$Z-$Transforms

Definition 1:

If the function $f(n)$ is defined for $n = 0, 1, 2, \cdots$ and $f(n) = 0$ for $n < 0$, then $f(0), f(1), f(2), \cdots$ is a sequence, denoted by $\{f(n)\}$. The $Z-$transform of the sequence $\{f(n)\}_{n=0}^{\infty}$ is defined as

$$\boldsymbol{Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}}, \text{ if the series converges.}$$

We denote the sum by F(z), where z is a complex variable.

Thus, $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$

Note

1. This $Z-$transform is called an one sided $Z-$transform.
2. The series $\sum_{n=0}^{\infty} f(n) z^{-n}$ is convergent for z such that $|Z| > R$, where $R = \lim_{n\to\infty} \left| \frac{f(n)}{f(n+1)} \right|$
3. If $f(t)$ is a continuous function, representing a continuous signal, then we sample at the time instances $0, T, 2T, \cdots, nT, \cdots$ get the sequence $f(0), f(T), f(2T), \cdots$ which is a discrete time function. T is called a sample period.

Definition 2:

If the continuous function $f(t)$ is defined for the sampled values $t = nT$, $n = 0, 1, 2, \cdots$, then the $Z-$transform of $f(t)$ is defined as $Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$. It is also denoted by $F(z)$.

Formula to remember!!!

1. $\boldsymbol{(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots}$ if $\boldsymbol{|z| < 1}$
2. $\boldsymbol{(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots}$ if $\boldsymbol{|z| < 1}$
3. $\boldsymbol{(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots}$ if $\boldsymbol{|z| < 1}$
4. $\boldsymbol{(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots}$ if $\boldsymbol{|z| < 1}$
5. $\boldsymbol{-\log_e(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots}$ if $\boldsymbol{|z| < 1}$
6. $\boldsymbol{e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$

Z-transforms of some standard sequences	Proof Tip!
1. Prove that $\boldsymbol{Z\,[1] = \dfrac{z}{z-1}, \qquad \lvert z\rvert > 1.}$	
Proof:	
Given $f(n) = 1$	
$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$	From the definition of Z-transform.
$\therefore\ Z[1] = \sum_{n=0}^{\infty} 1.z^{-n}$	Substitute the given f(n).
$= \sum_{n=0}^{\infty} z^{-1.n}$	$a^{m.n} = (a^m)^n$
$= \sum_{n=0}^{\infty} \left(z^{-1}\right)^n$	$z^{-1} = \left(\frac{1}{z}\right)$
$= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$	
$= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$	Running the summation by the below formula
$= \left[1 - \left(\frac{1}{z}\right)\right]^{-1} \quad \text{if } \left\lvert \frac{1}{z} \right\rvert < 1$	$\boldsymbol{(1-z)^{-1} = 1 + z + z^2 + \cdots \quad \text{if } \lvert z\rvert < 1}$
$= \left[\left(\frac{z-1}{z}\right)\right]^{-1} \quad \text{if } \lvert z\rvert > 1$	
$= \frac{z}{z-1} \quad \text{if } \lvert z\rvert > 1$	$\frac{1}{\lvert z\rvert} < 1 \Rightarrow \lvert z\rvert > 1$

Z-transforms of some standard sequences	Proof Tip!
2. Prove that $\boldsymbol{Z\left[a^{n}\right]=\frac{z}{z-a}, \qquad \|z\|>a.}$	
Proof:	
Given $f(n)=a^{n}$	
$Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$	From the definition of Z-transform.
$\therefore\ Z[a^{n}] = \sum_{n=0}^{\infty} a^{n}.z^{-n}$	Substitute the given f(n).
$= \sum_{n=0}^{\infty} a^{n}z^{-1.n}$	$a^{m.n}=(a^{m})^{n}$
$= \sum_{n=0}^{\infty} a^{n}\left(z^{-1}\right)^{n}$	$z^{-1}=\left(\frac{1}{z}\right)$
$= \sum_{n=0}^{\infty} a^{n}\left(\frac{1}{z}\right)^{n}$	$a^{n}.b^{n}=(ab)^{n}$
$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$	
$= 1+\left(\frac{a}{z}\right)+\left(\frac{a}{z}\right)^{2}+\left(\frac{a}{z}\right)^{3}+\cdots$	Running the summation by formula
$= \left[1-\left(\frac{a}{z}\right)\right]^{-1} \quad \text{if } \left\|\frac{a}{z}\right\|<1$	$\boldsymbol{(1-z)^{-1} = 1+z+z^{2}+\cdots}$ if $\boldsymbol{\|z\|<1}$
$= \left[\frac{z-a}{z}\right]^{-1} \quad \text{if } \|z\|>a$	$\left\|\frac{a}{z}\right\|<1 \Rightarrow \frac{\|a\|}{\|z\|}<1 \Rightarrow \|a\|<\|z\|$
$= \frac{z}{z-a} \quad \text{if } \|z\|>a$	$\because\ a>0, \|a\|<\|z\| \Rightarrow\ a<\|z\|$ $\Rightarrow\ \|z\|>a$

Z-transforms of some standard sequences	Proof Tip!
3. Prove that $\boldsymbol{Z[n] = \frac{z}{(z-1)^2}}, \quad \boldsymbol{|z| > 1,.}$	
Proof:	
Given $f(n) = n$	
$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$	From the definition of Z-transform.
$\therefore Z[n] = \sum_{n=0}^{\infty} n.z^{-n}$	Substitute the given f(n).
$= \sum_{n=0}^{\infty} n z^{-1.n}$	$a^{m.n} = (a^m)^n$
$= \sum_{n=0}^{\infty} n \left(z^{-1}\right)^n$	$z^{-1} = \left(\frac{1}{z}\right)$
$= \sum_{n=0}^{\infty} n \left(\frac{1}{z}\right)^n$	
$= \sum_{n=0}^{\infty} n \left(\frac{1}{z}\right)^n$	Running the summation
$= 0 + 1\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + 4\left(\frac{1}{z}\right)^4 + \cdots$	Take out the term $\left(\frac{1}{z}\right)$
$= \left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + 4\left(\frac{1}{z}\right)^4 + \cdots$	by formula
$= \left(\frac{1}{z}\right)\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \cdots\right]$	$\boldsymbol{(1-z)^{-2} = 1 + 2z + 3z^2 + \cdots}$ **if** $\boldsymbol{\lvert z\rvert < 1}$
$= \frac{1}{z}\left[1 - \left(\frac{1}{z}\right)\right]^{-2} \quad \text{if } \left\lvert\frac{1}{z}\right\rvert < 1$	
$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2} = \frac{1}{z}\frac{z^2}{(z-1)^2} \quad \text{if } \lvert z\rvert > 1$	$\left\lvert\frac{1}{z}\right\rvert < 1 \Rightarrow \frac{1}{\lvert z\rvert} < 1 \Rightarrow \lvert z\rvert > 1$
$= \frac{1}{\cancel{z}}\frac{\cancel{z^2}^{1}}{(z-1)^2} = \frac{z}{(z-1)^2} \quad \text{if } \lvert z\rvert > 1$	

Z-transforms of some standard sequences	Proof Tip!
4. Prove that $Z\left[\frac{1}{n}\right] = \log_e\left(\frac{z}{z-1}\right), \quad |z|>1, n>0.$	
Proof:	
Given $f(n) = \frac{1}{n}$	
$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$	From the definition of Z-transform.
$\therefore Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n}.z^{-n}$	Substitute the given f(n).
$= \sum_{n=1}^{\infty} \frac{1}{n} z^{-1.n}$	$a^{m.n} = (a^m)^n$
$= \sum_{n=1}^{\infty} \frac{1}{n} \left(z^{-1}\right)^n$	$z^{-1} = \left(\frac{1}{z}\right)$
$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{z}\right)^n$	Running the summation
$= \sum_{n=1}^{\infty} \frac{(1/z)^n}{n}$	Take out the term $\left(\frac{1}{z}\right)$
$= (1/z) + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \cdots$	by formula
$= -\log_e\left(1-\frac{1}{z}\right) \quad \text{if } \left|\frac{1}{z}\right| < 1$	$-\log_e(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$ if $|z|<1$
$= -\log_e\left(\frac{z-1}{z}\right) \quad \text{if } |z|>1$	
$= \log_e\left(\frac{z-1}{z}\right)^{-1} \quad \text{if } |z|>1$	$\left|\frac{1}{z}\right| < 1 \Rightarrow \frac{1}{|z|} < 1 \Rightarrow |z| > 1$
$= \log_e\left(\frac{z}{z-1}\right) \quad \text{if } |z|>1$	

