Flow through pipes

Flow through a pipe

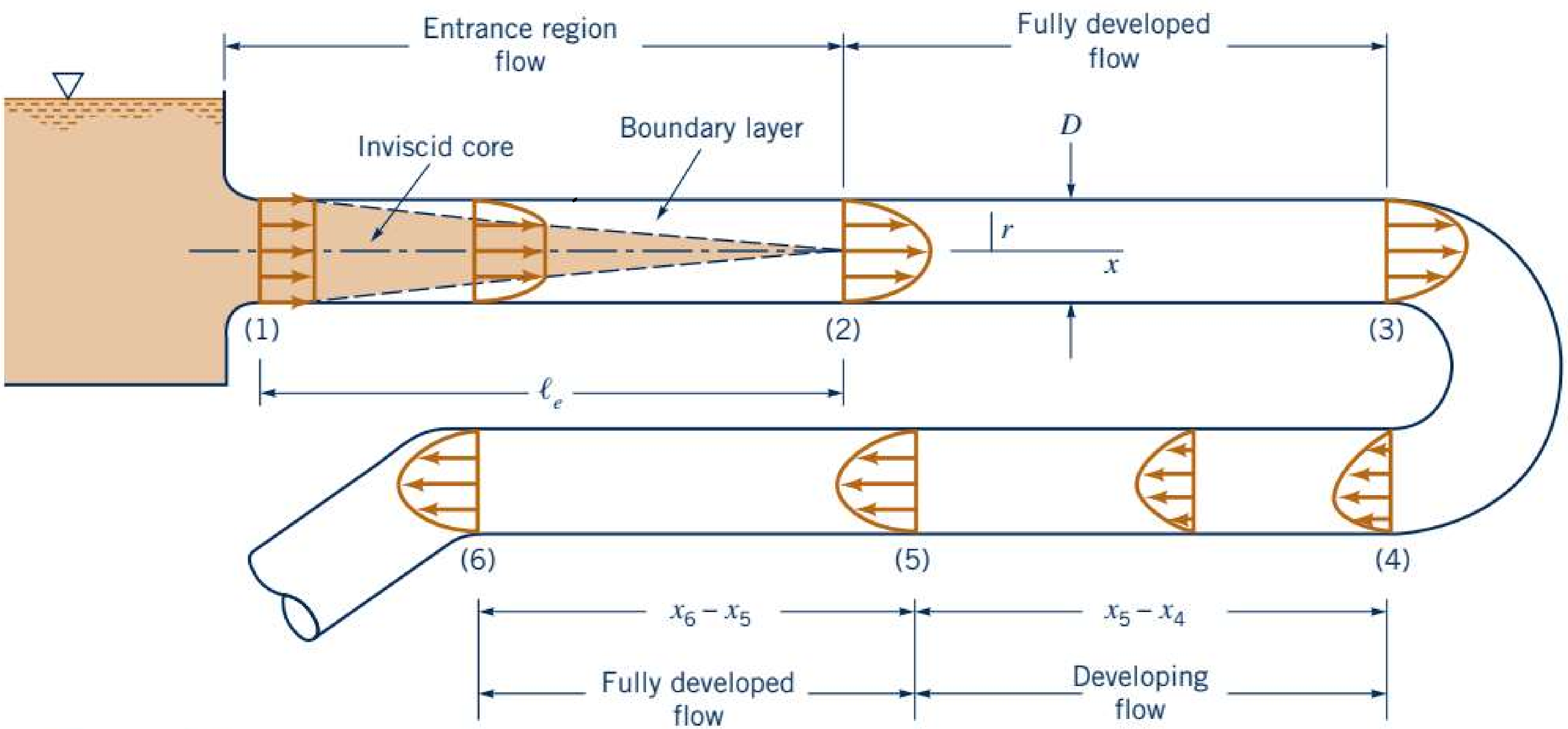


Figure 8.5 **Entrance region, developing flow, and fully developed flow in a pipe system.**

Entrance length

- Shape of the velocity profile changes in the axial direction
- At the pipe entrance, the velocity profile is uniform as shown
- As the fluid enters the pipe, the fluid velocity at the wall will be zero because of the no-slip condition
- Due to this, the layers of fluid that are adjacent to the wall will experience a deceleration and thereby slows down. This creates a velocity gradient adjacent to the walls. These layers adjacent to the wall experiences shear stress.
- As the fluid moves along the pipe, more and more adjacent layers gets decelerated, the region of velocity gradient spreads towards the pipe centerline.
- This leads to the formation of boundary layer in the pipe – a layer in which the velocity gradient is present.
- Outside the boundary layer, the velocity gradients are negligible and shear is zero (inviscid behavior).
- As the fluid near the walls decelerate, the fluid at the centerline accelerate along the pipe length to maintain a constant volume flow rate.
- This behavior is observed until the boundary layer reaches the centerline as shown in the figure.

