Unit- III. Applications of Partial Differential Equations.

Classification of Partial differential equations of the second Order :

The most general linear partial differential equation of second order can be written as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \longrightarrow (1)$$

ie, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

where A, B, C, D, E, F are in general functions of x & y.

The equation (1) of second order (linear) is said to

(i) elliptic if $B^2 - 4AC < 0$

(ii) hyperbolic if $B^2 - 4AC > 0$

(iii) parabolic if $B^2 - 4AC = 0$.

Examples :-

(1) Elliptic Type : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace equation in two dimension)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ (Poisson's equation)

(2) Parabolic type : $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (one-dimensional heat equation)

(3) Hyperbolic type : $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (one-dimensional wave equation).

Problems :- Classify the following equations :

① $x u_{xx} + u_{yy} = 0$

$A = x \quad B = 0 \quad C = 1$

$B^2 - 4AC = 0 - 4(1)(x)$

$= -4x$

The equation is elliptic if $x > 0$

The equation is hyperbolic if $x < 0$

The equation is parabolic if $x = 0$.

Note :- The same differential equation may be elliptic in one region, parabolic in another and hyperbolic in some other region.

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② $$\frac{\partial u}{\partial x^2} + 2\frac{\partial u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1, \quad B = 2, \quad C = 1$

$B^2 - 4AC = 4 - 4(1)(1) = 0$.

Hence, the equation is parabolic at all points.

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③ $x f_{xx} + y f_{yy} = 0, \quad x > 0, \; y > 0$

$A = x \quad B = 0 \quad C = y$

$B^2 - 4AC = 0 - 4xy$

$= -4xy$

$\therefore$ It is elliptic for all $x > 0, \; y > 0$.

④ $x^2 f_{xx} + (1-y^2) f_{yy} = 0$

here, $A = x^2 \quad B = 0 \quad C = 1-y^2$

$$B^2 - 4AC = 0 - 4(x^2)(1-y^2)$$
$$= -4x^2(1-y^2)$$
$$= 4x^2(y^2-1)$$

For all x, except $x=0$ (x^2 is positive)

If $-1<y<1$, y^2-1 is negative.

$\therefore$ For $-\infty<x<\infty$ $(x\neq 0)$, $-1<y<1$, the equation is elliptic.

For $-\infty<x<\infty$, $x\neq 0$, $y<-1$ or $y>1$, the equation is hyperbolic.

For $x=0$ for all y or for all x, $y=\pm 1$, the equation is parabolic.

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⑤ $u_{xx} + 4u_{xy} + (x^2+4y^2)u_{yy} = \sin(x+y)$.

here, $A=1$; $B=4$; $C = x^2+4y^2$

$$B^2 - 4AC = 16 - 4(1)(x^2+4y^2)$$
$$= 4(4-(x^2+4y^2))$$
$$= 4(4-x^2-4y^2).$$

The equation is elliptic if $4-x^2-4y^2 < 0$

$$-x^2-4y^2 < -4.$$
$$x^2+4y^2 > 4.$$
$$\frac{x^2}{4}+\frac{y^2}{1} > 1$$

$\therefore$ It is elliptic in the region outside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.

The equation is hyperbolic if $4-x^2-4y^2>0$

$$-x^2-4y^2 > -4$$
$$x^2+4y^2 < 4$$
$$\frac{x^2}{4}+\frac{y^2}{1} < 1$$

$\therefore$ It is hyperbolic in the region inside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

The equation is parabolic if $4 - x^2 - 4y^2 = 0$

$$-x^2 - 4y^2 = -4$$

$$x^2 + 4y^2 = 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

It is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

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(6) The Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

here, $A = 1$, $B = 0$, $C = 1$

$\therefore$ $B^2 - 4AC = 0 - 4 < 0$.

Hence the equation is elleptic

The poisson's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$

here also, $B^2 - 4AC = -4 < 0$.

Hence the equation is elleptic.

One dimensional heat equation $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

here, $A = \alpha^2$, $B = 0$, $C = 0$

$\therefore$ $B^2 - 4AC = 0 - 4(\alpha^2)(0) = 0$

Hence the equation is parabolic.

One dimensional wave equation $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

here, $A = \alpha^2$, $B = 0$, $C = -1$

$B^2 - 4AC = 0 - 4(\alpha^2)(-1) = 4\alpha^2 > 0$

$\therefore$ The equation is hyperbolic.

Exercise problems :-

① $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

② $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - 12\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 7u = x^2+y^2$

③ $(1+x^2)f_{xx} + (5+2x^2)f_{xy} + (4+x^2)f_{yy} = 2\sin(x+y)$

④ $(1-x^2)f_{xx} - 2xy f_{xy} + (1-y^2)f_{yy} = 0$.

⑤ prove $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is elliptic

⑥ prove $f_{xx} - 2f_{xy} + f_{yy} = 0$ and $u_{xx} = u_t$ are parabolic

Transverse Vibrations of a stretched elastic string

Consider small transverse vibrations of an elastic string of length l, which is stretched and then fixed at its two ends.

Now we will study the transverse vibration of the string when no external forces act on it.

Take an end of the string as the origin and the string in the equilibrium position as the x-axis and the line through the origin and perpendicular to the x-axis as the y-axis.

We make the following assumptions:

1) The motion takes place entirely in one plane. This plane is chosen as the xy plane.

2) In this plane, each particle of the string moves in a direction perpendicular to the equilibrium position of the string.

3) The tension T caused by stretching the string before fixing it at the end points is constant at all times at all points of the deflected string.

4) The tension T is very large compared with the weight of the string and hence the gravitational force may be neglected.

5) The effect of friction is negligible.

6) The string is perfectly flexible. It can transmit only tension but not bending or shearing forces.

7) The slope of the deflection curve is small at all points and at all times.

By using second law of Newton, we can derive one dimensional wave equation:

ie, $$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

where $a^2 = \frac{\text{Tension}}{\text{mass}} = \frac{T}{m}$ (positive).

<u>Note</u> :- The displacement $y(x,t)$ is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

Solution of the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

(By the method of seperation of variables)

Let $y = X(x) \cdot T(t)$ be a solution of equation (1), where $X(x)$ is a function of x only and $T(t)$ is a function of t only.

$$\frac{\partial y}{\partial x} = X'T \qquad \frac{\partial y}{\partial t} = XT'$$

$$\frac{\partial^2 y}{\partial x^2} = X''T \qquad \frac{\partial^2 y}{\partial t^2} = XT'' \quad \text{where}$$

$$X'' = \frac{d^2X}{dx^2} \ \& \ T'' = \frac{d^2T}{dt^2}.$$

Hence (1) becomes, $XT'' = a^2 X''T$

$$\text{ie, } \frac{X''}{X} = \frac{T''}{a^2 T} \longrightarrow (2)$$

The L.H.S of (2) is a function of x only, whereas the R.H.S is a function of t only. But x and t are independent variables. Hence (2) is true only if each is equal to a constant.

$$\therefore \frac{X''}{X} = \frac{T''}{a^2 T} = k \text{ (say) where } k \text{ is any constant.}$$

Hence $X'' - kX = 0$ & $T'' - a^2kT = 0$.

Solutions of these equations depend upon the nature of the value of k.

Case (i). Let $k = \lambda^2$, a positive value.

Now, the equation ③ are $X'' - \lambda^2 X = 0$ & $T'' - a^2\lambda^2 T = 0$.

$$X'' - \lambda^2 X = 0$$

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0,$$

$$\left(\frac{d^2}{dx^2} - \lambda^2\right) X = 0. \quad D = d/dx$$

$$(D^2 - \lambda^2) X = 0$$

The auxiliary eqn is

$$m^2 - \lambda^2 = 0$$

$$m^2 = \lambda^2$$

$$m = \pm\lambda.$$

$$T'' - a^2\lambda^2 T = 0.$$

$$\frac{d^2T}{dt^2} - a^2\lambda^2 T = 0.$$

$$\left(\frac{d^2}{dt^2} - a^2\lambda^2\right) T = 0 \quad D = d/dt$$

$$D^2 - a^2\lambda^2 = 0.$$

The auxiliary eqn is

$$m^2 - a^2\lambda^2 = 0$$

$$m^2 = a^2\lambda^2$$

$$m = \pm\lambda a$$

$$\therefore T = C_1 e^{\lambda a t} + D_1 e^{-\lambda a t}$$

$$\therefore X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}.$$

$$\therefore y = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{\lambda a t} + D_1 e^{-\lambda a t})$$

Case (ii) Let $k = -\lambda^2$, a negative value.

