$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s[k] = \sqrt{\frac{2}{\pi}} \int_0^\infty k \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \, k \int_0^\infty \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \, k \left[\frac{-\cos sx}{s}\right]_0^\infty$$

$\therefore$ $F_s[k]$ does not exist.

Since, $\cos \infty$ is undefined.

28. State the condition for the existence of Fourier cosine and sine transform of derivatives. **[A.U A/M 2019 R-17]**

Solution :

Let $f(x)$ be continuous and absolutely integrable on the x-axis,

Let $f'(x)$ be piecewise continuous on finite interval, and

let $f(x) \to 0$ as $x \to \infty$ then $F_c[f'(x)] = -\sqrt{\frac{2}{\pi}} f(0) + s F_s F(s)$

$$= s F_s(s) - \sqrt{\frac{2}{\pi}} f(0)$$

UNIT - V

Z - TRANSFORMS AND DIFFERENCE EQUATIONS

Z - Transforms - Elementary properties - Inverse Z - transform (using partial fraction and residues) - Convolution theorem - Formation of difference equations - Solution of difference equations using Z-transform.

Introduction

The Z - transform plays the same role for discrete systems as the Laplace transform does for continuous systems.

Applications of Z - transform **[A.U N/D 2018 R-17]**

Communication is one of the fields whose development is based on discrete analysis. Difference equations are also based on discrete system and their solutions and analysis are carried out by Z - transform.

In the system analysis area, the Z - transform converts convolutions to a product and difference equations to algebraic equations.

The stability of a discrete linear system can be determined by analyzing the transfer function H (z) given by the Z - transform.

Digital filters can be analyzed and designed using the Z - transform.

Digital control systems can be analyzed and designed using Z-transforms.

5.1 Z - transform, elementary properties of Z - transform

5.1.1 Z - transform

Definition : *Z - transform (two-sided or bilateral)*

Let $\{x(n)\}$ be a sequence defined for all integers then its Z-transform is defined to be $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

where z is an arbitrary complex number.

Definition : *Z - transform (one-sided or unilateral)*

Let $\{x(n)\}$ be a sequence defined for $n = 0, 1, 2, \ldots$ and $x(n) = 0$ for $n < 0$, then its Z - transform is defined to be

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \text{ (or) } \sum_{n=0}^{\infty} x_n z^{-n}$$

where z is an arbitrary complex number.

Definition : *Z - transform for discrete values of t*

If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n = 0, 1, 2, 3, \ldots$ T being the sampling period, then Z - transform of $f(t)$ is defined as

$$Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

Note : 1. Mostly we study one-sided Z - transform

2. If $f(t)$ given, then replace 't' by 'n T'

3. The double braces { } are used for a sequence. Sometimes we use [ ] or ( ).

I. Problems based on Z-transform of some basic functions

Find the Z-transform of the following functions :

1. 1 2. a^n 3. n 4. $\frac{1}{n}$ 5. $\frac{1}{n+1}$

6. $\frac{1}{n-1}$ 7. $\frac{1}{n!}$ 8. $\frac{1}{(n+1)!}$ 9. $(n+2)$ 10. $\frac{1}{n+2}$

1. **Prove that, $Z[1] = \frac{z}{z-1}$, $|z| > 1$.**

Solution : We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z[1] = \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \ldots$$

$$= \left[1 - \frac{1}{z}\right]^{-1}$$

$$= \left[\frac{z-1}{z}\right]^{-1}$$

$$= \frac{z}{z-1}, \ |z| > 1$$

$[\because (1-x)^{-1} = 1+x+x^2+\ldots]$ $\lvert x \rvert < 1$
Here, $x = \frac{1}{z}$, $\left\lvert\frac{1}{z}\right\rvert < 1$ i.e., $1 < \lvert z \rvert$ i.e., $\lvert z \rvert > 1$

2. **Prove that, $Z[a^n] = \frac{z}{z-a}$ if $|z| > |a|$**

[A.U. A/M 1999, A/M 2000, A/M 2008] [A.U A/M 2017 R-13, R-8]

Solution : We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} a^n \left[\frac{1}{z}\right]^n = \sum_{n=0}^{\infty} \left[\frac{a}{z}\right]^n$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \ldots$$

$$= \left[1 - \frac{a}{z}\right]^{-1}$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$= \frac{z}{z-a}, \quad |z| > |a|$$

$[\because (1-x)^{-1} = 1 + x + x^2 + \ldots]$ $\lvert x \rvert < 1$
Here, $x = \frac{a}{z}$ $\left\lvert\frac{a}{z}\right\rvert < 1$ i.e., $\lvert a\rvert < \lvert z\rvert$ i.e., $\lvert z\rvert > \lvert a\rvert$

3. Prove that, $Z(n) = \frac{z}{(z-1)^2}, \ |z| > 1$ **[A.U. 1998] [A.U. A/M 2008] [A.U CBT N/D 2010, M/J 2013] [A.U N/D 2014 R-13]**

Solution: We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

$$Z[n] = \sum_{n=0}^{\infty} n z^{-n}$$

Formula :
$(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$ $\lvert x \rvert < 1$
Here, $x = \frac{1}{z}$, $\left\lvert\frac{1}{z}\right\rvert < 1$ i.e., $\lvert 1\rvert < \lvert z\rvert$ i.e., $\lvert z\rvert > 1$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n} = \sum_{n=0}^{\infty} n \left(\frac{1}{z}\right)^n$$

$$= 0 + \left[\frac{1}{z}\right] + 2\left[\frac{1}{z}\right]^2 + \ldots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \ldots\right]$$

$$= \frac{1}{z}\left[\left(1 - \frac{1}{z}\right)^{-2}\right]$$

$$= \frac{1}{z}\left[\left(\frac{z-1}{z}\right)^{-2}\right]$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right]^2 = \frac{1}{z}\,\frac{z^2}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}, \quad |z| > 1$$

4. Prove that, $Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$ if $|z| > 1,\ n > 0$.

Solution : **[A.U M/J 2007, A.U. A/M 2008, A.U.T. CBT N/D 2011] [A.U N/D 2013]**

We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n z^n} = \sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{1}{z}\right]^n$$

Formula :
$-\log(1-x)$ $= x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$ $\lvert x\rvert < 1$
Here, $x = \frac{1}{z}$, $\left\lvert\frac{1}{z}\right\rvert < 1$ i.e., $\lvert 1\rvert < \lvert z\rvert$ i.e., $\lvert z\rvert > 1$
$\log a^p = p \log a$

$$= \frac{1}{z} + \frac{1}{2}\left[\frac{1}{z}\right]^2 + \frac{1}{3}\left[\frac{1}{z}\right]^3 + \ldots$$

$$= \left(\frac{1}{z}\right) + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \ldots$$

$$= -\log\left[1 - \frac{1}{z}\right] = -\log\left[\frac{z-1}{z}\right]$$

$$= (-1)\log\left[\frac{z-1}{z}\right] = \log\left[\frac{z-1}{z}\right]^{-1}$$

$$= \log\frac{z}{z-1}, \quad |z| > 1$$

5. Prove that, $Z\left[\dfrac{1}{n+1}\right] = z\log\dfrac{z}{z-1}$

[A.U. N/D 2005] [A.U. CBT Dec. 2008] [A.U Tvli M/J 2011] [A.U. Tvli. N/D 2011] [A.U A/M 2015 R-13]

Solution :

We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

$$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \ldots$$

Formula :
$-\log(1-x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots$
Here, $x = \dfrac{1}{z}$

$$= z\left[\frac{1}{z}\right]\left[1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \ldots\right]$$

$$= z\left[\frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \ldots\right]$$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$= z\left[-\log\left(\frac{z-1}{z}\right)\right]$$

$$= z\left[\log\frac{z}{z-1}\right] = z\log\frac{z}{z-1}$$

6. Prove that, $Z\left[\dfrac{1}{n-1}\right] = \dfrac{1}{z}\log\dfrac{z}{z-1}$, $n > 1$.

Solution : We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

$$Z\left[\frac{1}{n-1}\right] = \sum_{n=2}^{\infty} \frac{1}{n-1} z^{-n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{(n-1)\, z^n} = \sum_{n=2}^{\infty} \frac{1}{n-1}\left(\frac{1}{z}\right)^n$$

$$= \left[\frac{1}{z}\right]^2 + \frac{1}{2}\left[\frac{1}{z}\right]^3 + \ldots$$

$$= \frac{1}{z}\left[\frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \ldots\right]$$

$$= \frac{1}{z}\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$= \frac{1}{z}\left[-\log\left[\frac{z-1}{z}\right]\right]$$

$$= \frac{1}{z}\log\left[\frac{z}{z-1}\right]$$

7. Prove that, $Z\left[\dfrac{1}{n!}\right] = e^{1/z}$

[A.U Tvli N/D 2009, A.U.T. Trichy N/D 2011, A.U. N/D 2011] [A.U. M/J 2016 R-13]

Solution :

We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n$$

$\because e^x = 1 + \dfrac{x}{\lfloor 1} + \dfrac{x^2}{\lfloor 2} + \ldots$
Here, $x = \dfrac{1}{z}$

$$= 1 + \frac{1}{\lfloor 1}\left(\frac{1}{z}\right) + \frac{1}{\lfloor 2}\left(\frac{1}{z}\right)^2 + \ldots$$

$$= 1 + \frac{\left(\frac{1}{z}\right)}{\lfloor 1} + \frac{\left(\frac{1}{z}\right)^2}{\lfloor 2} + \ldots$$

$$= e^{1/z}$$

8. **Find** $Z\left[\frac{1}{(n+1)!}\right]$

[A.U. March, 1996] [A.U Tvli N/D 2009] [A.U.T Tvli N/D 2011]

Solution : We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z\left[\frac{1}{(n+1)!}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left[\frac{1}{z}\right]^{n}$$

$$= \frac{1}{\lfloor 1} \left[\frac{1}{z}\right]^{0} + \frac{1}{\lfloor 2} \left[\frac{1}{z}\right]^{1} + \frac{1}{\lfloor 3} \left[\frac{1}{z}\right]^{2} + \ldots$$

$$= 1 + \frac{[1/z]}{2!} + \frac{[1/z]^2}{3!} + \ldots$$

$$= z\left[\frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \ldots\right]$$

$$= z\left[e^{1/z} - 1\right]$$

$$= ze^{1/z} - z$$

Formula :

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots$$

$$e^x - 1 = \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots$$

9. **Find the Z-transform of (n + 2)** **[A.U. N/D 2006]**

Solution : $Z(n+2) = Z(n) + Z(2)$

$$= Z(n) + 2Z(1)$$

$$= \frac{z}{(z-1)^2} + 2\frac{z}{z-1}$$

Formula :

$$Z(1) = \frac{z}{z-1},$$

$$Z(n) = \frac{z}{(z-1)^2}$$

10. **Find** $Z\left[\frac{1}{n+2}\right]$

Solution : We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n+2} \left(\frac{1}{z}\right)^{n}$$

$$= \frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \ldots$$

$$= z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \frac{1}{4}\left(\frac{1}{z}\right)^4 + \ldots\right]$$

$$= z^2\left[-\log\left(1-\frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= z^2\left[-\log\left(\frac{z-1}{z}\right) - \frac{1}{z}\right]$$

$$= z^2\left[\log\left(\frac{z}{z-1}\right) - \frac{1}{z}\right]$$

$$= z^2 \log\frac{z}{z-1} - z$$

Formula :

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

Here, $x = \frac{1}{z}$, $\left|\frac{1}{z}\right| < 1$

i.e., $|1| < |z|$

i.e., $|z| > 1$

$$\log a^p = p \log a$$

5.1.2 Linearity property :

The Z - transform is linear

(i.e.,) $Z\{ax(n) + by(n)\} = a\,Z\{x(n)\} + b\,Z\{y(n)\}$

Proof : We know that, $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z\{ax(n) + by(n)\} = \sum_{n=0}^{\infty} [ax(n) + by(n)]\, z^{-n}$$

$$= a\sum_{n=0}^{\infty} x(n) z^{-n} + b\sum_{n=0}^{\infty} y(n) z^{-n}$$

$$= a\,Z\{x(n)\} + b\,Z\{y(n)\}$$

II. Problems based on $Z(1) = \frac{z}{z-1}$ and $Z[a^n] = \frac{z}{z-a}$ if $|z| > |a|$

Find the Z-transform of the following functions :

(1) K (2) $(-1)^n$ (3) $(-3)^n$ (4) $\frac{1}{3^n}$ (5) e^{an} (6) e^{-an}

(7) $\cos n\theta$, $\sin n\theta$ (8) $r^n \cos n\theta$ and $r^n \sin n\theta$ (9) t (10) e^{-at}

(11) $\cos an$ (12) a^{n-1} (13) $\cosh \alpha n$ (14) $\sinh 3n$

1. Find Z (K) **[A.U A/M 2019 R-13]**

Solution : We know that, $Z[1] = \frac{z}{z-1}$

$$Z[K] = Z[K \,.\, 1] = K\, Z[1] \text{ by linearity property}$$

$$= K\left[\frac{z}{z-1}\right]$$

2. Find Z [$(-1)^n$]

Solution : We know that, $Z[a^n] = \frac{z}{z-a}$

$$Z[(-1)^n] = \frac{z}{z-(-1)} \qquad \text{Here, } a = -1$$

$$= \frac{z}{z+1}$$

3. Find Z [$(-3)^n$]

Solution : We know that, $Z[a^n] = \frac{z}{z-a}$

$$Z[(-3)^n] = \frac{z}{z-(-3)} \qquad \text{Here, } a = -3$$

$$= \frac{z}{z+3}$$

4. Find Z $\left[\frac{1}{3^n}\right]$

Solution :

We know that, $Z[a^n] = \frac{z}{z-a}$

$$Z\left[\frac{1}{3^n}\right] = Z\left[\left(\frac{1}{3}\right)^n\right] = \frac{z}{z-\frac{1}{3}} \qquad \text{Here, } a = \frac{1}{3}$$

$$= \frac{3z}{3z-1}$$

5. Find Z [e^{an}].

Solution :

We know that, $Z[a^n] = \frac{z}{z-a}$

$$Z[e^{an}] = Z[(e^a)^n] = \frac{z}{z-e^a} \qquad \text{Here, } a = e^a$$

6. Find Z [e^{-an}] **[A.U CBT N/D 2010]**

Solution : We know that, $Z[a^n] = \frac{z}{z-a}$

$$Z[e^{-an}] = Z[(e^{-a})^n] = \frac{z}{z-e^{-a}} \qquad \text{Here, } a = e^{-a}$$

7. Find Z [$\cos n\theta$] and Z [$\sin n\theta$] **[A.U N/D 2010]**

Solution : Let $a = e^{i\theta}$ **[A.U N/D 2014 R-2013, M/J 2016 R-13]**

$$a^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

We know that, $Z[a^n] = \frac{z}{z-a}$, $|z| > |a|$

$$Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}} \qquad \text{Here, } a = e^{i\theta}$$

$$Z\,[e^{in\theta}] = \frac{z}{z-(\cos\theta + i\sin\theta)} \quad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$Z\,[\cos n\theta + i\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$Z\,[\cos n\theta] + i\,Z\,[\sin n\theta]$$

$$= \left[\frac{z}{(z-\cos\theta)-i\sin\theta}\right]\left[\frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}\right]$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$= \frac{z\,(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + i\,\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real and imaginary parts, we get

$$Z\,\{\cos n\theta\} = \frac{z\,(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1$$

$$Z\,\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1$$

8. Find $Z\,[r^n\cos n\theta]$ and $Z\,[r^n\sin n\theta]$

Solution : Let $a = re^{i\theta}$ **[A.U M/J 2014][A.U A/M 2015, R-08, 13]**

$$a^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$= r^n[\cos n\theta + i\sin n\theta]$$

$$= r^n\cos n\theta + i\,r^n\sin n\theta$$

We know that, $Z\,[a^n] = \dfrac{z}{z-a}$

$$Z\,[(re^{i\theta})^n] = \frac{z}{z-re^{i\theta}}$$

$$Z\,[r^n e^{in\theta}] = \frac{z}{z-r(\cos\theta + i\sin\theta)}$$

$$Z\,[r^n\cos n\theta + i\,r^n\sin n\theta] = \frac{z}{z - r\cos\theta - ir\sin\theta}$$

$$= \frac{z\,[(z-r\cos\theta) + ir\sin\theta]}{[(z-r\cos\theta) - ir\sin\theta)\,[(z-r\cos\theta) + ir\sin\theta)]}$$

$$= \frac{z\,(z-r\cos\theta) + irz\sin\theta}{(z-r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{z\,(z-r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \frac{z\,(z-r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$= \frac{z\,(z-r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} + i\,\frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Equating the real and imaginary parts, we get

$$Z\,\{r^n\cos n\theta\} = \frac{z\,(z-r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}, \quad |z| > r$$

$$Z\,\{r^n\sin n\theta\} = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}, \quad |z| > r$$

9. Find Z (t).

Solution : We know that, $Z\,\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\,z^{-n}$

$$Z\,\{t\} = \sum_{n=0}^{\infty} nT\,z^{-n}$$

$$= T\sum_{n=0}^{\infty} nz^{-n}$$

$$= T\,Z[n]$$

$$= T\frac{z}{(z-1)^2} = \frac{Tz}{(z-1)^2}$$

10. Find Z [e^{-at}]

Solution : We know that, $Z[f(t)] = \sum_{n=0}^{\infty} f(nT)\,z^{-n}$

$$Z[e^{-at}] = \sum_{n=0}^{\infty} e^{-anT} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(e^{-aT}\right)^n z^{-n}$$

$$= Z[(e^{-aT})^n]$$

$$= \frac{z}{z-e^{-aT}}$$

Formula :

$$Z[a^n] = \frac{z}{z-a}$$

11. Find Z [cos an] [A.U. 1998]

Solution : We know that, $Z[\cos n\theta] = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

Here, $\theta = a$

$$Z[\cos an] = \frac{z(z-\cos a)}{z^2-2z\cos a+1}$$

12. Find Z [a^{n-1}]

Solution :

We know that, $Z[a^n] = \dfrac{z}{z-a}$

$$Z[a^{n-1}] = z^{-1} Z[a^n], \quad \text{by shift property}$$

$$= \frac{1}{z}\left[\frac{z}{z-a}\right]$$

$$= \frac{1}{z-a}$$

13. Find Z [cosh α n]

Solution :

$$Z[\cosh\alpha n] = Z\left[\frac{e^{\alpha n}+e^{-\alpha n}}{2}\right] = \frac{1}{2}\left[Z[e^{\alpha n}+e^{-\alpha n}]\right]$$

$$= \frac{1}{2}Z\left[e^{\alpha n}\right] + \frac{1}{2}Z\left[e^{-\alpha n}\right]$$

$$= \frac{1}{2}Z\left[(e^{\alpha})^n\right] + \frac{1}{2}Z\left[(e^{-\alpha})^n\right]$$

$$= \frac{1}{2}\left(\frac{z}{z-e^{\alpha}}\right) + \frac{1}{2}\left(\frac{z}{z-e^{-\alpha}}\right)$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\alpha}} + \frac{1}{z-e^{-\alpha}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\alpha}+z-e^{\alpha}}{z^2-ze^{-\alpha}-ze^{\alpha}+1}\right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{\alpha}+e^{-\alpha})}{z^2-z(e^{\alpha}+e^{-\alpha})+1}\right]$$

$$= \frac{z}{2}\left[\frac{2z-2\cosh\alpha}{z^2-2z\cosh\alpha+1}\right]$$

$$= z\left[\frac{z-\cosh\alpha}{z^2-2z\cosh\alpha+1}\right]$$

14. Find Z [sinh 3n]

Solution :

$$Z[\sinh 3n] = Z\left[\frac{e^{3n}-e^{-3n}}{2}\right] = \frac{1}{2}\left[Z[e^{3n}]-Z[e^{-3n}]\right]$$

$$= \frac{1}{2} Z\left[(e^3)^n\right] - \frac{1}{2} Z\left[(e^{-3})^n\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^3}\right] - \frac{1}{2}\left[\frac{z}{z-e^{-3}}\right]$$

$$= \frac{z}{2}\left[\frac{1}{z-e^3} - \frac{1}{z-e^{-3}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-3}-z+e^3}{z^2 - z e^{-3} - z e^3 + 1}\right]$$

$$= \frac{z}{2}\left[\frac{e^3 - e^{-3}}{z^2 - z(e^3+e^{-3}) + 1}\right]$$

$$= \frac{z}{2}\left[\frac{2\sinh 3}{z^2 - 2z\cosh 3 + 1}\right]$$

$$= \frac{z\sinh 3}{z^2 - 2z\cosh 3 + 1}$$

III. Find the Z transform of the following :

1. $\cos\frac{n\pi}{2}$ 2. $\frac{1}{n(n+1)}, n > 0$ 3. $\frac{n-2}{n(n+1)}$

4. $\sin\left(\frac{n\pi}{2}+a\right)$ 5. $\sin^2\frac{n\pi}{2}$ 6. $\sin^3\frac{n\pi}{6}$ 7. $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

8. $4\left(3^n\right) + 2(-1)^n$ 9. $\frac{1}{(n+1)(n+2)}$ 10. $\frac{2n+3}{(n+1)(n+2)}$

1. Find $Z\left[\cos\frac{n\pi}{2}\right]$ **[A.U A/M 2008] [A.U N/D 2008] [A.U. M/J 2016 R-13] [A.U N/D 2018 R-13]**

Solution :

We know that, (i) $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

(ii) $Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1$

$$Z\left[\cos n\frac{\pi}{2}\right] = \frac{z\left[z - \cos\frac{\pi}{2}\right]}{z^2 - 2z\cos\frac{\pi}{2} + 1} \qquad \left[\text{Here, } \theta = \frac{\pi}{2}\right]$$

$$= \frac{z[z-0]}{z^2 - 2z(0) + 1}$$

$$= \frac{z^2}{z^2+1}, \quad |z| > 1$$

2. Find $Z\left[\frac{1}{n(n+1)}\right]$ **[A.U. Nov/Dec. End Semester 1996] [A.U N/D 2014 R-13] [A.U M/J 2016 R-13, N/D 2016 R-13] [A.U A/M 2019 R-13]**

Solution : $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ ... (1)

$$1 = A(n+1) + B(n)$$

Put $n = 0$, we get

$1 = A$

Put $n = -1$, we get

$1 = -B$

$\therefore B = -1$

$$(1) \Rightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad \text{... (2)}$$

We know that, $Z\left[\frac{1}{n}\right] = \log\frac{z}{z-1}$, $Z\left[\frac{1}{n+1}\right] = z\log\frac{z}{z-1}$

Take Z on both sides, we get

$$\therefore Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right] \text{ by linearity}$$

$$= \log\frac{z}{z-1} - z\log\frac{z}{z-1}$$

$$= (1-z)\log\frac{z}{z-1}$$

3. Find $Z\left[\frac{n-2}{n(n+1)}\right]$

Solution : $\frac{n-2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ ... (1)

$n - 2 = A(n+1) + Bn$

put $n = 0$, we get

$-2 = A$

$A = -2$

Put $n = -1$, we get

$-3 = -B$

$B = 3$

$$(1) \Rightarrow \frac{n-2}{n(n+1)} = \frac{-2}{n} + \frac{3}{n+1}$$

$$= -2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n+1}\right)$$

We know that, $Z\left[\frac{1}{n}\right] = \log\frac{z}{z-1}$, $Z\left[\frac{1}{n+1}\right] = z\log\frac{z}{z-1}$