Z-transforms of some standard sequences	Proof Tip!
5. Prove that $\boldsymbol{Z\left[\frac{1}{n!}\right] = e^{1/z}}$.	
Proof:	
Given $f(n) = \dfrac{1}{n!}$	
$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$	From the definition of Z-transform.
$\therefore\ Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} . z^{-n}$	Substitute the given f(n).
$= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-1.n}$	$a^{m.n} = (a^m)^n$
$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(z^{-1}\right)^n$	$z^{-1} = \left(\frac{1}{z}\right)$
$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n$	
$= \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!}$	Running the summation
$= 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots$	by formula
$= e^{1/z}$	$\boldsymbol{e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$

Z-transforms of some standard sequences	**Proof Tip!**
6. Prove that $\boldsymbol{Z}\left[\dfrac{a^n}{n!}\right] = e^{a/z}$.	
Proof:	
Given $f(n) = \dfrac{1}{n!}$	
$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$	From the definition of Z-transform.
$\therefore Z\left[\dfrac{a^n}{n!}\right] = \sum_{n=0}^{\infty} \dfrac{a^n}{n!}.z^{-n}$	Substitute the given f(n).
$= \sum_{n=0}^{\infty} \dfrac{a^n}{n!} z^{-1.n}$	$a^{m.n} = (a^m)^n$
$= \sum_{n=0}^{\infty} \dfrac{a^n}{n!} \left(z^{-1}\right)^n$	$z^{-1} = \left(\dfrac{1}{z}\right)$
$= \sum_{n=0}^{\infty} \dfrac{1}{n!} \left(\dfrac{a}{z}\right)^n$	
$= \sum_{n=0}^{\infty} \dfrac{(a/z)^n}{n!}$	Running the summation
$= 1 + \dfrac{(a/z)}{1!} + \dfrac{(a/z)^2}{2!} + \dfrac{(a/z)^3}{3!} + \cdots$	by formula
$= e^{a/z}$	$\boldsymbol{e^z = 1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots}$

Z-transforms of some standard sequences	Proof Tip!
7. Prove that $Z\left[\dfrac{1}{(n+1)!}\right] = z\left(e^{1/z}-1\right)$.	
Proof:	
Given $f(n) = \dfrac{1}{(n+1)!}$	
$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$	From the definition of Z-transform.
$\therefore Z\left[\dfrac{1}{(n+1)!}\right] = \sum_{n=0}^{\infty} \dfrac{1}{(n+1)!}.z^{-n}$	Substitute the given f(n).
$= \sum_{n=0}^{\infty} \dfrac{1}{(n+1)!} z^{-1.n}$	$a^{m.n} = (a^m)^n$
$= \sum_{n=0}^{\infty} \dfrac{1}{(n+1)!} \left(z^{-1}\right)^n$	$z^{-1} = \left(\dfrac{1}{z}\right)$
$= \sum_{n=0}^{\infty} \dfrac{1}{(n+1)!} \left(\dfrac{1}{z}\right)^n$	
$= \sum_{n=0}^{\infty} \dfrac{(1/z)^n}{(n+1)!}$	
$= \dfrac{1}{1!} + \dfrac{(1/z)}{2!} + \dfrac{(1/z)^2}{3!} + \dfrac{(1/z)^3}{4!} + \cdots$	Running the summation
$= \dfrac{z}{z}\left[\dfrac{1}{1!} + \dfrac{(1/z)}{2!} + \dfrac{(1/z)^2}{3!} + \dfrac{(1/z)^3}{4!} + \cdots\right]$	multiply by z/z
$= z\left[\dfrac{1/z}{1!} + \dfrac{(1/z)^2}{2!} + \dfrac{(1/z)^3}{3!} + \cdots\right]$	
$= z\left[1 + \dfrac{1/z}{1!} + \dfrac{(1/z)^2}{2!} + \dfrac{(1/z)^3}{3!} + \cdots - 1\right]$	Add and subtract 1
$= z\left[\left(1 + \dfrac{1/z}{1!} + \dfrac{(1/z)^2}{2!} + \cdots\right) - 1\right]$	by the below formula
$= z\left[e^{1/z} - 1\right]$	$e^z = 1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$

Properties of $Z-$Transforms

1. **Linearity Property:** $Z\left[af(n)+bg(n)\right] = aZ\left[f(n)\right] + bZ\left[g(n)\right]$

2. **Time shifting Property:**

   Shifting to the right:

   If $Z\left[f(n)\right] = F(z)$, then $Z\left[f(n-k)\right] = z^{-k}F(z),\ k>0$

   Shifting to the left:

   If $Z\left[f(n)\right] = F(z)$, then $Z\left[f(n+k)\right] = z^{k}\left[F(z) - f(0) - \dfrac{f(1)}{z} - \dfrac{f(2)}{z^2} + \cdots\right],\ k>0$

3. **Change of scale (in z domain) or Dumping rule:**

   If $Z\left[f(n)\right] = F(z)$, then $Z\left[a^{-n}f(n)\right] = \left[Z\left[f(n)\right]\right]_{z\to az}$

   If $Z\left[f(n)\right] = F(z)$, then $Z\left[a^{n}f(n)\right] = \left[Z\left[f(n)\right]\right]_{z\to z/a}$

4. **Multiplication by n (or) Differentiation in the z domain**

   If $Z\left[f(n)\right] = F(z)$, then $Z\left[nf(n)\right] = -z\dfrac{d}{dz}\left[Z\left[f(n)\right]\right]$

   Proof:

$$
\begin{aligned}
F(z) &= Z\left[f(n)\right] \\
&= \sum_{n=0}^{\infty} f(n)z^{-n}
\end{aligned}
$$

Diff. w. r. to z,

$$
\begin{aligned}
\frac{d}{dz}\left[F(z)\right] &= \frac{d}{dz}\left[\sum_{n=0}^{\infty} f(n)z^{-n}\right] \\
&= \sum_{n=0}^{\infty} f(n)\frac{d}{dz}\left[z^{-n}\right] \\
&= \sum_{n=0}^{\infty} f(n)\left[-nz^{-n-1}\right] \\
&= \sum_{n=0}^{\infty} f(n)\left[-nz^{-n}z^{-1}\right] \\
&= -z^{-1}\sum_{n=0}^{\infty} f(n)\left[nz^{-n}\right] \\
&= -\frac{1}{z}\sum_{n=0}^{\infty} nf(n)z^{-n} \\
&= Z\left[nf(n)\right] \\
\therefore\ Z\left[nf(n)\right] &= -z\frac{d}{dz}\left[F(z)\right]
\end{aligned}
$$

Initial value theorem on $Z-$transforms
If $Z[f(n)] = F(z)$, then $f(0) = \lim_{z\to\infty} F(z)$
Proof:
Given $Z[f(n)] = F(z)$

$$\begin{aligned}
F(z) &= Z[f(n)] \\
&= \sum_{n=0}^{\infty} f(n)z^{-n} \\
&= f(0)z^{0} + f(1)z^{-1} + f(2)z^{-2}\cdots \\
&= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2}\cdots \\
\lim_{z\to\infty} F(z) &= \lim_{z\to\infty}\left[f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2}\cdots\right] \\
&= f(0) \qquad \because \lim_{z\to\infty}\left[\frac{1}{z^n}\right] = 0
\end{aligned}$$