Fully developed flow:

- As the boundary layer reaches the centerline, the flow is said to be fully-developed
- Velocity profile is parabolic, and it does not change shape
- Velocity does not change along the pipe length
- Pressure gradient remains constant along the pipe length.

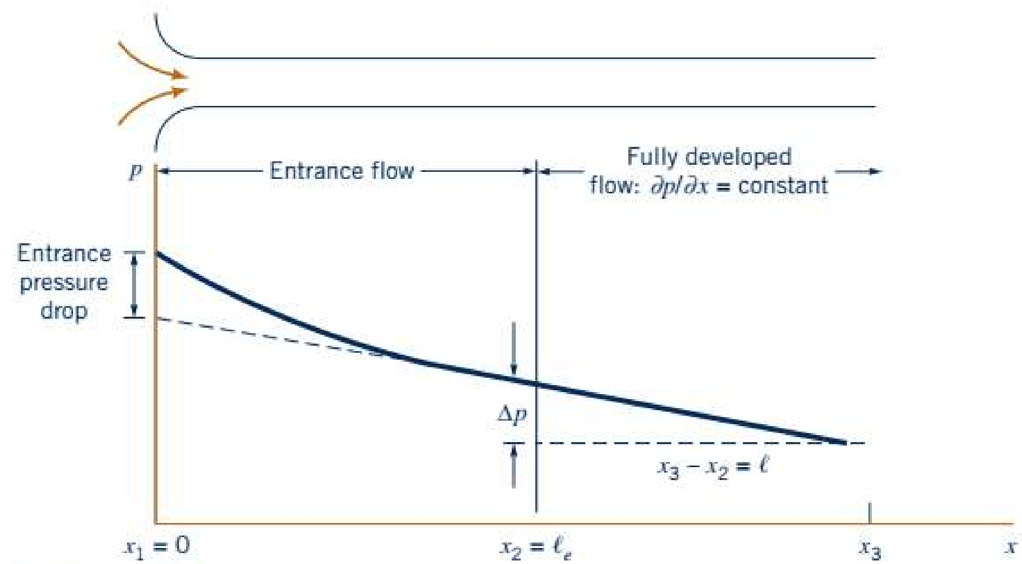
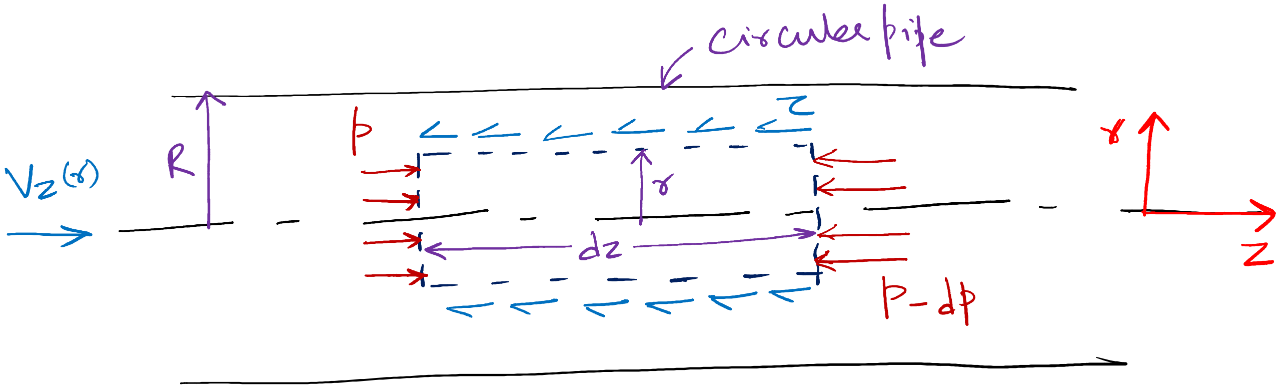


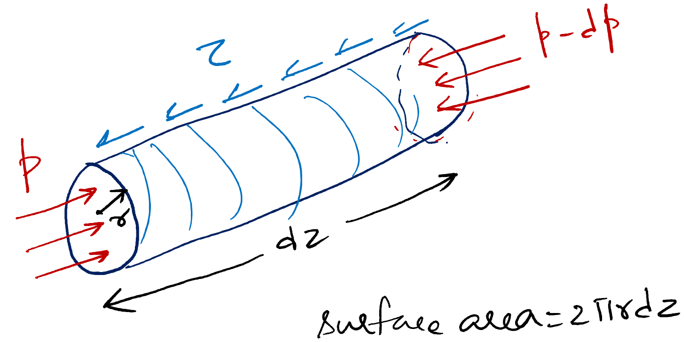
Figure 8.6 Pressure distribution along a horizontal pipe.

