Infinite Fourier Cosine and Sine transform:

Infinite Fourier Cosine transform:

$$F_c(s) = F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$f(x) = \sqrt{2/\pi} \int_0^\infty F_c(s) \cos sx \, ds \quad \text{(inversion formula)}$$

Infinite Fourier sine transform:

$$F_s(s) = F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds \quad \text{(inversion formula)}$$

Properties of Cosine and Sine transform:

① $F_c[a f(x) + b g(x)] = a F_c(f(x)) + b G_c(g(x))$

② $F_c[f(x) \sin ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$

③ $F_c[f(x) \cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

④ $F_c[f(ax)] = \frac{1}{a} F_c(s/a)$

⑤ $F_s[a f(x) + b g(x)] = a F_s(f(x)) + b F_s(g(x))$

⑥ $F_s[f(x) \sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

⑦ $F_s[f(x) \cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

⑧ $F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right)$.

① $F_c\left[af(x)+bg(x)\right] = aF_c(f(x)) + bF_c(g(x))$

Proof :-

$$F_c\left[af(x)+bg(x)\right] = \sqrt{2/\pi}\int_0^\infty (af(x)+bg(x))\cos sx\,dx \quad (\text{by definition})$$

$$= \underbrace{a\sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx}\; + \; b\sqrt{2/\pi}\int_0^\infty g(x)\cos sx\,dx$$

$$= aF_c(f(x)) + bF_c(g(x))$$

$$\therefore\; F_c\left(af(x)+bg(x)\right) = aF_c(f(x)) + bF_c(g(x)).$$

---

③ $F_c(f(x)\sin ax) = \frac{1}{2}\left[F_s(a+s) + F_s(a-s)\right]$

Proof :-

$$F_c(f(x)\sin ax) = \sqrt{2/\pi}\int_0^\infty f(x)\sin ax\cos sx\,dx \quad (\text{by definition})$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\left(\frac{\sin(a+s)x + \sin(a-s)x}{2}\right)dx$$

$$= \frac{1}{2}\left[\sqrt{2/\pi}\int_0^\infty f(x)\sin(a+s)x\,dx + \sqrt{2/\pi}\int_0^\infty f(x)\sin(a-s)x\,dx\right]$$

$$= \frac{1}{2}\left[F_s(a+s) + F_s(a-s)\right].$$

---

⑥ $F_s(f(x)\sin ax) = \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right]$

Proof :-

$$F_s(f(x)\sin ax) = \sqrt{2/\pi}\int_0^\infty f(x)\sin ax\sin sx\,dx \quad (\text{by definition})$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\sin ax\,dx$$

$$= \sqrt{2/\pi} \int_0^\infty f(x) \left(\frac{\cos(s-a)x - \cos(s+a)x}{2} \right] dx$$

$$= \frac{1}{2} \left[\sqrt{2/\pi} \int_0^\infty f(x) \cos(s-a)x \, dx - \sqrt{2/\pi} \int_0^\infty f(x) \cos(s+a) \, dx \right]$$

$$= \frac{1}{2} \left[F_c(s-a) - F_c(s+a) \right]$$

---

⑧ $F_s(f(ax)) = \frac{1}{a} F_s(s/a)$

Proof :-

$$F_s(f(ax)) = \sqrt{2/\pi} \int_0^\infty f(ax) \sin sx \, dx \quad (\text{ by defn.})$$

Put $ax = t \Rightarrow x = t/a$ when $x=0,\ t=0$

$a\,dx = dt$ $\quad x=\infty,\ t=\infty$.

$$= \sqrt{2/\pi} \int_0^\infty f(t) \sin\left(\frac{st}{a}\right) \frac{dt}{a}.$$

$$= \frac{1}{a} \sqrt{2/\pi} \int_0^\infty f(t) \sin\left(s/a\, t\right) dt$$

$$= \frac{1}{a} F_s(s/a)$$

Note : Remaining properties (try).

Identities :

If $F_c(s), G_c(s)$ are the Fourier cosine transforms and $F_s(s), G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

(1) $$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_c(s)\,G_c(s)\,ds$$

(2) $$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F_s(s)\,.\,G_s(s)\,ds$$

(3) $$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |F_s(s)|^2\,ds.$$

Exercise Problems.

① Find the Fourier transform of $f(x) = \begin{cases} x & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$

Solution: $i\sqrt{2/\pi}\left[\dfrac{\sin sa - as\cos sa}{s^2}\right]$

② Show that the Fourier transform of $f(x) = \begin{cases} |x| & \text{if } |x| < a \\ 0 & \text{if } |x| > a,\ a > 0 \end{cases}$

is $\sqrt{\dfrac{2}{\pi}}\left[\dfrac{as\sin sa + \cos sa - 1}{s^2}\right]$

③ Find the Fourier transform of the function $f(x)$ defined by

$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$ Hence prove that

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\left(s/2\right) ds = \frac{3\pi}{16}.$$

④. Find the Fourier transform of $f(x) = \begin{cases} a - |x| & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$

hence deduce that $\displaystyle\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$

⑤ Show that the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

is $2\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin as - as\cos as}{s^3}\right)$. Hence, deduce that

$\displaystyle\int_0^\infty \frac{\sin t - t\cos t}{t^3} dt = \frac{\pi}{4}$. Using Parseval's identity, Show that

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$$

Problems :-

(1) Find Fourier cosine and sine transforms of e^{-ax}, $a>0$ and hence deduce the inversion formula.