Final value theorem on $Z-$transforms
If $Z[f(n)] = F(z)$, then $\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z)$
Proof:

$$\begin{aligned}
\text{We know that} Z[f(n+1)] &= z[F(z) - f(0)] \\
&= zF(z) - zf(0) \\
\Rightarrow Z[f(n+1)] - F(z) &= zF(z) - F(z) - zf(0) \\
\Rightarrow Z[f(n+1)] - Z[f(n)] &= zF(z) - F(z) - zf(0) \\
\Rightarrow Z[f(n+1) - f(n)] &= (z-1)F(z) - zf(0) \\
\Rightarrow \sum_{n=0}^{\infty}[f(n+1) - f(n)]\,z^{-n} &= (z-1)F(z) - zf(0) \\
\therefore \lim_{z\to 1}[(z-1)F(z) - zf(0)] &= \sum_{n=0}^{\infty}[f(n+1) - f(n)] \\
&= \lim_{n\to\infty} S_n \;\; --\,\mathbf{(1)} \quad \text{where } S_n \text{ is the partial sum of n terms}
\end{aligned}$$

$$\begin{aligned}
S_n &= \cancel{f(1)} - f(0) \\
&\quad + \cancel{f(2)} - \cancel{f(1)} \\
&\quad + \cancel{f(3)} - \cancel{f(2)} \\
&\quad \vdots \\
&\quad + f(n) - \cancel{f(n-1)}
\end{aligned}$$

$$\begin{aligned}
\therefore\ S_n &= f(n) - f(0) \ \text{ Subs. in } \mathbf{(1)} \\
\therefore \lim_{z\to 1}\left[(z-1)F(z) - zf(0)\right] &= \lim_{n\to\infty}\left[f(n) - f(0)\right] \\
\Rightarrow \lim_{z\to 1}\left[(z-1)F(z)\right] - f(0) &= \lim_{n\to\infty}\left[f(n) - f(0)\right] \\
\therefore \lim_{z\to 1}\left[(z-1)F(z)\right] &= \lim_{n\to\infty} f(n)
\end{aligned}$$

Problems	Solving Tip!
8. Find $\boldsymbol{Z}\left[(\boldsymbol{n}+\boldsymbol{1})(\boldsymbol{n}+\boldsymbol{1})\right]$.	

Proof:

$$\begin{aligned}
Z\left[(n+1)(n+1)\right] &= Z\left[n(n+1)+(n+1)\right] \\
&= Z\left[n^2+n+n+1\right] \\
&= Z\left[n^2+2n+1\right] \\
&= Z\left[n^2\right]+2Z\left[n\right]+Z\left[1\right] \\
&= \frac{z^2+z}{(z-1)^3}+2.\frac{z}{(z-1)^2}+\frac{z}{z-1} \\
&= \frac{z^2+z+2z(z-1)+z(z-1)^2}{(z-1)^3} \\
&= \frac{z^2+z+2z^2-2z+z(z^2-2z+1)}{(z-1)^3} \\
&= \frac{z^2+z+2z^2-2z+z^3-2z^2+z}{(z-1)^3} \\
&= \frac{z^3+z^2}{(z-1)^3}
\end{aligned}$$

Solving Tip!

First simplify the $(n+1)(n+2)$ and then by the linearity property we get the 4th line. And substitute the formula of $Z[n^2]$, $Z[n]$ and $Z[1]$ and take the LCM.
whether you forgot the formula for $Z[n^2.]$ Find it from the property,

$$Z[nf(n)] = -z\frac{d}{dz}\left[Z[f(n)]\right].$$

i.e.,

$$\begin{aligned}
Z[n^2.] &= Z[n.n] \\
&= -z\frac{d}{dz}\left[Z[n]\right] \\
&= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]
\end{aligned}$$

Use

$$\left[\frac{u}{v}\right]' = \frac{vu'-uv'}{v^2}$$

and find it.

Problems	**Solving Tip!**
9. Find $Z-$transform of $\cos n\theta$ and $\sin n\theta$. Hence deduce the $Z-$transforms of $\cos(n+1)\theta$ and $a^n \sin n\theta$	
Solution:	
$$\text{We know that } Z\left[a^n\right] = \frac{z}{z-a}, \text{ if } \lvert z\rvert < 1$$	
Put $a = e^{i\theta}$ $\therefore a^n = e^{in\theta} = [\cos n\theta + i \sin n\theta]$	
$$Z\left[\cos n\theta + i\sin n\theta\right] = Z\left[e^{in\theta}\right]$$	$a^{mn} = (a^m)^n$
$$= Z\left[\left(e^{i\theta}\right)^n\right]$$	
$$= \frac{z}{z-(\cos\theta + i\sin\theta)} = \frac{z}{z-\cos\theta - i\sin\theta}$$	In the denominator, gather the real part and imaginary part separately.
$$= \frac{z}{(z-\cos\theta) - i\sin\theta}$$	
$$= \frac{z}{(z-\cos\theta) - i\sin\theta} \times \frac{(z-\cos\theta) + i\sin\theta}{(z-\cos\theta) + i\sin\theta}$$	If we have the complex number in Denominator, we take conjugate.
$$= \frac{z((z-\cos\theta) + i\sin\theta)}{(z-\cos\theta)^2 - (i\sin\theta)^2}$$	
$$= \frac{z((z-\cos\theta) + i\sin\theta)}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$	And simplify it use the formula $(a^2 - b^2) = (a-b)(a+b)$.
$$= \frac{z((z-\cos\theta) + i\sin\theta)}{z^2 - 2z\cos\theta + 1}$$	
$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$	
$$= \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + i\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$	

Problems

$$\Rightarrow Z\left[\cos\theta\right] + iZ\left[\sin\theta\right] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1} \cdots\cdots(1)$$

Write the L.H.S and R.H.S like as the real part + imaginary part.

Equating the real parts, we get

$$Z\left[\cos n\theta\right] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

Equating the imaginary parts, we get

$$Z\left[\sin n\theta\right] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

To find $Z\left[\cos(n+1)\theta\right]$,

we use the shifting property,

$$\begin{aligned} Z\left[f(n+1)\right] &= z\left[F(z)-f(0)\right] \\ &= z\left[Z\left[f(n)\right]-f(0)\right] \end{aligned}$$

Putting $k = 1$ in the property of shift to the left

$$\begin{aligned} \text{Here} f(n) &= \cos n\theta \quad \therefore\ f(0) = \cos 0 = 1 \\ f(n+1) &= \cos(n+1)\theta \end{aligned}$$

Write the terms what we need

$$\begin{aligned} Z\left[\cos(n+1)\theta\right] &= z\left[Z\left[\cos n\theta\right]-f(0)\right] \\ &= z\left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}-1\right] \\ &= z\left[\frac{z(z-\cos\theta)-(z^2-2z\cos\theta+1)}{z^2-2z\cos\theta+1}\right] \\ &= z\left[\frac{\cancel{z^2}-z\cos\theta-\cancel{z^2}+2z\cos\theta-1}{z^2-2z\cos\theta+1}\right] \\ &= z\left[\frac{z\cos\theta-1}{z^2-2z\cos\theta+1}\right] \\ &= \left[\frac{z(z\cos\theta-1)}{z^2-2z\cos\theta+1}\right] \end{aligned}$$

Substituting the corresponding values and simplify to get the required result.

To find $a^n \sin n\theta$, we use the property

$$\begin{aligned} Z\left[a^n f(n)\right] &= \left[Z\left[f(n)\right]\right]_{z\to z/a} \\ Z\left[a^n\sin\theta\right] &= \left[Z\left[\sin\theta\right]\right]_{z\to z/a} \\ &= \left[\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right]_{z\to z/a} \end{aligned}$$

using the dumping rule.