Then the equation ③ are $X'' + \lambda^2 X = 0$ & $T'' + a^2\lambda^2 T = 0$

$$X'' + \lambda^2 X = 0$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0.$$

$$(D^2 + \lambda^2) X = 0$$

The aux. eqn is

$$m^2 + \lambda^2 = 0$$

$$m^2 = -\lambda^2$$

$$m = \pm i\lambda$$

$$\therefore X = A_2 \cos\lambda x + B_2 \sin\lambda x$$

$$T'' + a^2\lambda^2 T = 0.$$

$$\frac{d^2T}{dt^2} + a^2\lambda^2 T = 0.$$

$$(D^2 + \lambda^2 a^2) T = 0$$

The aux. eqn is

$$m^2 + \lambda^2 a^2 = 0$$

$$m^2 = -\lambda^2 a^2$$

$$m = \pm i\lambda a$$

$$T = C_2 \cos\lambda at + D_2 \sin\lambda at$$

$$\therefore y = (A_2 \cos\lambda x + B_2 \sin\lambda x)(C_2 \cos\lambda at + D_2 \sin\lambda at)$$

Case (iii) let $k=0$.

Now the equations (3) are $X''=0$ & $T''=0$.

$X''=0$

(ie) $\frac{d^2X}{dx^2}=0$

integrating twice w r to x,

$\frac{dX}{dx} = A_3$

$X = A_3x + B_3$

$T''=0$

ie, $\frac{d^2T}{dt^2}=0$

int. w.r to 't', (twice)

$\frac{dT}{dt} = C_3$

$T = C_3t + D_3$.

$\therefore \quad y = (A_3x+B_3)(C_3t+D_3)$

The various possible solutions of the wave equation are

$y(x,t) = (A_1e^{\lambda x} + B_1e^{-\lambda x})(C_1e^{\lambda at} + D_1e^{-\lambda at})$

$y(x,t) = (A_2\cos\lambda x + B_2\sin\lambda x)(C_2\cos\lambda at + D_2\sin\lambda at)$

$y(x,t) = (A_3x + B_3)(C_3t + D_3)$

Note :- We can choose the correct solution as follows :

Out of the above three types of solutions we have to choose the correct one which is consistent with the physical nature of the problem. Since we are dealing with problem on vibrations, y must be a periodic function of x and t.

Therefore, we choose the solution which contains the trigonometric terms since sine and cosine functions are periodic in nature. Hence the correct solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)\ (C\cos\lambda at + D\sin\lambda at).$$

One dimensional heat flow:-

Consider the flow of heat and the accompanying variation of temperature with position and with time in conducting solids.

The following empirical laws are taken as the basis of investigation.

1. Heat flows from a higher to lower temperature.
2. The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change. This constant of proportionality is known as the specific heat (c) of the conducting material.
3. The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area. This constant of proportionality is known as the thermal conductivity (k) of the normal.

The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

where α^2 stands for $\frac{k}{\rho c}$.

k – thermal Conductivity

c – Specific heat

ρ – density

$\frac{k}{\rho c}$ is called the diffusivity ($cm^2/sec.$)

Solution of the heat equation by the method of Seperation of variables.

The one-dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

Assume a solution of the form $u(x,t) = X(x) \cdot T(t) \longrightarrow (2)$

where X is a function of x alone and T is a function of t alone.

differentiating (2) partially w.r to 't', we get

$$\frac{\partial u}{\partial t} = XT' \longrightarrow (3)$$

differentiating (2) partially w.r to 'x' twice, we get

$$\frac{\partial^2 u}{\partial x^2} = X''T \longrightarrow (4)$$

Substituting (3) & (4) in eqn (1), we get

$$XT' = \alpha^2 X''T$$

Seperating the variables, we get

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = k \text{ (say)}$$

ie, $\frac{T'}{\alpha^2 T} = k$ and $\frac{X''}{X} = k$

ie, $T' - k\alpha^2 T = 0$ → (6) and $X'' - kX = 0$. → (7)

The equations (6) & (7) are ordinary differential equations the solution of which depend on the value of k.

<u>Case (i)</u> : Let $k = \lambda^2$, a positive number.

∴ The differential equations (6) & (7) become

$X'' - \lambda^2 X = 0$

ie, $\frac{d^2X}{dx^2} - \lambda^2 X = 0$.

$(D^2 - \lambda^2)X = 0$ where $D = \frac{d}{dx}$

The auxiliary equation is $m^2 - \lambda^2 = 0$

$m^2 = \lambda^2$

$m = \pm\lambda$.

$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$

and $T' - \lambda^2\alpha^2 T = 0$.

ie, $\frac{dT}{dt} - \lambda^2\alpha^2 T = 0$.

$\frac{dT}{dt} = \lambda^2\alpha^2 T$

$\frac{dT}{T} = \lambda^2\alpha^2 dt$

ie, integrating on b.s, we get

$\log T = \lambda^2\alpha^2 t + \log c_1$

Taking exponential on b.s, we get

$T = c_1 e^{\lambda^2\alpha^2 t}$

Substituting the values of X and T in eqn (2), we get

$$u(x,t) = \left(A_1 e^{\lambda x} + B_1 e^{-\lambda x}\right)\left(c_1 e^{\lambda^2\alpha^2 t}\right)$$

<u>Case (ii)</u> : Let $k = -\lambda^2$, a negative number.

∴ The differential equations (6) & (7) becomes

$$X'' + \lambda^2 X = 0$$ and $$T' + \lambda^2 \alpha^2 T = 0.$$

ie, $$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$$\left(\frac{d^2}{dx^2} + \lambda^2\right) X = 0.$$

$$(D^2 + \lambda^2) X = 0$$

The auxiliary eqns is

$$m^2 + \lambda^2 = 0$$

$$m^2 = -\lambda^2$$

$$m = \pm i\lambda$$

$$\therefore X = A_2 \cos\lambda x + B_2 \sin\lambda x.$$

$$\frac{dT}{dt} + \lambda^2 \alpha^2 T = 0.$$

$$\frac{dT}{dt} = -\lambda^2 \alpha^2 T$$

$$\frac{dT}{T} = -\lambda^2 \alpha^2 dt$$

Int. on b.s, we get

$$\log T = -\lambda^2 \alpha^2 t + \log c_2$$

Taking exponential on b.s we get

$$T = c_2 e^{-\lambda^2 \alpha^2 t}$$

$$\therefore u(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x)\, c_2 e^{-\lambda^2 \alpha^2 t}.$$

Case (iii)

Let $k = 0$.

Then equations ⑥ & ⑦ becomes,

$$X'' = 0$$

$$\frac{d^2X}{dx^2} = 0$$

integrating w.r to 'x' twice, we get

$$X = A_3 x + B_3$$

$$T' = 0.$$

$$\frac{dT}{dt} = 0.$$

integrating w.r to 't', we get

$$T = c_3.$$

$$\therefore u(x,t) = (A_3 x + B_3)\, c_3.$$

Hence the possible solutions of (1) are

$$u(x,t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x}) \, c_1 e^{\alpha^2 \lambda^2 t}$$

$$u(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x) \, c_2 e^{-\alpha^2 \lambda^2 t}$$

$$u(x,t) = (A_3 x + B_3) \, c_3 .$$

Note :-

(1) Out of these three solutions we have to choose the correct solution which satisfies the physical nature of the problem.

here, we are dealing with the problem on heat conduction. According to the law of Thermodynamics, when time 't' increases the temperature $u(x,t)$ will not increase.

Now, Consider the solution

$$\underline{u(x,t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x}) \, c_1 e^{\alpha^2 \lambda^2 t}}$$

here, if t increases then $u(x,t)$ is also increases.

ie, if we allow $t \to \infty$ then $u(x,t) \to \infty$

This is in contradiction with the law of Thermodynamics. Hence this solution is not suitable for our problems on heat conduction.

(2) At steady state conditions only we can use the solution (ie, when the temperature no longer varies with time)

$$\underline{u(x,t) = (A_3 x + B_3) \, c_3 .}$$

(3) Therefore, the correct solution which is suitable for our problems on one dimensional heat flow is

$$\underline{u(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x) \, c_2 e^{-\alpha^2 \lambda^2 t} .}$$

Steady state conditions and non-zero boundary conditions.

(1) A bar, 10cm long, with insulated sides, has its ends A and B kept at 20° and 40°c., respectively, until steady-state conditions prevail, that is, until the temperature at any interior point no longer changes with time. The temperature at A is then suddenly raised to 50°C and at the same instant that at B is lowered to 10°C. Find the subsequent temperature function $u(x,t)$ at any time.

Soln :- The partial differential equation of one dimensional heat flow is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow$ ⓐ

In steady state conditions, the temperature at any particular point does not vary with time ie, u depends only on x and not on time t.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \left[\because \frac{\partial u}{\partial t} = 0 \text{ since } u \text{ is a function of } x \text{ only} \right]$$

Since u is a function of x only, the above equation can be written as $\frac{d^2u}{dx^2} = 0 \quad (\alpha \neq 0)$

Hence when steady state conditions prevails the heat flow equation becomes

$$\frac{d^2u}{dx^2} = 0 \longrightarrow \text{ⓑ}$$

integrating eqn ⓑ w.r to 'x' twice, we get

$$\boxed{u(x) = ax+b} \longrightarrow (*)$$

The boundary conditions are

(i) $u(0) = 20$

(ii) $u(10) = 40$.