Suggested watching:
https://www.youtube.com/watch?v=AFq6MDRsfu0

Suggested reading:
Section 8.1.2 in Munson

Fully Developed Laminar Flow through a pipe (Hagen–Poiseuille equation)





For the fluid element shown, $\sum F_z = 0$

$$p \times \pi r^2 - (p-dp) \times \pi r^2 - \tau \times 2\pi r\, dz = 0$$

$$\Rightarrow \quad dp \times \cancel{\pi} r^{\cancel{2}} = \tau \times 2\cancel{\pi}\cancel{r}\, dz$$

$$\Rightarrow \quad \frac{dp}{dz} \times r = 2\tau \qquad \text{— (1)}$$

We can write the shear stress τ as $\tau = \mu \dfrac{dV_z}{dr}$ — (2)

② in ① $\Rightarrow$ $$\frac{dp}{dz} \times r = 2 \times \mu \frac{dV_z}{dr}$$

$$\Rightarrow \frac{dV_z}{dr} = \frac{1}{2\mu} \frac{dp}{dz} \cdot r \quad \text{—— (A)}$$

By integrating w.r.t. r, we get

$$\int dV_z = \frac{1}{2\mu} \frac{dp}{dz} \int r \, dr$$

$$\Rightarrow V_z = \frac{1}{2\mu} \frac{dp}{dz} \frac{r^2}{2} + K \quad \text{—— ③}$$

K – Constant

Boundary condition at wall

at wall at $r = R \Rightarrow V_z = 0$ (NO-SLIP CONDITION)

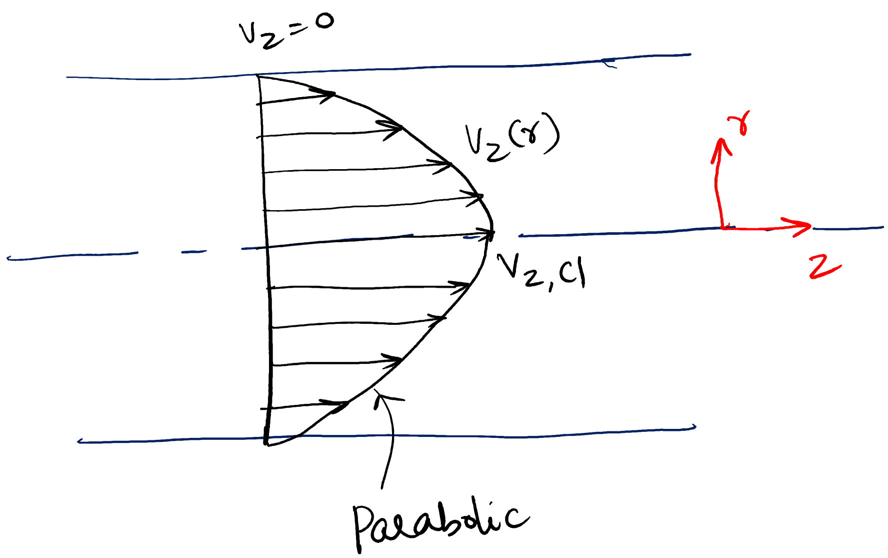
Substituting in ③ $\Rightarrow$ $$0 = \frac{1}{2\mu} \frac{dp}{dz} \frac{R^2}{2} + K$$

$$\Rightarrow K = \frac{-1}{2\mu} \frac{dp}{dz} \frac{R^2}{2} \Rightarrow K = \frac{-R^2}{4\mu} \frac{dp}{dz} \quad \text{—— ④}$$

④ in ③ $\Rightarrow$

$$\boxed{V_z(r) = \frac{R^2}{4\mu}\left(\frac{-dp}{dz}\right)\left(1-\frac{r^2}{R^2}\right)}$$

$\leftarrow$ Velocity profile for a fully-developed laminar flow in a circular pipe



on the wall i.e. at $r=R$

$$V_z(r=R)=0$$

on the centerline i.e. $r=0$

$$V_z(r=0) = V_{z,cl} = \frac{R^2}{4\mu}\left(\frac{-dp}{dz}\right)$$

Shear Stress

Shear stress $\tau = \mu \dfrac{dV_z}{dr}$

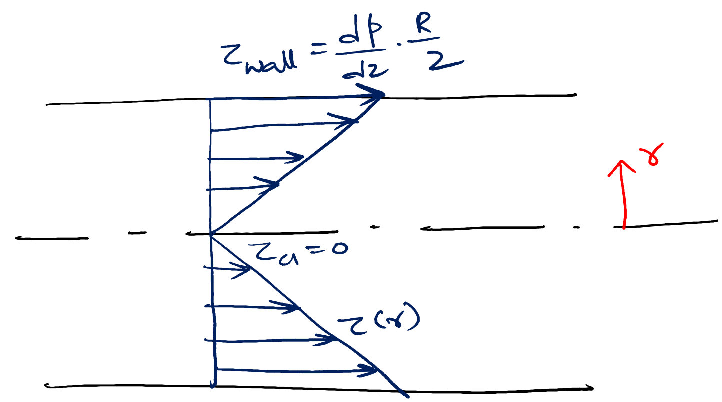
$$= \cancel{\mu} \frac{1}{2\cancel{\mu}} \frac{dp}{dz} \cdot r$$