$\therefore Z\left[\frac{n-2}{n(n+1)}\right] = -2\,Z\left[\frac{1}{n}\right] + 3\,Z\left[\frac{1}{n+1}\right]$, by linearity

$$= -2\log\frac{z}{z-1} + 3z\log\frac{z}{z-1}$$

$$= (-2+3z)\log\frac{z}{z-1}$$

4. Find $Z\left[\sin\left(\frac{n\pi}{2}+a\right)\right]$

Solution : $\sin\left(\frac{n\pi}{2}+a\right) = \sin\frac{n\pi}{2}\cos a + \cos\frac{n\pi}{2}\sin a$

since, $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$Z\left[\sin\left(\frac{n\pi}{2}+a\right)\right] = Z\left[\sin\frac{n\pi}{2}\cos a + \cos\frac{n\pi}{2}\sin a\right]$$

$$= \cos a\,Z\left[\sin\frac{n\pi}{2}\right] + \sin a\,Z\left[\cos\frac{n\pi}{2}\right] \quad ...(1)$$

by linearity

We know that, $Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$,

$Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

$$\therefore Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1} = \frac{z}{z^2+1}$$

$$Z\left[\cos\frac{n\pi}{2}\right] = \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1} = \left[\frac{z^2}{z^2+1}\right]$$

$$\therefore (1) \Rightarrow \quad = \cos a\left[\frac{z}{z^2+1}\right] + \sin a\left[\frac{z^2}{z^2+1}\right] = \frac{z\cos a + z^2\sin a}{z^2+1}$$

5. (a) Find $Z\left[\sin^2\frac{n\pi}{2}\right]$

Solution :

Formula :

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2\frac{n\pi}{2} = \frac{1-\cos n\pi}{2}$

$$Z\left[\sin^2\frac{n\pi}{2}\right] = Z\left[\frac{1-\cos n\pi}{2}\right]$$

$$= Z\left[\frac{1-(-1)^n}{2}\right]$$

$$= \frac{1}{2}Z[1-(-1)^n]$$

$$= \frac{1}{2}[Z(1) - Z(-1)^n] = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-(-1)}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z+1}\right] = \frac{z}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$

$$= \frac{z}{2}\left[\frac{z+1-z+1}{z^2-1}\right]$$

$$= \frac{z}{2}\left[\frac{2}{z^2-1}\right] = \frac{z}{z^2-1}$$

5. (b) Find $Z\left[\sin^2\frac{n\pi}{4}\right]$ **[A.U N/D 2018-A, R-17]**

Solution : $Z\left[\sin^2\frac{n\pi}{4}\right] = Z\left[\frac{1-\cos\left(\frac{n\pi}{2}\right)}{2}\right]$

$$= \frac{1}{2}\left[Z[1] - Z\left[\cos\frac{n\pi}{2}\right]\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1}\right]$$

6. Find Z $\left[\sin^3\left(\frac{n\pi}{6}\right)\right]$

Solution :

> **Formula :**
>
> $\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$
>
> $\sin^3\left(\frac{n\pi}{6}\right) = \frac{1}{4}\left[3\sin\frac{n\pi}{6} - \sin\frac{n\pi}{2}\right]$

$$Z\left[\sin^3\left(\frac{n\pi}{6}\right)\right] = \frac{3}{4}Z\left[\sin\frac{n\pi}{6}\right] - \frac{1}{4}Z\left[\sin\frac{n\pi}{2}\right]$$

$$= \frac{3}{4}\frac{z\sin\frac{\pi}{6}}{z^2-2z\cos\frac{\pi}{6}+1} - \frac{1}{4}\frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1}$$

$$= \frac{3}{4}\left[\frac{z\left(\frac{1}{2}\right)}{z^2-2z\left(\frac{\sqrt{3}}{2}\right)+1}\right] - \frac{1}{4}\left[\frac{z(1)}{z^2+1}\right]$$

$$= \frac{3z}{8(z^2-z\sqrt{3}+1)} - \frac{z}{4(z^2+1)}$$

$$= \frac{3}{8}\frac{z}{z^2-z\sqrt{3}+1} - \frac{1}{4}\frac{z}{z^2+1}$$

7. Find Z $\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right]$ **[A.U N/D 2018 R-17]**

Solution :

> **Formula :**
>
> $\cos(A+B) = \cos A\cos B - \sin A\sin B$
>
> $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) = \cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}$
>
> $= \cos\frac{n\pi}{2}\frac{1}{\sqrt{2}} - \sin\frac{n\pi}{2}\frac{1}{\sqrt{2}}$
>
> $= \frac{1}{\sqrt{2}}\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$

$$Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = \frac{1}{\sqrt{2}}Z\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}Z\left[\cos\frac{n\pi}{2}\right] - \frac{1}{\sqrt{2}}Z\left[\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}\frac{z^2}{z^2+1} - \frac{1}{\sqrt{2}}\frac{z}{z^2+1} = \frac{z}{\sqrt{2}}\left[\frac{z}{z^2+1} - \frac{1}{z^2+1}\right]$$

$$= \frac{z}{\sqrt{2}}\left[\frac{z-1}{z^2+1}\right] = \frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

8. Find Z $[4.3^n + 2(-1)^n]$

Solution : We know that, $Z[a^n] = \frac{z}{z-a}$

$$Z\left[4.3^n + 2.(-1)^n\right] = 4Z[3^n] + 2Z[(-1)^n]$$

$$= 4\left[\frac{z}{z-3}\right] + 2\left[\frac{z}{z+1}\right]$$

9. Find $Z\left[\frac{1}{(n+1)(n+2)}\right]$ [A.U., March, 1996] [A.U. N/D 2006] [A.U.T CBT N/D 2011, A.U.T. Trichy N/D 2011] [A.U N/D 2013] [A.U N/D 2018 R-13]

Solution : $$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \quad \text{... (1)}$$

$$1 = A(n+2) + B(n+1)$$

Put $n = -1$, we get

$1 = A$

$A = 1$

Put $n = -2$, we get

$1 = -B$

$B = -1$

$$\therefore (1) \Rightarrow \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$Z\left[\frac{1}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] - Z\left[\frac{1}{n+2}\right] \quad \text{... (2)}$$

We know that, $$Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$$

$$Z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$$

$$(2) \Rightarrow Z\left[\frac{1}{(n+1)(n+2)}\right] = z\log\left(\frac{z}{z-1}\right) - \left[z^2\log\left(\frac{z}{z-1}\right) - z\right]$$

$$= z\log\frac{z}{z-1} - z^2\log\left(\frac{z}{z-1}\right) + z$$

$$= (z - z^2)\log\frac{z}{z-1} + z$$

10. Find the Z-transform of $f(n) = \frac{2n+3}{(n+1)(n+2)}$ [A.U N/D 2...] [A.U A/M 2017 R-8, N/D 2016 R-8, A/M 2017 R-...]

Solution : Given : $$f(x) = \frac{2n+3}{(n+1)(n+2)}$$

$$\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \quad \text{... (1)}$$

$$2n+3 = A(n+2) + B(n+1)$$

Put $n = -1$, we get

$-2 + 3 = A$

$A = 1$

Put $n = -2$, we get

$-4 + 3 = -B$

$-1 = -B$

$B = 1$

$$(1) \Rightarrow \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$$

$$Z[f(n)] = Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] + Z\left[\frac{1}{n+2}\right] \quad \text{... (2)}$$

We know that, $$Z\left[\frac{1}{n+1}\right] = z\log\frac{z}{z-1}, \quad \text{... (3)}$$

$$Z\left[\frac{1}{n+2}\right] = z^2\left[\log\frac{z}{z-1}\right] - z \quad \text{... (4)}$$

$$\therefore (2) \Rightarrow Z[f(n)] = z\log\frac{z}{z-1} + z^2\log\frac{z}{z-1} - z \quad \text{by (3) \& (4)}$$

$$= z\log\frac{z}{z-1}[1+z] - z$$

$$= z(z+1)\log\frac{z}{z-1} - z$$

5.1.3. Differentiation in the Z-Domain

$$Z[n f(n)] = -z\frac{d}{dz}F(z)]$$

[A.U. April, 2000]

Proof :

Given : $F(z) = Z[f(n)]$

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} (-n) f(n) z^{-n-1} = -\sum_{n=0}^{\infty} n f(n) \frac{z^{-n}}{z}$$

$$z \frac{d}{dz} F(z) = -\sum_{n=0}^{\infty} n f(n) z^{-n} = -Z[n f(n)]$$

$$Z[n f(n)] = -z \frac{d}{dz} F(z)$$

IV. Find the Z-transform of the following : [Differentiation in the Z-Domain]

(1) n^2 (2) n^3 (3) n^k (4) $an^2 + bn + c$ (5) $n(n-1)$

(6) nC_2 (7) $n \cos n\theta$ (8) $n(n-1)(n-2)$

(9) Given that $F(Z) = \log(1 + az^{-1})$, for $|z| > |a|$ find $f(n)$ and also find $z[n f(n)]$

(10) nC_k (11) $k+n\,C_n$ (12) $(n+1)(n+2)$

1. Find Z (n^2)

[A.U, N/D 1996, A.U Tvli N/D 2010, A.U M/J 2014, N/D 2016 R-8]

Solution : We know that, $Z[n f(n)] = -z \frac{d}{dz} F(z)$

$$Z[n^2] = Z[n\,n] = -z \frac{d}{dz}[Z(n)]$$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right] = -z\left[\frac{-1-z}{(z-1)^3}\right]$$

$$= z\frac{(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$$

2. Find Z (n^3) **[A.U N/D 2016 R-13]**

Solution : We know that, $Z[n f(n)] = -z \frac{d}{dz} F(z)$

$$Z[n^3] = Z[n\,n^2] = -z \frac{d}{dz} Z[n^2]$$

$$= -z \frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\left[\frac{(z-1)^3(2z+1) - (z^2+z)\,3(z-1)^2}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)(2z+1) - 3(z^2+z)}{(z-1)^4}\right]$$

$$= -z\left[\frac{2z^2 - 2z + z - 1 - 3z^2 - 3z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$$

$$= \frac{z(z^2+4z+1)}{(z-1)^4}$$

3. Find Z (n^k)

Solution : We know that, $Z[n f(n)] = -z \frac{d}{dz} F(z)$

$$Z[n^k] = Z[n\,n^{k-1}]$$

$$= -z \frac{d}{dz} Z[n^{k-1}]$$

Which is a recurrence formula.

4. Find Z $[an^2 + bn + c]$

Solution : $Z[an^2 + bn + c]$

$$= a Z[n^2] + b Z[n] + c Z[1]$$

$$= a\frac{z^2+z}{(z-1)^3} + b\frac{z}{(z-1)^2} + C\frac{z}{z-1}$$

$$= \frac{z}{(z-1)^3}\left[a(z+1) + b(z-1) + c(z-1)^2\right]$$

$$= \frac{z}{(z-1)^3}\left[cz^2 + (a+b-2c)z + (a-b+c)\right]$$

5. Find Z [n (n – 1)]

Solution : $Z[n(n-1)] = Z[n^2-n]$

$$\therefore Z[n^2] - Z[n]$$

$$= \frac{z^2+z}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{1}{(z-1)^3}[z^2+z-z(z-1)]$$

$$= \frac{1}{(z-1)^3}[z^2+z-z(z-1)]$$

$$= \frac{1}{(z-1)^3}[z^2+z-z^2+z]$$

$$= \frac{2z}{(z-1)^3}$$

6. Find Z [nC$_2$]
[A.U A/M 2017 R-8]

Solution :

> **Formula :**
> $$nC_2 = \frac{n(n-1)}{2} = \frac{1}{2}[n^2-n] = \frac{1}{2}n^2 - \frac{1}{2}n$$

$$Z[nC_2] = \frac{1}{2}Z[n^2] - \frac{1}{2}Z[n]$$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3}\right] - \frac{1}{2}\frac{z}{(z-1)^2}$$

$$= \frac{z(z+1) - z(z-1)}{2(z-1)^3}$$

$$= \frac{z^2+z-z^2+z}{2(z-1)^3}$$

$$= \frac{2z}{2(z-1)^3} = \frac{z}{(z-1)^3}$$

Find Z [n cos nθ]

Solution : We know that, (i) $Z[nf(n)] = -z\frac{d}{dz}F(z)$

(ii) $Z[\cos n\theta] = \frac{z[z-\cos\theta]}{z^2-2z\cos\theta+1}$

$$Z[n\cos n\theta] = -z\frac{d}{dz}[Z(\cos n\theta)]$$

$$= -z\frac{d}{dz}\left[\frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}\right]$$

$$-z\left[\frac{(z^2-2z\cos\theta+1)(2z-\cos\theta)-(z^2-z\cos\theta)(2z-2\cos\theta)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$-z\left[\frac{2z^3-4z^2\cos\theta+2z-z^2\cos\theta+2z\cos^2\theta-\cos\theta-2z^3+2z^2\cos\theta+2z^2\cos\theta-2z\cos^2\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$$\frac{-z[2z-z^2\cos\theta-\cos\theta]}{(z^2-2z\cos\theta+1)^2}$$

$$\frac{z[z^2\cos\theta-2z+\cos\theta]}{(z^2-2z\cos\theta+1)^2}$$

8. Find Z [n (n − 1) (n − 2)] **[AU M/J 2012]**

Solution :

$$\boxed{\begin{aligned} n(n-1)(n-2) &= n[n^2-2n-n+2] \\ &= n[n^2-3n+2] \\ &= n^3-3n^2+2n \end{aligned}}$$

$$Z[n(n-1)(n-2)] = Z[n^3-3n^2+2n]$$

$$= Z[n^3] - 3Z[n^2] + 2Z[n]$$

$$= \frac{z[z^2+4z+1]}{(z-1)^4} - 3\frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2}$$

$$= \frac{z^3+4z^2+z-3z(z+1)(z-1)+2z(z-1)^2}{(z-1)^4}$$

$$= \frac{z^3+4z^2+z-3z(z^2-1)+2z(z^2-2z+1)}{(z-1)^4}$$

$$= \frac{z^3+4z^2+z-3z^3+3z+2z^3-4z^2+2z}{(z-1)^4}$$

$$= \frac{6z}{(z-1)^4}$$

9. Given that F (z) = log (1 + az^{-1}), for |z| > |a |, find f(n) and also find Z[n f(n)].

Solution : Given : $F(z) = \log(1+az^{-1})$

$$= az^{-1} - \frac{1}{2}(az^{-1})^2 + \frac{1}{3}(az^{-1})^3 - \ldots + (-1)^{n-1}\frac{1}{n}(az^{-1})^n + \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} (az^{-1})^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} (-1)^n (-1)^{-1} \frac{1}{n} a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{-1}\right] \frac{1}{n} a^n z^{-n}$$

$$F(z) = \sum_{n=1}^{\infty} - \frac{(-1)^n a^n}{n} z^{-n} = \sum_{n=1}^{\infty} \frac{-(-a)^n}{n} z^{-n}$$

$$\therefore f(n) = \frac{-(-a)^n}{n}$$

$$Z[n f(n)] = -z\frac{d}{dz}F(z)$$

$$= -z\frac{d}{dz}\log(1+az^{-1})$$

$$= -z\frac{1}{1+az^{-1}}[-az^{-2}] = -z\left[\frac{-az^{-2}}{1+az^{-1}}\right]$$

$$= \frac{az^{-1}}{1+az^{-1}} = \frac{\frac{a}{z}}{1+\frac{a}{z}} = \frac{a}{z+a}$$

10. Find the Z-transform of {nC$_k$}.

Solution : $Z\{nC_k\} = \sum_{k=0}^{n} nC_k z^{-k}$

$$= 1 + nC_1 z^{-1} + nC_2 z^{-2} + \ldots + nC_n z^{-n}$$

This is the expansion of binomial theorem.

$$= (1+z^{-1})^n$$

11. Find the Z-transform of $\{{}^{k+n}C_n\}$

Solution : $Z\{{}^{k+n}C_n\} = \sum_{k=0}^{\infty} {}^{k+n}C_n z^{-k} \qquad (k+n > n\,;\, k > 0$

$$= \sum_{k=0}^{\infty} {}^{k+n}C_k z^{-k} \quad (nC_r = nC_{n-r})$$

$$= 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + {}^{n+3}C_3 z^{-3} + \ldots$$

$$= (1 - z^{-1})^{-n-1} = (1 - z^{-1})^{-(n+1)}$$

12. Find the Z-transform of (n + 1) (n + 2).

[A.U. M/J 2006] [A.U. CBT Dec. 2008] [A.U. CBT Dec. 2009]

Solution : $Z[(n+1)(n+2)] = Z[n^2 + 2n + n + 2]$

$$= Z[n^2 + 3n + 2]$$

$$= Z[n^2] + 3Z[n] + 2Z[1]$$

$$= \frac{z^2+z}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\left[\frac{z}{z-1}\right]$$

$$= \frac{(z^2+z) + 3z(z-1) + 2z(z-1)^2}{(z-1)^3}$$

$$= \frac{2z^3}{(z-1)^3} = 2\left[\frac{z}{z-1}\right]^3$$

5.1.4 First Shifting theorem [Frequency shifting]:

Damping rule

[The geometric factor a^{-n} when $|a| < 1$, damps the function u_n. Hence we use the name damping rule]

(i) If $Z\{f(n)\} = F(z)$, then $Z\left\{a^n f(n)\right\} = F\left[\frac{z}{a}\right]$

[A.U N/D 2014 R-08, A/M 2015 R-13] [A.U N/D 2019, R-17]

(ii) If $Z\{f(n)\} = F(z)$, then $Z\left[a^{-n} f(n)\right] = F[az]$

Proof : (i) Given :. $F(z) = Z\{f(n)\}$

$$= \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z\left\{a^n f(n)\right\} = \sum_{n=0}^{\infty} a^n f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n} = F\left[\frac{z}{a}\right]$$

$$Z\left\{a^n f(n)\right\} = F[z] \text{ where } z \to \frac{z}{a}$$

(ii) Given : $F(z) = Z[f(n)]$

$$= \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z[a^{-n} f(n)] = \sum_{n=0}^{\infty} a^{-n} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\,(az)^{-n}$$

$$= F(az) = \left[F[z]\right]_{z \to az}$$

V. Problems based on first shifting theorem [Frequency shifting]

Find the Z-transform of the following :

(1) $a^n n$	(2) $\frac{a^n}{n!}$	(3) $\frac{a^n}{n}$	(4) $a^n \sin n\theta$
(5) $a^n \cos n\theta$	(6) $a^n r^n \cos n\theta$	(7) $(n-1)a^{n-1}$	(8) $a^n \cos n\pi$
(9) $a^n n^3$	(10) $a^{-n} n^2$	(11) $2^n n^2$	(12) $2^n \sinh 3n$
(13) $a^n \cosh \alpha n$	(14) $a^n \sin\frac{n\pi}{2}$		

1. Find Z [a^n n] [A.U, 1998], [A.U. April, 2001] [A.U N/D 2007] [A.U. A/M 2008] [A.U N/D 2018-A R-17]

Solution : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\,[a^n n] = \left[Z\,[n]\right]_{z \to z/a}$$

$$= \left[\frac{z}{(z-1)^2}\right]_{z \to z/a} \qquad \left[\because Z(n) = \frac{z}{(z-1)^2}\right]$$

$$= \frac{\frac{z}{a}}{\left(\frac{z}{a} - 1\right)^2}$$

$$= \frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{az}{(z-a)^2}$$

2. Find Z $\left[\dfrac{a^n}{n!}\right]$ [A.U. N/D 2005] [A.U Trichy N/D 2009] [A.U M/J 2012] [A.U A/M 2019 R-8]

Solution :

We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\left[a^n \frac{1}{n!}\right] = \left[Z\left[\frac{1}{n!}\right]\right]_{z \to z/a}$$

$$= \left[e^{1/z}\right]_{z \to z/a} \qquad \because Z\left[\frac{1}{n!}\right] = e^{1/z}$$

$$= e^{\frac{1}{(z/a)}}$$

$$= e^{\frac{a}{z}}$$

Find Z $\left[\dfrac{a^n}{n}\right]$

Solution: We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\left[a^n \frac{1}{n}\right] = \left[Z\left[\frac{1}{n}\right]\right]_{z \to z/a}$$

$$= \left[\log \frac{z}{z-1}\right]_{z \to z/a} \qquad \because Z\left[\frac{1}{n}\right] = \log\frac{z}{z-1}$$

$$= \left[\log\left(\frac{\frac{z}{a}}{\frac{z}{a} - 1}\right)\right] = \log\left[\frac{\frac{z}{a}}{\frac{z-a}{a}}\right]$$

$$= \log\left[\frac{z}{z-a}\right]$$

Find Z [$a^n \sin n\theta$] [A.U N/D 2010]

Solution: We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\,[a^n \sin n\theta] = \left[Z\,[\sin n\theta]\right]_{z \to z/a}$$

$$= \left[\frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}\right]_{z \to z/a}$$

$$= \frac{\frac{z}{a}\sin\theta}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos\theta + 1}$$

$$= \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

5. Find $Z\,[a^n \cos n\theta]$ [A.U. M/J 2007]

Solution : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\,[a^n \cos n\theta] = \Big[Z\,[\cos n\theta]\Big]_{z \to z/a}$$

$$= \left[\frac{z\,(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}\right]_{z \to z/a}$$

$$= \frac{\frac{z}{a}\left[\frac{z}{a} - \cos\theta\right]}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos\theta + 1}$$

$$= \frac{z\,[z - a\cos\theta]}{z^2 - 2az\cos\theta + a^2}$$

6. Find $Z\,[a^n r^n \cos n\theta]$

Solution :

We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\left[a^n r^n \cos n\theta\right] = \Big[Z\,[r^n \cos n\theta]\Big]_{z \to z/a}$$

$$= \left[\frac{z\,[z - r\cos\theta]}{z^2 - 2rz\cos\theta + r^2}\right]_{z \to \frac{z}{a}}$$

$$= \left[\frac{\frac{z}{a}\left[\frac{z}{a} - r\cos\theta\right]}{\frac{z^2}{a^2} - 2r\frac{z}{a}\cos\theta + r^2}\right]$$

$$= \frac{z\,(z - ar\cos\theta)}{z^2 - 2\,arz\cos\theta + a^2 r^2}$$

$$\ldots)^n \cos n\theta] = \Big[Z\,[\cos n\theta]\Big]_{z \to \frac{z}{ar}} = \left[\frac{z\,(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}\right]_{z \to \frac{z}{ar}}$$

$$= \frac{\frac{z}{ar}\left(\frac{z}{ar} - \cos\theta\right)}{\left(\frac{z}{ar}\right)^2 - 2\left(\frac{z}{ar}\right)\cos\theta + 1} = \frac{z\,(z - ar\cos\theta)}{z^2 - 2\,arz\cos\theta + (ar)^2}$$

Find $Z\,[(n-1)\,a^{n-1}]$

...tion : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z[(n-1)\,a^{n-1}] = \frac{1}{z} Z\,[n\,a^n] \text{ by shift property}$$

$$= \frac{1}{z}\Big[Z\,[n]\Big]_{z \to \frac{z}{a}} = \frac{1}{z}\left[\frac{z}{(z-1)^2}\right]_{z \to \frac{z}{a}}$$

$$= \frac{1}{z}\,\frac{z/a}{\left(z/a - 1\right)^2} = \frac{1}{z}\,\frac{az}{(z-a)^2} = \frac{a}{(z-a)^2}$$

Find $Z\,[a^n \cos n\pi]$

...tion : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\,[a^n \cos n\pi] = \Big[Z\,[\cos n\pi]\Big]_{z \to z/a}$$

$$= \left[\frac{z\,(z - \cos\pi)}{z^2 - 2z\cos\pi + 1}\right]_{z \to z/a}$$

$$= \left[\frac{z\,(z+1)}{z^2 + 2z + 1}\right]_{z \to z/a}$$

$$= \left[\frac{z\,(z+1)}{(z+1)^2}\right]_{z \to z/a}$$

$$= \left[\frac{z}{z+1}\right]_{z \to z/a}$$

$$= \frac{\frac{z}{a}}{\frac{z}{a}+1} = \frac{\frac{z}{a}}{\frac{z+a}{a}} = \frac{z}{z+a}$$