Applying b.c (i) in eqn: (*), we get

$$u(0) = a(0) + b = 20$$

$\therefore \boxed{b=20}$

Sub. in (*), we get $u(x) = ax + 20 \rightarrow$ (**).

Applying b.c (ii) in eqn (**), we get

$$u(10) = a(10) + 20 = 40.$$

$$10a = 20 \Rightarrow \boxed{a=2}$$

Sub $\boxed{a=2}$ in eqn (**), we get

$$\boxed{u(x) = 2x + 20}$$

When the temperatures at A and B are changed, the state is no longer steady. Then the temperature function $u(x,t)$ satisfies (a).

The boundary conditions in the second state are

$u(0,t) = 50 \quad \forall t > 0$

$u(10,t) = 10 \quad \forall t > 0$

The initial temperature of this state is the temperature in the previous steady-state. Hence the initial condition is

$$u(x,0) = 2x + 20 \text{ for } 0 < x < 10.$$

[Since non-zero boundary conditions have infinite number of values for A & B].

Therefore, in this case, we split the solution $u(x,t)$ into two parts.

ie, $u(x,t) = u_s(x) + u_t(x,t) \longrightarrow$ I

where $u_s(x)$ is a steady state solution of ⓐ and $u_t(x,t)$ is a transient solution which decreases with increase of t.

To find steady state Temperature $u_s(x)$:

The boundary conditions are

(i) $u_s(0) = 50$

(ii) $u_s(10,t) = 10$.

The steady state temperature is given by

$\boxed{u_s(x) = a_1 x + b_1} \longrightarrow$ II

Applying boundary condition (i) in eqn II, we get

$u_s(0) = a_1(0) + b_1 = 50$

$\boxed{b_1 = 50}$

Sub. b_1 in eqn II, we get $u_s(x) = a_1 x + 50 \longrightarrow$ III

Applying b.c (ii) in eqn III, we get

$u_s(10) = a_1(10) + 50 = 10$

$10 a_1 = -40$

$\boxed{a_1 = -4}$

Sub a_1 in III, we get $\boxed{u_s(x) = -4x + 50}$.

To find $u_t(x,t)$:

we assume that $u_t(x,t)$ is a transient solution of and satisfying the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,t) = u_s(x) + u_t(x,t)$$

$$\therefore u_t(x,t) = u(x,t) - u_s(x) \longrightarrow \text{IV}$$

we have to find the boundary conditions for $u_t(x,t)$

Putting $x=0$ in IV, we get

$$u_t(0,t) = u(0,t) - u_s(0) = 50 - 50 = 0.$$

$$\therefore u_t(0,t) = 0.$$

Putting $x=10$ in IV, we get

$$u_t(10,t) = u(10,t) - u_s(10) = 10-10 = 0.$$

$$\therefore u_t(10,t) = 0.$$

Putting $t=0$ in IV, we get

$$u_t(x,0) = u(x,0) - u_s(x)$$
$$= 2x+20 - (-4x+50)$$
$$= 2x+20+4x-50$$
$$= 6x-30$$

$$\therefore u_t(x,0) = 6x-30$$

Now for the function $u_t(x,t)$ we have the following boundary conditions.

(i) $u_t(0,t) = 0$ $\forall t$

(ii) $u_t(10,t) = 0$ $\forall t$

(iii) $u_t(x,0) = 6x-30$. for $0<x<10$

The suitable solution is

$$u_t(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow 1$$

Applying b.c (i) in eqn 1, we get

$$u_t(0,t) = A e^{-\alpha^2\lambda^2 t} = 0.$$

either $A=0$ or $e^{-\alpha^2\lambda^2 t} = 0$.

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0}$

Sub $A=0$ in eqn (1), we get

$$u(x,t) = B \sin\lambda x \, e^{-\alpha^2\lambda^2 t} \longrightarrow (2)$$

Applying b.c (ii) in eqn (2), we get

$$u(10,t) = B \sin 10\lambda \, e^{-\alpha^2\lambda^2 t} = 0.$$

here, $B \neq 0$ ($\because$ If $B=0$, we get a trivial solution)

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$).

$\sin 10\lambda = 0.$

$\sin 10\lambda = \sin n\pi$

$10\lambda = n\pi$

$\Rightarrow \boxed{\lambda = \frac{n\pi}{10}}$

Sub. $\lambda = \frac{n\pi}{10}$ in eqn (2), we get

$$u(x,t) = B \sin\frac{n\pi x}{10} \, e^{-\alpha^2 \frac{\pi^2 t}{100}}$$

$$u(x,t) = B_n \sin\frac{n\pi x}{10} \, e^{-\frac{\alpha^2 n^2\pi^2 t}{100}}$$ where $B = B_n$, & B_n is any constant.

The most general soln. is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} \, e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}} \longrightarrow (3).$$

Applying b.c (iii) in eqn (3), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} (1) = 6x - 30 \longrightarrow (4)$$

To find B_n expand $6x-30$ in half-range sine series in $(0,10)$

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$$6x-30 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \quad \text{where} \longrightarrow (5)$$

$$b_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx$$

From (4) & (5), we get $\boxed{B_n = b_n}$

$$\therefore B_n = \frac{2}{10}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 6\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$$

$$= \frac{1}{5}\left[(30)\left(\frac{10}{n\pi}\right)(-(-1)^n) - (-30)\left(\frac{10}{n\pi}\right)(-1)\right]$$

$$= \frac{1}{5}\left[\frac{-300}{n\pi}(-1)^n - \frac{300}{n\pi}\right]$$

$$= \frac{1}{5}\left[\frac{-300}{n\pi}\right]\left[(-1)^n+1\right]$$

$$= \frac{-60}{n\pi}\left(1+(-1)^n\right)$$

$\therefore$ sub in eqn (3), we get

$$u_t(x,t) = \sum_{n=1}^{\infty}\frac{-60}{n\pi}\left(1+(-1)^n\right)\sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{100}}.$$

$$\therefore u(x,t) = u_s(x) + u_t(x,t)$$

$$u(x,t) = 50-4x + \sum_{n=1}^{\infty}\frac{-60}{n\pi}\left(1+(-1)^n\right)\sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{100}}$$

which is temperature distribution.

Exercise Problem

1. A rod AB of length 10 cm. has its ends A and B kept at temperature 30°C and 100°C respectively until the steady-state conditions prevail. At sometime later, the temperature at A is lowered to 20°C and that at B to 40°C, and then these temperatures are maintained. Find the subsequent temperature distribution.

Problems :-

① Solve : $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. subject to

(i) $u(0,t) = 0$ for $t \geq 0$

(ii) $u(l,t) = 0$ for $t \geq 0$

(iii) $u(x,0) = \begin{cases} x & \text{for } 0 \leq x \leq l/2 \\ l-x & \text{for } l/2 \leq x \leq l \end{cases}$

soln :-

here the given equation is a one dimensional heat flow equation and therefore the correct solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow ①$$

The boundary conditions are

(i) $u(0,t) = 0$ for $t \geq 0$

(ii) $u(l,t) = 0$ for $t \geq 0$

(iii) $u(x,0) = \begin{cases} x & \text{for } 0 \leq x \leq l/2 \\ l-x & \text{for } l/2 \leq x \leq l \end{cases}$

Applying b.c (i) in eqn ①, we get

$$u(0,t) = A e^{-\alpha^2\lambda^2 t} = 0.$$

either $A = 0$ or $e^{-\alpha^2\lambda^2 t} = 0$.

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0}$

Substituting $\boxed{A=0}$ in eqn ①, we get

$$u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t} \longrightarrow ②$$

Applying b.c (ii) in eqn ②, we get

$$u(l,t) = B\sin\lambda l\, e^{-\alpha^2\lambda^2 t} = 0$$

here, $B \neq 0$ since if $B=0$ we get trivial solution.

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

hence, $\sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$, where n is an integer

$\lambda l = n\pi$

$\therefore \lambda = \frac{n\pi}{l}$

Substituting $\lambda = \frac{n\pi}{l}$ in equation ②, we get

$$u(x,t) = B \sin\frac{n\pi x}{l} \, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = B_n \sin\frac{n\pi x}{l} \, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \longrightarrow (*)$$

where B_n is any constant.

Since the partial differential equation (heat equation) is linear any linear combinations of solutions (or sum of the solutions) of the form $(*)$ with $n = 1,2,3,\dots$ is also a solution of the equation.

