UNIT - V

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PART-A QUESTIONS AND ANSWERS.

UNIT - I
PARTIAL DIFFERENTIAL EQUATIONS

Formation of partial differential equations - Singular integrals - Solutions of Standard types of first order partial differential equations - Lagrange's linear equation - Linear partial differential equations of second and higher order with constant coefficients of both homogeneous and non-homogeneous types.

1.0 INTRODUCTION

Partial differential equations arise in connection with various physical and geometrical problems. When the functions involved depend on two or more independent variables, usually on time t and one or several space variables. It is fair to say that only the simplest physical systems can be modeled by ordinary differential equations whereas most problems in fluid mechanics, elasticity, heat transfer, electromagnetic theory, quantum mechanics and other areas of physics lead to partial differential equations.

A partial differential equation is one which involves partial derivatives. The order of a partial differential equation is the order of the highest derivative occuring in it.

Throughout this chapter, we use the following notations : z will be taken as a dependent variable which depends on two independent variables x, y so that $z = f(x, y)$. We write

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t.$$

Thus, $p + qx = x + y$ is a partial differential equation of order 1 and $r + t = x^2 + y$ is a partial differential equation of order 2.

1.1. FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS BY ELIMINATION OF ARBITRARY CONSTANTS.

Consider an equation $f(x, y, z, a, b) = 0$ .... (1)

where a and b denote arbitrary constants.

Let z be regarded as function of two independent variables x and y.

Differentiating (1) with respect to x and y partially, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0 \quad \text{... (2) and}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0 \quad \text{... (3)}$$

Eliminating two constants a and b from three equations, we sha obtain an equation of the form $\varphi\,(x, y, z, p, q) = 0$

which is partial differential equation of the first order.

Note 1 : In a similar manner, it can be shown that if there a more arbitrary constants than the number of independent variables, t above procedure of elimination will give rise to partial different equations of higher order than the first.

Note 2 : $f(x, y, z, a, b) = 0$ is called the complete solution $\varphi\,(x, y, z, p, q) = 0$

§ Define 'a partial differential equation'.

Solution : A p.d.e is one which involves partial derivatives for instance $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$ are all partial differential equatio

§ Define the order of a p.d.e and its degree.

Solution: The order of a p.d.e is the order of the highest partial differential coefficient occurring in it.

... the degree of the p

§ When is a p.d.e said to be linear?

Solution : A p.d.e is said to be linear, if the dependent variable and the partial derivatives occur in the first degree only and separately.

§ Distinguish between homogeneous and non-homogeneous p.d.e.

Solution : An equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

is called a homogeneous linear p.d.e. of n^{th} order with constant co-efficients. It is called homogeneous because all the terms contain derivatives of the same order.

The linear differential equations which are not homogeneous, are called non-homogeneous linear equations.

Example : $3 \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} + 5 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = 0$

§ Explain how p.d.e. is formed.

Solution : P.d.e. can be obtained

(i) by eliminating the arbitrary constants that occur in the functional relation between the dependent and independent variables. (or)

(ii) by eliminating arbitrary functions from a given relation between the dependent and independent variables.

§ What is the essential difference between ordinary differential equation and p.d.e., when both are formed by eliminating arbitrary constants ?

Solution : The order of an ordinary differential equation will be the same, as the number of constants eliminated. The order of a p.d.e. will be one, in cases, when the number of constants to be eliminated is equal to the number of independent variables. But if the number of constants to be eliminated is more than the number of independent variables, the result of the elimination will, in general, be p.d.e. of second and higher orders.

§ **Explain the formation of p.d.e. by elimination of arbitrary constants.**

Solution : A p.d.e. is formed by eliminating the arbitrary constants that occur in the functional relation between the variables.

Let $f(x, y, z, a, b) = 0$ ... (1) be a relation

Connecting x, y, z and the arbitrary constants a, b, in (1), z is considered as the dependent variable.

diff (1) p.w.r. to x, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \text{ i.e., } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p = 0 \quad \text{... (2)} \qquad \left[\because \frac{\partial z}{\partial x} = p\right]$$

diff (1) p.w.r. to y, we get

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \text{ i.e., } \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q = 0 \quad \text{... (3)} \qquad \left[\because \frac{\partial z}{\partial y} = q\right]$$

From the three equations by eliminating a, b we can obtain a relation connecting x, y, z, p, q such as $F(x, y, z, p, q) = 0$ which is a p.d.e. of the first order.

I. Problems based on formation of p.d.e by elimination of arbitrary constants (a.c.)

1.1.(a) **Case (i)** If no. a.c. $\leq$ no. I.V then we get p.d.e of order 1. we use p and q only.

a.c. → arbitrary constant   I.V → Independent Variable

Example 1.1a(1) : Form the p.d.e. by eliminating the arbitrary constants a & b from z = ax + by.

Solution : Given : $z = ax + by$ ... (1)

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V ∴ we use p & q only	

differentiating (1) partially w.r. to 'x', we get

$\frac{\partial z}{\partial x} = a$ i.e., $p = a \Rightarrow a = p$ .. (2)

differentiating (1) partially w.r. to 'y', we get

$\frac{\partial z}{\partial y} = b$ i.e., $q = b \Rightarrow b = q$ ... (3)

Substituting (2) & (3) in (1), we get the required p.d.e $z = px + qy$

Example 1.1a(2) : Eliminate the arbitrary constants a & b from $z = ax + by + a^2 + b^2$. [A.U. A/M 2003 PT]

Solution: Given : $z = ax + by + a^2 + b^2$ ... (1)

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only	

differentiating (1) partially w.r. to 'x', we get

$$\frac{\partial z}{\partial x} = a$$

i.e., $p = a \Rightarrow a = p$ ... (2)

differentiating (1) partially w.r. to 'y', we get

$$\frac{\partial z}{\partial y} = b$$

i.e., $q = b \Rightarrow b = q$ ... (3)

∴ Substituting (2) & (3) in (1), we get the required p.d.e

$$z = px + qy + p^2 + q^2$$

Example 1.1a(3) : Form the p.d.e by eliminating the arbitrary constants from z = ax + by + ab [A.U. April/May 2003 PTMA]

Solution : Given : $z = ax + by + ab$ ... (1)

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only	

$$p = \frac{\partial z}{\partial x} = a \Rightarrow a = p \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = b \Rightarrow b = q \quad \text{... (3)}$$

Substituting (2) & (3) in (1), we get the required p.d.e.

i.e., $z = px + qy + pq$

Example 1.1a(4) : Form a p.d.e by eliminating the arbitrary constants a and b from $z = (x + a)^2 + (y - b)^2$ [A.U A/M 2001, A/M 2008]

Solution : Given : $z = (x + a)^2 + (y - b)^2$ ... (1)

$$p = \frac{\partial z}{\partial x} = 2(x + a) \text{ i.e., } x + a = \frac{p}{2} \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = 2(y - b) \text{ i.e., } y - b = \frac{q}{2} \quad \text{... (3)}$$

Substituting (2) & (3) in (1), we get

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V.	
∴ we use p & q only	

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$4z = p^2 + q^2$ which is the required p.d.e

Example 1.1a(5) : Eliminate the arbitrary constants a & b from $z = (x^2 + a)(y^2 + b)$ **[A.U. A/M 2004, PTMA, CBT N/D 2010]**

Solution : Given : $z = (x^2 + a)(y^2 + b)$ ... (1)

differentiating (1) partially w.r. to 'x', we get

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b) \quad \text{i.e.,} \quad p = 2x(y^2 + b)$$

$$\frac{p}{2x} = y^2 + b \quad \text{i.e.,} \quad \boxed{y^2 + b = \frac{p}{2x}} \quad \text{... (2)}$$

differentiating (1) partially w.r. to 'y', we get

$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a) \quad \text{i.e.,} \quad q = 2y(x^2 + a)$$

$$\frac{q}{2y} = x^2 + a \quad \text{i.e.,} \quad \boxed{x^2 + a = \frac{q}{2y}} \quad \text{... (3)}$$

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V.	
∴ we use p & q only	

Substituting (2) & (3) in (1), we get

$$z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right)$$

i.e., $4xyz = pq$ which is the required p.d.e

Example 1.1a(6) : Form the partial differential equation by eliminating a and b from $z = (x^2 + a^2)(y^2 + b^2)$ **[A.U. Nov/Dec. 2004]**
[A.U N/D 2018-A, R-17]

Solution : Given :

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V.	
∴ we use p & q only	

$$z = (x^2 + a^2)(y^2 + b^2) \quad \text{... (1)}$$

$$p = \frac{\partial z}{\partial x} = (2x)(y^2 + b^2)$$

$$\frac{p}{2x} = y^2 + b^2 \quad \text{i.e.,} \quad \boxed{(y^2 + b^2) = \frac{p}{2x}} \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = (x^2 + a^2)(2y)$$

$$\frac{q}{2y} = (x^2 + a^2) \quad \text{i.e.,} \quad \boxed{(x^2 + a^2) = \frac{q}{2y}} \quad \text{... (3)}$$

Substituting (2) & (3) in (1), we get the required p.d.e.

$$\text{i.e.,} \quad z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right) = \frac{pq}{4xy}$$

$$4xyz = pq$$

Example 1.1a(7) : Form the p.d.e. by eliminating the constants a and b from $z = ax^n + by^n$ **[A.U. A/M, 2005, A.U CBT Dec. 2008]**

Solution : Given: $z = ax^n + by^n$ ... (1)

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V.	
∴ we use p & q only	

$$p = \frac{\partial z}{\partial x} = a\,n\,x^{n-1}$$

$$xp = anx^n \Rightarrow \frac{xp}{n} = ax^n \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = bn\,y^{n-1}$$

$$yq = bn\,y^n \Rightarrow \frac{yq}{n} = by^n \quad \text{... (3)}$$

Substituting (2) & (3) in (1), we get the required p.d.e.

$$z = \frac{xp}{n} + \frac{yq}{n} \quad \Rightarrow xp + yq = nz$$

Example 1.1a(8) : Form a partial differential equation by eliminating the arbitrary constants *a* and *b* from the relation

$$z = (2x^2 + a)(3y - b).$$ **[A.U. CEG April/May 2004]**

Solution : $z = (2x^2 + a)(3y - b)$ ... (1)

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V.	
∴ we use p & q only	

$$p = \frac{\partial z}{\partial x} = (4x)(3y - b)$$

$$\text{i.e.,} \quad \boxed{(3y - b) = \frac{p}{4x}} \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = (2x^2 + a)(3)$$

i.e., $\boxed{2x^2 + a = \frac{q}{3}}$ ... (3)

Substituting (2) & (3) in (1), we get the required p.d.e.

$$\therefore z = \left(\frac{q}{3}\right)\left(\frac{p}{4x}\right)$$

$$12xz = pq$$

Example 1.1a(9) : Form a partial differential equation by eliminating the arbitrary constants from $z = a^2 x + ay^2 + b$

[A.U N/D 2008] [A.U N/D 2015 R.

Solution : Given : $z = a^2 x + ay^2 + b$ ... (1)

$$p = \frac{\partial z}{\partial x} = a^2 \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = 2ay \quad \text{... (3)}$$

Squaring on both sides, we get

$$q^2 = 4a^2y^2$$

$$q^2 = 4py^2 \quad \text{by (2)}$$

$4y^2 p = q^2$ which is the required p.d.e.

Example 1.1a(10) : Form the partial differential equation by eliminating a and b from

$z = a(x + y) + b$ [A.U N/D 2007] [A.U. CBT Dec. 20

Solution : Given : $z = a(x + y) + b$ ... (1)

$$p = \frac{\partial z}{\partial x} = a \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = a \quad \text{... (3)}$$

From (2) & (3), we get the required p.d.e

$$p = q$$

Example 1.1(a)(11) : Form the p.d.e. by eliminating the arbitrary constants a & b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$.

Solution :

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only	

Given : $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ ... (1)

Differentiating (1) p.w.r. to x, we get

$$\frac{\partial z}{\partial x} = ae^y$$

i.e., $p = ae^y$ i.e., $ae^y = p$ ... (2)

Differentiating (1) p.w.r. to y, we get

$$\frac{\partial z}{\partial y} = axe^y + a^2e^{2y}$$

i.e., $q = axe^y + (ae^y)^2$

$q = xp + p^2$ by (2)

$\therefore$ $q = xp + p^2$ is the required p.d.e.

Example 1.1(a)(12) : Form the partial differential equation by eliminating a and b from $(x - a)^2 + (y - b)^2 = z^2\cot^2\alpha$.

Solution : The given equation is **[A.U. N/D 2007, Tvli M/J 2011]**

$$(x - a)^2 + (y - b)^2 = z^2\cot^2\alpha \quad \text{... (1)}$$

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only α is a given constant no need to eliminate	

Differentiating (1) p.w.r. to x, we get

$$2(x - a) = 2z\frac{\partial z}{\partial x}\cot^2\alpha$$

$$(x - a) = zp\cot^2\alpha \quad \text{... (2)}$$

Differentiating (1) p.w.r. to y, we get

$$2(y - b) = 2z\frac{\partial z}{\partial y}\cot^2\alpha$$

$$(y - b) = zq\cot^2\alpha \quad \text{... (3)}$$

Substitute (2) & (3) in (1), we get

$$(zp\cot^2\alpha)^2 + (zq\cot^2\alpha)^2 = z^2\cot^2\alpha$$

$$z^2p^2\cot^4\alpha + z^2q^2\cot^4\alpha = z^2\cot^2\alpha$$
$$p^2\cot^2\alpha + q^2\cot^2\alpha = 1$$
$$p^2 + q^2 = \tan^2\alpha$$

which is the required p.d.e.

Example 1.1(a)(13) : Obtain partial differential equation by eliminating arbitrary constants a and b from $(x-a)^2 + (y-b)^2 + z^2 = 1$. **[A.U. Nov./Dec. 2003]**

Solution : Given : $(x-a)^2 + (y-b)^2 + z^2 = 1$ ... (1)

Here, a and b are the two arbitrary constants.

Differentiating (1) p.w.r.to x, we get

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$(x-a) + zp = 0 \qquad [\because p = \frac{\partial z}{\partial x}]$$

i.e., $(x-a) = -zp$ ... (2)

Differentiating (1) p.w.r.to y, we get

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$(y-b) + zq = 0 \qquad [\because q = \frac{\partial z}{\partial y}]$$

i.e., $y - b = -zq$ ... (3)

Substituting (2) & (3) in (1), we get

$\therefore$ (1) $\Rightarrow$ $(-zp)^2 + (-zq)^2 + z^2 = 1$

$$z^2p^2 + z^2q^2 + z^2 = 1$$
$$z^2[p^2 + q^2 + 1] = 1$$

which is the required p.d.e.

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. $\therefore$ we use p & q only	

Example 1.1(a)(14) : Find the p.d.e. of all planes through the origin. **[A.U M/J 2016 R-8]**

Solution : The general equation to a plane is

$$ax + by + cz + d = 0 \quad ... (1)$$

a, b, c, d being constants.

(1) passes through (0, 0, 0) then $d = 0$.

Therefore (1) $\Rightarrow ax + by + cz = 0$

$$\div c \Rightarrow \frac{a}{c}x + \frac{b}{c}y + z = 0$$

i.e., $a'x + b'y + z = 0$ ... (2)

$[\because a' = \frac{a}{c},\ b' = \frac{b}{c}$ and $c \neq 0$ are effective constants$]$

a.c.	I.V.
a, b, c	x, y
3	2
Here 'c' is automatically eliminated no. of a.c. = no. of I.V. $\therefore$ we use p & q only	

Differentiating p.w.r.to x, we get

$$a' + \frac{\partial z}{\partial x} = 0$$

i.e., $a' + p = 0$

$\Rightarrow$ $a' = -p$

Differentiating p.w.r.to y, we get

$$b' + \frac{\partial z}{\partial y} = 0$$

i.e., $b' + q = 0$

$\Rightarrow$ $b' = -q$

$\therefore$ (2) $\Rightarrow -px - qy + z = 0$

$\Rightarrow$ $z = px + qy$

Example 1.1(a)(15) : Find the p.d.e. of all sphere whose centres lie on the z axis. **[A.U N/D 2015 R-13]**

Solution : Let the centre of the sphere be $(0, 0, c)$ a point on the z axis and k its radius (arbitrary)

Its equation is $(x-0)^2 + (y-0)^2 + (z-c)^2 = k^2$

i.e., $x^2 + y^2 + (z-c)^2 = k^2$ ... (1)

Here, c & k are arbitrary constants

Differentiating (1) p.w.r. to x, we get

a.c.	I.V.
c, k	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only	

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

i.e., $$x + p(z-c) = 0$$

i.e., $$z - c = \frac{-x}{p} \quad \text{... (2)}$$

Differentiating (1) p.w.r. to y, we get

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$2y + 2q(z-c) = 0$$

$$y + q(z-c) = 0$$

i.e., $$z - c = \frac{-y}{q} \quad \text{... (3)}$$

Eliminate c from (2) & (3), we get

From (2) & (3), we get

$$\frac{-x}{p} = \frac{-y}{q}$$

$$qx = py$$

which is the required p.d.e

Example 1.1(a)(16) : Find the p.d.e. of all spheres of radius 'c' having their centres in the XOY plane.

Solution : Let the centre of the sphere be $(a, b, 0)$ a point in the XOY plane, 'c' is the given radius.

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only 'c' is a known constant no need to eliminate	

The equation to the sphere is

$$(x-a)^2 + (y-b)^2 + z^2 = c^2 \quad \text{... (1)}$$

Here, a, b are the two arbitrary constants.

Differentiating (1) p.w.r. to x, we get

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$2(x-a) + 2zp = 0$$

$$x - a + zp = 0$$

i.e., $$x - a = -zp \quad \text{... (2)}$$

Differentiating (1) p.w.r. to y, we get

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$y - b + qz = 0$$

i.e., $$y - b = -zq \quad \text{... (3)}$$

Eliminate a, b from (1), (2) & (3), we get

$$(1) \Rightarrow (-zp)^2 + (-qz)^2 + z^2 = c^2$$

$$z^2p^2 + q^2z^2 + z^2 = c^2$$

$$z^2(p^2 + q^2 + 1) = c^2$$ which is the required p.d.e.

Example 1.1(a)(17) : Find the PDE of all planes having equal intercepts on the x and y axis. [A.U N/D 2005] [A.U N/D 2009]

Solution : Intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Given : $a = b$ [∵ Equal intercepts on the x and y-axis]

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \quad \text{-- (1)}$$

Here, a and c are the two arbitrary constants.

Differentiating (1) p.w.r. to x, we get

$$\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0$$

$$\frac{1}{a} + \frac{1}{c}p = 0$$

$$\frac{1}{a} = \frac{-1}{c}p \quad \text{... (2)}$$

Differentiating (1) p.w.r.to y, we get

$$0+\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y} = 0$$

$$\frac{1}{a}+\frac{1}{c}q = 0$$

$$\frac{1}{a} = \frac{-1}{c}q \qquad \dots (3)$$

(2) and (3) $\Rightarrow$ $$\frac{-1}{c}p = \frac{-1}{c}q$$

$$p = q \quad \text{which is the required p.d.e}$$

Example 1.1(a)(18) : Form the partial differential equation by eliminating the arbitrary constants a & b from

$\log z = a\log x + \sqrt{1-a^2}\,\log y + b.$

Solution : Given : $\log z = a\log x + \sqrt{1-a^2}\,\log y + b \qquad \dots (1)$

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only	

Differentiating (1) p.w.r. to x, we get

$$\frac{1}{z}\frac{\partial z}{\partial x} = \frac{a}{x}$$

i.e., $$\frac{1}{z}p = \frac{a}{x}$$

i.e., $$a = \frac{xp}{z} \qquad \dots (2)$$

Differentiating (1) p.w.r. to y, we get

$$\frac{1}{z}\frac{\partial z}{\partial y} = \sqrt{1-a^2}\left[\frac{1}{y}\right]$$

$$\frac{1}{z}q = \frac{\sqrt{1-a^2}}{y}$$

i.e., $$\sqrt{1-a^2} = \frac{yq}{z} \qquad \dots (3)$$

Squaring on both sides, we get

$$1-a^2 = \frac{y^2q^2}{z^2}$$

$$1-\frac{x^2p^2}{z^2} = \frac{y^2q^2}{z^2} \quad \text{by (2)}$$

$$x^2p^2+y^2q^2 = z^2 \quad \text{which is the required p.d.e}$$

Example 1.1(a)(19) : Form the partial differential equation by eliminating the arbitrary constants a & b from $\sqrt{1+a^2}\,\log\left(z+\sqrt{z^2-1}\right) = x+ay+b.$

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only	

Solution : Given :

$$\sqrt{1+a^2}\,\log\left(z+\sqrt{z^2-1}\right) = x+ay+b \quad \dots (1)$$

Differentiating (1) p.w.r. to x, we get

$$\sqrt{1+a^2}\left(\frac{1}{z+\sqrt{z^2-1}}\right)\left[1+\frac{2z}{2\sqrt{z^2-1}}\right]\frac{\partial z}{\partial x} = 1 \qquad \dots (2)$$

Differentiating (1) p.w.r. to y, we get

$$\sqrt{1+a^2}\left(\frac{1}{z+\sqrt{z^2-1}}\right)\left[1+\frac{2z}{2\sqrt{z^2-1}}\right]\frac{\partial z}{\partial y} = a \qquad \dots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{\left(\frac{\partial z}{\partial x}\right)}{\left(\frac{\partial z}{\partial y}\right)} = \frac{1}{a}$$

$$\frac{p}{q} = \frac{1}{a} \qquad \dots (4)$$

$$(2) \Rightarrow \sqrt{1+a^2}\left[\frac{1}{z+\sqrt{z^2-1}}\right]\left[\frac{z+\sqrt{z^2-1}}{\sqrt{z^2-1}}\right]p = 1$$

$$\Rightarrow \sqrt{1+a^2}\,\frac{1}{\sqrt{z^2-1}}\,p = 1$$

Squaring on both sides, we get

$$(1 + a^2) \frac{1}{(z^2 - 1)} p^2 = 1 \quad \ldots (5)$$

$$(4) \Rightarrow \quad a = \frac{q}{p}, \quad 1 + a^2 = 1 + \frac{q^2}{p^2} = \frac{p^2 + q^2}{p^2}$$

$$(5) \Rightarrow \quad \left[\frac{p^2 + q^2}{p^2}\right] \frac{1}{(z^2 - 1)} p^2 = 1$$

$$\Rightarrow \quad p^2 + q^2 = z^2 - 1$$

$p^2 + q^2 + 1 = z^2$ which is the required p.d.e

Example 1.1(a)(20) : Form the p.d.e. by eliminating the arbitrary constants a & b from

$$z = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{1}{2} y \sqrt{y^2 - a^2} + \frac{a^2}{2} \log\left[\frac{x + \sqrt{x^2 + a^2}}{y + \sqrt{y^2 - a^2}}\right] + b.$$

Solution : Given :

a.c.	I.V.
a, b	x, y
2	2
no. of a.c. = no. of I.V. ∴ we use p & q only	

$$z = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{1}{2} y \sqrt{y^2 - a^2}$$
$$+ \frac{a^2}{2} \log (x + \sqrt{x^2 + a^2})$$
$$- \frac{a^2}{2} \log (y + \sqrt{y^2 - a^2}) + b \quad \ldots (1)$$

Differentiating (1) p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = \frac{1}{2}\left[x \frac{2x}{2\sqrt{x^2+a^2}} + \sqrt{x^2+a^2}\right] + 0 + \frac{a^2}{2} \frac{1}{x + \sqrt{x^2+a^2}} \left[1 + \frac{2x}{2\sqrt{x^2+a^2}}\right]$$

$$= \frac{x^2}{2\sqrt{x^2 + a^2}} + \frac{\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2\sqrt{x^2 + a^2}}$$

$$= \frac{x^2 + a^2}{2\sqrt{x^2 + a^2}} + \frac{\sqrt{x^2 + a^2}}{2}$$

$$= \frac{\sqrt{x^2 + a^2}}{2} + \frac{\sqrt{x^2 + a^2}}{2}$$

i.e., $p = \sqrt{x^2 + a^2} \quad \ldots (2)$

Similarly, Differentiating p.w.r. to y, we get

$$q = \sqrt{y^2 - a^2} \quad \ldots (3)$$

From (2) & (3), we get

$$p^2 - x^2 = a^2$$
$$q^2 - y^2 = -a^2$$

i.e., $y^2 - q^2 = a^2$

i.e., $p^2 - x^2 = y^2 - q^2$

$p^2 + q^2 = x^2 + y^2$ which is the required p.d.e.

Example 1.1(a)(21) : Find the partial differential equation of the family of spheres having their centres on the line x = y = z

[A.U. Oct./Nov. 2002]

Solution : General equation of the sphere is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = k^2$$

Here, centre is (a, b, c) and radius k

Centre lies on $x = y = z \Rightarrow a = b = c$

Equation of sphere is

$$(x - a)^2 + (y - a)^2 + (z - a)^2 = k^2 \quad \ldots (1)$$

a.c.	I.V.
a, k	x, y
2	2
no. of a.c.=no. of I.V. ∴ we use p & q only	

Differentiating (1) p.w.r.to x, we get

$$2(x - a) + 2(z - a) \frac{\partial z}{\partial x} = 0$$

$$(x - a) + (z - a) p = 0 \ldots (2)$$

$$x - a + zp - ap = 0$$

$$x + zp = a + ap$$
$$x + zp = a(1 + p)$$
$$\Rightarrow a = \frac{x + zp}{1 + p} \quad \dots (3)$$

Differentiating (1) p.w.r.to y, we get

$$2(y - a) + 2(z - a)\frac{\partial z}{\partial y} = 0$$
$$(y - a) + (z - a)q = 0$$
$$y - a + zq - aq = 0$$
$$y + zq = a + aq$$
$$y + zq = a(1 + q)$$
$$\Rightarrow a = \frac{y + zq}{1 + q} \quad \dots (4)$$

From (3) & (4), we get

$$\frac{x + zp}{1 + p} = \frac{y + zq}{1 + q}$$
$$x + xq + zp + zpq = y + yp + zq + zpq$$
$$x - y = (y - z)p + (z - x)q$$
$$(y - z)p + (z - x)q = x - y$$

which is the required p.d.e

EXERCISE 1.1(a)

1. Form the partial differential equations by eliminating the arbitr[ary] constants a & b

1. $z = (x + a)(y + b)$ **[Ans.** $z = pq$**]**
2. (i) $z = ax^3 + by^3$ **[A.U M/J 2014]** **[Ans.** $px + qy = 3z$ **]**
   (ii) $z = ax^2 + by^2$ **[A.U N/D 2013]** **[Ans.** $px + qy = 2z$ **]** **[A.U A/M 2017 R-8]**
3. $z = a^2x + b^2y + ab$ **[Ans.** $z = px + qy + \sqrt{pq}$

4. $z = a(x + \log y) - \frac{x^2}{2} - b$ **[Ans.** $p + x = qy$ **]**
5. $z = \frac{1}{2}[\sqrt{x + a} + \sqrt{y - a}] + b$ **[Ans.** $p^2 + q^2 = 16p^2q^2(x + y)$ **]**
6. $\log(az - 1) = x + ay + b$ **[Ans.** $p(q + 1) = qz$ **]**
7. $z = xy + y\sqrt{x^2 - a^2} + b$ [A.U. Dec. 1998] **[Ans.** $px + qy = pq$ **]**
8. $z = ax - \frac{a}{a+1}y + b$ **[Ans.** $-pq = p + q$ **]**
9. $3z = ax^3 + 2\sqrt{a - 1}\, y^{3/2} + b$ **[Ans.** $yp - x^2q^2 = x^2y$**]**
10. $z = (x - a)^2 + (y - b)^2 + 1$ **[Ans.** $4z = p^2 + q^2 + 4$**]** **[A.U N/D 2018 R-13]**
11. $z = a(x + y)$ **[Ans.** $z = q(x + y)$ **]**
12. $z = ax + h(a)y + c$ **[Ans.** $q = h(p)$ **]**
13. $z = ax + by + \sqrt{a^2 + b^2}$ **[Ans.** $z = px + qy + \sqrt{p^2 + q^2}$ **]**
14. $z = ax + by + \frac{a}{b}$ **[Ans.** $z = px + qy + \frac{p}{q}$ **]**
15. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ **[Ans.** $2z = xp + yq$**]**
16. $4z(1 + a^2) = (x + ay + b)^2$ **[Ans.** $p^2 + q^2 = z$**]**

2. Find the differential equation of all the planes which are at constant distance k from the origin. **[Ans.** $z = px + qy + k\sqrt{p^2 + q^2 + 1}$ **]** **[A.U N/D 2016 R-13]**

3. Find the PDE of all spheres whose centres lie on the x-axis. **[A.U N/D 2016 R-13]** **[Ans.** $y + zq = 0$**]**

1.1.(b) **Case (ii)** a.c. > I.V, p.d.e of order > 1.
$\therefore$ we use p, q, r, s, t

Example 1.1(b)(1) : Obtain the p.d.e. by eliminating a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution : Given : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Differentiating p.w.r. to x, we get

a.c.	I.V.
a, b, c	x, y
3	2
no. of a.c.>no. of I.V. ∴ we use p, q, r, s & t	

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$$

$$\frac{x}{a^2} + \frac{z}{c^2}p = 0 \quad \ldots (1)$$

Differentiating (1) p.w.r. to y, we get

$$0 + \frac{1}{c^2}\left[z\frac{\partial^2 z}{\partial y\,\partial x} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right] = 0$$

$$\Rightarrow \quad z\frac{\partial^2 z}{\partial y\,\partial x} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 0$$

$$zs + pq = 0 \text{ which is a p.d.e.}$$

> Notations : $z = f(x, y)$
>
> $\frac{\partial z}{\partial x} = p, \ \frac{\partial z}{\partial y} = q, \ \frac{\partial^2 z}{\partial x^2} = r, \ \frac{\partial^2 z}{\partial y^2} = t, \ \frac{\partial^2 z}{\partial x\,\partial y} = s$

> **Note :** The answer is not unique, we can get two other equivalent partial differential equations.

Example 1.1(b)(2) : Find the differential equation of all spheres whose radii are the same. **[A.U N/D 2016 R-8]**

Sol. The equation of all spheres with equal radius can be taken as

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = k^2 \quad \ldots (1)$$

where a, b, c are arbitrary constants and k is a given constant.

Differentiating (1) p.w.r. to x, we get

$$2(x-a) + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$(x-a) + (z-c)\frac{\partial z}{\partial x} = 0 \quad \ldots (2)$$

Differentiating (2) p.w.r. to x, we get

$$1 + (z-c)\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right) = 0$$

a.c.	I.V.
a, b, c	x, y
3	2
no. of a.c.>no. of I.V. ∴ we use p, q, r, s & t 'k' is a given constant no need to eliminate	

$$\Rightarrow \quad 1 + (z-c)r + p^2 = 0$$

$$\Rightarrow \quad (z-c)r = -[1+p^2] \quad \ldots (3)$$

Differentiating (1) p.w.r. to y, we get

$$2(y-b) + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$(y-b) + (z-c)\frac{\partial z}{\partial y} = 0 \quad \ldots (4)$$

Differentiating (4) p.w.r. to y, we get

$$1 + (z-c)\frac{\partial^2 z}{\partial y^2} + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \quad 1 + (z-c)t + q^2 = 0$$

$$\Rightarrow \quad (z-c)t = -[1+q^2] \quad \ldots (5)$$

$$\frac{(3)}{(5)} \Rightarrow \quad \frac{r}{t} = \frac{1+p^2}{1+q^2}$$

$$\Rightarrow \quad r(1+q^2) = t(1+p^2)$$

which is the required p.d.e

> **Note :** The answer is not unique, we can get other equivalent partial differential equations.

Example 1.1(b)(3) : Form a partial differential equation by eliminating the arbitrary constants a, b & c from z = ax + by + c xy

Solution : Given : $z = ax + by + cxy \quad \ldots (A)$

$$p = \frac{\partial z}{\partial x} = a + cy \quad \ldots (1)$$

$$q = \frac{\partial z}{\partial y} = b + cx \qquad \text{... (2)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = 0 \qquad \text{... (3)}$$

> **Note :** The answer is not unique, we can get other equivalent partial differential equations.

EXERCISE 1.1(b)

Form the partial differential equations by eliminating the arbitrary constants a, b, c as the case may be

1. $ax + by + cz = 1$ **[Ans.** $r = 0$ (or) $s = 0$ (or) $t = 0$**]**
2. $z = (x + a)^2 + (y + b)^2 + c^2$ **[Ans.** $r = 2$ (or) $t = 2$ (or) $s = 0$ **]**

1.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS BY ELIMINATION OF ARBITRARY FUNCTIONS

§ **Explain the formation of p.d.e. by elimination of arbitrary functions.**

Solution : Suppose two functions u & v are connected by the relation

$\phi(u, v) = 0$ ... (1) where ϕ is arbitrary

and $u = u(x, y, z),\ v = v(x, y, z)$ ... (2)

Let us eliminate ϕ diff (1) p.w.r. to x and y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \qquad \text{... (3)}$$

and
$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \qquad \text{... (4)}$$

eliminating $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ from (3) & (4) by using determinants, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0$$

i.e.,
$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

which, on simplification gives a p.d.e. of the form

$$Pp + Qq = R \qquad \text{... (5) where}$$

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = \frac{\partial(u, v)}{\partial(y, z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = \frac{\partial(u, v)}{\partial(z, x)}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial(u, v)}{\partial(x, y)}$$

Equation (5) is a first order p.d.e. linear in p and q.

§ **What can you say about the order of p.d.e. got by eliminating arbitrary functions?**

Solution : The elimination of one arbitrary function will result in a p.d.e. of the first order. The elimination of two arbitrary functions will result in equations of second order and so on.

II. Problems based on Formation of p.d.e. by elimination of arbitrary functions.

1.2.(a) **Case (i)** a.f. = 1, p.d.e order = 1

Example 1.2(a)(1) : Eliminate f from $z = f(x^2 + y^2)$.

[A.U. N/D 2016 R-8][A.U N/D 2018 R-8]

Solution : Given : $z = f(x^2 + y^2)$ ... (1)

Differentiating (1) p.w.r. to x and y, we get

$$\frac{\partial z}{\partial x} = f'(x^2 + y^2)\,[2x]$$

i.e., $$p = f'(x^2 + y^2)\,[2x] \qquad \text{... (2)}$$

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\,[2y]$$

i.e., $$q = f'(x^2+y^2)\,(2y) \quad \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{x}{y}$$

$$yp - xq = 0$$ which is the required p.d.e.

a.f
f
1
∴ we use p & q only

Example 1.2(a)(2) : Eliminate f from $z = x + y + f(xy)$.

[A.U. N/D 200

Solution : Given : $z = x + y + f(xy)$ ... (1)

Differentiating (1) p.w.r. to x

$$p = \frac{\partial z}{\partial x} = 1 + f'(xy)\,y$$

$$p - 1 = y f'(xy) \quad \ldots (2)$$

Differentiating (1) p.w.r. to y, we get

$$q = \frac{\partial z}{\partial y} = 1 + f'(xy)\,x$$

$$q - 1 = x f'(xy) \quad \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p-1}{q-1} = \frac{y}{x}$$

$$xp - x = yq - y$$

$$xp - yq = x - y$$ is the required p.d.e.

a.f
f
1
∴ we use p & q only

Example 1.2(a)(3) : Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$ and form a partial differential equation.

[A.U. Trichy N/D 2009][A.U N/D 2012][A.U N/D 2014, R-13]

[A.U A/M 2019 R13]

Solution : Given : $z = f\left(\frac{y}{x}\right)$ ... (1)

Differentiating (1) p.w.r. to x, we get

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad \ldots (2)$$

Differentiating (1) p.w.r. to y, we get

a.f
f
1
∴ we use p & q only

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{-y}{x}$$

Therefore $xp = -yq$

i.e., $xp + yq = 0$ is the required p.d.e.

Example 1.2(a)(4) : Form the p.d.e. by eliminating f from $z = f(x + y)$.