soln:-

$$F_c(f(x)) = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx \quad \text{[Fourier cosine transform]}$$

$$F_c(e^{-ax}) = \sqrt{2/\pi}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\left[\int_0^\infty e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

here $a=-a$, $b=s$

$$= \sqrt{2/\pi}\left[\frac{e^{-ax}}{(-a)^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{2/\pi}\left[0-\left(\frac{1}{a^2+s^2}(-a)\right)\right]$$

$$= \sqrt{2/\pi}\left(\frac{a}{a^2+s^2}\right) \text{ if } a>0.$$

By inversion formula,

$$f(x) = \sqrt{2/\pi}\int_0^\infty F_c(s)\cos sx\,ds$$

$$e^{-ax} = \sqrt{2/\pi}\int_0^\infty \sqrt{2/\pi}\,\frac{a}{a^2+s^2}\cos sx\,ds$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos sx}{a^2+s^2}\,ds.$$

$$\therefore \int_0^\infty \frac{\cos sx}{a^2+s^2}\,ds = \frac{\pi}{2a}\cdot e^{-ax},\ a>0.$$

Fourier sine transform:

$$F_s(f(x)) = \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s(e^{-ax}) = \sqrt{2/\pi} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\left[\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b\cos bx) \right]$$

here $a = -a$, $b = s$.

$$= \sqrt{2/\pi} \left[\frac{e^{-ax}}{(-a)^2+s^2} (-a \sin sx - s\cos sx) \right]_0^\infty$$

$$= \sqrt{2/\pi} \left[0 - \left(\frac{1}{a^2+s^2} (-s) \right) \right]$$

$$= \sqrt{2/\pi} \left[\frac{s}{a^2+s^2} \right].$$

By inversion formula,

$$f(x) = \sqrt{2/\pi} \int_0^\infty F_s(s) \sin sx \, ds$$

$$e^{-ax} = \sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi} \left(\frac{s}{a^2+s^2} \right) \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, ds.$$

$$\therefore \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, dx = \frac{\pi}{2} e^{-ax}, \quad a > 0.$$

2) Using Parseval's identity, evaluate

(i) $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ and (ii) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$ if $a>0$.

soln:-

we know that, If $f(x) = e^{-ax}$ then $F_s(s) = \sqrt{2/\pi}\,\frac{s}{a^2+s^2}$

and $F_c(s) = \sqrt{2/\pi}\,\frac{a}{a^2+s^2}$

(i) Using Parseval's identity,

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds.$$

$$\int_0^\infty (e^{-ax})^2\,dx = \int_0^\infty \left(\sqrt{2/\pi}\,\frac{a}{a^2+s^2}\right)^2 ds.$$

$$\int_0^\infty e^{-2ax}\,dx = \int_0^\infty \frac{2}{\pi}\frac{a^2}{(a^2+s^2)^2}\,ds.$$

$$\left.\frac{e^{-2ax}}{-2a}\right|_0^\infty = \frac{2}{\pi}a^2\int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

$$0-\left(-1/2a\right) = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

$$\int_0^\infty \frac{ds}{(a^2+s^2)^2} = \frac{\pi}{2a^2}\left(\frac{1}{2a}\right)$$

$$= \frac{\pi}{4a^3},\ a>0.$$

$$\therefore \int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3},\ a>0.$$

(ii) using Parseval's identity,

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_s(s)|^2\,ds$$

$$\int_0^\infty (e^{-ax})^2\,dx = \int_0^\infty \left(\sqrt{2/\pi}\,\frac{s}{a^2+s^2}\right)^2 ds$$

$$\Rightarrow \int_0^\infty e^{-2ax}\,dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(a^2+s^2)^2}\,ds.$$

$$\frac{1}{2a} = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(a^2+s^2)^2}\,ds.$$

$$\therefore \int_0^\infty \frac{s^2}{(a^2+s^2)^2}\,ds = \frac{1}{2a}\times\frac{\pi}{2}$$

$$= \frac{\pi}{4a}$$

$$\therefore \int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a},\ a>0.$$

---

(3) Evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ using transform methods.

Soln: Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$ then

$$F_c(s) = \sqrt{2/\pi}\,\frac{a}{a^2+s^2} \quad \& \quad G_c(s) = \sqrt{2/\pi}\,\frac{b}{b^2+s^2}$$

$\therefore$ Using $\int_0^\infty F_c(s)\cdot G_c(s)\,ds = \int_0^\infty f(x)\,g(x)\,dx.$

$$\int_0^\infty \sqrt{2/\pi}\,\frac{a}{a^2+s^2}\cdot\sqrt{2/\pi}\,\frac{b}{b^2+s^2}\,ds = \int_0^\infty e^{-ax}\cdot e^{-bx}\,dx$$

$$\int_0^\infty \frac{2}{\pi}\,\frac{ab}{(a^2+s^2)(b^2+s^2)}\,ds = \int_0^\infty e^{-(a+b)x}\,dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left.\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= 0-\left(\frac{1}{-(a+b)}\right) = \frac{1}{a+b}$$

$$\therefore \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab}\cdot\frac{1}{a+b}, \quad a,b>0.$$

$$\therefore \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)} \quad \text{if } a,b>0.$$

---

(4) Find Fourier sine transform of $1/x$.