$\therefore$ The most general solution of $(*)$ can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \longrightarrow ③$$

Applying boundary condition (iii) in eqn ③, we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} (1) = f(x) \longrightarrow ④$$

where $f(x) = \begin{cases} x & \text{for } 0 < x < l/2 \\ l - x & \text{for } l/2 < x < l \end{cases}$

To find B_n expand $f(x)$ in $(0,l)$ in a half-range Fourier sine series we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \longrightarrow (5)$$

From equation (4) & (5), we get

$$\boxed{B_n = b_n}$$

$$\therefore B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{l/2} +$$

$$\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\left[\frac{l}{2}\left(\frac{l}{n\pi}\right)\left(-\cos\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + 0 - \left(\left(\frac{l}{2}\right)\left(\frac{l}{n\pi}\right)\left(-\cos\frac{n\pi}{2}\right) - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{l}\left[\cancel{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{l}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Substitute the value of B_n in equation (3), we get

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}.$$

2) Find the temperature $u(x,t)$ in a silver bar of length 10cm, constant cross-section of $1\ cm^2$ area, density $10.6\ gm/cm^3$, thermal conductivity 1.04 cal/cm deg. sec; specific heat 0.056 cal/gm. deg.) which is perfectly insulated laterally, if the ends are kept at 0°c, and if initially, the temperature is 5°c at the centre of the bar and falls uniformly to zero at its ends.

Soln: In this problem, $\alpha^2 = k/\rho c = \frac{1.04}{(10.6)(0.056)} = 1.75$

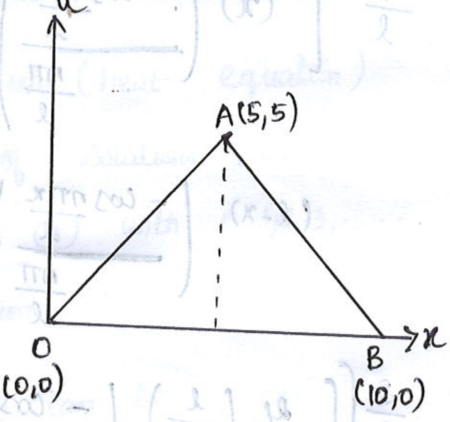
The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

(i) $u(0,t) = 0$ for all $t \geq 0$

(ii) $u(10,t) = 0$ for all $t \geq 0$.



The equation of line along OA is $\underset{x_1\, u_1}{(0,0)}$ $\underset{x_2\, u_2}{(5,5)}$

$$\frac{u-u_1}{u_2-u_1} = \frac{x-x_1}{x_2-x_1}$$

$$\Rightarrow \frac{u-0}{5-0} = \frac{x-0}{5-0} \Rightarrow \frac{u}{5} = \frac{x}{5}$$

$$\Rightarrow \boxed{u = x}, \quad 0 \leq x \leq 5.$$

The equation of line along AB is $\underset{x_1\, u_1}{(5,5)}$ $\underset{x_2\, u_2}{(10,0)}$

$$\Rightarrow \frac{u-5}{0-5} = \frac{x-5}{10-5}$$

$\Rightarrow$

$\Rightarrow \frac{u-5}{-5} = \frac{x-5}{5}$ $\Rightarrow u-5 = -x+5$

$\Rightarrow u = -x+10, \quad 5 \le x \le 10.$

$\Rightarrow \cancel{-u+5} = \cancel{x-5}$

$\therefore$ (ii) $u(x,10) = \begin{cases} x & \text{in } 0 \le x \le 5 \\ -x+10 & \text{in } 5 \le x \le 10. \end{cases}$

The suitable solution is

$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow (1)$

Applying b.c (i) in eqn (1), we get

$u(0,t) = A(1)\, e^{-\alpha^2\lambda^2 t} = 0.$

either $A=0$ or $e^{-\alpha^2\lambda^2 t} = 0.$

$e^{-\alpha^2\lambda^2 t} \ne 0$ ($\because$ it is defined for all t)

$\boxed{\therefore A=0}$

Sub $A=0$ in eqn (1), we get

$u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t} \longrightarrow (2)$

Applying b.c (ii) in eqn (2), we get

$u(10,t) = B\sin 10\lambda\, e^{-\alpha^2\lambda^2 t} = 0.$

here, $B \ne 0$ since if $B=0$ we get trivial solution.

$e^{-\alpha^2\lambda^2 t} \ne 0$ ($\because$ it is defined for all t)

hence, $\sin 10\lambda = 0$

$\sin 10\lambda = \sin n\pi$, n is an integer

$10\lambda = n\pi$

$\boxed{\lambda = \frac{n\pi}{10}}$

Sub. $\lambda = \frac{n\pi}{10}$ in eqn (2), we get

$$u(x,t) = B \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

$$u(x,t) = B_n \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$ where $B = B_n$, B_n is any constant.

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}} \longrightarrow (3)$$

Applying b.c (iii) in eqn (3), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = f(x) \longrightarrow (4)$$ where

$$f(x) = \begin{cases} x & \text{in } 0 \le x \le 5 \\ 10-x & \text{in } 5 \le x \le 10 \end{cases}$$

To find B_n expand $f(x)$ in half-range sine series in the interval $(0, 10)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \longrightarrow (5)$$ where $b_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx$

From (4) & (5), we get $\boxed{B_n = b_n}$

$$\therefore B_n = \frac{2}{10}\left[\int_0^5 x\sin\frac{n\pi x}{10}\,dx + \int_5^{10}(10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$= \frac{1}{5}\left[\left[(x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^5 + \right.$$

$$\left.\left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_5^{10}\right]$$

$$= \frac{1}{5}\left[\left[(5)\left(\frac{10}{n\pi}\right)\left(-\cos\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right] +\right.$$

$$\left[0-\left((5)\left(\frac{10}{n\pi}\right)\left(-\cos\frac{n\pi}{2}\right)-\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow \frac{1}{5}\left[\cancel{-\frac{50}{n\pi}\cos\frac{n\pi}{2}}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}+\cancel{\frac{50}{n\pi}\cos\frac{n\pi}{2}}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{1}{5}\left[\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$\Rightarrow \frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$\therefore$ sub. the value of B_n in eqn ③, we get

$$u(x,t)=\sum_{n=1}^{\infty}\frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{10}\,e^{-\frac{\alpha^2 n^2\pi^2 t}{100}}$$

Exercise problems.

① Solve the equation $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=x$.

② A homogeneous rod of conducting material of length 'l' units has ends kept at zero temperature and the temperature at the centre is T and falls uniformly to zero at the two ends. Find $u(x,t)$.

Steady state Conditions and zero boundary Conditions.

1. A rod 30cm long has its ends A and B kept at 20°c and 80°c respectively, until steady state Conditions prevail. The temperature at each end is then suddenly reduced to 0°c. and kept so. Find the resulting temperature function $u(x,t)$ taking $x=0$ at A.

Soln: The P.D.E of one dimensional heat flow is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (a)$$

In steady state Conditions, the temperature at any particular point does not vary with time. ie, u depends only on x and not on time t.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \left[\because \frac{\partial u}{\partial t} = 0 \text{ since } u \text{ is a function of } x \text{ only}\right]$$

Since u is a function of x only, the above equation can be written as $\frac{d^2u}{dx^2} = 0$. $(\alpha \neq 0)$

Hence when ~~steady~~ steady state Conditions prevails the heat flow equation becomes

$$\frac{d^2u}{dx^2} = 0. \longrightarrow (2)$$

int. eqn (2) w.r.to 'x' twice, we get

$$\boxed{u(x) = ax+b} \longrightarrow (*)$$

The boundary conditions in steady state are

(i) $u(0) = 20$

(ii) $u(30) = 80$

Applying b.c (i) in eqn ⊛, we get

$$u(0) = a(0) + b = 20$$

$$\therefore \boxed{b = 20}$$

Sub. $b = 20$ in eqn ⊛ we get

$$u(x) = ax + 20 \longrightarrow ⊛⊛.$$

Applying b.c (ii) in eqn ⊛⊛, we get

$$u(30) = a(30) + 20 = 80$$

$$30a = 60$$

$$\boxed{a = 2}.$$

Sub $a = 2$ in eqn ⊛⊛, we get

$$\therefore \boxed{u(x) = 2x + 20}$$

When the temperature at A and B are reduced to zero, the temperature distribution changes and the state is no more steady state. For this transient state, the boundary conditions are

(i) $u(0,t) = 0 \quad \forall\ t \geq 0$

(ii) $u(30,t) = 0 \quad \forall\ t \geq 0.$

The initial temperature of this state is the temperature in the prerious steady-sate. Hence the initial condition is

(iii) $u(x,0) = 2x + 20$, for $0 < x < 30$.