$$\Rightarrow \boxed{\tau = \frac{1}{2} \frac{dp}{dz} \cdot r}$$

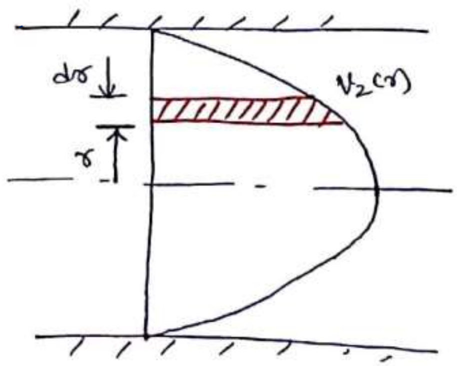
at centerline, $r = 0 \Rightarrow \tau_{cl} = 0$

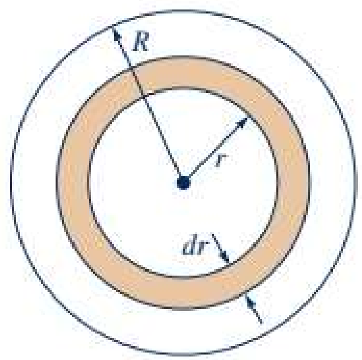
at wall $r = R \Rightarrow \tau_w = \dfrac{dp}{dz} \cdot \dfrac{R}{2}$

τ varies linearly with r



Volume flow rate/discharge through a pipe





$dA = 2\pi r\, dr$

- Consider a radial fluid element as shown
- If dA is the element area, the volume flow rate through the element is

volume flow rate = Area × velocity

$$dQ = dA \times V_z(r)$$

$$= V_z(r) \times 2\pi r\, dr$$

The volume flow rate through the pipe is given by

$$Q = \int_0^R dQ = \int_0^R 2\pi r\, V_z(r)\, dr$$

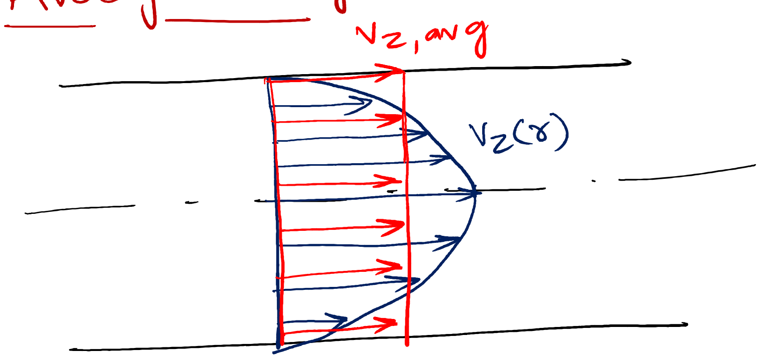
$$= 2\pi \int_0^R r \cdot \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(1 - \frac{r^2}{R^2}\right) dr$$

$$\Rightarrow Q = 2\pi \frac{R^2}{4\mu}\left(\frac{-dp}{dz}\right)\int_0^R \left(r - \frac{r^3}{R^2}\right)dr$$

$$= \frac{2\pi R^2}{4\mu}\left(\frac{-dp}{dz}\right)\left[\frac{r^2}{2}-\frac{r^4}{4R^2}\right]_0^R = \frac{2\pi R^2}{4\mu}\left(\frac{-dp}{dz}\right)\left[\frac{R^2}{2}-\frac{R^4}{4R^2}\right]$$