Solution : $z = f(x+y)$ ... (1)

Differentiating (1) p.w.r. to x, we get

$$p = \frac{\partial z}{\partial x} = f'(x+y) \quad \ldots (2)$$

Differentiating (1) p.w.r. to y, we get

$$q = \frac{\partial z}{\partial y} = f'(x+y) \quad \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = 1$$

i.e., $p = q$

(or) $p - q = 0$ which is the required p.d.e.

a.f
f
1
∴ we use p & q only

Example 1.2(a)(5) : Eliminate f from $z = f(x^2 + y^2 + z^2)$.

Solution : Given : $z = f(x^2+y^2+z^2)$ ... (1)

$$p = \frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left[2x + 2z\frac{\partial z}{\partial x}\right]$$

$$= f'(x^2+y^2+z^2)\,2\,[x + pz] \quad \ldots (2)$$

$$q = \frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left[2y + 2z\frac{\partial z}{\partial y}\right]$$

$$= f'(x^2+y^2+z^2)\,2\,[y + qz] \quad \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{x+pz}{y+qz}$$

a.f
f
1
∴ we use p & q only

$$\Rightarrow \quad yp + zpq = xq + zpq$$

$$yp = xq$$

i.e., $yp - xq = 0$ which is the required p.d.e.

Example 1.2(a)(6) : Eliminate f from $z = f\left(\dfrac{xy}{z}\right)$. **[A.U A/M 2004]**

Solution : Given : $z = f\left(\dfrac{xy}{z}\right)$ ... (1)

a.f
f
1
∴ we use p & q only

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\frac{\partial}{\partial x}\left[\frac{xy}{z}\right]$$

$$= f'\left(\frac{xy}{z}\right) y\left[\frac{(z)(1) - x\dfrac{\partial z}{\partial x}}{z^2}\right]$$

$$= yf'\left(\frac{xy}{z}\right)\frac{[z - px]}{z^2} \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right) x\left[\frac{(z)(1) - y\dfrac{\partial z}{\partial y}}{z^2}\right]$$

$$= xf'\left(\frac{xy}{z}\right)\frac{[z - qy]}{z^2} \quad \text{... (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \quad \frac{p}{q} = \frac{y}{x}\,\frac{z - px}{z - qy}$$

i.e., $\quad pxz - pqxy = qzy - pqxy$

$$\Rightarrow \quad pxz = qzy$$

i.e., $\quad xp - yq = 0$ is the required p.d.e.

Example 1.2(a)(7) : Eliminate ϕ from $xyz = \phi(x + y + z)$.

Solution : Given : $xyz = \phi(x + y + z)$ ... (1)

a.f
ϕ
1
∴ we use p & q only

Differentiating (1) p.w.r. to x, we get

$$y\frac{\partial}{\partial x}(xz) = \phi'(x + y + z)\frac{\partial}{\partial x}(x + y + z)$$

$$y\left[z + x\frac{\partial z}{\partial x}\right] = \phi'(x + y + z)\left[1 + \frac{\partial z}{\partial x}\right]$$

i.e., $\quad y(z + xp) = \phi'(x + y + z)(1 + p)$ ... (2)

Differentiating (1) p.w.r. to y, we get

$$x\frac{\partial}{\partial y}(yz) = \phi'(x + y + z)\frac{\partial}{\partial y}(x + y + z)$$

$$x\left[z + y\frac{\partial z}{\partial y}\right] = \phi'(x + y + z)\left[1 + \frac{\partial z}{\partial y}\right]$$

i.e., $x(z + qy) = \phi'(x + y + z)(1 + q)$ ... (3)

$$\frac{(2)}{(3)} \Rightarrow \quad \frac{y(z + px)}{x(z + qy)} = \frac{1 + p}{1 + q}$$

i.e., $(yz + pxy)(1 + q) = (xz + qxy)(1 + p)$

$$(xy - xz)p + (yz - xy)q = xz - yz$$

(or) $\; x(y - z)p + y(z - x)q = z(x - y)$ is the required p.d.e.

Example 1.2(a)(8) : Find the p.d.e. by eliminating the arbitrary function ϕ from $\phi(x^2 + y^2) = y^2 + z^2$.

Solution : Given $y^2 + z^2 = \phi(x^2 + y^2)$ ... (1)

a.f
ϕ
1
we use p & q only

Differentiating (1) p.w.r. to x, we get

$$2z\frac{\partial z}{\partial x} = \phi'(x^2 + y^2)\,2x$$

$$zp = x\phi'(x^2 + y^2) \quad \text{... (2)}$$

Differentiating (1) p.w.r. to y, we get

$$2y + 2z\frac{\partial z}{\partial y} = \phi'(x^2 + y^2)(2y)$$

$$y + qz = y\phi'(x^2 + y^2) \quad \text{... (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \quad \frac{pz}{y + qz} = \frac{x}{y} \quad \Rightarrow yzp = xy + xzq$$

$yzp - xzq = xy$ is the required p.d.e.

Example 1.2(a)(9) : Form the p.d.e. by eliminating f from $z = xy + f(x^2 + y^2 + z^2)$.

Solution : Given : $z = xy + f(x^2 + y^2 + z^2)$ ... (1)

a.f
f
1
∴ we use p & q only

Differentiating (1) p.w.r. to x, we get

$$\frac{\partial z}{\partial x} = y + f'(x^2 + y^2 + z^2)\left[2x + 2z\frac{\partial z}{\partial x}\right]$$

i.e., $p - y = f'(x^2 + y^2 + z^2)\; 2(x + pz)$ ... (2)

Differentiating (1) p.w.r. to y, we get

$$\frac{\partial z}{\partial y} = x + f'(x^2 + y^2 + z^2)\left[2y + 2z\frac{\partial z}{\partial y}\right]$$

i.e., $q - x = f'(x^2 + y^2 + z^2)\; 2[y + qz]$ ... (3)

$$\frac{(2)}{(3)} \Rightarrow \frac{p - y}{q - x} = \frac{x + pz}{y + qz}$$

$\Rightarrow (y + xz)p - (yz + x)q = y^2 - x^2$ is the required p.d.e.

Example 1.2(a)(10) : Form the p.d.e. by eliminating ϕ from $\phi(x^2 + y^2 + z^2, x + y + z) = 0$. **[A.U M/J 2016 R8]**

Solution : Rewriting the given equation as

a.f
f
1
∴ we use p & q only

$$x^2 + y^2 + z^2 = f(x + y + z) \quad \text{... (1)}$$

Differentiating (1) p.w.r. to x, we get

$$2x + 2zp = f'(x + y + z)(1 + p) \quad \text{... (2)}$$

Differentiating (1) p.w.r. to y, we get

$$2y + 2zq = f'(x + y + z)(1 + q) \quad \text{... (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{2(x + zp)}{2(y + zq)} = \frac{1 + p}{1 + q}$$

i.e., $(z - y)p + (x - z)q = y - x$ is the required p.d.e.

i.e., $(y - z)p + (z - x)q = x - y$

Example 1.2(a)(11) : Form the p.d.e. by eliminating f from $xy + yz + zx = f\left(\dfrac{z}{x + y}\right)$.

a.f
f
1
∴ we use p & q only

Solution : Given : $xy + yz + zx = f\left(\dfrac{z}{x + y}\right)$ ... (1)

Differentiating (1) p.w.r. to x, we get

$$y + yp + xp + z = f'\left(\frac{z}{x + y}\right)\left[\frac{(x + y)p - z}{(x + y)^2}\right] \quad \text{... (2)}$$

Differentiating (1) p.w.r. to y, we get

$$x + yq + z + xq = f'\left(\frac{z}{x + y}\right)\left[\frac{(x + y)q - z}{(x + y)^2}\right] \quad \text{... (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{y + yp + xp + z}{x + yq + z + xq} = \frac{(x + y)p - z}{(x + y)q - z}$$

$$\frac{(y + z) + (x + y)p}{(x + z) + (x + y)q} = \frac{(x + y)p - z}{(x + y)q - z}$$

$$(x + y)(y + z)q + (x + y)^2pq - z(y + z) - z(x + y)p$$
$$= (x + z)(x + y)p - z(x + z) + (x + y)^2pq - z(x + y)q$$

$$[-z(x + y) - (x + z)(x + y)]p + [(x + y)(y + z) + z(x + y)]q$$
$$= z(y + z) - z(x + z)$$

$$-(x + y)(x + 2z)p + (x + y)(y + 2z)q = z[y + z - x - z]$$

$$-(x + y)(x + 2z)p + (x + y)(y + 2z)q = z(y - x)$$

i.e., $(x + y)(x + 2z)p - (x + y)(y + 2z)q = z(x - y)$

is the required p.d.e.

Example 1.2(a)(12) : Obtain p.d.e. from $z = f(\sin x + \cos y)$.

Solution : Given : $z = f(\sin x + \cos y)$ ... (1)

$$p = \frac{\partial z}{\partial x} = f'(\sin x + \cos y)[\cos x] \quad \text{... (2)}$$

$$q = \frac{\partial z}{\partial y} = f'(\sin x + \cos y)[-\sin y] \quad \text{... (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \quad \frac{p}{q} = \frac{\cos x}{-\sin y}$$

a.f
f
1
we use p & q only

$$-(\sin y) p = (\cos x) q$$

$(\sin y) p + (\cos x) q = 0$ is the required p.d.e.

Example 1.2(a)(13) : Form the p.d.e. by eliminating f from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$. **[AU, April 2000] [A.U M/J 2014]**

Solution : Given : $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

a.f
f
1
we use p & q only

$$p = \frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left[\frac{-1}{x^2}\right]$$

$$= \frac{-2}{x^2} f'\left(\frac{1}{x} + \log y\right)$$

$$x^2 p = -2f'\left(\frac{1}{x} + \log y\right) \quad \ldots (1)$$

$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left[\frac{1}{y}\right]$$

$$= 2y + \frac{2}{y} f'\left(\frac{1}{x} + \log y\right)$$

$$q - 2y = \frac{2}{y} f'\left(\frac{1}{x} + \log y\right)$$

$$(q - 2y) y = 2f'\left(\frac{1}{x} + \log y\right) \quad \ldots (2)$$

$$\frac{(1)}{(2)} \Rightarrow \quad \frac{x^2 p}{(q - 2y) y} = -1$$

$$\Rightarrow \quad x^2 p = -yq + 2y^2$$

$\Rightarrow \quad x^2 p + yq = 2y^2$ is the required p.d.e.

Example 1.2(a)(14) : Form the PDE by eliminating the arbitrary function from $\phi\left[z^2 - xy, \frac{x}{z}\right] = 0$.

[A.U. M/J 2006, N/D 2010, ND 2012]

Solution : Given : $\phi\left[z^2 - xy, \frac{x}{z}\right] = 0$

a.f
f
1
we use p & q only

Rewriting the given equation as

$$z^2 - xy = f\left(\frac{x}{z}\right) \quad \ldots (1)$$

Differentiating (1) p. w.r.t. x, we get

$$2z\frac{\partial z}{\partial x} - y = f'\left(\frac{x}{z}\right)\left[\frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}\right]$$

$$2z\, p - y = f'\left(\frac{x}{z}\right)\left[\frac{z - xp}{z^2}\right] \quad \ldots (2)$$

Differentiating (1) p w.r.t. y we get

$$2z\frac{\partial z}{\partial y} - x = f'\left(\frac{x}{z}\right)\left[\frac{z(0) - x\frac{\partial z}{\partial y}}{z^2}\right]$$

$$2zq - x = f'\left(\frac{x}{z}\right)\left[\frac{-xq}{z^2}\right] \quad \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \quad \frac{2zp - y}{2zq - x} = \frac{z - xp}{-xq}$$

$$-2xzpq + xyq = 2z^2 q - 2xzpq - xz + x^2 p$$

$$xyq = 2z^2 q - xz + x^2 p$$

$$x^2 p + 2z^2 q - xyq = xz$$

$x^2 p - (xy - 2z^2) q = xz$ which is the required p.d.e.

Second Method :

Let $u = z^2 - xy, \quad v = \frac{x}{z}$

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2zp - y & \dfrac{z - px}{z^2} \\ 2zq - x & \dfrac{-xq}{z^2} \end{vmatrix} = 0$$

$$\Rightarrow px^2 - q(xy - 2z^2) = xz$$

$$\Rightarrow x^2 p - (xy - 2z^2) q = xz$$

EXERCISE 1.2(a)

Form a p.d.e of eliminating the arbitrary function from

1. $\phi(x - y, x + y + z) = 0$ [A.U. May 2000] [**Ans.** $p + q = -2$]
   [A.U. N/D 2016 R-8]
2. $z = f(x^2 + y^2)$ [A.U. April 2001] [**Ans.** $yp - xq = 0$]
3. $f(xy + z^2, x + y + z) = 0$ [A.U. A/M 2003 PTMA, T+B M/J 2011]
   [**Ans.** $(2z - x)p - (2z - y)q = (x - y)$]
4. $z = yf\left(\dfrac{x}{y}\right)$ [**Ans.** $xp + yq = z$]
5. $z = \phi(x^3 - y^3)$ [**Ans.** $y^2 p + x^2 q = 0$]
6. $z = f(bx - ay)$ [**Ans.** $ap + bq = 0$]
7. $z = x^2 + 2f\left(\dfrac{1}{y} + \log x\right)$ [**Ans.** $xp + y^2 q = 2x^2$]
8. $z = e^{ay} f(x + by)$ [**Ans.** $bp - q = -az$] [A.U. A/M 2017 R-13]
9. $lx + my + nz = f(x^2 + y^2 + z^2)$ [A.U. CBT A/M 201...]
   [**Ans.** $(ny - mz)p + (lz - nx)q = mx - ly$]
10. $f(z - xy, x^2 + y^2) = 0$ [A.U. M/J 2016 R13]
    [A.U. N/D 2018 R17] [**Ans.** $yp - xq = y^2 - x^2$]
11. $z = xy + f(x^2 + y^2)$ [**Ans.** $yp - xq = y^2 - x^2$]
12. $xyz = f(x^2 + y^2 + z^2)$
    [**Ans.** $x(y^2 - z^2)p - y(z^2 - x^2)q = z(x^2 - y^2)$]

Problems based on Formation of p.d.e by elimination of arbitrary functions

1.2.(b) Case (ii) a.f = 2, p.d.e order = 2

Example 1.2(b)(1) : Form the p.d.e. by eliminating the arbitrary functions f and ϕ from $z = f(x + ct) + \phi(x - ct)$.

[A.U.T CBT N/D 2011][A.U N/D 2019, R17]

Solution :

Given $\quad z = f(x + ct) + \phi(x - ct) \quad \ldots (1)$

a.f
f, ϕ
2
we use p, q, r, s & t

$$\frac{\partial z}{\partial x} = f'(x + ct) + \phi'(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ct) + \phi''(x - ct) \quad \ldots (2)$$

$$\frac{\partial z}{\partial t} = cf'(x + ct) - c\phi'(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 [f''(x + ct) + \phi''(x - ct)] \quad \ldots (3)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \text{ by (2) \& (3)}$$

Example 1.2(b)(2) : Form the p.d.e. by eliminating f and ϕ from $z = f(y) + \phi(x + y + z)$. [A.U. Nov./Dec. 2004]

Solution : Given

$$z = f(y) + \phi(x + y + z) \quad \ldots (1)$$

a.f
f, ϕ
2
we use p, q, r, s & t

$$\frac{\partial z}{\partial x} = 0 + \phi'(x + y + z)\left(1 + 0 + \frac{\partial z}{\partial x}\right)$$

$$= \phi'(x + y + z)\left(1 + \frac{\partial z}{\partial x}\right) \quad (2)$$

Differentiating (2) p.w.r. to x, we get

$$\frac{\partial^2 z}{\partial x^2} = \phi'(x + y + z)\frac{\partial^2 z}{\partial x^2} + \left(1 + \frac{\partial z}{\partial x}\right)\phi''(x + y + z)\left[1 + 0 + \frac{\partial z}{\partial x}\right]$$

$$r = \phi'(x + y + z)r + (1 + p)\phi''(x + y + z)(1 + p)$$

$$r - \phi'(x+y+z)\,r = (1+p)^2 \phi''(x+y+z)$$

$$r\,[1 - \phi'(x+y+z)] = (1+p)^2 \phi''(x+y+z) \qquad \dots (3)$$

Differentiating (2) p.w.r. to y, we get

$$\frac{\partial^2 z}{\partial y\,\partial x} = \phi'(x+y+z)\frac{\partial^2 z}{\partial y\,\partial x} + \left(1+\frac{\partial z}{\partial x}\right)\phi''(x+y+z)\left[0+1+\frac{\partial z}{\partial y}\right]$$

$$s = \phi'(x+y+z)\,s + (1+p)\,\phi''(x+y+z)\,(1+q)$$

$$s\,[1 - \phi'(x+y+z)] = (1+p)(1+q)\,\phi''(x+y+z) \qquad \dots (4)$$

$$\frac{(3)}{(4)} \Rightarrow \qquad \frac{r}{s} = \frac{1+p}{1+q}$$

$\Rightarrow$ $\quad r(1+q) = s(1+p)$ which is the required p.d.e

Example 1.2(b)(3) : Eliminate the arbitrary functions f and g from $z = f(x+iy) + g(x-iy)$ to obtain a partial differential equation involving z, x, y. **[A.U. N/D 2004]**

a.f
f, g
2
∴ we use p, q, r, s & t

Solution : Given : $z = f(x+iy) + g(x-iy) \qquad \dots (1)$

$$p = \frac{\partial z}{\partial x} = f'(x+iy) + g'(x-iy) \qquad \dots (2)$$

$$q = \frac{\partial z}{\partial y} = i f'(x+iy) - i g'(x-iy) \qquad \dots (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy) + g''(x-iy) \qquad \dots (4)$$

$$t = \frac{\partial^2 z}{\partial y^2} = -f''(x+iy) - g''(x-iy) \qquad \dots (5)$$

$r + t = 0$ is the required p.d.e. by (4) & (5)

Example 1.2(b)(4) : Form the p.d.e. by eliminating f and ϕ from $z = xf\left(\frac{y}{x}\right) + y\phi(x)$. **[A.U N/D 2016 R-13]**

a.f
f, ϕ
2
∴ we use p, q, r, s & t

Solution :

Given: $z = xf\left(\frac{y}{x}\right) + y\phi(x) \qquad \dots (1)$

$$p = \frac{\partial z}{\partial x} = xf'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) + y\phi'(x) \qquad \dots (2)$$

$$q = \frac{\partial z}{\partial y} = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \phi(x) = f'\left(\frac{y}{x}\right) + \phi(x) \qquad \dots (3)$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = f''\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + \phi'(x) \qquad \dots (4)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \qquad \dots (5)$$

$$px + qy = -yf'\left(\frac{y}{x}\right) + xf\left(\frac{y}{x}\right) + xy\phi'(x) + yf'\left(\frac{y}{x}\right) + y\phi(x)$$

$$= xy\phi'(x) + xf\left(\frac{y}{x}\right) + y\phi(x)$$

$$px + qy = xy\phi'(x) + z \qquad \dots (6)$$

$$z = xp + yq - xy\phi'(x) \qquad \dots (7)$$

$$(4) \times xy \Rightarrow \quad xys = xyf''\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + xy\phi'(x)$$

$$= -y^2\left[f''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)\right] + xy\phi'(x)$$

$$= -y^2 t + xy\phi'(x) \quad \text{by (5)}$$

$$xy\phi'(x) = xys + y^2 t$$

$$(7) \Rightarrow \quad z = px + qy - xys - y^2 t$$

which is the required p.d.e.

Example 1.2(b)(5) : Form the differential equation by eliminating the arbitrary functions f and g in $z = f(x^3 + 2y) + g(x^3 - 2y)$

[A.U. Oct./Nov. 2002] [A.U. N/D 2018-A R17]

Solution : $z = f(x^3 + 2y) + g(x^3 - 2y)$ ... (1)

$$p = \frac{\partial z}{\partial x} = f'(x^3 + 2y)(3x^2) + g'(x^3 - 2y)(3x^2) \quad \text{... (2)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = f'(x^3 + 2y)(6x) + (3x^2) f''(x^3 + 2y)(3x^2)$$

$$+ g'(x^3 - 2y)(6x) + (3x^2) g''(x^3 - 2y)(3x^2)$$

$$= 9x^4 [f''(x^3 + 2y) + g''(x^3 - 2y)]$$

$$+ 6x [f'(x^3 + 2y) + g'(x^3 - 2y)] \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = f'(x^3 + 2y)(2) + g'(x^3 - 2y)(-2) \quad \text{... (4)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(x^3 + 2y)(4) + g''(x^3 - 2y)(4)$$

$$= 4 \left[f''(x^3 + 2y) + g''(x^3 - 2y)\right] \quad \text{... (5)}$$

(3) ⇒ $\therefore r = 9x^4 \left[\frac{t}{4}\right] + 6x \left[\frac{1}{3x^2} p\right]$ by (5) & (2)

$$r = \frac{9}{4} x^4 t + \frac{2}{x} p \Rightarrow 4xr = 9x^5 t + 8p$$

which is the required p.d.e.

Example 1.2(b)(6) : Form the partial differential equation by eliminating the arbitrary functions f and g in $z = x^2 f(y) + y^2 g(x)$.

[A.U. A/M 2003, N/D 2013]

Solution : Given : $z = x^2 f(y) + y^2 g(x)$ ... (1)

$$p = \frac{\partial z}{\partial x} = 2x f(y) + y^2 g'(x) \quad \text{... (2)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = 2f(y) + y^2 g''(x) \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = x^2 f'(y) + 2y g(x) \quad \text{... (4)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = x^2 f''(y) + 2g(x) \quad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2x f'(y) + 2y g'(x) \quad \text{... (6)}$$

(2) × x ⇒ $px = 2x^2 f(y) + xy^2 g'(x)$

(4) × y ⇒ $qy = x^2 y f'(y) + 2y^2 g(x)$

$$px + qy = 2[x^2 f(y) + y^2 g(x)] + xy[y g'(x) + x f'(y)]$$

$$= 2z + xy \left[\frac{s}{2}\right] \text{ by (1) and (6)}$$

$$2px + 2qy = 4z + xys$$

$$4z = 2px + 2qy - sxy$$

which is the required p.d.e.

Example 1.2(b)(7): Obtain the partial differential equation by eliminating f and g from $z = f(2x + y) + g(3x - y)$.

[A.U N/D. 2003 PT]

Solution : Given : $z = f(2x + y) + g(3x - y)$

$$\frac{\partial z}{\partial x} = f'(2x + y)\, 2 + g'(3x - y)\, 3$$

$$\frac{\partial^2 z}{\partial x^2} = 4f''(2x + y) + 9g''(3x - y)$$

$$\frac{\partial z}{\partial y} = f'(2x+y) + g'(3x-y)(-1)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(2x+y) + g''(3x-y)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= f''(2x+y)\,2 - 3g''(3x-y)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} - 6\frac{\partial^2 z}{\partial y^2}$$

$$= 4f''(2x+y) + 9g''(3x-y) + 2f''(2x+y) - 3g''(3x-y)$$

$$-6f''(2x+y) - 6g''(3x-y)$$

$$= 6f''(2x+y) + 6g''(3x-y) - 6f''(2x+y) - 6g''(3x-y)$$

$$= 0$$

(i.e.,) The required solution is $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

Example 1.2(b)(8): Form a partial differential equation by eliminating arbitrary functions from $z = x f(2x+y) + g(2x+y)$.

[A.U. A/M 2008] [A.U. CBT Dec. 2008]

Solution :

Given : $z = x f(2x+y) + g(2x+y)$

$$\frac{\partial z}{\partial x} = x[f'(2x+y)\,2] + f(2x+y)(1) + g'(2x+y)(2)$$

$$= 2[x f'(2x+y) + g'(2x+y)] + f(2x+y) \quad \dots (1)$$

$$\frac{\partial z}{\partial y} = x f'(2x+y) + g'(2x+y) \quad \dots (2)$$

$$(1) \Rightarrow \frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + f(2x+y) \quad \dots (3) \text{ by } (2)$$

Differentiating (3) p.w.r. to x, we get

$$\frac{\partial^2 z}{\partial x^2} = 2\frac{\partial^2 z}{\partial x\,\partial y} + f'(2x+y)\,2 \quad \dots (4)$$

Differentiating (3) p.w.r. to y, we get

$$\frac{\partial^2 z}{\partial y\,\partial x} = 2\frac{\partial^2 z}{\partial y^2} + f'(2x+y) \quad \dots (5)$$

$$(4) - 2\times(5) \Rightarrow \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial y\,\partial x} = 2\frac{\partial^2 z}{\partial x\,\partial y} - 4\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} - 4\frac{\partial^2 z}{\partial x\,\partial y} = 0$$

which is the required p.d.e.

EXERCISE 1.2(b)

Form the p.d.e by eliminating the arbitrary functions

1. $z = f(x) + g(y)$ **[Ans.** $s = 0$]
2. $z = x f(y) + \phi(y) - \sin x$ **[Ans.** $r = \sin x$]
3. $z = y f(x) + \phi(x) - \cos y$ **[Ans.** $t = \cos y$]
4. $z = f(ax + by) + g(\alpha x + \beta y)$ [A.U. N/D 2018 R13]

   [Ans. $b\beta \frac{\partial^2 z}{\partial x^2} - (b\beta + b\alpha)\frac{\partial^2 z}{\partial y\,\partial x} + a\alpha\frac{\partial^2 z}{\partial y^2} = 0$]
5. $u = f(x^2 + y) + \phi(x^2 - y)$ **[Ans.** $\frac{\partial^2 u}{\partial x^2} - \frac{1}{x}\frac{\partial u}{\partial x} - 4x^2\frac{\partial^2 u}{\partial y^2} = 0$]
6. $z = f(x + y + z) + \phi(x - y)$

   [Ans. $(1+q)r + (q-p)s - (1+p)t = 0$]
7. $z = f(\sqrt{x} + y) + g(\sqrt{x} - y)$ **[Ans.** $4x\,r - t + 2p = 0$]

8. $z = f(2x+3y) + g(2x+y)$ [Ans. $3r - 8s + 4t = 0$]

9. $z = x f_1(x+t) + f_2(x+t)$ [Ans. $\frac{\partial^2 z}{\partial x^2} = 2\frac{\partial^2 z}{\partial t\,\partial x} - \frac{\partial^2 z}{\partial t^2}$]

10. $z = f(x+4t) + g(x-4t)$ [Ans. $16\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0$]

11. $z = f(2x-3y) + x g(2x-3y)$ [Ans. $4\frac{\partial^2 z}{\partial y^2} + 12\frac{\partial^2 z}{\partial x\,\partial y} + 9\frac{\partial^2 z}{\partial x^2} = 0$]

12. $z = f(2x+3y) + y g(2x+3y)$ [Ans. $\frac{\partial^2 z}{\partial y^2} + \frac{9}{4}\frac{\partial^2 z}{\partial x^2} = 3\frac{\partial^2 z}{\partial x\,\partial y}$]

13. $z = f(x+t) + g(x-t)$ [A.U.T Chennai N/D 2011]
[Ans. $r - t = 0$]

14. $z = f(x+y)\,\phi(x-y)$ [Ans. $p^2 - q^2 = z(r-t)$]

15. $z = f(x) + e^y g(x)$ [Ans. $q = t$]

16. $z = y f(x) + x g(y)$ [A.U A/M 2017 R-8]
[Ans. $z = xp + yq - xys$]

1.3 (1) SINGULAR INTEGRALS - SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

III.(a) Problems based on General solution of p.d.e in ordinary cases:

Example 1.3(a)(1): Solve $\frac{\partial z}{\partial x} = \sin x$.

Solution : Given : $\frac{\partial z}{\partial x} = \sin x$

...ating p.w.r.to x on both sides

$$z = -\cos x + f(y),$$

where $f(y)$ is an arbitrary function.

$$\therefore z = -\cos x + f(y).$$

Example 1.3(a)(2): Find the general solution of $\frac{\partial^2 z}{\partial y^2} = 0$.

Solution : Given : $\frac{\partial^2 z}{\partial y^2} = 0$ [A.U. N/D 2007]

i.e., $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 0$

Integrating p.w.r.to y on both sides

$$\frac{\partial z}{\partial y} = f(x)$$

Again Integrating p.w.r.to y on both sides

$$z = f(x)\,y + F(x)$$

(or) $z = yf(x) + F(x)$ where both $f(x)$ and $F(x)$ are arbitrary.

Example 1.3(a)(3): Solve $\frac{\partial^2 z}{\partial x\,\partial y} = 0$.

[A.U. CBT Dec. 2008]
[A.U A/M 2017 R-8]

Solution : Given : $\frac{\partial^2 z}{\partial x\,\partial y} = 0$ i.e., $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0$

Integrating p.w.r.to x, we get

$$\frac{\partial z}{\partial y} = f(y)$$

Again Integrating p.w.r.to y, we get

$$z = \int f(y)\,dy + \phi(x) = F(y) + \phi(x)$$

Example 1.3(a)(4): Solve $\frac{\partial^2 z}{\partial x\,\partial y} = \sin x$.

Solution : Given : $\frac{\partial^2 z}{\partial x\,\partial y} = \sin x$

i.e., $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x$

Integrating p.w.r.to x, we get

$$\frac{\partial z}{\partial y} = -\cos x + f(y)$$

Integrating p.w.r.to y, we get

$$z = -y\cos x + \int f(y)\,dy + \phi(x)$$
$$= -y\cos x + F(y) + \phi(x)$$

Here both $F(y)$ and $\phi(x)$ are arbitrary.

Example 1.3(a)(5): Solve : $\dfrac{\partial^2 z}{\partial x^2} = \sin y.$ **[A.U. M/J 2007]**

[A.U Tvli M/J 2011]

Solution : $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \sin y$

Integrating p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = (\sin y)x + f(y)$$

Again integrating p.w.r.to x, we get

$$z = (\sin y)\frac{x^2}{2} + f(y)x + F(y)$$

$$\therefore \quad z = \frac{x^2}{2}\sin y + xf(y) + F(y)$$

where both $f(y)$ and $F(y)$ are arbitrary functions.

Example 1.3(a)(6): Solve $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$ **given that** $x = 0, \dfrac{\partial z}{\partial x} = a\sin y$ **and** $\dfrac{\partial z}{\partial y} = 0.$ **[AU, April 2000][A.U N/D 2014 R-08]**

Solution : z is a function of x alone

Given : $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$

$$\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$$

A.E. is $m^2 - a^2 = 0$; $\quad m^2 = a^2$; $\quad m = \pm a$

$$z = Ae^{ax} + Be^{-ax}$$

Since z is a function of x and y. Therefore A & B will be the functions of y alone. Hence

$$z = f(y)e^{ax} + \phi(y)e^{-ax} \quad \text{... (1)}$$

where $f(y)$ & $\phi(y)$ are functions of y alone.

$$\frac{\partial z}{\partial x} = af(y)e^{ax} - a\phi(y)e^{-ax} \quad \text{... (2)}$$

$$\frac{\partial z}{\partial y} = f'(y)e^{ax} + \phi'(y)e^{-ax} \quad \text{... (3)}$$

Case (i) Given : $\dfrac{\partial z}{\partial x} = a\sin y$; when $x = 0$

$$(2) \Rightarrow a\sin y = af(y) - a\phi(y) = a[f(y) - \phi(y)]$$
$$\sin y = f(y) - \phi(y)$$

i.e., $f(y) - \phi(y) = \sin y$ ... (4)

Case (ii) Given : $\dfrac{\partial z}{\partial y} = 0$; when $x = 0$

i.e., $(3) \Rightarrow 0 = f'(y) + \phi'(y)$ i.e., $f'(y) + \phi'(y) = 0$

Integrating we get

$$f(y) + \phi(y) = k \quad \text{... (5)}$$

$$(4) + (5) \Rightarrow f(y) = \frac{1}{2}[\sin y + k]$$

$$(5) - (4) \quad \phi(y) = \frac{1}{2}[k - \sin y]$$

$$z = \frac{1}{2}[\sin y + k]e^{ax} + \frac{1}{2}[k - \sin y]e^{-ax}$$

$$z = \frac{1}{2}\sin y\, e^{ax} + \frac{1}{2}k\,e^{ax} + \frac{1}{2}ke^{-ax} - \frac{1}{2}\sin y\, e^{-ax}$$

$$= \frac{1}{2}\sin y\,[e^{ax} - e^{-ax}] + \frac{1}{2}k\,[e^{ax} + e^{-ax}]$$

$$= \sin y\left[\frac{e^{ax} - e^{-ax}}{2}\right] + k\left[\frac{e^{ax} + e^{-ax}}{2}\right]$$

$z = \sin y \sinh ax + k\cosh ax$ where k is any constant.

EXERCISE 1.3(a)

I. Solve the following partial differential equation.

(1) $\dfrac{\partial^2 z}{\partial x^2} = \cos x$ (2) $\dfrac{\partial^2 z}{\partial x\,\partial y} = x^2 + y^2$ (3) $\dfrac{\partial z}{\partial x} = 0$ (4) $\dfrac{\partial^2 z}{\partial x\,\partial y} =$

(5) Solve $\dfrac{\partial^2 z}{\partial x\,\partial y} = e^{-y}\cos x$ given that $z = 0$ when $y = 0$ and $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$.

(6) $\dfrac{\partial^3 z}{\partial x^2\,\partial y} = \cos(ax + by)$ (7) $x\dfrac{\partial z}{\partial x} = 2x + y + 3z$

(8) By interchanging the independent variables by the relations $z = x + iy$, $\bar{z} = x - iy$, show that the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ transforms into $\dfrac{\partial^2 u}{\partial z\,\partial \bar{z}} = 0$. Hence obtain a general solution o the equation.

(9) With the help of the substitution $u = x + \alpha y$, $v = x + \beta y$ when α, β are suitable constants, transform the equation $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x\,\partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = 0$ to the form $\dfrac{\partial^2 z}{\partial u\,\partial v} = 0$ and henc obtain its general solution.

(10) By obtaining the general solution or otherwise solve $\dfrac{\partial^2 z}{\partial x^2} + z =$ and $z = e^y$, $\dfrac{\partial z}{\partial x} = 1$ when $x = 0$

(11) Solve : $\dfrac{\partial^3 z}{\partial x^2\,\partial y} + 18xy^2 + \sin(2x - y) = 0$

(12) Solve : $\dfrac{\partial z}{\partial x} = 3x - y$; $\dfrac{\partial z}{\partial y} = \cos y - x$

(13) Solve $\dfrac{\partial^2 z}{\partial x^2} = xy$

(14) Solve $\dfrac{\partial z}{\partial x} = 6x + 3y$, $\dfrac{\partial z}{\partial y} = 3x - 4y$.

(15) Solve $\dfrac{\partial^2 u}{\partial x\,\partial t} = e^{-t}\cos x$, given that $u = 0$ when $t = 0$ and $\dfrac{\partial u}{\partial t} = 0$ when $x = 0$. Show that as $t \to \infty$, $u \to \sin x$.

(16) By changing the independent variables by the relations $r = x + at$, $s = x - at$, show that the equation $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$ gets transformed into the equation $\dfrac{\partial^2 y}{\partial r\,\partial s} = 0$. Hence find a general solution of the partial differential equation.

ANSWERS

1. $z = -\cos x + x f(y) + g(y)$
2. $z = \dfrac{x^3 y}{3} + \dfrac{xy^3}{3} + f(x) + g(y)$
3. $z = f(y)$
4. $z = \left[\dfrac{x^2}{2} + f(y)\right] + g(x)$
5. $z = \sin x\,(1 - e^{-y})$
6. $z = \dfrac{-1}{a^2 b}\sin(ax + by) + x f(y) + g(y) + \phi(x)$
7. $z = -x - \dfrac{y}{3} + x^3 f(y)$
8. $u = \phi(x + iy) + f(x - iy)$
9. $z = f(2x + y) + \phi(3x + y)$
10. $z = e^y\cos x + \sin x$
11. $z = \dfrac{1}{4}\cos(2x - y) - x^3 y^3 + x f(y) + g(x) + \phi(y)$
12. $z = \dfrac{3x^2}{2} - xy + \sin y + c$
13. $z = \dfrac{x^3 y}{6} + x f(y) + g(y)$
14. $z = 3x^2 + 3xy - 2y^2 + k$
15. $u(x, t) = \sin x\,[1 - e^{-t}]$
16. $y = f(x - at) + g(x + at)$

1.3(2) Methods to solve the first order partial differential equatio

The general form of a first order partial differential equation

$f(x, y, z, p, q) = 0$, where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

§ Mention three types of solution of a p.d.e.