Problems	**Solving Tip!**
$$= \left[\frac{(z/a)\sin\theta}{(z/a)^2 - 2(z/a)\cos\theta + 1}\right]$$ $$= \left[\frac{\frac{z\sin\theta}{\cancel{a}}}{\frac{z^2 - 2az\cos\theta + a^2}{a^{\cancel{2}1}}}\right] = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$	
10. Find the $Z-$transform of $\frac{1}{(n+1)(n+2)}$	
Solution:	
We use a partial fraction method,	
Let $\frac{1}{(n+1)(n+2)} = \frac{A}{(n+1)} + \frac{B}{(n+2)} \cdots\cdots (1)$	
$$\frac{1}{\cancel{(n+1)(n+2)}}\cancel{(n+1)(n+2)} = \frac{A}{\cancel{(n+1)}}\cancel{(n+1)}(n+2) + \frac{B}{\cancel{(n+2)}}(n+1)\cancel{(n+2)}$$	
$$\Rightarrow 1 = A(n+2) + B(n+1)$$	
Put $n = -2$ $1 = A(-2+2) + B(-2+1)$ $1 = 0 - B$ $B = -1$; Put $n = -1$ $1 = A(-1+2) + B(-1+1)$ $1 = A + 0$ $A = 1$	
Subs. in (1), we get,	
$$\frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)} + \frac{-1}{(n+2)} = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$	
$$\therefore\ Z\left[\frac{1}{(n+1)(n+2)}\right] = Z\left[\frac{1}{(n+1)} - \frac{1}{(n+2)}\right] = Z\left[\frac{1}{(n+1)}\right] - Z\left[\frac{1}{(n+2)}\right] \cdots (2)$$	Partial fraction of Type I

Problems	**Solving Tip!**
But	
$$Z\left[\frac{1}{(n+1)}\right] = \sum_{n=0}^{\infty}\frac{1}{(n+1)}.z^{-n}$$	
$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)}z^{-1.n}$$	
$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)}\left(z^{-1}\right)^{n}$$	
$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)}\left(\frac{1}{z}\right)^{n}$$	
$$= \sum_{n=0}^{\infty}\frac{(1/z)^n}{(n+1)}$$	
$$= \frac{1}{1}+\frac{(1/z)}{2}+\frac{(1/z)^2}{3}+\frac{(1/z)^3}{4}+\cdots$$	Multiply and divide the z, for making one of the known equation from the page number 5.1
$$= \frac{z}{z}\left[\frac{1}{1}+\frac{(1/z)}{2}+\frac{(1/z)^2}{3}+\frac{(1/z)^3}{4}+\cdots\right]$$	
$$= z\left[\frac{1/z}{1}+\frac{(1/z)^2}{2}+\frac{(1/z)^3}{3}+\cdots\right]$$	
$$= z\left[-\log(1-(1/z))\right] \quad \text{if } \lvert z\rvert > 1$$	
$$= z\left[-\log\left(1-\frac{1}{z}\right)\right] \quad \text{if } \lvert z\rvert > 1$$	
$$= z\left[-\log\left(\frac{z-1}{z}\right)\right] \quad \text{if } \lvert z\rvert > 1$$	
$$= z\log\left(\frac{z}{z-1}\right) \quad \text{if } \lvert z\rvert > 1$$	
and	
$$Z\left[\frac{1}{(n+2)}\right] = \sum_{n=0}^{\infty}\frac{1}{(n+2)}.z^{-n}$$	
$$= \sum_{n=0}^{\infty}\frac{1}{(n+2)}z^{-1.n}$$	
$$= \sum_{n=0}^{\infty}\frac{1}{(n+2)}\left(z^{-1}\right)^{n}$$	

Problems	**Solving Tip!**
$= \sum_{n=0}^{\infty} \frac{1}{(n+2)} \left(\frac{1}{z}\right)^n$ $= \sum_{n=0}^{\infty} \frac{(1/z)^n}{(n+2)}$ $= \frac{1}{2} + \frac{(1/z)}{3} + \frac{(1/z)^2}{4} + \frac{(1/z)^3}{5} + \cdots$ $= z^2 . \frac{1}{z^2} \left[\frac{1}{2} + \frac{(1/z)}{3} + \frac{(1/z)^2}{4} + \cdots\right]$ $= z^2 \left[\frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \frac{(1/z)^4}{4} + \cdots\right]$ $= z^2 \left[\left(\frac{(1/z)}{1} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \frac{(1/z)^4}{4} + \cdots\right) - \frac{(1/z)}{1}\right]$ $= z^2 \left[-\log(1-(1/z)) - \frac{(1/z)}{1}\right]$ $= z^2 \left[-\log\left(1-\frac{1}{z}\right) - \frac{1}{z}\right]$ if $\|z\| > 1$ $= \left[z^2 \log\left(\frac{z-1}{z}\right)^{-1} - z^2 \frac{1}{z}\right]$ if $\|z\| > 1$ $= z^2 \log\left(\frac{z}{z-1}\right) - z$ if $\|z\| > 1$ Subs. in (2), we get $Z\left[\frac{1}{(n+1)(n+2)}\right] = z\log\left(\frac{z}{z-1}\right) - \left[z^2 \log\left(\frac{z}{z-1}\right) - z\right]$ $= (z - z^2)\log\left(\frac{z}{z-1}\right) + z$ **Exercise:** 11. Find the $Z-$transform of $\frac{2n+3}{(n+1)(n+2)}$	Add and subtract the term $\frac{(1/z)}{1}$ by the below formula $-\log_e(1-z)$ $= z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$ if if $\|z\| < 1$

Inverse Z–Transforms

Definition 1:

If $Z\left[f(n)\right] == F(z)$, then $F(z)$ is unique and the inverse Z-transform is $Z^{-1}\left[F(Z)\right] = f(n)$
Inverse Z–transform is found by one of the following methods.

1. Long division method.
2. Partial fraction method.
3. Residue method.
4. Convolution method.

Formulae!!!

$Z[f(n)] = F(z)$	$Z^{-1}\left[F(z)\right] = f(n)$
$Z[1] = \dfrac{z}{z-1}$	$Z^{-1}\left[\dfrac{z}{z-1}\right] = 1$
$Z[a^n] = \dfrac{z}{z-a}$	$Z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$
$Z[a^{n-1}] = \dfrac{1}{z-a}$	$Z^{-1}\left[\dfrac{1}{z-a}\right] = a^{n-1}$
$Z[na^n] = \dfrac{az}{(z-a)^2}$	$Z^{-1}\left[\dfrac{az}{(z-a)^2}\right] = na^n$
$Z[na^{n-1}] = \dfrac{z}{(z-a)^2}$	$Z^{-1}\left[\dfrac{z}{(z-a)^2}\right] = na^{n-1}$
$Z[n] = \dfrac{z}{(z-1)^2}$	$Z^{-1}\left[\dfrac{z}{(z-1)^2}\right] = n$
$Z[n(n-1)] = \dfrac{2z}{(z-1)^3}$	$Z^{-1}\left[\dfrac{2z}{(z-1)^3}\right] = n(n-1)$
$Z\left[a^n \cos\dfrac{n\pi}{2}\right] = \dfrac{z^2}{z^2+a^2}$	$Z^{-1}\left[\dfrac{z^2}{z^2+a^2}\right] = a^n \cos\dfrac{n\pi}{2}$
$Z\left[a^n \sin\dfrac{n\pi}{2}\right] = \dfrac{az}{z^2+a^2}$	$Z^{-1}\left[\dfrac{az}{z^2+a^2}\right] = a^n \sin\dfrac{n\pi}{2}$

Procedure for Long division method:
Step1: We rewrite $F(z)$ in terms of negative powers of z.
Step2: Perform the division.
Step3: Write $F(z) =$ quotient and from that deduce $f(n)$.

Procedure for residue method:
Step1: Multiply $F(z)$ with z^{n-1}.
Step2: Find poles and the residue of it.
Step3: Write $f(n) =$ sum of the residues.

Procedure for partial fraction method:
Step1: First find $\frac{F(z)}{z}$.
Step2: Do Partial fraction for $\frac{F(z)}{z}$.
Step3: Take Z^{-1} for $F(z)$.

Formula to remember!!!

