac{\delta B_{sys}}{\delta t} = \frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t}$$

$$= \frac{B_{CV}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t) - B_{sys}(t)}{\delta t}$$

$$= \frac{B_{CV}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t) - B_{CV}(t)}{\delta t}$$

$$\Rightarrow \frac{\delta B_{sys}}{\delta t} = \frac{B_{CV}(t+\delta t) - B_{CV}(t)}{\delta t} + \frac{B_{II}(t+\delta t)}{\delta t} - \frac{B_{I}(t+\delta t)}{\delta t}$$

Simplifying the above equation in the limit of $\delta t \to 0$

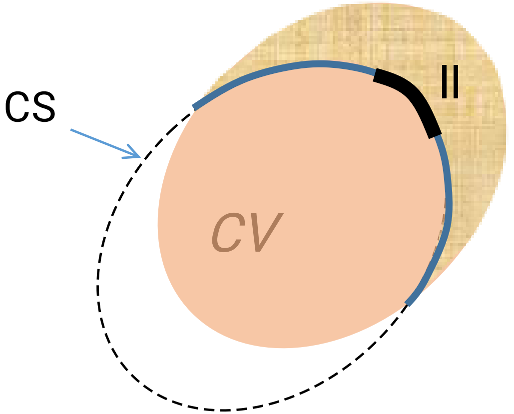
LHS:

$$\lim_{\delta t \to 0} \frac{\delta B_{sys}}{\delta t} = \frac{dB_{sys}}{dt} = \frac{DB_{sys}}{Dt}$$

First term on RHS:

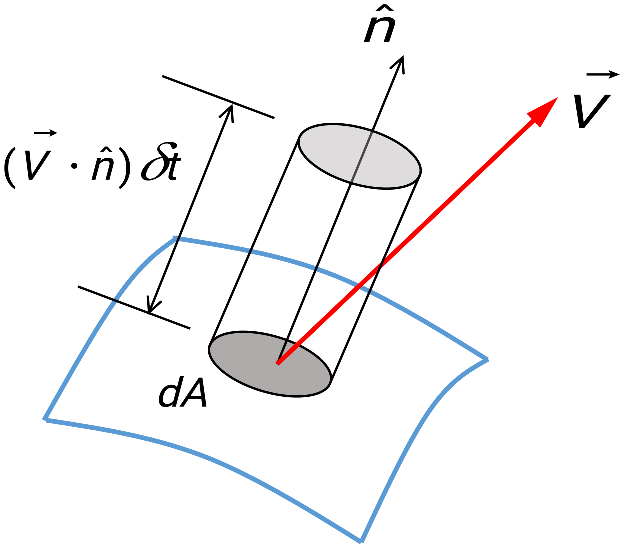
$$\lim_{\delta t \to 0} \frac{B_{CV}(t+\delta t) - B_{CV}(t)}{\delta t} = \frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t} \iiint_{CV} \rho\, b\, d\forall$$

Second term on RHS:



$$\lim_{\delta t \to 0} \frac{B_{II}(t+\delta t)}{\delta t} = \dot{B}_{out}$$

the rate at which the extensive parameter B flows from the control volume across the control surface

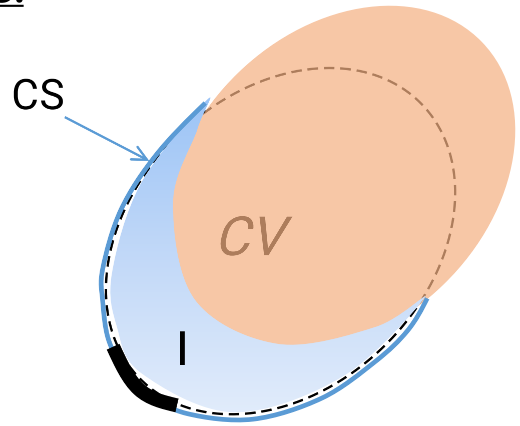


$$B_{II}(t+\delta t) = \iint_{CS_{out}} \rho\, b\, (\vec{V}\cdot\hat{n})\,\delta t\, dA$$

$$\frac{B_{II}(t+dt)}{\delta t} = \iint_{CS_{out}} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA$$