$$\Rightarrow \boxed{Q = \frac{\pi R^4}{8\mu}\left(\frac{-dp}{dz}\right)}$$

Average velocity :-



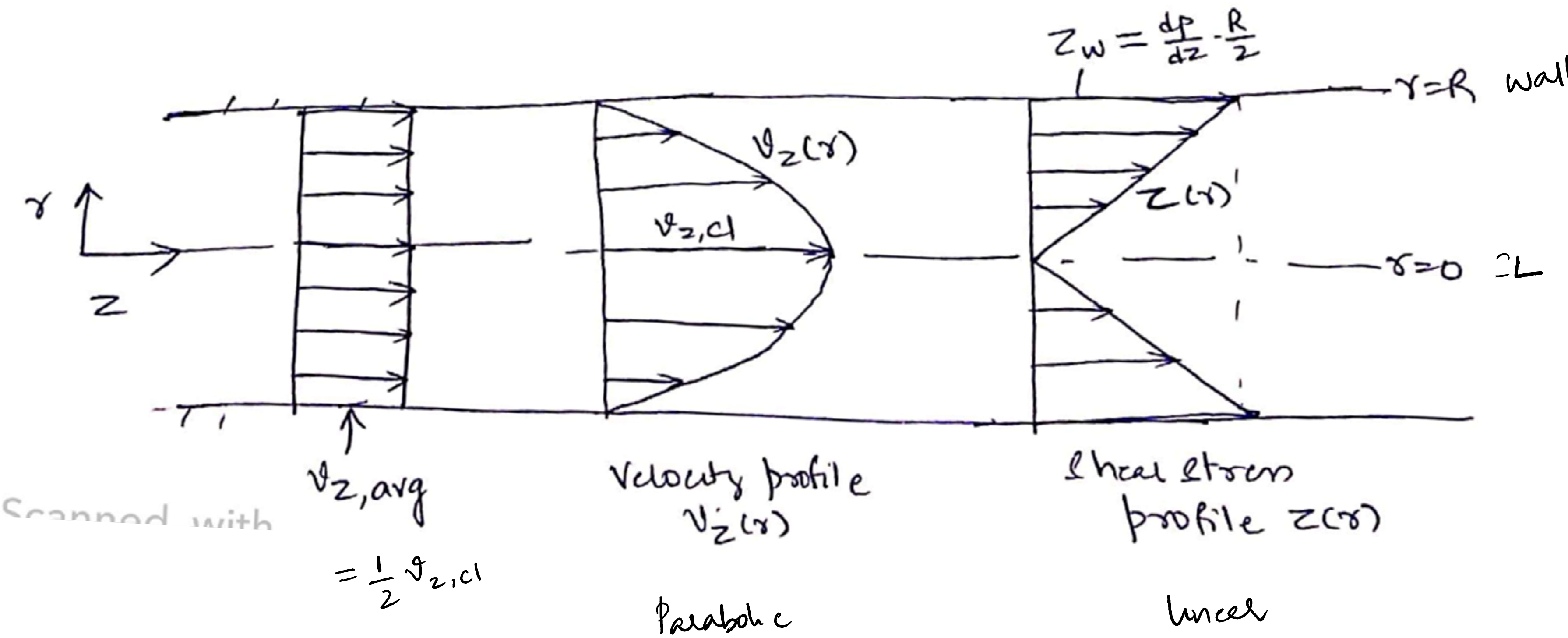
Average velocity is given by

$$Q = A\, V_{z,avg}$$

$$V_{z,avg} = \frac{Q}{A} = \frac{\frac{\cancel{\pi} \cancel{R^4}^{\,2}}{8\mu}\left(\frac{-dp}{dz}\right)}{\cancel{\pi R^2}} = \frac{R^2}{8\mu}\left(\frac{-dp}{dz}\right)$$

$$= \frac{V_{z,cl}}{2}$$

$$\boxed{V_{z,avg} = \frac{V_{z,cl}}{2}}$$


$\tau_w = \frac{dp}{dz} \cdot \frac{R}{2}$
r=R wall
r=0 CL
r
z
$v_z(r)$
$v_{z,cl}$
$\tau(r)$
$v_{z,avg}$
$= \frac{1}{2} v_{z,cl}$
Velocity profile
$v_z(r)$
Parabolic
Shear stress
profile $\tau(r)$
linear

Numericals on Flow Thru Pipes

Q1 A fluid of viscosity 0.7 Ns/m² and specific gravity 1.3 is flowing through a circular pipe of diameter 100 mm. The maximum shear stress at pipe wall is given as 196.2 N/m². Find (i) pressure gradient (ii) Average velocity and (iii) Reynolds number of the flow

Sol:

$\mu = 0.7 \ Ns/m^2$

$SG = 1.3 \Rightarrow \rho = SG \times \rho_{water} = 1.3 \times 1000 = 1300 \ kg/m^3$

$D = 100 \ mm \Rightarrow R = \frac{D}{2} = 50 \ mm$

$\tau_{max} = \tau_w = -196.2 \ N/m^2$ (-ve as shear always acts opposite to the flow direction)

(i) pressure gradient: $\left(\frac{dp}{dz}\right) = ?$

$$\tau_w = \frac{dp}{dz} \cdot \frac{R}{2}$$

$$\Rightarrow \frac{dp}{dz} = \frac{2\tau_w}{R} = \frac{2 \times (-196.2)}{0.05} = -7848 \ N/m^3$$