Types	Partial fraction(it should be proper fraction)
I	$\frac{Nr}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
II	$\frac{Nr}{(x-a)(x-b)^n} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2} + \cdots + \frac{O}{(x-b)^n}$
III	$\frac{Nr}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$ here (x^2+bx+c) is an irreducible

Problem based on inverse Z-transforms by long division method	**Solving Tip!**

1. Find $Z^{-1}\left[\dfrac{z^2+z}{(z-1)^3}\right]$ by long division method

Solution:

Given $F(z) = \dfrac{z^2+z}{(z-1)^3}$

$$
\begin{aligned}
Z[F(z)] &= \frac{z^2+z}{(z-1)^3} = \frac{z^2+z}{(z^3-3z^2+3z-1)} \\
&= \frac{z^2\left(1+\frac{1}{z}\right)}{z^3(1-3\frac{1}{z}+3\frac{1}{z^2}-\frac{1}{z^3})} \\
&= \frac{\cancel{z^2}\left(1+z^{-1}\right)}{\cancel{z^2}(1-3z^{-1}+3z^{-2}-z^{-3})} \\
&= \frac{\cancel{z^2}\left(1+z^{-1}\right)}{\cancel{z^3}\,1(1-3z^{-1}+3z^{-2}-z^{-3})} \\
&= \frac{z^{-1}\left(1+z^{-1}\right)}{(1-3z^{-1}+3z^{-2}-z^{-3})} \\
&= \frac{\left(z^{-1}+z^{-2}\right)}{(1-3z^{-1}+3z^{-2}-z^{-3})}
\end{aligned}
$$

Step1: To perform long division method we rewrite $F(z)$ in terms of negative powers of z.
For that, Take out the higher degree terms of z from the Numerator and the Denomenator and simplify it!

Now, we perform the long division method

Step2 Perform the division.

$$
\begin{array}{r|llllll}
 & z^{-1} & +4z^{-2} & +9z^{-3} & +16z^{-4} & & +\cdots \\
\hline
1-3z^{-1}+3z^{-2}-z^{-3} & z^{-1} & +z^{-2} & & & & \\
 & z^{-1} & -3z^{-2} & +3z^{-3} & -z^{-4} & & \\
\cline{2-5}
 & & 4z^{-2} & -3z^{-3} & +z^{-4} & & \\
 & & 4z^{-2} & -12z^{-3} & +12z^{-4} & -4z^{-5} & \\
\cline{3-6}
 & & & 9z^{-3} & -11z^{-4} & +4z^{-5} & \\
 & & & 9z^{-3} & -27z^{-4} & +27z^{-5} & -9z^{-6} \\
\cline{4-7}
 & & & & +16z^{-4} & -23z^{-5} & +9z^{-6}
\end{array}
$$

Problem based on inverse Z-transforms by long division method	**Solving Tip!**

$$\therefore F(z) = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \cdots$$

$$\sum_{n=0}^{\infty} f(n)z^{-n} = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \cdots$$

$$f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + f(4)z^{-4} + \cdots = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \cdots$$

Step3
Write $F(z) =$ quotient and from that deduce $f(n)$.

Running the summation

Equating the like coefficients, we get
$f(0) = 0,\ f(1) = 1,\ f(2) = 4,\ f(3) = 9,\ f(4) = 16, \cdots$

$$\therefore f(n) = \begin{cases} 0 \text{ if } n = 0 \\ n^2 \text{ if } n = 1, 2, 3, \cdots \end{cases}$$

$$\therefore Z^{-1}\left[\frac{z^2 + z}{(z-1)^3}\right] = f(n) = \begin{cases} 0 \text{ if } n = 0 \\ n^2 \text{ if } n = 1, 2, 3, \cdots \end{cases}$$

2. Find $\boldsymbol{Z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]}$ by long division method

Solution: Given $F(z) = \frac{10z}{(z-1)(z-2)} = \frac{10z}{z^2 - 3z + 2}$

$$Z[F(z)] = \frac{10z}{z^2 - 3z + 2} = \frac{10\cancel{z}}{z^{\cancel{2}1}\left(1 - 3\frac{1}{z} + \frac{2}{z^2}\right)}$$

$$= \frac{10z^{-1}}{\left(1 - 3z^{-1} + 2z^{-2}\right)}$$

Step1: To perform long division method we rewrite $F(z)$ in terms of negative powers of z.
For that, Take out the higher degree terms of z from the Numerator and the Denomenator and simplify it!

<table>
<tr><th>Problem based on inverse Z-transforms by long division method</th><th>Solving Tip!</th></tr>
<tr><td>

Now, we perform the long division method,

$$\begin{array}{r|lllll} & 10z^{-1} & +30z^{-2} & +70z^{-3} & & +\cdots \\ \hline 1-3z^{-1}+2z^{-2} & 10z^{-1} & & & & \\ & 10z^{-1} & -30z^{-2} & +20z^{-3} & & \\ \hline & & 30z^{-2} & 20z^{-3} & & \\ & & 30z^{-2} & -90z^{-3} & +60z^{-4} & \\ \hline & & & 70z^{-3} & -60z^{-4} & \\ & & & 70z^{-3} & -210z^{-4} & +40z^{-5} \\ \hline & & & & +150z^{-4} & -140z^{-5} \end{array}$$

</td><td></td></tr>
<tr><td>

$$\begin{aligned} \therefore\ F(z) &= 10z^{-1}+30z^{-2}+70z^{-3}+\cdots \\ \sum_{n=0}^{\infty} f(n)z^{-n} &= 10z^{-1}+30z^{-2}+70z^{-3}+\cdots \\ f(0)+f(1)z^{-1} &+ f(2)z^{-2}+f(3)z^{-3}+f(4)z^{-4}+\cdots \\ &= 10z^{-1}+30z^{-2}+70z^{-3}+\cdots \end{aligned}$$

Equating the like coefficients, we get
$f(0)=0,\ f(1)=10,\ f(2)=30,\ f(3)=70,\cdots$

$$\therefore\ f(n) = 10(2^n-1) \text{ if } n=0,1,2,3,\cdots$$

$$\therefore\ Z^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = f(n) = 10(2^n-1),\ n=1,2,3,\cdots$$

</td><td>Running the summation</td></tr>
</table>

Problem based on inverse Z-transforms by residue method	Solving Tip!

1. Find $Z^{-1}\left[\dfrac{z^2+z}{(z-1)^3}\right]$ by residue method

Solution:

Given $F(z)=\dfrac{z^2+z}{(z-1)^3}=\dfrac{z(z+1)}{(z-1)^3}$

$$\begin{aligned}\therefore\ F(z).z^{n-1} &= \frac{z(z+1)}{(z-1)^3}z^{n-1}\\ &= \frac{z^n(z+1)}{(z-1)^3}\end{aligned}$$

The poles are given by

Consider the denominator term of $F(z)z^{n-1}$ and equate it to zero and solving it we get the values of z.

$$\begin{aligned}(z-1)^3 &= 0\\ (z-1)(z-1)(z-1) &= 0\\ z &= 1,1,1 \text{ which is a pole of order 3.}\end{aligned}$$

For the repeated values, we use the residue formula for pole of order m. Here $m=3$ and $a=1$

$$\begin{aligned}\therefore\ R(a) &= \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m.\left(F(z).z^{n-1}\right)\right]\\ R(1) &= \lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-1)^3.\left(\frac{z^n(z+1)}{(z-1)^3}\right)\right]\\ &= \lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left[\cancel{(z-1)^3}.\left(\frac{z^n(z+1)}{\cancel{(z-1)^3}}\right)\right]\\ &= \lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left[z^{n+1}+z^n\right]\\ &= \lim_{z\to 1}\frac{1}{2!}\frac{d}{dz}\left[(n+1)z^n+nz^{n-1}\right]\\ &= \lim_{z\to 1}\frac{1}{2}\left[(n+1)nz^{n-1}+n(n-1)z^{n-2}\right]\\ &= \frac{1}{2}\left[(n+1)n+n(n-1)\right]\\ &= \frac{n}{2}\left[(n+\cancel{1}+n-\cancel{1})\right]\\ &= \frac{n}{\cancel{2}}\left[\cancel{2}n\right]=n^2\end{aligned}$$