Soln:-

$$F_s(f(x)) = \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s(1/x) = \sqrt{2/\pi}\int_0^\infty \frac{1}{x}\cdot\sin sx\,dx.$$

Put $sx=\theta \Rightarrow x=\theta/s$. when $x=0,\ \theta=0$

$s\,dx=d\theta \Rightarrow dx=\frac{d\theta}{s}$. $\quad x=\infty,\ \theta=\infty$.

$$= \sqrt{2/\pi}\int_0^\infty \frac{\sin\theta}{\theta/s}\cdot\frac{d\theta}{s}.$$

$$= \sqrt{2/\pi}\int_0^\infty \frac{\cancel{s}\sin\theta}{\theta}\frac{d\theta}{\cancel{s}}$$

$$= \sqrt{2/\pi}\int_0^\infty \frac{\sin\theta}{\theta}\,d\theta \quad \left(\because \int_0^\infty \frac{\sin\theta}{\theta}d\theta=\frac{\pi}{2}\right)$$

$$= \sqrt{2/\pi}\times\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}.$$

---

(5) Show that (6)

(i) $F_s(x.f(x)) = -\frac{d}{ds} F_c(s)$ (ii) $F_c[x.f(x)] = \frac{d}{ds} F_s(s)$ and

hence find Fourier cosine and sine transform of xe^{-ax}.

Soln:-

(i) $F_c(s) = \sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx$.

diff. b.s with respect to 's', we get

$$\frac{d}{ds} F_c(s) = \frac{d}{ds}\left[\sqrt{2/\pi}\int_0^\infty f(x) \cos sx \, dx\right]$$

$$= \sqrt{2/\pi}\int_0^\infty f(x) \frac{\partial}{\partial s}(\cos sx)\, dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)(-\sin sx \cdot x)\, dx.$$

$$= -\sqrt{2/\pi}\int_0^\infty x f(x) \sin sx \, dx$$

$$= -F_s[x.f(x)]$$

$$\therefore F_s[xf(x)] = -\frac{d}{ds} F_c(s).$$

Fourier sine transform of xe^{-ax}.

$(\because F_c(s) = F_c(f(x)))$

ie, $F_s[x.f(x)] = -\frac{d}{ds} F_c(f(x))$

$$F_s[xe^{-ax}] = -\frac{d}{ds} F_c(e^{-ax}).$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}\right]$$

$$= -\sqrt{2/\pi}\left[\frac{(a^2+s^2)(0) - a(2s)}{(a^2+s^2)^2}\right]$$

$$= \sqrt{2/\pi}\ \frac{2as}{(a^2+s^2)^2}.$$

$$\therefore\ F_s(xe^{-ax}) = \sqrt{2/\pi}\ \frac{2as}{(a^2+s^2)^2}.$$

(ii)

$$F_s(s) = \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx \quad (\text{definition}).$$

diff. both sides w. r to 's', we get

$$\frac{d}{ds}F_s(s) = \frac{d}{ds}\left[\sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx\right]$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\frac{\partial}{\partial s}(\sin sx)\,dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)(\cos sx\cdot x)\,dx$$

$$= \sqrt{2/\pi}\int_0^\infty x f(x)\cos sx\,dx.$$

$$= F_c[x.f(x)]$$

$$\therefore\ F_c(xf(x)) = \frac{d}{ds}F_s(s).$$

<u>Fourier cosine transform of xe^{-ax}:</u>

ie, $F_c(xf(x)) = \frac{d}{ds}F_s(f(x))$.

$$F_c(xe^{-ax}) = \frac{d}{ds}F_s(e^{-ax}).$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\ \frac{s}{a^2+s^2}\right]$$

$$= \sqrt{2/\pi}\ \frac{d}{ds}\left[\frac{s}{a^2+s^2}\right]$$

$$= \sqrt{2/\pi}\left[\frac{(a^2+s^2)(1)-s(2s)}{(a^2+s^2)^2}\right]$$

$$= \sqrt{2/\pi}\left[\frac{a^2+s^2-2s^2}{(a^2+s^2)^2}\right]$$

$$= \sqrt{2/\pi}\left[\frac{a^2-s^2}{(a^2+s^2)^2}\right]$$

$$\therefore\ F_c(xe^{-ax}) = \sqrt{2/\pi}\left(\frac{a^2-s^2}{(a^2+s^2)^2}\right).$$

---

6) Find Fourier cosine transform of $e^{-a^2x^2}$ and hence evaluate Fourier sine transform of $xe^{-a^2x^2}$.

Soln:-

$$F_c(f(x)) = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx.$$

$$F_c(e^{-a^2x^2}) = \sqrt{2/\pi}\int_0^\infty e^{-a^2x^2}\cos sx\,dx$$

$$= \sqrt{2/\pi}\times\frac{1}{2}\int_{-\infty}^\infty e^{-a^2x^2}\cos sx\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}\cos sx\,dx.$$

$$= \text{Real part of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}\cdot e^{isx}\,dx.$$

$$= R.P \text{ of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx}\,dx.$$

we know that,

$$\int_{-\infty}^\infty f(x)\,dx = 2\int_0^\infty f(x)\,dx.$$

$$e^{isx} = \cos sx + i\sin sx.$$

Real part of $e^{isx} = \cos sx$.

$$= \text{R.P. of } \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}}\, dx.$$