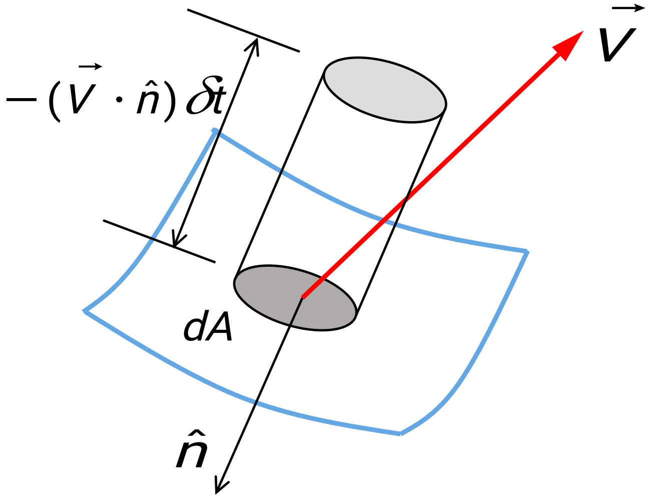
$$\Rightarrow \dot{B}_{out} = \iint_{CS_{out}} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA$$

Third term on RHS:



$$\lim_{\delta t \to 0} \frac{B_I(t+\delta t)}{\delta t} = \dot{B}_{in}$$

the rate at which the extensive parameter B flows into the control volume across the control surface



$$B_I(t+\delta t) = -\iint_{CS_{in}} \rho\, b\, (\vec{V}\cdot\hat{n})\,\delta t\; dA$$

$$\Rightarrow \dot{B}_{in} = -\iint_{CS_{in}} \rho\, b\, (\vec{V}\cdot\hat{n})\; dA$$

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \dot{B}_{out} - \dot{B}_{in}$$

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \iint_{CS_{out}} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA - \left(- \iint_{CS_{in}} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA \right)$$

$$\boxed{\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \oiint_{CS} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA}$$

Reynolds transport theorem – Relates System to Control Volume

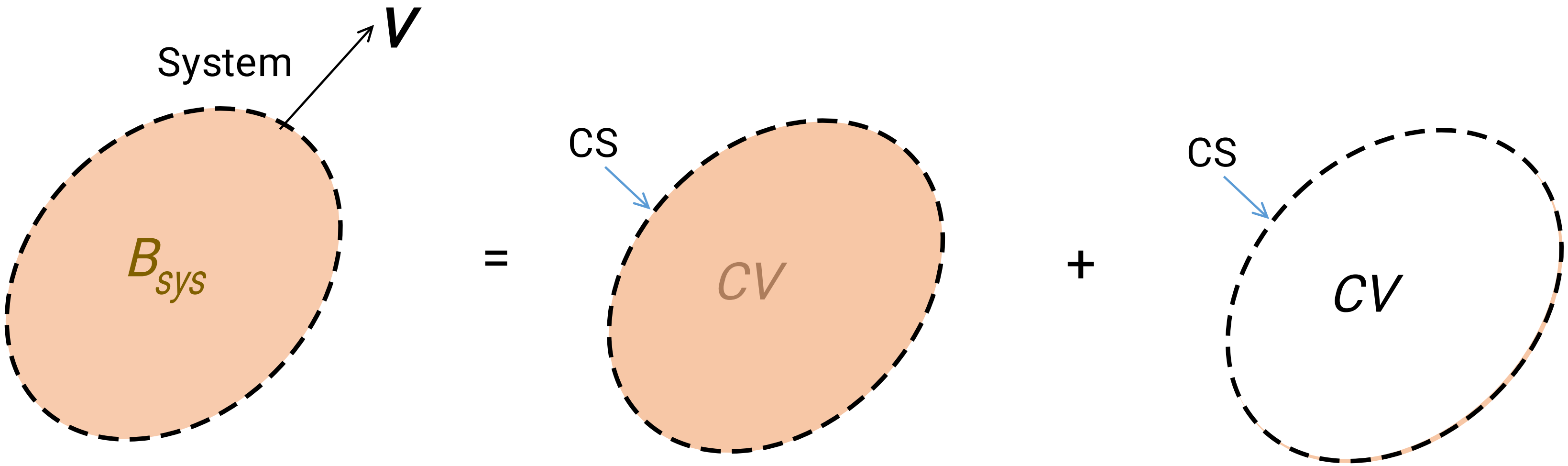
$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \oiint_{CS} \rho\, b\, (\vec{V} \cdot \hat{n})\, dA$$