(or) Define P.I, general and complete integrals of a p.d.e.

1. Complete integral (or) Complete solution

A solution which contains as many arbitrary constants as the are independent variables is called a complete integral (or) comple solution. (number of a.c. = number of I.V)

2. Particular integral (or) Particular solution

A solution obtained by giving particular values to the arbitra constants in a complete integral is called a particular integral (o particular solution.

3. General integral (or) General solution

A solution of a p.d.e. which contains the maximum possible numb of arbitrary functions is called a general integral (or) general soluti

Example (i) : $z = ax + by + ab$

a.c.	I.V
a, b	x, y
2	2

Number of a.c = Number of I.V

Hence, $z = ax + by + ab$ is a complete integral

Example (ii) : $z = 2x + 3y + 6$ is a particular integral.

Since, $a = 2, b = 3$ are the particular values of the compl integral $z = ax + by + ab$

Example (iii) : $z = f(x^2 - y^2)$ ... (1) is the solution of p.d.e

$yp + xq = 0$ ... (2)

Here, $yp + xq = 0$ is a p.d.e of order 1.

So, the maximum possible number of a.f. = 1

$\therefore z = f(x^2 - y^2)$ is a general solution of (2).

§ Show how to find the general integral of the p.d.e. $f(x, y, z, p, q) = 0$.

Solution :

Let the p.d.e. be $f(x, y, z, p, q) = 0$ ... (1)

Let the complete integral of (1) be $\phi(x, y, z, a, b) = 0$ ... (2)

where a & b are arbitrary constants.

Suppose, in (2) one of the constants is a function of the other say $b = f(a)$. Then (2) becomes

$$\phi[x, y, z, a, f(a)] = 0 \quad \text{... (3)}$$

Differentiating (3) p.w.r. to a we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial u}{\partial a} f'(a) = 0 \quad \text{... (4)}$$

The elimination of 'a' between (3) & (4) if it exists, is called the general integral of (1).

§ Define singular integral. **[A.U. N/D 2018 R17]**

Solution : Let $f(x, y, z, p, q) = 0$ ... (1)

Let the complete integral be $\phi(x, y, z, a, b) = 0$ ... (2)

Differentiating (2) p.w.r. to a and b in turn we get,

$$\frac{\partial \phi}{\partial a} = 0 \quad \text{... (3) and}$$

$$\frac{\partial \phi}{\partial b} = 0 \quad \text{... (4)}$$

The elimination of a & b from the three equations (2), (3) & (4) if it exists, is called the singular integral.

Standard types of first order p.d.e

Type 1 : $f(p,q) = 0$

Type 2 : $z = px + qy + f(p,q)$ [Clairaut's form]

Type 3 : $f(z,p,q) = 0$, $f(x,p,q) = 0$, $f(y,p,q) = 0$

Type 4 : $f(x,p) = \phi(y,q)$

Type 5 : $f(x^m p, y^n q) = 0$ and $f(z, x^m p, y^n q) = 0$

Type 6 : $f(z^m p, z^m q) = 0$ and $f_1(x, z^m p) = f_2(y, z^m q)$

Type 1 f(p, q) = 0

Suppose that $z = ax + by + c$

then $p = a, \quad q = b$

we get, $f(a, b) = 0$

solving for b, we get $b = \phi(a)$

$$z = ax + \phi(a)y + c \qquad \text{... (1)}$$

which is the required complete integral.

Differentiating (1) p. w.r.to c, we get $0 = 1$ which is absurd th[at] there is no singular integral.

Put $c = f(a)$ in (1) we get

$$z = ax + \phi(a)y + f(a) \qquad \text{... (2)}$$

Differentiating (2) p. w.r.to a we get

$$0 = x + \phi'(a)y + f'(a) \qquad \text{... (3)}$$

eliminating 'a' between (2) and (3), we get the general solutio[n]

III(b) Problems based on First order p.d.e

Problems based on Type 1 : $f(p, q) = 0$ [A.U A/M 2019 R-13]

Example 1.3(b)(1) : [Type 1] [A.U M/J 2016 R-8]

Solve : $\sqrt{p} + \sqrt{q} = 1$ [A.U. Oct. 1996] [A.U. CBT Dec. 2008]

Solution : Given : $\sqrt{p} + \sqrt{q} = 1$ ... (1)

This equation is of the form $f(p, q) = 0$ ... (2)

Hence, the trial solution is $z = ax + by + c$ ... (3)

To get the complete integral (solution)

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

$$\therefore (1) \Rightarrow \sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1 - \sqrt{a})^2$$

Hence, the complete solution is

$$\therefore (3) \Rightarrow \quad z = ax + (1 - \sqrt{a})^2 y + c \qquad \text{... (4)}$$

Since, number of a.c. = number of I.V.

To find the singular integral :

Differentiating (4) p.w.r.to c we get,

$$0 = 1 \qquad \text{[absurd]}$$

Hence, there is no singular integral.

To get the general integral (solution)

Put, $c = f(a)$ in (4), we get

$$z = ax + (1 - \sqrt{a})^2 y + f(a) \qquad \text{... (5)}$$

Differentiating partially w.r.to 'a', we get i.e., $x + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right)y + f'(a) = 0$ ... (6) $\left[\because \frac{\partial z}{\partial a} = \right.$

Eliminating 'a' between (5) and (6), we get the general solu[tion] of the given p.d.e.

Example 1.3b(2) : [Type 1]

Solve p + q = pq **[A.U. April 2003, A.U.T. Tvli N/D 20..] [A.U M/J 20..]**

Solution : Given : $p + q = pq$ ... (1)

This equation is of the form $f(p, q) = 0$ ... (2)

Hence, the trial solution is $z = ax + by + c$ ... (3)

To get the complete integral (solution)

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

$\therefore$ (1) $\Rightarrow a + b = ab$

$\Rightarrow a = ab - b$

$\Rightarrow a = b(a-1)$

$\Rightarrow b = \frac{a}{a-1}$

Hence, the complete solution is

$\therefore$ (3) $\Rightarrow z = ax + \frac{a}{a-1}y + c$ ... (4)

Since, number of a.c. = number of I.V

To find the singular integral :

Differentiating (4) p.w.r.to c we get,

$0 = 1$ [absurd]

Hence, there is no singular integral.

To get the general integral (solution)

Put $c = f(a)$ in (4), we get

$$z = ax + \frac{a}{a-1}y + f(a) \quad \text{... (5)}$$

differentiating partially w.r.to 'a', we get

i.e. $x + \frac{(a-1)(1)-(a)(1)}{(a-1)^2}y + f'(a) = 0$ $\left[\because \frac{\partial z}{\partial a} = 0\right]$

$$x - \frac{1}{(a-1)^2}y + f'(a) = 0 \quad \text{... (6)}$$

Eliminating 'a' between (5) & (6), we get the general solution of the given p.d.e.

Example 1.3b(3) : [Type 1] Solve : pq = k

Solution : Given : $pq = k$ ... (1)

This equation is of the form $f(p, q) = 0$ ... (2)

Hence, the trial solution is $z = ax + by + c$ ... (3)

To get the complete integral :

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

(1) $\Rightarrow ab = k$

$b = \frac{k}{a}$ Hence, the complete solution is

$\therefore$ (3) $\Rightarrow z = ax + \frac{k}{a}y + c$ ... (4)

Since, number of a.c. = number of I.V.

To find the singular integral

Differentiating (4) p.w.r.to c we get,

$0 = 1$ [absurd]

There is no singular integral.

To get the general integral (solution)

Put $c = f(a)$ in (4), we get

$$z = ax + \frac{k}{a}y + f(a) \quad \dots (5)$$

differentiating p.w.r to 'a', we get

i.e., $$x - \frac{k}{a^2}y + f'(a) = 0 \quad \dots (6) \quad \left[\because \frac{\partial z}{\partial a} = 0\right]$$

Eliminating 'a' between (5) & (6), we get the general solution of the given p.d.e.

Example 1.3b(4) : [Type 1] Solve : $p^2 + q^2 = npq$

Solution : Given : $p^2 + q^2 = npq$ ... (1)

This equation is of the form $f(p, q) = 0$ ... (2)

Hence, the trial solution is $z = ax + by + c$ ... (3)

To get the complete integral (solution) :

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

$\therefore$ (1) $\Rightarrow$ $a^2 + b^2 = n\,ab$

$$b^2 - n\,ab + a^2 = 0$$

$$b = \frac{na \pm \sqrt{a^2 n^2 - 4a^2}}{2} = \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]$$

Hence, the complete solution is

(3) $\Rightarrow$ $$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + c \quad \dots (4)$$

Since, number of a.c. = number of I.V

To find the singular integral :

Differentiating (4) p.w.r.to c we get,

$0 = 1$ [absurd]

There is no singular integral.

To get the general integral (solution)

Put $c = \phi(a)$ in (4), we get

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + \phi(a) \quad \dots (5)$$

Differentiating p.w.r. to 'a', we get

i.e. $$x + \frac{1}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + \phi'(a) = 0 \quad \dots (6) \quad \left[\because \frac{\partial z}{\partial a} = 0\right]$$

Eliminating 'a' between (5) & (6), we get the general solution of the given p.d.e.

> **Note :** Solve $p^2 + q^2 - 4pq = 0$ **[A.U. N/D 2007]**
>
> In the above problem put $n = 4$ we get the solution.

Example 1.3b(5) : [Type 1] Find the complete integral of $p - q = 0$ **[A.U. N/D 2008]**

Solution : Given $p - q = 0$ ... (1)

This equation is of the form $f(p, q) = 0$ ... (2)

Hence, the trial solution is $z = ax + by + c$ ... (3)

To get the complete integral (solution)

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

$\therefore$ (1) $\Rightarrow$ $a - b = 0$

$b = a.$

Hence, the complete integral is $z = ax + ay + c$.

Hence, number of a.c. = number of I.V.

[We can get general solution easily by using Lagrange's idea]

EXERCISE 1.3(b) Type 1.

Find the complete integral of the following equations.

1. $pq = 1$ [Ans. $z = ax + \frac{1}{a}y + c$]

2. $pq = 4$ [Ans. $z = ax + \frac{4}{a}y + c$]

3. $pq + p + q = 0$ [Ans. $z = ax - \frac{a}{a+1}y + c$]

4. $p^2 + q^2 = 4$ [Ans. $z = ax \pm (\sqrt{4-a^2})\ y + c$]

5. $p^2 + q^2 - 4pq = 0$ [Ans. $z = ax + a(2 \pm \sqrt{3})y + c$]

6. $3p^2 - 2q^2 = 4pq$ [Ans. $z = ax + a\left[\frac{-1 \pm \sqrt{10}}{2}\right]y + c$]

7. $p^2 + q^2 = m^2$ [Ans. $ax \pm (\sqrt{m^2 - a^2})y + c$]

8. $p^2 - 2pq + 3q = 5$ [Ans. $z = ax + \left(\frac{5-a^2}{3-2a}\right)y + c$]

Problems based on Type 2.

TYPE 2. Clairaut's form $z = px + qy + f(p, q)$

Example 1.3b(6) : Solve $z = px + qy + pq$ and classify the following integrals (i) $z = ax + by + ab$; (ii) $z = 2x + 3y + 6$ (iii) $z + xy = 0$

[A.U. N/D 2008]

Solution : Given : $z = px + qy + pq$

This equation is of the form $z = px + qy + f(p,q)$

Therefore, the complete integral is $z = ax + by + f(a,b)$

$$\text{i.e., } z = ax + by + ab \quad \dots (1)$$

To find the singulár integral

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + b \qquad [\because \frac{\partial z}{\partial a} = 0]$$

$$\Rightarrow \quad b = -x \quad \dots (2)$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + a \qquad [\because \frac{\partial z}{\partial b} = 0]$$

$$\Rightarrow \quad a = -y \quad \dots (3)$$

$$\therefore (1) \Rightarrow \quad z = (-y)x + (-x)y + (-y)(-x)$$

$$= -xy - xy + xy$$

$$z = -xy$$

i.e., $z + xy = 0$ ... (4) which is the singular integral.

To get the general integral,

put $b = \phi(a)$ in (1), we get

$$z = ax + \phi(a)y + a\phi(a) \quad \dots (5)$$

differentiating (5) p.w.r.to a, we get

$$0 = x + \phi'(a)y + a\phi'(a) + \phi(a) \quad \dots (6)$$

Eliminating 'a' between (5) & (6), we get the general solution.

Classification of integrals

(i) $z = ax + by + ab$ is a complete integral.

(ii) $z = 2x + 3y + 6$ is a particular integral. Since, Here $a = 2$, $b = 3$

(iii) $z + xy = 0$ is a singular integral.

Example 1.3b(7) : Solve $z = px + qy + p^2 - q^2$. **[AU, April 2001]**

[A.U. N/D 2003, A/M 2008, M/J 2006]

[A.U Tvli N/D 2010, Trichy N/D 2010][A.U N/D 2014, R-13]

Solution : Given : $z = px + qy + p^2 - q^2$

This equation is of the form $z = px + qy + f(p,q)$ [Clairaut's type]

Therefore, the complete integral is $z = ax + by + a^2 - b^2$ ... (1)

Since the number of a.c. = number of I.V

To find the singular integral.

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + 2a - 0 \qquad [\because \frac{\partial z}{\partial a} = 0]$$

$$\Rightarrow \quad a = \frac{-x}{2} \qquad \text{... (2)}$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + 0 - 2b \qquad [\because \frac{\partial z}{\partial b} = 0]$$

$$\Rightarrow \quad b = \frac{y}{2} \qquad \text{... (3)}$$

$$\therefore (1) \Rightarrow \quad z = \left(\frac{-x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$4z = -2x^2 + 2y^2 + x^2 - y^2$$

$4z = -x^2 + y^2$ is the singular solution.

To get the general integral.

put $b = \phi(a)$ in (1), we get

$$z = ax + \phi(a)y + a^2 - [\phi(a)]^2 \qquad \text{... (4)}$$

differentiating (4) p.w.r. to 'a', we get

$$0 = x + \phi'(a)y + 2a - 2\phi(a)\phi'(a) \qquad \text{... (5)}$$

Eliminating 'a' between (4) & (5), we get the general solution.

Example 1.3b(8) : Find the complete integral of $(p - q)(z - px - qy) = 1$.

Solution : Given : $(p - q)(z - px - qy) = 1$

$$z - px - qy = \frac{1}{p - q}$$

$$z = px + qy + \frac{1}{p - q}$$

This equation is of the form $z = px + qy + f(p, q)$ [Clairaut's equation]

Therefore, the complete integral is

$$z = ax + by + \frac{1}{a - b}$$

Since, the number of a.c. = number of I.V

Example 1.3b(9) : Find the singular integral of $z = px + qy + p^2$

[A.U. CBT N/D 2011]

Solution : Given : $z = px + qy + p^2$

This equation is of the form $z = px + qy + f(p, q)$ [Clairaut's type]

Therefore, the complete integral is $z = ax + by + a^2$ ... (1)

Since the number of a.c. = number of I.V

To get the singular integral.

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + 2a \qquad [\because \frac{\partial z}{\partial a} = 0]$$

$$\Rightarrow \quad a = -\frac{x}{2} \qquad \text{... (2)}$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + 0 \qquad [\because \frac{\partial z}{\partial b} = 0]$$

$$\Rightarrow \quad y = 0$$

$$\Rightarrow \quad by = 0 \quad \text{... (3)}$$

$$(1) \Rightarrow \quad z = \left(\frac{-x}{2}\right)x + 0 + \left(\frac{-x}{2}\right)^2$$

$$z = \frac{-x^2}{2} + \frac{x^2}{4}$$

$$z = -\frac{x^2}{4}$$

$$4z = -x^2$$

$4z + x^2 = 0$ is the singular integral.

Example 1.3b(10) : Solve $z = px + qy + p^2q^2$. **[A.U N/D 2009]**

[A.U A/M 2015 R-13][A.U N/D 2018-A R-17][A.U A/M 2019 R-13, R-8]

Solution : Given : $z = px + qy + p^2q^2$ **[A.U N/D 2018 R-8]**

This is Clairaut's form,

The complete solution is

$$z = ax + by + a^2b^2 \quad \text{... (1)}$$

Since, the number of a.c. = number of I.V

To get the singular integral,

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + 2ab^2 \qquad \left[\because \frac{\partial z}{\partial a} = 0\right]$$

$$x = -2ab^2 \quad \text{... (2)}$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + 2a^2b \qquad \left[\because \frac{\partial z}{\partial b} = 0\right]$$

$$y = -2a^2b \quad \text{... (3)}$$

$$(1) \Rightarrow \quad z = ax + by + a^2b^2$$

$$= a(-2ab^2) + b(-2a^2b) + a^2b^2$$

$$= -2a^2b^2 - 2a^2b^2 + a^2b^2$$

$$z = -3a^2b^2$$

$$\Rightarrow \quad z^3 = -27a^6b^6 \quad \text{... (4)}$$

To find : a^6b^6

$$x^2y^2 = (-2ab^2)^2(-2a^2b)^2 = (4a^2b^4)(4a^4b^2)$$

$$= 16a^6b^6 \quad \text{... (5)}$$

$$a^6b^6 = \frac{1}{16}x^2y^2$$

$$(4) \Rightarrow \quad z^3 = -27\left(\frac{x^2y^2}{16}\right)$$

$$\Rightarrow \quad 16z^3 = -27x^2y^2$$

$16z^3 + 27x^2y^2 = 0$ is the singular solution.

To get the general integral.

put, $b = \phi(a)$ in (1)

$$z = ax + \phi(a)y + a^2[\phi(a)]^2 \quad \text{... (6)}$$

differentiating (6) w.r.to a, we get

$$0 = x + \phi'(a)y + a^2[2\phi(a)\phi'(a)] + [\phi(a)]^2 2a \quad \text{... (7)}$$

eliminate 'a' between (6) & (7), we get the general solution.

Example 1.3b(11) : Solve $z = px + qy + \sqrt{1 + q^2 + p^2}$.

[A.U. A/M 2004 PT][A.U. Tvli N/D 2009][A.U. CBT Dec. 2009]

[A.U. N/D 2011][A.U M/J 2013, N/D 2013][A.U N/D 2015 R-8]

Solution : Given : $z = px + qy + \sqrt{1+q^2+p^2}$ **[A.U. M/J 2016 R13]**

This is of the form $z = px + qy + f(p,q)$

[Clairaut's form]

Hence, the **complete integral** is $z = ax + by + \sqrt{1 + a^2 + b^2}$ ... (1)

where a and b are arbitrary constants.

Since the number of a.c. = number of I.V

To get the singular integral :

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + \frac{1}{2\sqrt{1+a^2+b^2}}(2a) \qquad \left[\because \frac{\partial z}{\partial a} = 0\right]$$

i.e.,
$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}} \Rightarrow \frac{a}{\sqrt{1+a^2+b^2}} = -x$$

$$\Rightarrow \boxed{a = -x\sqrt{1+a^2+b^2}} \quad \ldots (2)$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + \frac{1}{2\sqrt{1+a^2+b^2}}(2b) \qquad \left[\because \frac{\partial z}{\partial b} = 0\right]$$

i.e.,
$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}} \Rightarrow \frac{b}{\sqrt{1+a^2+b^2}} = -y$$

$$\Rightarrow \boxed{b = -y\sqrt{1+a^2+b^2}} \quad \ldots (3)$$

$$(1) \Rightarrow z = -x^2\sqrt{1+a^2+b^2} - y^2\sqrt{1+a^2+b^2} + \sqrt{1+a^2+b^2}$$

$$z = (1-x^2-y^2)\sqrt{1+a^2+b^2} \quad \ldots (4)$$

To find : $\sqrt{1+a^2+b^2}$

$$(1+a^2+b^2) = 1 + x^2(1+a^2+b^2) + y^2(1+a^2+b^2) \quad \text{by (2) \& (3)}$$

$$(1+a^2+b^2) - x^2(1+a^2+b^2) - y^2(1+a^2+b^2) = 1$$

$$(1+a^2+b^2)(1-x^2-y^2) = 1$$

$$1+a^2+b^2 = \frac{1}{1-x^2-y^2}$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\therefore (4) \Rightarrow z = (1-x^2-y^2)\frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$x^2+y^2+z^2 = 1$ is the singular solution.

To get the general integral,

put $b = \phi(a)$ in (1), we get

$$z = ax + \phi(a)y + \sqrt{1+a^2+[\phi(a)]^2} \quad \ldots (5)$$

differentiating (5) p.w.r. to 'a', we get

$$0 = x + \phi'(a)y + \frac{[2a + 2\phi(a)\phi'(a)]}{2\sqrt{1+a^2+[\phi(a)]^2}} \quad \ldots (6)$$

eliminate 'a' between (5) & (6), we get the general solution.

Example 1.3.b(12) : Solve $z = px + qy + \sqrt{1-p^2-q^2}$

Solution : Given : $z = px + qy + \sqrt{1-p^2-q^2}$

This is of the form $z = px + qy + f(p,q)$ [Clairaut's form]

Hence, the complete integral is $z = ax + by + \sqrt{1-a^2-b^2}$ ... (1)

where a and b are arbitrary constants.

To get the singular integral :

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + \frac{1}{2\sqrt{1-a^2-b^2}}(-2a) \qquad \left[\because \frac{\partial z}{\partial a} = 0\right]$$

$$0 = x - \frac{a}{\sqrt{1-a^2-b^2}}$$

i.e.,
$$\boxed{a = x\sqrt{1-a^2-b^2}} \quad \ldots (2)$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + \frac{1}{2\sqrt{1-a^2-b^2}}(-2b) \qquad \left[\because \frac{\partial z}{\partial b} = 0\right]$$

$$0 = y - \frac{b}{\sqrt{1-a^2-b^2}}$$

i.e., $$\boxed{b = y\sqrt{1-a^2-b^2}} \quad \text{.. (3)}$$

(1) $\Rightarrow$ $$z = x^2\sqrt{1-a^2-b^2} + y^2\sqrt{1-a^2-b^2} + \sqrt{1-a^2-b^2}$$

$$z = (1+x^2+y^2)\sqrt{1-a^2-b^2} \quad \text{... (4)}$$

To find : $\sqrt{1-a^2-b^2}$

$(1-a^2-b^2) = 1 - x^2(1-a^2-b^2) - y^2(1-a^2-b^2)$ by (2) & (3)

$(1+x^2+y^2)(1-a^2-b^2) = 1$

$$1 - a^2 - b^2 = \frac{1}{1+x^2+y^2}$$

$$\Rightarrow \quad \sqrt{1-a^2-b^2} = \frac{1}{\sqrt{1+x^2+y^2}}$$

(4) $\Rightarrow$ $$z = (1+x^2+y^2)\frac{1}{\sqrt{1+x^2+y^2}}$$

$$z = \sqrt{1+x^2+y^2}$$

$$\Rightarrow \quad z^2 = 1 + x^2 + y^2$$

$\Rightarrow$ $z^2 - x^2 - y^2 = 1$ is the singular solution.

To get the general integral,

put, $b = \phi(a)$ in (1), we get

$$z = ax + \phi(a)y + \sqrt{1-a^2-[\phi(a)]^2} \quad \text{... (4)}$$

differentiating (4) p.w.r. to 'a', we get

$$0 = x + \phi'(a)y + \frac{[-2a - 2\phi(a)\phi'(a)]}{2\sqrt{1-a^2-[\phi(a)]^2}} \quad \text{... (5)}$$

eliminate 'a' between (4) & (5), we get the general solution.

Example 1.3b(13) : Obtain the complete solution of the equation $z = px + qy - 2\sqrt{pq}$. **[A.U. O/N 1996][A.U. CBT Dec. 2008]**

Solution : Given : $z = px + qy - 2\sqrt{pq}$ **[A.U. Tvli N/D 2011]**

This is of the form $z = px + qy + f(p, q)$ [Clairaut's form]

Hence, the complete solution is

$$z = ax + by - 2\sqrt{ab}$$ where a and b are arbitrary constants.

Since, the number of a.c. = number of I.V

Example 1.3b(14) : Solve $z = px + qy + (pq)^{3/2}$. **[A.U N/D 2016 R-13]**

(OR) Solve $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$ **[A.U, May, 2000]**

[A.U M/J 2007][A.U CBT Dec. 2008]

Solution : Given : $z = px + qy + (pq)^{3/2}$

This is of the form $z = px + qy + f(p, q)$ [Clairaut's form]

Hence, the **complete integral** is

$$z = ax + by + (ab)^{3/2} \quad \text{... (1)}$$

Since, the number of a.c. = number of I.V

To find the singular integral,

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + \frac{3}{2}a^{1/2}b^{3/2} \qquad \left[\because \frac{\partial z}{\partial a} = 0\right]$$

$$\Rightarrow \quad x = -\frac{3}{2}a^{1/2}b^{3/2} \quad \text{... (2)}$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + \frac{3}{2}a^{3/2}b^{1/2} \qquad \left[\because \frac{\partial z}{\partial b} = 0\right]$$

$$\Rightarrow \quad y = -\frac{3}{2}a^{3/2}b^{1/2} \qquad \ldots (3)$$

$$(1) \Rightarrow \quad z = ax + by + a^{3/2}b^{3/2}$$

$$= a\left(\frac{-3}{2}a^{1/2}b^{3/2}\right) + b\left(\frac{-3}{2}a^{3/2}b^{1/2}\right) + a^{3/2}b^{3/2}$$

$$= -\frac{3}{2}a^{3/2}b^{3/2} - \frac{3}{2}a^{3/2}b^{3/2} + a^{3/2}b^{3/2}$$

$$= -3a^{3/2}b^{3/2} + a^{3/2}b^{3/2}$$

$$= -2a^{3/2}b^{3/2}$$

$$z^2 = 4a^3b^3$$

$$= 4(ab)^3 \qquad \ldots (4)$$

To find : $(ab)^3$

$$xy = \frac{9}{4}a^2b^2 \qquad \text{by (2) \& (3)}$$

$$(xy)^{1/2} = \frac{3}{2}ab$$

$$\Rightarrow \quad ab = \frac{2}{3}(xy)^{1/2}$$

$$\Rightarrow \quad (ab)^3 = \frac{8}{27}(xy)^{3/2}$$

$$(4) \Rightarrow \quad z^2 = 4\left(\frac{8}{27}\right)(xy)^{3/2}$$

$$z^2 = \frac{32}{27}x^{3/2}y^{3/2}$$ is the singular integral.

Example 1.3b(15) : Find the singular integral of

$$z = px + qy + p^2 + q^2$$ **[A.U A/M 2001] [A.U N/D 2016 R-8]**

Solution : $z = px + qy + p^2 + q^2 \qquad \ldots (1)$

This equation is of the form $z = px + qy + f(p, q)$ [Clairaut's type]

Therefore, the complete integral is $z = ax + by + a^2 + b^2$

Since, the number of a.c. = number of I.V

To get the singular integral differentiating (1) p.w.r.to 'a' & 'b' we get

$0 = x + 2a$	$0 = y + 2b$
$\Rightarrow a = \frac{-x}{2}$	$\Rightarrow b = \frac{-y}{2}$

$$(1) \Rightarrow \quad z = \frac{-x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4} = \frac{-x^2}{4} - \frac{y^2}{4}$$

$$x^2 + y^2 = -4z$$ is the singular solution.

Example 1.3b(16) : Find the complete integral of the partial differential equation $(1 - x)p + (2 - y)q = 3 - z$ **[A.U N/D 2006]**

Solution : Given

$$(1 - x)p + (2 - y)q = 3 - z$$
$$p - px + 2q - qy = 3 - z$$
$$z = px + qy - p - 2q + 3$$

This equation is of the form $z = px + qy + f(p, q)$ [Clairaut's type]

Hence, the complete integral is

$$z = ax + by - a - 2b + 3$$

Since, the number of a.c. = number of I.V

Example 1.3b(17) : Find the singular integral of

$$z = px + qy + p^2 + pq + q^2$$ **[A.U N/D 2006, N/D 2012]**

Solution : Given : $z = px + qy + p^2 + pq + q^2$ **[A.U A/M 2017 R-13]**

i.e., $z = px + qy + f(p, q)$

This equation is of the Clairaut's form.

Hence, the complete solution is

$$z = ax + by + a^2 + ab + b^2 \qquad \ldots (1)$$

Since, the number of a.c. = number of I.V

To find the singular integral.

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + 2a + b + 0 \qquad [\because \frac{\partial z}{\partial a} = 0]$$

$$0 = x + 2a + b$$

$$\Rightarrow \quad x = -2a - b \qquad \dots (2)$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + 0 + a + 2b \qquad [\because \frac{\partial z}{\partial b} = 0]$$

$$\Rightarrow \quad y = -a - 2b \qquad \dots (3)$$

$$(1) \Rightarrow \quad z = ax + by + a^2 + ab + b^2$$

$$z = a(-2a - b) + b(-a - 2b) + (a^2 + ab + b^2)$$

$$= -2a^2 - ab - ab - 2b^2 + (a^2 + ab + b^2)$$

$$= -2a^2 - 2ab - 2b^2 + (a^2 + ab + b^2)$$

$$= -2(a^2 + ab + b^2) + (a^2 + ab + b^2)$$

$$= -(a^2 + ab + b^2) \qquad \dots (4)$$

To find : $a^2 + ab + b^2$

$$x^2 + y^2 = (4a^2 + b^2 + 4ab) + (a^2 + 4b^2 + 4ab) \text{ by (2) \& (3)}$$

$$= 5(a^2 + ab + b^2) + 3ab$$

$$xy = 2a^2 + 4ab + ab + 2b^2 \qquad \text{by (2) \& (3)}$$

$$= 2(a^2 + ab + b^2) + 3ab$$

$$x^2 + y^2 - xy = 3(a^2 + ab + b^2)$$

$$\Rightarrow a^2 + ab + b^2 = \frac{1}{3}(x^2 + y^2 - xy)$$

$$(4) \Rightarrow \quad z = -\frac{1}{3}(x^2 + y^2 - xy)$$

$$3z = -x^2 - y^2 + xy$$

$$3z + x^2 + y^2 - xy = 0 \text{ is the singular integral.}$$

Example 1.3b(18) : Solve the equation

$(pq - p - q)(z - px - qy) = pq$

Solution : Rewriting the given equation as

$$z = px + qy + \frac{pq}{pq - p - q}$$

This is of the form $z = px + qy + f(p, q)$ [Clairaut's form]

Hence, the complete solution is

$$z = ax + by + \frac{ab}{ab - a - b} \qquad \dots (1)$$

Since, the number of a.c. = number of I.V

To find the singular solution of (1).

differentiating (1) p.w.r. to 'a', we get

$$0 = x + 0 + \frac{(ab - a - b)(b) - ab(b - 1)}{(ab - a - b)^2} \qquad [\because \frac{\partial z}{\partial a} = 0]$$

$$0 = x + \frac{ab^2 - ab - b^2 - ab^2 + ab}{(ab - a - b)^2}$$

$$\Rightarrow \quad x = \frac{b^2}{(ab - a - b)^2}$$

$$\Rightarrow \quad \sqrt{x} = \frac{b}{ab - a - b} \qquad \dots (2)$$

differentiating (1) p.w.r. to 'b', we get

$$0 = 0 + y + \frac{(ab - a - b)(a) - ab(a - 1)}{(ab - a - b)^2} \qquad [\because \frac{\partial z}{\partial b} = 0]$$

$$y = \frac{a^2}{(ab - a - b)^2} \Rightarrow \sqrt{y} = \frac{a}{ab - a - b} \qquad \dots (3)$$

$$(1) \Rightarrow z = ax + by + \frac{ab}{ab-a-b}$$

$$= a\left[\frac{b^2}{(ab-a-b)^2}\right] + b\left[\frac{a^2}{(ab-a-b)^2}\right] + \frac{ab}{ab-a-b}$$

$$= \frac{ab^2 + ba^2 + ab(ab-a-b)}{(ab-a-b)^2}$$

$$= \frac{ab^2 + ba^2 + a^2b^2 - a^2b - ab^2}{(ab-a-b)^2}$$

$$z = \frac{a^2b^2}{(ab-a-b)^2} = \left[\frac{ab}{(ab-a-b)}\right]^2 \quad \text{.. (4)}$$

To find : $\dfrac{\mathbf{ab}}{\mathbf{ab-a-b}}$

$$1 + \sqrt{x} + \sqrt{y} = 1 + \frac{b}{ab-a-b} + \frac{a}{ab-a-b} \quad \text{by (2) \& (3)}$$

$$= \frac{ab-a-b+b+a}{ab-a-b}$$

$$= \frac{ab}{ab-a-b}$$

$$(4) \Rightarrow z = (1 + \sqrt{x} + \sqrt{y})^2$$

i.e., $z = (1 + \sqrt{x} + \sqrt{y})^2$ is the singular integral.

EXERCISE 1.3(b) [Type 2]

Find the complete solution and singular integral of the following

1. $z = px + qy = \log pq$

   [**Ans.** C.S is $z = ax + by + \log ab$
   S.I is $z = -2 - \log(xy)$]

2. $z = px + qy + 3(pq)^{1/3}$

   [**Ans.** C.S is $z = ax + by + 3(ab)^{1/3}$
   S.I is $xyz = 1$]

3. $z = px + qy + 2\sqrt{pq}$

   [**Ans.** C.S is $z = ax + by + 2\sqrt{ab}$
   S.I is $xy = 1$]

4. $z = px + qy + \dfrac{p}{q} - p$

   [**Ans.** C.S is $z = ax + by + \dfrac{a}{b} - a$
   S.I is $z = \dfrac{y}{1-x}$]

5. (i) $z = px + qy + \sqrt{p^2 + q^2 + 16}$ **[A.U O/N 2002]**

   [**Ans.** C.S is $z = ax + by + \sqrt{a^2 + b^2 + 16}$
   S.I is $x^2 + y^2 + \dfrac{z^2}{16} = 1$]

5. (ii) $z = px + qy + \sqrt{p^2 + q^2 + k^2}$

   Ans. C.S is $z = ax + by + \sqrt{a^2 + b^2 + k^2}$
   S.I is $x^2 + y^2 + \dfrac{z^2}{k^2} = 1$

5. (iii) $z = px + qy + \sqrt{1 + p^2 - q^2}$

   Ans. C.S is $z = ax + by + \sqrt{1 + a^2 - b^2}$
   S.I is $x^2 - y^2 + z^2 = 1$

5. (iv) $z = px + qy + \sqrt{1 - p^2 + q^2}$

   Ans. C.S is $z = ax + by + \sqrt{1 - a^2 + b^2}$
   S.I is $z^2 - x^2 + y^2 = 1$

6. Verify that $z = ax + by + \dfrac{a}{b}$ is a complete solution of the equation $z = px + qy + \dfrac{p}{q}$. Show that the singular solution is $zx + y = 0$

7. $z = px + qy + \frac{q}{p} - p$

[Ans. C.S is $z = ax + by + \frac{b}{a} - a$

S.I. is $yz = 1 - x$]

8. $z = px + qy + c\sqrt{1 + p^2 + q^2}$

[Ans. C.S is $z = ax + by + c\sqrt{1 + a^2 + b^2}$

S.I. is $x^2 + y^2 + z^2 = c^2$]

Type 3 (a) Equation of the type f (z, p, q) = 0

i.e., equations not containing x and y.

Let z be a function of u where

$$u = x + ay$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

Then $$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}(a) = a\frac{dz}{du}$$

Substitute the values of p and q in the given equation $f(z, p, q) = 0$ it becomes

$$f\left[z, \frac{dz}{du}, a\frac{dz}{du}\right] = 0$$ which is an ordinary differential equation the first order

Solving for $\frac{dz}{du}$, we obtain $\frac{dz}{du} = \phi(z, a)$ say

$$\frac{dz}{\phi(z, a)} = du$$

$$\int \frac{dz}{\phi(z, a)} = \int du$$

$$f(z, a) = u + b$$
$$= x + ay + b$$

This is the complete integral, singular and general integral are found out as usual.

Rule : Assume $u = x + ay$; replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ in the given equation and then solve the ordinary diff. Equation obtained.