$2! = 2\times 1 = 2$

$\dfrac{d}{dz}[z^n]=nz^{n-1}$

$$\therefore\ f(n) = \text{sum of residues} = n^2,\ n=0,1,2,\cdots$$

Problem based on inverse Z-transforms by residue method	**Solving Tip!**
2. Find $Z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$ by residue method.	
Solution: Given $F(z)=\dfrac{10z}{(z-1)(z-2)}$	
$$\therefore\ F(z).z^{n-1} = \frac{10z}{(z-1)(z-2)}z^{n-1} = \frac{10z^n}{(z-1)(z-2)}$$ The poles are given by	
$$(z-1)(z-2)=0$$ $$z = 1,2 \text{ which are simple pole.}$$	Consider the denominator term of $F(z)z^{n-1}$ and equate it to zero and solving it we get the values of z.
$$\therefore\ R(a) = \lim_{z\to a}\left[(z-a).\left(F(z).z^{n-1}\right)\right]$$ $$\therefore\ R(1) = \lim_{z\to 1}\left[(z-1).\left(\frac{10z^n}{(z-1)(z-2)}\right)\right] = \lim_{z\to 1}\left[\cancel{(z-1)}.\left(\frac{10z^n}{\cancel{(z-1)}(z-2)}\right)\right] = \left[\left(\frac{10(1)^n}{(1-2)}\right)\right] = \left[\left(\frac{10(1)^n}{(-1)}\right)\right] = -10$$	For the non repeated values, we use the residue formula for simple pole. $R(a)=\lim_{z\to a}\left[(z-a).\left(F(z).z^{n-1}\right)\right]$
and $$R(2) = \lim_{z\to 2}\left[(z-2).\left(\frac{10z^n}{(z-1)(z-2)}\right)\right] = \lim_{z\to 2}\left[\cancel{(z-2)}.\left(\frac{10z^n}{(z-1)\cancel{(z-2)}}\right)\right] = \left[\left(\frac{10(2)^n}{(2-1)}\right)\right] = \left[\left(\frac{10(2)^n}{1}\right)\right] = 10(2^n)$$	Replace z by 2
$\therefore\ f(n) = \text{sum of residues} = (-10)+(10(2^n)) = 10\left[2^n-1\right],$ $n=0,1,2,\cdots$	

Problem based on inverse Z-transforms by convolution method	**Solving Tip!**
1. Find $\boldsymbol{Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]}$ by convolution method.	

Solution:

$$
\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] &= Z^{-1}\left[\frac{z}{(z-a)}.\frac{z}{(z-b)}\right] \\
&= Z^{-1}\left[\frac{z}{(z-a)}\right] * Z^{-1}\left[\frac{z}{(z-b)}\right] \\
&= a^n * b^n \\
&= \sum_{m=0}^{n} a^m.b^{n-m} \\
&= \sum_{m=0}^{n} a^m.b^n b^{-m} \\
&= b^n \sum_{m=0}^{n} a^m.b^{-m} \\
&= b^n \sum_{m=0}^{n} \frac{a^m}{b^m} \\
&= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m \\
&= b^n\left[\left(\frac{a}{b}\right)^0 + \left(\frac{a}{b}\right)^1 + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right] \\
&= b^n\left[1 + \left(\frac{a}{b}\right)^1 + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right] \\
&= b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\left(\frac{a}{b}\right) - 1}\right] \\
&= b^n\left[\frac{\dfrac{\left(a^{n+1} - b^{n+1}\right)}{b^{n+1}}}{\left(\dfrac{a-b}{b}\right)}\right] \\
&= \frac{\left(a^{n+1} - b^{n+1}\right)}{a-b}, \quad n = 0, 1, 2, \cdots
\end{aligned}
$$

Geometric Progression: $a + ar + ar^2 + \cdots + ar^n$

$= a\left[\dfrac{r^{n+1} - 1}{r-1}\right]$ if $r > 1$

(or)

$= a\left[\dfrac{1 - r^{n+1}}{1-r}\right]$ if $r < 1$

Problem based on inverse Z-transforms by convolution method	**Solving Tip!**
2. Find $\boldsymbol{Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]}$ by convolution method.	
Solution:	
$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] &= Z^{-1}\left[\frac{z}{(z-a)}.\frac{z}{(z-a)}\right] \\ &= Z^{-1}\left[\frac{z}{(z-a)}\right] * Z^{-1}\left[\frac{z}{(z-a)}\right] \\ &= a^n * a^n \\ &= \sum_{m=0}^{n} a^m.a^{n-m} \\ &= \sum_{m=0}^{n} a^m.a^n a^{-m} \\ &= a^n \sum_{m=0}^{n} a^m.a^{-m} \\ &= a^n \sum_{m=0}^{n} a^{m-m} \\ &= a^n \sum_{m=0}^{n} (1)^m \\ &= a^n\left[(1)^0+(1)^1+(1)^2+\cdots+(1)^n\right] \\ &= a^n\,[1+1+1+\cdots+1] \\ &= a^n\,[(n+1).1] \\ &= (n+1)a^n,\ n=0,1,2,\cdots \end{aligned}$$	Here 1 is $n+1$ times occured

Problem based on inverse Z-transforms by convolution method	**Solving Tip!**
3. Find $\boldsymbol{Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z+1)}\right]}$ by convolution method.	

Given

$$F(z) = \left[\frac{8z^2}{(2z-1)(4z+1)}\right]$$

$$= \left[\frac{8z^2}{2.(z-\frac{1}{2})4.(z+\frac{1}{4})}\right] = \left[\frac{\cancel{8}z^2}{\cancel{8}.(z-\frac{1}{2})(z+\frac{1}{4})}\right]$$

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}.\frac{z}{\left(z+\frac{1}{4}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right] * Z^{-1}\left[\frac{z}{\left(z-\left(-\frac{1}{4}\right)\right)}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(-\frac{1}{4}\right)^n$$

$$= \sum_{m=0}^{n}\left(\frac{1}{2}\right)^m.\left(-\frac{1}{4}\right)^{n-m}$$

$$= \left(-\frac{1}{4}\right)^n\sum_{m=0}^{n}\left(\frac{1}{2}\right)^m.\left(-\frac{1}{4}\right)^{-m}$$

$$= \left(-\frac{1}{4}\right)^n\sum_{m=0}^{n}\left(-4\frac{1}{2}\right)^m$$

$$= \left(-\frac{1}{4}\right)^n\sum_{m=0}^{n}(-2)^m$$

$$= \left(-\frac{1}{4}\right)^n\left[(-2)^0+(-2)^1+(-2)^2+\cdots+(-2)^n\right]$$

Problem based on inverse Z-transforms by convolution method	**Solving Tip!**
$= \left(-\frac{1}{4}\right)^n \left[1 + (-2)^1 + (-2)^2 + \cdots + (-2)^n\right]$ $= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{(1-(-2))}\right]$ $= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{(1-(-2))}\right]$ $= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{(1+2)}\right]$ $= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)(-2)^{n}}{(3)}\right]$ $= \left(-\frac{1}{4}\right)^n \left[\frac{1+2(-2)^{n}}{(3)}\right]$ $= \left[\frac{\left(-\frac{1}{4}\right)^n + 2(-2)^n}{(3)}\right]$ $= \left[\frac{1}{3}\left(-\frac{1}{4}\right)^n + \frac{2}{3}\left(-\frac{1}{4}.-2\right)^n\right]$ $= \left[\frac{1}{3}\left(-\frac{1}{4}\right)^n + \frac{2}{3}\left(\frac{1}{2}\right)^n\right] \quad n = 0, 1, 2, \cdots$	Geometric Progression: $a + ar + ar^2 + \cdots + ar^n$ $= a\left[\frac{r^{n+1}-1}{r-1}\right]$ if $r > 1$ (or) $= a\left[\frac{1-r^{n+1}}{1-r}\right]$ if $r < 1$

Problem based on inverse Z-transforms by convolution method	**Solving Tip!**

4. Find $Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z-1)}\right]$ by convolution method.

Given

$$\begin{aligned}
F(z) &= \left[\frac{8z^2}{(2z-1)(4z-1)}\right]\\
&= \left[\frac{8z^2}{2.(z-\frac{1}{2})4.(z-\frac{1}{4})}\right] = \left[\frac{\cancel{8}z^2}{\cancel{8}.(z-\frac{1}{2})(z-\frac{1}{4})}\right]
\end{aligned}$$

$$\begin{aligned}
Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] &= Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]\\
&= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}.\frac{z}{\left(z-\frac{1}{4}\right)}\right]\\
&= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right] * Z^{-1}\left[\frac{z}{\left(z-\frac{1}{4}\right)}\right]\\
&= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n\\
&= \sum_{m=0}^{n}\left(\frac{1}{2}\right)^m.\left(\frac{1}{4}\right)^{n-m}\\
&= \left(\frac{1}{4}\right)^n\sum_{m=0}^{n}\left(\frac{1}{2}\right)^m.\left(\frac{1}{4}\right)^{-m}\\
&= \left(\frac{1}{4}\right)^n\sum_{m=0}^{n}\left(4\frac{1}{2}\right)^m\\
&= \left(\frac{1}{4}\right)^n\sum_{m=0}^{n}(2)^m\\
&= \left(\frac{1}{4}\right)^n\left[(2)^0+(2)^1+(2)^2+\cdots+(2)^n\right]
\end{aligned}$$