(ii) Average velocity:

$$V_{z,avg} = \frac{1}{2} V_{z,cl}$$

$$V_z(r) = \frac{R^2}{4\mu}\left(\frac{-dp}{dz}\right)\left(1 - \frac{r^2}{R^2}\right)$$

at centerline $r = 0$ ⇒ centerline velocity $V_{z,cl} = V_z(r=0)$

$$= \frac{R^2}{4\mu} \cdot \left(\frac{-dp}{dz}\right)$$

$$= \frac{0.05^2}{4 \times 0.7} \times (7848) = 7 \text{ m/s}$$

Average velocity $V_{z,avg} = \frac{1}{2} V_{z,cl} = \frac{1}{2} \times 7 = 3.5$ m/s

(iii) Reynolds number

$$Re = \frac{\rho V_{z,avg} D}{\mu} = \frac{1300 \times 3.5 \times 0.1}{0.7}$$

⇒ $Re = 650$

For pipe flow

if $Re < 2000$ – laminar

$2000 < Re < 4000$ – Transition

$Re > 4000$ – Turbulent

Q2: An oil of specific gravity 0.9 and viscosity 10 poise is flowing through a pipe of diameter 100 mm. The velocity at the center of the pipe is 2 m/s. Find the (i) pressure gradient in the flow direction (ii) shear stress at the pipe wall (iii) Reynolds number (iv) velocity at a distance of 20 mm from the wall.

Sol:

$$SG = 0.9 \Rightarrow \rho = SG \times \rho_{water} = 0.9 \times 1000 = 900 \ kg/m^3$$

$$\mu = 10 \text{ poise} = 1 \ Ns/m^2$$

$$D = 100 \text{ mm} \Rightarrow R = 50 \text{ mm}$$

$$V_{z,cl} = V_z(r=0) = 2 \ m/s$$

(i) Pressure gradient :-

$$V_z(r) = \frac{R^2}{4\mu}\left(\frac{-dp}{dz}\right)\left(1 - \frac{r^2}{R^2}\right)$$

Centerline velocity $V_{z,cl} = V_z(r=0) = \frac{R^2}{4\mu}\left(\frac{-dp}{dz}\right)$

$$\Rightarrow \quad 2 = \frac{0.05^2}{4 \times 1}\left(\frac{-dp}{dz}\right) \Rightarrow \frac{dp}{dz} = -3200 \ N/m^3$$

(ii) Shear Stress at pipe wall τ_w

$$\tau_w = \frac{dp}{dz} \cdot \frac{R}{2}$$

$$= -3200 \times \frac{0.05}{2}$$

$$\tau_w = -80 \ N/m^2$$

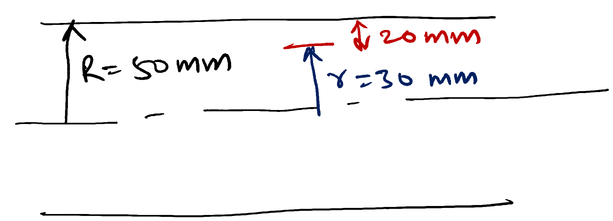
(iii) Reynolds number

$$Re = \frac{\rho V_{z,avg} D}{\mu}$$

$$V_{z,avg} = \frac{1}{2} V_{z,cl} = 1 \ m/s$$

$$= \frac{900 \times 1 \times 0.1}{1} = 90$$

(iv) velocity at a distance of 20mm from wall



$$V_z(r) = \frac{R^2}{4\mu}\left(\frac{-dp}{dz}\right)\left(1 - \frac{r^2}{R^2}\right)$$

$$V_z(r=0.3) = \frac{0.05^2}{4 \times 1} \times 3200 \times \left(1 - \frac{0.03^2}{0.05^2}\right) = 1.28 \ m/s$$

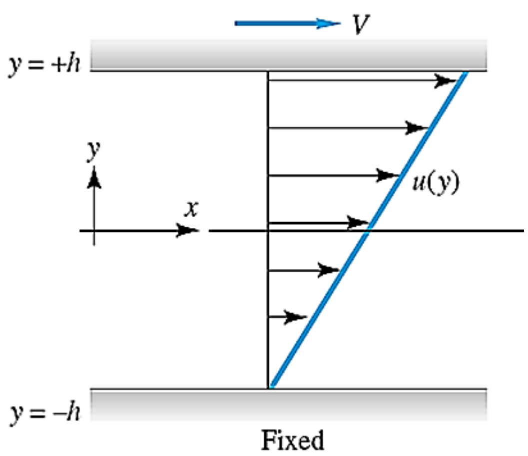
Unit-3

Exact Solutions to Navier-Stokes Equations

Couette and Poiseuille Flows

Couette Flow between a Fixed and a Moving Plate

- Couette Flow is the laminar flow of viscous fluid in the space between two parallel plates, one of which is moving relative to the other.
- Consider two-dimensional incompressible plane viscous flow between parallel plates as shown in Fig. below.