Time rate of change of a parameter B of a system	=	Time rate of change of parameter B within the control volume as the fluid flows through it	+	Net flux of the parameter B across the control surface

End of Reynolds Transport Theorem Derivation

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{cv} \rho\, b\; d\forall + \oiint_{cs} \rho\, b\, (\vec{V}\cdot\hat{n})\; dA$$

Time rate of change of a parameter B of a system = Time rate of change of parameter B within the control volume as the fluid flows through it + Net flux of the parameter B across the control surface



Governing equations expressed in Control Volume Concept:

RTT:

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \oiint_{CS} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA \qquad b = \frac{B}{m}$$

Mass conservation/Continuity:

Substituting $B = m \Rightarrow b = 1$ in RTT

$$\frac{Dm}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, d\forall + \oiint_{CS} \rho\, (\vec{V}\cdot\hat{n})\, dA$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\iiint_{CV} \rho\, d\forall + \oiint_{CS} \rho\, (\vec{V}\cdot\hat{n})\, dA = 0$$

Momentum conservation:

Substituting $B = m\vec{V} \quad \Rightarrow \quad b = \vec{V}$ in RTT

$$\frac{D(m\vec{V})}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho \vec{V}\, d\forall + \oiint_{CS} \vec{V}\, \rho(\vec{V}\cdot\hat{n})\, dA$$

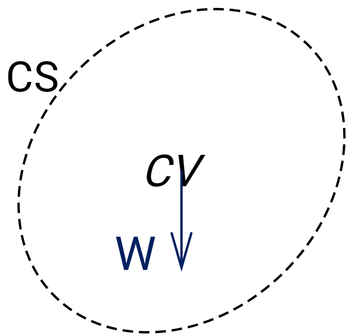
Newton's second law: $\dfrac{D(m\vec{V})}{Dt} = \sum \vec{F}$

$$\Rightarrow \quad \frac{\partial}{\partial t}\iiint_{CV} \rho \vec{V}\, d\forall + \oiint_{CS} \vec{V}\, \rho(\vec{V}\cdot\hat{n})\, dA = \sum \vec{F}$$

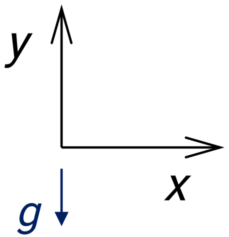
$$\Rightarrow \quad \frac{\partial}{\partial t}\iiint_{CV} \rho \vec{V}\, d\forall + \oiint_{CS} \vec{V}\, \rho(\vec{V}\cdot\hat{n})\, dA = \vec{F}_{body} + \vec{F}_{surface} + \vec{F}_{external}$$