Now, we have to find $u(x,t)$ satisfying the conditions (i), (ii) and (iii) and the P.D.E. (a). The correct solution of ① is of the form

$$u(x,t) = (A\cos \lambda x + B\sin \lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow ①$$

Applying b.c (i) in eqn ①, we get

$$u(0,t) = A(1)\, e^{-\alpha^2\lambda^2 t} = 0.$$

either $A=0$ or $e^{-\alpha^2\lambda^2 t} = 0$.

but $e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ $\boxed{A=0}$.

Sub. $A=0$ in eqn ①, we get

$$u(x,t) = B\sin \lambda x\, e^{-\alpha^2\lambda^2 t} \longrightarrow ②$$

Applying b.c (ii) in eqn ②, we get

$$u(30,t) = B\sin 30\lambda\, e^{-\alpha^2\lambda^2 t} = 0.$$

$B \neq 0$ ($\because$ If $B=0$, we get trivial solution).

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\sin 30\lambda = 0$

$\sin 30\lambda = \sin n\pi$, where n is an integer

$30\lambda = n\pi$

$$\boxed{\lambda = \frac{n\pi}{30}}$$

Sub. $\lambda = \frac{n\pi}{30}$ in eqn ②, we get

$$u(x,t) = B\sin \frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{900}}$$

$$u(x,t) = B_n \sin \frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{900}}$$ where $B = B_n$, B_n is any constant.

The most general solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{900}} \longrightarrow (3)$$

Applying b.c (iii) in eqn (3), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}\,(1) = 2x+20. \longrightarrow (4)$$

To find B_n, expand $2x+20$ in half-range sine series in the interval $(0,30)$.

$$2x+20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{30} \text{ where} \longrightarrow (5)$$

$$b_n = \frac{2}{30}\int_0^{30} f(x) \sin\frac{n\pi x}{30}\, dx$$

From (4) & (5), we get

$$\therefore \boxed{B_n = b_n}$$

$$B_n = \frac{2}{30}\int_0^{30} (2x+20) \sin\frac{n\pi x}{30}\, dx$$

$$= \frac{1}{15}\left[(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right) - (2)\left(\frac{-\sin\frac{n\pi x}{30}}{\left(\frac{n\pi}{30}\right)^2}\right)\right]_0^{30}$$

$$= \frac{1}{15}\left[(80)\left(\frac{30}{n\pi}\right)(-(-1)^n) + \frac{2(900)}{n^2\pi^2}(0) - (20)\left(\frac{30}{n\pi}\right)(-1) - 2\left(\frac{900}{n^2\pi^2}\right)(0)\right].$$

$$= \frac{1}{15}\left[\frac{2400}{n\pi}(-1)^{n+1} + \frac{600}{n\pi}\right]$$

$$= \frac{1}{15} \times \frac{600}{n\pi}\left[4(-1)^{n+1} + 1\right]$$

$$= \frac{40}{n\pi}\left[1 + 4(-1)^{n+1}\right].$$

Sub. in eqn 'sub. the value of B_n in eqn ⑤,

we get

The temperature distribution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left(1+4(-1)^{n+1}\right) \sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{900}} \text{ degrees.}$$

---

2) A rod of length 'l' has its ends A and B kept at 0°C and 100°C until steady state condition prevail. If the temperature at B is reduced suddenly to 0°C and kept so while that of A is maintained, find the temperature $u(x,t)$ at a distance x from A and at time t.

Soln:-

The partial differential equation of one dimensional heat flow is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow ⓐ$$

In steady state conditions, the temperature at any particular point does not vary with time. ie, u depends only on x and not on time t.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \left[\because \frac{\partial u}{\partial t} = 0 \text{ since } u \text{ is a function of } x \text{ only}\right]$$

Since u is a function of x only, the above equation can be written as $\frac{d^2u}{dx^2} = 0$ $(\alpha \neq 0)$

Hence when steady state conditions prevails the heat flow equation becomes

$$\frac{d^2u}{dx^2} = 0 \longrightarrow (2)$$

integrating eqn (2). w.r to 'x' twice, we get

$$u(x) = ax+b \longrightarrow (*)$$

The boundary conditions are

(i) $u(0) = 0$

(ii) $u(l) = 100$

Applying b.c (i) in eqn (*), we get

$$u(0) = a(0) + b = 0$$

$$\therefore \boxed{b=0}$$

Sub $b=0$ in eqn (*), $u(x) = ax+0 \longrightarrow (*)(*)$

Applying b.c (ii) in eqn (*)(*), we get

$$u(l) = a(l) = 100$$

$$a = \frac{100}{l}.$$

Sub $a = \frac{100}{l}$ in eqn (*)(*), we get

$$\boxed{u(x) = \frac{100x}{l}}$$

When the temperature at B is reduced to zero, the temperature distribution changes and the state is no more steady state. For this transient state, the boundary conditions are

(i) $u(0,t) = 0 \quad \forall t \geq 0$

(ii) $u(l,t) = 0 \quad \forall t \geq 0$

The initial temperature of this state is the temperature in the previous steady-state. Hence the initial condition is

$$\text{(iii)} \quad u(x,0) = \frac{100x}{l} \quad \text{for} \quad 0 \le x \le l.$$

Now, we have to find $u(x,t)$ satisfying the conditions (i), (ii) and (iii) and the P.D.E (a).

The suitable solution of ① is of the form

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow ①$$

Applying b.c (i) in eqn ①, we get

$$u(0,t) = A(1)\, e^{-\alpha^2\lambda^2 t} = 0.$$

either $A=0$ or $e^{-\alpha^2\lambda^2 t} = 0.$

$$e^{-\alpha^2\lambda^2 t} \neq 0 \; (\because \text{it is defined } \forall t)$$

$$\therefore \boxed{A=0}$$

sub $A=0$ in eqn ①, we get

$$u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t} \longrightarrow ②$$

Applying b.c (ii) in eqn ②, we get

$$u(l,t) = B\sin\lambda l\, e^{-\alpha^2\lambda^2 t} = 0.$$

here, $B \neq 0$ ($\because$ If $B=0$, we get trivial solution)

$$e^{-\alpha^2\lambda^2 t} \neq 0 \; (\because \text{it is defined for all } t)$$

$$\sin\lambda l = 0$$

$$\therefore \sin\lambda l = \sin n\pi, \text{ where } n \text{ is an integer}$$

$$\lambda l = n\pi$$

$$\therefore \boxed{\lambda = \frac{n\pi}{l}}$$

Sub $\lambda = \frac{n\pi}{l}$ in eqn ②, we get

$$u(x,t) = B \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$ where $B = B_n$, B_n is any constant.

The most general solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \longrightarrow ③$$

Applying b.c (iii) in eqn ③, we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} (1) = \frac{100x}{l}.$$ $\longrightarrow$ ④

To find B_n expand $\frac{100x}{l}$ in half-range sine series in $(0,l)$.

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$ where $\longrightarrow$ ⑤

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx.$$

From ④ & ⑤, we get $\boxed{B_n = b_n}$

$$B_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2(100)}{l^2}\int_0^l x \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[l\left(\frac{l}{n\pi}\right)(-\cos n\pi)\right]$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}(-1)^n\right] = \frac{200}{n\pi}(-1)^{n+1}$$

sub. the value of B_n in eqn ③, we get

The temperature distribution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}.$$

Exercise problems:

① A rod of length l has its ends A and B kept at 0°C and 120°C respectively, until steady state conditions prevail. If the temperature at B is reduced to 0°C and kept so while that of A is maintained. find the temperature distribution in the rod.

Ans: $$u(x,t) = \frac{240}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}.$$

Problems on vibrating string with non-zero initial velocity:

① A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l-x)$, find the displacement.

Soln:-

The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

From the given problem we get the following boundary and initial conditions,

(i) $y(0,t) = 0$ for $t \geq 0$

(ii) $y(l,t) = 0$ for $t \geq 0$

(iii) $y(x,0) = 0$ for $0 \leq x \leq l$

(iv) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 3x(l-x)$ for $0 \leq x \leq l$.

Now, the suitable solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow ①$$

Applying boundary condition (i) in eqn, we get

$$y(0,t) = (A(1))(C\cos\lambda at + D\sin\lambda at) = 0.$$

either $A=0$ (or) $C\cos\lambda at + D\sin\lambda at = 0$.

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$, $\boxed{A=0}$

Substitute $A=0$ in eqn ①, we get

$$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow ②$$

Applying boundary condition (ii) in eqn ②, we get

$$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0.$$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ either $B=0$ or $\sin\lambda l = 0$.

Suppose, we take $B=0$ and already we have $A=0$ then we get a trivial solution.

$\therefore B \neq 0$.

The only possibility is $\sin\lambda l = 0$.