Type 3 (b) Equation of the type f (x, p, q) = 0 ... (1)

i.e., equations not containing y and z.

Let z is a function of x and y

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy$$

Assume that, $q = a$

Then the equation becomes $f(x, p, a) = 0$

Solving for p, we obtain $p = \phi(x, a)$

$$dz = \phi(x, a)\,dx + a\,dy$$

Integrating we get,

$$z = \int \phi(x, a)\,dx + \int a\,dy$$

$$z = f(x, a) + ay + b \quad \text{... (2)}$$

equation (2) is the complete integral of (1) since it contains two arbitrary constants a & b.

Type 3 (c) Equation of the type f (y, p, q) = 0 ... (1)

i.e., equations not containing x and z.

Assume, $p = a$ and proceed as Type 3 (b)

The complete integral will be of the form

$$z = ax + f(y, a) + b$$

Problems based on types 3(a) $f(z, p, q) = 0$

Example 1.3b(18) : Solve $p(1+q) = qz$.

[A.U. Dec., 1998] [A.U. M/J 2007] [A.U CBT N/D 2010]

Solution : Given : $p(1+q) = qz$ ... (1)

This equation is of the form $f(z,p,q) = 0$ **[Type 3(a)]**

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

$$\therefore (1) \Rightarrow \quad \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$1 + a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{dz}{du} = \frac{az-1}{a}$$

$$\Rightarrow \quad \frac{du}{dz} = \frac{a}{az-1}$$

$$du = \frac{a}{az-1}dz$$

Integrating on both sides, we get

$$u = \log(az-1) - b$$

$$\Rightarrow u + b = \log(az-1)$$

Hence, the complete solution is

$$x + ay + b = \log[(az-1)]$$

Since, the number of a.c. = number of I.V

General integral can be found out in a usual way.

Example 1.3b(19) : Solve $p(1+q^2) = q(z-a)$. **[AU, Oct., 1996]**

[A.U Trichy N/D 2009]

Solution : Given : $p(1+q^2) = q(z-a)$ ... (1)

This equation is of the form $f(z,p,q) = 0$ **[Type 3(a)]**

Let $u = x + by$

$$p = \frac{dz}{du}$$

$$q = b\frac{dz}{du}$$

substituting these values of p & q in (1), we have

$$\frac{dz}{du}\left[1 + b^2\left(\frac{dz}{du}\right)^2\right] = b\frac{dz}{du}(z-a)$$

$$1 + b^2\left(\frac{dz}{du}\right)^2 = b(z-a)$$

$$b^2\left(\frac{dz}{du}\right)^2 = bz - ab - 1$$

$$\frac{dz}{du} = \frac{1}{b}\sqrt{bz-ab-1}$$

$$\int \frac{b\,dz}{\sqrt{bz-ab-1}} = \int du$$

$$2\sqrt{bz-ab-1} = u + c$$

$$4(bz-ab-1) = (u+c)^2$$

Hence, the complete solution is

$$4(bz-ab-1) = (x+by+c)^2$$

[Here, a is a given constant, b and c are arbitrary constants]

Since, the number of a.c. = number of I.V

General integral can be found out in a usual way.

Example 1.3b(20) : Solve $z^2 = 1 + p^2 + q^2$. **[AU, April, 1996] [A.U. A/M 2003][A.U. CBT Dec. 2008]**

Solution : Given : $z^2 = 1 + p^2 + q^2$ ... (1)

The given problem is of the type $f(z, p, q) = 0$ **[Type 3(a)]**

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

substitute in (1), we get

$$z^2 = 1 + \left[\frac{dz}{du}\right]^2 + a^2\left[\frac{dz}{du}\right]^2$$

i.e., $$z^2 - 1 = \left[\frac{dz}{du}\right]^2 [1 + a^2]$$

$$\left[\frac{dz}{du}\right]^2 = \frac{z^2 - 1}{1 + a^2}$$

$$\frac{dz}{du} = \sqrt{\frac{z^2 - 1}{1 + a^2}}$$

$$\frac{dz}{\sqrt{z^2 - 1}} = \frac{du}{\sqrt{1 + a^2}}$$

Integrating on both sides, we get

$$\cosh^{-1} z = \frac{1}{\sqrt{1 + a^2}} u + b$$

Hence, the complete solution is

$$\cosh^{-1} z = \frac{1}{\sqrt{1 + a^2}}(x + ay) + b$$

Since, the number of a.c. = number of I.V

General integral can be found out in a usual way.

Example 1.3b(21) : Solve $9(p^2z + q^2) = 4$. **[A.U. A/M. 2003 P.T] [A.U N/D 2014 R-2008]**

Solution : Given : $9(p^2 z + q^2) = 4$ ... (1)

This equation is of the form $f(z, p, q) = 0$ **[Type 3(a)]**

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

substituting in (1), we get

$$9\left[\left(\frac{dz}{du}\right)^2 z + \left(a\frac{dz}{du}\right)^2\right] = 4$$

$$\Rightarrow \left[\frac{dz}{du}\right]^2 = \frac{4}{9(z + a^2)}$$

$$\frac{dz}{du} = \frac{2}{3}\frac{1}{\sqrt{z + a^2}}$$

$$3\sqrt{z + a^2}\, dz = 2\, du$$

Integrating on both sides, we get

$$3\frac{(z + a^2)^{3/2}}{(3/2)} = 2u + 2b$$

$$\Rightarrow (z + a^2)^{3/2} = x + ay + b$$

Hence, the complete solution is

$$\Rightarrow (z + a^2)^3 = (x + ay + b)^2$$

Since, the number of a.c. = number of I.V

General integral can be found out in a usual way.

Example 1.3b(22) : Solve $z = p^2 + q^2$. **[A.U A/M 2017 R-8]**

Solution : Given : $z = p^2 + q^2$ ... (1)

This equation is of the form $f(z, p, q) = 0$ **[Type 3(a)]**

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

substituting in (1), we get

$$z = \left[\frac{dz}{du}\right]^2 + a^2\left[\frac{dz}{du}\right]^2$$

$$z = \left[\frac{dz}{du}\right]^2 [1 + a^2]$$

$$\frac{z}{1 + a^2} = \left[\frac{dz}{du}\right]^2$$

$$\frac{dz}{du} = \sqrt{\frac{z}{1 + a^2}}$$

$$\frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{1 + a^2}}$$

Integrating on both sides, we get

$$2\sqrt{z} = \frac{u}{\sqrt{1 + a^2}} + b$$

Hence, the complete solution is

$$2\sqrt{z} = \frac{x + ay}{\sqrt{1 + a^2}} + b$$

Since, the number of a.c. = number of I.V

General integral can be found out in a usual way.

Example 1.3b(23) : Solve $Ap + Bq + Cz = 0$.

Solution : Given $Ap + Bq + Cz = 0$ ...(1)

This is of the form $f(z, p, q) = 1$ **[Type 3(a)]**

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

substituting in (1), we get

$$A\frac{dz}{du} + Ba\frac{dz}{du} + Cz = 0$$

$$\frac{dz}{du} = \frac{-Cz}{A + Ba}$$

$$\frac{dz}{z} = -\frac{C}{A + Ba}du$$

Integrating on both sides, we get

$$\log z = -\frac{C}{A + Ba}(x + ay) + b$$

is the complete solution. [Here, A, B, C are given constants]

Since, the number of a.c. = number of I.V

General integral can be found out in a usual way.

Example 1.3b(24) : Solve $p(1-q^2) = q(1-z)$. **[A.U. M/J 2006]** **[A.U CBT Dec. 2009]**

Solution : Given : $p(1-q^2) = q(1-z)$ ... (1)

This equation is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

$$(1) \Rightarrow \frac{dz}{du}\left[1 - a^2\left(\frac{dz}{du}\right)^2\right] = a\frac{dz}{du}[1-z]$$

$$1 - a^2\left(\frac{dz}{du}\right)^2 = a(1-z)$$

$$1 - a(1-z) = a^2\left(\frac{dz}{du}\right)^2$$

$$1 - a + az. = a^2\left(\frac{dz}{du}\right)^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{1}{a^2}[1 - a + az]$$

$$\frac{dz}{du} = \frac{1}{a}\sqrt{1-a+az}$$

$$\int \frac{a}{\sqrt{1-a+az}}\,dz = \int du$$

$$2\sqrt{1-a+az} = u + b$$

Squaring in both sides, we get

$$4(1-a+az) = (u+b)^2$$

$$4(1-a+az) = (x+ay+b)^2$$

which gives the complete integral of the given equation.

Since, the number of a.c. = number of I.V

General integral can be found out in a usual way.

EXERCISE 1.3(b) - Type 3 (Case i) f(z, p, q) = 0

Find the complete integrals of the following :

1. $p^3 + q^3 = 8z$ **[Ans.** $(1+a^3)z^2 = \frac{64}{27}(x+ay+b)^3$ **]**

2. $z^2(p^2+q^2+1) = 1$ **[Ans.** $-\sqrt{1-z^2} = \frac{1}{\sqrt{1+a^2}}(x+ay)+b$ **]**

3. $p^2 + pq = z^2$ **[Ans.** $\log z = \frac{1}{\sqrt{1+a}}(x+ay)+b$ **]**

4. $pz = 1 + q^2$

   [Ans. $\frac{1}{4a^2}\left[\frac{z^2}{2} - \frac{z\sqrt{z^2-4a^2}}{2} + \frac{4a^2}{2}\cosh^{-1}\left(\frac{z}{2a}\right)\right] = \frac{x+ay}{2a^2} + b$ **]**

5. $z = pq$ **[Ans.** $(x+ay+b)^2 = 4az$ **]**

6. $z = p^2 - q^2$ **[Ans.** $2\sqrt{z} = \frac{1}{\sqrt{1-a^2}}(x+ay)+b$ **]**

7. $z = p + q$ **[Ans.** $x + ay = (1+a)\log z + b$ **]**

8. $q^2 = z^2p^2(1-p^2)$ **[Ans.** $z^2 = a^2 + (x+ay+b)^2$ **]**

9. $p^3 = qz$ **[Ans.** $4z = a(x+ay-b)^2$ **]**

10. $z^2(p^2+q^2+1) = c^2$ **[Ans.** $(1+a^2)(c^2-z^2) = (x+ay+b)^2$**]**

11. $9pqz^4 = 4(1+z^3)$ **[Ans.** $a(1+z^3) = (x+ay+b)^2$ **]**

Problems based on Type 3(b) $f(x, p, q) = 0$

Example 1.3b(25) : Solve $p = 2qx$. [A.U N/D 2018 ... [A.U N/D 2019, ...

Solution: Given: $p = 2qx$, this equation is of the form $f(x, p, q) = 0$ [Type 3...

Let $q = a$

Then $p = 2ax$

But, $dz = pdx + qdy$

$$dz = 2ax \, . \, dx + ady$$

Integrating on both sides, we get

$$z = ax^2 + ay + b \quad \dots (1)$$

equation (1) is the complete integral of the given equation.

Since, the number of a.c. = number of I.V

Differentiating partially w.r.to b, we get $1 = 0$.

Hence, there is no singular integral.

General integral can be found out in the usual way.

[We can solve this easily by using Lagrange's idea]

Example 1.3b(26) : Solve $q = px + p^2$.

Solution : Given : $q = px + p^2 \quad \dots (1)$

This equation is of the form $f(x, p, q) = 0$ [Type 3...

Assume $q = a$ (constant)

Then $p^2 + px - a = 0$

$$p = \frac{-x \pm \sqrt{x^2 + 4a}}{2}$$

We know that, $dz = pdx + qdy$

$$= \left[\frac{-x \pm \sqrt{x^2 + 4a}}{2}\right] dx + ady$$

Integrating on both sides, we get

$$z = \frac{-x^2}{4} \pm \frac{1}{2} \int \sqrt{x^2 + 4a} \, dx + ay + b$$

$$= \frac{-x^2}{4} \pm \frac{1}{2} \left[2a \sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) + \frac{x}{2}\sqrt{x^2 + 4a}\right] + ay + b$$

is the complete solution.

Since, the number of a.c. = number of I.V

Singular integral does not exist

General integral can be found out in the usual way.

Example 1.3b(27) : Solve $\sqrt{p} + \sqrt{q} = \sqrt{x}$.

Solution : Given : $\sqrt{p} + \sqrt{q} = \sqrt{x} \quad \dots (1)$

This equation is of the form $f(x, p, q) = 0$ **[Type 3(b)]**

Assume $q = a$ (constant)

$$(1) \Rightarrow \sqrt{p} + \sqrt{a} = \sqrt{x}$$

$$\sqrt{p} = \sqrt{x} - \sqrt{a}$$

$$p = (\sqrt{x} - \sqrt{a})^2$$

$$= x + a - 2\sqrt{ax}$$

We know that, $dz = pdx + qdy$

$$dz = [x + a - 2\sqrt{a}\sqrt{x}]\, dx + ady$$

Integrating on both sides, we get

$$z = \frac{x^2}{2} + ax - 2\sqrt{a}\,\frac{x^{3/2}}{(3/2)} + ay + b$$

Hence, the complete solution is

$$z = \frac{x^2}{2} + ax - \frac{4\sqrt{a}}{3} x^{3/2} + ay + b$$

Since, the number of a.c. = number of I.V

Example 1.3b(28) : Find the complete integral of $q = 2px$

[A.U. April, 2001, A.U CBT N/D 2010, A/M 2015 R-13]

Solution : Given : $q = 2px$

This equation is of the form $f(x, p, q) = 0$

Let $q = a$ Then $p = \frac{a}{2x}$

$$dz = p\,dx + q\,dy$$

But, $dz = \frac{a}{2x}dx + a\,dy$

Integrating on both sides, we get

$$\int dz = \int \frac{a}{2x}dx + \int a\,dy$$

Hence, the complete solution is

$$z = \frac{a}{2}\log x + ay + b$$

Since the number of a.c. = number of I.V

[We can solve this easily by Lagrange's idea to get the general integral

EXERCISE 1.3.b [Type 3 - Case (ii)] f(x, p, q) = 0

Find the complete integrals of the following :

1. $\sqrt{p} + \sqrt{q} = 2x$ [Ans. $z = \frac{(a+2x)^2}{6} + a^2y + b$]

Problems based on Type 3(c) f(y, p, q) = 0

Example 1.3b(29) : Solve pq = y.

Solution : Given : $pq = y$ ... (1)

This equation is of the form $f(y, p, q) = 0$ [Type 3(c)]

Assume $p = a$ (constant)

Then $aq = y$

$$q = \frac{y}{a}$$

$$dz = pdx + qdy$$

$$= adx + \frac{y}{a}dy$$

Integrating, $z = ax + \frac{y^2}{2a} + b$ is the complete solution

Since, the number of a.c. = number of I.V

differentiating p.w.r. to b, we get $0 = 1$ (absurd)

There is no singular integral

put $b = \phi(a)$,

$$z = ax + \frac{y^2}{2a} + \phi(a) \quad \text{... (2)}$$

differentiating (2) p.w.r. to a, we get

$$0 = x - \frac{y^2}{2a^2} + \phi'(a) \quad \text{... (3)}$$

Eliminate 'a' between (2) and (3) to get general solution.

Example 1.3b(30) : Solve $q = py + p^2$.

Solution : Given : $q = py + p^2$ ... (1)

This equation is of the form $f(y, p, q) = 0$ [Type 3(c)]

Assume $p = a$ (constant)

$\therefore$ (1) $\Rightarrow q = ay + a^2$

Since, $dz = pdx + qdy$

$$dz = adx + (ay + a^2)\,dy$$

Integrating on both sides, we get

$$z = ax + \frac{ay^2}{2} + a^2y + b \quad \text{... (2)}$$

differentiating p.w.r. to b, we get $0 = 1$ (absurd)

There is no singular integral

Let $b = \phi(a)$.

$$z = ax + \frac{ay^2}{2} + a^2y + \phi(a) \quad \text{... (3)}$$

differentiating (3) p.w.r. to a, we get

$$0 = x + \frac{y^2}{2} + 2ay + \phi'(a) \quad \dots (4)$$

Eliminate 'a' between (3) and (4), we get general solution.

EXERCISE 1.3.b [Type 3 - Case (iii)] f(y, p, q) = 0

Find the complete solution of the following :

1. $\sqrt{p} + \sqrt{q} = \sqrt{y}$ [**Ans.** $z = ax + \frac{y^2}{2} + ay - \frac{4}{3}\sqrt{a}\, y^{3/2} + b$]

2. $p = 2qy$ [**Ans.** $z = ax + a \log$

Type 4. Separable equations.

First order partial differential equations are separable.

It can be written as $f(x,p) = \phi(y,q)$

put $f(x,p) = \phi(y,q) = a$ say

Solving for p and q, we get $p = f_1(x,a)$ and $q = \phi_1(y,a)$

But $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

Hence $dz = pdx + qdy$

$$= f_1(x,a)\,dx + \phi_1(y,a)\,dy$$

Integrating on both sides, we get

$$z = \int f_1(x,a)\,dx + \int \phi_1(y,a)\,dy + b$$

This equation contains two arbitrary constants and hence it is the complete integral. The singular and general integrals are found out as usual.

Problems based on Type 4. Separable equations.

Example 1.3b(31) : Solve $p^2y(1+x^2) = qx^2$.

[A.U. A/M 2008, A.U Tvli M/J 201

Solution : Given : $p^2y(1+x^2) = qx^2 \quad \dots (1)$

The equation is separable

$$p^2 \frac{(1+x^2)}{x^2} = \frac{q}{y} = a, \text{ where } a \text{ is an arbitrary constant.}$$

Thus

$$p^2 \frac{1+x^2}{x^2} = a$$

$$p = \frac{x\sqrt{a}}{\sqrt{1+x^2}}$$

$$q = ay$$

We know that, $dz = pdx + qdy$

$$= \frac{x\sqrt{a}}{\sqrt{1+x^2}}\,dx + ay\,dy$$

Integrating on both sides, we get

$$z = \sqrt{a}\int \frac{x}{\sqrt{1+x^2}}\,dx + a\int y\,dy$$

$$= \sqrt{a}\sqrt{1+x^2} + \frac{1}{2}ay^2 + b$$

This is the complete integral where a and b are arbitrary constants.

Differentiating partially w.r.to b, we find that there is no singular integral.

Example 1.3b(32) : Find the complete solution of $p + q = \sin x + \sin y$.

Solution : Given : $p + q = \sin x + \sin y$ **[AU, Oct. 1996]**

The given differentiating equation can be written as

$$p - \sin x = \sin y - q$$

it is of the form $f(x,p) = \phi(y,q)$.

Let $p - \sin x = \sin y - q = a$ (say)

$$p - \sin x = a, \qquad \sin y - q = a$$

$$p = a + \sin x, \qquad q = \sin y - a$$

$$dz = pdx + qdy$$

$$dz = (a + \sin x)\,dx + (\sin y - a)\,dy$$

Integrating on both sides

$$z = ax - \cos x + (-\cos y - ay) + b$$

$$z = a(x - y) - \cos x - \cos y + b$$

is the complete integral.

[We can solve this easily by Lagrange's idea]

Example 1.3b(33) : Find the complete integral of pq = xy.

[A.U N/D Trichy 2009]

[A.U N/D 2015 R-8]

Solution : Given : $pq = xy$

Hence $\frac{p}{x} = \frac{y}{q}$

It is of the form $f(x,p) = \phi(y,q)$

Let $\frac{p}{x} = \frac{y}{q} = a$ (a is an arbitrary constant)

$\therefore p = ax$ and $q = \frac{y}{a}$

Hence, $dz = pdx + qdy$

$$dz = ax\,dx + \frac{y}{a}\,dy$$

Integrating on both sides, we get

$$z = a\frac{x^2}{2} + \frac{y^2}{2a} + b$$

$2az = a^2x^2 + y^2 + 2ab$ is the required complete integral.

Example 1.3b(34) : Find the complete integral of $\sqrt{p} + \sqrt{q} = 2x$.

Solution : Given : $\sqrt{p} + \sqrt{q} = 2x$

The given equation can be written as

$$\sqrt{p} - 2x = -\sqrt{q}$$

This is of the form $f(x,p) = \phi(y,q)$

Let $\sqrt{p} - 2x = -\sqrt{q} = a$ say

$$\sqrt{p} = a + 2x, \quad \sqrt{q} = -a$$

$$p = (a + 2x)^2, \quad q = a^2$$

Now, $dz = pdx + qdy$

$$= (a + 2x)^2\,dx + a^2\,dy$$

$z = \frac{(a + 2x)^3}{6} + a^2y + b$ is the complete integral.

Since, the number of a.c. = number of I.V

Example 1.3b(35) : Find the complete integral of $p^{-1}x + q^{-1}y = 1$.

Solution : Given : $p^{-1}x + q^{-1}y = 1$

The given equation can be written as

$$p^{-1}x - 1 = -q^{-1}y$$

This is of the form $f(x,p) = \phi(y,q)$

Let $p^{-1}x - 1 = -q^{-1}y = a$

$\therefore p = \frac{x}{a+1}$ and $q = \frac{-y}{a}$

Now, $dz = pdx + qdy$

$$dz = \frac{x}{a+1}\,dx - \frac{y}{a}\,dy$$

Integrating on both sides, we get

$z = \frac{x^2}{2(a+1)} - \frac{y^2}{2a} + b$ is the complete integral.

Since, the number of a.c. = number of I.V

Example 1.3b(36) : Solve the equation $yp = 2yx + \log q$.

[A.U., March, 1996 & May, 1996, P.T.]

Solution : Given : $yp = 2yx + \log q$

$$yp - 2yx = \log q$$

$$y(p - 2x) = \log q$$

$$(p - 2x) = \frac{1}{y}\log q = a \text{ say} \quad \ldots (1)$$

$$p - 2x = a \text{ and } \frac{1}{y}\log q = a$$

$$p = 2x + a \text{ and } \log q = ay$$

$$q = e^{ay}$$

Now, $dz = p\,dx + q\,dy$

$$dz = (2x + a)\,dx + e^{ay}\,dy \quad \ldots (2)$$

$$\int dz = \int (2x + a)\,dx + \int e^{ay}\,dy$$

$$z = 2\frac{x^2}{2} + ax + \frac{1}{a}e^{ay} + b$$

$$z = x^2 + ax + \frac{1}{a}e^{ay} + b \quad \ldots (3)$$

where a and b are arbitrary constants equation (3) is the complete solution of the given equation.

Since, the number of a.c. = number of I.V

differentiating (3) p. w.r.to b, we get

$$0 = 1 \text{ [absurd]}$$

there is no singular integral.

Put $b = \phi(a)$ in (3)

$$z = x^2 + ax + \frac{1}{a}e^{ay} + \phi(a) \quad \ldots (4)$$

differentiating partially. w.r.to a, we get

$$0 = 0 + x + \frac{1}{a}\left[\frac{e^{ay}}{y}\right] + e^{ay}\left[\frac{-1}{a^2}\right] + \phi'(a)$$

$$0 = x + \frac{1}{ay}e^{ay} - \frac{1}{a^2}e^{ay} + \varphi'(a) \quad \ldots (5)$$

Eliminate 'a' between (4) and (5), we get the general solution.

EXERCISE 1.3.b [Type 4]

Find the complete integral of the following :

1. $p^2 + q^2 = x + y$ [A.U. A/M. 2001]
   [Ans. $z = \frac{2}{3}(x + a)^{3/2} + \frac{2}{3}(y - a)^{3/2} + b$]
2. $p + q = x + y$ [A.U N/D 2018 R-13]
   [Ans. $2z = 2ax + x^2 + y^2 - 2ay + b$]
3. $p + q = x - y$ [A.U. Dec. 1996] [Ans. $z = ax + \frac{x^2}{2} - \frac{y^2}{2} - ay + b$]
4. $\sqrt{p} + \sqrt{q} = x + y$ [Ans. $z = \frac{(x + a)^3}{3} + \frac{(y - a)^3}{3} + b$
5. $p - x^2 = q + y^2$ [A.U N/D 2019, R17]
   [Ans. $z = ax + \frac{x^3}{3} - \frac{y^3}{3} + ay + b$]
6. $p + x = qy$ [Ans. $z = a(x + \log y) - \frac{x^2}{2} - b$]
7. $q = xyp^2$ [Ans. $z = 2\sqrt{ax} + \frac{ay^2}{2} + b$]
8. $pq + qx = y$ [Ans. $2az = 2a^2x - ax^2 + y^2 + b$]

1.3(3) Equations reducible to standard types :

A partial differential equation which is one of the four types.

1. $f(p, q) = 0$
2. $z = px + qy + f(p, q)$
3. $f(z, p, q) = 0$ (or) $f(x, p, q) = 0$ (or) $f(y, p, q) = 0$
4. $f(x, p) = \phi(y, q)$

We will see below a few types of equations, reducible in each case to one of the standard types.

III (c) Problems based on equations reducible to standard form :

Type 5

Equation of the type of $f(x^m p, y^n q) = 0$ ... (1) and

$f(z, x^m p, y^n q) = 0$ ... (2)

Case (i) If $m \neq 1$ and $n \neq 1$, then put $X = x^{1-m}$, $Y = y^{1-n}$

$$X = x^{1-m}$$

$$\frac{\partial X}{\partial x} = (1-m)x^{-m}$$

$$P = \frac{\partial z}{\partial X}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$p = P[1-m][x^{-m}]$$

$$\boxed{x^m p = P(1-m)}$$

$$Y = y^{1-n}$$

$$\frac{\partial Y}{\partial y} = (1-n)y^{-n}$$

$$Q = \frac{\partial z}{\partial Y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$$

$$\boxed{y^n q = Q(1-n)}$$

Substitute in (1), we get

$f(P, Q) = 0$ [Type 1]

Substitute in (2), we get

$f(z, P, Q) = 0$ [Type 3]

Case (ii) If $m = n = 1$, then put $X = \log x$, $Y = \log y$

$$X = \log x$$

$$\frac{\partial X}{\partial x} = \frac{1}{x}$$

$$P = \frac{\partial z}{\partial X}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$p = P\frac{1}{x}$$

$$\boxed{xp = P}$$

$$Y = \log y$$

$$\frac{\partial Y}{\partial y} = \frac{1}{y}$$

$$Q = \frac{\partial z}{\partial Y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$$

$$q = Q\frac{1}{y}$$

$$\boxed{yq = Q}$$

Substitute in equation (1), we get $f(P, Q) = 0$ [Type 1

Substitute in equation (2), we get $f(z, P, Q) = 0$ [Type

Problem based on Type 5

Formula : For $f(x^m p, y^n q) = 0$ (or) $f(z, x^m p, y^n q) = 0$

Case (i) If $m \neq 1$, $n \neq 1$, put $X = x^{1-m}$, $Y = y^{1-n}$
then $x^m p = P(1-m)$, $y^n q = Q(1-n)$

Case (ii) If $m = 1$, $n = 1$, put $X = \log x$, $Y = \log y$
then $xp = P$, $yq = Q$

Example 1.3c(1) : **Solve : $x^4 p^2 + y^2 zq = 2z^2$**

Solution : Given : $x^4 p^2 + y^2 zq = 2z^2$ **Type 5, case (i)**

This is of the form $f(z, x^m p, y^n q) = 0$

$$(x^2 p)^2 + (y^2 q)z = 2z^2 \quad \text{... (1) Here, } m = 2, n = 2$$

Here, $X = x^{1-m}$, $m \neq 1$	$Y = y^{1-n}$, $n \neq 1$
$x^m p = P(1-m)$	$\therefore y^n q = Q(1-n)$
$x^2 p = P(1-2)$, $m = 2$	$y^2 q = Q(1-2)$, $n = 2$
$x^2 p = -P$	$y^2 q = -Q$

$$\therefore (1) \Rightarrow (-P)^2 + (-Q)z = 2z^2$$

$$\text{i.e.,} \quad P^2 - Qz = 2z^2 \quad \text{... (2)}$$

This is of the form $f(z, P, Q) = 0$

(i.e.,) Type 3 case (i)

Let $u = X + aY$

$$\frac{\partial u}{\partial X} = 1, \quad \frac{\partial u}{\partial Y} = a$$

$$P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial X} = \frac{dz}{du}$$

$$\Rightarrow P = \frac{dz}{du}$$

$$Q = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial Y} = \frac{dz}{du}a$$

$$\Rightarrow Q = a\frac{dz}{du}$$

$$(2) \Rightarrow \left(\frac{dz}{du}\right)^2 - a\frac{dz}{du}z = 2z^2$$

$$\left(\frac{dz}{du}\right)^2 - az\frac{dz}{du} = 2z^2$$

$$\left(\frac{dz}{du}\right)^2 - az\frac{dz}{du} - 2z^2 = 0$$

$$\frac{dz}{du} = \frac{az \pm \sqrt{(az)^2 - 4(1)(-2z^2)}}{2}$$

$$= \frac{az \pm z\sqrt{a^2+8}}{2}$$

$$= z\left[\frac{a \pm \sqrt{a^2+8}}{2}\right]$$

$$\frac{dz}{z} = \frac{a \pm \sqrt{a^2+8}}{2}du$$

$$\int \frac{dz}{z} = \int \frac{a \pm \sqrt{a^2+8}}{2}du$$

$$\log z = \frac{a \pm \sqrt{a^2+8}}{2}u + b$$

$$\log z = \frac{a \pm \sqrt{a^2+8}}{2}[X + aY] + b$$

$$\log z = \frac{a \pm \sqrt{a^2+8}}{2}\left[\frac{1}{x} + \frac{a}{y}\right] + b$$

which is the complete integral. Singular and general solutions are found out as usual.

Example 1.3c(2) : Solve : $p^2 + x^2y^2q^2 = x^2z^2$ **[A.U. A/M 2005]** **[A.U A/M 2015 R-13] [A.U M/J 2016 R-8]**

Solution : Given : $p^2 + x^2y^2q^2 = x^2z^2$

$$\frac{p^2}{x^2} + y^2q^2 = z^2$$

$$x^{-2}p^2 + y^2q^2 = z^2$$

$$(x^{-1}p)^2 + (yq)^2 = z^2 \quad \dots (1)$$

This is of the form $f(x^m p, y^n q, z) = 0$

[Type 5 case (i) and (ii)]

$$(x^{-1}p)^2 + (yq)^2 = z^2 \quad \dots (1), \text{ Here } m = -1, n = 1$$

Here, $X = x^{1-m}$, $m \neq 1$ $x^m p = P(1-m)$	$Y = \log y$, $n = 1$ $yq = Q$
$x^{-1}p = P(1+1)$, $m = -1$ $x^{-1}p = 2P$	

$$(1) \Rightarrow (2P)^2 + Q^2 = z^2 \quad \dots (2)$$

This is of the form $f(P, Q, z) = 0$

Type 3 case (i)

Let $u = X + aY$

$$\frac{\partial u}{\partial X} = 1,$$

$$\frac{\partial u}{\partial Y} = a$$

$$P = \frac{\partial z}{\partial X} = \frac{dz}{du}\frac{\partial u}{\partial X}$$

$$\Rightarrow P = \frac{dz}{du}$$

$$Q = \frac{\partial z}{\partial Y} = \frac{dz}{du}\frac{\partial u}{\partial Y}$$

$$\Rightarrow Q = a\frac{dz}{du}$$

$$(2) \Rightarrow \left(2\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2$$

$$(4+a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{4+a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{4+a^2}}$$

$$\frac{dz}{z} = \frac{1}{\sqrt{4+a^2}}\,du$$

$$\int\frac{dz}{z} = \int\frac{1}{\sqrt{4+a^2}}\,du$$

$$\log z = \frac{1}{\sqrt{4+a^2}}\,u + b$$

$$\log z = \frac{1}{\sqrt{4+a^2}}\,(X + aY) + b$$

$$= \frac{1}{\sqrt{4+a^2}}\,[x^2 + a\log y] + b$$

which is the complete integral. Singular and general solutions are found out as usual.

Example 1.3c(3) : Solve : $x^2 p^2 + y^2 q^2 = z^2$

[A.U. N/D 2005] [A.U N/D 2015 R-8]

Solution : Given : $x^2 p^2 + y^2 q^2 = z^2$ **[A.U.T Trichy N/D 2011]**

$$(xp)^2 + (yq)^2 = z^2 \quad \dots (1),$$ **[A.U M/J 2014]**

Here $m = 1, n = 1$

This equation is of the form $f(z, x^m p, y^n q) = 0$

Here, $X = \log x,\ m = 1$ put $Y = \log y,\ n = 1$

$\therefore\ xp = P \qquad \therefore\ yq = Q$

Substitute in eqn. (1), we get

$$P^2 + Q^2 = z^2 \quad \dots (2)$$

This equation is of the form $f(z, P, Q) = 0$ [Type 3 case (i)]

Let $u = X + aY$

$$\frac{\partial u}{\partial X} = 1, \quad \frac{\partial u}{\partial Y} = a$$

$$P = \frac{\partial z}{\partial X} = \frac{dz}{du}\frac{\partial u}{\partial X}$$

$$\Rightarrow P = \frac{dz}{du}$$

$$Q = \frac{\partial z}{\partial Y} = \frac{dz}{du}\frac{\partial u}{\partial Y}$$

$$\Rightarrow Q = a\frac{dz}{du}$$

Substitute in (2) we get

$$\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2 \quad \text{... (3)}$$

$$\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$(1+a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{1+a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

$$\frac{1}{z}\,dz = \frac{1}{\sqrt{1+a^2}}\,du$$

$$\int \frac{1}{z}\,dz = \int \frac{1}{\sqrt{1+a^2}}\,du$$

$$\log z = \frac{1}{\sqrt{1+a^2}}\,u + b$$

$$\log z = \frac{1}{\sqrt{1+a^2}}\,(X + a\,Y) + b$$

$$\log z = \frac{1}{\sqrt{1+a^2}}\,[\log x + a\log y] + b$$

which is the complete integral. The singular and general solutions are found as usual.