$\underline{\vec{F}_{body}\ :}$

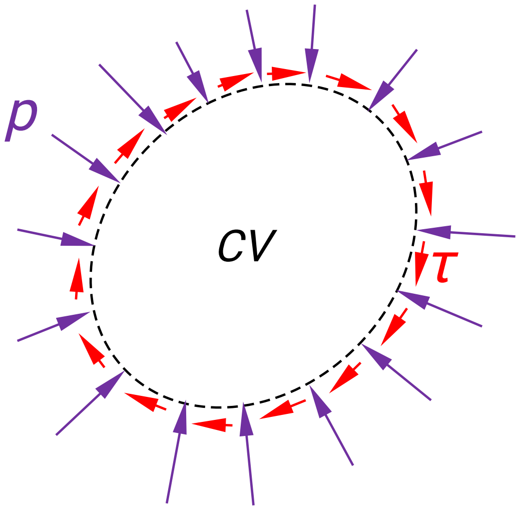
Body force – Weight of the control volume



$$\vec{F}_{body} = \vec{W} = \iiint_{CV} \rho\, \vec{g}\, d\forall$$



$\underline{\vec{F}_{surface}\ :}$

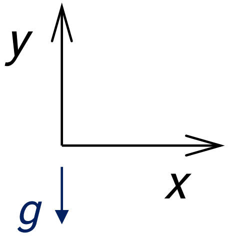


$$\vec{F}_{surface} = \vec{F}_{pressure} + \vec{F}_{shear}$$

$$\vec{F}_{surface} = -\oiint_{CS} p\, dA + \oiint_{CS} \tau\, dA$$

Governing equations for control volume analysis

$$\vec{V} = u\hat{i} + v\hat{j}$$



$$F_{x,body} = 0$$

$$F_{y,body} = -W$$

Continuity equation:

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, d\forall + \oiint_{CS} \rho \, (\vec{V} \cdot \hat{n}) \, dA = 0$$

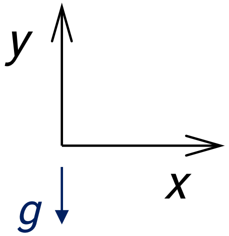
x-momentum equation:

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, u \, d\forall + \oiint_{CS} u \, \rho \, (\vec{V} \cdot \hat{n}) \, dA = F_{x,body} + F_{x,\,press} + F_{x,\,shear} + F_{x,\,ext}$$

y-momentum equation:

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, v \, d\forall + \oiint_{CS} v \, \rho \, (\vec{V} \cdot \hat{n}) \, dA = F_{y,body} + F_{y,\,press} + F_{y,\,shear} + F_{y,\,ext}$$

If the flow properties are uniform over the area
The equations can be expressed in simplified form as....



Continuity equation:

$$\frac{\partial m_{CV}}{\partial t} + \dot{m}_{out} - \dot{m}_{in} = 0$$

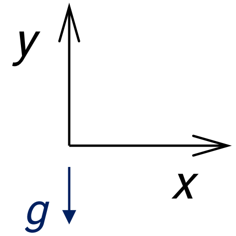
x-momentum equation:

$$\frac{\partial (mu)_{CV}}{\partial t} + (\dot{m}u)_{out} - (\dot{m}u)_{in} = F_{x,\,press} + F_{x,\,shear} + F_{x,\,ext}$$

y-momentum equation:

$$\frac{\partial (mv)_{CV}}{\partial t} + (\dot{m}v)_{out} - (\dot{m}v)_{in} = -W + F_{y,\,press} + F_{y,\,shear} + F_{y,\,ext}$$

If the flow properties are uniform over the area and If the flow is Steady
The equations can be expressed in simplified form as….



Continuity equation:

$$\dot{m}_{out} = \dot{m}_{in}$$

x-momentum equation:

$$(\dot{m}u)_{out} - (\dot{m}u)_{in} = F_{x,\,press} + F_{x,\,shear} + F_{x,\,ext}$$

y-momentum equation:

$$(\dot{m}v)_{out} - (\dot{m}v)_{in} = -W + F_{y,\,press} + F_{y,\,shear} + F_{y,\,ext}$$