$\sin\lambda l = \sin n\pi$, where n is any integer.

$$\lambda l = n\pi$$

$$\boxed{\lambda = \frac{n\pi}{l}}$$

Sub. $\lambda = \frac{n\pi}{l}$ in eqn ②, we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow ③$$

Applying boundary condition (iii) in eqn ③, we get.

$$y(x,0) = B\sin\frac{n\pi x}{l}\,(C(1) + D(0)) = 0.$$

$$\Rightarrow B\sin\frac{n\pi x}{l}\,C = 0$$

here, $B \neq 0$ ($\because$ If $B=0$, we already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

$\therefore \boxed{C=0}$

Substitute $c=0$ in equation ③, we get

$$y(x,t) = B \sin\frac{n\pi x}{l} \; D \sin\frac{n\pi at}{l}.$$

$$= B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \quad \text{where } BC = B_n, \text{ where}$$

n is any integer & B_n is any Constant.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \longrightarrow ④$$

Before applying b.c (iv), differentiate ④ partially w.r to 't', we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi at}{l} \cdot \frac{n\pi a}{l}$$

Now, applying b.c (iv), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{l} \cdot (1) \frac{n\pi a}{l} = 3x(l-x) \longrightarrow ⑤$$

To find B_n, expand $3x(l-x)$ in a half-range Fourier sine series in the interval $(0,l)$.

$$3x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where} \longrightarrow ⑥$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} \, dx$$

From ⑤ & ⑥, we get

$$B_n \cdot \frac{n\pi a}{l} = b_n.$$

$\therefore$

$$\therefore \quad B_n \cdot \frac{n\pi a}{l} = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l 3x(l-x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{6}{l}\int_0^l (lx-x^2) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{6}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{6}{l}\left[(-2)\frac{l^3}{n^3\pi^3}(-1)^n - \left((-2)\left(\frac{l^3}{n^3\pi^3}\right)(1)\right)\right]$$

$$\Rightarrow \frac{6}{l} \times \frac{-2l^3}{n^3\pi^3}\left((-1)^n - 1\right) = \frac{-12l^3}{l n^3\pi^3}\left((-1)^n - 1\right)$$

$$\Rightarrow \frac{12l^2}{n^3\pi^3}\left(1-(-1)^n\right).$$

$$B_n = \frac{l}{n\pi a} \cdot \frac{12l^2}{n^3\pi^3}\left(1-(-1)^n\right).$$

$$B_n = \frac{12l^3}{n^4\pi^4 a}\left(1-(-1)^n\right)$$

Substitute the value of B_n in eqn ④, we get

$$y(x,t) = \sum_{n=1}^{\infty} \frac{12l^3}{an^4\pi^4}\left(1-(-1)^n\right) \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}.$$

2) If a string of length l is initially at rest in equilibrium position and each point of it is given the velocity $\left.\frac{\partial y}{\partial t}\right|_{t=0} = v_0 \sin^3\frac{\pi x}{l}$, $0<x<l$, determine the transverse displacement $y(x,t)$.

Soln:- The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

From the given problem, we get the following boundary and initial conditions,

(i) $y(0,t)=0$ for $t\geq 0$

(ii) $y(l,t)=0$ for $t\geq 0$

(iii) $y(x,0)=0$ for $0<x<l$

(iv) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = v_0 \sin^3\frac{\pi x}{l}$, $0<x<l$.

Now, the suitable solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow (1)$$

Applying b.c (i) in eqn (1), we get

$$y(x,t) = A(C\cos\lambda at + D\sin\lambda at)=0$$

either $A=0$ (or) $C\cos\lambda at + D\sin\lambda at = 0$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0}$

Sub $A=0$ in eqn ①, we get

$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow$ ②

Applying b.c (ii) in eqn ②, we get

$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0.$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ either $B=0$ or $\sin\lambda l = 0$.

Suppose, we take $B=0$ and already we have $A=0$ then we get a trivial solution.

$\therefore B \neq 0$.

The only possibility is $\sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$, where n is any integer.

$\lambda l = n\pi$

$$\boxed{\lambda = \frac{n\pi}{l}}$$

Sub. $\lambda = \frac{n\pi}{l}$ in eqn ②, we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow ③$$

Applying b.c (iii) in eqn ③, we get

$y(x,0) = B\sin\frac{n\pi x}{l}\,(C(1) + D(0)) = 0$

$B\sin\frac{n\pi x}{l}\,C = 0.$

here $B \neq 0$ ($\because$ If $B=0$, we already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

$\therefore \boxed{C=0}$

Substitute $C=0$ in eqn. ③, we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\, D\sin\frac{n\pi at}{l}$$

$$= BD\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \quad \text{where } BD = B_n,$$

n is any integer
B_n is any constant.

$\therefore$ The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \longrightarrow ④$$

Before applying b.c (iv) in eqn ④, differentiate ④ partially w.r to 't', we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \cdot \frac{n\pi a}{l}$$

Now, applying bc (iv), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}(1)\cdot\frac{n\pi a}{l} = v_0\sin^3\frac{\pi x}{l}.$$

[we know that $\sin 3A = 3\sin A - 4\sin^3 A$

$\Rightarrow \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$]

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l} = v_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

$$\Rightarrow B_1\sin\frac{\pi x}{l}\cdot\frac{\pi a}{l} + B_2\sin\frac{2\pi x}{l}\cdot\frac{2\pi a}{l} + B_3\sin\frac{3\pi x}{l}\cdot\frac{3\pi a}{l} + \cdots = \frac{3v_0}{4}\sin\frac{\pi x}{l} - \frac{v_0}{4}\sin\frac{3\pi x}{l}$$

equating the co-efficient on both sides.

$$B_1 \cdot \frac{\pi a}{l} = \frac{3v_0}{4} \quad ; \quad B_3 \cdot \frac{3\pi a}{l} = \frac{-v_0}{4},$$

$$B_n = 0 \ , \ n \neq 1,3 .$$

$$B_1 = \frac{3v_0 l}{4\pi a} \quad , \quad B_3 = \frac{-v_0}{4} \times \frac{l}{3\pi a}$$

$$B_3 = \frac{-v_0 l}{12\pi a}$$

Sub these values in ④, we get

$$y(x,t) = \frac{3v_0 l}{4\pi a} \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a} \sin\frac{3\pi x}{l} \sin\frac{3\pi a t}{l}.$$

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③ A string is stretched between two fixed points at a distance $2l$ apart and the points of the string are given initial velocities v where

$$v = \begin{cases} \dfrac{cx}{l} & \text{in } 0<x<l \\ \dfrac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases}$$

x being the distance from an end point. Find the displacement of the string at any subsequent time.

Solution:

The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

From the given problem, we get the following boundary and initial conditions, equating the co-efficient on both sides.

$$B_1 \cdot \frac{\pi a}{l} = \frac{3v_0}{4} \; ; \quad B_3 \cdot \frac{3\pi a}{l} = \frac{-v_0}{4},$$

$$B_n = 0, \quad n \neq 1,3.$$

$$B_1 = \frac{3v_0 l}{4\pi a}, \quad B_3 = \frac{-v_0}{4} \times \frac{l}{3\pi a}$$

$$B_3 = \frac{-v_0 l}{12\pi a}$$

Sub these values in ④, we get

$$y(x,t) = \frac{3v_0 l}{4\pi a} \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a} \sin\frac{3\pi x}{l} \sin\frac{3\pi a t}{l}.$$

---

③ A string is stretched between two fixed points at a distance $2l$ apart and the points of the string are given initial velocities v where
$$v = \begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases}$$

x being the distance from an end point. Find the displacement of the string at any subsequent time.

solution:

The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

From the given problem, we get the following boundary and initial conditions,

(i) $y(0,t) = 0$ for $t \geq 0$

(ii) $y(2l, t) = 0$ for $t \geq 0$

(iii) $y(x,0) = 0$ for $0 \leq x \leq 2l$

(iv) $$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \begin{cases} \frac{cx}{l} & \text{in } 0 < x < l \\ \frac{c}{l}(2l-x) & \text{in } l < x < 2l. \end{cases}$$

Now, the suitable solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow (1)$$

Applying b.c (i) in eqn (1), we get

$$y(0,t) = A(C\cos\lambda at + D\sin\lambda at) = 0$$

either $A=0$ (or) $C\cos\lambda at + D\sin\lambda at = 0$.