EXERCISE 1.3.c

Find the complete integral of the following :

1. $z^2(p^2x^2+q^2) = 1$ **[Ans.** $z^2 = a\log x + \sqrt{4-a^2}\,y + b$**]**
2. $x^2p^2 + xpq = z^2$ **[Ans.** $\sqrt{1+a}\log z = \log x + ay + b$**]**
3. $x^4p^2 - yzq = z^2$ **[Ans.** $\log z = \frac{a}{x} + (a^2-1)\log y + b$**]**

Type 6

Equation of the type $f(z^m p, z^m q) = 0$ ... (1)

and $f_1(x, z^m p) = f_2(y, z^m q)$ ... (2)

Case (i) If $m \neq -1$ then

put $Z = z^{m+1}$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}$$

$$P = (m+1)\,z^m p$$

$$\boxed{\frac{P}{m+1} = z^m p} \Rightarrow z^m p = \frac{P}{m+1} \quad \text{... (3)}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y}$$

$$Q = (m+1)\,z^n q$$

$$\boxed{\frac{Q}{m+1} = z^n q} \Rightarrow z^n q = \frac{Q}{m+1} \quad \text{... (4)}$$

Substitute in equation (1), we get

$$f[P, Q] = 0 \qquad \text{[Type 1]}$$

Substitute in equation (2), we get

$$f_1(x, P) = f_2(y, Q) \quad \text{[Type 4]}$$

Case (ii) If $m = -1$ then

put $\quad Z = \log z$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} \qquad \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y}$$

$$P = \frac{1}{z}p \qquad Q = \frac{1}{z}q$$

i.e., $\boxed{z^{-1}p = P}$ $\quad$ i.e., $\boxed{z^{-1}q = Q}$

Substitute in equation (1), we get $f[P, Q] = 0$ [Typ

Substitute in equation (2), we get $f_1(x, P) = f_2(y, Q)$ [Typ

Problem based on Type 6 :

Formula for $f(z^m p, z^m q) = 0$

Case (i) If $m \neq -1$, put $Z = z^{m+1}$, then
$$z^m p = \frac{P}{m+1}, \quad z^m q = \frac{Q}{m+1}$$

Case (ii) If $m = -1$, put $Z = \log z$, then
$$z^{-1}p = P, \quad z^{-1}q = Q$$

Example 1.3c(8) : Solve. $z^2(p^2+q^2) = x^2+y^2$ **[Anna, March 19 [A.U. CBT A/M 2011][A.U.T CH N/D 2011][A.U N/D 2015 R [A.U A/M 2019 R**

Solution :

Given: $\quad z^2(p^2+q^2) = x^2+y^2$

$$(zp)^2 + (zq)^2 = x^2+y^2 \qquad \dots (1)$$

This equation is of the form $f_1(x, z^m p) = f_2(y, z^m q)$ [Type 6]

Here $\quad m \neq -1$, put $Z = z^{m+1}$

then $z^m p = \dfrac{P}{m+1}, \quad z^m q = \dfrac{Q}{m+1}$

$$zp = \frac{P}{1+1}, \quad zq = \frac{Q}{1+1}$$

$$zp = \frac{P}{2}, \quad zq = \frac{Q}{2}$$

Substitute in equation (1), we get

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x^2+y^2$$

$$P^2 + Q^2 = 4(x^2+y^2)$$

$$P^2 - 4x^2 = 4y^2 - Q^2$$

This equation is of the form $f_1(x, P) = f_2(y, Q)$ [Type 4]

$$\therefore \quad P^2 - 4x^2 = 4y^2 - Q^2 = 4a^2 \text{ (say)}$$

$$P^2 = 4a^2 + 4x^2, \qquad Q^2 = 4y^2 - 4a^2$$

$$P = 2\sqrt{a^2+x^2}, \qquad Q = 2\sqrt{y^2-a^2}$$

$$dZ = P\,dx + Q\,dy$$

$$dZ = 2\sqrt{a^2+x^2}\,dx + 2\sqrt{y^2-a^2}\,dy$$

$$\int dZ = 2\int \sqrt{a^2+x^2}\,dx + 2\int \sqrt{y^2-a^2}\,dy$$

$$Z = 2\left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a}\right] + b$$

$$z^2 = x\sqrt{x^2+a^2} + a^2\sinh^{-1}\frac{x}{a} + y\sqrt{y^2-a^2} - a^2\cosh^{-1}\frac{y}{a} + b$$

$$= x\sqrt{x^2+a^2} + y\sqrt{y^2-a^2} + a^2\left[\sinh^{-1}\frac{x}{a} - \cosh^{-1}\frac{y}{a}\right] + b$$

Example 1.3c(9) : Solve $p^2+q^2 = z^2(x^2+y^2)$

Solution :

Given : $p^2+q^2 = z^2(x^2+y^2)$ ... (1)

$$\left[\frac{p}{z}\right]^2 + \left[\frac{q}{z}\right]^2 = x^2+y^2$$

This equation is of the form $f_1(x, z^m p) = f_2(y, z^m q)$ Type 6

Here $m = -1$

put $Z = \log z$, then $z^{-1}p = P$, $z^{-1}q = Q$

Substitute in equation (1), we get

$$P^2+Q^2 = x^2+y^2$$

$$P^2-x^2 = y^2-Q^2$$

This equation is of the form $f_1(x, P) = f_2(y, Q)$ Type 4

$\therefore$ $P^2-x^2 = y^2-Q^2 = a^2$ say

$P^2-x^2 = a^2$;	$y^2-Q^2 = a^2$
$P = \sqrt{a^2+x^2}$	$Q = \sqrt{y^2-a^2}$

$$dZ = P\,dx + Q\,dy$$

$$dZ = \sqrt{a^2+x^2}\,dx + \sqrt{y^2-a^2}\,dy$$

$$\int dZ = \int\sqrt{a^2+x^2}\,dx + \int\sqrt{y^2-a^2}\,dy$$

$$Z = \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + b$$

$$\log z = \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + b$$

Example 1.3c(10) : Solve : $(zp+x)^2 + (zq+y)^2 = 1$

Solution :

Given : $(zp+x)^2 + (zq+y)^2 = 1$ ... (1)

This equation is of the form $f_1(x, z^m p) = f_2(y, z^m q)$ [Type 6]

Here $m \neq -1$ put $Z = z^{m+1}$

then $z^m p = \dfrac{P}{m+1}$, $z^n q = \dfrac{Q}{m+1}$

$$zp = \frac{P}{2}, \qquad zq = \frac{Q}{2}$$

Substitute in (1), we get

$$\left(\frac{P}{2}+x\right)^2 + \left(\frac{Q}{2}+y\right)^2 = 1 \quad \text{... (2)}$$

$$\left(\frac{P}{2}+x\right)^2 = 1 - \left(\frac{Q}{2}+y\right)^2$$

This equation is of the form $f_1(P, x) = f_2(Q, y)$

$$\therefore \left(\frac{P}{2}+x\right)^2 = 1 - \left(\frac{Q}{2}+y\right)^2 = a^2 \text{ say}$$

$$\left[\frac{P}{2}+x\right]^2 = a^2 \qquad\qquad 1-\left(\frac{Q}{2}+y\right)^2 = a^2$$

$$\frac{P}{2}+x = a \qquad\qquad \frac{Q}{2}+y = \sqrt{1-a^2}$$

$$\frac{P}{2} = a-x \qquad\qquad \frac{Q}{2} = \sqrt{1-a^2}-y$$

$$P = 2(a-x) \qquad\qquad Q = 2[\sqrt{1-a^2}-y]$$

we know that, $dZ = P\,dx + Q\,dy$

$$dZ = 2(a-x)\,dx + 2[\sqrt{1-a^2}-y]\,dy$$

$$\int dZ = 2\int (a-x)\,dx + 2\int (\sqrt{1-a^2}-y)\,dy$$

$$Z = 2ax - x^2 + 2\sqrt{1-a^2}\,y - y^2 + b$$

$$z^2 = 2ax - x^2 + 2\sqrt{1-a^2}\,y - y^2 + b$$

EXERCISE [Type 6]

Find the complete integral of the following :

1. $z(p^2-q^2) = x^2-y^2$

[Ans. $z^{3/2} = \frac{3}{2}\left[\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{y}{a}+\frac{y\sqrt{y^2+a^2}}{2}\right]+b$]

2. $4zq^2 = y + 2zp - x$

[Ans. $z^2 = \frac{2}{3}(y+a)^{3/2}+\frac{(x+a)^2}{2}+b$]

1.4 LAGRANGE'S LINEAR EQUATION

The equation of the form $Pp + Qq = R$ is known as Lagrange's equation when P, Q & R are functions of x, y and z.

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\ ;\ Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\ ;\ R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

To solve this equation it is enough to solve the subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Working Rule

First step : Write down the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Second step : Solve the above subsidiary equations.

Let the two solutions be $u = a$ and $v = b$

Third step : Then $f(u, v) = 0$ or $u = \phi(v)$ is the required solution of $Pp + Qq = R$

Generally, the subsidiary equation can be solved in two ways.

1. Method of Grouping 2. Method of Multipliers

1. Method of Grouping

In the subsidiary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ if the variables can be separated in any pair of equations, then we get a solution of the form $u(x, y) = a$ and $v(x, y) = b$.

2. Method of Multipliers

Choose any three multipliers l, m, n which may be constants or function of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$

then $l\,dx + m\,dy + n\,dz = 0$

If $l\,dx + m\,dy + n\,dz$ is an exact differential then on integration we get a solution $u = a$.

The multipliers l, m, n are called lagrangian multipliers.

1.4a. Problems based on Lagrange's linear equation - method of Grouping :

Example 1.4a(1) : Solve $px + qy = z$. **[A.U CBT Dec. 2008]**

Solution : Given : $px + qy = z$ (i.e.,) $xp + yq = z$ ... (1)

This equation is of the form $Pp + Qq = R$

where $P = x$, $Q = y$, $R = z$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e., $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Take, $\frac{dx}{x} = \frac{dy}{y}$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x = \log (y c_1)$$

$$x = yc_1$$

$$\frac{x}{y} = c_1$$

i.e., $u = \frac{x}{y}$

Take, $\frac{dx}{x} = \frac{dz}{z}$

$$\int \frac{dx}{x} = \int \frac{dz}{z}$$

$$\log x = \log z + \log c_2$$

$$\log x = \log (z c_2)$$

$$x = zc_2$$

$$\frac{x}{z} = c_2$$

i.e., $v = \frac{x}{z}$

Hence, the general solution is $f(u, v) = 0$

i.e., $f\left(\frac{x}{y}, \frac{x}{z}\right) = 0$, where f is arbitrary.

Example 1.4a(2) : Write the solution of $px^2 + qy^2 = z^2$.

Solution : Given : $px^2 + qy^2 = z^2$ **[M.U. April, 97] [A.U. N/D 2005]**

i.e., $x^2 p + y^2 q = z^2$

This equation is of the form $P p + Q q = R$

where $P = x^2$, $Q = y^2$, $R = z^2$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{i.e., } \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \quad \text{... (1)}$$

Take, $\frac{dx}{x^2} = \frac{dy}{y^2}$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$-\frac{1}{x} = \frac{-1}{y} - c_1$$

$$\frac{1}{y} - \frac{1}{x} = -c_1$$

$$c_1 = \frac{1}{x} - \frac{1}{y} \quad \text{i.e., } u = \frac{1}{x} - \frac{1}{y}$$

Take, $\frac{dy}{y^2} = \frac{dz}{z^2}$

$$\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$$

$$-\frac{1}{y} = \frac{-1}{z} - c_2$$

$$c_2 = \frac{1}{y} - \frac{1}{z}$$

i.e., $v = \frac{1}{y} - \frac{1}{z}$

Hence, the general solution is $f(u, v) = 0$

i.e., $f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$, where f is arbitrary.

Note : (1) $\int \frac{1}{x} dx = \log x + c$ (2) $\int \frac{1}{x^2} dx = -\frac{1}{x} + c$

(3) $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$ (4) $\log (ab) = \log a + \log b$

Example 1.4a(3) : Find the solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.

Solution : Given : $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ **[M.U. Oct 96]**

i.e., $\sqrt{x}\, p + \sqrt{y}\, q = \sqrt{z}$

This equation is of the form $P p + Q q = R$

where $P = \sqrt{x}$, $Q = \sqrt{y}$, $R = \sqrt{z}$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{i.e., } \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \quad \text{... (1)}$$

Take, $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$

$$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$$

$$2\sqrt{x} = 2\sqrt{y} + 2c_1$$

$$\sqrt{x} = \sqrt{y} + c_1$$

$$c_1 = \sqrt{x} - \sqrt{y}$$

i.e., $u = \sqrt{x} - \sqrt{y}$

Take, $\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

$$\int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$2\sqrt{y} = 2\sqrt{z} + 2c_2$$

$$\sqrt{y} = \sqrt{z} + c_2$$

$$c_2 = \sqrt{y} - \sqrt{z}$$

i.e., $v = \sqrt{y} - \sqrt{z}$

Hence, the general solution is $f(u, v) = 0$

i.e., $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$, where f is arbitrary.

Example 1.4a(4) : Find the general solution of $p\tan x + q\tan y = \tan z$. [A.U N/D 2016 R...

Solution : Given : $p\tan x + q\tan y = \tan z$

i.e., $(\tan x)p + (\tan y)q = \tan z$

This equation is of the form $Pp + Qq = R$

where $P = \tan x$, $Q = \tan y$, $R = \tan z$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Take, $\frac{dx}{\tan x} = \frac{dy}{\tan y}$

$$\int\frac{dx}{\tan x} = \int\frac{dy}{\tan y}$$

$$\int \cot x\,dx = \int \cot y\,dy$$

$$\log\sin x = \log\sin y + \log a$$

$$\log\sin x = \log(a\sin y)$$

$$\sin x = a\sin y$$

$$a = \frac{\sin x}{\sin y}$$

i.e., $u = \frac{\sin x}{\sin y}$,

Take, $\frac{dy}{\tan y} = \frac{dz}{\tan z}$

$$\int\frac{dy}{\tan y} = \int\frac{dz}{\tan z}$$

$$\int \cot y\,dy = \int \cot z\,dz$$

$$\log\sin y = \log\sin z + \log b$$

$$\log\sin y = \log(b\sin z)$$

$$\sin y = b\sin z$$

i.e., $b = \frac{\sin y}{\sin z}$

$$v = \frac{\sin y}{\sin z}$$

Hence, the general solution is $f(u,v) = 0$

i.e., $f\left[\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right] = 0$, where f is arbitrary.

Example 1.4a(5) : Write the general integral of $pyz + qzx = xy$. [M.U. Ap. 19...

Solution : Given : $pyz + qzx = xy$ (i.e.,) $yz\,p + zx\,q = xy$

This equation is of the form $Pp + Qq = R$

where $P = yz$, $Q = zx$, $R = xy$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Take, $\frac{dx}{yz} = \frac{dy}{zx}$

i.e., $\frac{dx}{y} = \frac{dy}{x}$

$$x\,dx = y\,dy$$

$$\int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + \frac{c_1}{2}$$

$$x^2 = y^2 + c_1$$

$$c_1 = x^2 - y^2$$

i.e., $u = x^2 - y^2$

Take, $\frac{dy}{zx} = \frac{dz}{xy}$

$$\frac{dy}{z} = \frac{dz}{y}$$

$$y\,dy = z\,dz$$

$$\int y\,dy = \int z\,dz$$

$$\frac{y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

$$y^2 = z^2 + c_2$$

$$c_2 = y^2 - z^2$$

i.e., $v = y^2 - z^2$

Hence, the general solution is $f(u,v) = 0$

i.e., $f(x^2 - y^2, y^2 - z^2) = 0$, where f is arbitrary.

Example 1.4a(6) : Find the general integral of $p - q = \log(x+y)$.

[AU, April 1996] [M.U. OCT 95]

Solution : Given : $p - q = \log(x+y)$ (i.e.,) $(1)p + (-1)q = \log(x+y)$

This equation is of the form $Pp + Qq = R$,

where $P = 1$, $Q = -1$, $R = \log(x+y)$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)} \quad \dots (1)$$

Take, $\frac{dx}{1} = \frac{dy}{-1}$

$$dx = -dy$$

$$\int dx = \int -dy$$

$$x = -y + c_1$$

$$c_1 = x + y$$

i.e., $u = x + y \quad \ldots (2)$

Take, $\frac{dx}{1} = \frac{dz}{\log(x+y)}$

$$\frac{dx}{1} = \frac{dz}{\log(c_1)} \text{ by}$$

$$(\log c_1)\, dx = dz$$

$$\int (\log c_1)\, dx = \int dz$$

[since, $\log c_1$ is constan

$$(\log c_1)\, x = z + c_2$$

$$x \log(x+y) - z = c_2$$

i.e., $v = x\log(x+y) - z$

Hence, the general solution is

$$f(x+y, x\log(x+y) - z) = 0$$

or $x\log(x+y) - z = f(x+y)$, where f is arbitrary.

Example 1.4a(7) : Obtain the general solution of $pzx + qzy =$

[A.U A/M 2001] [A.U. CBT Dec.

Solution : Given : $pzx + qzy = xy$ (i.e.,) $zxp + zyq = xy$

This equation is of the form $Pp + Qq = R$

Here, $P = zx$, $Q = zy$, $R = xy$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{zx} = \frac{dy}{zy} = \frac{dz}{xy}$$

Take, $\frac{dx}{zx} = \frac{dy}{zy}$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log \frac{x}{y} = \log c_1$$

Take, $\frac{dy}{zy} = \frac{dz}{xy}$

$$\Rightarrow \frac{dy}{z} = \frac{dz}{x}$$

$$\Rightarrow \frac{dy}{z} = \frac{dz}{yc_1} \text{ by (1)}$$

$$\therefore yc_1\, dy = z\, dz$$

$$\frac{x}{y} = c_1 \quad \ldots (1)$$

i.e., $u = \frac{x}{y}$

Integrating,

$$\frac{y^2}{2} c_1 = \frac{z^2}{2} + \frac{c_2}{2}$$

i.e., $c_1 y^2 - z^2 = c_2$

i.e., $c_2 = \left(\frac{x}{y}\right)(y^2) - z^2$ by (1)

i.e., $v = xy - z^2$

Hence, the general solution is

i.e., $f(u, v) = 0$

$$f\left(\frac{x}{y}, xy - z^2\right) = 0 \text{ where } f \text{ is arbitraty.}$$

Example 1.4a(8) : Solve : $y^2 p - xyq = x(z - 2y)$ **[A.U. N/D 2004]**

Solution : Given: $y^2 p - xyq = x(z - 2y)$ **[A.U Tvli M/J 2011]**

i.e., $y^2 p + (-xy)\, q = x(z - 2y)$

This equation of the form $Pp + Qq = R$

Here, $P = y^2$, $Q = -xy$, $R = x(z - 2y)$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{i.e.,} \quad \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Take,

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\, dx = -y\, dy$$

$$\int x\, dx = -\int y\, dy$$

$$\frac{x^2}{2} = \frac{-y^2}{2} + \frac{c_1}{2}$$

Take,

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dy}{-y} = \frac{dz}{z - 2y}$$

$$(z - 2y)\, dy = -y\, dz$$

$$z\, dy - 2y\, dy = -y\, dz$$

$$y\, dz + z\, dy = 2y\, dy$$

$$x^2 = -y^2 + c_1$$

$$x^2 + y^2 = c_1$$

i.e., $u = x^2 + y^2$

$$d\,[yz] = 2y\,dy$$

$$\int d\,(yz) = \int 2y\,dy$$

$$yz = 2\frac{y^2}{2} + c_2$$

$$yz = y^2 + c_2$$

$$yz - y^2 = c_2$$

i.e., $v = yz - y^2$

Hence, the general solution is $f(u, v) = 0$

i.e., $f(x^2 + y^2,\ yz - y^2) = 0$, where f is arbitrary.

Example 1.4.a(9) : Solve $x^2 p + y^2 q = z$

Solution : Given : $x^2 p + y^2 q = z$

This equation is of the form $P\,p + Q\,q = R$,

where $P = x^2$, $Q = y^2$, $R = z$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{z}$

Take, $\dfrac{dx}{x^2} = \dfrac{dy}{y^2}$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} - c_1$$

$$c_1 = \frac{1}{x} - \frac{1}{y}$$

i.e., $u = \dfrac{1}{x} - \dfrac{1}{y}$

Take, $\dfrac{dx}{x^2} = \dfrac{dz}{z}$

$$\int \frac{dx}{x^2} = \int \frac{dz}{z}$$

$$-\frac{1}{x} = \log z - c_2$$

$$c_2 = \frac{1}{x} + \log z$$

i.e., $v = \dfrac{1}{x} + \log z$

Hence, the general solution is

$$f(u, v) = 0$$

i.e., $f\left(\dfrac{1}{x} - \dfrac{1}{y},\ \dfrac{1}{x} + \log z\right) = 0,$

where f is arbitrary.

EXERCISES 1.4.a

Solve the following :

1. $2p + 3q = 1$ **[Ans.** $f(3x - 2y,\ y - 3z) = 0$**]**
2. $zp + x = 0$ **[Ans.** $f(x^2 + z^2, y) = 0$**]**
3. $p \cos x\ q \cot y = \cot z$ **[Ans.** $f\left(\dfrac{\cos y}{\cos x},\ \dfrac{\cos z}{\cos y}\right) = 0$**]**
4. $xp + yq = x$ **[Ans.** $f\left(\dfrac{x}{y},\ x - z\right) = 0$**]**
5. $\dfrac{y^2}{x} zp + xzq = y^2$ **[Ans.** $f(x^3 - y^3,\ x^2 - z^2) = 0$**]**
6. $p + q = 1$ **[Ans.** $f(x - z,\ y - z) = 0$**]**
7. $xp + yq = 0$ **[Ans.** $\phi\left(z,\ \dfrac{x}{y}\right) = 0$**]**
8. $(x + 2)\,p + 2\,yq = 2z$ **[Ans.** $\phi\left(\dfrac{x+2}{y},\ \dfrac{y}{z}\right) = 0$**]**
9. $4\,(p + q) = z$ **[Ans.** $f\left(x - y,\ \dfrac{z}{e^{y/4}}\right) = 0$**]**
10. $xp - yq = xz$ **[Ans.** $f(xy,\ x - \log z) = 0$**]**
11. $x^2 p + y^2 q + z^2 = 0$ **[Ans.** $f\left(\dfrac{1}{x} - \dfrac{1}{y},\ \dfrac{1}{y} + \dfrac{1}{z}\right) = 0$**]**
12. $x^2 p + y^2 q = z\,(x + y)$ [A.U A/M 2015 R-08]
    [Ans. $f\left(\dfrac{1}{x} - \dfrac{1}{y},\ \dfrac{x - y}{z}\right) = 0$**]**

1.4.b. Problems based on Lagrange's method of multipliers

Example 1.4b(1) : Solve $x(y-z)p + y(z-x)q = z(x-y)$.

(OR) Solve $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$ **[A.U. Oct. 200..]**

[A.U. Tvli. N/D 2009] [A.U. N/D 2008][A.U Tvli N/D 201..]
[A.U N/D 2011][A.U N/D 2014, R-1..]

Solution : Given : $x(y-z)p + y(z-x)q = z(x-y)$

This equation is of the form $Pp + Qq = R$

where $P = x(y-z)$, $Q = y(z-x)$, $R = z(x-y)$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad \ldots (1)$$

Taking the Lagrangian multipliers are 1, 1, 1, we get each ratio in (1)

$$= \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{d(x+y+z)}{0}$$

Hence, $d(x+y+z) = 0$ Integrating, we get

$$x+y+z = a$$

Taking the Lagrangian multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, **we get each ratio in (1)**

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z)+(z-x)+(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

i.e., $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

$$xyz = b$$

Hence, the general solution is $f(a,b) = 0$

i.e., $f(x+y+z, xyz) = 0$, where f is arbitrary.

Example 1.4b(2) : Solve $(mz-ny)p + (nx-lz)q = ly - mx$.

[A.U. A/M 2004][A.U CBT N/D 2010][A.U N/D 2016 R-08]
[A.U A/M 2019 R-08]

Solution : Given : $(mz-ny)p + (nx-lz)q = ly - mx$

This equation is of the form $Pp + Qq = R$

where $P = mz-ny$, $Q = nx-lz$, $R = ly-mx$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e., $$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \quad \ldots (1)$$

Taking the Lagrangian multipliers are x, y, z, we get each ratio in (1)

$$= \frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{xdx+ydy+zdz}{0}$$

Hence, $xdx + ydy + zdz = 0$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \frac{a}{2}$$

i.e., $x^2+y^2+z^2 = a$

Taking the Lagrangian multipliers are l, m, n, we get each ratio in (1)

$$= \frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)} = \frac{ldx+mdy+ndz}{0}$$

Hence, $ldx + mdy + ndz = 0$

Integrating, we get

$$lx+my+nz = b$$

$$lx+my+nz = b$$

Hence, the general solution is $f(a, b) = 0$

i.e., $f(x^2 + y^2 + z^2, lx + my + nz) = 0$, where f is arbitrary.

Example 1.4b(3) : Sove $(3z - 4y)\,p + (4x - 2z)\,q = 2y - 3x$

[A.U N/D 2003, N/D 2006, A.U.T CBT N/D 2011, A.U.T Tvli N/D 2... [A.U N/D 2018 R...

Solution : Given $(3z - 4y)\,p + (4x - 2z)\,q = 2y - 3x$

This equation is of the form $P\,p + Q\,q = R$

where $P = 3z - 4y$, $Q = 4x - 2z$, $R = 2y - 3x$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e., } \frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x} \quad \text{... (1)}$$

Use Lagrangian multipliers x, y, z ,

we get each ratio in (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(3z - 4y) + y(4x - 2z) + z(2y - 3x)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e., $x\,dx + y\,dy + z\,dz = 0$ (by method of multipliers formula)

Integrating, we get $\int x\,dx + \int y\,dy + \int z\,dz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2} \qquad \text{i.e., } x^2 + y^2 + z^2 = ...$$

Again use Lagrangian multipliers 2, 3, 4,

we get each ratio in (1)

$$= \frac{2\,dx + 3dy + 4dz}{6z - 8y + 12x - 6z + 8y - 12x} = \frac{2\,dx + 3dy + 4...}{0}$$

i.e., $2\,dx + 3dy + 4dz = 0$

Integrating, we get $\int 2dx + \int 3\,dy + \int 4\,dz = 0$

$$2x + 3y + 4z = b$$

Hence, the general solution is $f(a, b) = 0$

i.e., $f(x^2 + y^2 + z^2, 2x + 3y + 4z) = 0$, where f is arbitrary.

Example 1.4b(4) : Find the general solution of **[AU, Oct. 1996, Dec. 1998, April, 2001]**

$x(y^2 - z^2)\,p + y(z^2 - x^2)\,q = z(x^2 - y^2)$

(or) $x(z^2 - y^2)\,p + y(x^2 - z^2)\,q = z(y^2 - x^2)$ **[A.U N/D 2016 R-13, A.U M/J 2016 R-8]**

[A.U. A/M 2002, A/M 2008, N/D 2008] [A.U Trichy N/D 2009] [A.U CBT Dec. 2009][A.U.T Chennai N/D 2011][A.U M/J 2013] [A.U N/D 2015 R8][A.U M/J 2016 R8][A.U N/D 2018 R-13]

Solution : Given $x(y^2 - z^2)\,p + y(z^2 - x^2)\,q = z(x^2 - y^2)$ ... (1)

This equation is of the form $P\,p + Q\,q = R$

where $P = x(y^2 - z^2)$, $Q = y(z^2 - x^2)$, $R = z(x^2 - y^2)$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{i.e., } \frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \quad \text{... (2)}$$

Use Lagrange multipliers x, y, z, we get each ratio in (2)

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e., $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}$$

i.e., $x^2 + y^2 + z^2 = a$

Use Lagrange multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get

each ratio in (2)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

i.e., $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$\int \frac{1}{x}\,dx + \int \frac{1}{y}\,dy + \int \frac{1}{z}\,dz = 0$$

$$\log x + \log y + \log z = \log b$$

$$\log (xyz) = \log b$$

$$\text{i.e., } xyz = b$$

Hence, the general solution is $f(a, b) = 0$ i.e., $f(x^2 + y^2 + z^2, xyz) = 0$, where f is arbitrary.

Example 1.4b(5) : Solve $(x^2 - yz)\, p + (y^2 - zx)\, q = (z^2 - xy)$.
[A.U., May 96] [M.U. Oct. 99] [A.U. A/M 2005] [A.U A/M 2015 R-13] [A.U M/J 2016 R-13] [A.U N/D 2018 R-17] [A.U A/M 2019 R-17] [A.U N/D 2019, R-17]

Solution : Given : $(x^2 - yz)\, p + (y^2 - zx)\, q = (z^2 - xy)$ ... (1)

This equation is of the form $P\, p + Q\, q = R$

where $P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e., } \frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \quad \text{... (2)}$$

Method of grouping is not possible

Using two sets of multipliers x, y, z ; $1, 1, 1$ each of the ratio in (2)

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$= \frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$= \frac{xdx + ydy + zdz}{x + y + z} = \frac{dx + dy + dz}{1}$$

$$xdx + ydy + zdz = (x + y + z)\, d(x + y + z)$$

Integrating on both sides, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x + y + z)^2}{2} + \frac{a}{2}$$

$$x^2 + y^2 + z^2 = (x + y + z)^2 + a$$

$$a = x^2 + y^2 + z^2 - (x + y + z)^2$$

$$= x^2 + y^2 + z^2 - x^2 - y^2 - z^2 - 2xy - 2yz - 2zx$$

$$= -2(xy + yz + zx)$$

$$\text{i.e., } xy + yz + zx = -\frac{a}{2} = u \text{ [constant]}$$

Using two sets of multipliers $1, -1, 0$; $0, 1, -1$ each of the ratio in (2)

$$= \frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}$$

$$\frac{d(x - y)}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy}$$

$$\frac{d(x - y)}{(x^2 - y^2) + z(x - y)} = \frac{d(y - z)}{(y^2 - z^2) + x(y - z)}$$

$$\frac{d(x - y)}{(x - y)(x + y) + z(x - y)} = \frac{d(y - z)}{(y - z)(y + z) + x(y - z)}$$

$$\frac{d(x - y)}{(x - y)(x + y + z)} = \frac{d(y - z)}{(y - z)(x + y + z)}$$

$$\frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z}$$

Integrating on both sides, we get

$$\log (x - y) = \log (y - z) + \log b$$

$$\log (x - y) = \log [b(y - z)]$$

$$x - y = b(y - z)$$

$$\frac{x - y}{y - z} = b = v$$

Hence, the general solution is $f(u, v) = 0$

i.e., $f\left(xy + yz + zx, \dfrac{x - y}{y - z}\right) = 0$, where f is arbitrary.

Aliter : Using Lagrange multipliers $y+z$, $z+x$, $x+y$ we get each ratio in (2)

$$= \frac{(y+z)\,dx + (z+x)\,dy + (x+y)\,dz}{(y+z)(x^2-yz) + (z+x)(y^2-zx) + (x+y)(z^2-xy)}$$

$$= \frac{(y+z)\,dx + (z+x)\,dy + (x+y)\,dz}{0}$$

$$\Rightarrow (y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$$

$$y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz = 0$$

$$(x\,dy + y\,dx) + (y\,dz + z\,dy) + (z\,dx + x\,dz) = 0$$

$$d(xy) + d(yz) + d(zx) = 0$$

Integrating, we get

$$\int d(xy) + \int d(yz) + \int d(zx) = a$$

$$xy + yz + zx = a = u$$

Note : The author wishes to thank Dr. B. Jothiram, formerly Asst. Prof. and HOD Maths, Govt. College of Engg., Salem, for having drawn his attention to this elegant method of getting one of the independent solutions.

Example 1.4b(6): Solve $(x^2+y^2+yz)\,p + (x^2+y^2-xz)\,q = z(x+y)$. **[A.U. N/D 2005]**

Solution : Given : $(x^2+y^2+yz)\,p + (x^2+y^2-xz)\,q = z(x+y)$

This equation is of the form $P\,p + Q\,q = R$

where $P = x^2+y^2+yz$; $Q = x^2+y^2-xz$; $R = z(x+y)$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-xz} = \frac{dz}{z(x+y)} \quad \dots (1)$$

Use Lagrange multipliers 1, −1, −1, we get each ratio in (1)

$$= \frac{dx - dy - dz}{x^2+y^2+yz-x^2-y^2+xz-zx-zy} = \frac{dx-dy-dz}{0}$$

$$\Rightarrow dx - dy - dz = 0$$

Integrating, we get

$$x - y - z = a$$

$$\text{i.e., } u = x - y - z$$

Take, $$\frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-xz}$$

$$\frac{x\,dx + y\,dy}{x(x^2+y^2) + xyz + y(x^2+y^2) - xyz} = \frac{x\,dx + y\,dy}{(x+y)(x^2+y^2)} \quad \dots (2)$$

Take, $$\frac{x\,dx + y\,dy}{(x+y)(x^2+y^2)} = \frac{dz}{z(x+y)}$$

$$\frac{x\,dx + y\,dy}{x^2+y^2} = \frac{dz}{z} \Rightarrow \frac{1}{2}\,\frac{2x\,dx + 2y\,dy}{x^2+y^2} = \frac{dz}{z}$$

$$\frac{1}{2}\,\frac{[d(x^2+y^2)]}{x^2+y^2} = \frac{dz}{z}$$

Integrating, we get

$$\frac{1}{2}\log(x^2+y^2) = \log z + \log b$$

$$\log(x^2+y^2) = 2\log z + 2\log b$$

$$= \log z^2 + \log b^2$$

$$\log b^2 = \log(x^2+y^2) - \log z^2$$

$$= \log(x^2+y^2) - \log z^2$$

$$= \log\left(\frac{x^2+y^2}{z^2}\right)$$

$$b^2 = \left(\frac{x^2+y^2}{z^2}\right)$$

(i.e.,) $v = \dfrac{x^2+y^2}{z^2}$ $\quad [\because b^2 = \text{constant}]$

Hence, the general solution is $f(u, v) = 0$

(i.e.,) $f\left(x-y-z, \dfrac{x^2+y^2}{z^2}\right) = 0$, where f is arbitrary.

Example 1.4b(7) : Solve $(y^2+z^2)\,p - xyq + xz = 0$

[A.U M/J 2006] [A.U. CBT Dec. 2008] [A.U N/D 2011]

Solution : Given : $(y^2+z^2)p - xyq + xz = 0$

This equation is of the form $P\,p + Q\,q = R$

Where $P = y^2+z^2$, $\quad Q = -xy$, $\quad R = -xz$

The Lagrange's subsidiary equations are $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz} \qquad \dots (A)$$

... (1) ↲ ... (2) ↲ ... (3) ↲

Take, $\dfrac{dy}{-xy} = \dfrac{dz}{-xz}$ [Method of grouping for (2) & (3)]

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log C_1$$

$$\log y = (\log z\, C_1)$$

$$y = z\, C_1$$

$C_1 = \dfrac{y}{z}$ $\quad$ (i.e.,) $\quad u = \dfrac{y}{z}$

Use Lagrangian multipliers x, y, z,

we get each ratio in (A)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2) - xy^2 - xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

(i.e.,) $x\,dx + y\,dy + z\,dz = 0$

(by Lagrangian multiplier formula)

$$\int x\,dx + \int y\,dx + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{C_2}{2}$$

$$x^2+y^2+z^2 = C_2$$

(i.e.,) $v = x^2+y^2+z^2$

Hence, the general solution is $f(u, v) = 0$

(i.e.,) $f\left(\dfrac{y}{z}, x^2+y^2+z^2\right) = 0$, where f is arbitrary.

Example 1.4b(8) : Solve : $x(y^2+z)\,p + y(x^2+z)\,q = z(x^2-y^2)$.

[A.U. M/J 2007] [A.U.T Trichy N/D 2011] [A.U N/D 2018-A R-17]

Solution : This equation is of the form $P\,p + Q\,q = R$

where $P = x(y^2+z)$, $\quad Q = y(x^2+z)$, $\quad R = z(x^2-y^2)$.

The Lagrange's subsidiary equations are

$\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ we get

$$\frac{dx}{x(y^2+z)} = \frac{dy}{y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Taking the Lagrangian multipliers $\frac{1}{x}, -\frac{1}{y}, \frac{1}{z}$, we get

$$\frac{dx}{x(y^2+z)} = \frac{dy}{y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{z}dz}{(y^2+z)-(x^2+z)+(x^2-y^2)}$$

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2}$$

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

Hence, $\frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$\log x - \log y + \log z = \log a$$

$$\log\left[\frac{x}{y}\right] + \log z = \log a$$

$$\log\left[\frac{xz}{y}\right] = \log a$$

$$\frac{xz}{y} = a$$

Taking the Lagrangian multipliers $x, -y, -1$, we get

$$\frac{dx}{x(y^2+z)} = \frac{dy}{y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx - y\,dy - dz}{x^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)}$$

$$= \frac{x\,dx - y\,dy - dz}{x^2y^2+x^2z-y^2x^2-y^2z-zx^2+zy^2}$$

$$= \frac{x\,dx - y\,dy - dz}{0}$$

Hence, $x\,dx - y\,dy - dz = 0$

Integrating, we get

$$\int x\,dx - \int y\,dy - \int dz = c$$

$$\frac{x^2}{2} - \frac{y^2}{2} - z = c$$

$$\frac{x^2-y^2}{2} - z = c$$

$$x^2 - y^2 - 2z = 2c$$

$$\text{(i.e.,)}\ x^2 - y^2 - 2z = b$$

Hence, the general solution is $f(a, b) = 0$

i.e., $f\left(\frac{xz}{y}, x^2-y^2-2z\right) = 0$, where f is arbitrary.

Example 1.4b(9): Solve the equation : $(x^2-y^2-z^2)\,p + 2\,xyq = 2\,zx$

[A.U N/D 2007]

Solution : Given : $(x^2-y^2-z^2)\,p + 2xyq = 2zx$

This equation is of the form $Pp + Qq = R$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} \quad \ldots (1)$$

Taking $\frac{dy}{2xy} = \frac{dz}{2zx}$ [by using method grouping idea]

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log C_1$$

$$\log y = \log (z C_1)$$

$$y = zC_1$$

i.e., $C_1 = \dfrac{y}{z}$

Choosing the method of multiplier idea x, y, z as multipliers.

$$(1) = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + 2xy^2 + 2z^2x} = \frac{dz}{2zx}$$

$$\frac{2(x\,dx + y\,dy + z\,dz)}{x[x^2 - y^2 - z^2 + 2y^2 + 2z^2]} = \frac{dz}{zx}$$

$$\frac{d[x^2 + y^2 + z^2]}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

$$\int \frac{d[x^2 + y^2 + z^2]}{x^2 + y^2 + z^2} = \int \frac{dz}{z}$$

$$\log (x^2 + y^2 + z^2) = \log z + \log c_2$$

$$\log (x^2 + y^2 + z^2) = \log (z C_2)$$

i.e., $zC_2 = x^2 + y^2 + z^2$

$$C_2 = \frac{x^2 + y^2 + z^2}{z}$$

Hence, the general solution is $f\left[\dfrac{y}{z}, \dfrac{x^2 + y^2 + z^2}{z}\right] = 0$, whe

is arbitrary.