Problem based on inverse Z-transforms by convolution method	**Solving Tip!**
$= \left(\frac{1}{4}\right)^n \left[1 + (2)^1 + (2)^2 + \cdots + (2)^n\right]$ $= \left(\frac{1}{4}\right)^n \left[\frac{(2)^{n+1} - 1}{(2-1)}\right]$ $= \left(\frac{1}{4}\right)^n \left[\frac{(2)^{n+1} - 1}{(1)}\right]$ $= \left(\frac{1}{4}\right)^n \left[2\,(2)^n - 1\right]$ $= \left[2\left(\frac{1}{4}\right)^n (2)^n - \left(\frac{1}{4}\right)^n\right]$ $= \left[2\left(2.\frac{1}{4}\right)^n - \left(\frac{1}{4}\right)^n\right]$ $= \left[2\left(\cancel{2}.\frac{1}{\cancel{4}2}\right)^n - \left(\frac{1}{4}\right)^n\right]$ $= \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] \quad n = 0, 1, 2, \cdots$	Geometric Progression: $a + ar + ar^2 + \cdots + ar^n$ $= a\left[\frac{r^{n+1} - 1}{r - 1}\right]$ if $r > 1$ (or) $= a\left[\frac{1 - r^{n+1}}{1 - r}\right]$ if $r < 1$

Problem based on inverse Z-transforms by convolution method	**Solving Tip!**
5. Find $\boldsymbol{Z^{-1}\left[\left(\frac{z}{z-4}\right)^3\right]}$ by convolution method. **Solution:** $$\begin{aligned} Z^{-1}\left[\left(\frac{z}{z-4}\right)^3\right] &= Z^{-1}\left[\frac{z}{z-4}.\frac{z^2}{(z-4)^2}\right] \\ &= Z^{-1}\left[\frac{z}{z-4}\right] * Z^{-1}\left[\frac{z^2}{(z-4)^2}\right] \\ &= 4^n * (n+1)\,4^n \\ &= (n+1)\,4^n * 4^n \\ &= \sum_{m=0}^{n} (m+1)\,4^m.\,(4)^{n-m} \\ &= \sum_{m=0}^{n} (m+1)\,4^m.4^n.4^{-m} \\ &= 4^n \sum_{m=0}^{n} (m+1)\,4^m.4^{-m} \\ &= 4^n \sum_{m=0}^{n} (m+1) \\ &= 4^n\,[1+2+3+\cdots+(n+1)] \\ &= 4^n\left[\frac{(n+1)(n+2)}{2}\right], \end{aligned}$$ $n = 0, 1, 2, \cdots$	$1+2+3+\cdots+n$ $= \frac{n(n+1)}{2}$

Problem based on inverse Z-transforms by convolution method	**Solving Tip!**

6. Find $Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right]$ by convolution method.

Solution:

$$\begin{aligned}
Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right] &= Z^{-1}\left[\frac{z^2}{(z-2)^2}.\frac{z}{(z-3)}\right] \\
&= Z^{-1}\left[\frac{z^2}{(z-2)^2}\right] * Z^{-1}\left[\frac{z}{(z-3)}\right] \\
&= (n+1)\,2^n * 3^n \\
&= \sum_{m=0}^{n}(m+1)\,2^m.\,(3)^{n-m} \\
&= \sum_{m=0}^{n}(m+1)\,2^m.3^n.3^{-m} \\
&= 3^n\sum_{m=0}^{n}(m+1)\,2^m.3^{-m} \\
&= 3^n\sum_{m=0}^{n}(m+1)\left(\frac{2}{3}\right)^m \\
&= 3^n\left[1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2\right. \\
&\qquad\left.+\cdots+(n+1)\left(\frac{2}{3}\right)^n\right]
\end{aligned}$$

Let $S=\left[1+2x+3x^2+\cdots+(n+1)x^n\right]$, where $x=\frac{2}{3}$

This is an arithmetic-geometric series. To find its sum, multiply by the common ratio of the geometric progression and subtract from the series.

$$\begin{aligned}
S &= 1+2x+3x^2+\cdots+(n+1)x^n \\
xS &= \qquad x+2x^2+3x^3+\cdots+(n+1)x^n+(n+1)x^{n+1} \\
\hline
S-xS &= 1+x+x^2+x^3+\cdots+x^n-(n+1)x^{n+1}
\end{aligned}$$

$$\begin{aligned}
(1-x)S &= (1+x+x^2+x^3+\cdots+x^n)-(n+1)x^{n+1}\\
&= \left[\frac{1-x^{n+1}}{1-x}\right]-(n+1)x^{n+1}\\
\therefore\ S &= \left[\frac{1-x^{n+1}}{(1-x)^2}\right]-\frac{(n+1)x^{n+1}}{(1-x)},
\end{aligned}$$

Since $x=\dfrac{2}{3}$ and $1-x=1-\dfrac{2}{3}=\dfrac{1}{3}$

$$\begin{aligned}
\therefore\ S &= \left[\frac{1-\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{1}{3}\right)^2}\right]-\frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{1}{3}\right)}\\
&= 9\left[1-\left(\frac{2}{3}\right)^{n+1}\right]-3(n+1)\left(\frac{2}{3}\right)^{n+1}\\
&= 9-9\left(\frac{2}{3}\right)^{n+1}-3(n+1)\left(\frac{2}{3}\right)^{n+1}\\
&= 9-[9+3n+3]\left(\frac{2}{3}\right)^{n+1}\\
&= 9-[12+3n]\left(\frac{2}{3}\right)^{n+1}\\
&= 9-3\,[n+4]\left(\frac{2}{3}\right)^{n+1}
\end{aligned}$$

$$\begin{aligned}
Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right] &= 3^n\left[9-3\,[n+4]\left(\frac{2}{3}\right)^{n+1}\right]\\
&= 3^n\left[3^2-3\,[n+4]\left(\frac{2}{3}\right)^{n+1}\right]\\
&= \left[3^{n+2}-3^{n+1}\,[n+4]\,\frac{2^{n+1}}{3^{n+1}}\right]\\
&= \left[3^{n+2}-[n+4]\,2^{n+1}\right],\ n=0,1,2,\cdots
\end{aligned}$$

Problem based on inverse Z-transforms by partial fraction method	**Solving Tip!**
1. Find $\boldsymbol{Z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]}$ by partial fraction method.	
Solution: Given $F(z) = \frac{10z}{(z-1)(z-2)}$	
Now $\frac{F(z)}{z} = \frac{1}{z}\frac{10z}{(z-1)(z-2)} = \frac{10}{(z-1)(z-2)}$	Step 1
Let $\frac{10}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$	We use partial fraction of Type I
$$\frac{10}{(z-1)(z-2)}(z-1)(z-2) = \frac{A}{(z-1)}(z-1)(z-2) + \frac{B}{(z-2)}(z-1)(z-2)$$	Multiply both sides by the denominator term of the left hand side
$\Rightarrow 10 = A(z-2) + B(z-1)$	
Put $z=1$: $10 = A(-1)+0$, $10 = -A$, $A = -10$ Put $z=2$: $10 = 0 + B(1)$, $10 = B$, $B = 10$	
$$\begin{aligned} \therefore \frac{10}{(z-1)(z-2)} &= \frac{A}{(z-1)} + \frac{B}{(z-2)} \\ &= \frac{-10}{(z-1)} + \frac{10}{(z-2)} \\ \frac{F(z)}{z} &= -\frac{10}{(z-1)} + \frac{10}{(z-2)} \\ F(z) &= -10\frac{z}{(z-1)} + 10\frac{z}{(z-2)} \end{aligned}$$	
Taking Z^{-1} on both sides, we get	
$$\begin{aligned} f(n) &= -10Z^{-1}\left[\frac{z}{(z-1)}\right] + 10Z^{-1}\left[\frac{z}{(z-2)}\right] \\ &= -10(1) + 10(2)^n \\ &= -10 + 10(2)^n,\ n = 0, 1, 2, \cdots \end{aligned}$$	