9. Find $Z[a^n n^3]$

Solution : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z[a^n n^3] = \left[Z[n^3]\right]_{z \to z/a}$$

$$= \left[\frac{z(z^2+4z+1)}{(z-1)^4}\right]_{z \to z/a}$$

$$= \frac{\frac{z}{a}\left(\frac{z^2}{a^2}+4\frac{z}{a}+1\right)}{\left(\frac{z}{a}-1\right)^4}$$

$$= \frac{\left(\frac{z}{a}\right)\left(\frac{z^2+4az+a^2}{a^2}\right)}{\left(\frac{z-a}{a}\right)^4}$$

$$= \frac{za(z^2+4az+a^2)}{(z-a)^4}$$

10. Find $Z\left[a^{-n} n^2\right]$

Solution : We know that, $\boxed{Z\left[a^{-n} f(n)\right] = \left[F(z)\right]_{z \to az}}$

$$Z\left[a^{-n} n^2\right] = \left[Z[n^2]\right]_{z \to az}$$

$$= \left[\frac{z^2+z}{(z-1)^3}\right]_{z \to az} \qquad \left[\because Z[n^2] = \frac{z^2+z}{(z-1)^3}\right]$$

$$= \frac{(az)^2+(az)}{(az-1)^3} = \frac{a^2z^2+az}{(az-1)^3}$$

Find $Z\left[2^n n^2\right]$

Solution : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\left[2^n n^2\right] = \left[Z[n^2]\right]_{z \to \frac{z}{2}} \qquad [\because a = 2]$$

$$= \left[\frac{z^2+z}{(z-1)^3}\right]_{z \to \frac{z}{2}} \qquad \left[\because Z[n^2] = \frac{z^2+z}{(z-1)^3}\right]$$

$$= \frac{(z/2)^2+z/2}{\left(\frac{z}{2}-1\right)^3} = \frac{\frac{z^2}{4}+\frac{z}{2}}{\left(\frac{z-2}{2}\right)^3} = \frac{\left[\frac{z^2+2z}{4}\right]}{\left[\frac{(z-2)^3}{8}\right]}$$

$$= \left[\frac{z^2+2z}{(z-2)^3}\right](2) = \frac{2z(z+2)}{(z-2)^3}$$

Find $Z\left[2^n \sinh 3n\right]$

Solution : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\left[2^n \sinh 3n\right] = \left[Z[\sinh 3n]\right]_{z \to \frac{z}{2}} \qquad [\because a = 2]$$

$$= \left[\frac{z \sinh 3}{z^2 - 2z\cosh 3 + 1}\right]_{z \to \frac{z}{2}}$$

$$= \frac{\left(\frac{z}{2}\right)\sinh 3}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cosh 3 + 1}$$

$$= \frac{\left[\frac{z \sinh 3}{2}\right]}{\left[\frac{z^2 - 4z\cosh 3 + 4}{4}\right]}$$

$$= \frac{2z\sinh 3}{z^2 - 4z\cosh 3 + 4}$$

13. Find $Z\left[a^n \cosh \alpha n\right]$

Solution : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\left[a^n \cosh \alpha n\right] = \left[Z\left[\cosh \alpha n\right]\right]_{z \to \frac{z}{a}}$$

$$= \left[z\left(\frac{z - \cosh \alpha}{z^2 - 2z\cosh \alpha + 1}\right)\right]_{z \to \frac{z}{a}}$$

$$= \frac{z}{a}\left[\frac{\frac{z}{a} - \cosh \alpha}{\frac{z^2}{a^2} - 2\frac{z}{a}\cosh \alpha + 1}\right]$$

$$= \frac{z}{a}\frac{\left[\frac{z - a\cosh \alpha}{a}\right]}{\left[\frac{z^2 - 2az\cosh \alpha + a^2}{a^2}\right]}$$

$$= \frac{z(z - a\cosh \alpha)}{z^2 - 2az\cosh \alpha + a^2}$$

(i) Find $Z\left[a^n \sin \frac{n\pi}{2}\right]$ **[A.U A/M 2008]**

Solution : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\left[a^n \sin \frac{n\pi}{2}\right] = \left[Z\left(\sin \frac{n\pi}{2}\right)\right]_{z \to \frac{z}{a}} \quad \dots (1)$$

Formula : $\boxed{Z\left[\sin n\theta\right] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}}$

$$Z\left[\sin \frac{n\pi}{2}\right] = \frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1} = \frac{z}{z^2 + 1} \qquad \left[\begin{array}{l}\sin 90^\circ = 1 \\ \cos 90^\circ = 0\end{array}\right]$$

$$\Rightarrow Z\left[a^n \sin \frac{n\pi}{2}\right] = \left[\frac{z}{z^2 + 1}\right]_{z \to \frac{z}{a}} = \frac{\frac{z}{a}}{\left(\frac{z}{a}\right)^2 + 1} = \frac{\frac{z}{a}}{\frac{z^2 + a^2}{a^2}} = \frac{az}{a^2 + z^2}$$

(ii) Find $Z\left[a^n \cos \frac{n\pi}{2}\right]$

Solution : We know that, $\boxed{Z\left[a^n f(n)\right] = \left[F(z)\right]_{z \to \frac{z}{a}}}$

$$Z\left[\cos n\theta\right] = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$Z\left[\cos \frac{n\pi}{2}\right] = \frac{z\left(z - \cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1} = \frac{z^2}{z^2 + 1} \qquad [\because \cos 90^\circ = 0]$$

$$Z\left[a^n \cos \frac{n\pi}{2}\right] = \left[\frac{z^2}{z^2 + 1}\right]_{z \to \frac{z}{a}} = \frac{\left(\frac{z}{a}\right)^2}{\left(\frac{z}{a}\right)^2 + 1} = \frac{z^2}{z^2 + a^2}$$

5.1.5. Second shifting theorem [Time Shifting]

1. $Z[f(n+1)] = zF(z) - zf(0)$ **[A.U. N/D 2007, CBT N/D 2010] [A.U.T. CBT N/D 2011]**

Proof : $Z[f(n+1)] = \sum_{n=0}^{\infty} f(n+1) z^{-n} = z \sum_{n=0}^{\infty} f(n+1)\, z^{-(n+1)}$

$$= z \sum_{m=1}^{\infty} f(m)\, z^{-m} \text{ where } m = n+1$$

$$= z\left[\sum_{m=0}^{\infty} f(m) z^{-m} - f(0)\right] = zF(z) - zf(0)$$

2. $Z[f(n+2)] = z^2[F(z) - f(0) - f(1) z^{-1}]$

Proof : $Z[f(n+2)] = \sum_{n=0}^{\infty} f(n+2)\, z^{-n} = z^2 \sum_{n=0}^{\infty} f(n+2)\, z^{-(n+2)}$

$$= z^2 \sum_{m=2}^{\infty} f(m) z^{-m} \text{ where } m = n+2$$

$$= z^2\left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - f(0) - f(1) z^{-1}\right]$$

$$= z^2[F(z) - f(0) - f(1) z^{-1}]$$

Note : (i) $Z[f(n+k)] = z^n f(z) - \sum_{i=0}^{k-1} f(i) z^{k-i},\ n \geq -k$

Note : (ii) $Z[f(n-1)] = z^{-1} F(z)$

3. $Z[f(n-k)] = z^{-n} F(z)$ **[A.U Trichy N/D 2010]**

Proof : $Z[f(n-k)] = \sum_{k=0}^{\infty} f(n-k)\, z^{-k} = z^{-n} \sum_{k=0}^{\infty} f(n-k)\, z^{-(n-k)}$

$$= z^{-n} \sum_{m=-n}^{\infty} f(m) z^{-m} \text{ where } n-k = m$$

$$= z^{-n} \sum_{m=0}^{\infty} f(m)\, z^{-m} = z^{-n} F(z)$$

VI. Problems based on Time shifting

Find the Z-transform of the following :

(1) $\dfrac{1}{(n+2)!}$ (2) $\cos(n+1)\theta$ (3) $\sin(n-1)\theta$

1. Find $Z\left[\dfrac{1}{(n+2)!}\right]$

Solution :

$$Z[f(n+2)] = z^2[F(z) - f(0) - f(1) z^{-1}] \quad \ldots (1)$$

Let $f(n+2) = \dfrac{1}{(n+2)!}$

$\Rightarrow f(n) = \dfrac{1}{n!}$ i.e., $f(0) = \dfrac{1}{0!} = 1,\ f(1) = \dfrac{1}{1!} = 1$

$$F(z) = Z[f(n)] = Z\left[\frac{1}{n!}\right] = e^{1/z}$$

$$(1) \Rightarrow Z[f(n+2)] = z^2\left[e^{1/z} - 1 - z^{-1}\right]$$

$$= z^2 e^{1/z} - z^2 - z$$

2. Find $Z[\cos(n+1)\theta]$

Solution :

$$Z[f(n+1)] = zF(z) - zf(0) \quad \ldots (1)$$

Let $f(n+1) = \cos(n+1)\theta$

$\Rightarrow f(n) = \cos n\theta,\quad f(0) = \cos 0 = 1$

$$F(z) = Z[f(n)] = Z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$(1) \Rightarrow Z[f(n+1)] = z\left[\frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}\right] - z(1)$$

$$= z\left[\frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1} - 1\right]$$

$$= z\left[\frac{z^2 - z\cos\theta - z^2 + 2z\cos\theta - 1}{z^2 - 2z\cos\theta + 1}\right]$$

$$= z\left[\frac{z\cos\theta - 1}{z^2 - 2z\cos\theta + 1}\right]$$

3. Find Z $\left[\sin(n-1)\theta\right]$

Solution :

$$\boxed{Z\left[f(n-1)\right] = z^{-1}F(z)} \quad \dots (1)$$

Let $f(n-1) = \sin(n-1)\theta$

$$\Rightarrow \quad f(n) = \sin n\theta$$

$$F[z] = Z[f(n)] = Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$(1) \Rightarrow \quad Z[f(n-1)] = \frac{1}{z}\left[\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right]$$

$$= \frac{\sin\theta}{z^2 - 2z\cos\theta + 1}$$

5.1.6. Unit impulse sequence and unit step sequence.

Definition : *Unit impulse sequence*

The unit impulse sequence $\delta(n)$ is defined as the sequence with values

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

(1) Z-Transfrom of unit impulse sequence is 1. i.e., Z $[\delta(n)] = 1$

[A.U A/M 2019 R-17]

Proof : We know that, $Z\{x(n)\} = \sum\limits_{n=0}^{\infty} x(n) z^{-n}$

$$Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n}$$

$$= 1 + 0 + 0 + \dots \text{ by definition of } \delta(n)$$

$$= 1$$

Definition : *Unit step sequence*

The unit step sequence $u(n)$ has values

$$u(n) = \begin{cases} 1 & \text{for } n > 0 \\ 0 & \text{for } n < 0 \end{cases}$$

(2) Z-transform of unit step sequence i.e., Z $\{u(n)\} = \frac{z}{z-1}$

Proof : We know that, $Z\{x(n)\} = \sum\limits_{n=0}^{\infty} x(n) z^{-n}$ **[A.U N/D 2008]**

$$Z\{u(n)\} = \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} \text{ by definition of } u(n)$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \dots$$

$$= \left[1 - \frac{1}{z}\right]^{-1}$$

$$= \left[\frac{z-1}{z}\right]^{-1}$$

$$= \frac{z}{z-1}$$

(3) Z-transform of $a^n u(n)$ is $\frac{z}{z-a}$ is $|z| > a$

Proof : We know that, $Z\{x(n)\} = \sum\limits_{n=0}^{\infty} x(n) z^{-n}$

$$Z\left\{a^n u(n)\right\} = \sum_{n=0}^{\infty} a^n z^{-n} \qquad [\because \text{ by def. of } u(n)]$$

$$= \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \ldots$$

$$= \left[1 - \frac{a}{z}\right]^{-1} = \left[\frac{z-a}{z}\right]^{-1}$$

$$= \frac{z}{z-a}, \ |z| > |a|$$

VII. Find the Z-transform of the following.

(based on unit impulse sequence and unit step sequence)

(1) $\delta(n-k)$ (2) $a^n \delta(n-k)$ (3) $2^n \delta(n-2)$ (4) $3^n \delta(n-1)$

(5) $u(n-1)$ (6) $\cos\frac{n\pi}{2}\, u(n)$ (7) $-a^n u(-n-1)$ (8) $\left(\frac{1}{2}\right)^n u(n)$

1. Find $Z[\delta(n-k)]$

Solution : $Z[\delta(n-k)] = \sum_{n=0}^{\infty} \delta(n-k)\, z^{-n}$ ... (1)

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$$

$$(1) \Rightarrow Z[\delta(n-k)] = \frac{1}{z^k}$$

$$Z[\delta(n-1)] = \frac{1}{z}$$

Find $Z[a^n \delta(n-k)]$

Solution : $Z[a^n \delta(n-k)] = Z[\delta(n-k)]_{z \to z/a}$

$$= \left[\frac{1}{z^k}\right]_{z \to \frac{z}{a}} = \frac{1}{\left(\frac{z}{a}\right)^k}$$

$$= \frac{a^k}{z^k} = \left(\frac{z}{a}\right)^{-k}$$

Find $Z[2^n \delta(n-2)]$

Solution : $Z[2^n \delta(n-2)] = Z[\delta(n-2)]_{z \to z/2}$

$$= \left[\frac{1}{z^2}\right]_{z \to z/2} = \frac{1}{\left(\frac{z}{2}\right)^2} = \frac{4}{z^2}$$

Find $Z[3^n \delta(n-1)]$

Solution : $Z[3^n \delta(n-1)] = Z[\delta(n-1)]_{z \to z/3}$

$$= \left[\frac{1}{z}\right]_{z \to z/3} = \frac{1}{\left(\frac{z}{3}\right)} = \frac{3}{z}$$

Find $Z[u(n-1)]$ **[A.U N/D 2018 R-8]**

Solution : $Z[u(n-1)] = \sum_{n=1}^{\infty} 1 . z^{-n}$

$$= \frac{1}{z} + \frac{1}{z^2} + \ldots \infty$$

$$= \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \ldots\right]$$

$$= \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} = \frac{1}{z}\left[\frac{z-1}{z}\right]^{-1}$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right] = \frac{1}{z-1}$$

6. Find $Z\left[\cos\frac{n\pi}{2}\,u(n)\right]$

Solution : $Z\left[\cos\frac{n\pi}{2}\,u(n)\right] = Z\left[\cos\frac{n\pi}{2}\right]$

$$= \sum_{n=0}^{\infty} \cos\frac{n\pi}{2}\, z^{-n}$$

$$= 1 - \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^4 - \dots$$

$$= 1 - \left(\frac{1}{z^2}\right) + \left(\frac{1}{z^2}\right)^2 - \dots$$

$$= \left[1 + \frac{1}{z^2}\right]^{-1}$$

$$= \left[\frac{z^2+1}{z^2}\right]^{-1} = \frac{z^2}{z^2+1}$$

7. Find $Z\left[\left(\frac{1}{2}\right)^n u(n)\right]$

Solution : We know that,

$$Z\,[a^n u(n)] = \frac{z}{z-a}$$

$$Z\left[\left(\frac{1}{2}\right)^n u(n)\right] = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}$$

Initial value theorem and final value theorem.

Initial value theorem

If $Z[f(n)] = F(z)$, then $f(0) = \lim_{z\to\infty} F(z)$

[A.U. M/J 2006, A.U CBT N/D 2010] [A.U N/D 2016 R-8]
[A.U N/D 2019, R-17]

Pf: $Z[f(n)] = \sum_{n=0}^{\infty} f(n)\,z^{-n} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots$

$$\lim_{z\to\infty} Z[f(n)] = \lim_{z\to\infty}\left[f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots\right]$$

$$\lim_{z\to\infty} F(z) = f(0)$$

Final value theorem. **[A.U, End Semester Nov/Dec. 1996]**

If $Z\{f(n)\} = F(z)$, then $\lim_{n\to\infty} [f(n)] = \lim_{z\to 1} (z-1)\,F(z)$

[A.U. N/D 2006] [A.U. M/J 2007] [A.U CBT N/D 2010]
[A.U N/D 2014 R-2008] [A.U N/D 2015 R-13]
[A.U A/M 2017 R-8, N/D 2016 R-8]
[A.U N/D 2019, R-17]

Pf: $Z\{f(n+1)\} = \sum_{n=0}^{\infty} f(n+1)\, z^{-n}$

Put $n+1 = m$

$$Z\{f(m)\} = \sum_{m=1}^{\infty} f(m)\, z^{-m+1}$$

$$= z\,[F(z) - f(0)]$$

$$\therefore zF(z) - zf(0) - F(z) = Z[f(n+1)] - Z[f(n)]$$

$$(z-1)F(z) - zf(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]\,z^{-n}$$

Taking limits as $z \to 1$

$$(z-1)F(z) - f(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]$$

$$= \lim_{n\to\infty} [f(1) - f(0)] + [f(2) - f(1)] + [f(3) - f(2)] + \dots + [f(n+1) - f(n)]$$

$$= \lim_{n \to \infty} [f(1) - f(0)] + [f(2) - f(1)] + [f(3) - f(2)] + \ldots + [f(n+1) - f(n)]$$

$$= \lim_{n \to \infty} [f(n+1) - f(0)]$$

$$= \lim_{n \to \infty} f(n) - f(0) = \lim_{z \to 1} (z-1) F(z) = \lim_{n \to \infty} f(n)$$

$$\lim_{n \to \infty} f(n) = \lim_{z \to 1} [(z-1) F(z)]$$

VIII. Problems based on initial value theorem and final value theorem

(1) If $F(z) = \dfrac{5z}{(z-2)(z-3)}$ find $f(0)$ and $\lim\limits_{t \to \infty} f(t)$

(2) If $U(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, evaluate u_2 and u_3.

[A.U N/D 2018-A R-17][A.U A/M 2019 R-8]

(3) If $U(z) = \dfrac{z^3 + z}{(z-1)^3}$, find the value of u_0, u_1 and u_2.

[A.U N/D 2015-R13]

1. If $F(z) = \dfrac{5z}{(z-2)(z-3)}$ find $f(0)$ and $\lim\limits_{t \to \infty} f(t)$

Solution : *By Initial value theorem*

$$f(0) = \lim_{z \to \infty} F(z)$$

$$= \lim_{z \to \infty} \frac{5z}{(z-2)(z-3)} = \frac{\infty}{\infty}$$

$$= \lim_{z \to \infty} \frac{5}{(z-2)(1) + (z-3)(1)} \text{ by L'Hospital's rule}$$

$$= \lim_{z \to \infty} \frac{5}{2z-5} = 0$$

Final value theorem we get

$$\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$$

$$= \lim_{z \to 1} (z-1) \frac{5z}{(z-2)(z-3)} = \frac{(0)}{(-1)(-2)} = 0$$

2. If $U(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, evaluate u_2 and u_3.

[A.U.T. CH N/D 2011][A.U N/D 2015 R-8][A.U M/J 2016 R-8] [A.U N/D 2018-A, R-17] [A.U A/M 2019 R-8]

Solution : Given : $U(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$

$$= \frac{z^2 \left[2 + \frac{5}{z} + \frac{14}{z^2}\right]}{z^4 \left[1 - \frac{1}{z}\right]^4} = \frac{1}{z^2} \frac{[2 + 5z^{-1} + 14z^{-2}]}{(1 - z^{-1})^4}$$

Initial value theorem :

$$u_0 = \underset{z \to \infty}{Lt}\, U(z) = 0$$

$$u_1 = \underset{z \to \infty}{Lt} \left[z \left(U(z) - u_0\right)\right] = 0$$

$$u_2 = \underset{z \to \infty}{Lt} \left[z^2 \left(U(z) - u_0 - u_1 z^{-1}\right)\right] = 2 - 0 - 0 = 2$$

$$u_3 = \underset{z \to \infty}{Lt}\, z^3 \left(U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}\right)$$

$$= \underset{z \to \infty}{Lt}\, z^3 [U(z) - 0 - 0 - 2z^{-2}]$$

$$= \underset{z \to \infty}{Lt}\, z^3 \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2}\right] = \underset{z \to \infty}{Lt}\, z^3 \left[\frac{13z^3 + 2z^2 + 8z - 2}{z^2 (z-1)^4}\right] = 13$$

3. If $U(z) = \dfrac{z^3 + z}{(z-1)^3}$, find the value of u_0, u_1 and u_2.

[A.U N/D 2015-R13]

Solution : Given : $U(z) = \dfrac{z^3 + z}{(z-1)^3} = \dfrac{z^3 [1 + z^{-2}]}{z^3 \left(1 - \frac{1}{z}\right)^3} = \dfrac{1 + z^{-2}}{\left(1 - \frac{1}{z}\right)^3}$

By Initial value theorem :

$$u_0 = \lim_{z\to\infty} U(z) = \lim_{z\to\infty} \frac{1+z^{-2}}{\left(1-\frac{1}{z}\right)^3} = 1$$

$$u_1 = \lim_{z\to\infty} [z\,U(z) - z\,u_0] = \lim_{z\to\infty}\left[\frac{3z^3-2z^2+z}{(z-1)^3}\right] = 3$$

$$u_2 = \lim_{z\to\infty} [z^2 U(z) - z^2 u_0 - z\,u_1] = \lim_{z\to\infty}\left[\frac{7z^3-8z^2+8z}{(z-1)^3}\right] = 7$$

5.1.8. Differentiation :

1. If $Z\{x(n)\} = X(z)$ **then** $Z[n\,x(n)] = z^{-1}\dfrac{dX}{dz^{-1}}$

Solution : Given $Z\{x(n)\} = X(z)$

i.e., $$X(z) = \sum_{n=0}^{\infty} x(n)\,z^{-n} = \sum_{n=0}^{\infty} x(n)\,(z^{-1})^n$$

differentiate both sides w.r.to z^{-1} we get

$$\frac{dX(z)}{dz^{-1}} = \sum_{n=0}^{\infty} x(n)\,n\,(z^{-1})^{n-1} = \sum_{n=0}^{\infty} n\,x(n)\,(z^{-1})^n (z^{-1})^{-1}$$

$$= z\sum_{n=0}^{\infty} nx(n)\,z^{-n} = z\,Z[nx(n)]$$

$$z^{-1}\frac{dX(z)}{dz^{-1}} = Z[n\,x(n)] \qquad \text{i.e., } Z[nx(n)] = z^{-1}\frac{d}{dz^{-1}}X(z)$$

Similarly, $$Z[n(n-1)\,x(n)] = z^{-2}\frac{d^2}{d(z^{-1})^2}X(z)$$

IX. Problems based on $Z[n\,x(n)] = z^{-1}\dfrac{dX(z)}{dz^{-1}}$ and

$$Z[n(n-1)\,x(n)] = z^{-2}\frac{d^2}{d(z^{-1})^2}X(z)$$

Find the Z-transform of (i) $n\,a^n u(n)$ **; (ii)** $n(n-1)\,a^n u(n)$

Solution : (i) $$Z[n\,a^n u(n)] = z^{-1}\frac{d}{dz^{-1}}\left[\frac{z}{z-a}\right] \quad \text{by definition of } u(n)$$

$$= z^{-1}\frac{d}{dz^{-1}}[1-az^{-1}]^{-1}$$

$$= z^{-1}(-1)(1-az^{-1})^{-2}[-a]$$

$$= az^{-1}(1-az^{-1})^{-2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \qquad \ldots (1)$$

(ii) $$Z[n(n-1)\,a^n u(n)] = z^{-2}\frac{d^2}{d(z^{-1})^2}[1-az^{-1}]^{-1}$$

$$= z^{-2}\frac{d}{d(z^{-1})}\left[\frac{d}{dz^{-1}}[1-az^{-1}]^{-1}\right]$$

$$= z^{-2}\frac{d}{dz^{-1}}\left[\frac{a}{(1-az^{-1})^2}\right]$$

$$= z^{-2}\left[\frac{0-a\,2(1-az^{-1})(-a)}{(1-az^{-1})^4}\right]$$

$$= z^{-2}\left[\frac{2a^2}{(1-az^{-1})^3}\right] = \frac{2a^2z^{-2}}{(1-az^{-1})^3} \qquad \ldots (2)$$

Note : Put $a = 1$, we get

(2) ⇒ $$Z[n(n-1)\,1^n u(n)] = \frac{2z^{-2}}{(1-z^{-1})^3}$$

(1) ⇒ $$Z[n\,u(n)] = \frac{z^{-1}}{(1-z^{-1})^2}$$

5.2 INVERSE Z-TRANSFORM

Def. Inverse Z-transform

If $Z[x(n)] = X(z)$ then $Z^{-1}[X(z)] = [x(n)]$

$Z^{-1}[X(z)]$ can be found out by any one of the following methods

5.2.1 (METHOD I). PARTIAL FRACTIONS METHOD

X. Problems based on Inverse Z-transform

Find the inverse Z-transform of

(1) $\dfrac{10z}{(z-1)(z-2)}$ [A.U N/D 2009]