[Refer: (9) Show that the transform of $e^{-s^2/2}$ is $e^{-s^2/2}$ by finding the fourier transform of $e^{-a^2x^2}$, $a>0$]

$$= \text{R.P of } \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$$

$$\therefore Fc(e^{-a^2x^2}) = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$$

Fourier sine transform of $xe^{-a^2x^2}$

WKT, $F_s[xf(x)] = -\frac{d}{ds} Fc(s)$ $\quad \because Fc(s) = Fc(f(x))$

$$Fs[\underline{xe^{-a^2x^2}}] = \frac{-d}{ds} Fc(e^{-a^2x^2})$$

$$= \frac{-d}{ds}\left[\frac{1}{a\sqrt{2}} e^{-s^2/4a^2}\right]$$

$$= -\left[\frac{1}{a\sqrt{2}} e^{-s^2/4a^2} \cdot \frac{-2s}{4a^2}\right]$$

$$= \frac{s}{2a^3\sqrt{2}} e^{-s^2/4a^2}$$

---

(7) Solve for $f(x)$ from the integral equation

$$\int_0^\infty f(x) \sin sx \, dx = \begin{cases} 1 & \text{for } 0 \le s < 1 \\ 2 & \text{for } 1 \le s < 2 \\ 0 & \text{for } s > 2 \end{cases} \longrightarrow (*)$$

Proof:- Multiplying (*) by $\sqrt{2/\pi}$, both sides

$$\underbrace{\sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx}_{\downarrow \text{(definition)}} = \sqrt{2/\pi} \begin{cases} 1 & \text{for } 0 \le s < 1 \\ 2 & \text{for } 1 \le s < 2 \\ 0 & \text{for } s > 2 \end{cases}$$

$$Fs(f(x)) = \begin{cases} \sqrt{2/\pi} & \text{for } 0 \le s < 1 \\ 2\sqrt{2/\pi} & \text{for } 1 \le s < 2 \\ 0 & \text{for } s > 2 \end{cases}$$

(10)

$$\therefore \quad f(x) = F_s^{-1}\left(\begin{cases} \sqrt{2/\pi} & \text{for } 0 \le s < 1 \\ 2\sqrt{2/\pi} & \text{for } 1 \le s < 2 \\ 0 & \text{for } s \geq 2 \end{cases}\right) \qquad f(x) = \sqrt{2/\pi}\int_0^\infty F_s(s)\sin sx\, ds$$

$$= \sqrt{2/\pi}\left[\int_0^1 \sqrt{2/\pi}\sin sx\, ds + \int_1^2 2\sqrt{2/\pi}\sin sx\, ds + \cancel{\int_2^\infty 0 \sin sx\, ds}\right]$$

$$= \frac{2}{\pi}\int_0^1 \sin sx\, ds + \frac{4}{\pi}\int_1^2 \sin sx\, ds$$

$$= \frac{2}{\pi}\left[-\frac{\cos sx}{x}\right)_0^1 + \frac{4}{\pi}\left(-\frac{\cos sx}{x}\right)_1^2$$

$$= \frac{2}{\pi}\left[-\frac{\cos x}{x} + \frac{1}{x}\right) + \frac{4}{\pi}\left(-\frac{\cos 2x}{x} + \frac{\cos x}{x}\right)$$

$$= \frac{2}{\pi}\left[\frac{1}{x} - \frac{\cos x}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right)$$

$$f(x) = \frac{2}{\pi x}\left(1 + \cos x - 2\cos 2x\right)$$

---

8) Find the function if its sine transform is $\frac{e^{-as}}{s}$.

Soln:

Let $F_s(f(x)) = \frac{e^{-as}}{s}$.

$$f(x) = \sqrt{2/\pi}\int_0^\infty F_s(s)\sin sx\, ds.$$

$$f(x) = \sqrt{2/\pi}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds \longrightarrow (1)$$

$$\therefore \frac{df}{dx} = \frac{d}{dx}\left[\sqrt{2/\pi}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\frac{\partial}{\partial x}(\sin sx)\, ds$$

$$= \sqrt{2/\pi} \int_0^\infty \frac{e^{-as}}{\cancel{s}} \cdot \cos sx \cdot \cancel{s}\, ds$$

$$= \sqrt{2/\pi} \int_0^\infty e^{-as} \cos sx \, ds . \qquad a = -a \quad b = x$$

$$= \sqrt{2/\pi} \left[\frac{e^{-as}}{(-a)^2 + x^2} \left(-a \cos xs + x \sin xs \right) \right]_0^\infty$$

$$= \sqrt{2/\pi} \left(0 - \left(\frac{-a}{a^2+x^2} \right) \right)$$

$$= \sqrt{2/\pi} \left(\frac{a}{a^2+x^2} \right)$$

integrating w.r to 'x' on both sides,

$$\therefore \int \frac{df}{dx} dx = \sqrt{2/\pi} \int \frac{a}{a^2+x^2} dx$$

$$f(x) = \sqrt{2/\pi}\, \cancel{a} \cdot \frac{1}{\cancel{a}} \tan^{-1}\left(\frac{x}{a}\right) + c .$$

$$f(x) = \sqrt{2/\pi} \tan^{-1}(x/a) + c \longrightarrow (2) .$$

At $x = 0$, $f(0) = 0$ using (1)

using this in equation (2), $f(0) = 0 + c$.

ie, $\boxed{0 = c}$

$$\therefore f(x) = \sqrt{2/\pi} \tan^{-1}(x/a) .$$

---

9) Find Fourier sine and cosine transform of x^{n-1}.