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0}$

Sub $A=0$ in eqn (1), we get

$$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow (2)$$

Applying b.c (ii) in eqn (2), we get

$$y(2l,t) = B\sin 2l\lambda\,(C\cos\lambda at + D\sin\lambda at) = 0.$$

$C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ either $B=0$ (or) $\sin 2l\lambda = 0$.

but $B \neq 0$ (suppose we take $B=0$ and already we have $A=0$, then we get a trivial solution)

The only possibility is $\sin 2l\lambda = 0$

$\sin 2\lambda l = \sin n\pi$ where n is any integer

$$2\lambda l = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{2l}}$$

Sub $\boxed{\lambda = \frac{n\pi}{2l}}$ in eqn ②, we get

$$y(x,t) = B \sin\frac{n\pi x}{2l}\left(C\cos\frac{n\pi at}{2l} + D\sin\frac{n\pi at}{2l}\right) \longrightarrow ③.$$

Applying b.c (iii) in eqn ③, we get

$$y(x,0) = B\sin\frac{n\pi x}{2l}\left(C(1) + D(0)\right) = 0.$$

$$B\sin\frac{n\pi x}{2l}\, C = 0.$$

here, $B \neq 0$ ($\because$ If $B=0$, we already explained)

$\sin\frac{n\pi x}{2l} \neq 0$ ($\because$ it is defined for all x)

$$\therefore \boxed{C=0}$$

Sub $C=0$ in eqn ③, we get

$$y(x,t) = B\sin\frac{n\pi x}{2l}\, D\sin\frac{n\pi at}{2l}$$

$$= B_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l},$$ where $BD = B_n$, n is any integer, B_n is any constant.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l} \longrightarrow ④$$

Before applying b.c (iv), differentiate ④ partially w.r to 't', we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{2l} \cos\frac{n\pi a t}{2l} \cdot \frac{n\pi a}{2l}$$

Now, applying b.c (iii), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l} (1) \frac{n\pi a}{2l} = \begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases} \longrightarrow (5)$$

To find B_n, expand $\begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases}$ in a half-range sine series in the interval $(0, 2l)$.

$$\begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l} \text{ where} \longrightarrow (6)$$

$$b_n = \frac{2}{(2l)} \int_0^{2l} f(x) \sin\frac{n\pi x}{2l}\, dx.$$

From (5) & (6), we get

$$\boxed{B_n \cdot \frac{n\pi a}{2l} = b_n}$$

$$\therefore B_n \cdot \frac{n\pi a}{2l} = \frac{2}{2l}\int_0^{2l} f(x) \sin\frac{n\pi x}{2l}\, dx$$

$$= \frac{2}{2l}\left[\int_0^{l} \frac{cx}{l} \sin\frac{n\pi x}{2l}\, dx + \int_l^{2l} \frac{c}{l}(2l-x) \sin\frac{n\pi x}{2l}\, dx\right]$$

$$= \frac{c}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_0^l +$$

$$(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\Bigg]_l^{2l}$$

$$= \frac{c}{l^2}\left[l\left(\frac{2l}{n\pi}\right)\left(-\cos\frac{n\pi(l)}{2l}\right) + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi(l)}{2l} - 0\right] +$$

$$\left[0 - \left((l)\left(\frac{2l}{n\pi}\right)\left(-\cos\frac{n\pi(l)}{2l}\right) - \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi(l)}{2l}\right)\right]\Bigg]$$

$$= \frac{c}{l^2}\left[\cancel{-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{c}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] \Rightarrow \frac{8c}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$\therefore\ B_n \cdot \frac{n\pi a}{2l} = \frac{8c}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$B_n = \frac{16cl}{n^3\pi^3 a}\sin\frac{n\pi}{2}.$$

Sub. the value of B_n in eqn (4), we get

$$y(x,t) = \sum_{n=1}^{\infty}\frac{16lc}{n^3\pi^3 a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi a t}{2l}.$$

Exercise problems

① If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $\left[\frac{\partial y}{\partial t}\right]_{t=0} = v_0 \sin\frac{\pi x}{l}$, $0<x<l$, determine the displacement $y(x,t)$.

② A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l-x)$, show that

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi a t}{l}$$

3) If a string of length l is initially at rest in its equilibrium position and each of its points is given a velocity v such that $v = \begin{cases} cx & \text{for } 0<x\le l/2 \\ c(l-x) & \text{for } l/2<x<l, \end{cases}$

show that the displacement $y(x,t)$ at any time t is given by

$$y(x,t) = \frac{4l^2c}{\pi^3 a}\left[\sin\frac{\pi x}{l}\sin\frac{\pi a t}{l} - \frac{1}{3^3}\sin\frac{3\pi x}{l}\sin\frac{3\pi a t}{l} + \cdots\right]$$

Problems on vibrating string with zero-initial velocity:

① A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any time and at any distance from the end $x=0$.

Solution:- The displacement y of the particle at a distance x from the end $x=0$ at time t is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

From the given problem, we get the following boundary conditions.

(i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(l,t) = 0$ for all $t \geq 0$

(iii) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ for $0 \leq x \leq l$

(iv) $y(x,0) = y_0 \sin^3\frac{\pi x}{l}$, for $0 \leq x \leq l$.

Now the correct solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow ①$$

Applying the boundary condition (i) in ①, we get

$y(0,t) = A(C\cos\lambda at + D\sin\lambda at) = 0.$

either $A=0$ or $C\cos\lambda at + D\sin\lambda at = 0.$

$\ldots D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0.}$

Sub $A=0$ in eqn ①, we get

$$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow ②$$

Applying boundary condition (ii) in eqn ②, we get

$$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0.$$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined for all t)

$\therefore$ either $B=0$ (or) $\sin\lambda l = 0$.

but, $B \neq 0$ (suppose if we take $B=0$ and already we have $A=0$, then we get a trivial solution)

$\therefore$ we take $\sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$ where n is any integer

$\lambda l = n\pi$

$\therefore \boxed{\lambda = \frac{n\pi}{l}}$

Substitute $\lambda = \frac{n\pi}{l}$ in eqn ②, we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow ③.$$

Before applying boundary condition (iii), diff ③ partially w.r.to 't',

$$\frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{l}\left(C\left(-\sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right) + D\left(\cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)\right)$$

Applying boundary condition (iii), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = B\sin\frac{n\pi x}{l}\left(D\cdot\frac{n\pi a}{l}\right) = 0.$$

here $B \neq 0$ ($\because$ Suppose $B=0$, we have already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

$\cancel{n\pi x}$ $\frac{n\pi a}{l} \neq 0$ ($\because$ all are constants).

$\therefore \boxed{D=0}$.

$\therefore$ Substituting $D=0$ in eqn ③, we get

$$y(x,t) = B \sin\frac{n\pi x}{l} \, C \cos\frac{n\pi a t}{l}$$

$$y(x,t) = B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} \quad \text{where } BC = B_n,$$

n is any integer

$\rightarrow$ (*) & B_n is any constant.

Since the partial differential equation (wave equation) is linear any linear combination of solutions (or sum of the solutions) of the form (*) with $n = 1, 2, 3, \ldots$ is also a solution of the equation.

$\therefore$ The most general solution of (*) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}. \longrightarrow ④$$

Applying the boundary condition (iv) in eqn ④, we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l}.$$

[we know that $\sin 3A = 3\sin A - 4\sin^3 A$.

$\sin 3A - 3\sin A = -4\sin^3 A$.

$-\sin 3A + 3\sin A = 4\sin^3 A$

$\therefore \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$]

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = y_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

$$\sum B_n \sin\frac{n\pi x}{l} = \frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}.$$

ie, $B_1 \sin\frac{\pi x}{l} + B_2 \sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l} + \dots = \frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}$

By equating like coefficients on either side, we get

$B_1 = \frac{3y_0}{4}$, $B_3 = \frac{-y_0}{4}$ and $B_n = 0$, for $n \neq 1, 3$.

Substituting these values of in eqn ④, we get

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}.$$

---

2) A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t=0$. Find the displacement of any point of the string at a distance x from one end at any time t.

Soln:-

The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

(i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(l,t) = 0$ for all $t \geq 0$

(iii) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ for $0 \leq x \leq l$.

(iv) $y(x,0) = k(lx - x^2)$ for $0 \leq x \leq l$.