(2) $\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}$ [A.U N/D 2006, A.U CBT N/D 2011]

(3) $\dfrac{z^2+2z}{z^2+2z+5}$ (4) $\dfrac{z^2}{(z+2)(z^2+4)}$ [A.U N/D 2007, A.U Tvli. M/J 2011]

(5) $\dfrac{z}{z^2+4z+3}$ [A.U A/M 2015 R-2008]

(6) $\dfrac{2z^2-10z+13}{(z-3)^2(z-2)}$, $2 < |z| < 3$ [A.U A/M 2019 R-17]

(7) $\dfrac{z^3}{(z-1)^2(z-2)}$ [A.U. N/D 2005]

(8) $\dfrac{z}{z^2+7z+10}$ [A.U M/J 2007, CBT N/D 2010]

1. Find $Z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$ [A.U N/D 2009]

Solution : Let $X(z) = \dfrac{10z}{(z-1)(z-2)}$

$$\frac{X(z)}{z} = \frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \quad \ldots (1)$$

$$10 = A(z-2) + B(z-1)$$

put $z = 1$ we get

$10 = A(1-2) + 0$

$10 = -A$

$A = -10$

put $z = 2$ we get

$10 = 0 + B$

$B = 10$

$$\therefore (1) \Rightarrow \frac{X(z)}{z} = \frac{-10}{z-1} + \frac{10}{z-2}$$

$$\text{i.e.,} \quad X(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$Z\{x(n)\} = 10\left[\frac{z}{z-2}\right] - 10\left[\frac{z}{z-1}\right]$$

$$x(n) = 10Z^{-1}\left[\frac{z}{z-2}\right] - 10Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= 10(2^n) - 10(1^n) = 10(2^n - 1^n)$$

2. Find $Z^{-1}\left[\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$ **[A.U. N/D 2006] [A.U. CBT N/D 2011]**

Solution : Let $X(z) = \dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}$

$$\frac{X(z)}{z} = \frac{z^2-z+2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2} \quad \ldots (1)$$

$$z^2 - z + 2 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

put $z = -1$, we get

$1+1+2 = 4A$

$4 = 4A$

$A = 1$

put $z = 1$, we get

$1-1+2 = 0+0+2C$

$2 = 2C$

$C = 1$

put $z = 0$, we get

$2 = A - B + C$

$2 = 1 - B + 1$

$2 = 2 - B$

$B = 0$

$$\therefore (1) \Rightarrow \frac{X(z)}{z} = \frac{1}{z+1} + \frac{0}{z-1} + \frac{1}{(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{1}{z+1} + \frac{1}{(z-1)^2}$$

$$X(z) = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

$$Z\{x(n)\} = \frac{z}{z+1} + \frac{z}{(z-1)^2} \qquad [\because X(z) = Z\{x(n)\}]$$

$$x(n) = Z^{-1}\left[\frac{z}{z+1}\right] + Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= Z^{-1}\left[\frac{z}{z-(-1)}\right] + Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= (-1)^n + n$$

3. Find $Z^{-1}\left[\dfrac{z^2+2z}{z^2+2z+5}\right]$

Solution : Let $X(z) = \dfrac{z^2+2z}{z^2+2z+5}$

$$X(z) = \frac{z(z+2)}{z^2+2z+5}$$

$$\frac{X(z)}{z} = \frac{z+2}{z^2+2z+5} \qquad \dots (1)$$

$$z^2+2z+5 = 0$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

$$= -1+2i, -1-2i$$

$$z^2+2z+5 = [z-(-1+2i)]\,[z-(-1-2i)]$$

$$(1) \Rightarrow \frac{X(z)}{z} = \frac{z+2}{z^2+2z+5} = \frac{A}{z-(-1+2i)} + \frac{B}{z-(-1-2i)} \qquad \dots (2)$$

$$z+2 = A\,[z-(-1-2i)] + B\,[z-(-1+2i)]$$

Put $z = -1+2i$, we get

$$-1+2i+2 = A\,[-1+2i+1+2i] + 0$$

$$1+2i = A\,[4i]$$

$$A = \frac{1}{4i}[1+2i] = \frac{-i}{4}[1+2i]$$

$$= \frac{1}{4}[-i+2] = \frac{1}{4}[2-i]$$

Put $z = -1-2i$, we get

$$-1-2i+2 = 0 + B\,[-1-2i+1-2i]$$

$$1-2i = B\,[-4i]$$

$$B = \frac{-1}{4i}[1-2i] = \frac{-1}{4i} + \frac{2}{4} = \frac{i}{4} + \frac{2}{4} = \frac{1}{4}[2+i]$$

$$(2) \Rightarrow \frac{X(z)}{z} = \frac{\frac{1}{4}(2-i)}{z-(-1+2i)} + \frac{\frac{1}{4}(2+i)}{z-(-1-2i)}$$

$$X(z) = \frac{1}{4}(2-i)\frac{z}{z-(-1+2i)} + \frac{1}{4}(2+i)\frac{z}{[z-(-1-2i)]}$$

$$Z\{x(n)\} = \frac{2-i}{4}\,\frac{z}{z-(-1+2i)} + \frac{2+i}{4}\,\frac{z}{z-(-1-2i)}$$

$$x(n) = \frac{2-i}{4}Z^{-1}\left[\frac{z}{z-(-1+2i)}\right] + \frac{2+i}{4}Z^{-1}\left[\frac{z}{z-(-1-2i)}\right]$$

$$= \frac{2-i}{4}(-1+2i)^n + \frac{2+i}{4}(-1-2i)^n$$

$$\boxed{\text{by formula } Z[a^n] = \frac{z}{z-a}}$$

4. Find $Z^{-1}\left[\dfrac{z^2}{(z+2)(z^2+4)}\right]$ **[A.U. N/D 2007] [A.U Tvli M/J 2011]**

Solution : Let $X(z) = \dfrac{z^2}{(z+2)(z^2+4)}$

$$\frac{X(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4} \quad \dots (1)$$

$$z = A(z^2+4) + (Bz+c)(z+2)$$

put $z=-2$ we get	put $z=0$ we get	equating z^2 on both sides
$-2 = A\,8$	$0 = 4A + 2C$	$0 = A + B$
$A = \dfrac{-1}{4}$	$2C = -4A$	$B = -A$
	$C = -2A$	$B = \dfrac{1}{4}$
	$= -2\left(\dfrac{-1}{4}\right)$	
	$C = \dfrac{1}{2}$	

$$\frac{X(z)}{z} = \frac{\frac{-1}{4}}{z+2} + \frac{\frac{1}{4}z + \frac{1}{2}}{z^2+4}$$

$$\frac{X(z)}{z} = \frac{-1}{4}\frac{1}{z-(-2)} + \frac{1}{4}\frac{z}{z^2+4} + \frac{1}{2}\frac{1}{z^2+4}$$

$$Z[x(n)] = X(z) = \frac{-1}{4}\left[\frac{z}{z-(-2)}\right] + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{2}\frac{z}{z^2+4}$$

$$x(n) = \frac{-1}{4}Z^{-1}\left[\frac{z}{z-(-2)}\right] + \frac{1}{4}Z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{4}Z^{-1}\left[\frac{2z}{z^2+4}\right]$$

$$= \frac{-1}{4}(-2)^n + \frac{1}{4}2^n \cos\frac{n\pi}{2} + \frac{1}{4}2^n \sin\frac{n\pi}{2}$$

Formula

$$Z\left[a^n \cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+a^2}$$

$$Z\left[a^n \sin\frac{n\pi}{2}\right] = \frac{az}{z^2+a^2}$$

Find $Z^{-1}\left[\dfrac{z}{z^2+4z+3}\right]$. **[A.U A/M 2015 R-08]**

Solution : Let $X(z) = \dfrac{z}{z^2+4z+3}$

$$\frac{X(z)}{z} = \frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$$

$$\frac{X(z)}{z} = \frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} \quad \dots (1)$$

$$1 = A(z+3) + B(z+1)$$

put $z = -1$, we get	put $z = -3$, we get
$1 = A(-1+3) + 0$	$1 = 0 + B(-3+1)$
$1 = 2A$	$1 = -2B$
$A = \dfrac{1}{2}$	$B = \dfrac{-1}{2}$

$$(1) \Rightarrow \frac{X(z)}{z} = \frac{\frac{1}{2}}{z+1} + \frac{\left(\frac{-1}{2}\right)}{z+3}$$

$$Z[x(n)] = X(z) = \frac{1}{2}\frac{z}{z+1} - \frac{1}{2}\frac{z}{z+3}$$

$$x(n) = \frac{1}{2}Z^{-1}\left[\frac{z}{z+1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z+3}\right]$$

$$= \frac{1}{2}(-1)^n - \frac{1}{2}(-3)^n$$

6. Find $Z^{-1}\left[\dfrac{2z^2 - 10z + 13}{(z-3)^2(z-2)}\right]$ when $2 < |z| < 3$. **[A.U A/M 2019 R-17]**

Solution : Let $X(z) = \dfrac{2z^2 - 10z + 13}{(z-3)^2(z-2)}$

$$Z[x(n)] = X(z) = \frac{2z^2 - 10z + 13}{(z-3)^2(z-2)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2} \quad \dots (1)$$

$$2z^2 - 10z + 13 = A(z-3)^2 + B(z-2)(z-3) + C(z-2)$$

put $z = 2$, we get	put $z = 3$, we get	put $z = 0$, we get
$8 - 20 + 13 = A$	$18 - 30 + 13 = C$	$13 = 9A + 6B - 2C$
$A = 1$	$1 = C$	$13 = 9 + 6B - 2$
	$C = 1$	$13 = 7 + 6B$
		$6B = 6$; $B = 1$

$$\therefore (1) \Rightarrow X(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{z}\frac{1}{1-\frac{2}{z}} - \frac{1}{3\left[1-\frac{z}{3}\right]} + \frac{1}{9}\frac{1}{\left(1-\frac{z}{3}\right)^2}$$

$$= \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1} - \frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} + \frac{1}{9}\left[1-\frac{z}{3}\right]^{-2}$$

> Given : $2 < |z| < 3$
>
> (i) $2 < |z|$ (ii) $|z| < 3$
>
> $\dfrac{2}{|z|} < 1$ $\left|\dfrac{z}{3}\right| < 1$
>
> $\left|\dfrac{2}{z}\right| < 1$

$$Z[x(n)] = X(z) = \frac{1}{z}\left[1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots\right] - \frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{9} + \dots\right] + \frac{1}{9}\left[1 + \frac{2z}{3} + \frac{3z^2}{9} + \dots\right]$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty}(n+1)\left(\frac{1}{3}\right)^{n+2} z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}} - (n+1)\cdot\frac{1}{3^{n+2}}\right) z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty}\frac{1}{3^{n+2}}(3 - n - 1) z^n$$

$$Z[f(n)] = \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty}(2-n)\left(\frac{1}{3}\right)^{n+2} z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty}(2-n)3^{-n-2} z^n$$

Change the second term in terms of negative powers.

Put $-n = m$ $\quad \therefore m \le 0$ as $n \ge 0$

$$\therefore \sum_{n=0}^{\infty}(2-n)3^{-n-2}z^n = \sum_{m\le 0}(2+m)3^{m-2}z^{-m}$$

This can be written as $\sum\limits_{n\le 0}(2+n)3^{n-2}z^{-n}$ [Replace m by n]

$$\therefore Z[f(n)] = \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n\le 0}(2+n)3^{n-2}z^{-n}$$

$$\therefore f(n) = 2^{n-1} \text{ if } n \ge 1$$

and $\quad f(n) = -(2+n)3^{n-2}$ if $n \le 0$

7. **Find** $Z^{-1}\left[\dfrac{z^3}{(z-1)^2(z-2)}\right]$ **using partial fraction.**

[A.U. N/D 2005]

Solution : Let $X(z) = \dfrac{z^3}{(z-1)^2(z-2)}$

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$z^2 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

put $z = 1$, we get

$$1 = 0 + B(-1)$$

$$\boxed{B = -1}$$

put $z = 2$, we get

$$4 = 0 + 0 + C$$

$$\boxed{C = 4}$$

put $z = 0$, we get

$$0 = A(-1)(-2) + B(-2) + C(-1)^2$$

$$0 = 2A - 2B + C$$

$$0 = 2A + 2 + 4$$

$$= A + 1 + 2$$

$$\boxed{A = -3}$$

$$\therefore \frac{X(z)}{z} = \frac{-3}{z-1} - \frac{1}{(z-1)^2} + \frac{4}{z-2}$$

$$\therefore X(z) = \frac{-3z}{z-1} - \frac{z}{(z-1)^2} + 4\frac{z}{z-2}$$

$$Z\{x(n)\} = X(z) = -3\frac{z}{z-1} - \frac{z}{(z-1)^2} + 4\frac{z}{z-2}$$

$$x(n) = -3Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -3(1)^n - n + 4(2)^n$$

Evaluate $Z^{-1}\left(\dfrac{z}{z^2+7z+10}\right)$.

[A.U. M/J 2007, CBT N/D 2010]

Solution :

$$Z^{-1}\left[\frac{z}{z^2+7z+10}\right] = Z^{-1}\left[\frac{z}{(z+5)(z+2)}\right]$$

Let $X(z) = \dfrac{z}{(z+5)(z+2)}$

$$\frac{X(z)}{z} = \frac{1}{(z+2)(z+5)} = \frac{A}{z+2} + \frac{B}{z+5}$$

$$1 = A(z+5) + B(z+2)$$

put $z = -2$, we get

$$1 = 3A$$

$$A = \frac{1}{3}$$

put $z = -5$, we get

$$1 = -3B$$

$$B = \frac{-1}{3}$$

$$\therefore \frac{X(z)}{z} = \frac{1}{3}\frac{1}{z+2} - \frac{1}{3}\frac{1}{z+5}$$

$$X(z) = \frac{1}{3}\frac{z}{z+2} - \frac{1}{3}\frac{z}{z+5}$$

$$Z\{x(n)\} = \frac{1}{3}\frac{z}{z+2} - \frac{1}{3}\frac{z}{z+5}$$

$$x(n) = \frac{1}{3}Z^{-1}\left[\frac{z}{z+2}\right] - \frac{1}{3}Z^{-1}\left[\frac{z}{z+5}\right]$$

$$\therefore Z^{-1}\left[\frac{z}{z^2+7z+10}\right] = \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

5.2.2 (Method : II). Inverse of Z-transform by Inverse integral method. (Cauchy's residue theorem)

From the relation between the Z-transform and Fourier transform of a sequence we get

$$x(n) = \frac{1}{2\pi i} \int_C X(z)\, z^{n-1}\, dz$$

By Cauchy's residue theorem

$$\int_C X(z)\, z^{n-1}\, dz = 2\pi i \text{ [sum of the residues of } X(z)\, z^{n-1} \text{ at the isolated singularities]}$$

i.e., $x(n)$ = Sum of the residues of $X(z)\, z^{n-1}$ at the isolated singularities.

Note : Take the contour C such that all the poles of the function $X(z)\, z^{n-1}$ lie within the contour.

Find the inverse Z-transform of

(1) $\dfrac{10z}{(z-1)(z-2)}$ (2) $\dfrac{z^2}{(z-a)(z-b)}$ (3) $\dfrac{z(z^2-z+2)}{(z-1)(z+1)^2}$

(4) $\dfrac{z}{z^2+2z+2}$ (5) $\dfrac{z(z+1)}{(z-1)^3}$ (6) $\dfrac{3z^2+z}{z^3-3z^2+4}$

(7) $\dfrac{z^2}{(z+2)(z^2+4)}$ [A.U N/D 2015 R-13] (8) $\dfrac{9z^3}{(3z-1)^2(z-2)}$ [A.U A/M 2015 R-08]

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1. Find $Z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$ **[A.U N/D 2009, CBT A/M 2011]**

Solution : Let $X(z) = \dfrac{10z}{(z-1)(z-2)}$

$$X(z)\, z^{n-1} = \frac{10z}{(z-1)(z-2)} z^{n-1} = \frac{10z^n}{(z-1)(z-2)}$$

$z = 1$ is a simple pole and $z = 2$ is a simple pole

Let us consider a contour in $|z| > 2$

$$\underset{z=1}{\text{Res}}\, X(z)\, z^{n-1} = \underset{z \to 1}{Lt}\, (z-1) \frac{10z^n}{(z-1)(z-2)}$$

$$= \underset{z \to 1}{Lt} \frac{10z^n}{z-2} = \frac{10(1)^n}{1-2} = -10$$

$$\underset{z=2}{\text{Res}}\, X(z)\, z^{n-1} = \underset{z \to 2}{Lt}\, (z-2) \frac{10z^n}{(z-1)(z-2)}$$

$$= \underset{z \to 2}{Lt} \frac{10z^n}{z-1} = \frac{10(2^n)}{2-1} = (10)(2^n)$$

$\therefore x(n)$ = Sum of the residues

$$= (10)(2^n) - 10 = 10(2^n - 1)$$

Find $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$.

Solution : Let $X(z) = \dfrac{z^2}{(z-a)(z-b)}$

$$X(z)\, z^{n-1} = \frac{z^2}{(z-a)(z-b)} z^{n-1} = \frac{z^{n+1}}{(z-a)(z-b)}$$

$z = a$ is a simple pole and $z = b$ is a simple pole.

Let us consider the contour C, sufficiently large.

$$\underset{z=a}{\text{Res}}\, X(z)\, z^{n-1} = \underset{z \to a}{Lt}\, (z-a) \frac{z^{n+1}}{(z-a)(z-b)}$$

$$= \underset{z \to a}{Lt} \frac{z^{n+1}}{z-b} = \frac{a^{n+1}}{a-b}$$

$$\operatorname*{Res}_{z=b} X(z)\, z^{n-1} = \operatorname*{Lt}_{z \to b} (z-b) \frac{z^{n+1}}{(z-a)(z-b)}$$

$$= \operatorname*{Lt}_{z \to b} \frac{z^{n+1}}{z-a} = \frac{b^{n+1}}{b-a}$$

$$\therefore x(n) = \text{Sum of the residues}$$

$$= \frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}$$

$$= \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right]$$

3. **Find** $Z^{-1}\left[\dfrac{z(z^2 - z + 2)}{(z-1)(z+1)^2}\right]$

Solution : $X(z) = \dfrac{z(z^2 - z + 2)}{(z-1)(z+1)^2}$

$$X(z)\, z^{n-1} = \frac{z(z^2 - z + 2)}{(z-1)(z+1)^2} z^{n-1}$$

$$= \frac{z^n (z^2 - z + 2)}{(z-1)(z+1)^2}$$

$z = 1$ is a simple pole

$z = -1$ is a pole of order 2

Let us consider the contour in $|z| > 1$

$$\operatorname*{Res}_{z=1} X(z)\, z^{n-1} = \operatorname*{Lt}_{z \to 1} (z-1) \frac{z^n (z^2 - z + 2)}{(z-1)(z+1)^2}$$

$$= \operatorname*{Lt}_{z \to 1} \frac{z^n (z^2 - z + 2)}{(z+1)^2}$$

$$= \frac{1^n (1 - 1 + 2)}{(1+1)^2} = \frac{2}{2^2} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=-1} X(z)\, z^{n-1} = \operatorname*{Lt}_{z \to -1} \frac{d}{dz}\left[(z+1)^2 \frac{z^n (z^2 - z + 2)}{(z-1)(z+1)^2}\right]$$

$$= \operatorname*{Lt}_{z \to -1} \frac{d}{dz}\left[\frac{z^n (z^2 - z + 2)}{z-1}\right]$$

$$= \operatorname*{Lt}_{z \to -1} \frac{(z-1)\left[z^n (2z-1) + (z^2 - z + 2)\, n z^{n-1}\right] - \left[z^n (z^2 - z + 2)\right][1]}{(z-1)^2}$$

$$= \frac{(-2)\left[(-1)^n (-3) + (1+1+2)\, n\, (-1)^{n-1}\right] - (-1)^n (1+1+2)}{(-1-1)^2}$$

$$= (-2) \frac{\left[-3(-1)^n + 4n(-1)^{n-1} + 2(-1)^n\right]}{4}$$

$$= \frac{-1}{2}\left[-3(-1)^n - 4n(-1)^n + 2(-1)^n\right]$$

$$= \frac{-(-1)^n}{2}\left[-3 - 4n + 2\right]$$

$$= -\frac{(-1)^n}{2}(-1 - 4n) = (4n+1)\frac{(-1)^n}{2}$$

$$x(n) = \text{Sum of the residues}$$

$$= \frac{1}{2} + \frac{1}{2}(-1)^n (4n+1)$$

4. **Find** $Z^{-1}\left[\dfrac{z}{z^2+2z+2}\right]$.

[A.U. N/D 2013]

[A.U N/D 2019, R-17]

Solution : Let $\quad X(z) = \dfrac{z}{z^2+2z+2}$

$$X(z)\,z^{n-1} = \frac{z}{z^2+2z+2}\,z^{n-1}$$

$$= \frac{z^n}{z^2+2z+2} \quad \ldots (1)$$

To get singularities put Dr $= 0$

$$z^2+2z+2 = 0$$

$$z = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$(1) \Rightarrow X(z)\,z^{n-1} = \frac{z^n}{[z-(-1+i)]\,[z-(-1-i)]}$$

$z = -1+i$ is a simple pole

$z = -1-i$ is a simple pole

$$\therefore \operatorname*{Res}_{z=-1+i} X(z)\,z^{n-1} = \operatorname*{Lt}_{z\to -1+i} [z-(-1+i)]\,\frac{z^n}{[z-(-1+i)][z-(-1-i)]}$$

$$= \operatorname*{Lt}_{z\to -1+i} \frac{z^n}{z-(-1-i)}$$

$$= \frac{(-1+i)^n}{-1+i+1+i}$$

$$= \frac{(-1+i)^n}{2i}$$

$$\operatorname*{Res}_{z=-1-i} X(z)\,z^{n-1} = \operatorname*{Lt}_{z\to -1-i} [z-(-1-i)]\,\frac{z^n}{[z-(-1+i)]\,[z-(-1-i)]}$$

$$= \operatorname*{Lt}_{z\to -1-i} \frac{z^n}{z-(-1+i)}$$

$$= \frac{(-1-i)^n}{-1-i+1-i}$$

$$= \frac{(-1-i)^n}{-2i}$$

$$= -\frac{1}{2i}(-1-i)^n$$

$$x(n) = \text{Sum of the residues}$$

$$= \frac{1}{2i}(-1+i)^n - \frac{1}{2i}(-1-i)^n$$

$$= \frac{1}{2i}[(-1+i)^n - (-1-i)^n]$$

$$= \frac{(\sqrt{2})^n}{2i}\left\{\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]^n - \left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right]^n\right\}$$

$$= \frac{(\sqrt{2})^n}{2i}\left[\cos\frac{n\,3\pi}{4} + i\sin\frac{n\,3\pi}{4} - \cos\frac{n\,3\pi}{4} + i\sin\frac{n\,3\pi}{4}\right]$$

$$= \frac{(\sqrt{2})^n}{2i}\,2i\sin\frac{3n\pi}{4}$$

$$= (\sqrt{2})^n \sin\frac{3n\pi}{4}, \quad n = 0, 1, 2, \ldots$$

5. **Find** $Z^{-1}\left[\dfrac{z(z+1)}{(z-1)^3}\right]$

[A.U. M/J 2007] [A.U Trichy N/D 2009]

[A.U N/D 2010]

Solution : Let $\quad X(z) = \dfrac{z(z+1)}{(z-1)^3}$

$$X(z)\, z^{n-1} = \frac{z(z+1)}{(z-1)^3}\, z^{n-1}$$

$$= \frac{z^n (z+1)}{(z-1)^3}$$

$z = 1$ is a pole of order 3.

$$\underset{z=1}{Res}\, X(z)\, z^{n-1} = \underset{z \to 1}{Lt}\ \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-1)^3 \frac{z^n (z+1)}{(z-1)^3}\right]$$

$$= \underset{z \to 1}{Lt}\ \frac{1}{2}\frac{d^2}{dz^2}[z^n (z+1)]$$

$$= \frac{1}{2}\underset{z \to 1}{Lt}\ \frac{d}{dz}[z^n (1) + (z+1)\, n z^{n-1}]$$

$$= \frac{1}{2}\underset{z \to 1}{Lt}\ \frac{d}{dz}[z^n + n(z+1) z^{n-1}]$$

$$= \frac{1}{2}\underset{z \to 1}{Lt}\ [n z^{n-1} + n(z+1)(n-1) z^{n-2} + n z^{n-1}(1)]$$

$$= \frac{1}{2}[n(1)^{n-1} + n\,2\,(n-1)(1)^{n-2} + n(1)^{n-1}]$$

$$= \frac{1}{2}[2n(1)^{n-1} + 2n(n-1)(1)^{n-2}]$$

$$= \frac{1}{2}[2n + 2n(n-1)]$$

$$= \frac{1}{2}[2n + 2n^2 - 2n]$$

$$= \frac{1}{2}(2n^2)$$

$$= n^2$$

$\therefore x(n)$ = Sum of the residues

$$= n^2$$

6. **Find** $Z^{-1}\left[\dfrac{3z^2 + z}{z^3 - 3z^2 + 4}\right]$ **[A.U. CBT A/M 2011]**

Solution : Let $X(z) = \dfrac{3z^2 + z}{z^3 - 3z^2 + 4}$

$$X(z)\, z^{n-1} = \frac{z(3z+1)}{z^3 - 3z^2 + 4}\, z^{n-1}$$

$$= \frac{z^n (3z+1)}{(z+1)(z-2)^2}$$

$\therefore z = -1$ is a simple pole

$z = 2$ is a pole of order 2.