Soln:-

we know that, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$, $n > 0$

Replace x by ax, $a > 0$.

$$= \int_0^\infty e^{-ax} (-ax)^{n-1} a\,dx$$

$$= \int_0^\infty e^{-ax} \cdot a^{n-1} x^{n-1} a \, dx$$

$$= a^n \int_0^\infty e^{-ax} x^{n-1} \, dx$$

$$\therefore \int_0^\infty e^{-ax} x^{n-1} \, dx = \frac{\Gamma(n)}{a^n}, \quad n>0, \ a>0.$$

we can prove the result even if a is complex.

setting $a = is$

$$\int_0^\infty e^{-isx} x^{n-1} \, dx = \frac{\Gamma(n)}{(is)^n}$$

$$\frac{1}{i^n} = \left(\frac{1}{i}\right)^n = \left(\frac{1}{i} \times \frac{i}{i}\right)^n = \left(\frac{i}{i^2}\right)^n = (-i)^n$$

$$= \frac{(-i)^n \Gamma(n)}{s^n}$$

$$= \frac{e^{-\pi/2 \, ni} \, \Gamma(n)}{s^n}$$

$$-i = e^{-\pi/2 \, i}$$

Equating real and imaginary parts on both sides, we get

ie, $$\int_0^\infty x^{n-1} \cos sx \, dx = \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2}$$

$$\int_0^\infty x^{n-1} \sin sx \, dx = \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2}$$

Multiply $\sqrt{2/\pi}$ on both sides, we get

$$\sqrt{2/\pi} \int_0^\infty x^{n-1} \cos sx \, dx = \sqrt{2/\pi} \, \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2}$$

$$\sqrt{2/\pi} \int_0^\infty x^{n-1} \sin sx \, dx = \sqrt{2/\pi} \, \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2}$$

$$\therefore \quad F_c(x^{n-1}) = \sqrt{2/\pi}\ \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \qquad (1)$$

$$F_s(x^{n-1}) = \sqrt{2/\pi}\ \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}. \qquad (2)$$

Taking $n = 1/2$, in (1)

$$F_c\left(x^{1/2-1}\right) = \sqrt{2/\pi}\ \frac{\Gamma 1/2}{s^{1/2}}\cos\frac{\pi}{4}$$

$$F_c\left(x^{-1/2}\right) = \sqrt{\cancel{2}/\cancel{\pi}}\ \frac{\cancel{\sqrt{\pi}}}{s^{1/2}}\cdot\frac{1}{\cancel{\sqrt{2}}} \qquad \left[\because \Gamma 1/2 = \sqrt{\pi}\right]$$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{s^{1/2}}$$

ie, $F_c\left(1/\sqrt{x}\right) = \dfrac{1}{\sqrt{s}}$

III^{ly} $F_s\left(1/\sqrt{x}\right) = \dfrac{1}{\sqrt{s}}$

Note :- $\dfrac{1}{\sqrt{x}}$ is self- reciprocal under Fourier sine and cosine transform.

① Show that the transform of $e^{-x^2/2}$ is $e^{-s^2/2}$ by finding the Fourier transform of $e^{-a^2x^2}$, $a>0$.

Solution:

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$F(e^{-a^2x^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-((ax)^2 - isx)} \, dx$$

$\Big[$ $A^2 + B^2 - 2AB = (A-B)^2$

$(ax)^2 - isx$

$A = ax$, $\quad 2AB = isx$

$$B = \frac{isx}{2A} = \frac{isx}{2(ax)} = \frac{is}{2a}$$

$$\Rightarrow \underbrace{(ax)^2 - isx + \left(\frac{is}{2a}\right)^2}- \left(\frac{is}{2a}\right)^2$$

$$= \left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2 \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax - is/2a)^2} \, e^{(is/2a)^2} \, dx$$

$$= e^{-s^2/4a^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax - is/2a)^2} \, dx$$

Put $t = ax - \frac{is}{2a}$

$dt = a\,dx$ when $x = \infty,\ t = \infty$

$\Rightarrow dx = \dfrac{dt}{a}$. $x = -\infty,\ t = -\infty$.

$$= e^{-s^2/4a^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= e^{-s^2/4a^2} \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= e^{-s^2/4a^2} \frac{1}{a\sqrt{2\cancel{\pi}}} \cdot \sqrt{\cancel{\pi}} \quad \left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \right]$$

$$= \frac{1}{\sqrt{2}\,a} e^{-s^2/4a^2}$$

$$\therefore F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2}\,a} e^{-s^2/4a^2}$$

Put $a = \dfrac{1}{\sqrt{2}}$

$$F\left[e^{-x^2/2}\right] = \frac{1}{\cancel{\sqrt{2}} \cdot \frac{1}{\cancel{\sqrt{2}}}} e^{-s^2/4\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= e^{-s^2/4 \times 2/1} = e^{-s^2/2}$$

$$\therefore F\left[e^{-x^2/2}\right) = e^{-s^2/2}$$

ie, $e^{-x^2/2}$ is self-reciprocal under Fourier transform.

Exercise Problems.