The suitable solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow (1)$$

Applying boundary condition (i) in eqn (1), we get

$$y(0,t) = A\,(C\cos\lambda at + D\sin\lambda at) = 0.$$

either $A = 0$ (or) $C\cos\lambda at + D\sin\lambda at = 0$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$. ($\because$ it is defined $\forall t$)

$$\therefore \boxed{A = 0.}$$

Substitute $A=0$ in eqn (1), we get

$$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow (2)$$

Applying boundary condition (ii) in eqn (2), we get

$$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0$$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$, either $B = 0$ or $\sin\lambda l = 0$.

but $B \neq 0$ (suppose, we take $B = 0$ and already we have $A = 0$, then we get a trivial solution)

$\therefore$ we take $\sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$, where n is any integer

$$\lambda l = n\pi$$

$$\therefore \lambda = \frac{n\pi}{l}$$

Substituting $\lambda = \frac{n\pi}{l}$ in eqn (2), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow (3)$$

Before applying condition (iii), diff (3) partially w.r to 't', we get

$$\frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{l}\left[C\left(-\sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right) + D\left(\cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)\right]$$

Now applying b.c (iii), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = B \sin\frac{n\pi x}{l}\left(D \cdot \frac{n\pi a}{l}\right) = 0.$$

Here, $B \neq 0$ ($\because$ if $B=0$, we already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

and $\frac{n\pi a}{l} \neq 0$ ($\because$ all are constants)

$\therefore$ $D=0$

Substituting $D=0$ in eqn ③, we get

$$y(x,t) = B \sin\frac{n\pi x}{l} \cdot C \cos\frac{n\pi at}{l}.$$

$$y(x,t) = B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$ where $BC = B_n$, n is any integer & B_n is any constant.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}. \longrightarrow ④$$

Applying the boundary condition (iv) in eqn ④, we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = k(lx - x^2) \longrightarrow ⑤$$

To find B_n, expand $k(lx-x^2)$ in a half-range Fourier sine series in the interval $(0,l)$.

$$k(lx-x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$ where $\longrightarrow ⑥$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

From ⑤ & ⑥, we get $B_n = b_n$

$$\therefore B_n = \frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx .$$

$$= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[-0+0-2\left(\frac{l^3}{n^3\pi^3}\right)(-1)^n - \left(-0+0-2\left(\frac{l^3}{n^3\pi^3}\right)(1)\right)\right]$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2k}{l}\left(\frac{+2l^3}{n^3\pi^3}\right)\left(-(-1)^n+1\right)$$

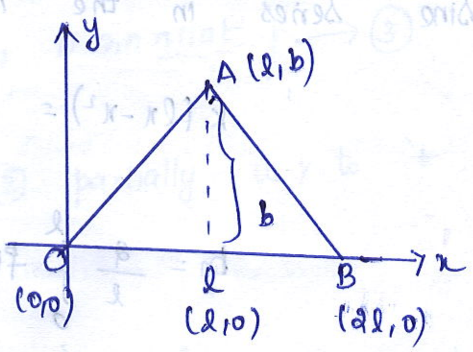
$$= \frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right).$$

$\therefore$ substitute the value of B_n in eqn ④, we get

$$y(x,t) = \sum_{n=1}^{\infty}\frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}.$$

---

3) A taut string of length $2l$ is fastened at both ends. The mid point of the string is taken to a height b and then released from the rest in that position. Find the displacement of the string.



Soln:-

The boundary conditions are

(i) $y(0,t)=0, \quad t \geq 0$

(ii) $y(2l,t)=0, \quad t \geq 0$

Equation of line along OA, $\underset{x_1\ y_1}{(0,0)}$ $\underset{x_2\ y_2}{(l,b)}$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-0}{b-0} = \frac{x-0}{l-0}$$

$$\frac{y}{b} = \frac{x}{l}.$$

$$\Rightarrow y = \frac{bx}{l}, \quad 0 \le x \le l.$$

Equation of line along AB, $\underset{x_1\ y_1}{(l,b)}$ & $\underset{x_2\ y_2}{(2l,0)}$

$$\frac{y-b}{0-b} = \frac{x-l}{2l-l} \Rightarrow \frac{y-b}{-b} = \frac{x-l}{l}.$$

$$y-b = \frac{-b(x-l)}{l}$$

$$y = b - b\frac{(x-l)}{l} \Rightarrow y = b\left(\frac{2l-x}{l}\right), \quad l \le x \le 2l.$$

$\therefore$ (iv). $y(x,0) = \begin{cases} \dfrac{bx}{l}, & 0 \le x \le l \\ b\left(\dfrac{2l-x}{l}\right), & l \le x \le 2l. \end{cases}$

The suitable solutions which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow (1).$$

Applying boundary condition (i) in eqn (1), we get

$$y(x,0) = A(C\cos\lambda at + D\sin\lambda at) = 0.$$

either $A=0$ or $C\cos\lambda at + D\sin\lambda at = 0$.

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ $A=0$

Substitute $A=0$ in eqn (1), we get

$$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow (2)$$

Applying b.c (ii) in eqn (2), we get

$$y(2l,t) = B\sin\lambda 2l\,(C\cos\lambda at + D\sin\lambda at) = 0.$$

$$y(2l,t) = B\sin 2l\lambda\,(C\cos\lambda at + D\sin\lambda at) = 0.$$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$).

Therefore, either $B=0$ or $\sin 2l\lambda = 0$.

Suppose, we take $B=0$ and already we have $A=0$.

then we get a trivial solution.

$\therefore\ B \neq 0$

The only possibility is $\sin 2l\lambda = 0$.

ie, $\sin 2l\lambda = \sin n\pi$, where n is any integer.

$2l\lambda = n\pi$

$$\boxed{\lambda = \frac{n\pi}{2l}}$$

Substitute $\lambda = \frac{n\pi}{2l}$ in eqn (2), we get

$$y(x,t) = B\sin\frac{n\pi x}{2l}\left(C\cos\frac{n\pi at}{2l} + D\sin\frac{n\pi at}{2l}\right) \longrightarrow (3).$$

Before applying boundary condition (iii), diff. eqn (3) partially w.r to t, we get

$$\frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{2l}\left(C\left(-\sin\frac{n\pi at}{2l}\cdot\frac{n\pi a}{2l}\right) + D\left(\cos\frac{n\pi at}{2l}\cdot\frac{n\pi a}{2l}\right.\right.$$

Now, applying boundary condition (iii), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = B\sin\frac{n\pi x}{2l}\ D\cdot\frac{n\pi a}{2l} = 0.$$

$B \neq 0$ (Since suppose $B=0$, we have already explained)

$\sin \frac{n\pi x}{2l} \neq 0$ ($\because$ it is defined $\forall x$)

$\frac{n\pi x}{2l} \neq 0$ ($\because$ all are constants).

$\therefore \boxed{D=0}$

Sub. $D=0$ in eqn ③, we get.

$$y(x,t) = B \sin \frac{n\pi x}{2l} \, C \cos \frac{n\pi a t}{2l}$$

$$= BC \sin \frac{n\pi x}{2l} \cos \frac{n\pi a t}{2l}$$

$$= B_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi a t}{2l}$$ where $BC = B_n$, n is any integer and B_n is any constant.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi a t}{2l} \longrightarrow ④$$

Applying b.c (iv) in eqn ④, we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} = \begin{cases} \frac{bx}{l}, & 0 \le x \le l \\ b\left(\frac{2l-x}{l}\right), & l \le x \le 2l. \end{cases} \longrightarrow ⑤$$

To find B_n, expand $\begin{cases} \frac{bx}{l}, & 0 \le x \le l \\ b\left(\frac{2l-x}{l}\right), & l \le x \le 2l \end{cases}$ in a half-range Fourier sine series in the interval $(0, 2l)$.

$$\begin{cases} \frac{bx}{l}, & 0 \le x \le l \\ b\left(\frac{2l-x}{l}\right), & l \le x \le 2l \end{cases} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2l},$$ where

$$b_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx$$

From ⑤ & ⑥, we get $B_n = b_n$

$$B_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx.$$

$$= \frac{1}{l}\left[\int_0^{l}\frac{bx}{l}.\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}\frac{b(2l-x)}{l}\sin\frac{n\pi x}{2l}\,dx\right]$$

$$= \frac{1}{l}\left[\frac{b}{l}\right]\left[\int_0^{l} x\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}(2l-x)\sin\frac{n\pi x}{2l}\,dx\right].$$

$$= \frac{b}{l^2}\left[(x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_0^l +$$

$$(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\Bigg]_l^{2l}$$

$$= \frac{b}{l^2}\left[l.\frac{2l}{n\pi}\left[-\cos\frac{n\pi(l)}{2l}\right] + \frac{4l^2}{n^2\pi^2}\left(\sin\frac{n\pi(l)}{2l}\right) - 0\right] +$$

$$\left[0 - \left((l).\frac{2l}{n\pi}\left(-\cos\frac{n\pi(l)}{2l}\right) + \left(\frac{4l^2}{n^2\pi^2}\left(-\sin\frac{n\pi(l)}{2l}\right)\right)\right)\right].$$

$$= \frac{b}{l^2}\left[\frac{2l^2}{n\pi}\left(-\cos\frac{n\pi}{2}\right) + \frac{4l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right.$$

$$\left. - \left[\frac{2l^2}{n\pi}\left(-\cos\frac{n\pi}{2}\right) - \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\right]$$

$$= \frac{b}{l^2}\left[\cancel{-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}}\right.$$

$$\left. + \frac{4l^2}{\quad}\sin\frac{n\pi}{\quad}\right]$$

$$= \frac{b}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Sub. value of B_n in eqn (4), we get.

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi a t}{2l}.$$

Exercise problems.

(1) A tightly stretched string with end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin\frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x,t)$ at any point of the string.

(2) A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = 3(lx - x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at any time t.

(3) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = k\left(\sin\frac{\pi x}{l} - \sin\frac{2\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any distance x from one end at any time t.