Let us consider the contour C in $|z| > 2$

$$\underset{z=-1}{Res}\, X(z)\, z^{n-1} = \underset{z \to -1}{Lt}(z+1)\frac{z^n (3z+1)}{(z+1)(z-2)^2}$$

$$= \underset{z \to -1}{Lt}\ \frac{z^n (3z+1)}{(z-2)^2} = \frac{(-1)^n (-3+1)}{(-1-2)^2}$$

$$= \frac{(-1)^n (-2)}{(-3)^2} = -\frac{2}{9}(-1)^n$$

$$\underset{z=2}{Res}\ X(z)\, z^{n-1} = \underset{z \to 2}{Lt}\ \frac{d}{dz}(z-2)^2 \frac{z^n (3z+1)}{(z+1)(z-2)^2}$$

$$= \underset{z \to 2}{Lt}\ \frac{d}{dz}\frac{z^n (3z+1)}{z+1}$$

$$= \underset{z \to 2}{Lt}\ \frac{(z+1)[z^n (3) + (3z+1)\, n z^{n-1}] - z^n (3z+1)(1)}{(z+1)^2}$$

$$= \underset{z \to 2}{Lt}\ \frac{(z+1)[3z^n + n3z^n + nz^{n-1}] - z^n (3z+1)}{(z+1)^2}$$

$$= \frac{3[3(2^n) + 3n\, 2^n + n\, 2^{n-1}] - 2^n (7)}{9}$$

$$= \frac{9(2^n) + 9n\,2^n + 3n\,(2^{n-1}) - 7(2^n)}{9}$$

$$= \frac{2^{n-1}[18 + 18n + 3n - 14]}{9}$$

$$= \frac{2^{n-1}[18 + 21n - 14]}{9} = \frac{2^{n-1}}{9}[21\,n + 4]$$

$x(n)$ = Sum of the residues

$$= \frac{-2}{9}(-1)^n + \frac{2^{n-1}}{9}[21\,n + 4]$$

7. **Find** $Z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$ **[A.U N/D 2015 R-13]**

Solution :

Let $U(z) = \dfrac{z^2}{(z+2)(z^2+4)}$

$$U(z)z^{n-1} = \frac{z^2}{(z+2)(z+2i)(z-2i)}z^{n-1}$$

$$\underset{z=-2}{Res}\ U(z)z^{n-1} = \lim_{z\to -2}(z+2)\frac{z^2}{(z+2)(z^2+4)}z^{n-1}$$

$$= \lim_{z\to -2}\frac{z^2}{z^2+4}z^{n-1}$$

$$= \frac{(-2)^2}{4+4}(-2)^{n-1} = \frac{1}{2}(-2)^{n-1}$$

$$\underset{z=-2i}{Res}\ U(z)z^{n-1} = \lim_{z\to -2i}(z+2i)\frac{z^2}{(z+2)(z+2i)(z-2i)}z^{n-1}$$

$$= \lim_{z\to -2i}\frac{z^2}{(z+2)(z-2i)}z^{n-1}$$

$$= \frac{(-2i)^2}{(-2i+2)(-4i)}(-2i)^{n-1}$$

$$= \frac{-4}{2(1-i)(-4i)}(-2i)^{n-1}$$

$$= \frac{1}{2i(1-i)}(-2i)^{n-1}$$

$$\underset{z=2i}{Res}\ U(z)z^{n-1} = \lim_{z\to 2i}(z-2i)\frac{z^2}{(z+2)(z+2i)(z-2i)}z^{n-1}$$

$$= \lim_{z\to 2i}\frac{z^2}{(z+2)(z+2i)}z^{n-1}$$

$$= \frac{(2i)^2}{(2i+2)(4i)}(2i)^{n-1}$$

$$= \frac{-4}{2(1+i)(4i)}(2i)^{n-1}$$

$$= \frac{-1}{2i(1+i)}(2i)^{n-1}$$

$\therefore x(n)$ = sum of the residues

$$= \frac{1}{2}(-2)^{n-1} + \frac{1}{2i(1-i)}(-2i)^{n-1} + \frac{(-1)}{2i(1+i)}(2i)^{n-1}$$

$$u_n = -\frac{1}{4}(-2)^n + \frac{(-2)^n i^n}{4(1-i)} + \frac{2^n i^n}{4(1+i)}$$

8. **Find** $Z^{-1}\left[\frac{9z^3}{(3z-1)^2(z-2)}\right]$ **[A.U A/M 2015 R-2008]**

[A.U N/D 2016 R-2013]

Solution : Let $X(z) = \dfrac{9z^3}{(3z-1)^2(z-2)}$

$$= \frac{9z^3}{9\left[z - \frac{1}{3}\right]^2 [z-2]} = \frac{z^3}{\left(z - \frac{1}{3}\right)^2 (z-2)}$$

$$X(z)\, z^{n-1} = \frac{z^3 z^{n-1}}{\left(z - \frac{1}{3}\right)^2 (z-2)} = \frac{z^{n+2}}{\left(z - \frac{1}{3}\right)^2 (z-2)}$$

$z = \frac{1}{3}$ is a pole of order 2

$z = 2$ is a simple pole

Let us consider the contour C in $|z| > 2$

$$\therefore \operatorname*{Res}_{z=2} X(z)\, z^{n-1} = \operatorname*{Lt}_{z \to 2} (z-2) \frac{z^{n+2}}{\left(z - \frac{1}{3}\right)^2 (z-2)} = \operatorname*{Lt}_{z \to 2} \frac{z^{n+2}}{\left(z - \frac{1}{3}\right)^2}$$

$$= \frac{(2)^{n+2}}{\left(2 - \frac{1}{3}\right)^2} = \frac{2^{n+2}}{\left(\frac{5}{3}\right)^2} = \frac{9}{25} 2^{n+2}$$

$$\operatorname*{Res}_{z=\frac{1}{3}} X(z)\, z^{n-1} = \operatorname*{Lt}_{z \to \frac{1}{3}} \frac{d}{dz} \left[\left(z - \frac{1}{3}\right)^2 \frac{z^{n+2}}{\left(z - \frac{1}{3}\right)^2 (z-2)} \right]$$

$$= \operatorname*{Lt}_{z \to \frac{1}{3}} \frac{d}{dz} \left[\frac{z^{n+2}}{z-2} \right]$$

$$= \operatorname*{Lt}_{z \to \frac{1}{3}} \frac{(z-2)(n+2) z^{n+1} - z^{n+2}(1)}{(z-2)^2}$$

$$= \frac{\left(\frac{1}{3} - 2\right)(n+2)\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{3}\right)^{n+2}}{\left(\frac{1}{3} - 2\right)^2}$$

$$= \frac{\left(\frac{-5}{3}\right)(n+2)\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{3}\right)^{n+2}}{\left(\frac{-5}{3}\right)^2}$$

$$= \frac{(-5)(n+2)\left(\frac{1}{3}\right)^{n+2} - \left(\frac{1}{3}\right)^{n+2}}{\left(\frac{25}{9}\right)}$$

$$= \frac{9}{25}\left(\frac{1}{3}\right)^{n+2} [-5(n+2) - 1]$$

$$= \frac{9}{25}\left(\frac{1}{3}\right)^{n+2} [-5n - 10 - 1]$$

$$= \frac{9}{25}\left(\frac{1}{3}\right)^{n+2} [-5n - 11]$$

$$= \frac{-1}{25}\left(\frac{1}{3}\right)^{n} [5n + 11]$$

$\therefore x(n)$ = Sum of the residues

$$= \frac{9}{25} 2^{n+2} - \frac{1}{25}\left(\frac{1}{3}\right)^{n} (5n + 11)$$

5.3 CONVOLUTION THEOREM

The convolution theorem plays an important role in the solution of difference equations and in probability problems involving sums of two independent random variables.

Definition : Convolution of sequences :

1. The convolution of two sequences $\{x(n)\}$ and $\{y(n)\}$ is defined as

(i) $\{x(n) * y(n)\} = \sum_{K=-\infty}^{\infty} f(K)\, g(n-K)$ if the sequences are non-causal

and

(ii) $\{x(n) * y(n)\} = \sum_{K=0}^{n} f(K)\, g(n-K)$ if the sequences are causal.

2. The convolution of two functions $f(t)$ and $g(t)$ is defined as

$f(t) * g(t) = \sum_{K=0}^{n} f(KT)\, g[(n-K)T]$, where T is the sampling period.

State and prove convolution theorem on Z-transform.

[A.U. April 2000, Nov/Dec. 1996, March 1996, April/May 1999]

Statement : **[A.U. CBT Dec. 2008] [A.U M/J 2014]**

If $Z^{-1}[X(z)] = x_n$ and $Z^{-1}[Y(z)] = y_n$, then

$Z^{-1}[X(z)\,Y(z)] = \sum_{m=0}^{n} x_m y_{n-m} = x_n * y_n$, where $*$ denotes the convolution operation. **[A.U N/D 2016 R-13]**

Proof :

We know that,

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} \text{ and } Y(z) = \sum_{n=0}^{\infty} y_n z^{-n}$$

$$X(z)\,Y(z) = \left(\sum_{n=0}^{\infty} x_n z^{-n}\right)\left(\sum_{n=0}^{\infty} y_n z^{-n}\right)$$

$$= \left(x_0 + x_1 z^{-1} + x_2 z^{-2} + \dots + x_n z^{-n} + \dots\right)\left(y_0 + y_1 z^{-1} + y_2 z^{-2} + \dots + y_n z^{-n} + \dots\right)$$

$$= \sum_{n=0}^{\infty} [x_0 y_n + x_1 y_{n-1} + x_2 y_{n-2} + \dots + x_n y_0]\, z^{-n}$$

$$= Z\left[x_0 y_n + x_1 y_{n-1} + x_2 y_{n-2} + \dots + x_n y_0\right]$$

$$\Rightarrow Z^{-1}\left[X(z)\,Y(z)\right] = x_0 y_n + x_1 y_{n-1} + \dots + x_n y_0$$

$$= \sum_{m=0}^{n} x_m\, y_{n-m}$$

$$= x_n * y_n \text{ [by convolution definition]}$$

Note : $Z[x_n * y_n] = X(z)\,Y(z)$

$= Z^{-1}(x_n)\, Z^{-1}(y_n)$

Aliter : Proof :

(i) $$Z\{x(n) * y(n)\} = Z\left[\sum_{K=-\infty}^{\infty} x(K)\, y(n-K)\right]$$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{K=-\infty}^{\infty} x(K)\, y(n-K)\right] z^{-n}$$

$$= \sum_{K=-\infty}^{\infty} x(K) \sum_{n=-\infty}^{\infty} y(n-K)\, z^{-n}$$

by changing the order of summation,

$$= \sum_{K=-\infty}^{\infty} x(K)\left[\sum_{m=-\infty}^{\infty} y(m)\, z^{-(m+K)}\right] \text{ by putting } n - K = m$$

$$= \sum_{K=-\infty}^{\infty} x(K)\, z^{-K}\left[\sum_{m=-\infty}^{\infty} y(m)\, z^{-m}\right]$$

$$= \sum_{K=-\infty}^{\infty} x(K)\, z^{-K} \sum_{m=-\infty}^{\infty} y(m)\, z^{-m}$$

$$= X(z)\,.\,Y(z)$$

XI. Z-transform of $f(n) * g(n)$ type.

1. **Find the Z-transform of the convolution of $x(n) = u(n)$ and $y(n) = a^n u(n)$**

Solution :

$$X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1} \text{ if } |z| > 1 \quad [\because \text{ by definition of } u(n)]$$

$$Y(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{z}{z-a} \text{ if } |z| > |a| \quad [\because \text{ by definition of } u(n)]$$

The Z-transform of the convolution of $x(n)$ and $y(n)$ is

$$F(z) = \frac{z}{z-1} \cdot \frac{z}{z-a} = \frac{z^2}{(z-1)(z-a)}$$

2. **Find the Z-transform of $f(n) * g(n)$, where $f(n) = u(n)$ and $g(n) = \delta(n) + \left(\frac{1}{2}\right)^n u(n)$.**

Solution : By convolution theorem,

$$Z\{f(n) * g(n)\} = F(z)\, G(z)$$

$$F(z) = Z\{u(n)\} = \frac{z}{z-1}$$

$$G(z) = Z\{\delta(n)\} + Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\}$$

$$= 1 + \frac{z}{z-\frac{1}{2}} \text{ by definition of } u(n) \text{ and } \delta(n)$$

$$= 1 + \frac{2z}{2z-1} = \frac{4z-1}{2z-1}$$

$$Z\{f(n) * g(n)\} = \left(\frac{z}{z-1}\right)\left(\frac{4z-1}{2z-1}\right)$$

Find the Z-transform of $f(n) * g(n)$ where $f(n) = \left(\frac{1}{2}\right)^n$ and $g(n) = \cos n\pi$

Solution : By convolution theorem,

$$Z\{f(n) * g(n)\} = F(z)\, G(z)$$

$$F(z) = Z\left\{\left(\frac{1}{2}\right)^n\right\} = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}$$

$$G(z) = Z\{\cos n\pi\} = Z\{(-1)^n\} = \frac{z}{z+1}$$

$$\therefore Z\{f(n) * g(n)\} = F(z)\, G(z)$$

$$= \left[\frac{2z}{2z-1}\right]\left[\frac{z}{z+1}\right]$$

$$= \frac{2z^2}{2z^2+z-1}$$

XII. Use convolution theorem to find the inverse Z-transform of

(a) $\dfrac{z^2}{(z-a)(z-b)}$ [A.U. April 2001, M/J 2014 R-8, M/J 2013 R-8, A/M 2015 R-8, N/D 2015 R-8]

(b) $\dfrac{8z^2}{(2z-1)(4z-1)} = \dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

[A.U A/M 2015 R-13, N/D 2007, N/D 2014 R-8, M/J 2012 R-8]

(c) $\dfrac{8z^2}{(2z-1)(4z+1)} = \dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$ [A.U N/D 2014 R-8] [A.U N/D 2019, R-17] [A.U A/M 2017 R-13] [A.U N/D 2018 R-13]

(d) $\dfrac{z^2}{(z-1)(z-3)}$ [A.U M/J & N/D 2006, N/D 2013 R-8, A/M 2011 R-8] [A.U N/D 2018-A R-17]

1(e) $\dfrac{z^2}{(z-3)(z-4)}$ [A.U N/D 2015 R-13, N/D 2009 R-8]

2. $\dfrac{z^3}{(z-2)^2(z-3)}$ [A.U. April 2000]

3. (a) $\dfrac{z^2}{(z-a)^2}$ 3. (b) $\dfrac{z^2}{(z+a)^2}$ [A.U N/D 2007, N/D 2012]

[A.U A/M 2017 R-8]

1(a) Find $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$

[A.U. April 2001][A.U. CBT Dec. 08]
[A.U CBT A/M 2011][A.U.T CH N/D 2011][A.U A/M 2015 R-08]
[A.U N/D 2015 R-13] [A.U N/D 2016 R-13]
[A.U N/D 2018 R-17][A.U A/M 2019 R-8]

Solution : $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\dfrac{z}{z-a}\cdot\dfrac{z}{z-b}\right]$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n = \sum_{m=0}^{n} a^m b^{n-m}$$

$$= \sum_{m=0}^{n} a^m b^n b^{-m}$$

$$= b^n \sum_{m=0}^{n} a^m \frac{1}{b^m} = b^n \sum_{m=0}^{n}\left(\frac{a}{b}\right)^m$$

$$= b^n\left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \dots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n\left[\frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}}\right] = b^n\left[\frac{1-\frac{a^{n+1}}{b^{n+1}}}{1-\frac{a}{b}}\right] \text{ being a G.P}$$

$$= \frac{b^n\left[\frac{b^{n+1}-a^{n+1}}{b^{n+1}}\right]}{\left[\frac{b-a}{b}\right]}$$

$$= \frac{b^{n+1}-a^{n+1}}{b-a}$$

> **Formula :**
> $a + ar + ar^2 + \dots + ar^n$
> $= \dfrac{a[1-r^{n+1}]}{1-r}$, if $r < 1$
> Here, $a = 1$

1(b) Find $Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z-1)}\right]$ **(or)** $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]$

[A.U A/M 2015 R-13, N/D 2007, N/D 2014 R-8, M/J 2012 R-8]

Solution :

> We first prove that
> $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right] = \dfrac{b^{n+1}-a^{n+1}}{b-a}$
> See Example 1(a)

Here, $a = \frac{1}{2}$, $b = \frac{1}{4}$

$$\therefore Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = \frac{\left(\frac{1}{4}\right)^{n+1} - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{4}-\frac{1}{2}}$$

$$= \frac{\left(\frac{1}{4}\right)^{n+1} - \left(\frac{1}{2}\right)^{n+1}}{(-1/4)}$$

$$= -\left(\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^{n-1}$$

1(c) Find $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right]$ **[A.U M/J 2016 R-08] [A.U A/M 2017 R-13]**

Solution : We first prove that

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1}-a^{n+1}}{b-a}$$

See Example 1(a)

Here, $a = \frac{1}{2},\ b = -\frac{1}{4}$

$$\therefore\ Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left[z-\left(\frac{-1}{4}\right)\right]}\right] = \frac{\left(\frac{-1}{4}\right)^{n+1}-\left(\frac{1}{2}\right)^{n+1}}{-\frac{1}{4}-\frac{1}{2}}$$

$$= \frac{\left(\frac{-1}{4}\right)^{n+1}-\left(\frac{1}{2}\right)^{n+1}}{(-3/4)}$$

$$= -\frac{4}{3}\left[\left(\frac{-1}{4}\right)^{n+1}-\left(\frac{1}{2}\right)^{n+1}\right]$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^{n}+\frac{1}{3}\left(\frac{-1}{4}\right)^{n}$$

1(d) Using convolution theorem evaluate inverse Z-transform of $\left[\dfrac{z^2}{(z-1)(z-3)}\right]$ **[A.U N/D 2018-A, R-17] [A.U. M/J 2006] [A.U CBT N/D 2010] [A.U.T CBT N/D 2011] [A.U N/D 2013]**

Solution : We first prove that

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1}-a^{n+1}}{b-a}$$

See Example 1(a)

Here, $a = 1,\ b = 3$

$$\therefore\ Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{3^{n+1}-1^{n+1}}{3-1} = \frac{3^{n+1}-1}{2}$$

(e) Find $Z^{-1}\left[\dfrac{z^2}{(z-3)(z-4)}\right]$ **[A.U N/D 2015 R-13, N/D 2009 R-8]**

Solution : We first prove that

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1}-a^{n+1}}{b-a}$$

See Example 1(a)

Here, $a = 3,\ b = 4$

$$\therefore\ Z^{-1}\left[\frac{z^2}{(z-3)(z-4)}\right] = \frac{4^{n+1}-3^{n+1}}{4-3}$$

$$= 4^{n+1}-3^{n+1}$$

Find $Z^{-1}\left[\dfrac{z^3}{(z-2)^2(z-3)}\right]$ **[A.U. April 2000]**

Solution : $Z^{-1}\left[\dfrac{z^3}{(z-2)^2(z-3)}\right] = Z^{-1}\left[\dfrac{z^2}{(z-2)^2}\cdot\dfrac{z}{z-3}\right]$

$$= Z^{-1}\left[\frac{z^2}{(z-2)^2}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (n+1)\,2^n * 3^n$$

$$= \sum_{m=0}^{n}(m+1)\,2^m\,3^{n-m}$$

$$= 3^n\sum_{m=0}^{n}(m+1)\,2^m\,3^{-m}$$

$$= 3^n \sum_{m=0}^{n} (m+1) \left(\frac{2}{3}\right)^m$$

$$= 3^n \left\{ 1 + 2 . \left(\frac{2}{3}\right) + 3 . \left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)^3 + \ldots + (n+1)\left(\frac{2}{3}\right)^n \right\}$$

Let $S = 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n$, where $x = \frac{2}{3}$

$$S = 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} + (n+1)x^n$$

$$\therefore xS = x + 2x^2 + 3x^3 + \ldots + nx^n + (n+1)x^{n+1}$$

$$\therefore (1-x)S = (1 + x + x^2 + x^3 + \ldots + x^n) - (n+1)x^{n+1}$$

$$= \frac{1 - x^{n+1}}{1-x} - (n+1)x^{n+1}$$

$$\therefore S = \frac{1 - x^{n+1}}{(1-x)^2} - \frac{(n+1)x^{n+1}}{1-x}$$

Since, $x = \frac{2}{3}$, $1 - x = \frac{1}{3}$

$$\therefore S = 9\left[1 - \left(\frac{2}{3}\right)^{n+1}\right] - 3(n+1)\left(\frac{2}{3}\right)^{n+1}$$

$$= 9 - \left(\frac{2}{3}\right)^{n+1} [9 + 3n + 3]$$

$$= 9 - \left(\frac{2}{3}\right)^{n+1} (12 + 3n)$$

$$= 9 - 3(n+4)\left(\frac{2}{3}\right)^{n+1}$$

$$\therefore Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right] = 3^n \left\{ 9 - 3(n+4)\left(\frac{2}{3}\right)^{n+1} \right\}$$

$$= 9.3^n - 3^{n+1}(n+4) . \frac{2^{n+1}}{3^{n+1}}$$

$$= 3^{n+2} - (n+4)\, 2^{n+1}$$

(a) **Find** $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$ **[A.U CBT Dec. 2009][A.U M/J 2016 R13] [A.U A/M 20019 R-13] [A.U N/D 2018 R-8]**

Solution : $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{(z-a)} \cdot \frac{z}{z-a}\right]$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{m=0}^{n} a^{n-m} a^m, \quad \text{by convolution theorem}$$

$$= \sum_{m=0}^{n} a^n = a^n \sum_{m=0}^{n} (1)^m$$

$$= a^n [1^0 + 1^1 + 1^2 + \ldots + 1^n]$$

$$= a^n (n+1) = (n+1)a^n$$

(b) **Find** $Z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$ **[A.U. N/D 2007]**

Solution : Prove that $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n$

Replace a by $-a$ in 3 (a), we get

$$Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = (-a)^n (n+1)$$

5.4 Formation of difference equations :

Def. Difference equations : A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Thus $\quad \Delta y_{(n+1)} + y_{(n)} = 2 \quad \dots (1)$

and $\Delta y_{(n+1)} + \Delta^2 y_{(n-1)} = 1 \quad \dots (2)$

are difference equations.