① Find the Fourier transform of $f(x) = \begin{cases} x & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$

Solution: $i\sqrt{2/\pi}\left[\dfrac{\sin sa - as\cos sa}{s^2}\right]$

② Show that the Fourier transform of $f(x) = \begin{cases} |x| & \text{if } |x| < a \\ 0 & \text{if } |x| > a,\ a > 0 \end{cases}$

is $\sqrt{\dfrac{2}{\pi}}\left[\dfrac{as\sin sa + \cos sa - 1}{s^2}\right]$

③ Find the Fourier transform of the function $f(x)$ defined by

$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$ Hence prove that

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\left(s/2\right) ds = \frac{3\pi}{16}.$$

④. Find the Fourier transform of $f(x) = \begin{cases} a - |x| & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$

hence deduce that $\displaystyle\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$

⑤ Show that the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

is $2\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin as - as\cos as}{s^3}\right)$. Hence, deduce that

$\displaystyle\int_0^\infty \frac{\sin t - t\cos t}{t^3} dt = \frac{\pi}{4}.$ Using Parseval's identity, Show that

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$$

Unit- 4

Fourier Transforms.

Complex Fourier Transform (Infinite)

Let $f(x)$ be a function defined in $(-\infty, \infty)$ and be piece-wise Continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$. Then the complex Fourier Transform of $f(x)$ is defined by

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Inversion Theorem for Complex Fourier Transform :-

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

Fourier Integral theorem :

If $f(x)$ is piece-wise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i(x-t)s} dt\, ds.$$

Properties of Fourier Transforms :

Thm: 1 Fourier transform is linear.

ie, $F[af(x) + bg(x)] = aF(f(x)) + bF[g(x)]$

where F stands for Fourier transform.

Proof :- $F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af(x) + bg(x)) e^{isx} dx$ [by definition]

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} a f(x) e^{isx} dx + \int_{-\infty}^{\infty} b g(x) e^{isx} dx \right]$$

$$= a \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx + b \cdot \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

$$= a F[f(x)] + b F[g(x)].$$

Theorem : 2 Shifting theorem

If $F[f(x)] = F(s)$, then $F(f(x-a)) = e^{isa} F(s)$.

Proof :

$$F(f(x-a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx \quad [\text{by definition}]$$

Put $x - a = t \Rightarrow x = a + t$

$dx = dt$

when $x = -\infty$, $t = \infty$

$x = \infty$, $t = \infty$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i(a+t)s} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{ias} \cdot e^{its} ds.$$

$$= e^{ias} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{its} ds}. \quad [\because \text{`}t\text{' is a dummy variable}]$$

$$= e^{ias} F(s).$$

Theorem : 3 Change of scale property.

If $F\{f(x)\} = F(s)$, then $F\{f(ax)\} = \frac{1}{|a|} F\left(s/a\right)$, where $a \neq 0$.

Proof :

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx \quad [\text{by definition}]$$

Case (i) Put $ax = t \Rightarrow x = t/a$ when $x = \infty,\ t = \infty$

and $a > 0$, $a\,dx = dt$ $\quad x = -\infty,\ t = -\infty$

$$dx = \frac{dt}{a}$$

$$\therefore \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t} \frac{dt}{a}$$

$$= \frac{1}{a} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t}\, dt}_{} \quad \text{(by the definition)}$$

$$= \frac{1}{a} \cdot F(s/a), \quad \text{where } a > 0.$$

Case (ii).

Put $ax = t$ & $a < 0$.

$x = t/a$ & $dx = \frac{dt}{a}$

when $x = \infty,\ t = -\infty$

$x = -\infty,\ t = \infty.$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{i(s/a)t} \frac{dt}{a}.$$

$$= -\frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t}\, dt$$

$$= \frac{-1}{a} F(s/a), \quad \text{where } a < 0.$$

$$\therefore \quad F\{f(ax)\} = \frac{1}{|a|} F(s/a).$$

Theorem: 4

$$F\{e^{iax} f(x)\} = F(s+a)$$

Proof:- $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$ (by the definition)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx$$

$$= F(s+a).$$

Theorem: 5 **Modulation theorem.**

If $F\{f(x)\} = F(s)$, then $F\{f(x)\cos ax\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$

Proof:

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2} \right) e^{isx}\, dx$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(x)\, e^{iax} . e^{isx}\, dx \right] + \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(x)\, e^{-iax} e^{isx}\, dx \right] \right]$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx \right]$$

$$= \frac{1}{2} \left[F(s+a) + F(s-a) \right]$$

Theorem: 6 If $F(f(x)) = F(s)$, then $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$.

Proof: By the definition, $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$.

differentiating w.r to 's' both sides, n times

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n f(x) e^{isx}\, dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n f(x) e^{isx}\, dx$$

$$= (i)^n F\{x^n f(x)\}$$

$$F\{x^n f(x)\} = \frac{1}{(i)^n} \frac{d^n F(s)}{ds^n}$$

$$F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s). \quad \left[\because \frac{1}{(i)^n} = \left(\frac{1}{i}\right)^n = \left(\frac{1}{i} \times \frac{i}{i}\right)^n = \left(\frac{i}{-1}\right)^n = (-i)^n\right.$$

<u>Theorem : 7</u>

$$F\{f'(x)\} = -is F(s) \text{ if } f(x) \to 0 \text{ as } x \to \pm\infty.$$

<u>Proof :</u>

$$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} d(f(x))$$

$u = e^{isx}$ $\quad$ $v = f(x)$ $\qquad$ $\int u\, dv = uv - \int v\, du.$

$\frac{du}{dx} = (is)e^{isx}$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ e^{isx} f(x) \right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) (is) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - is \int_{-\infty}^{\infty} f(x) e^{isx} dx \right]$$

$[\because \text{if } f(x) \to 0 \text{ as } x \to \pm\infty]$

$$= -is \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx}$$

$$= -is F(s)$$

<u>Theorem : 8</u>

$$F\left\{ \int_a^x f(x) dx \right\} = \frac{F(s)}{-is}$$

Let $\quad \phi(x) = \int_a^x f(x) dx$

Then $\quad \phi'(x) = f(x)$

$F(\phi'(x)) = (-is) F[\phi(x)]$ $\quad$ (by Thm. 7)

$F[f(x)] = -is\, F[\phi(x)]$

$F[\phi(x)] = \frac{F(s)}{-is}$ //

Problems :-

① Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$.