Example 1.4b(10) : Solve $(z^2 - 2yz - y^2)\,p + (xy + zx)\,q = xy - z$
[A.U, March, 1996][A.U N/D 2015 R-13][A.U A/M 2017 R

Solution : Given : $(z^2 - 2yz - y^2)\,p + (xy + zx)\,q = xy - zx$

This equation is of the form $Pp + Qq = R$ where

$$P = z^2 - 2yz - y^2, \quad Q = xy + zx, \quad R = xy - zx$$

The Lagrange's subsidiary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

take last two ratios,

$$\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$(y - z)\,dy = (y + z)\,dz$$

$$y\,dy - z\,dy = y\,dz + z\,dz$$

$$y\,dy - z\,dy - y\,dz - z\,dz = 0$$

$$y\,dy - d(yz) - z\,dz = 0$$

Integrating, we get

$$\int y\,dy - \int d(yz) - \int z\,dz = \frac{C_1}{2}$$

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = \frac{C_1}{2}$$

$$y^2 - 2yz - z^2 = C_1$$

Choose **the multipliers x, y, z, we get**

$$= \frac{x\,dx + y\,dy + z\,dz}{xz^2 - 2xyz - xy^2 + xy^2 + xyz + xyz - z^2x}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e., $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\int x\,dx + \int y\,dy + \int z\,dz = \frac{C_2}{2}$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \frac{C_2}{2}$$

$$\text{i.e., } C_2 = x^2+y^2+z^2$$

Hence, the general solution is

$$f(y^2-2yz-z^2,\ x^2+y^2+z^2) = 0, \text{ where } f \text{ is arbitrary}$$

Example 1.4b(11) : Solve $(y+z)p+(z+x)q = x+y$.

[A.U. Dec., 1998] [A.U. N/D

Solution : Given $(y+z)p+(z+x)q = x+y$

This equation is of the form $Pp+Qq=R$

where P $= y+z$, Q $= z+x$, R $= x+y$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

$$= \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$= \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}$$

Take, $\dfrac{d(x-y)}{-(x-y)} = \dfrac{d(y-z)}{-(y-z)}$

$$\int\frac{d(x-y)}{x-y} = \int\frac{d(y-z)}{y-z}$$

$$\log(x-y) = \log(y-z)+\log c_1$$

$$\log\left(\frac{x-y}{y-z}\right) = \log c_1$$

$$c_1 = \frac{x-y}{y-z}, \quad \text{i.e., } \boxed{u = \frac{x-y}{y-z}}$$

Take, $\dfrac{d(y-z)}{-(y-z)} = \dfrac{d(x+y+z)}{2(x+y+z)}$

$$-2\frac{d(y-z)}{y-z} = \frac{d(x+y+z)}{x+y+z}$$

$$-2\int\frac{d(y-z)}{y-z} = \int\frac{d(x+y+z)}{x+y+z}$$

$$-2\log(y-z) = \log(x+y+z)-\log c_2$$

$$\log c_2 = \log(x+y+z)+2\log(y-z)$$

$$= \log(x+y+z)+\log(y-z)^2$$

$$= \log[(x+y+z)(y-z)^2]$$

$$c_2 = (x+y+z)(y-z)^2$$

$$v = (x+y+z)(y-z)^2$$

Hence, the general solution is $f(u,v) = 0$

$$\text{i.e., } f\left[\left(\frac{x-y}{y-z}\right),\ (y-z)^2(x+y+z)\right] = 0, \text{ where } f \text{ is arbitrary.}$$

Example 1.4b(12) : Solve $(y-xz)p+(yz-x)q = (x+y)(x-y)$

[A.U. Oct./Nov. 2002]

Sol. Given: $(y-xz)p+(yz-x)q = (x+y)(x-y)$ **[A.U N/D 2009]**

This equation is of the form $Pp+Qq=R$

where $P=(y-xz)$, $Q=yz-x$, $R=(x+y)(x-y)$

The Lagrange's subsidiary equations are $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\text{i.e., } \frac{dx}{y-xz} = \frac{dy}{yz-x} = \frac{dz}{(x+y)(x-y)} \quad \dots (1)$$

Choose Lagrangian multipliers x, y, z ,

$$= \frac{x\,dx+y\,dy+z\,dz}{xy-x^2z+y^2z-yx+z(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{xy - x^2 z + y^2 z - yx + zx^2 - zy^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

Hence, $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\int x\,dx + \int y\,dy + \int z\,dz = a$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}$$

$$x^2 + y^2 + z^2 = a$$

Choose Lagrangian multipliers y, x, 1 ,

$$= \frac{y\,dx + x\,dy + dz}{y^2 - xyz + xyz - x^2 + x^2 - y^2}$$

$$= \frac{y\,dx + x\,dy + dz}{0}$$

Hence, $y\,dx + x\,dy + dz = 0$

$$d(xy) + dz = 0$$

Integrating, we get

$$xy + z = b$$

Hence, the general solution is

$$f(a, b) = 0$$

i.e., $f(x^2 + y^2 + z^2,\ xy + z) = 0$, where f is arbitrary.

Example 1.4b(13) : Solve $(y - z)\,p - (2x + y)\,q = 2x + z$.

[A.U. April/May 20

Solution : Given : $(y - z)\,p - (2x + y)\,q = 2x + z$

This equation is of the form $P p + Q q = R$

where $P = y - z$, $Q = -(2x + y)$, $R = 2x + z$

The Lagrange's subsidiary equations are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{y - z} = \frac{dy}{-(2x + y)} = \frac{dz}{2x + z}$$

Choose Lagrangian multipliers 1, 1, 1,

$$= \frac{dx + dy + dz}{0}$$

Hence, $d(x + y + z) = 0$

i.e., $x + y + z = a \quad \dots (1)$

Take, $$\boxed{\frac{dx}{y - z} = \frac{dz}{2x + z}}$$

$$\frac{dx}{a - z - x - z} = \frac{dz}{2x + z} \quad [\text{by (1) } y = a - x - z]$$

$$\frac{dx}{a - x - 2z} = \frac{dz}{2x + z}$$

$$(2x + z)\,dx = (a - x - 2z)\,dz$$

$$2x\,dx + z\,dx = a\,dz - x\,dz - 2z\,dz$$

$$2x\,dx + z\,dx + x\,dz - a\,dz + 2z\,dz = 0$$

$$2x\,dx + d(xz) - a\,dz + 2z\,dz = 0$$

Integrating, we get

$$2\frac{x^2}{2} + xz - az + \frac{2z^2}{2} = b$$

$$x^2 + xz - (x + y + z)z + z^2 = b \quad \text{by (1)}$$

$$x^2 + xz - xz - yz - z^2 + z^2 = b$$

$$x^2 - yz = b$$

Hence, the general solution is $f(a, b) = 0$

i.e., $f(x + y + z,\ x^2 - yz) = 0$, where f is arbitrary.

Example 1.4b(14) : Show that the integral surface of the equation $2y(z-3)p + (2x - z)q = y(2x-3)$ that passes through the circle $x^2 + y^2 = 2x$, $z = 0$ is $x^2 + y^2 - z^2 - 2x + 4z = 0$.

Solution :

Given : $2y(z-3)p + (2x-z)q = y(2x-3)$

This equation is of the form $Pp + Qq = R$

where $P = 2y(z-3)$, $Q = 2x - z$, $R = y(2x-3)$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \quad \dots (1)$$

Take, $$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$$(2x-3)\,dx = (2z-6)\,dz$$

Integrating, we get

$$2\frac{x^2}{2} - 3x = 2\frac{z^2}{2} - 6z + a$$

$$x^2 - 3x = z^2 - 6z + a$$

$$x^2 - z^2 - 3x + 6z = a \quad \dots (2)$$

using the multipliers 1, $2y$, -2 each of the ratio in (1)

$$= \frac{dx + 2y\,dy - 2dz}{0}$$

i.e., $dx + 2y\,dy - 2dz = 0$

Integrating, we get

$$x + y^2 - 2z = b \quad \dots (3)$$

The required surface has to pass through

$$x^2 + y^2 = 2x \quad \dots (4)$$

and $$z = 0 \quad \dots (5)$$

using (5) in (2) and (3), we get

$$x^2 - 3x = a \quad \dots (6)$$

$$x + y^2 = b \quad \dots (7)$$

from (6) & (7), we get

$$x^2 + y^2 - 2x = a + b \quad \dots (8)$$

using (4) in (8), we have

$$a + b = 0 \quad \dots (9)$$

substitute for a and b from (2) and (3) in (9), we get

$x^2 + y^2 - z^2 - 2x + 4z = 0$ which is the equation of the required integral surface.

EXERCISES 1.4.b

Find the general solution of

1. $z(x-y) = x^2p - y^2q$ [**Ans.** $f\left(\frac{1}{x} + \frac{1}{y}, \frac{z}{x+y}\right) = 0$]

2. $(x-y)p + (y - x - z)q = z$ [**Ans.** $f\left(x + y + z, \frac{z^2}{x - y + z}\right) = 0$]

3. $(1+y)p + (1+x)q = z$ [**Ans.** $f\left[z(y-x), \frac{x+y+2}{z}\right] = 0$]

4. $\left(\frac{b-c}{a}\right) yzp + \left(\frac{c-a}{b}\right) xzq = \left(\frac{a-b}{c}\right) xy$

   [**Ans.** $f(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2) = 0$]

5. $y^2p - xyq = x(z - 2y)$ [**Ans.** $f(x^2 + y^2, yz - y^2) = 0$]

6. $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$ **[A.U N/D 2008]**

[Ans. $f[x^2+y^2-2z\,,\, xyz] = 0$ **]**

7. $z(xp+yq) = y^2-x^2$ **[Ans.** $f\left[\dfrac{x}{y},\, x^2-y^2+z^2\right] = 0$**]**

8. $(3x+y-z)p + (x+y-z)q = 2(z-y)$

[Ans. $f\left[x-3y-z\,,\, \dfrac{(x-y+z)^2}{x+y-z}\right] = 0$

9. $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$ **[A.U CBT A/M 201**

[A.U M/J 2012, A.U A/M 2017 R

[Ans. $f\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z},\, xyz\right) = 0$ **]** **[A.U A/M 2019 R-1**

10. $x(y^2+z^2)p + y(z^2+x^2)q = z(y^2-x^2)$

[Ans. $f\left[\dfrac{yz}{x},\, x^2-y^2+z^2\right] = 0$**]**

11. $(x-2z)p + (2z-y)q = y-x$

[Ans. $f[x+y+z\,,\, xy+z^2] = 0$**]** **[AU N/D 201**

12. $(p-q)z = z^2 + (x+y)$

[Ans. $f\left(x+y\,,\, 2x-\log(x+y+z^2)\right) = 0$**]**

13. $(y+zx)p - (x+yz)q = x^2-y^2$

[Ans. $f\left((x+y)^2-(z-1)^2\,,\, (x-y)^2-(z+1)^2\right) = 0$ **]**

14. $x^2p + y^2q = z(x+y)$

[Ans. $f\left(\dfrac{x-y}{xy},\, \dfrac{x-y}{z}\right) = 0$**]**

15. Solve the Lagrange's equation $(x+2z)p + (2xz-y)q = x^2+y$ **[A.U M/J 201**

[Ans. $f[xy-z^2, x^2-2y-2z] = 0$ **]**

1.5 LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS OF BOTH HOMOGENEOUS AND NON-HOMOGENEOUS TYPES

1.5.1 Partial differential equation of Higher order

Linear partial differential equations of higher order with constant co-efficients may be divided into two categories as given below.

(i) homogeneous p.d.e. with constant co-efficients.

(ii) non-homogeneous p.d.e. with constant co-efficients.

We shall use the differential operators D and D′ to denote $D = \dfrac{\partial}{\partial x}$ and $D' = \dfrac{\partial}{\partial y}$

Example : (i) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = x$

(ii) $\dfrac{\partial^3 z}{\partial x^3} + 3\dfrac{\partial^3 z}{\partial x^2\,\partial y} + 2\dfrac{\partial^3 z}{\partial y^3} = e^x$

The above equations can be written as

(i) $[D^2 + D'^2]z = x$

(ii) $[D^3 + 3D^2D' + 2D'^3]z = e^x$

Definition :

A linear p.d.e. with constant co-efficients in which all the partial derivatives are of the same order is called homogeneous ; otherwise it is called non-homogeneous.

1.5.2. Homogeneous linear equation

A homogeneous linear partial differential equation of *n*th order with constant co-efficients is of the form

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + a_n\frac{\partial^n z}{\partial y^n} = F(x,y) \quad \text{... (i)}$$

where a_i's are constants.

equation (i) can be written as

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n] z = F(x, y) \quad \dots (ii)$$

$$\text{(or)} \qquad f(D, D')z = F(x, y) \quad (iii)$$

The solution of $f(D, D')z = 0$ is called the complementary function C.F of (iii).

We find a particular integral (P.I.) of (iii) which is given by $\frac{1}{f(D, D')} F(x, y)$. Then

z = C.F. + P.I is the complete solution of (iii) (or) general solution.

Method of finding C.F.

1. To get the auxiliary equation of $f(D, D') = F(x, y)$ put $D = m$ and $D' = 1$.
2. The auxiliary equation is $f(D, D') = 0$
   i.e., $f(m, 1) = 0$

$$\text{i.e., } a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0 \quad \dots (2)$$

Let $m_1, m_2, \dots m_n$ be the roots of (2)

Case (i) The roots $m_1, m_2, \dots m_n$ are distinct

Then C.F. $= \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \dots + \phi_n(y + m_n x)$

where $\phi_1, \phi_2, \dots \phi_n$ are arbitrary functions.

Case (ii) The auxiliary equation has repeated roots

Suppose $m_1 = m_2 = \dots = m_r = m$

Then C.F. $= \phi_1(y + mx) + x\phi_2(y + mx) + \dots + x^{r-1}\phi_r(y + mx)$

where $\phi_1, \phi_2, \dots \phi_r$ are arbitrary functions.

To find the complementary functions :

	Replace D by m and D$'$ by 1

	Roots of A.E	C.F
1.	Roots are different $m_1 \neq m_2 \neq m_3 \neq m_4$	$\phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x) + \phi_4(y + m_4 x)$
2.	Roots are equal $m_1 = m_2 = m_3 = m_4 = m$(say)	$\phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + x^3\phi_4(y + mx)$
	Note :- $\alpha + i\beta$ and $\alpha - i\beta$ are different roots.	

To find the particular integral : P.I $= \frac{1}{f(D, D')} F(x, y)$

	F (x, y)	P.I
1.	e^{ax+by}	P.I $= \frac{1}{f(D, D')} e^{ax+by}$ $= e^{ax+by} \frac{1}{f(a, b)}$, $f(a, b) \neq 0$ $= x e^{ax+by} \frac{1}{f'(a, b)}$, $f(a, b) = 0$, $f'(a, b) \neq 0$ [Here, differentiate Dr. p.w.r.to D]
2.	$x^r y^s$	P.I $= \frac{1}{f(D, D')} x^r y^s = [f(D, D')]^{-1} x^r y^s$ Expand $[f(D, D')]^{-1}$ in ascending powers of D or D$'$ and then operate.
3.	$\sin(ax + by)$ (or) $\cos(ax + by)$	P.I $= \frac{1}{f(D, D')} [\sin(ax + by)$ (or) $\cos(ax + by)]$ Replace D^2 by $-a^2$, DD' by $-ab$, D'^2 by $-b^2$
4.	$e^{ax+by}\phi(x, y)$	P.I $= \frac{1}{f(D, D')} e^{ax+by}\phi(x, y)$ $= e^{ax+by} \frac{1}{f(D + a, D' + b)} \phi(x, y)$

5.	$\sin ax \sin by$ (or) $\cos ax \cos by$	P.I $= \frac{1}{f(D^2, D'^2)}$ [$\sin ax \sin by$ (or) $\cos ax \cos by$] Replace D^2 by $-a^2$, D'^2 by $-b^2$

Note : To find :

(1) $\frac{1}{D - mD'} f(x,y) = \int f(x, y - mx)\, dx$

Treat y as constant and in the resulting integral, change y to $y + mx$

(2) $\frac{1}{D + mD'} f(x,y) = \int f(x, y + mx)\, dx$

Treat y as constant and in the resulting integral, change y to $y - mx$

Problems based on Homogeneous equations

Example 1.5.1 : Solve $(D^2 - 4DD' + 3D'^2)\, z = 0$.

Solution : Given $[D^2 - 4DD' + 3D'^2] z = 0$

The auxiliary equation is $m^2 - 4m + 3 = 0$

[Replace D by m and D′ by 1]

Solving, $m^2 - 3m - m + 3 = 0$

$m(m-3) - 1(m-3) = 0$

$(m-1)(m-3) = 0$

$m = 1, m = 3$

Here, the roots are distinct

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y + 3x)$

Since, R.H.S. is zero, there is no particular integral

Hence, the general solution is z = C.F.

$= \phi_1(y + x) + \phi_2(y + 3x)$

Example 1.5.2 : Solve $[D^2 - 2DD' + D'^2]\, z = 0$.

Solution : $[D^2 - 2DD' + D'^2] z = 0$.

The auxiliary equation is $m^2 - 2m + 1 = 0$

[Replace D by m and D′ by 1]

i.e., $(m-1)^2 = 0$

$m = 1, 1$

Here, the roots are equal

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x)$.

Since R.H.S. is zero, there is no particular integral

Hence, the general solution is z = C.F

$z = \phi_1(y + x) + x\phi_2(y + x)$.

Example 1.5.3 : Solve $[D^3 + DD'^2 - D^2D' - D'^3]\, z = 0$

Solution : Given $[D^3 + DD'^2 - D^2D' - D'^3] z = 0$

The auxiliary equation is $m^3 - m^2 + m - 1 = 0$

[Replace D by m and D′ by 1]

$m^2(m-1) + (m-1) = 0$

$(m-1)(m^2+1) = 0$

$m = 1, m^2 + 1 = 0$

i.e., $m = 1, m = \pm i$

i.e., $m = 1, m = i, m = -i$

Here, the roots are distinct

$\therefore$ C.F $= \phi_1(y + x) + \phi_2(y + ix) + \phi_3(y - ix)$

Since R.H.S. is zero, there is no particular integral

Hence, the general solution is

$$z = \phi_1 (y + x) + \phi_2 (y + ix) + \phi_3 (y - ix)$$

Example 1.5.4 : Solve 2r + 5s – 3t = 0.

Solution : The given differential equation can be written as

$$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x\, \partial y} - 3\frac{\partial^2 z}{\partial y^2} = 0$$

i.e., $(2D^2 + 5DD' - 3D'^2) z = 0$

The auxiliary equation is $2m^2 + 5m - 3 = 0$

[Replace D by m and D′ by 1]

$$2m^2 + 6m - m - 3 = 0$$

$$2m (m + 3) - 1 (m + 3) = 0$$

$$(m + 3)(2m - 1) = 0$$

$$m = -3,\ m = \frac{1}{2}$$

Here the roots are distinct.

$$\text{C.F.} = \phi_1 (y - 3x) + \phi_2 \left(y + \frac{1}{2}x\right)$$

Since R.H.S. is zero, there is no particular integral

Hence, the general solution is

$$\therefore\ z = \phi_1 (y - 3x) + \phi_2 \left(y + \frac{1}{2}x\right)$$

Example 1.5.5 : Solve $(D^4 - D'^4) z = 0$ [A.U CBT A/M

Solution : Given $(D^4 - D'^4) z = 0$ [A.U M/J

The auxiliary equation is $m^4 - 1 = 0$
[Replace D by m and D' by 1]

$$\text{i.e., } (m^2)^2 - 1^2 = 0$$

Solving, $(m^2 - 1)(m^2 + 1) = 0$

$$m^2 - 1 = 0,\ m^2 + 1 = 0$$

$$m^2 = 1,\ m^2 = -1$$

$$m = \pm 1,\ m = \pm\sqrt{-1} = \pm i$$

$$\text{i.e., } m = 1, -1, i, -i$$

Here, the roots are distinct

$$\therefore\ \text{C.F} = \phi_1 (y + x) + \phi_2 (y - x) + \phi_3 (y + ix) + \phi_4 (y - ix)$$

Since R.H.S. is zero, there is no particular integral

∴ Hence, the general solution is

$$z = \phi_1 (y + x) + \phi_2 (y - x) + \phi_3 (y + ix) + \phi_4 (y - ix)$$

EXERCISES

I. Solve : [R.H.S = 0, unequal roots]

1. $[D^2 - 7DD' + 6D'^2] z = 0$ **[AU M/J 2012]** **[Ans.** $z = \phi_1 (y + 6x) + \phi_2 (y + x)$**]**
2. $[D^2 - DD' - 6D'^2] z = 0$ **[Ans.** $z = \phi_1 (y - 2x) + \phi_2 (y + 3x)$**]**
3. $[D^2 + 3DD' + 2D'^2] z = 0$ **[Ans.** $z = \phi_1 (y - x) + \phi_2 (y - 2x)$**]**
4. $[D^2 - 3DD' + 2D'^2] z = 0$ **[Ans.** $z = \phi_1 (y + x) + \phi_2 (y + 2x)$**]**
5. $(D^2 - DD') z = 0$ **[Ans.** $z = \phi_1 (y + 0x) + \phi_2 (y + x)$**]**
6. $[D^2 + DD' - 6D'^2] z = 0$ **[Ans.** $z = \phi_1 (y + 2x) + \phi_2 (y - 3x)$**]**
7. $[D^2 - DD' - 30D'^2] z = 0$ **[Ans.** $z = \phi_1 (y + 6x) + \phi_2 (y - 5x)$**]**
8. $[D^2 - 2DD'] z = 0$ **[Ans.** $z = \phi_1 (y + 0x) + \phi_2 (y + 2x)$**]**
9. $[2D^2 - 5DD' + 2D'^2] z = 0$ **[Ans.** $z = \phi_1 (y + 2x) + \phi_2 (y + \frac{1}{2}x)$**]**
10. $[D^2 - DD' - 20D'^2] z = 0$ **[Ans.** $z = \phi_1 (y + 5x) + \phi_2 (y - 4x)$**]**
11. $[D^2 + 4DD' - 5D'^2] z = 0$ **[Ans.** $z = \phi_1 (y + x) + \phi_2 (y - 5x)$**]**
12. $[D^2 - D'^2] z = 0$ **[Ans.** $z = \phi_1 (y + x) + \phi_2 (y - x)$**]**

13. $[D^2 - DD' - 2D'^2]z = 0$ **[Ans.** $z = \phi_1(y + 2x) + \phi_2(y - x)$**]**

14. $[D^2 + 3DD' - 4D'^2]z = 0$ **[Ans.** $z = \phi_1(y + x) + \phi_2(y - 4x)$**]**

15. $[D^2 - 6DD' + 5D'^2]z = 0$ **[Ans.** $z = \phi_1(y + 5x) + \phi_2(y + x)$**]**

16. $[5D^2 - 12DD' - 9D'^2]z = 0$ **[Ans.** $z = \phi_1\left(y - \frac{3}{5}x\right) + \phi_2(y +$

II. Solve : [R.H.S = 0, unequal roots of Higher order].

1. $[D^3 - 4D^2D' + 4DD'^2]z = 0$ **[A.U A/M 2015 R-2008]**
   [Ans. $z = \phi_1(y + 0x) + \phi_2(y + 2x) + x\phi_3(y + 2x)$**]**

2. $[D^3 - 7DD'^2 - 6D'^3]z = 0$ **[Ans.** $z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y +$

3. $[D^3 - 3DD'^2 + 2D'^3]z = 0$ **[Ans.** $z = \phi_1(y + x) + x\,\phi_2(y + x) + \phi_3(y -$
   [A.U CBT N/D 2010]

4. $[D^3 + D^2D' - DD'^2 - D'^3]z = 0$ **[Ans.** $z = \phi_1(y + x) + \phi_2(y - x) + x\phi_3(y -$

III. Solve : [R.H.S = 0, equal roots]

1. $(D^2 - 4DD' + 4D'^2)z = 0$ **[Ans.** $z = \phi_1(y + 2x) + x\,\phi_2(y +$

2. $(D + D')^2 z = 0$ **[Ans.** $z = \phi_1(y - x) + x\,\phi_2(y - x$

3. $[4D^2 - 12DD' + 9D'^2)z = 0$ **[Ans.** $z = \phi_1(y + 3/2\,x) + x\,\phi_2(y + 3$

IV. Solve : [R.H.S = 0, different types of roots]

1. $\dfrac{\partial^2 z}{\partial t^2} - a^2 \dfrac{\partial^2 z}{\partial x^2} = 0$ **[Ans.** $z(x, t) = \phi_1\left(x + \frac{1}{a}t\right) + \phi_2\left(x -\right.$

2. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\,\partial y} = 0$ **[Ans.** $z = \phi_1(y + 0x) + \phi_2(y - x)$**]**

3. $r = a^2 t$ **[Ans.** $z = \phi_1(y + ax) + \phi_2(y - ax)$**]**

4. $2r + 5s + 2t = 0$ **[Ans.** $z = \phi_1(y - 2x) + \phi_2(y - x)$**]**

5. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2\,\partial y} - 4\dfrac{\partial^3 z}{\partial x\,\partial y^2} + 8\dfrac{\partial^3 z}{\partial y^3} = 0$
   [Ans. $z = \phi_1(y + 2x) + x\,\phi_2(y + 2x) + \phi_3(y - 2$

Note : To solve $[DD' - D'^2]z = 0$

A.E is $m - m^2 = 0$ [Replace D by 1 and D' by m]

$m(m - 1) = 0 \Rightarrow m = 0, m = 1$

$\therefore$ The solution is $z = \phi_1[x + 0y] + \phi_2[x + (1)(y)]$

i.e., $z = \phi_1(x) + \phi_2(x + y)$

R.H.S = e^{ax+by} Replace D by a and D′ by b

Example 1.5.6 : Solve : $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x\,\partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = e^{x+y}$. **[M.U. AP 88]**

Solution : Given : $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x\,\partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = e^{x+y}$

i.e, $[D^2 - 5DD' + 6D'^2]z = e^{x+y}$

The auxiliary equation is $m^2 - 5m + 6 = 0$

[Replace D by m and D′ by 1]

$m = 2, 3$. Here, the roots are distinct.

$$\text{C.F} = \phi_1(y + 2x) + \phi_2(y + 3x)$$

$$\text{P.I} = \frac{1}{D^2 - 5DD' + 6D'^2}e^{x+y} = e^{x+y}\left[\frac{1}{1 - 5 + 6}\right],$$

[Replace D by 1 and D′ by 1]

$$= e^{x+y}\left[\frac{1}{2}\right] = \frac{1}{2}e^{x+y}$$

Hence, the general solution is

$$z = \phi_1(y + 2x) + \phi_2(y + 3x) + \frac{1}{2}e^{x+y}$$

Example 1.5.7 : Find the P.I of $[D^2 + 4DD']\,z = e^x$.

[M.U. APRIL 96] [M.S.U. OCT...

Solution :
$$\text{P.I} = \frac{1}{D^2 + 4DD'} e^x$$
$$= \frac{1}{D^2 + 4DD'} e^{x+0y}$$
$$= e^x \left[\frac{1}{1 + 4(1)(0)}\right], \quad \text{[Replace D by 1 and D' by ...}$$
$$= e^x \left[\frac{1}{1}\right] = e^x$$

Example 1.5.8 : Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\, \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x-y}$

Solution : Given : $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\, \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x-y}$

i.e., $[D^2 - 4DD' + 4D'^2]\,z = e^{2x-y}$

The auxiliary equation is

$$m^2 - 4m + 4 = 0 \quad \text{[Replace D by } m \text{ and D' ...}$$
$$(m-2)^2 = 0$$
$$m = 2, 2$$

$$\text{C.F} = \phi_1(y+2x) + x\,\phi_2(y+2x)$$

$$\text{P.I.} = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x-y}$$
$$= e^{2x-y}\left[\frac{1}{4 - 4(2)(-1) + 4(-1)^2}\right] \quad \text{[Replace D by 2 and D' ...}$$
$$= e^{2x-y}\left[\frac{1}{16}\right] = \frac{1}{16} e^{2x-y}$$

$\therefore$ Hence, the general solution is z = C.F + P.I

$$z = \phi_1(y+2x) + x\,\phi_2(y+2x) + \frac{1}{16} e^{2x-y}$$

Example 1.5.9 : Solve $(D + D')^2 z = e^{x-y}$

Solution : Given : $(D + D')^2 z = e^{x-y}$

The auxiliary equation is

$$(m+1)^2 = 0 \quad \text{[Replace D by } m \text{ and D' by 1]}$$
$$m = -1, -1$$

Here, the roots are equal.

$$\therefore \quad \text{C.F} = \phi_1(y-x) + x\,\phi_2(y-x)$$

$$\text{P.I} = \frac{1}{(D+D')^2} e^{x-y} \quad \text{... (1)}$$
$$= \frac{1}{[1+(-1)]^2} e^{x-y} \quad \left[\begin{array}{l}\text{Replace D by 1} \\ \text{and D' by } -1\end{array}\right]$$
$$= \frac{1}{0} e^{x-y} \quad \text{[Ordinary rule fails]}$$
$$= x\,\frac{1}{2(D+D')} e^{x-y} \quad \text{... (2)}$$

[In (1), multiply x in the Nr. and differentiate Dr. p.w.r.to D]

$$= \frac{x}{2}\,\frac{1}{[1+(-1)]} e^{x-y} \quad \left[\begin{array}{l}\text{Replace D by 1} \\ \text{and D' by } -1\end{array}\right]$$
$$= \frac{x}{2}\,\frac{1}{0} e^{x-y} \quad \text{[Again Ordinary rule fails]}$$
$$= x^2\,\frac{1}{2} e^{x-y} \quad \text{[In (2), multiply x in the Nr. and differentiate Dr. p.w.r.to D]}$$
$$= \frac{x^2}{2} e^{x-y}$$

Hence, the general solution is

$$z = \text{C.F} + \text{P.I}$$
$$= \phi_1(y-x) + x\,\phi_2(y-x) + \frac{x^2}{2} e^{x-y}$$

Example 1.5.10 : Find the P.I of $(D^2 + DD')z = e^{x-y}$. [M.U. Oc...

Solution : Given : $(D^2 + DD')z = e^{x-y}$

$$P.I = \frac{1}{D^2 + DD'} e^{x-y} \quad \dots (1)$$

$$= \frac{1}{1^2 + (1)(-1)} e^{x-y} \quad \begin{bmatrix}\text{Replace D by 1} \\ \text{and D}' \text{ by } -1\end{bmatrix}$$

$$= \frac{1}{1-1} e^{x-y}$$

$$= \frac{1}{0} e^{x-y} \quad \text{[Ordinary rule fails]}$$

$$= x \frac{1}{2D + D'} e^{x-y} \quad \text{[In (1), multiply } x \text{ in the ... differentiate Dr. p.w.r. to ...]}$$

$$= x \frac{1}{2(1) + (-1)} e^{x-y} \quad \begin{bmatrix}\text{Replace D by 1} \\ \text{and D}' \text{ by } -1\end{bmatrix}$$

$$= x \frac{1}{2-1} e^{x-y}$$

$$= x\, e^{x-y}$$

Example 1.5.11 : Solve $(D^3 - 3DD'^2 + 2D'^3)z = e^{2x-y} + e^{x+y}$

Solution : Given : $(D^3 - 3DD'^2 + 2D'^3)z = e^{2x-y} + e^{x+y}$

The auxiliary equation is $m^3 - 3m + 2 = 0$

$$m = 1, 1, -2$$

$$C.F = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x)$$

$$PI_1 = \frac{1}{D^3 - 3DD'^2 + 2D'^3} e^{2x-y} \quad \dots (1)$$

$$= \frac{1}{2^3 - 3(2)(-1)^2 + 2(-1)^3} e^{2x-y} \quad \begin{bmatrix}\text{Replace D by 2} \\ \text{and D}' \text{ by } -1\end{bmatrix}$$

$$= \frac{1}{8-6-2} e^{2x-y}$$

$$= \frac{1}{0} e^{2x-y} \quad \text{[Ordinary rule fails]}$$

$$= x \frac{1}{3D^2 - 3D'^2} e^{2x-y} \quad \text{[In (1), multiply } x \text{ in the Nr and differentiate Dr. p.w.r. to } D]$$

$$= x \frac{1}{3(2)^2 - 3(-1)^2} e^{2x-y} \quad \begin{bmatrix}\text{Replace D by 2} \\ \text{and D}' \text{ by } -1\end{bmatrix}$$

$$= x \frac{1}{12-3} e^{2x-y}$$

$$= \frac{x}{9} e^{2x-y}$$

$$PI_2 = \frac{1}{D^3 - 3DD'^2 + 2D'^3} e^{x+y} \quad \dots (2)$$

$$= \frac{1}{1^3 - 3(1)(1) + 2(1)^3} e^{x+y} \quad \begin{bmatrix}\text{Replace D by 1} \\ \text{and D}' \text{ by } 1\end{bmatrix}$$

$$= \frac{1}{1-3+2} e^{x+y}$$

$$= \frac{1}{0} e^{x+y} \quad \text{[Ordinary rule fails]}$$

$$= x \frac{1}{3D^2 - 3D'^2} e^{x+y} \quad \dots (3) \quad \text{[In (2), multiply } x \text{ in the Nr and differentiate Dr. p.w.r. to } D]$$

$$= x \frac{1}{3(1^2) - 3(1)^2} e^{x+y} \quad \begin{bmatrix}\text{Replace D by 1} \\ \text{and D}' \text{ by } 1\end{bmatrix}$$

$$= x \frac{1}{0} e^{x+y} \quad \text{[Again ordinary rule fails]}$$

$$= x^2 \frac{1}{6D} e^{x+y} \quad \text{[In (3), multiply } x \text{ in the Nr and differentiate Dr. p.w.r. to } D]$$

$$= x^2 \frac{1}{6(1)} e^{x+y} \qquad \begin{bmatrix} \text{Replace D by 1} \\ \text{and D' by 1} \end{bmatrix}$$

$$= \frac{x^2}{6} e^{x+y}$$

Hence, the general solution is

$$\therefore z = CF + PI_1 + PI_2$$

$$= \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x) + \frac{x}{9} e^{2x-y} + \frac{x^2}{6} e^{x+y}$$

EXERCISES

Find the P.I of the following :

1. $[D^2 - 4DD' + 4D'^2]z = e^{2x+y}$ [A.U. M/J 2006]

   [Ans. P.I $= \frac{x^2}{2} e^{2x+y}$] [A.U N/D 2018 R-17]

2. (a) $\frac{\partial^3 z}{\partial x^3} - 3\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) + 4\frac{\partial^3 z}{\partial y^3} = e^{2x+3y}$

   [Ans. P.I $= \frac{1}{80} e^{2x+3y}$]

2. (b) $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

   [Ans. P.I $= \frac{1}{27} e^{x+2y}$]

3. $[D^2 - 2DD' + D'^2]z = 8e^{x+2y}$ [A.U. A/M 2001]

   [Ans. P.I $= 8e^{x+2y}$]

4. $[D^2 - 3DD' + 2D'^2]z = e^x \cosh y$

   [Ans. P.I $= -\frac{1}{2} x e^{x+y} + \frac{1}{12} e^{x-y}$]

5. $[D^2 - 3DD' + 2D'^2]z = 2\cosh(3x+4y)$

   [Ans. P.I $= \frac{2}{5}\cosh(3x+4y)$]

6. $[D^4 - 2D^3D' + 2DD'^3 - D'^4]z = e^{2x+3y}$

   [Ans. P.I $= -\frac{1}{5} e^{2x+3y}$]

R.H.S $= x^r y^s$

Example 1.5.12 : Solve $[D^2 - 7DD' + 6D'^2]z = xy$

Solution : Given : $[D^2 - 7DD' + 6D'^2]z = xy$

The auxiliary equation is

$$m^2 - 7m + 6 = 0 \qquad [\text{Replace } D \text{ by } m \text{ and } D' \text{ by } 1]$$

$$(m-6)(m-1) = 0$$

$$m = 6,\ m = 1 \qquad \text{i.e., } m = 1,\ m = 6$$

$$\text{C.F} = \phi_1(y+x) + \phi_2(y+6x)$$

$$\text{P.I} = \frac{1}{D^2 - 7DD' + 6D'^2} xy$$

$$= \frac{1}{D^2} \frac{1}{1 - \left(\frac{7D'}{D} - \frac{6D'^2}{D^2}\right)} xy$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{7D'}{D} - \frac{6D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{7D'}{D} - \frac{6D'^2}{D^2}\right)\right] xy$$

$$[\because (1-x)^{-1} = 1 + x + x^2 + \ldots]$$

$$= \frac{1}{D^2} \left[1 + \frac{7D'}{D}\right] xy \qquad [\text{Omit } D'^2 \text{ and higher powers}]$$

$$= \frac{1}{D^2} \left[xy + \frac{7}{D}(x)\right]$$

$$= \frac{1}{D^2}xy + \frac{7}{D^3}(x)$$

$$= y\frac{1}{D^2}(x) + \frac{7}{D^3}(x)$$

$$\boxed{\begin{array}{l} \frac{1}{D}(x) = \int x\,dx = \frac{x^2}{2}, \quad \frac{1}{D^2}(x) = \int \frac{x^2}{2}dx = \frac{x^3}{6} \\ \frac{1}{D^3}(x) = \int \frac{x^3}{6}dx = \frac{x^4}{24} \end{array}}$$

$$= y\left(\frac{x^3}{6}\right) + 7\left(\frac{x^4}{24}\right) = \frac{x^3}{24}[4y + 7x]$$

Hence, the general solution is

$$z = \text{C.F} + \text{P.I}$$

$$= \phi_1(y + x) + \phi_2(y + 6x) + \frac{x^3}{24}(4y + 7x)$$

Example 1.5.13 : Solve $[D^2 + 3DD' + 2D'^2]z = x + y$

[AU Trichy N/D 20...