Def : *Order of a difference equation :*

The order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

Def : *Solution of a difference equation :*

The solution of a difference equation is an expression for $y(n)$ which satisfies the given difference equation.

Def : *The general solution of a difference equation* : The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

Def : *The particular solution of a difference equation :*

A particular solution is that the solution which is obtained from the general solution by giving particular values to the constants.

XII. Formation of difference equations :

1. Form the difference equation corresponding to the family of curves $y = ax + bx^2$ [A.U. March 1996] [A.U N/D 2018 R-17]

Solution :

$$y_x = ax + bx^2 \quad \dots (1)$$

$$y_{x+1} = a(x+1) + b(x+1)^2 \quad \dots (2)$$

$$y_{x+2} = a(x+2) + b(x+2)^2 \quad \dots (3)$$

Eliminating a and b from (1), (2) and (3), we get

$$\begin{vmatrix} y_x & x & x^2 \\ y_{x+1} & x+1 & (x+1)^2 \\ y_{x+2} & x+2 & (x+2)^2 \end{vmatrix} = 0$$

Hint : Expand the determinant through column

$$y_x[(x+1)(x+2)^2 - (x+2)(x+1)^2] - y_{x+1}[x(x+2)^2 - x^2(x+2)] + y_{x+2}[x(x+1)^2 - x^2(x+1)] = 0$$

$$y_x(x+1)(x+2)[x+2-x-1] - y_{x+1}[x(x+2)][x+2-x] + y_{x+2}x(x+1)[x+1-x] = 0$$

$$y_x(x+1)(x+2) - y_{x+1}2x(x+2) + y_{x+2}x(x+1) = 0$$

$$(x^2+3x+2)y_x - 2(x^2+2x)y_{x+1} + (x^2+x)y_{x+2} = 0$$

2. From $y_n = a2^n + b(-2)^n$, derive a difference equation by eliminating the arbitrary constants. [A.U. April, 2001]

Solution : Given : $\quad y_n = a2^n + b(-2)^n \quad \dots (1)$

$$y_{n+1} = a\,2^{n+1} + b(-2)^{n+1} = 2a\,2^n - 2b(-2)^n \quad \dots (2)$$

$$y_{n+2} = a\,2^{n+2} + b(-2)^{n+2}$$

$$= a(2^n)4 + b(-2)^n(-2)^2 = 4a\,2^n + 4b(-2)^n \quad \dots (3)$$

Eliminating $a(2)^n$ & $b(-2)^n$ from (1), (2) and (3), we get

$$\Rightarrow \begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -2 \\ y_{n+2} & 4 & 4 \end{vmatrix} = 0$$

$$y_n(8+8) - y_{n+1}(4-4) + y_{n+2}(-2-2) = 0$$

$$16y_n - (0)\,y_{n+1} - 4y_{n+2} = 0$$

$$y_{n+2} - 4y_n = 0$$

which is the desired difference equation.

3. **Derive the difference equation from**

$y_n = (A + Bn)(-3)^n$. **[A.U. End semester Nov/Dec. 1996]**

Solution :

Given : $y_n = (A + Bn)(-3)^n = A(-3)^n + Bn(-3)^n$ ... (1)

$$y_{n+1} = [A + B(n+1)](-3)^{n+1}$$

$$= A(-3)^n(-3) + B(n+1)(-3)^n(-3)$$

$$= -3A(-3)^n - 3B(n+1)(-3)^n \quad \text{... (2)}$$

$$y_{n+2} = [A + B(n+2)](-3)^{n+2}$$

$$= A(-3)^{n+2} + B(n+2)(-3)^{n+2}$$

$$= 9A(-3)^n + 9B(n+2)(-3)^n \quad \text{... (3)}$$

Eliminating $A(-3)^n$ and $B(-3)^n$ from (1), (2) & (3), we get

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & -3 & -3(n+1) \\ y_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0$$

$$y_n[-27(n+2) + 27(n+1)] - y_{n+1}[9(n+2) - 9n]$$

$$+ y_{n+2}[-3(n+1) + 3n] = 0$$

$$-27y_n - 18y_{n+1} - 3y_{n+2} = 0$$

$$y_{n+2} + 6y_{n+1} + 9y_n = 0$$

4. **Derive the difference equation from**

$u_n = A2^n + Bn$ **[A.U. End semester Nov/Dec. 1996]**

Solution :

Given : $u_n = A2^n + Bn$ ... (1)

$$u_{n+1} = A2^{n+1} + B(n+1)$$

$$= 2A2^n + (n+1)B \quad \text{... (2)}$$

$$u_{n+2} = A2^{n+2} + B(n+2)$$

$$= 4A2^n + (n+2)B \quad \text{... (3)}$$

Eliminating $A2^n$ and B from (1), (2) and (3), we get

$$\begin{vmatrix} u_n & 1 & n \\ u_{n+1} & 2 & n+1 \\ u_{n+2} & 4 & n+2 \end{vmatrix} = 0$$

$$u_n[2(n+2) - 4(n+1)] - u_{n+1}[n+2-4n]$$

$$+ u_{n+2}[n+1-2n] = 0$$

$$u_n[-2n] - u_{n+1}[-3n+2] + u_{n+2}[-n+1] = 0$$

$$(1-n)\,u_{n+2} + (3n-2)\,u_{n+1} - 2n\,u_n = 0$$

5. Derive the difference equation from

$y_n = (A + Bn)\, 2^n$ [A.U. A/M 2000] [A.U. CBT Dec. 2008] [A.U N/D 2013]

Solution :

Given : $y_n = (A + Bn)\, 2^n$

$$y_n = A2^n + Bn2^n \quad \text{... (1)}$$

$$y_{n+1} = A2^{n+1} + B(n+1)\,2^{n+1}$$

$$= 2A2^n + 2B(n+1)\,2^n \quad \text{... (2)}$$

$$y_{n+2} = A\,2^{n+2} + B\,(n+2)\,2^{n+2}$$

$$= 4A2^n + 4B\,(n+2)\,2^n \quad \text{... (3)}$$

Eliminating $A\,2^n$ and $B\,2^n$ from (1), (2) and (3), we get

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$$y_n[8(n+2) - 8(n+1)] - y_{n+1}[4(n+2) - 4n] + y_{n+2}[2(n+1) - 2n] = 0$$

$$y_n[8] - y_{n+1}[8] + y_{n+2}[2] = 0$$

$$y_{n+2} - 4y_{n+1} + 4y_n = 0$$

Note : Formulae

$$\Delta y_r = y_{r+1} - y_r$$

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

$$\Delta^3 y_r = \Delta^2 y_{r+1} - \Delta^2 y_r$$

In general, $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$

6. Write the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ in the subscript notation :

Solution : $\Delta^3 y_x = \Delta^2 y_{x+1} - \Delta^2 y_x$

$$\Delta^2 y_x = \Delta y_{x+1} - \Delta y_x$$

$$\Delta y_x = y_{x+1} - y_x$$

$$\Delta y_{x+1} = y_{x+2} - y_{x+1}$$

$$\Delta^2 y_{x+1} = \Delta y_{x+2} - \Delta y_{x+1}$$

$$\Delta y_{x+2} = y_{x+3} - y_{x+2}$$

Given : $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$

$$(\Delta^2 y_{x+1} - \Delta^2 y_x) + \Delta^2 y_x + \Delta y_x + y_x = 0$$

$$\Delta^2 y_{x+1} + \Delta y_x + y_x = 0$$

$$\Delta y_{x+2} - \Delta y_{x+1} + y_{x+1} - y_x + y_x = 0$$

$$y_{x+3} - y_{x+2} - y_{x+2} + y_{x+1} + y_{x+1} = 0$$

$$y_{x+3} - 2y_{x+2} + 2y_{x+1} = 0$$

7. Find the difference equation satisfied by $y = ax^2 - bx$.

Solution :

Given : $y = ax^2 - bx$

i.e., $y_x = ax^2 - bx \quad \text{... (1)}$

$$y_{x+1} = a(x+1)^2 - b(x+1) \quad \text{... (2)}$$

$$y_{x+2} = a(x+2)^2 - b(x+2) \quad \text{... (3)}$$

Eliminating a and b from (1), (2) and (3), we get

$$\begin{vmatrix} y_x & x^2 & -x \\ y_{x+1} & (x+1)^2 & -(x+1) \\ y_{x+2} & (x+2)^2 & -(x+2) \end{vmatrix} = 0$$

$$y_x\left[-(x+1)^2(x+2)+(x+2)^2(x+1)\right]$$
$$-y_{x+1}\left[-x^2(x+2)+x(x+2)^2\right]$$
$$+y_{x+2}\left[-x^2(x+1)+x(x+1)^2\right] = 0$$

$$y_x(x+1)(x+2)[-x-1+x+2]-y_{x+1}x(x+2)[-x+x+2]$$
$$+y_{x+2}x(x+1)[-x+x+1] = 0$$

$$\Rightarrow y_x(x+1)(x+2)-y_{x+1}x(x+2)2+y_{x+2}x(x+1) = 0$$

$$\Rightarrow (x+1)(x+2)y_x-2x(x+2)y_{x+1}+x(x+1)y_{x+2} = 0$$

8. Form the difference equation generated by $y_x = ax + b\,2^x$

Solution : Given : $y_x = ax + b2^x$ ... (1)

$$y_{x+1} = a(x+1)+b2^{x+1}$$
$$= (x+1)a + 2b2^x \quad \text{... (2)}$$
$$y_{x+2} = a(x+2)+b2^{x+2}$$
$$= (x+2)a + 4b\,2^x \quad \text{... (3)}$$

Eliminating a and $b\,2^x$ from (1), (2) and (3), we get

$$\begin{vmatrix} y_x & x & 1 \\ y_{x+1} & x+1 & 2 \\ y_{x+2} & x+2 & 4 \end{vmatrix} = 0$$

$$y_x[4(x+1)-2(x+2)]-y_{x+1}[4x-(x+2)]$$
$$+y_{x+2}[2x-(x+1)] = 0$$

$$y_x[2x]-y_{x+1}[3x-2]+y_{x+2}[x-1] = 0$$

$$(x-1)y_{x+2}+(-3x+2)y_{x+1}+2xy_x = 0$$

9. Form the difference equation generated by $y_x = a2^x + b3^x + c$.

Solution : Given : $y_x = a2^x + b3^x + c$ ... (1)

$$y_{x+1} = a2^{x+1}+b3^{x+1}+c = 2a\,2^x+3b\,3^x+c \quad \text{... (2)}$$
$$y_{x+2} = a2^{x+2}+b3^{x+2}+c = 4a\,2^x+9b\,3^x+c \quad \text{... (3)}$$
$$y_{x+3} = 8a\,2^x+27b\,3^x+c \quad \text{... (4)}$$

Eliminating $a\,2^x, b\,3^x, c$ from (1), (2), (3) and (4), we get

$$\begin{vmatrix} y_x & 1 & 1 & 1 \\ y_{x+1} & 2 & 3 & 1 \\ y_{x+2} & 4 & 9 & 1 \\ y_{x+3} & 8 & 27 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} y_x & 1 & 1 & 1 \\ y_{x+1}-y_x & 1 & 2 & 0 \\ y_{x+2}-y_x & 3 & 8 & 0 \\ y_{x+3}-y_x & 7 & 26 & 0 \end{vmatrix} = 0 \qquad \begin{matrix} R_2 \Rightarrow R_2 - R_1 \\ R_3 \Rightarrow R_3 - R_1 \\ R_4 \Rightarrow R_4 - R_1 \end{matrix}$$

i.e., $$-1\begin{vmatrix} y_{x+1}-y_x & 1 & 2 \\ y_{x+2}-y_x & 3 & 8 \\ y_{x+3}-y_x & 7 & 26 \end{vmatrix} = 0$$

i.e., $$\begin{vmatrix} y_{x+1}-y_x & 1 & 2 \\ y_{x+2}-y_x & 3 & 8 \\ y_{x+3}-y_x & 7 & 26 \end{vmatrix} = 0$$

$$(y_{x+1}-y_x)[78-56]-1[26(y_{x+2}-y_x)-8(y_{x+3}-y_x)]$$
$$+2[7(y_{x+2}-y_x)-3(y_{x+3}-y_x)] = 0$$

$$22\,y_{x+1} - 22y_x - 26y_{x+2} + 26y_x + 8y_{x+3} - 8y_x$$
$$+\ 14y_{x+2} - 14y_x - 6y_{x+3} + 6y_x = 0$$

$$2y_{x+3} - 12y_{x+2} + 22y_{x+1} - 12y_x = 0$$

i.e., $$y_{x+3} - 6y_{x+2} + 11y_{x+1} - 6y_x = 0$$

10. Form the difference equation from $y_n = a + b\,3^n$.

[A.U A/M 2008] [A.U N/D 2010]

Solution : Given : $y_n = a + b\,3^n$ ... (1)

$y_{n+1} = a + b\,3^{n+1} = a + (b)(3)\,3^n$ ... (2)

$y_{n+2} = a + b\,3^{n+2} = a + (b)(9)\,3^n$ ... (3)

Eliminating a and $b\,3^n$ from (1), (2) and (3) we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n[9-3] - y_{n+1}[9-1] + y_{n+2}[3-1] = 0$$

$$6\,y_n - 8\,y_{n+1} + 2\,y_{n+2} = 0$$

$$y_{n+2} - 4y_{n+1} + 3y_n = 0$$

11. Form the difference equation from $u_n = a\,2^{n+1}$

Solution : **[A.U N/D 2015, R-8]**

Given : $u_n = a\,2^{n+1}$ ... (1)

$u_{n+1} = a\,2^{(n+1)+1} = a\,2^{n+1}\,2$ ... (2)

Eliminating $a\,2^{n+1}$ from (1) & (2) we get

$$\begin{vmatrix} u_n & 1 \\ u_{n+1} & 2 \end{vmatrix} = 0$$

i.e., $$2\,u_n - u_{n+1} = 0$$

i.e., $$u_{n+1} - 2\,u_n = 0$$

5.5 Solution of difference equations using Z-transform.

We know that Laplace Transforms are very useful to solve linear differential equations

The Z-transforms are useful to solve linear difference equations.

Formula :

(1) $Z[y_n] = Y(z)$

(2) $Z[y_{n+1}] = z\,Y(z) - z\,y(0)$

(3) $Z[y_{n+2}] = z^2\,Y(z) - z^2\,y(0) - z\,y(1)$

(4) $Z[y_{n+3}] = z^3\,Y(z) - z^3\,y(0) - z^2\,y(1) - z\,y(2)$

(5) $Z[y_{n-1}] = z^{-1}\,Y(z)$

Standard formulae :

(1) $Z[a^n] = \dfrac{z}{z-a}$	(5) $Z[a^n \cos n\theta] = \dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$
(2) $Z[n] = \dfrac{z}{(z-1)^2}$	(6) $Z[a^n \sin n\theta] = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$
(3) $Z[a^n n] = \dfrac{az}{(z-a)^2}$	(7) $Z\left[\cos\dfrac{n\pi}{2}\right] = \dfrac{z^2}{z^2+1}$
(4) $Z[n(n-1)] = \dfrac{2z}{(z-1)^3}$	(8) $Z\left[\sin\dfrac{n\pi}{2}\right] = \dfrac{z}{z^2+1}$

XIV. Problems based on Solution of the difference equations using Z-transform

Solve :

1. $y_{n+1} - 2y_n = 0$ given $y_0 = 3$ **[A.U A/M 2000]**
2. $y_{n+2} - 4y_n = 0$ given $y_0 = 0,\ y_1 = 2$
3. $y_{n+2} - 4y_n = 0$
4. $u_{n+2} + 3u_{n+1} + 2u_n = 0$ given $u_0 = 1,\ u_1 = 2$ **[A.U A/M 1999, A.U CBT A/M 2011]**
5. $y(n+3) - 3y(n+1) + 2y(n) = 0$ given $y(0) = 4, y(1) = 0, y(2) = 8$ **[A.U N/D 2014, R-08] [A.U. N/D 2007, N/D 2012]**
6. $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = 0$ given $y_0 = 1,\ y_1 = \cos\alpha$
7. $y(k+2) - 4y(k+1) + 4y(k) = 0$ given $y(0) = 1,\ y(1) = 0$ **[A.U N/D 2005, N/D 2006]**
8. $y(n) + 3y(n-1) - 4y(n-2) = 0$ given $y(0)=3, y(1)=-2,\ n \geq 2$ **[AU M/J 2006]**
9. $x(n+1) - 2x(n) = 1$, given $x(0) = 0$ **[A.U N/D 2018 R-8]**
10. $y_{n+2} + y_n = 2$ given $y_0 = y_1 = 0$ **[A.U M/J 2007][A.U M/J 2016 R13]**
11. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$ **[A.U. A/M 2008, N/D 2008, A/M 2009, N/D 2009, N/D 2012] [A.U M/J 2016 R-8] [A.U N/D 2016 R-13][A.U N/D 2018 R-17]**
12. $u_{n+2} + 4u_{n+1} + 3u_n = 2^n$ given $u_0 = 0,\ u_1 = 1$ **[A.U N/D 2010] [A.U N/D 2015 R-8]**
13. $u_{n+2} - 5u_{n+1} + 6u_n = 4^n$ given $u_0 = 0, u_1 = 1$ **[A.U A/M 2001, A.U CBT Dec. 2008, A.U CBT N/D 2010]**
14. $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ given $y_0 = 0, y_1 = 1$ **[A.U. N/D 2011][A.U A/M 2015 R-08][A.U N/D 2015 R-13] [A.U A/M 2019 R-17]**

15. $y_{n+2} + y_n = n\,2^n$ **[A.U M/J 2012]**
16. $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$ given $y_0 = 3, y_1 = -5$ **[A.U. A/M 2000], A.U.T. CH N/D 2011]**
17. $y(n) - y(n-1) = u(n) + u(n-1)$ given $u(n) = n$, $u(n-1) = n-1$
18. $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, $y_0 = 0,\ y_1 = 1,\ u_n = 1$ **[A.U N/D 2018-A R-17]**
19. $x_{n+1} = 5x_{n+7}$, $y_{n+1} = x_n + 2y_n$, $x_0 = 0$, $y_0 = 1$
20. $x_{n+1} = 7x_n + 10y_n$, $y_{n+1} = x_n + 4y_n$, $x_0 = 3,\ y_0 = 2$ **[A.U. N/D 1996]**

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1. Solve $y_{n+1} - 2y_n = 0$ given $y_0 = 3$ **[A.U. April/May 2000]**

Solution : Given : $y_{n+1} - 2y_n = 0$

Taking Z-transform on both sides of the difference equation, we get

$$Z[y_{n+1}] - 2Z[y_n] = Z(0)$$

$$\left[z\,Y(z) - zy(0)\right] - 2\,Y(z) = 0$$

$$\left[z\,Y(z) - (z)(3)\right] - 2\,Y(z) = 0$$

Given : $y_0 = 3$
$\Rightarrow y(0) = 3$

$$(z-2)\,Y(z) - 3z = 0$$

$$Y(z) = \frac{3z}{z-2}$$

$$\Rightarrow Z[y_n] = \frac{3z}{z-2}$$

$$y_n = Z^{-1}\left[\frac{3z}{z-2}\right] = 3Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= 3\left(2^n\right)$$

Formula : $Z[a^n] = \dfrac{z}{z-a}$

2. Using Z-transform, solve $y_{n+2} - 4y_n = 0$, given that $y_0 = 0, y_1 = 2$.

Solution : Given : $y_{n+2} - 4y_n = 0$

$$Z[y_{n+2}] - 4Z[y_n] = Z[0]$$

$$\left[z^2 Y(z) - z^2 y(0) - zy(1)\right] - 4Y(z) = 0$$

$$z^2 Y(z) - 0 - 2z - 4Y(z) = 0 \quad [\because y(0) = 0, y(1) = 2]$$

$$(z^2 - 4) Y(z) = 2z$$

$$Y(z) = \frac{2z}{z^2 - 4} = \frac{2z}{z^2 - 2^2}$$

$$Y(z) = \frac{2z}{(z-2)(z+2)}$$

Let $\frac{Y(z)}{z} = \frac{2}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$ ... (1)

$$\Rightarrow 2 = A(z+2) + B(z-2)$$

Put $z = 2$, we get

$$2 = 4A$$

$$A = \frac{1}{2}$$

Put $z = -2$, we get

$$2 = -4B$$

$$B = -\frac{1}{2}$$

$$\therefore (1) \Rightarrow \frac{Y(z)}{z} = \frac{1/2}{z-2} - \frac{1/2}{z+2}$$

$$\Rightarrow Y(z) = \frac{1}{2}\left(\frac{z}{z-2}\right) - \frac{1}{2}\left(\frac{z}{z+2}\right)$$

$$Z[y_n] = \frac{1}{2}\left(\frac{z}{z-2}\right) - \frac{1}{2}\left(\frac{z}{z+2}\right)$$

$$y(n) = \frac{1}{2} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{2} Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n$$

$$= \frac{1}{2} 2^n \left[1 - (-1)^n\right]$$

Formula : $Z[a^n] = \frac{z}{z-a}$

Solve $y_{n+2} - 4y_n = 0$

Solution : Given $y_{n+2} - 4y_n = 0$

$$Z[y_{n+2}] - 4Z[y_n] = 0$$

$$\left[z^2 Y(z) - z^2 y(0) - zy(1)\right] - 4Y(z) = 0 \quad \text{... (1)}$$

y_0 and y_1 are not given.