Solution :-

$$F(s) = F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \Bigg[\underbrace{\int_{-\infty}^{-1} f(x)\, e^{isx}\, dx}_{(0)} + \int_{-1}^{1} f(x)\, e^{isx}\, dx + \underbrace{\int_{1}^{\infty} f(x)\, e^{isx}\, dx}_{(0)} \Bigg]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \Bigg[\int_{-1}^{1} \underbrace{\underbrace{(1-x^2)}_{\text{even}} \underbrace{\cos sx}_{\text{even}}}_{\text{even}}\, dx + i \int_{-1}^{1} \underbrace{\underbrace{(1-x^2)}_{\text{even}} \underbrace{\sin sx}_{\text{odd}}}_{\text{odd}}\, dx \Bigg]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_0^1 (1-x^2) \cos sx\, dx + 0 \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x^2) \cos sx\, dx$$

$$= \sqrt{2/\pi} \left[(1-x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) - 2\left(\frac{-\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{2/\pi} \left[\frac{-2\cos s}{s^2} + 2\frac{\sin s}{s^3} \right]$$

$$= \sqrt{2/\pi} \left[\frac{-2s\cos s + 2\sin s}{s^3} \right]$$

$$= \frac{-2\sqrt{2}}{\sqrt{\pi}}\left[\frac{s\cos s - \sin s}{s^3}\right]$$

Using inversion formula:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{-2\sqrt{2}}{\sqrt{\pi}}\left[\frac{s\cos s - \sin s}{s^3}\right] e^{-isx}\, ds$$

$$= \frac{-2}{\pi}\int_{-\infty}^{\infty}\left[\frac{s\cos s - \sin s}{s^3}\right] e^{-isx}\, ds$$

$$\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)(\cos sx - i\sin sx)\, ds = -\pi/2\, f(x)$$

$$= -\pi/2 \begin{cases} (1-x^2) & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

$$= \begin{cases} -\pi/2\,(1-x^2) & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

Equating real parts,

$$\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\, ds = \begin{cases} -\pi/2\,(1-x^2) & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

Put $x = 1/2$

$$\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos s/2\, ds = -\frac{\pi}{2}\left(1-1/4\right)$$

$$\int_{-\infty}^{\infty}\underbrace{\underset{\text{even}}{\underbrace{\left(\frac{s\cos s - \sin s}{s^3}\right)}}\ \underset{\text{even}}{\underbrace{\cos s/2}}}_{\text{even}}\, ds = \frac{-3\pi}{8}$$

$$2\int_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos s/_2 \, ds = -\frac{3\pi}{8}.$$

$$\therefore \int_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos s/_2 \, ds = -\frac{3\pi}{16}.$$

---

2) Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$ and $\int_{-\infty}^\infty \frac{\sin as \cos sx}{s}\, ds$.

Solution:-

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\cancel{\int_{-\infty}^{-a} f(x) e^{isx}\, dx}}_{(0)} + \int_{-a}^{a} f(x) e^{isx}\, dx + \underbrace{\cancel{\int_{a}^{\infty} f(x) e^{isx}\, dx}}_{(0)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-a}^{a} \cos sx\, dx}_{\text{even function}} + i \underbrace{\int_{-a}^{a} \sin sx\, dx}_{\substack{\text{odd function} \\ (0)}} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^a \cos sx\, dx \right]$$

$$= \sqrt{2/\pi} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{2/\pi}\ \frac{\sin as}{s}$$

Using inversion formula, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin as}{s}\, e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s}\, e^{-isx}\, ds.$$

$$\int_{-\infty}^{\infty} \frac{\sin as}{s}\, e^{-isx}\, ds = \pi . f(x)$$

$$\int_{-\infty}^{\infty} \frac{\sin as}{s}\, (\cos sx - i \sin sx)\, ds = \pi \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{\sin as}{s}\, (\cos sx - i \sin sx)\, ds = \begin{cases} \pi & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

Equating real parts,

$$\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \begin{cases} \pi & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

Put $x = 0$.

$$\int_{-\infty}^{\infty} \underbrace{\frac{\sin as}{s}}_{\text{even function}}\, ds = \pi$$

$$2 \int_{0}^{\infty} \frac{\sin as}{s}\, ds = \pi$$

$$\int_{0}^{\infty} \frac{\sin as}{s}\, ds = \pi/2$$

Put $as = \theta \Rightarrow s = \theta/a$

$a\, ds = d\theta \Rightarrow ds = \frac{d\theta}{a}$

when $s = 0,\ \theta = 0$

$s = \infty,\ \theta = \infty$

$$\int_0^\infty \frac{\sin\theta}{\theta/a}\frac{d\theta}{a} = \pi/2$$

$$\int_0^\infty \frac{\alpha\sin\theta}{\theta}\frac{d\theta}{\alpha} = \pi/2$$

$$\int_0^\infty \frac{\sin\theta}{\theta}\,d\theta = \pi/2.$$

Convolution Theorem or Faltung Theorem:

Definition: The Convolution of two functions $f(x)$ and $g(x)$ is defined as $f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt.$

Theorem: The Fourier transform of the Convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

ie, $F\{f(x)*g(x)\} = F(s)\cdot G(s) = F\{f(x)\}\cdot F\{g(x)\}$

Parseval's identity:

If $F(s)$ is the Fourier transform of $f(x)$

then, $\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$

(3) Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$$ and prove that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$

Solution:-

Refer Previous problem, we know that

$$F(s) = \sqrt{\frac{2}{\pi}}\ \frac{\sin as}{s}.$$

<u>Using Parseval's identity,</u>

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-a}^{a} (1)\, dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\ \frac{\sin as}{s}\right)^2 ds$$

$$(a-(-a)) = \frac{2}{\pi} \int_{-\infty}^{\infty} \underbrace{\left(\frac{\sin as}{s}\right)^2}_{\text{even function.}} ds$$

$$\cancel{2}a = \frac{\cancel{2}}{\pi}\left[2\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)^2 ds\right]$$

$$\frac{a \times \pi}{2} = \int_{0}^{\infty}\left(\frac{\sin as}{s}\right)^2 ds$$

Put $as = \theta \Rightarrow s = \theta/a$ when $s = 0,\ \theta = 0$

$a\, ds = d\theta$ $\qquad s = \infty,\ \theta = \infty.$

$ds = \frac{d\theta}{a}.$

$$\int_{0}^{\infty}\left(\frac{\sin\theta}{\theta/a}\right)^2 \frac{d\theta}{a} = \frac{\pi a}{2}$$

$$\int_{0}^{\infty} a^{\cancel{2}}\left(\frac{\sin\theta}{\theta}\right)^2 \frac{d\theta}{\cancel{a}} = \frac{\pi \cancel{a}}{2}$$

$$\therefore \int_{0}^{\infty}\left(\frac{\sin\theta}{\theta}\right)^2 d\theta = \frac{\pi}{2}.$$

4) Find the Fourier transform of $f(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$

and hence find the value $\int_0^\infty \frac{\sin^4 t}{t^4}\, dt$.

soln:-

$$F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\underbrace{\int_{-\infty}^{-1} f(x) e^{isx}\, dx}_{(0)} + \int_{-1}^{1} f(x) e^{isx}\, dx + \underbrace{\int_{1}^{\infty} f(x) e^{isx}\, dx}_{(0)} \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\int_{-1}^{1} \underbrace{\underbrace{(1-|x|)}_{\text{even}} \underbrace{\cos sx}_{\text{even}}}_{\text{even}}\, dx + i \underbrace{\int_{-1}^{1} \underbrace{\underbrace{(1-|x|)}_{\text{e.f}} \underbrace{\sin sx}_{\text{O.f}}}_{\text{O.f}}\, dx}_{(0)} \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \Big[2 \int_0^1 (1-|x|) \cos sx\, dx \Big]$$

$$= \sqrt{2/\pi} \int_0^1 (1-x) \cos sx\, dx \quad \Big[\because 1-|x| = 1-x \text{ in interval } (0,1) \Big]$$

$$= \sqrt{2/\pi} \Big\{ \Big[(1-x) \Big(\frac{\sin sx}{s} \Big) - (-1) \Big(\frac{-\cos sx}{s^2} \Big) \Big]_0^1$$

$$= \sqrt{2/\pi} \Big(\frac{-\cos s}{s^2} + \frac{1}{s^2} \Big) = \sqrt{2/\pi} \Big(\frac{1-\cos s}{s^2} \Big)$$

Using Parseval's identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-|x|)^2 dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \left(\frac{1-\cos s}{s^2} \right) \right)^2 ds$$

$$2\int_{0}^{1} (1-x)^2 dx = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds$$ $\longrightarrow$ even function

$$\cancel{2}\left(\frac{(1-x)^3}{-3} \right)_0^1 = \frac{\cancel{2}}{\pi} \left[2\int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds \right]$$

$$1/3 = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds.$$

$$\frac{\pi}{6} = \int_0^{\infty} \left(\frac{2\sin^2 s/2}{s^2} \right)^2 ds.$$

$$\frac{\pi}{6} = 4\int_0^{\infty} \left(\frac{\sin^2 s/2}{s^2} \right)^2 ds.$$

[$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\sin^2\theta/2 = \dfrac{1-\cos\theta}{2}$

$2\sin^2\theta/2 = 1-\cos\theta.$

$\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a.n+1}$]

Put $t = s/2 \Rightarrow s = 2t$

$ds = 2dt$

when $s=0, \quad t=0$

$s=\infty, \quad t=\infty.$

$$\frac{\pi}{24} = \int_0^{\infty} \left(\frac{\sin^2 t}{(2t)^2} \right)^2 2dt$$

$$\int_0^{\infty} \frac{\sin^4 t}{16t^4} 2dt = \frac{\pi}{24}$$

$$\int_0^{\infty} \frac{\sin^4 t}{8t^4} dt = \frac{\pi}{24} \Rightarrow \int_0^{\infty} \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3}.$$