Solution : Given : $[D^2 + 3DD' + 2D'^2]z = x + y$

The auxiliary equation is

$$m^2 + 3m + 2 = 0 \qquad \text{[Replace } D \text{ by } m \text{ and } D' \text{ b...}$$

$$(m + 1)(m + 2) = 0$$

$$m = -1, m = -2$$

$$\text{C.F} = \phi_1(y - x) + \phi_2(y - 2x)$$

$$\text{P.I} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$$

$$= \frac{1}{D^2}\left[\frac{1}{1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)}\right](x + y)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(x + y)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right](x + y)$$

$$[\because (1 + x)^{-1} = 1 - x + x^2 - \ldots]$$

$$= \frac{1}{D^2}\left[1 - \frac{3D'}{D}\right](x + y)$$

[Omit D'^2 and higher powers]

$$= \frac{1}{D^2}\left[x + y - \frac{3}{D}(1)\right] \qquad [\because D'(x + y) = 0 + 1]$$

$$= \frac{1}{D^2}\left[x + y - \frac{3}{D}\right]$$

$$= \frac{1}{D^2}(x) + \frac{1}{D^2}(y) - \frac{3}{D^3}$$

$$= \frac{1}{D^2}(x) + y\frac{1}{D^2}(1) - 3\frac{1}{D^3}(1)$$

$$\boxed{\begin{array}{l} \frac{1}{D}(x) = \int x\,dx = \frac{x^2}{2}, \quad \frac{1}{D^2}(x) = \int \frac{x^2}{2}dx = \frac{x^3}{6} \\ \hline \frac{1}{D}(1) = \int dx = x, \quad \frac{1}{D^2}(1) = \int x\,dx = \frac{x^2}{2} \\ \frac{1}{D^3}(1) = \int \frac{x^2}{2}dx = \frac{x^3}{6} \end{array}}$$

$$= \frac{x^3}{6} + y\frac{x^2}{2} - 3\frac{x^3}{6} = \frac{x^2y}{2} - \frac{x^3}{3} = \frac{x^2}{6}[3y - 2x]$$

Hence, the general solution is

$$z = \text{C.F} + \text{P.I}$$

$$= \phi_1(y - x) + \phi_2(y - 2x) + \frac{x^2}{6}(3y - 2x)$$

Example 1.5.14 : Solve $[D^2 + 4DD' - 5D'^2]\, z = x + y^2 + \pi$

Solution : Given : $[D^2 + 4DD' - 5D'^2] z = x + y^2 + \pi$

The auxiliary equation is

$$m^2 + 4m - 5 = 0$$

[Replace D by m and D' by 1]

$$(m+5)(m-1) = 0$$

$$m = -5,\ m = 1$$

Here, the roots are distinct.

$$\therefore \quad C.F = \phi_1(y - 5x) + \phi_2(y + x)$$

$$P.I = \frac{1}{D^2 + 4DD' - 5D'^2}(x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[\frac{1}{1 + \frac{4D'}{D} - \frac{5D'^2}{D^2}}\right](x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right)\right]^{-1}(x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right) + \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right)^2 - \ldots\right](x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 - \frac{4D'}{D} + \frac{5D'^2}{D^2} + \frac{16D'^2}{D^2} + \frac{25D'^4}{D^4} - \frac{40D'^3}{D^3} - \ldots\right](x + y^2 + \pi)$$

[R.H.S $= x + y^2 + \pi$, $\therefore$ Omitting D'^3 and higher powers]

$$= \frac{1}{D^2}\left[1 - \frac{4D'}{D} + \frac{5D'^2}{D^2} + \frac{16D'^2}{D^2}\right](x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 - \frac{4D'}{D} + \frac{21}{D^2}D'^2\right](x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[(x + y^2 + \pi) - \frac{4}{D}(2y) + \frac{21}{D^2}(2)\right]$$

$$= \frac{1}{D^2}\left[x + y^2 + \pi - \frac{8}{D}y + \frac{42}{D^2}\right]$$

$$= \frac{1}{D^2}(x) + \frac{1}{D^2}(y^2) + \frac{1}{D^2}(\pi) - \frac{8}{D^3}(y) + \frac{42}{D^4}(1)$$

$$= \frac{1}{D^2}(x) + y^2\frac{1}{D^2}(1) + \pi\frac{1}{D^2}(1) - 8y\frac{1}{D^3}(1) + 42\frac{1}{D^4}(1)$$

$\frac{1}{D}(x) = \int x\,dx = \frac{x^2}{2}$,	$\frac{1}{D^2}(x) = \int \frac{x^2}{2}dx = \frac{x^3}{6}$
$\frac{1}{D}(1) = \int dx = x$,	$\frac{1}{D^2}(1) = \int x\,dx = \frac{x^2}{2}$
$\frac{1}{D^3}(1) = \int \frac{x^2}{2}dx = \frac{x^3}{6}$,	$\frac{1}{D^4}(1) = \int \frac{x^3}{6}dx = \frac{x^4}{24}$

$$= \frac{x^3}{6} + y^2\left(\frac{x^2}{2}\right) + \pi\left(\frac{x^2}{2}\right) - 8y\left(\frac{x^3}{6}\right) + 42\left(\frac{x^4}{24}\right)$$

$$= \frac{x^3}{6} + \frac{x^2y^2}{2} + \frac{\pi x^2}{2} - \frac{4x^3y}{3} + \frac{7x^4}{4}$$

Hence, the general solution is

$$z = \phi_1(y - 5x) + \phi_2(y + x) + \frac{x^3}{6} + \frac{x^2}{2}(y^2 + \pi) - \frac{4}{3}x^3y + \frac{7}{4}x^4$$

EXERCISES

Find the Particular Integral of

1. $(D^2 + 2DD' + D'^2)z = x^2y$ [A.U. A/M 1996]

   [Ans. P.I $= \frac{x^4y}{12} - \frac{x^5}{30}$]

2. $[D^2 - DD' - 6D'^2]z = x^2y$ [Ans. P.I $= \frac{x^4}{60}(5y + x)$]

3. $[D^3 - 7DD'^2 - 6D'^3]z = x^2y$ [Ans. P.I $= \frac{x^5y}{60}$]

4. $[D^2 - 6DD' + 5D'^2]z = xy$ [Ans. P.I $= \frac{x^4}{4} + \frac{x^3y}{6}$]

5. $[D^2 + DD' - 6D'^2]z = x^2y$ [Ans. P.I $= \frac{x^4y}{12} - \frac{x^5}{60}$]

6. Find the general solution of $[D^2 + D'^2]z = x^2y^2$

[Ans. $z = \phi_1(y - ix) + \phi_2(y + ix) + \frac{x^4y^2}{12} - \frac{x^6}{180}$]

[A.UN/D 2015 R-13]

R.H.S $= e^{ax+by} + x^r y^s$

Example 1.5.15 : Solve : $[D^2 - DD' - 6D'^2]z = x^2y + e^{3x+y}$.

Solution : Given : $[D^2 - DD' - 6D'^2]z = x^2y + e^{3x+y}$

The auxiliary equation is

$$m^2 - m - 6 = 0 \quad \text{[Replace D by } m \text{ and D' by 1]}$$

$$m = -2, 3$$

$$\text{C.F} = \phi_1(y - 2x) + \phi_2(y + 3x)$$

$$\text{P.I}_1 = \frac{1}{D^2 - DD' - 6D'^2}x^2y = \frac{1}{D^2}\left[\frac{1}{1 - \left(\frac{D'}{D} + \frac{6D'^2}{D^2}\right)}\right]x^2y$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} + \frac{6D'^2}{D^2}\right)\right]^{-1}x^2y$$

$$= \frac{1}{D^2}\left[1 + \frac{D'}{D} + \frac{6D'^2}{D^2}\right]x^2y = \frac{1}{D^2}\left[x^2y + \frac{x^2}{D}\right]$$

$$= \frac{1}{D^2}(x^2y) + \frac{1}{D^3}(x^2)$$

$$= y\frac{1}{D^2}(x^2) + \frac{1}{D^3}(x^2)$$

$$\boxed{\begin{array}{l}\frac{1}{D}(x^2) = \int x^2\,dx = \frac{x^3}{3}, \quad \frac{1}{D^2}(x^2) = \int \frac{x^3}{3}\,dx = \frac{x^4}{12} \\ \frac{1}{D^3}(x^2) = \int \frac{x^4}{12}\,dx = \frac{x^5}{60}\end{array}}$$

$$= y\left(\frac{x^4}{12}\right) + \frac{x^5}{60}$$

$$= \frac{x^4}{60}[5y + x]$$

$$\text{P.I}_2 = \frac{1}{D^2 - DD' - 6D'^2}e^{3x+y}$$

$$= \frac{1}{9 - (3)(1) - 6(1)}e^{3x+y} \quad \text{[Replace } D \text{ by 3 and } D' \text{ by 1]}$$

$$= \frac{1}{0}e^{3x+y} \quad \text{[Ordinary rule fails]}$$

$$= x\frac{1}{2D - D'}e^{3x+y}$$

$$= x\frac{1}{6 - 1}e^{3x+y} \quad \text{[Replace } D \text{ by 3 and } D' \text{ by 1]}$$

$$= \frac{x}{5}e^{3x+y}$$

Hence, the general solution is

$$z = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

i.e., $z = \phi_1(y - 2x) + \phi_2(y + 3x) + \frac{x^4}{60}(x + 5y) + \frac{x}{5}e^{3x+y}$

Example 1.5.16 : Solve $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$.

[A.U. N/D 2005]

Solution : The auxiliary equation is $m^2 + m - 6 = 0$

$$m = 2, -3$$

$$\text{C.F} = \phi_1(y + 2x) + \phi_2(y - 3x)$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 + DD' - 6D'^2} x^2 y$$

$$= \frac{1}{D^2\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]} x^2 y$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]^{-1} x^2 y$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right) + \ldots\right] x^2 y$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D}\right] x^2 y \quad [\text{Omitting } D'^2 \text{ and higher po}$$

$$= \frac{1}{D^2}\left[x^2 y - \frac{1}{D}(x^2)\right]$$

$$= \frac{1}{D^2}(x^2 y) - \frac{1}{D^3}(x^2)$$

$$= y\frac{1}{D^2}(x^2) - \frac{1}{D^3}(x^2)$$

$$\frac{1}{D}(x^2) = \int x^2\,dx = \frac{x^3}{3}, \qquad \frac{1}{D^2}(x^2) = \int \frac{x^3}{3}\,dx = \frac{x^4}{12}$$

$$\frac{1}{D^3}(x^2) = \int \frac{x^4}{12}\,dx = \frac{x^5}{60}$$

$$= y\left[\frac{x^4}{12}\right] - \frac{x^5}{60}$$

$$= \frac{x^4 y}{12} - \frac{x^5}{60}$$

$$P.I_2 = \frac{1}{D^2 + DD' - 6D'^2} e^{3x+y}$$

$$= \frac{1}{9+3-6} e^{3x+y} \quad [\text{Replace D by 3 amd D' by 1}]$$

$$= \frac{1}{6} e^{3x+y}$$

Hence, the general solution is

$$\therefore z = C.F + P.I_1 + P.I_2$$

$$\text{i.e., } z = \phi_1(y+2x) + \phi_2(y-3x) + \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6}e^{3x+y}$$

Example 1.5.17 : Solve $(D^2 - 6DD' + 5D'^2)\, z = e^x \sinh y + xy$

[A.U. M/J 2006] [A.U N/D 2018 R-13]

Solution : Given : $[D^2 - 6DD' + 5D'^2]z = e^x \sinh y + xy$

$$= e^x\left[\frac{e^y - e^{-y}}{2}\right] + xy$$

$$= \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} + xy$$

The auxiliary equation is $m^2 - 6m + 5 = 0$

$$m^2 - 5m - m + 5 = 0$$

$$m(m-5) - 1(m-5) = 0$$

$$(m-5)(m-1) = 0$$

$$m = 5, m = 1$$

$$\text{i.e., } m = 1, m = 5$$

$$\therefore C.F = \phi_1(y+x) + \phi_2(y+5x)$$

$$P.I_1 = \frac{1}{D^2 - 6DD' + 5D'^2} \frac{e^{x+y}}{2}$$

$$= \frac{1}{2}\frac{1}{D^2 - 6DD' + 5D'^2} e^{x+y}$$

$$= \frac{1}{2}\,\frac{1}{1-6+5}\,e^{x+y}$$ [Replace D by 1 and D′ by
[ordinary rule fails]

$$= \frac{1}{2}x\,\frac{1}{2D-6D'}\,e^{x+y}$$

$$= \frac{x}{2}\,\frac{1}{2-6}e^{x+y}$$ [Replace D by 1 and D′ by

$$= \frac{-x}{8}\,e^{x+y}$$

$$P.I_2 = \frac{1}{D^2-6DD'+5D'^2}\,\frac{e^{x-y}}{2}$$

$$= \frac{1}{2}\,\frac{1}{D^2-6DD'+5D'^2}\,e^{x-y}$$

$$= \frac{1}{2}\,\frac{1}{1-6(1)(-1)+5(-1)^2}\,e^{x-y}$$ [Replace D by 1 and D′ by -

$$= \left(\frac{1}{2}\right)\left(\frac{1}{1+6+5}\right)e^{x-y}$$

$$= \frac{1}{2}\left(\frac{1}{12}\right)e^{x-y} = \left(\frac{1}{24}\right)e^{x-y}$$

$$P.I_3 = \frac{1}{D^2-6DD'+5D'^2}\,xy$$

$$= \frac{1}{D^2}\left[\frac{1}{1-\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)}\right]xy$$

$$= \frac{1}{D^2}\left[1-\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)\right]^{-1}xy$$

$$= \frac{1}{D^2}\left[1+\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)+\dots\right]xy$$

$$= \frac{1}{D^2}\left[1+\frac{6D'}{D}\right]xy$$ [Omitting D'^2 and higher powers]

$$= \frac{1}{D^2}\left[xy+\frac{6}{D}(x)\right]$$

$$= \frac{1}{D^2}(xy)+\frac{6}{D^3}(x)$$

$$= y\frac{1}{D^2}(x)+6\frac{1}{D^3}(x)$$

$$\boxed{\begin{aligned}\frac{1}{D}(x) &= \int x\,dx = \frac{x^2}{2}, \quad \frac{1}{D^2}(x) = \int\frac{x^2}{2}dx = \frac{x^3}{6}\\ \frac{1}{D^3}(x) &= \int\frac{x^3}{6}dx = \frac{x^4}{24}\end{aligned}}$$

$$= y\left(\frac{x^3}{6}\right)+6\left(\frac{x^4}{24}\right) = \frac{x^3y}{6}+\frac{x^4}{4}$$

$$\therefore\ z = C.F + P.I_1 - P.I_2 + P.I_3$$

Hence, the general solution is

$$z = \phi_1(y+x)+\phi_2(y+5x)-\frac{x}{8}e^{x+y}-\frac{1}{24}e^{x-y}+\frac{x^3y}{6}+\frac{x^4}{4}$$

Example 1.5.18 : Solve $(D^2+2DD'+D'^2)z = x^2y+e^{x-y}$

Solution : **[A.U. N/D 2006]**
[A.U A/M 2017 R-8] [A.U A/M 2017 R-13]

$$[D^2+2DD'+D'^2]z = x^2y$$

The auxiliary equation is

$$m^2+2m+1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1, -1$$

$$C.F. = \phi_1(y-x)+x\phi_2(y-x)$$

$$P.I_1 = \frac{1}{(D+D')^2} x^2 y$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)^2} x^2 y = \frac{1}{D^2}\left[1+\frac{D'}{D}\right]^{-2} x^2 y$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D} + \frac{3D'^2}{D^2}\right](x^2 y)$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right](x^2 y)$$ [Omitting D'^2 and higher power

$$= \frac{1}{D^2}\left[x^2 y - \frac{2}{D}(x^2)\right]$$

$$= \frac{1}{D^2}(x^2 y) - \frac{2}{D^3}(x^2)$$

$$= y\frac{1}{D^2}(x^2) - 2\frac{1}{D^3}(x^2)$$

$$\frac{1}{D}(x^2) = \int x^2 dx = \frac{x^3}{3}, \quad \frac{1}{D^2}(x^2) = \int \frac{x^3}{3} dx =$$

$$\frac{1}{D^3}(x^2) = \int \frac{x^4}{12} dx = \frac{x^5}{60}$$

$$= y\left(\frac{x^4}{12}\right) - 2\left(\frac{x^5}{60}\right)$$

$$= \frac{x^4 y}{12} - \frac{x^5}{30}$$

$$P.I_2 = \frac{1}{D^2 + 2DD' + D'^2} e^{x-y}$$

$$= \frac{1}{1-2+1} e^{x-y}$$ [Replace D by 1 and D' by

[ordinary rule fails]

$$= \frac{1}{0} e^{x-y}$$

$$= x\frac{1}{2D+2D'} e^{x-y} = \frac{x}{2}\frac{1}{D+D'} e^{x-y}$$

$$= \frac{x}{2}\frac{1}{1-1} e^{x-y}$$ [Replace D by 1 an D' by −1]

[ordinary rule fails]

$$= \frac{x}{2}\frac{1}{0} e^{x-y}$$

$$= \left(\frac{x}{2}\right) x \frac{1}{1} e^{x-y} = \frac{x^2}{2} e^{x-y}$$

$$\therefore z = C.F. + P.I_1 + P.I_2$$

$$= \phi_1(y-x) + x\phi_2(y-x) + \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2}e^{x-y}$$

EXERCISES

Find the Particular Integral of the following :

1. $[D^2 - DD' - 2D'^2]z = 2x + 3y + e^{3x+4y}$ [A.U. N/D 2007]

   [Ans. P.I $= \frac{5}{6}x^3 + \frac{3}{2}x^2 y - \frac{1}{35}e^{3x+4y}$] [A.U. N/D 2013]

2. $[D^2 - DD' - 30D'^2]z = xy + e^{6x+y}$

   [Ans. P.I $= \frac{x^3 y}{6} + \frac{x^4}{24} + \frac{x}{11}e^{6x+y}$]

3. $[D^2 - 2DD']z = x^3 y + e^{2x}$ [A.U. A/M. 2001, N/D 2004]

   [Ans. P.I $= \frac{x^5 y}{20} + \frac{x^6}{60} + \frac{1}{4}e^{2x}$] [A.U A/M 2019 R-13]

4. $(D^2 - 2DD')z = e^{2x} + x^3 y$

   [Ans. P.I $= \frac{1}{4}e^{2x} + \frac{x^5 y}{20} + \frac{x^6}{60}$]

5. $(D^3 - 2D^2 D')z = 2e^{2x} + 3x^2 y$ [A.U M/J 2016, R-13] [A.U N/D 2011 R-08] [A.U A/M 2019 R-17]

   [Ans. P.I $= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$]

6. $(D^2 - 4DD' + 4D'^2)z = xy + e^{2x+y}$

   [Ans. P.I $= \frac{x^2}{2}e^{2x+y} + \frac{x^3 y}{6} + \frac{x^4}{6}$]

R.H.S = sin (ax + by) or cos (ax + by)

Replace D^2 by $-a^2$, DD' by $-ab$, D'^2 by $-b^2$

Formulae

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

Example 1.5.19 : Solve $[D^2 - 2DD' + 2D'^2]\, z = \sin(x - y)$.

Solution : Given $[D^2 - 2DD' + 2D'^2]\, z = \sin(x-y)$

The auxiliary equation is $m^2 - 2m + 2 = 0$

[Replace D by m and D' by 1]

$$m = 1 \pm i$$

$$C.F = \phi_1[y + (1+i)x] + \phi_2[y + (1-i)x]$$

$$P.I = \frac{1}{D^2 - 2DD' + 2D'^2}\sin(x-y)$$

$$= \frac{\sin(x-y)}{-1 - 2(1) + 2(-1)}$$

$$= -\frac{1}{5}\sin(x-y)$$

Replace D^2 by $-1^2 = -1$

D'^2 by $-(-1)^2 = -1$

DD' by $-(1)(-1) = 1$

Hence, the general solution is $z = C.F + P.I$

$$z = \phi_1[y + (1+i)x] + \phi_2[y + (1-i)x] - \frac{1}{5}\sin(x-y)$$

Example 1.5.20 : Solve $[D^3 - 4D^2D' + 4DD'^2]\, z = 6\sin(3x + 6y)$.

Solution : Given : $[D^3 - 4D^2D' + 4DD'^2]\, z = 6\sin(3x+6y)$

The auxiliary equation is $m^3 - 4m^2 + 4m = 0$

$$\Rightarrow m(m^2 - 4m + 4) = 0$$

$$m = 0, \ (m-2)^2 = 0$$

$$m = 0, 2, 2$$

$$C.F = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

$$P.I = \frac{1}{D^3 - 4D^2D' + 4DD'^2}\, 6\sin(3x+6y)$$

$$= 6 \text{ I.P of } \frac{1}{D^3 - 4D^2D' + 4DD'^2}\, e^{i(3x+6y)}$$

$$= 6 \text{ I.P of } \frac{1}{(3i)^3 - 4(3i)^2(6i) + 4(3i)(6i)^2}\, e^{i(3x+6y)}$$

Replace D by $3i$, D' by $6i$

$$= 6 \text{ I.P of } \frac{1}{-27i + 216i - 432i}\, e^{i(3x+6y)}$$

$$= 6 \text{ I.P of } \frac{1}{-243i}\, e^{i(3x+6y)}$$

$$= \frac{2}{81} \text{ I.P of } \{i[\cos(3x+6y) + i\sin(3x+6y)]\}$$

$$= \frac{2}{81}\cos(3x+6y)$$

Hence, the general solution is

$$z = C.F + P.I$$

$$z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{2}{81}\cos(3x+6y)$$

Example 1.5.21 : Solve $[D^2 - 2DD' + D'^2]\, z = \cos(x - 3y)$.

Solution : Given : $[D^2 - 2DD' + D'^2]z = \cos(x-3y)$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1,\ 1$$

$$\text{C.F} = \phi_1(y+x) + x\phi_2(y+x)$$

$$\text{P.I} = \frac{1}{D^2 - 2DD' + D'^2}\cos(x-3y)$$

Replace D^2 by $-1^2 = -1$
DD' by $-(1)(-3) = 3$
D'^2 by $-(-3)^2 = -9$

$$= \frac{\cos(x-3y)}{-1-2(3)-9} = \frac{-1}{16}\cos(x-3y)$$

∴ Hence, the general solution is

$$z = \phi_1(y+x) + x\phi_2(y+x) - \frac{1}{16}\cos(x-3y)$$

Example 1.5 22 : Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} = \cos 2x \cos y$.

Solution : Given : $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} = \cos 2x \cos y$

i.e., $[D^2 + DD']\,z = \cos 2x \cos y$

The auxiliary equation is $m^2 + m = 0$

$$m(m+1) = 0$$

$$m = 0,\ m = -1$$

$$\therefore\ \text{C.F.} = \phi_1(y) + \phi_2(y-x)$$

$$\text{P.I} = \frac{1}{D^2+DD'}\cos 2x\cos y$$

$$= \frac{1}{D^2+DD'}\left[\frac{\cos(2x+y)+\cos(2x-y)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+DD'}\cos(2x+y) + \frac{1}{D^2+DD'}\cos(2x-y)\right]$$

$$= \frac{1}{2}\left[\frac{\cos(2x+y)}{-2^2-2(1)}\right] + \frac{1}{2}\left[\frac{\cos(2x-y)}{-2^2-2(-1)}\right]$$

$$= -\frac{\cos(2x+y)}{12} - \frac{\cos(2x-y)}{4}$$

∴ The complete integral is $z = \text{C.F} + \text{P.I}$

$$z = \phi_1(y_1) + \phi_2(y-x) - \frac{1}{12}\cos(2x+y) - \frac{1}{4}\cos(2x-y)$$

Example 1.5.23 : Solve $[D^2 - 3DD' + 2D'^2]\,z = \sin x\cos y$

Solution :

Given : $[D^2 - 3DD' + 2D'^2]\,z = \sin x\cos y$

The auxiliary equation is $m^2 - 3m + 2 = 0$

$$m = 1,\ 2$$

$$\text{C.F} = \phi_1(y+x) + \phi_2(y+2x)$$

$$\text{P.I} = \frac{1}{D^2 - 3DD' + 2D'^2}\sin x\cos y$$

$$\text{P.I} = \frac{1}{D^2 - 3DD' + 2D'^2}\,\frac{1}{2}\,[\sin(x+y) + \sin(x-y)]$$

$$\text{P.I}_1 = \frac{1}{D^2 - 3DD' + 2D'^2}\,\frac{1}{2}\sin(x+y)$$

$$= \frac{1}{2}\ \text{I.P of}\ \frac{1}{D^2 - 3DD' + 2D'^2}e^{i(x+y)}$$

$$= \frac{1}{2}\ \text{I.P of}\ \frac{1}{(i)^2 - 3(i)(i) + 2(i)^2}e^{i(x+y)}$$

[Replace D by i and D′ by i]

$$= \frac{1}{2}\ \text{I.P of}\ \frac{1}{-1+3-2}e^{i(x+y)}$$

$= \frac{1}{2}$ I.P of $\frac{1}{0} e^{i(x+y)}$ [Ordinary rule fails]

$= \frac{1}{2} x$ I.P of $\frac{1}{2D - 3D'} e^{i(x+y)}$

$= \frac{x}{2}$ I.P of $\frac{1}{2(i) - 3(i)} e^{i(x+y)}$

[Replace D by i and D' by i]

$= \frac{x}{2}$ I.P of $\frac{1}{-i} e^{i(x+y)}$

$= \frac{x}{2}$ I.P of $\{i[\cos(x+y) + i\sin(x+y)]\}$

$= \frac{x}{2}\cos(x+y)$

$$\text{P.I}_2 = \frac{1}{2}\left[\frac{1}{D^2 - 3DD' + 2D'^2}\sin(x-y)\right]$$

$$= \frac{1}{2}\frac{1}{-1 - 3(1) + 2(-1)}\sin(x-y)$$

Replace
D^2 by $-1^2 = -1$
D'^2 by $-(-1)^2 = -1$
DD' by $-(1)(-1) = 1$

$$= \frac{1}{2}\frac{1}{-4-2}\sin(x-y)$$

$$= \frac{1}{-12}\sin(x-y)$$

Hence, the general solution is

$$z = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

$$= \phi_1(y+x) + \phi_2(y+2x) + \frac{1}{2}[x\cos(x+y)] - \frac{1}{12}\sin(x-y)$$

Example 1.5.24 : Solve $(D^2 - DD')z = \sin x \sin 2y$

Solution : Given : $(D^2 - DD')z = \sin x \sin 2y$

The auxiliary equation is $m^2 - m = 0$

$$m(m-1) = 0$$

$$m = 0, m = 1$$

$$\text{C.F} = f_1(y) + f_2(y+x)$$

$$\text{P.I} = \frac{1}{D^2 - DD'}\sin x \sin 2y$$

$$= \frac{1}{D^2 - DD'}\left[\frac{\cos(x-2y) - \cos(x+2y)}{2}\right]$$

$$= \frac{1}{2}\frac{1}{D^2 - DD'}\cos(x-2y) - \frac{1}{2}\frac{1}{D^2 - DD'}\cos(x+2y)$$

$$= \frac{1}{2}\left[\frac{\cos(x-2y)}{-1+1(-2)}\right] - \frac{1}{2}\left[\frac{\cos(x+2y)}{-1+(1)(2)}\right]$$

$$= \frac{1}{2}\left[\frac{\cos(x-2y)}{-3}\right] - \frac{1}{2}\left[\frac{\cos(x+2y)}{1}\right]$$

$$= \frac{-1}{6}\cos(x-2y) - \frac{1}{2}\cos(x+2y)$$

Hence, the general solution is z = C.F. + P.I

$$z = f_1(y) + f_2(y+x) - \frac{1}{6}\cos(x-2y) - \frac{1}{2}\cos(x+2y)$$

EXERCISES

Find the P.I of the following :

1. $[2D^2 - 5DD' + 2D'^2]z = 5\sin(2x+y)$ **[A.U. March 96, M/J 07]**

   [**Ans.** P.I $= -\frac{5}{3}x\cos(2x+y)$]

2. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 8\sin(x+3y)$ **[A.U A/M. 2004 P.T]**

   [**Ans.** P.I $= -\frac{4}{5}\sin(x+3y)$]

3. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(3x + 2y)$

[Ans. P.I $= \dfrac{1}{9}\cos(3x + 2y)$]

4. $(D^2 - 2DD')\, z = \sin x \,.\, \cos 2y$

[Ans. P.I $= \dfrac{1}{6}\sin(x + 2y) - \dfrac{1}{10}\sin(x - 2y)$]

R.H.S = sin (ax + by) + e^{ax+by}

Example 1.5.25: Solve $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(x + 2y) + e^{3x}$
[A.U, April, 2000] [A.U. A/M. 2004] [N/D 2014, R-
[A.U N/D 2016 R

Solution : Given : $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(x + 2y) + e^{3x+y}$

The auxiliary equation is $m^3 - 7m - 6 = 0$

$$(m + 1)(m + 2)(m - 3) = 0$$

$$m = -1, -2, 3$$

$$C.F = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x)$$

$$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x + 2y)$$

$$= \text{I.P of } \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{i(x+2y)}$$

$$= \text{I.P of } \frac{1}{(i)^3 - 7(i)(2i)^2 - 6(2i)^3} e^{i(x+2y)}$$

[Replace D by i and D′ by $2i$]

$$= \text{I.P of } \frac{1}{-i + 28i + 48i} e^{i(x+2y)}$$

$$= \text{I.P of } \frac{1}{75i} e^{i(x+2y)}$$

$$= \text{I.P of } -\frac{1}{75}\, i\, e^{i(x+2y)}$$

$$= -\frac{1}{75}\text{ I.P of } \{i[\cos(x + 2y) + i\sin(x + 2y)]\}$$

$$= -\frac{1}{75}\cos(x + 2y)$$

$$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{3x+y}$$

$$= \frac{1}{(3)^3 - 7(3)(1^2) - 6(1^3)} e^{3x+y}$$ [Replace D by 3, and D′ by 1]

$$= \frac{1}{27 - 21 - 6} e^{3x+y}$$

$$= \frac{1}{0} e^{3x+y}$$ [Ordinary rule fails]

$$= x\frac{1}{3D^2 - 7D'^2} e^{3x+y}$$

$$= x\frac{1}{3(3)^2 - 7(1)^2} e^{3x+y}$$

[Replace D by 3 and D′ by 1]

$$= x\frac{1}{27 - 7} e^{3x+y}$$

$$= \frac{x}{20} e^{3x+y}$$

Hence, the general solution is

$$z = C.F + P.I_1 + P.I_2$$

$$= \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) - \frac{1}{75}\cos(x + 2y) + \frac{x}{20} e^{3x+y}$$

Example 1.5.26 : Solve : $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = e^{x+2y} + 4\sin(x+y)$

[A.U. A/M 2008] [A.U N/D 2...

Sol. Given : $[D^3 - 2D^2 D']z = e^{x+2y} + 4\sin(x+y)$

The auxiliary equation is $m^3 - 2m^2 = 0$

Replace 'D' by 'm' and D' by 1

$$m^2(m-2) = 0$$

$$m = 0, 0 \qquad m = 2$$

$$C.F = \phi_1(y+0x) + x\phi_2(y+0x) + \phi_3(y+2x)$$

$$= \phi_1(y) + x\phi_2(y) + \phi_3(y+2x)$$

$$P.I_1 = \frac{1}{D^3 - 2D^2 D'} e^{x+2y}$$

$$= \frac{1}{(1)^3 - 2(1)^2(2)} e^{x+2y} \quad \text{Replace } D \text{ by 1 and } D' \text{ by ...}$$

$$= \frac{1}{1-4} e^{x+2y} = -\frac{1}{3} e^{x+2y}$$

$$P.I_2 = \frac{1}{D^3 - 2D^2 D'} 4\sin(x+y)$$

$$= \text{I.P of } 4 \frac{1}{D^3 - 2D^2 D'} e^{i(x+y)}$$

$$= 4 \text{ I.P of } \frac{1}{(i)^3 - 2(i)^2(i)} e^{i(x+y)}$$

[Replace D by i and D' by i]

$$= 4 \text{ I.P of } \frac{1}{-i+2i} e^{i(x+y)}$$

$$= 4 \text{ I.P of } \frac{1}{i} e^{i(x+y)}$$

$$= 4 \text{ I.P of } \{-i[\cos(x+y) + i\sin(x+y)]\}$$

$$= -4\cos(x+y)$$

Hence, the general solution is

$$z = C.F + P.I_1 + P.I_2$$

$$z = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) - \frac{1}{3}e^{x+2y} - 4\cos(x+y)$$

EXERCISES

Find the P.I. of the following :

1. $[D^2 + DD' - 6D'^2]z = \cos(2x+y) + e^{x-y}$ [A.U. O/N 1996]

   [Ans. $P.I = \frac{x}{5}\sin(2x+y) - \frac{1}{6}e^{x-y}$]

2. $[2D^2 - 5DD' + 2D'^2]z = \sin(2x+y) + e^{2x+y}$ [A.U. March 1996]

   [Ans. $P.I = -\frac{x}{3}\cos(2x+y) + \frac{x}{3}e^{2x+y}$]

3. $[D^2 - DD' - 20D'^2]z = e^{5x+y} + \sin(4x-y)$

   [A.U. A/M. 2003, N/D 2005]

   [Ans. $P.I = \frac{1}{9}x e^{5x+y} - \frac{x}{9}\cos(4x-y)$]

4. $[D^3 + D^2 D' - DD'^2 - D'^3]z = e^{2x+y} + \cos(x+y)$

   [A.U. A/M. 2003, A/M. 2005]

   [Ans. $P.I = \frac{1}{9}e^{2x+y} - \frac{x}{4}\cos(x+y)$]

5. $[D^2 + 4DD' - 5D'^2]z = 3e^{2x-y} + \sin(x-2y)$ [A.U. N/D 2003]

   [Ans. $P.I = -\frac{1}{3}e^{2x-y} + \frac{1}{27}\sin(x-2y)$]

6. $[D^3 - 7DD'^2 - 6D'^3]z = \cos(x+2y) + 4$ [A.U. N/D 2008]

R.H.S $= x^r y^s + \sin(ax+by)$ or $\cos(ax+by)$

Example 1.5.27 : Solve $\left(D^2 + 3DD' - 4(D')^2\right) z = x + \sin y$

[A.U. N/D 2007] [A.U.T. Trichy N/D 20..]