We assume,

- that the plates are very wide and very long, so that the flow is essentially axial, $u \neq 0$, but $v = w = 0$.
- the upper plate moves at velocity *V*, but there is no pressure gradient.
- also, we neglect gravity effects.

From the continuity equation,

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 \qquad \text{or} \qquad u = u(y) \text{ only}}$$

Thus, there is a single non-zero axial-velocity component which varies only across the channel. The flow is said to be *fully developed* (far downstream of the entrance).

Couette Flow between a Fixed and a Moving Plate

Substitute, $u = u(y)$, into the x-component of the Navier-Stokes momentum equation for two-dimensional (x, y) flow,

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho(0 + 0) = 0 + 0 + \mu\left(0 + \frac{d^2 u}{dy^2}\right)$$

Most of the terms drop out, and the momentum equation simply reduces to

$$\frac{d^2 u}{dy^2} = 0 \qquad \text{or} \qquad u = C_1 y + C_2$$

The two constants are found by applying the no-slip condition at the upper and lower plates:

At $y = +h$: $\quad u = V = C_1 h + C_2$

At $y = -h$: $\quad u = 0 = C_1(-h) + C_2$

or $\quad C_1 = \dfrac{V}{2h} \quad$ and $\quad C_2 = \dfrac{V}{2}$

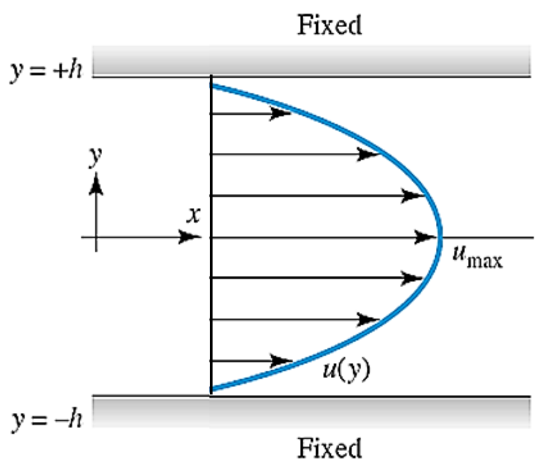
Therefore the solution for this case (a), flow between plates with a moving upper wall, is

$$u = \frac{V}{2h}y + \frac{V}{2} \qquad -h \le y \le +h$$

This is *Couette flow* due to a moving wall: a linear velocity profile with no-slip at each wall,

Poiseuille Flow due to Pressure Gradient between Two Fixed Plates

- Poiseuille Flow is the laminar flow of viscous fluid in the space between two parallel plates, due to pressure gradient.
- Consider two-dimensional incompressible plane viscous flow between parallel plates as shown in Fig. below.



We assume,

- that the plates are very wide and very long, so that the flow is essentially axial, $u \neq 0$, but $v = w = 0$.
- both plates are fixed, but the pressure varies in the x direction.
- gravity is neglected.

From the continuity equation,

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 \qquad \text{or} \qquad u = u(y) \text{ only}}$$

Thus, there is a single non-zero axial-velocity component which varies only across the channel. The flow is said to be *fully developed* (far downstream of the entrance).

Poiseuille Flow due to Pressure Gradient between Two Fixed Plates

Substitute, $u = u(y)$ and $\frac{\partial p}{\partial x} \neq 0$, into the x-component of the Navier-Stokes momentum equation for two-dimensional (x, y) flow,

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\boxed{\mu \frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x}} \qquad \text{Eqn.1}$$

Also, since $v = w = 0$ and gravity is neglected, the y- and z-momentum equations lead to

$$\frac{\partial p}{\partial y} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} = 0 \quad \text{or} \quad p = p(x) \text{ only}$$

Thus the pressure gradient in Eq. (1),) is the total and only gradient:

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} = \text{const} < 0 \qquad \text{Eqn.2}$$

Poiseuille Flow due to Pressure Gradient between Two Fixed Plates

The solution to Eq. (2), is accomplished by double integration:

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

The constants are found from the no-slip condition at each wall:

At $y = \pm h$: $u = 0$ or $C_1 = 0$ and $C_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}$

Thus the solution to case (b), flow in a channel due to pressure gradient, is

$$u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right)$$

The flow forms a *Poiseuille* parabola of constant negative curvature. The maximum velocity occurs at the centerline $y = 0$:

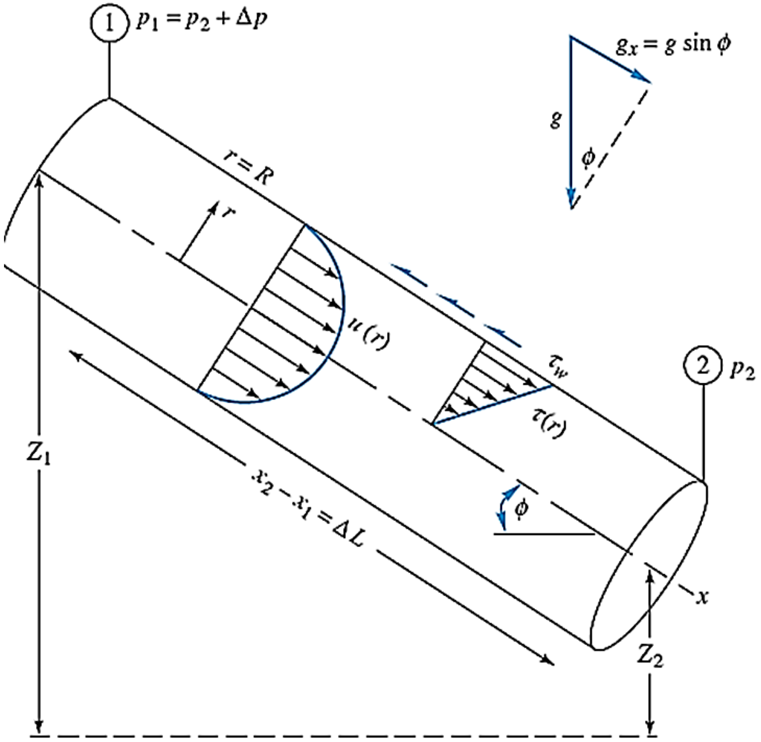
$$u_{\max} = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

Unit-3

Darcy-Weisbach Equation

Darcy-Weisbach Equation

- Consider an incompressible, fully developed viscous flow of fluid through a pipe of radius 'R'. The *x*-axis is taken in the flow direction and is inclined to the horizontal at an angle 'Φ'.



By volume conservation,

$$Q_1 = Q_2 = \text{const}$$

$$V_1 = \frac{Q_1}{A_1} = V_2 = \frac{Q_2}{A_2}$$

Darcy-Weisbach Equation

By applying incompressible Bernoulli equation, we have

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 + h_f$$

The velocities $V_1 = V_2$, since the flow is fully developed. Also, the pipe is of constant area, so $A_1 = A_2$. The head loss in the pipe due to friction, h_f is,

$$h_f = \left(z_1 + \frac{p_1}{\rho g}\right) - \left(z_2 + \frac{p_2}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g} \qquad \text{———— Eqn. 1}$$

Thus, the head loss (h_f) in the pipe due to friction is equal to the sum of the change in gravity head and pressure head.

By momentum balance, we get

$$\Delta p\, \pi R^2 + \rho g(\pi R^2)\Delta L \sin\theta - \tau_w(2\pi R)\Delta L = \dot{m}(V_1 - V_2) = 0$$

Dividing throughout by $(\pi R^2)\rho g$, we get

$$\frac{\Delta p}{\rho g} + \Delta L\,\sin\theta = \frac{2\tau_w}{\rho g}\frac{\Delta L}{R}$$

But $\Delta L \sin\theta = \Delta z$. Thus, $\displaystyle \frac{\Delta p}{\rho g} + \Delta z = \frac{2\tau_w}{\rho g}\frac{\Delta L}{R}$

Darcy-Weisbach Equation

Using Eqn. 1, we obtain

$$\frac{\Delta p}{\rho g} + \Delta z = h_f = \frac{2\tau_w}{\rho g}\frac{\Delta L}{R} \qquad \text{———— Eqn. 2}$$

Functionally, we can assume the wall shear stress 'τ_w' as

$$\tau_w = F(\rho,\ V,\ \mu,\ d,\ \epsilon)$$

Where, d is the pipe diameter, μ is the dynamic viscosity, and ε is the wall-roughness height. By dimensional analysis, we can obtain

$$\frac{8\tau_w}{\rho V^2} = f = F\left(\mathrm{Re}_d,\ \frac{\epsilon}{d}\right) \qquad \text{———— Eqn. 3}$$

Where, *f* is called the *Darcy friction factor*, which is a dimensionless parameter.

Combining Eqns. 2 and 3, we obtain

$$\boxed{h_f = f\,\frac{L}{d}\,\frac{V^2}{2g}}$$

For noncircular duct, d is replaced by hydraulic diameter, D_h

$$D_h = \frac{4A}{\mathcal{P}} = \frac{4 \times \text{area}}{\text{wetted perimeter}} = 4R_h$$

This is called the *Darcy-Weisbach equation*, valid for duct flows of any cross section and for laminar and turbulent flow.

Moody's Chart

Moody chart – Friction factor as a function of Reynolds number and relative roughness for round pipes