Choose $y_0 = A$ and $y_1 = B$,

$$(1) \Rightarrow (z^2 - 4) Y(z) = Az^2 + Bz$$

$$Y(z) = \frac{Az^2}{z^2 - 4} + \frac{Bz}{z^2 - 4}$$

$$Y(z) = \frac{Az^2 + Bz}{(z-2)(z+2)}$$

$$= A\left[\frac{z^2}{(z-2)(z+2)}\right] + B\left[\frac{z}{(z-2)(z+2)}\right]$$

$$= \frac{A}{2}\left[\frac{z}{z-2} + \frac{z}{z+2}\right] + \frac{B}{4}\left[\frac{z}{z-2} - \frac{z}{z+2}\right]$$

[Using short cut idea]

$$= \left(\frac{A}{2} + \frac{B}{4}\right)\left(\frac{z}{z-2}\right) + \left(\frac{A}{2} - \frac{B}{4}\right)\left(\frac{z}{z+2}\right)$$

$$Z[y(n)] = \left(\frac{A}{2} + \frac{B}{4}\right)\left(\frac{z}{z-2}\right) + \left(\frac{A}{2} - \frac{B}{4}\right)\left(\frac{z}{z+2}\right)$$

$$y(n) = \left(\frac{A}{2} + \frac{B}{4}\right) Z^{-1}\left[\frac{z}{z-2}\right] + \left(\frac{A}{2} - \frac{B}{4}\right) Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= C(2)^n + D(-2)^n$$

Formula : $Z[a^n] = \frac{z}{z-a}$

4. Using Z-transform, solve
$u_{n+2} + 3u_{n+1} + 2u_n = 0$ **given** $u_0 = 1, u_1 = 2.$

[A.U. April/May 1999] [A.U CBT A/M 2011]

Solution : Given : $u_{n+2} + 3u_{n+1} + 2u_n = 0$

$$Z[u_{n+2}] + 3Z[u_{n+1}] + 2Z[u_n] = 0$$

$$[z^2 U(z) - z^2 u(0) - z u(1)] + 3[z U(z) - z u(0)] + 2U(z) = 0$$

$$(z^2 + 3z + 2) U(z) - z^2 - 2z - 3z = 0 \quad [\because u(0) = 1, u(1) = 2]$$

$$(z^2 + 3z + 2) U(z) - z^2 - 5z = 0$$

$$U(z) = \frac{z^2 + 5z}{z^2 + 3z + 2}$$

$$U(z) = \frac{z(z+5)}{(z+1)(z+2)}$$

$$\frac{U(z)}{z} = \frac{z+5}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} \quad \ldots (1)$$

$$\Rightarrow \quad z + 5 = A(z+2) + B(z+1)$$

Put $z = -1$, we get	Put $z = -2$, we get
$4 = A$	$3 = -B$
$A = 4$	$B = -3$

$$\therefore (1) \Rightarrow \frac{U(z)}{z} = \frac{4}{z+1} - \frac{3}{z+2}$$

$$\Rightarrow U(z) = 4\left[\frac{z}{z+1}\right] - 3\left[\frac{z}{z+2}\right]$$

$$Z[u(n)] = 4\left[\frac{z}{z+1}\right] - 3\left[\frac{z}{z+2}\right]$$

$$u(n) = 4Z^{-1}\left[\frac{z}{z+1}\right] - 3Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= 4(-1)^n - 3(-2)^n$$

$$= [4 - 3(2)^n](-1)^n$$

Formula : $Z[a^n] = \frac{z}{z-a}$

5. Solve the difference equation
$y(n+3) - 3y(n+1) + 2y(n) = 0$ **given that** $y(0) = 4, y(1) = 0$ **and** $y(2) = 8.$ **[A.U. N/D 2007, N/D 2012] [A.U N/D 2018 R-13]**

Solution : Given : $y(n+3) - 3y(n+1) + 2y(n) = 0$

$$Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] = 0$$

$$[z^3 Y(z) - z^3 y(0) - z^2 y(1) - zy(2)] - 3[zY(z) - zy(0)] + 2Y(z) = 0$$

$$[z^3 Y(z) - 4z^3 - 8z] - 3[zY(z) - 4z] + 2Y(z) = 0$$

$$[\because y(0) = 4, y(1) = 0, y(2) = 8]$$

$$(z^3 - 3z + 2) Y(z) - 4z^3 - 8z + 12z = 0$$

$$(z^3 - 3z + 2) Y(z) - 4z^3 + 4z = 0$$

$$(z-1)^2 (z+2) Y(z) = 4z^3 - 4z$$

$$Y(z) = \frac{4z^3 - 4z}{(z-1)^2(z+2)}$$

$$= \frac{4z[z^2 - 1]}{(z-1)^2(z+2)}$$

$$= \frac{4z[z+1][z-1]}{(z-1)^2(z+2)}$$

$$= \frac{4z(z+1)}{(z-1)(z+2)}$$

$$\frac{Y(z)}{z} = \frac{4(z+1)}{(z+1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2} \quad \ldots (1)$$

Let $4(z+1) = A(z+2) + B(z-1)$

4. Using Z-transform, solve
$u_{n+2} + 3u_{n+1} + 2u_n = 0$ given $u_0 = 1, u_1 = 2$.

[A.U. April/May 1999] [A.U CBT A/M 2011]

Solution : Given :

$$u_{n+2} + 3u_{n+1} + 2u_n = 0$$

$$Z[u_{n+2}] + 3Z[u_{n+1}] + 2Z[u_n] = 0$$

$$[z^2 U(z) - z^2 u(0) - z u(1)] + 3[z U(z) - z u(0)] + 2U(z) = 0$$

$$(z^2 + 3z + 2) U(z) - z^2 - 2z - 3z = 0 \quad [\because u(0) = 1, u(1) = 2]$$

$$(z^2 + 3z + 2) U(z) - z^2 - 5z = 0$$

$$U(z) = \frac{z^2 + 5z}{z^2 + 3z + 2}$$

$$U(z) = \frac{z(z+5)}{(z+1)(z+2)}$$

$$\frac{U(z)}{z} = \frac{z+5}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} \quad \ldots (1)$$

$$\Rightarrow \quad z + 5 = A(z+2) + B(z+1)$$

Put $z = -1$, we get	Put $z = -2$, we get
$4 = A$	$3 = -B$
$A = 4$	$B = -3$

$$\therefore (1) \Rightarrow \frac{U(z)}{z} = \frac{4}{z+1} - \frac{3}{z+2}$$

$$\Rightarrow U(z) = 4\left[\frac{z}{z+1}\right] - 3\left[\frac{z}{z+2}\right]$$

$$Z[u(n)] = 4\left[\frac{z}{z+1}\right] - 3\left[\frac{z}{z+2}\right]$$

$$u(n) = 4Z^{-1}\left[\frac{z}{z+1}\right] - 3Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= 4(-1)^n - 3(-2)^n$$

$$= [4 - 3(2)^n](-1)^n$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

5. Solve the difference equation
$y(n+3) - 3y(n+1) + 2y(n) = 0$ given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$. **[A.U. N/D 2007, N/D 2012] [A.U N/D 2018 R-13]**

Solution : Given : $y(n+3) - 3y(n+1) + 2y(n) = 0$

$$Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] = 0$$

$$[z^3 Y(z) - z^3 y(0) - z^2 y(1) - zy(2)] - 3[zY(z) - zy(0)] + 2Y(z) = 0$$

$$[z^3 Y(z) - 4z^3 - 8z] - 3[zY(z) - 4z] + 2Y(z) = 0$$

$$[\because y(0) = 4, y(1) = 0, y(2) = 8]$$

$$(z^3 - 3z + 2) Y(z) - 4z^3 - 8z + 12z = 0$$

$$(z^3 - 3z + 2) Y(z) - 4z^3 + 4z = 0$$

$$(z-1)^2 (z+2) Y(z) = 4z^3 - 4z$$

$$Y(z) = \frac{4z^3 - 4z}{(z-1)^2 (z+2)}$$

$$= \frac{4z[z^2 - 1]}{(z-1)^2 (z+2)}$$

$$= \frac{4z[z+1][z-1]}{(z-1)^2 (z+2)}$$

$$= \frac{4z(z+1)}{(z-1)(z+2)}$$

$$\frac{Y(z)}{z} = \frac{4(z+1)}{(z+1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2} \quad \ldots (1)$$

Let $4(z+1) = A(z+2) + B(z-1)$

Put $z = 1$, we get

$$8 = 3A$$

$$A = \frac{8}{3}$$

Put $z = -2$, we get

$$-4 = -3B$$

$$B = \frac{4}{3}$$

$$\therefore (1) \Rightarrow \quad \frac{Y(z)}{z} = \frac{8/3}{z-1} + \frac{4/3}{z+2}$$

$$\Rightarrow \quad Y(z) = \frac{8}{3}\left(\frac{z}{z-1}\right) + \frac{4}{3}\left(\frac{z}{z+2}\right)$$

$$Z[y(n)] = \frac{8}{3}\left(\frac{z}{z-1}\right) + \frac{4}{3}\left(\frac{z}{z+2}\right)$$

$$y(n) = \frac{8}{3}Z^{-1}\left[\frac{z}{z-1}\right] + \frac{4}{3}Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= \frac{8}{3}(1)^n + \frac{4}{3}(-2)^n$$

$$= \frac{4}{3}[2 + (-2)^n]$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

6. Solve $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = 0$ given that $y_0 = 1$, $y_1 = \cos\alpha$.

Solution : Given : $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = 0$

$$Z[y_{n+2}] - 2\cos\alpha\, Z[y_{n+1}] + Z[y_n] = Z[0]$$

$$[z^2 Y(z) - z^2 y(0) - z y(1)] - 2\cos\alpha[zY(z) - zy(0)] + Y(z) = 0$$

$$[z^2 - 2z\cos\alpha + 1]\, Y(z) - z^2 - z\cos\alpha + 2z\cos\alpha = 0$$

$$[z^2 - 2z\cos\alpha + 1]\, Y(z) = z^2 - z\cos\alpha = 0$$

$$[z^2 - 2z\cos\alpha + 1]\, Y(z) = z^2 - z\cos\alpha$$

$$Y(z) = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$$

$$Z[y(n)] = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$$

$$y(n) = Z^{-1}\left[\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}\right]$$

$$= (1)^n \cos n\alpha$$

$$\boxed{\text{Formula : } Z[a^n \cos n\theta] = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}}$$

7. Solve the difference equation $y(k+2) - 4y(k+1) + 4y(k) = 0$ where $y(0) = 1$, $y(1) = 0$.

Solution : **[A.U. N/D 2005, A.U. N/D 2006, Tvli M/J 2011] [A.U.T. CBT N/D 2011]**

Given : $y(k+2) - 4y(k+1) + 4y(k) = 0$

Taking Z transform on both sides, we get

$$Z[y(k+2)] - 4Z[y(k+1)] + 4Z[y(k)] = 0$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] - 4[zY(z) - zy(0)] + 4Y(z) = 0$$

$$[z^2 Y(z) - z^2 - z(0)] - 4[zY(z) - z] + 4Y(z) = 0$$

$$Y(z)\,[z^2 - 4z + 4] - z^2 + 4z = 0$$

$$Y(z) = \frac{z^2 - 4z}{z^2 - 4z + 4} = \frac{z(z-4)}{z^2 - 4z + 4}$$

$$\frac{Y(z)}{z} = \frac{z-4}{z^2 - 4z + 4} = \frac{z-4}{(z-2)^2} = \frac{A}{z-2} + \frac{B}{(z-2)^2} \quad \ldots (1)$$

Let $z - 4 = A(z-2) + B$

Put $z = 2$, we get	Put $z = 0$, we get
$-2 = B$	$-4 = -2A + B$
$\boxed{B = -2}$	$-4 = -2A - 2$
	$-2 = -2A$
	$\boxed{A = 1}$

$$(1) \Rightarrow \frac{Y(z)}{z} = \frac{1}{z-2} - \frac{2}{(z-2)^2}$$

$$Y(z) = \frac{z}{z-2} - 2\frac{z}{(z-2)^2}$$

$$Z[y(k)] = \frac{z}{z-2} - 2\frac{z}{(z-2)^2}$$

$$y(k) = Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= (2)^k - (2)^k k = 2^k[1-k]$$

$$\boxed{\text{Formula : (1) } Z[a^n] = \frac{z}{z-a} \quad \text{(2) } Z[a^n n] = \frac{az}{(z-a)^2}}$$

8. Using Z-transform solve $y(n) + 3y(n-1) - 4y(n-2) = 0$, $n \geq 2$ given that $y(0) = 3$ and $y(1) = -2$. **[A.U M/J 2006]**

Solution : Given : $y(n) + 3y(n-1) - 4y(n-2) = 0,\ n \geq 2$

Replace n by $n+2$, we get

$$y(n+2) + 3y(n+1) - 4y(n) = 0, \qquad n+2 \geq 2,$$

$$\text{(i.e.,)} \quad y(n+2) + 3y(n+1) - 4y(n) = 0, \qquad n \geq 0$$

$$Z[y_{n+2}] + 3Z[y_{n+1}] - 4Z[y_n] = 0$$

$$[z^2Y(z) - z^2y(0) - zy(1)] + 3[zY(z) - zy(0)] - 4Y(z) = 0$$

$$[z^2Y(z) - 3z^2 + 2z] + 3[zY(z) - 3z] - 4Y(z) = 0$$

$$[z^2 + 3z - 4]Y(z) - 3z^2 + 2z - 9z = 0$$

$$[z^2 + 3z - 4]Y(z) = 3z^2 + 7z$$

$$Y(z) = \frac{3z^2 + 7z}{z^2 + 3z - 4} = \frac{z(3z+7)}{z^2 + 3z - 4}$$

$$\frac{Y(z)}{z} = \frac{3z+7}{z^2+3z-4} = \frac{3z+7}{(z+4)(z-1)} = \frac{A}{z+4} + \frac{B}{z-1} \quad \dots (1)$$

Let $3z + 7 = A(z-1) + B(z+4)$

put $z = 1$, we get	put $z = -4$, we get
$10 = 5B$	$-5 = -5A$
$\boxed{B = 2}$	$\boxed{A = 1}$

$$\therefore (1) \Rightarrow \frac{Y(z)}{z} = \frac{1}{z+4} + \frac{2}{z-1}$$

$$\Rightarrow Y(z) = \frac{z}{z+4} + 2\frac{z}{z-1}$$

$$Z[y(n)] = \frac{z}{z+4} + 2\frac{z}{z-1}$$

$$y(n) = Z^{-1}\left[\frac{z}{z+4}\right] + 2Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= (-4)^n + 2(1)^n$$

$$= 2 + (-4)^n \qquad \boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

9. Solve $x(n+1) - 2x(n) = 1$, given $x(0) = 0$

Solution : Given : $x(n+1) - 2x(n) = 1$ **[A.U N/D 2018 R-8]**

$$Z[x(n+1)] - 2Z[x(n)] = Z[1]$$

$$zX(z) - zx(0) - 2X(z) = \frac{z}{z-1}$$

$$zX(z) - 2X(z) = \frac{z}{z-1} \qquad [\because x(0) = 0]$$

$$(z-2)X(z) = \frac{z}{z-1}$$

$$X(z) = \frac{z}{(z-1)(z-2)}$$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \quad \dots (1)$$

$$\text{Let } 1 = A(z-2) + B(z-1)$$

Put $z = 1$, we get

$$1 = -A$$

i.e., $A = -1$

Put $z = 2$, we get

$$1 = B$$

i.e., $B = 1$

$$\therefore (1) \Rightarrow \frac{X(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\Rightarrow X(z) = -\frac{z}{z-1} + \frac{z}{z-2}$$

$$Z[x(n)] = -\frac{z}{z-1} + \frac{z}{z-2}$$

$$x(n) = -Z^{-1}\left[\frac{z}{z-1}\right] + Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -(1)^n + (2)^n$$

$$= 2^n - 1$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

10. Using Z-transform method solve $y_{n+2} + y_n = 2$ given that $y_0 = y_1 = 0$. [A.U. M/J 2007][A.U M/J 2016 R-13]

Solution : Given : $y_{n+2} + y_n = 2$ and $y_0 = y_1 = 0$

$$Z[y_{n+2}] + Z[y_n] = Z[2]$$

$$[z^2 Y(z) - z^2 y(0) - z y(1)] + Y(z) = 2Z[1]$$

$$z^2 Y(z) - 0 - 0 + Y(z) = 2\frac{z}{z-1} \quad [\because y(0) = 0, y(1) = 0]$$

$$Y(z)[z^2 + 1] = 2\frac{z}{z-1}$$

$$Y(z) = \frac{2z}{(z-1)(z^2+1)}$$

$$\frac{Y(z)}{z} = \frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz + C}{z^2+1} \quad \ldots (1)$$

$$2 = A(z^2+1) + (Bz + C)(z-1)$$

put $z = 1$, we get

$$2 = 2A$$

$$\boxed{A = 1}$$

put $z = 0$, we get

$$2 = A - C$$

$$C = A - 2$$

$$C = 1 - 2$$

$$\boxed{C = -1}$$

equating the coefficients of z^2 on both sides, we get

$$0 = A + B$$

$$\boxed{B = -1}$$

$$\therefore (1) \Rightarrow \frac{Y(z)}{z} = \frac{(1)}{z-1} + \frac{-z-1}{z^2+1}$$

$$Y(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}$$

$$Z[y(n)] = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}$$

$$y(n) = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right] - Z^{-1}\left[\frac{z}{z^2+1}\right]$$

$$= (1)^n - \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}$$

Formula : (1) $Z[a^n] = \frac{z}{z-a}$ (2) $Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$

(3) $Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$

11. Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$ [A.U. A/M 2008, A.U. N/D 2008, A.U A/M 2009][A.U Tvli N/D 2009] [A.U. N/D 2009] [A.U.T. Tvli N/D 2011][A.U M/J 2016 R-13] [A.U N/D 2012][A.U N/D 2018 R-17][A.U A/M 2019 R-17]

Solution : Given : $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] + 6[zY(z) - zy(0)] + 9Y(z) = \frac{z}{z-2}$$

$$z^2 Y(z) + 6z\, Y(z) + 9\, Y(z) = \frac{z}{z-2} \quad [\because y(0) = 0,\ y(1) = 0]$$

$$[z^2 + 6z + 9]\, Y(z) = \frac{z}{z-2}$$

$$(z+3)^2\, Y(z) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

$$= \frac{A}{z-2} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2} \quad \ldots (1)$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put $z = 2$,	Put $z = -3$,	equating z^2 co-eff.
we get	we get	on both sides, we get
$1 = 25A$	$1 = -5C$	$0 = A + B$
$\boxed{A = \frac{1}{25}}$	$\boxed{C = \frac{-1}{5}}$	$B = -A$
		$\boxed{B = -\frac{1}{25}}$

$$\therefore (1) \Rightarrow \frac{Y(z)}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\cdot\frac{1}{(z+3)^2}$$

$$Y(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$y(n) = \frac{1}{25} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$\text{i.e., } y(n) = \frac{1}{25} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} Z^{-1}\left[\frac{z}{z-(-3)}\right] + \frac{1}{15} Z^{-1}\left[\frac{-3z}{[z-(-3)]^2}\right]$$

$$= \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n\, n$$

Formula : (1) $Z[a^n] = \dfrac{z}{z-a}$ (2) $Z[a^n n] = \dfrac{az}{(z-a)^2}$

12. Using the Z-transform, solve

$u_{n+2} + 4u_{n+1} + 3u_n = 2^n$ **with** $u_0 = 0,\ u_1 = 1$ **[A.U N/D 2010]**

[A.U N/D 2015 R-8]

Solution : Given : $u_{n+2} + 4u_{n+1} + 3u_n = 2^n$

$$Z[u_{n+2}] + 4Z[u_{n+1}] + 3Z[u_n] = Z[2^n]$$

$$[z^2 U(z) - z^2 u(0) - z\,u(1)] + 4\,[zU(z) - zu(0)] + 3U(z) = \frac{z}{z-2}$$

$$z^2 U(z) - z + 4z\,u(z) + 3U(z) = \frac{z}{z-2}$$

$$(z^2 + 4z + 3)\, U(z) = \frac{z}{z-2} + z$$

$$(z+3)(z+1)\, U(z) = \frac{z + z^2 - 2z}{z-2}$$

$$U(z) = \frac{z^2 - z}{(z-2)(z+1)(z+3)}$$

$$= \frac{z(z-1)}{(z-2)(z+1)(z+3)}$$

$$\frac{U(z)}{z} = \frac{z-1}{(z-2)(z+1)(z+3)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z+3} \quad \ldots (1)$$

Let $z - 1 = A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1)$

Put $z = -1$, we get	put $z = 2$, we get	put $z = -3$, we get
$-2 = -6B$	$1 = 15A$	$-4 = 10C$
$\boxed{B = \frac{1}{3}}$	$\boxed{A = \frac{1}{15}}$	$\boxed{C = \frac{-2}{5}}$

$$(1) \Rightarrow \frac{U(z)}{z} = \frac{1/15}{z-2} + \frac{1/3}{z+1} + \frac{(-2/5)}{z+3}$$

$$U(z) = \frac{1}{15}\left(\frac{z}{z-2}\right) + \frac{1}{3}\left(\frac{z}{z+1}\right) - \frac{2}{5}\left(\frac{z}{z+3}\right)$$

$$Z[u(n)] = \frac{1}{15}\left(\frac{z}{z-2}\right) + \frac{1}{3}\left(\frac{z}{z+1}\right) - \frac{2}{5}\left(\frac{z}{z+3}\right)$$

$$u(n) = \frac{1}{15}Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{3}Z^{-1}\left[\frac{z}{z+1}\right] - \frac{2}{5}Z^{-1}\left[\frac{z}{z+3}\right]$$

$$= \frac{1}{15}(2)^n + \frac{1}{3}(-1)^n - \frac{2}{5}(-3)^n$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

13. Using Z-transform, solve $u_{n+2} - 5u_{n+1} + 6u_n = 4^n$ given that $u_0 = 0, u_1 = 1$. **[A.U. APril, 2001] [A.U. CBT Dec. 2008] [A.U CBT N/D 2010]**

Solution : Given : $u_{n+2} - 5u_{n+1} + 6u_n = 4^n$

$$Z[u_{n+2}] - 5Z[u_{n+1}] + 6Z[u_n] = Z[4^n]$$

$$\left[z^2 U(z) - z^2 u(0) - zu(1)\right]$$

$$-5[zU(z) - zu(0)] + 6U(z) = \frac{z}{z-4}$$

$$[z^2 - 5z + 6]U(z) - z = \frac{z}{z-4} \qquad [\because u(0) = 0, u(1) = 1]$$

$$(z^2 - 5z + 6)U(z) = z + \frac{z}{z-4}$$

$$(z-3)(z-2)U(z) = \frac{z^2 - 4z + z}{z-4}$$

$$(z-3)(z-2)U(z) = \frac{z^2 - 3z}{z-4}$$

$$U(z) = \frac{z(z-3)}{(z-3)(z-2)(z-4)}$$

$$U(z) = \frac{z}{(z-2)(z-4)}$$

$$\frac{U(z)}{z} = \frac{1}{(z-2)(z-4)} = \frac{A}{z-2} + \frac{B}{z-4} \qquad \dots (1)$$

$$1 = A(z-4) + B(z-2)$$

Put $z = 2$, we get	Put $z = 4$, we get
$1 = -2A,$	$1 = 2B$
$\boxed{A = -\frac{1}{2}}$	$\boxed{B = \frac{1}{2}}$

$$\therefore (1) \Rightarrow \frac{U(z)}{z} = \frac{\frac{-1}{2}}{z-2} + \frac{\frac{1}{2}}{z-4}$$

$$\Rightarrow U(z) = -\frac{1}{2}\left(\frac{z}{z-2}\right) + \frac{1}{2}\left(\frac{z}{z-4}\right)$$

$$Z[U(n)] = -\frac{1}{2}\left(\frac{z}{z-2}\right) + \frac{1}{2}\left(\frac{z}{z-4}\right)$$

$$u(n) = -\frac{1}{2}Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{2}Z^{-1}\left[\frac{z}{z-4}\right]$$

$$= -\frac{1}{2}(2)^n + \frac{1}{2}(4)^n$$

$$= -2^{n-1} + \frac{1}{2}2^{2n}$$

$$= -2^{n-1} + 2^{2n-1}$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

14. $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ **with** $y_0 = 0,\ y_1 = 1$

[A.U Tvli N/D 2010, A.U. N/D 2011] [A.U A/M 2015 R-08] [A.U A/M 2019 R-17]

Solution : Given : $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z[3^n]$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] + 4[zY(z) - zy(0)] + 3Y(z) = \frac{z}{z-3}$$

$$(z^2 + 4z + 3)Y(z) - z = \frac{z}{z-3}$$

$$(z+1)(z+3)Y(z) = \frac{z}{z-3} + z$$

$$= \frac{z + z(z-3)}{z-3}$$

$$= \frac{z^2 - 2z}{z-3}$$

$$Y(z) = \frac{z(z-2)}{(z+1)(z+3)(z-3)}$$

$$\frac{Y(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3} \quad \text{... (1)}$$

Let $z - 2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3)$

Put $z = -1$, we get	put $z = -3$, we get	put $z = 3$, we get
$-3 = -8A$	$-5 = 12B$	$1 = 24C$
$\boxed{A = \frac{3}{8}}$	$\boxed{B = \frac{-5}{12}}$	$\boxed{C = \frac{1}{24}}$

$$\therefore (1) \Rightarrow \frac{Y(z)}{z} = \frac{(3/8)}{z+1} + \frac{(-5/12)}{z+3} + \frac{(1/24)}{z-3}$$

$$Y(z) = \frac{3}{8}\left(\frac{z}{z+1}\right) - \frac{5}{12}\left(\frac{z}{z+3}\right) + \frac{1}{24}\left(\frac{z}{z-3}\right)$$

$$Z[y(n)] = \frac{3}{8}\left(\frac{z}{z+1}\right) - \frac{5}{12}\left(\frac{z}{z+3}\right) + \frac{1}{24}\left(\frac{z}{z-3}\right)$$

$$y(n) = \frac{3}{8}Z^{-1}\left[\frac{z}{z+1}\right] - \frac{5}{12}Z^{-1}\left[\frac{z}{z+3}\right] + \frac{1}{24}Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n + \frac{1}{24}(3)^n$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

15. Solve $y_{n+2} + y_n = n2^n$.

[A.U M/J 2012]