Solution : Given : $[D^2 + 3DD' - 4D'^2]z = x + \sin y$

The auxiliary equation is

$$m^2 + 3m - 4 = 0 \qquad \text{[Replace D by } m \text{ and } D' \text{ by ...]}$$

$$(m+4)(m-1) = 0$$

$$m = 1,\ m = -4 \qquad \text{i.e., } m = -4,\ m = 1$$

$$\therefore \quad \text{C.F.} = \phi_1(y-4x) + \phi_2(y+x)$$

$$\text{P.I}_1 = \frac{1}{D^2 + 3DD' - 4D'^2}\, x$$

$$= \frac{1}{D^2\left[1 + \dfrac{3DD' - 4D'^2}{D^2}\right]}\, x$$

$$= \frac{1}{D^2\left[1 + \dfrac{3D'}{D} - \dfrac{4D'^2}{D^2}\right]}\, x$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{3D'}{D} - \frac{4D'^2}{D^2}\right)\right]^{-1} x$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} - \frac{4D'^2}{D^2}\right) + \left(\frac{3D'}{D} - \frac{4D'^2}{D^2}\right)^2 - \ldots\right] x$$

$$= \frac{1}{D^2}[x] \qquad \text{[Omitting } D' \text{ and higher powers]}$$

$$= \frac{1}{D}\left[\frac{x^2}{2}\right] = \frac{x^3}{6}$$

$$\text{P.I}_2 = \frac{1}{D^2 + 3DD' - 4D'^2} \sin(0x + y)$$

$$= \frac{1}{0 + 0 - 4(-1)} \sin y \qquad \text{[Replace } D^2 \text{ by } 0,\ D'^2 \text{ by } -1^2 \text{ and } DD' \text{ by } 0]$$

$$= \frac{1}{4} \sin y$$

Hence, the general solution is

$$z = C.F + P.I_1 + P.I_2$$

$$= \phi_1(y-4x) + \phi_2(y+x) + \frac{x^3}{6} + \frac{1}{4}\sin y$$

Example 1.5.28 : Solve the equation

$[D^3 - 7DD'^2 - 6D'^3]\, z = \cos(x+2y) + x$

[A.U, March, 1996] [A.U.T CH. N/D 2011]

Solution : Given : $[D^3 - 7D'D'^2 - 6D'^3]z = \cos(x+2y) + x$

The auxiliary equation is

$$m^3 - 7m - 6 = 0$$

Put $m = 1$, we get $1 - 7 - 6 \neq 0$

Put $m = -1$, we get $-1 + 7 - 6 = 0$

$\therefore\ m = -1$ is a root

-1	1	0	-7	-6
	0	-1	$+1$	6
	1	-1	-6	0

$$\Rightarrow \quad m^2 - m - 6 = 0$$

i.e., $m + 1 = 0$, $\quad m^2 - m - 6 = 0$

$$(m-3)(m+2) = 0$$

i.e., $m = -1,\ m = -2,\ m = 3$

$$\therefore \quad \text{C.F.} = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$$

$$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \cos(x + 2y)$$

$$= R.P \text{ of } \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{i(x+2y)}$$

$$= R.P \text{ of } \frac{1}{i^3 - 7(i)(2i)^2 - 6(2i)^3} e^{i(x+2y)}$$

[Replace D by i and D' by $2i$]

$$= R.P \text{ of } \frac{1}{-i + 28i + 48i} e^{i(x+2y)}$$

$$= R.P \text{ of } \frac{1}{75i} e^{i(x+2y)}$$

$$= \frac{1}{75} R.P \text{ of } \left\{-i\left[\cos(x+2y) + i\sin(x+2y)\right]\right\}$$

$$= \frac{1}{75} \sin(x + 2y)$$

$$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} x = \frac{1}{D^3\left[1 - \frac{7D'^2}{D^2} - \frac{6D'^3}{D^3}\right]} x$$

$$= \frac{1}{D^3}\left[1 - \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right]^{-1} x$$

$$= \frac{1}{D^3}\left[1 + \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right) + \ldots\right] x$$

[Omitting D' and higher powers]

$$= \frac{1}{D^3} x = \frac{1}{D^2}\left[\frac{x^2}{2}\right] = \frac{1}{D}\left[\frac{x^3}{6}\right] = \frac{x^4}{24}$$

Hence, the general solution is

$$z = C.F + P.I_1 + P.I_2$$

$$= \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) + \frac{1}{75}\sin(x + 2y) + \ldots$$

R.H.S = $e^{ax+by}\,\phi(x, y)$

Replace D by D + a, D' by D' + b

Example 1.5.29 : Solve $[D^2 - 2DD' + D'^2]\, z = x^2 y^2 e^{x+y}$

Solution : Given $[D^2 - 2DD' + D'^2]\, z = x^2 y^2 e^{x+y}$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m - 1)^2 = 0$$

$$m = 1, 1$$

$$C.F = \phi_1(y + x) + x\,\phi_2(y + x)$$

$$P.I = \frac{1}{(D - D')^2} x^2 y^2 e^{x+y}$$

$$= e^{x+y} \frac{1}{[(D+1) - (D'+1)]^2} x^2 y^2$$ [Replace D by $D + 1$, D' by $D' + 1$]

$$= e^{x+y} \frac{1}{(D - D')^2} x^2 y^2$$

$$= e^{x+y} \frac{1}{D^2\left[1 - \frac{D'}{D}\right]^2} x^2 y^2 = e^{x+y} \frac{1}{D^2}\left[1 - \frac{D'}{D}\right]^{-2} x^2 y^2$$

$$= e^{x+y} \frac{1}{D^2}\left[1 + 2\frac{D'}{D} + \frac{3D'^2}{D^2}\right] x^2 y^2$$ [Omitting D'^3 and higher powers]

$$= e^{x+y} \frac{1}{D^2}\left[x^2 y^2 + \frac{2}{D}(2x^2 y) + \frac{3}{D^2}[2x^2]\right]$$

$$= e^{x+y}\left[y^2 \frac{1}{D^2}(x^2) + 4y\frac{1}{D^3}(x^2) + 6\frac{1}{D^4}(x^2)\right]$$

$$\boxed{\begin{array}{ll} \frac{1}{D}(x^2) = \int x^2\,dx = \frac{x^3}{3}, & \frac{1}{D^2}(x^2) = \int \frac{x^3}{3}\,dx = \frac{x^4}{12} \\ \frac{1}{D^3}(x^2) = \int \frac{x^4}{12}\,dx = \frac{x^5}{60}, & \frac{1}{D^4}(x^2) = \int \frac{x^5}{60}\,dx = \frac{x^6}{36} \end{array}}$$

$$= e^{x+y}\left[y^2\left[\frac{x^4}{12}\right] + 4y\left[\frac{x^5}{60}\right] + 6\left[\frac{x^6}{360}\right]\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12} + \frac{x^5y}{15} + \frac{x^6}{60}\right]$$

Hence, the general solution is

$$z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+y}\left[\frac{x^4y^2}{12} + \frac{x^5y}{15} + \frac{x^6}{60}\right]$$

Example 1.5.30 : Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$.
[AU, Dec. 1998] [A.U CBT N/D 2010]

Solution : Given : $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$

The auxiliary equation is $m^3 + m^2 - m - 1 = 0$

$$(m+1)^2(m-1) = 0; \quad m = 1, -1, -1$$

$$\text{C.F} = \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x)$$

$$\text{P.I} = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^x \cos 2y$$

$$= e^x \frac{1}{(D+1)^3 + (D+1)^2D' - (D+1)D'^2 - D'^3} \cos 2y$$

$$= e^x \text{ Real part of } \frac{1}{(D+1)^3 + (D+1)^2D' - (D+1)D'^2 - D'^3} e^{i2y}$$

$$= e^x \text{ R.P of } \frac{1}{1 + (1)(2i) - (1)(2i)^2 - (2i)^3} e^{i2y}$$

[Replce D by 0 and D' by $2i$]

$$= e^x \text{ R.P of } \frac{1}{1 + 2i + 4 + 8i} e^{i2y}$$

$$= e^x \text{ R.P of } \frac{1}{5 + 10i} e^{i2y} = \frac{e^x}{5} \text{ R.P of } \frac{1}{1 + 2i} e^{i2y}$$

$$= \frac{e^x}{5} \text{ R.P of } \frac{1}{1+2i}\,\frac{1-2i}{1-2i} e^{i2y} = \frac{e^x}{5} \text{ R.P of } \frac{1-2i}{1+4} e^{i2y}$$

$$= \frac{e^x}{25} \text{ R.P of } \left[(1-2i)[\cos 2y + i \sin 2y]\right]$$

$$= \frac{e^x}{25}[\cos 2y + 2 \sin 2y]$$

The general solution is $z = \text{C.F} + \text{P.I}$

$$z = \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x) + \frac{e^x}{25}[\cos 2y + 2\sin 2y]$$

EXERCISES

1. Solve : $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = e^{x-y} \sin(2x + 3y)$

[**Ans.** $z = \phi_1(y+x) + \phi_2(y-x) - \dfrac{e^{x-y}}{25}[2\cos(2x+3y) - \sin(2x+3y)]$]

2. Solve : $[D^2 - 3DD' + 2D'^2]z = (4x+2)e^{x+2y}$ **[A.U A/M 2015, R-13]**

[**Ans.** $z = \phi_1(y+x) + \phi_2(y+2x) + \frac{1}{3}\left(4x + \frac{22}{3}\right)e^{x+2y}$]

R.H.S. = sin ax sin by (or) cos ax cos by

Example 1.5.31 : Solve $[D^2 - D'^2]z = \sin 2x \sin 3y$ [A.U, April, 1996]

Solution : Given : $[D^2 - D'^2]z = \sin 2x \sin 3y$

The auxiliary equation is $m^2 - 1 = 0$

$$m^2 = 1$$

$$m = \pm 1 \quad \text{i.e., } m = 1, m = -1$$

$$\text{C.F.} = \phi_1(y+x) + \phi_2(y-x)$$

$$\text{P.I.} = \frac{1}{D^2 - D'^2} \sin 2x \sin 3y$$

$$= \frac{1}{(-4) - (-9)} \sin 2x \sin 3y \quad \left[\text{Replace } D^2 \text{ by } -2^2 = -4,\ D'^2 \text{ by } -3^2 = -9\right]$$

$$= \frac{1}{-4+9} \sin 2x \sin 3y$$

$$= \frac{1}{5} \sin 2x \sin 3y$$

Hence, the general solution is

$$z = \text{C.F} + \text{P.I.}$$

$$= \phi_1(y+x) + \phi_2(y-x) + \frac{1}{5}\sin 2x \sin 3y$$

R.H.S = y cos x (or) y sin x

Formulae :

$$\frac{1}{D - mD'} f(x,y) = \int F(x, c - mx)\, dx \quad \text{where } y = c - mx$$

$$\frac{1}{D + mD'} f(x,y) = \int F(x, c + mx)\, dx \quad \text{where } y = c + mx$$

Example 1.5.32 : Solve $[D^2 + DD' - 6D'^2] z = y \cos x$.
(or) $(r + s - 6t) = y \cos x$
[A.U. CBT Dec. 2008] [A.U M/J 2013] [A.U A/M 2015 R-20..]

Solution : Given : $[D^2 + DD' - 6D'^2] z = y \cos x$ **[A.U M/J 20..]**

The auxiliary equation is $m^2 + m - 6 = 0$ **[A.U N/D 2016 R..]**

$m = 2, -3$ **[A.U A/M 2018 R..]**

$$C.F = \phi_1 (y + 2x) + \phi_2 (y - 3x)$$

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} y \cos x$$

$$= \frac{1}{(D - 2D')(D + 3D')} y \cos x$$

$$= \frac{1}{D - 2D'} \int (c + 3x) \cos x\, dx \text{ when } y = c + 3x$$

$$= \frac{1}{D - 2D'} [(c + 3x) \sin x - 3 \int \sin x\, dx] \text{ when } c = y - 3x$$

$$= \frac{1}{D - 2D'} [y \sin x + 3 \cos x]$$

$$= \int [(c - 2x) \sin x + 3 \cos x]\, dx \text{ when } y = c - 2x$$

$$= [(c - 2x)(-\cos x) - (-2)(-\sin x)] + 3 \sin x \text{ when } c = y + 2x$$

$$= -y \cos x + \sin x$$

Hence, the general solution is

$$\therefore z = C.F + P.I$$

$$= \phi_1 (y + 2x) + \phi_2 (y - 3x) - y \cos x + \sin x$$

Aliter : To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} y \cos x$$

$$= \frac{1}{D^2} \left[\frac{1}{1 + \frac{D'}{D} - \frac{6D'^2}{D^2}} \right] y \cos x$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2} \right) \right]^{-1} y \cos x$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{D'}{D} - \frac{6D'^2}{D^2} \right) + \dots \right] y \cos x$$

$$= \frac{1}{D^2} \left[1 - \frac{D'}{D} \right] y \cos x \qquad \text{[omitting } D'^2 \text{ and higher powers]}$$

$$= \frac{1}{D^2} \left[y \cos x - \frac{1}{D} \cos x \right] = \frac{1}{D^2} [y \cos x - \sin x]$$

$$= \frac{1}{D} [y \sin x + \cos x] = -y \cos x + \sin x$$

EXERCISES

1. Solve $(D^2 - 5DD' + 6D'^2) z = y \sin x$ **[A.U N/D 2006]**

   Ans. $z = \phi_1 (y + 2x) + \phi_2 (y + 3x) + 5 \cos x - y \sin x$

2. Solve $[D^2 + DD' - 6D'^2] z = 6x \sin y$

   Ans. $z = \phi_1 (y + 2x) + \phi_2 (y - 3x) + x \sin y - \frac{1}{6} \cos y$

3. Sol : $[D^2 - 5DD' + 6D'^2] z = 6x \sin y$

   Ans. $z = \phi_1 (y + 2x) + \phi_2 (y + 3x) - x \sin y - \frac{5}{6} \cos y$

4. Find the general solution of
   $[D^2 + 2DD' + D'^2] z = 2 \cos y - x \sin y$ **[A.U N/D 2015 R-13]**

   Ans. $z = Q_1 (y - x) + x Q_2 (y - x) + x \sin y$

1.5.3 Non-homogeneous linear equation

The linear differential equations which are not homogeneous... called Non-homogeneous linear equations.

For example : $3\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x\,\partial y} + 4\dfrac{\partial^2 z}{\partial y^2} + 5\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} + z = 0$

$$f(D, D') = f_1(x, y)$$

Its solution $z = \text{C.F} + \text{P.I}$

Complementary function :

Let the non-homogeneous equation be $(D - mD' - a)z = 0$

$$\text{or} \quad \frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} - az = 0$$

$$p - mq = az$$

The Lagrange's subsidiary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{az}$

From first two ratios $-mdx = dy$; $\quad dy + mdx = 0$

$$y + mx = c_1 \quad \dots (1$$

From First and third ratios

$$\frac{dx}{1} = \frac{dz}{az} \quad \Rightarrow \quad a\,dx = \frac{dz}{z}$$

$$\int a\,dx = \int \frac{1}{z}\,dz$$

$$\Rightarrow \quad ax = \log z - \log c_2$$

$$\Rightarrow \quad ax = \log\left(\frac{z}{c_2}\right) ; \quad e^{ax} = \frac{z}{c_2}$$

$$c_2 = \frac{z}{e^{ax}} \quad \dots (2$$

From (1) & (2), we get $f\left(y + mx, \dfrac{z}{e^{ax}}\right) = 0$

$$\Rightarrow \frac{z}{e^{ax}} = \phi(y + mx)$$

$$\Rightarrow \quad z = e^{ax}\phi(y + mx)$$

we have, $z = e^{ax}\phi(y + mx)$

Similarly, the solution of $(D - mD' - a)^2 z = 0$ is

$$z = e^{ax}\phi_1(y + mx) + xe^{ax}\phi_2(y + mx)$$

Working rule

Case (i) If $(D - mD' - c)z = 0$, then $z = e^{cx} f(y + mx)$

* If $(D' - mD - c)z = 0$, then $z = e^{cy} f(x + my)$

Case (ii) If $(D - m_1 D' - c_1)(D - m_2 D' - c_2) \dots (D - m_n D' - c_n)z = 0$,

then $z = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x) + \dots + e^{c_n x} f_n(y + m_n x)$

* If $(D' - m_1 D - c_1)(D' - m_2 D - c_2) \dots (D' - m_n D - c_n)z = 0$,

then $z = e^{c_1 y} f_1(x + m_1 y) + e^{c_2 y} f_2(x + m_2 y) + \dots + e^{c_n y} f_n(x + m_n y)$

Case (iii) If $(D - mD' - c)^r z = 0$, then

$$z = e^{cx} f_1(y + mx) + xe^{cx} f_2(y + mx) + x^2 e^{cx} f_3(y + mx) + \dots + x^{r-1} e^{cx} f_r(y + mx)$$

* If $(D' - mD - c)^r z = 0$, then

$$z = e^{cy} f_1(x + my) + ye^{cy} f_2(x + my) + y^2 e^{cy} f_3(x + my) + \dots + y^{r-1} e^{cy} f_r(x + my)$$

Problems based on Non-homogeneous linear equation

Example 1.5.34 : Solve $(D + D' - 2)z = 0$

Solution : Given : $(D + D' - 2)z = 0$

$$\text{i.e., } [D - (-1)D' - 2]\,z = 0$$

We know that, working rule case (i) is

If $(D - m D' - c)z = 0$, then $z = e^{cx} f(y + mx)$

where f is arbitrary.

Here, $m = -1, c = 2$

$$\therefore z = e^{2x} f[y + (-1)x] = e^{2x} f(y - x)$$

Example 1.5.35 : Solve $(D + D' - 2)(D + 4D' - 3) z = 0$

Solution : Given : $(D + D' - 2)(D + 4D' - 3) z = 0$

i.e., $[D - (-1) D' - 2][D - (-4) D' - 3] z = 0$

We know that, working rule case (ii) is

If $(D - m_1 D' - c_1)(D - m_2 D' - c_2) \dots (D - m_n D' - c_n) z = 0,$

then $z = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x) + \dots + e^{c_n x} f_n(y + m_n x)$

Here, $m_1 = -1, \quad c_1 = 2$

$m_2 = -4, \quad c_2 = 3$

$$\therefore z = e^{2x} f_1(y + (-1)x) + e^{3x} f_2(y + (-4)x)$$

$$= e^{2x} f_1(y - x) + e^{3x} f_2(y - 4x)$$

Example 1.5.36 : Solve : $(D^2 - DD' + D' - 1) z = 0$ **[A.U N/D 2018-A R-17]**

Solution :

Given : $(D^2 - DD' + D' - 1) z = 0$

$$[D^2 - 1 - DD' + D'] z = 0$$

$$[(D - 1)(D + 1) - D'(D - 1)] z = 0$$

$$(D - 1)[D + 1 - D'] z = 0$$

$$(D - 1)[D - D' + 1] z = 0$$

$$[D - 0D' - 1][D - D' - (-1)] z = 0$$

by working rule case (ii)

$m_1 = 0, \quad c_1 = 1$

$m_2 = 1, \quad c_2 = -1$

$$\therefore z = e^x f_1(y + 0x) + e^{-x} f_2(y + x)$$

$$= e^x f_1(y) + e^{-x} f_2(y + x)$$

Another method :

Given : $(D^2 - DD' + D' - 1) z = 0$

Let $z = ce^{hx + ky}$ be a trial solution

Then $h^2 - hk + k - 1 = 0$ (replacing D by h and D′ by k)

(i.e.,) $h^2 - 1 - k(h - 1) = 0$

(i.e.,) $(h + 1)(h - 1) - k(h - 1) = 0$

(i.e.,) $(h - 1)[h - k + 1] = 0$

$h = 1$ or $h = k - 1$

$\therefore$ The complete solution is

$$z = \Sigma c_1 e^{x + ky} + \Sigma c_2 e^{(k - 1)x + ky}$$

$$= \Sigma c_1 e^x e^{ky} + \Sigma c_2 e^{-x} e^{k(y + x)}$$

$$= e^x \phi_1(y) + e^{-x} \phi_2(y + x)$$

Example 1.5.37 : Solve : $(2DD' + D'^2 - 3D') z = 3\cos(3x - 2y)$

Solution : To find the complementary function

Take $(2DD' + D'^2 - 3D') z = 0$

$$D'[D' + 2D - 3] z = 0$$

$$[D' - 0D - 0][D' - (-2)D - 3] z = 0$$

Here, $m_1 = 0, \quad c_1 = 0$

$m_2 = -2, \quad c_2 = 3$

$$\text{C.F.} = e^{0y} f_1(x + 0y) + e^{3y} f_2[x + (-2)y]$$

$$= f_1(x) + e^{3y} f_2(x - 2y)$$

$$\text{P.I.} = \frac{1}{(2DD' - D'^2 - 3D')} 3\cos(3x - 2y)$$

$$= 3 \text{ R.P} \frac{1}{2DD' - D'^2 - 3D'} e^{i(3x - 2y)}$$

$$= 3\ \text{R.P}\ \frac{1}{2(3i)(-2i)-(-2i)^2-3(-2i)} e^{i(3x-2y)} \quad \text{Replace } D \text{ by } 3i,\ D' \text{ by } -2i$$

$$= 3\ \text{R.P}\ \frac{1}{12+4+6i} e^{i(3x-2y)}$$

$$= 3\ \text{R.P} \left(\frac{1}{16+6i}\right)\left(\frac{16-6i}{16-6i}\right) e^{i(3x-2y)} = 3\ \text{R.P}\ \frac{(16-6i)}{256+36} e^{i(3x-2y)}$$

$$= 3\ \text{R.P}\ \frac{(16-6i)}{292} [\cos(3x-2y) + i \sin x (3x-2y)]$$

$$= \frac{3}{292} [16\cos(3x-2y) + 6\sin(3x-2y)]$$

$$= \frac{3}{146} [8\cos(3x-2y) + 3\sin(3x-2y)]$$

$\therefore$ The complete integral is $z = \text{C.F} + \text{P.I}$

$$z = f_1(x) + e^{3y} f_2(x-2y) + \frac{3}{146} [8\cos(3x-2y) + 3\sin(3x-2y)]$$

Another method to find C.F.

Let $z = ce^{hx+ky}$ be a trial solution of

$$(2DD' + D'^2 - 3D')z = 0$$

Then $\quad 2hk + k^2 - 3k = 0$

i.e., $\quad k(2h + k - 3) = 0$

$$k = 0,\ k = 3 - 2h$$

$$\text{C.F} = \Sigma c_1 e^{hx+0y} + \Sigma c_2 e^{hx+(3-2h)y}$$

$$= \Sigma c_1 e^{hx} + \Sigma c_2 e^{3y} e^{h(x-2y)}$$

$$= f_1(x) + e^{3y} f_2(x-2y)$$

Example 1.5.38 : Solve $(D - D' - 1)(D - D' - 2)z = e^{2x-y}$

Solution : **[A.U N/D 2011]**

Given : $(D - D' - 1)(D - D' - 2)z = e^{2x-y}$

To find C.F. take

$$(D - D' - 1)(D - D' - 2)z = 0$$

by working rule case (ii)

$$m_1 = 1, \quad c_1 = 1$$
$$m_2 = 1, \quad c_2 = 2$$

$$\text{C.F.} = e^x f_1(y+x) + e^{2x} f_2(y+x)$$

$$\text{P.I.} = \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$$

$$= \frac{1}{(2+1-1)(2+1-2)} e^{2x-y} \quad \text{Replacing D by 2, D' by } -1$$

$$= \frac{1}{2} e^{2x-y}$$

$$z = \text{C.F} + \text{P.I.}$$

$$= e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{2} e^{2x-y}$$

Example 1.5.39 : Solve $(D - D' - 1)(D - D' - 2)z = e^{2x+y}$.

Solution : Given : $(D - D' - 1)(D - D' - 2)z = e^{2x+y}$

To find C.F.

$$(D - D' - 1)(D - D' - 2)z = 0$$

by working rule case (ii)

$$m_1 = 1,\ c_1 = 1,\ c_2 = 2,\ m_2 = 1$$

$$\text{C.F} = e^x f_1(y+x) + e^{2x} f_2(y+x)$$

$$\text{P.I} = \frac{1}{(D-D'-1)(D-D'-2)} e^{2x+y} = \frac{1}{(D-D'-1)} \frac{e^{2x+y}}{(2-1-2)}$$

$$= -\frac{1}{(D-D'-1)} e^{2x+y} \quad \text{Replacing D by 2 and D' by 1}$$

$$= -xe^{2x+y}$$

$\therefore$ The general solution is

$$z = e^x f_1(y+x) + e^{2x} f_2(y+x) - xe^{2x+y}$$

Example 1.5.40 : Solve $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y.$

Solution :

Given : $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$

To find C.F.

$$(D + D' - 1)(D + 2D' - 3)z = 0$$

$$(D - (-1)D' - 1)(D - (-2)D' - 3)z = 0$$

by working rule case (ii)

$m_1 = -1,\ c_1 = 1,\ m_2 = -2,\ c_2 = 3$

$$\text{C.F} = e^{x} f_1(y - x) + e^{3x} f_2(y - 2x) \quad \dots (1)$$

$$\text{P.I} = \frac{1}{(D + D' - 1)(D + 2D' - 3)} 4 + 3x + 6y$$

$$= \frac{1}{3[1 - (D + D')]\left[1 - \frac{D + 2D'}{3}\right]} 4 + 3x + 6y$$

$$= \frac{1}{3}[1 - (D + D')]^{-1}\left[1 - \frac{D + 2D'}{3}\right]^{-1}(4 + 3x + 6y)$$

$$= \frac{1}{3}[1 + (D + D') + (D + D')^2 + \dots]$$
$$\left[1 + \frac{1}{3}(D + 2D') + \frac{1}{9}(D + 2D')^2 + \dots\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[1 + \frac{4}{3}D + \frac{5}{3}D' + \dots\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[4 + 3x + 6y + \frac{4}{3}(3) + \frac{5}{3}(6)\right]$$

$$= x + 2y + 6$$

The general solution is

$$z = e^{x} f_1(y - x) + e^{3x} f_2(y - 2x) + x + 2y + 6$$

Example 1.5.41 : Solve $(D + 3D' + 4)^2 z = 0$

Solution : Given : $(D + 3D' + 4)^2 z = 0$

$$[D - (-3)D' - (-4)]^2 z = 0$$

by working rule case (iii)

$$m = -3,\ c = -4$$

$$z = e^{-4x} f_1[y + (-3)x] + x e^{-4x} f_2[y + (-3)x]$$
$$= e^{-4x} f_1(y - 3x) + x e^{-4x} f_2(y - 3x)$$

Example 1.5.42 : Find the general solution of $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = \sin(2x + y)$

Solution : Given : $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = \sin(2x + y)$

To find C.F $\quad (D - D')(D - 2D' + 2)z = 0$

$$(D - D')[D - 2D' - (-2)]z = 0$$

Here, $\quad m_1 = 1,\ c_1 = 0,\ m_2 = 2,\ c_2 = -2$

$$\text{C.F} = e^{0x} f_1(y + x) + e^{-2x} f_2(y + 2x)$$
$$= f_1(y + x) + e^{-2x} f_2(y + 2x)$$

$$\text{P.I} = \text{I.P} \frac{1}{(D - D')(D - 2D' + 2)} e^{(2x + y)i}$$

$$= \text{I.P} \frac{1}{(2i - i)(2i - 2i + 2)} e^{(2x + y)i} \quad \text{[Replace } D \text{ by } 2i \text{ and } D' \text{ by } i]$$

$$= \text{I.P} \frac{1}{2i} e^{(2x + y)i}$$

$$= \frac{1}{2}\text{I.P}\left[(-i)[\cos(2x + y) + i\sin(2x + y)]\right]$$

$$= \frac{1}{2}\text{I.P}\left[-i\cos(2x + y) + \sin(2x + y)\right] = -\frac{1}{2}\cos(2x + y)$$

$$\therefore Z = \text{C.F} + \text{P.I}$$

$$= f_1(y + x) + e^{-2x} f_2(y + 2x) - \frac{1}{2}\cos(2x + y)$$

EXERCISES

1. Solve : $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$
2. Solve : $(D^2 - DD' + D' - 1)z = e^{y - x} + \cos(x + 2y)$
3. Solve : $(D^2 + DD' + D' - 1)z = \cos(x - y)$
4. Solve : $(D^2 - D'^2 - 3D + 3D')z = xy + 7$

5. Solve : $(D^2 + DD' + D' - 1)\ z = e^{-x}$

6. Solve : $(D^2 + 2DD' + D'^2 - 2D - 2D')\ z = \cosh (x - y)$

7. Solve : $(D^2 - DD' + D' - 1)\ z = e^{2x + 3y} + \cos^2 (x + 2y)$

8. Solve : $(2D^2 - DD' - D'^2 + 6D + 3D')\ z = xe^y + ye^x$

9. Solve : $(D - D' - 1)(D - D' - 2)\ z = e^{2x + y}$

10. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\, \partial y} + \frac{\partial z}{\partial x} = 0$ i.e., $(D^2 - DD' + D)\, z = 0$ **[A.U N/D 201**

11. Solve : $(D^2 + 2DD' + D'^2 - 2D - 2D')\, z = \sin (x + 2y)$
[A.U N/D 2015 R-8][A.U M/J 2016 - R13][A.U N/D 2018 R-

12. Solve : $(D + D' - 1)(D - 2D' + 3)\, z = 0$ **[A.U N/D 2015 - R**

Answers

1. $z = f_1 (y - x) + e^{-2x} f_2 (y + 2x)$

2. $z = e^x f_1 (y) + e^{-x} f_2 (x + y) + \frac{1}{2} \sin (x + 2y) + \frac{1}{2} e^{y - x}$

3. $z = e^{-x} f_1 (y) + e^x f_2 (y - x) - \frac{1}{2} [\sin (x - y) + \cos (x - y)]$

4. $z = f_1 (x + y) + e^{3x} f_2 (y - x) - \frac{1}{3}\left[\frac{x^2 y}{2} + \frac{xy}{3} + 7x + \frac{x^2}{3} + \frac{x}{3} + \frac{y}{9} + \frac{x^3}{6} + \ldots\right.$

5. $z = e^{-x} f_1 (y) + e^x f_2 (y - x) - \frac{1}{2} xe^{-x}$

6. $z = f_1 (y - x) + e^{2x} f_2 (y - x) - \frac{x}{2} \cosh (x - y)$

7. $z = e^2 f_1 (y) + e^{-x} f_2 (y + x) + xe^{2x + 3y} - \frac{1}{2}$
$+ \frac{1}{50} [4 \sin (2x + 4y) + 3 \cos (2x + 4y)]$

8. $z = f_1 (2y - x) + e^{-3x} f_2 (y + x) + \frac{1}{4} (2x - 5)\, e^y + \frac{1}{32} (4y - 1) e^{\ldots}$

9. $z = e^x f_1 (y + x) + e^{2x} f_2 (y + x) - xe^{2x + y}$

10. $z = f_1 (y) + e^{-x} f_2 (y + x)$

11. $z = f_1 (y - x) + e^{2x} f_2 (y - x) + \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)]$

12. $z = e^x f_1 (y - x) + e^{-3x} f_2 (y + 2x)$

PART-A QUESTIONS AND ANSWERS

1. Explain how p.d.e are formed.

2. Form the p.d.e by eliminating the arbitrary constants a and b from $z = ax + by$.

[Ans. $z = px + qy$**]**

3. Eliminate the arbitrary constants a and b from $z = ax + by + a^2 + b^2$.

[Ans. $z = px + qy + p^2 + q^2$**]**

4. Find the p.d.e. of all planes through the origin.

[Ans. $z = px + qy$**]**

5. Form the p.d.e by eliminating the arbitrary function from $z = f(x^2 - y^2)$.

[Ans. $py + qx = 0$**]**

6. Obtain p.d.e. from $z = f(\sin x + \cos y)$

[Ans. $p \sin y + q \cos x = 0$**]**

7. Solve : $\frac{\partial z}{\partial x} = \sin x$

[Ans. $z = -\cos x + f(y)$**]**

8. Define general and complete integrals of a p.d.e.

9. Define complete integral and singular integral.

10. Solve : $\sqrt{p} + \sqrt{q} = 1$. **[Ans.** $z = ax + (1 - \sqrt{a})^2 y + c$**]**

11. Solve : $p = 2qx$ **[Ans.** $z = ax^2 + cy + b$**]**

12. Solve : $z = px + qy + p^2$ **[Ans.** $z = ax + by + a^2$**]**

13. Find the complete integral of $(p - q)(z - px - qy) = 1$.

[Ans. $z = ax + by + \frac{1}{a - b}$ **]**

14. Find the complete integral of $pq = xy$

[Ans. $2az = a^2x^2 + y^2 + b$]

15. Solve : $px + qy = z$

[Ans. $\phi\left(\frac{x}{y}, \frac{x}{z}\right) = 0$]

16. Solve : $(D^2 - 4DD' + 3D'^2)\, z = 0$

[Ans. $z = \phi_1(y + x) + \phi_2(y + 3x)$]

17. Find the P.I. of $[D^2 + 4DD']\, z = e^x$.

[Ans. P.I $= e^x$]

18. Find the P.I of $(D^2 + DD')\, z = e^{x-y}$

[Ans. P.I $= xe^{x-y}$]

19. Solve : $[D^2 - 2DD' + D'^2]z = \cos(x - 3y)$

[Ans. $z = f_1(y + x) + xf_2(y + x) - \frac{1}{16}\cos(x - 3y)$]

20. Solve : $[D^2 - DD' + 6D'^2]\, z = 0$

[Ans. $z = \phi_1(y - 2x) + \phi_2(y + 3x)$]

21. Solve : $2r + 5s - 3t = 0$

[Ans. $z = \phi_1(y - 3x) + \phi_2(2y + x)$]

22. Find the general solution of $p\tan x + q\tan y = \tan z$

[Ans. $f\left[\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right] = 0$]

23. Solve : $p - x^2 = q + y^2$ [Ans. $z = ax + \frac{x^3}{3} - \frac{y^3}{3} + ay + b$]

24. Solve : $pq = y$ [Ans. $z = ax + \frac{y^2}{2a} + b$]

25. Solve : $Ap + Bq + Cz = 0$ [Ans. $\log z = -\frac{C}{A + Ba}(x + ay)$

26. Solve : $p^2 + q^2 = npq$ [Ans. $z = ax + \frac{a}{2}[n \pm \sqrt{n^2 - 4}$

UNIT - II

FOURIER SERIES

Drichlet's conditions - General Fourier series - Odd and even functions - Half range sine series - Half range cosine series - Complex form of Fourier Series - Parseval's identity - Harmonic Analysis.

2.0 INTRODUCTION

Fourier series, is named after the French Mathematician cum physicist **Jean - Baptiste Joseph Fourier (1768 - 1830).** He introduced Fourier Series in 1822, while he was investigating the problem of heat conduction. The series of sines and cosines are known after him.

Fourier Series are series of cosine and sine terms and arise in the important practical task of representing general periodic functions. They constitute a very important tool in solving problems that involve ordinary and partial differential equations.

§ Use of Fourier Series :

Fourier series are particularly suitable for expansion of periodic functions. We come across many periodic functions in voltage, current, flux, density, applied force, potential and electromagnetic force in electricity. Hence, Fourier Series are very useful in electrical engineering problems.

§ PERIODIC FUNCTION :

Definition : **Periodic Function**

A function $f(x)$ is said to be periodic, if and only if $f(x + p) = f(x)$ is true for some value of p and every value of x. The smallest value of p for which this equation is true for every value of x will be called the period of the function $f(x)$.