Problem based on inverse Z-transforms by partial fraction method	Solving Tip!
2. Find $\boldsymbol{Z^{-1}\left[\dfrac{z}{(z-2)(z+3)^2}\right]}$ by partial fraction method.	
Solution: Given $F(z) = \dfrac{z}{(z-2)(z+3)^2}$	
Now $\dfrac{F(z)}{z} = \dfrac{1}{z}\dfrac{z}{(z-2)(z+3)^2} = \dfrac{1}{(z-2)(z+3)^2}$	Step 1
Let $\dfrac{1}{(z-2)(z+3)^2} = \dfrac{A}{(z-2)} + \dfrac{B}{(z+3)} + \dfrac{C}{(z+3)^2}$	We use partial fraction of Type II
$\dfrac{1}{(z-2)(z+3)^2}(z-2)(z+3)^2 = \dfrac{A}{(z-2)}(z-2)(z+3)^2 + \dfrac{B}{(z+3)}(z-2)(z+3)^2 + \dfrac{C}{(z+3)^2}(z-2)(z+3)^2$	Multiply both sides by the denomenator term of the left hand side
$\Rightarrow 1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$	

Put $z=2$	Put $z=-3$	Equating the coeffi. of z^2
$1 = A(5)^2 + 0 + 0$	$1 = 0 + 0 + C(-5)$	$0 = A + B + 0$
$1 = 25A$	$1 = -5C$	$B = -A$
$A = \dfrac{1}{25}$	$C = -\dfrac{1}{5}$	$B = -\dfrac{1}{25}$

$$\begin{aligned}
\therefore \frac{1}{(z-2)(z+3)^2} &= \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2}\\
&= \frac{1/25}{(z-2)} + \frac{-1/25}{(z+3)} + \frac{-1/5}{(z+3)^2}\\
\frac{F(z)}{z} &= \frac{1}{25}\left[\frac{1}{(z-2)}\right] - \frac{1}{25}\left[\frac{1}{z-(-3)}\right]\\
&\quad -\frac{1}{5}\left[\frac{1}{(z-(-3))^2}\right]\\
F(z) &= \frac{1}{25}\left[\frac{z}{(z-2)}\right] - \frac{1}{25}\left[\frac{z}{z-(-3)}\right]\\
&\quad -\frac{1}{5}\left[\frac{z}{(z-(-3))^2}\right]
\end{aligned}$$

Taking Z^{-1} on both sides, we get

$$\begin{aligned}
f(n) &= \frac{1}{25}Z^{-1}\left[\frac{z}{(z-2)}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{z-(-3)}\right]\\
&\quad -\frac{1}{5}Z^{-1}\left[\frac{z}{(z-(-3))^2}\right]
\end{aligned}$$

Application: Solving Difference equation by $Z-$transforms	**Solving Tip!**
$$\begin{aligned} f(n) &= \frac{1}{25}2^n - \frac{1}{25}(-3)^n + \frac{1}{15}Z^{-1}\left[\frac{(-3)z}{(z-(-3))^2}\right] \\ &= \frac{1}{25}2^n - \frac{1}{25}(-3)^n + \frac{1}{15}\left(n(-3)^n\right),\ n = 0,1,2,\cdots \end{aligned}$$	
1. Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$ by $Z-$transforms.	Sometime difference equation will be asked like $y(n+2)+6y(n+1)+9y(n) = 2^n$
Solution:	
Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$	
Applying $Z-$transforms on both sides, we get	
$$\begin{aligned} & Z\left[y_{n+2}\right] + 6Z\left[y_{n+1}\right] + 9Z\left[y_n\right] = Z\left[2^n\right] \\ \Rightarrow\ & z^2\left[F(z) - y_0 - \frac{y_1}{z}\right] + 6z\left[F(z) - y_0\right] + 9F(z) = \frac{z}{z-2} \\ \Rightarrow\ & z^2\left[F(z) - 0 - 0\right] + 6z\left[F(z) - 0\right] + 9F(z) = \frac{z}{z-2} \\ & \qquad\qquad \because\ y_0 = y_1 = 0 \\ \Rightarrow\ & F(z)\left[z^2 + 6z + 9\right] = \frac{z}{z-2} \\ \Rightarrow\ & F(z) = \frac{z}{(z-2)(z^2+6z+9)} \\ \Rightarrow\ & F(z) = \frac{z}{(z-2)(z+3)^2} \\ \Rightarrow\ & \frac{F(z)}{z} = \frac{1}{(z-2)(z+3)^2} \end{aligned}$$	We use partial fraction of Type II
Let $\dfrac{1}{(z-2)(z+3)^2} = \dfrac{A}{(z-2)} + \dfrac{B}{(z+3)} + \dfrac{C}{(z+3)^2}$	
$$\begin{aligned} \frac{1}{(z-2)(z+3)^2}(z-2)(z+3)^2 &= \frac{A}{(z-2)}(z-2)(z+3)^2 \\ &+ \frac{B}{(z+3)}(z-2)(z+3)^2 \\ &+ \frac{C}{(z+3)^2}(z-2)(z+3)^2 \end{aligned}$$	Multiply both sides by the denomenator term of the left hand side
$\Rightarrow 1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$	

Application: Solving Difference equation by $Z-$transforms	**Solving Tip!**

First, see the common factors on the right hand side and make it as a zero after putting the corresponding z value. If there is no common factor to eliminate, we compare the higher degree

Put $z = 2$	Put $z = -3$	Equating the coeffi. of z^2
$1 = A(5)^2 + 0 + 0$	$1 = 0 + 0 + C(-5)$	$0 = A + B + 0$
$1 = 25A$	$1 = -5C$	$B = -A$
$A = \frac{1}{25}$	$C = -\frac{1}{5}$	$B = -\frac{1}{25}$

$$\begin{aligned}\therefore \quad \frac{1}{(z-2)(z+3)^2} &= \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2} \\ &= \frac{1/25}{(z-2)} + \frac{-1/25}{(z+3)} + \frac{-1/5}{(z+3)^2}\end{aligned}$$

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{25}\left[\frac{1}{(z-2)}\right] - \frac{1}{25}\left[\frac{1}{z-(-3)}\right] - \frac{1}{5}\left[\frac{1}{(z-(-3))^2}\right] \\ F(z) &= \frac{1}{25}\left[\frac{z}{(z-2)}\right] - \frac{1}{25}\left[\frac{z}{z-(-3)}\right] - \frac{1}{5}\left[\frac{z}{(z-(-3))^2}\right]\end{aligned}$$

Taking Z^{-1} on both sides, we get

$$\begin{aligned}f(n) &= \frac{1}{25}Z^{-1}\left[\frac{z}{(z-2)}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{z-(-3)}\right] \\ &\qquad -\frac{1}{5}Z^{-1}\left[\frac{z}{(z-(-3))^2}\right] \\ &= \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5}Z^{-1}\left[\frac{1}{(-3)}\frac{(-3)z}{(z-(-3))^2}\right] \\ &= \frac{1}{25}2^n - \frac{1}{25}(-3)^n + \frac{1}{15}Z^{-1}\left[\frac{(-3)z}{(z-(-3))^2}\right] \\ &= \frac{1}{25}2^n - \frac{1}{25}(-3)^n + \frac{1}{15}\left(n(-3)^n\right), \ n = 0, 1, 2, \cdots\end{aligned}$$

$$Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

$$Z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n$$

Exercise:

1. Solve $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ with $y_0 = 0, y_1 = 1$ by $Z-$transforms.

2. Solve $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$ with $y_0 = 3, y_1 = -5$ by $Z-$transforms.

3. Solve $y_n + 3y_{n-1} - 4y_{n-2} = 0$ with $y_0 = 3, y_1 = -2$ by $Z-$transforms.

Replace n by (n+2) and then proceed it as before.

4. Solve $y_{n+2} - y_{n+1} + 4y_n = 0$, given $y(0) = 1$ and $y(1) = 0$, using $Z-$transforms.