Solution : Given : $y_{n+2} + y_n = n2^n$

$$Z[y_{n+2}] + Z[y_n] = Z[2^n n]$$

$$\left[z^2 Y(z) - z^2 y(0) - zy(1)\right] + Y(z) = \frac{2z}{(z-2)^2} \quad \left[\because Z[na^n] = \frac{az}{(z-a)^2}\right]$$

$$(z^2+1)\ Y(z) - z^2 S - zT = \frac{2z}{(z-2)^2}$$

when $S = y(0)$, $T = y(1)$

$$(z^2+1)Y(z) = \frac{2z}{(z-2)^2} + z^2 S + zT$$

$$(z^2+1)Y(z) = \frac{2z}{(z-2)^2} + Sz^2 + Tz$$

$$Y(z) = \frac{2z}{(z-2)^2(z^2+1)} + \frac{Sz^2}{z^2+1} + \frac{Tz}{z^2+1}$$

$$\frac{Y(z)}{z} = \frac{2}{(z-2)^2(z^2+1)} + \frac{Sz}{z^2+1} + \frac{T}{z^2+1} \quad \text{... (1)}$$

Take $$\frac{2}{(z-2)^2(z^2+1)} = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{Cz+D}{z^2+1} \quad \text{... (2)}$$

$$2 = A(z-2)(z^2+1) + B(z^2+1) + (Cz+D)(z-2)^2$$

Put $z = 2$, we get

$$2 = 5B$$

$$\boxed{B = \frac{2}{5}}$$

equating z^3 on bothsides, we get

$$0 = A + C$$

$$A = -C$$

$$\text{(or) } C = -A \quad \dots (3)$$

put $z = 0$, we get

$$2 = -2A + B + 4D$$

$$2 = -2A + \frac{2}{5} + 4D$$

$$2 - \frac{2}{5} = -2A + 4D$$

$$\frac{8}{5} = -2A + 4D \quad \dots (4)$$

equating z^2 on both sides, we get

$$0 = -2A + B - 4C + D$$

$$0 = -2A + \frac{2}{5} + 4A + D$$

$$-\frac{2}{5} = 2A + D \quad \dots (5)$$

$$(4) + (5) \Rightarrow \quad 5D = \frac{6}{5}$$

$$\boxed{D = \frac{6}{25}}$$

$$(5) \quad \Rightarrow \quad 2A = \frac{-2}{5} - D$$

$$2A = -\frac{2}{5} - \frac{6}{25}$$

$$2A = -\frac{16}{25}$$

$$\boxed{A = -\frac{8}{25}}$$

$$(3) \quad \Rightarrow \quad C = -A = \frac{8}{25}$$

$$\boxed{C = \frac{8}{25}}$$

$$(2) \Rightarrow \frac{2}{(z-2)^2(z^2+1)} = -\frac{8}{25}\left(\frac{1}{z-2}\right) + \frac{2}{5}\frac{1}{(z-2)^2} + \frac{\frac{8}{25}z + \frac{6}{25}}{z^2+1}$$

$$\text{l)} \Rightarrow \frac{Y(z)}{z} = -\frac{8}{25}\left(\frac{1}{z-2}\right) + \frac{2}{5}\frac{1}{(z-2)^2} + \frac{8}{25}\left(\frac{z}{z^2+1}\right) + \frac{6}{25}\left(\frac{1}{z^2+1}\right) + S\left(\frac{z}{z^2+1}\right) + T\left(\frac{1}{z^2+1}\right)$$

$$\frac{(z)}{z} = -\frac{8}{25}\left(\frac{1}{z-2}\right) + \frac{2}{5}\frac{1}{(z-2)^2} + \left(\frac{8}{25} + S\right)\left(\frac{z}{z^2+1}\right) + \left(\frac{6}{25} + T\right)\left(\frac{1}{z^2+1}\right)$$

$$(z) = -\frac{8}{25}\left(\frac{z}{z-2}\right) + \frac{2}{5}\left[\frac{z}{(z-2)^2}\right] + \left(\frac{8}{25} + S\right)\left(\frac{z^2}{z^2+1}\right) + \left(\frac{6}{25} + T\right)\left(\frac{z}{z^2+1}\right)$$

$$y(n)] = -\frac{8}{25}\left(\frac{z}{z-2}\right) + \frac{2}{5}\left[\frac{z}{(z-2)^2}\right] + \left(\frac{8}{25} + S\right)\left(\frac{z^2}{z^2+1}\right) + \left(\frac{6}{25} + T\right)\left(\frac{z}{z^2+1}\right)$$

$$\text{n)} = -\frac{8}{25}Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{5}Z^{-1}\left[\frac{2z}{(z-2)^2}\right] + \left(\frac{8}{25} + S\right)Z^{-1}\left(\frac{z^2}{z^2+1}\right) + \left(\frac{6}{25} + T\right)Z^{-1}\left(\frac{z}{z^2+1}\right)$$

$$= -\frac{8}{25}(2)^n + \frac{1}{5}2^n n + E\cos\frac{n\pi}{2} + F\sin\frac{n\pi}{2}$$

$$\text{where } E = \frac{8}{25} + S,\ F = \frac{6}{25} + T$$

Formula : (1) $Z[a^n] = \dfrac{z}{z-a}$ (2) $Z[a^n n] = \dfrac{az}{(z-a)^2}$

(3) $Z\left[\cos\dfrac{n\pi}{2}\right] = \dfrac{z^2}{z^2+1}$ (4) $Z\left[\sin\dfrac{n\pi}{2}\right] = \dfrac{z}{z^2+1}$

16. **Using Z-transform, solve $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$ given that $y_0 = 3$ and $y_1 = -5$.**

[A.U. April, 2000] [A.U.T. CH N/D 2011]

Solution : Given : $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$

$$Z[y_{n+2}] + 4Z[y_{n+1}] - 5Z[y_n] = 24Z(n) - 8Z[1]$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] + 4[z Y(z) - zy(0)] - 5Y(z) = 24\frac{z}{(z-1)^2} - 8\frac{z}{z-1}$$

$$[z^2 + 4z - 5]\, Y(z) - 3z^2 + 5z - 12z = \frac{24z - 8z(z-1)}{(z-1)^2}$$

$$(z^2 + 4z - 5)\, Y(z) - 3z^2 - 7z = \frac{24z - 8z^2 + 8z}{(z-1)^2}$$

$$(z+5)(z-1)\, Y(z) = \frac{32z - 8z^2}{(z-1)^2} + 3z^2 + 7z$$

$$= \frac{32z - 8z^2 + (3z^2 + 7z)[z^2 - 2z + 1]}{(z-1)^2}$$

$$= \frac{32z - 8z^2 + 3z^4 - 6z^3 + 3z^2 + 7z^3 - 14z^2 + 7z}{(z-1)^2}$$

$$= \frac{3z^4 + z^3 - 19z^2 + 39z}{(z-1)^2}$$

$$= \frac{z[3z^3 + z^2 - 19z + 39]}{(z-1)^2}$$

$$Y(z) = \frac{z[3z^3 + z^2 - 19z + 39]}{(z-1)^3(z+5)}$$

$$\frac{Y(z)}{z} = \frac{3z^3 + z^2 - 19z + 39}{(z+5)(z-1)^3}$$

$$= \frac{A}{z+5} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3}$$

$$3z^3 + z^2 - 19z + 39 = A(z-1)^3 + B(z+5)(z-1)^2 + C(z-1)(z+5) + D(z+5)$$

put $z = -5$, we get

$$-375 + 25 + 95 + 39 = A(-6)^3$$
$$-216 = -216A$$
$$A = 1$$

put $z = 1$, we get

$$3 + 1 - 19 + 39 = D(6)$$
$$24 = D(6)$$
$$D = 4$$

equating z^3 on both sides, we get

$$3 = A + B$$
$$3 = 1 + B$$
$$B = 2$$

put $z = 0$, we get

$$0 + 0 - 0 + 39 = -A + 5B - 5C + 5D$$
$$39 = -1 + 10 - 5C + 20$$
$$39 = 29 - 5C$$
$$5C = 29 - 39$$
$$5C = -10$$
$$C = -2$$

$$\frac{Y(z)}{z} = \frac{1}{z+5} + \frac{2}{z-1} - \frac{2}{(z-1)^2} + \frac{4}{(z-1)^3}$$

$$Y(z) = \frac{z}{z+5} + 2\frac{z}{z-1} - 2\frac{z}{(z-1)^2} + 4\frac{z}{(z-1)^3}$$

$$y(n) = Z^{-1}\left[\frac{z}{z+5}\right] + 2Z^{-1}\left[\frac{z}{z-1}\right] - 2Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4Z^{-1}\left[\frac{z}{(z-1)^3}\right]$$

$$= (-5)^n + 2(1)^n - 2n + \frac{4}{2}Z^{-1}\left[\frac{2z}{(z-1)^3}\right]$$

$$= (-5)^n + 2 - 2n + 2n(n-1)$$

$$= (-5)^n + 2 - 2n + 2n^2 - 2n$$

$$= (-5)^n + 2n^2 - 4n + 2, \quad n = 0, 1, 2, \ldots$$

Formula : 1. $Z[a^n] = \dfrac{z}{z-a}$

2. $Z[n] = \dfrac{z}{(z-1)^2}$

3. $Z[n(n-1)] = \dfrac{2z}{(z-1)^3}$

17. Solve $y(n) - y(n-1) = u(n) + u(n-1)$ when $u(n) = n$, $u(n-1) = n-1$

Solution : Given : $y(n) - y(n-1) = u(n) + u(n-1)$

$$Z[y(n)] - Z[y(n-1)] = Z[u(n)] + Z[u(n-1)]$$

$$Y(z) - z^{-1}Y(z) = \frac{z}{z-1} + \frac{1}{z-1}$$

$$(1 - z^{-1})Y(z) = \frac{z}{z-1} + \frac{1}{z-1}$$

$$\left[1 - \frac{1}{z}\right]Y(z) = \frac{z+1}{z-1}$$

$$\left[\frac{z-1}{z}\right]Y(z) = \frac{z+1}{z-1}$$

$$Y(z) = \frac{(z+1)z}{(z-1)^2}$$

$$\frac{Y(z)}{z} = \frac{z+1}{(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} \quad \ldots (1)$$

Let $z + 1 = A(z-1) + B$

Put $z = 1$, we get

$2 = B$

i.e., $\boxed{B = 2}$

equation of coefficient of z on bothsides, we get

$\boxed{A = 1}$

$$\therefore (1) \Rightarrow \frac{Y(z)}{z} = \frac{1}{z-1} + \frac{2}{(z-1)^2}$$

$$Y(z) = \frac{z}{z-1} + 2\frac{z}{(z-1)^2}$$

$$Z[y(n)] = \frac{z}{z-1} + 2\frac{z}{(z-1)^2}$$

$$y(n) = Z^{-1}\left[\frac{z}{z-1}\right] + 2Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= (1)^n + 2(n)$$

$$= 2n + 1$$

Formula : $Z[a^n] = \dfrac{z}{z-a}$

18. Find the response of the system : $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, with $y_0 = 0$, $y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, \ldots$ by Z-transform method. [A.U N/D 2018-A R-17]

Solution : Given : $y_{n+2} - 5y_{n+1} + 6y_n = u_n$

Taking Z-transform of both sides of the given equation we get

$$Z[y_{n+2}] - 5Z[y_{n+1}] + 6Z[y_n] = Z[u_n]$$

$$[z^2Y(z) - z^2y(0) - zy(1)] - 5[zY(z) - zy(0)] + 6Y(z) = \frac{z}{z-1}$$

$$(z^2 - 5z + 6)\, Y(z) - z = \frac{z}{z-1}$$

$$(z^2 - 5z + 6)\, Y(z) = \frac{z}{z-1} + z$$

$$(z-2)(z-3)\, Y(z) = \frac{z + z(z-1)}{z-1}$$

$$(z-2)(z-3)\, Y(z) = \frac{z^2}{z-1}$$

$$Y(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3} \qquad \dots (1)$$

Let $z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$

Put $z = 1$, we get	put $z = 2$, we get	put $z = 3$
$1 = 2A$	$2 = -B$	$3 = 2C$
$\boxed{A = \frac{1}{2}}$	$\boxed{B = -2}$	$\boxed{C = \frac{3}{2}}$

$$(1) \Rightarrow \frac{Y(z)}{z} = \frac{(1/2)}{z-1} - \frac{2}{z-2} + \frac{(3/2)}{z-3}$$

$$Y(z) = \frac{1}{2}\left(\frac{z}{z-1}\right) - 2\left(\frac{z}{z-2}\right) + \frac{3}{2}\left(\frac{z}{z-3}\right)$$

$$Z[y(n)] = \frac{1}{2}\left(\frac{z}{z-1}\right) - 2\left(\frac{z}{z-2}\right) + \frac{3}{2}\left(\frac{z}{z-3}\right)$$

$$y(n) = \frac{1}{2} Z^{-1}\left[\frac{z}{z-1}\right] - 2Z^{-1}\left[\frac{z}{z-2}\right] + \frac{3}{2} Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= \frac{1}{2}(1)^n - 2(2)^n + \frac{3}{2}(3)^n$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

19. Solve the simultaneous difference equations $x_{n+1} = 5x_{n+7}$, $y_{n+1} = x_n + 2y_n$ given that $x_0 = 0$, and $y_0 = 1$.

Solution : Given : (i) $x_{n+1} = 5x_n + 7$

$$Z[x_{n+1}] = 5Z[x_n] + Z[7]$$

$$zX(z) - zx(0) = 5X(z) + 7Z[1]$$

$$zX(z) - 5X(z) = 7\left[\frac{z}{z-1}\right]$$

$$(z-5)X(z) = 7\left[\frac{z}{z-1}\right]$$

$$X(z) = 7\frac{z}{(z-1)(z-5)}$$

$$\frac{X(z)}{z} = \frac{7}{(z-1)(z-5)} = \frac{A}{(z-1)} + \frac{B}{(z-5)} \qquad \dots (1)$$

$$7 = A(z-5) + B(z-1)$$

Put $z = 1$, we get	put $z = 5$, we get
$7 = -4A$	$7 = 4B$
$\boxed{A = -\frac{7}{4}}$	$\boxed{B = \frac{7}{4}}$

$$(1) \Rightarrow \frac{X(z)}{z} = \frac{(-7/4)}{z-1} + \frac{7/4}{z-5}$$

$$X(z) = -\frac{7}{4}\frac{z}{z-1} + \frac{7}{4}\frac{z}{z-5}$$

$$Z[x(n)] = -\frac{7}{4}\frac{z}{z-1} + \frac{7}{4}\frac{z}{z-5}$$

$$x(n) = -\frac{7}{4}Z^{-1}\left[\frac{z}{z-1}\right] + \frac{7}{4}Z^{-1}\left[\frac{z}{z-5}\right]$$

$$= -\frac{7}{4}(1)^n + \frac{7}{4}(5)^n$$

$$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$$

(ii) $$y_{n+1} = x_n + 2y_n$$

$$Z[y_{n+1}] = Z[x_n] + 2Z[y_n]$$

$$zY(z) - zy(0) = X(z) + 2Y(z)$$

$$(z-2)Y(z) - z = X(z)$$

$$(z-2)Y(z) = X(z) + z$$

$$= \frac{7z}{(z-1)(z-5)} + z$$

$$Y(z) = \frac{7z}{(z-1)(z-2)(z-5)} + \frac{z}{z-2}$$

$$\frac{Y(z)}{z} = \frac{7}{(z-1)(z-2)(z-5)} + \frac{1}{z-2} \quad \ldots (2)$$

Take :

$$\frac{7}{(z-1)(z-2)(z-5)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-5} \quad \ldots (3)$$

$$7 = A(z-2)(z-5) + B(z-1)(z-5) + C(z-1)(z-2)$$

Put $z = 1$, we get	put $z = 2$, we get	put $z = 5$, we get
$7 = 4A$	$7 = -3B$	$7 = 12C$
$\boxed{A = \frac{7}{4}}$	$\boxed{B = -\frac{7}{3}}$	$\boxed{C = \frac{7}{12}}$

$$(3) \Rightarrow \frac{7}{(z-1)(z-2)(z-5)} = \frac{(7/4)}{z-1} + \frac{(-7/3)}{z-2} + \frac{(7/12)}{z-5}$$

$$(2) \Rightarrow \frac{Y(z)}{z} = \frac{7}{4}\frac{1}{z-1} - \frac{7}{3}\frac{1}{z-2} + \frac{7}{12}\frac{1}{z-5} + \frac{1}{z-2}$$

$$= \frac{7}{4}\frac{1}{z-1} - \frac{4}{3}\frac{1}{z-2} + \frac{7}{12}\frac{1}{z-5}$$

$$Y(z) = \frac{7}{4}\frac{z}{z-1} - \frac{4}{3}\frac{z}{z-2} + \frac{7}{12}\frac{z}{z-5}$$

$$Z[y(n)] = \frac{7}{4}\frac{z}{z-1} - \frac{4}{3}\frac{z}{z-2} + \frac{7}{12}\frac{z}{z-5}$$

$$y(n) = \frac{7}{4}Z^{-1}\left[\frac{z}{z-1}\right] - \frac{4}{3}Z^{-1}\left[\frac{z}{z-2}\right] + \frac{7}{12}Z^{-1}\left[\frac{z}{z-5}\right]$$

$$= \frac{7}{4}(1)^n - \frac{4}{3}(2)^n + \frac{7}{12}(5)^n$$

$\boxed{\text{Formula : } Z[a^n] = \frac{z}{z-a}}$

20. Solve the system using Z-transform

$$x_{n+1} = 7x_n + 10y_n$$

$$y_{n+1} = x_n + 4y_n$$ **given that** $x_0 = 3, y_0 = 2.$ **[A.U. Nov/Dec. 1996]**

Solution : Given : $$x_{n+1} = 7x_n + 10y_n$$

$$Z[x_{n+1}] = 7Z[x_n] + 10Z[y_n]$$

$$zX(z) - zx(0) = 7X(z) + 10Y(z)$$

$$zX(z) - 3z = 7X(z) + 10Y(z)$$

$$(z-7)X(z) - 3z = 10Y(z) \quad \ldots (1)$$

$$y_{n+1} = x_n + 4y_n$$

$$Z[y_{n+1}] = Z[x_n] + 4Z[y_n]$$

$$zY(z) - zy(0) = X(z) + 4Y(z)$$

$$zY(z) - 2z = X(z) + 4Y(z)$$

$$(z-4)Y(z) - 2z = X(z) \quad \ldots (2)$$

$$(1) \Rightarrow (z-7)X(z) - 10Y(z) = 3z \quad \ldots (3)$$

$$(2) \Rightarrow X(z) - (z-4)Y(z) = -2z \quad \ldots (4)$$

$$(3) \Rightarrow (z-7)X(z) - 10Y(z) = 3z$$

$$(4) \times (z-7) \quad (z-7)X(z) - (z-4)(z-7)Y(z) = -2z(z-7)$$

$$(-) \quad [(z-4)(z-7) - 10]Y(z) = 2z(z-7) + 3z$$

$$[z^2 - 11z + 28 - 10]Y(z) = 2z^2 - 14z + 3z$$

$$[z^2 - 11z + 18]Y(z) = 2z^2 - 11z$$

$$Y(z) = \frac{2z^2 - 11z}{(z-2)(z-9)} = \frac{z[2z-11]}{(z-2)(z-9)}$$

$$Y(z)z^{n-1} = \frac{z^n(2z-11)}{(z-2)(z-9)}$$

$z = 2$ is a simple pole

$z = 9$ is a simple pole

$$\underset{z=2}{\text{Res}}\, Y(z)\, z^{n-1} = \underset{z \to 2}{Lt}\, (z-2)\, \frac{z^n\,(2z-11)}{(z-2)\,(z-9)}$$

$$= \underset{z \to 2}{Lt}\, \frac{z^n\,(2z-11)}{z-9}$$

$$= \frac{2^n\,(-7)}{(-7)} = 2^n$$

$$\underset{z=9}{Res}\, Y(z)\, z^{n-1} = \underset{z \to 9}{Lt}\, \frac{(z-9)\, z^n\,(2z-11)}{(z-2)\,(z-9)}$$

$$= \underset{z \to 9}{Lt}\, \frac{z^n\,(2z-11)}{(z-2)}$$

$$= \frac{9^n\,(7)}{7} = 9^n$$

$y(n)$ = sum of the residues

$$= 2^n + 9^n$$

(3) $\times (z-4) \Rightarrow (z-7)(z-4)\, X(z) - 10(z-4)\, Y(z) = 3(z-4)\, z$

(4) $\times$ 10 $\Rightarrow 10\, X(z) \qquad - 10(z-4)\, Y(z) = -20z$

(–) $\quad [(z-7)(z-4) - 10]\, X(z) = 3(z-4)\, z + 20z$

$$[z^2 - 11z + 28 - 10]\, X(z) = 3z^2 - 12z + 20z$$

$$(z^2 - 11z + 18)\, X(z) = 3z^2 + 8z$$

$$X(z) = \frac{z\,[3z+8]}{(z-2)\,(z-9)}$$

$$X(z)\, z^{n-1} = \frac{z^n\,[3z+8]}{(z-2)\,(z-9)}$$

$z = 2$ is a simple pole

$z = 9$ is a simple pole

$$\underset{z=2}{\text{Res}}\, X(z)\, z^{n-1} = \underset{z \to 2}{Lt}\, (z-2)\, \frac{z^n\,[3z+8]}{(z-2)\,(z-9)}$$

$$= \underset{z \to 2}{Lt}\, \frac{z^n\,[3z+8]}{z-9}$$

$$= \frac{2^n\,[14]}{-7}$$

$$= -2\,(2^n)$$

$$= -2^{n+1}$$

$$\underset{z=9}{Res}\, X(z)\, z^{n-1} = \underset{z \to 9}{Lt}\, (z-9)\, \frac{z^n\,[3z+8]}{(z-2)\,(z-9)}$$

$$= \underset{z \to 9}{Lt}\, \frac{z^n\,[3z+8]}{z-2}$$

$$= \frac{9^n\,[27+8]}{7}$$

$$= 5(9^n)$$

$x(n)$ = sum of the residues

$$= -2^{n+1} + 5\,(9^n)$$

EXERCISE - 5.1

I. Find the Z-transform of the following sequences.

1. $n\,2^n$ [Ans. $\dfrac{2z}{(z-2)^2}$]

2. $3^n \cos\dfrac{n\pi}{2}$ [Ans. $\dfrac{z^2}{z^2+9}$]

3. $2^n \sin\dfrac{n\pi}{2}$ [Ans. $\dfrac{2z}{z^2+4}$]

4. $u(n-2)$ [Ans. $\dfrac{1}{z(z-1)}$]

5. $2^n\, u(n-1)$ [Ans. $\dfrac{2}{z-2}$]

6. $u(n-k)$ [Ans. $z^{-k}\dfrac{z}{z-1}$]

7. $2^n\, u(n-k)$ [Ans. $\dfrac{\left(\frac{z}{2}\right)^{-k}\left(\frac{z}{2}\right)}{\left(\frac{z}{2}\right)^{-1}}$]

8. $2^n\, \delta(n-1)$ [Ans. $\dfrac{2}{z}$]

9. $3\,2^n + 4(-1)^n$ [Ans. $z\left[\dfrac{3}{z-2}+\dfrac{4}{z+1}\right]$]

10. $\sin\dfrac{n\pi}{3}$ [Ans. $\dfrac{\sqrt{3}\,z/2}{z^2-z+1}$]

11. $\cos\dfrac{n\pi}{4}$ [Ans. $\dfrac{z(z-1/\sqrt{2})}{z^2-z\sqrt{2}+1}$]

12. $a^n\, U(n) + b^n\, U(-n-1)$ [Ans. $\dfrac{z}{z-a}+\dfrac{z}{z-b}$]

13. $2^{n-1} + \dfrac{1}{2}4^n - 3^n$ [Ans. $\dfrac{z(z^2-6z+6)}{2(2-z)(3-z)(4-z)}$]

14. $a^n \sinh an$ [Ans. $\dfrac{az\sinh a}{z^2-2z\cosh a+a^2}$]

15. $2^n \cosh 5n$ [Ans. $\dfrac{z(z-2\cosh 5)}{z^2-2z\cosh 5+4}$]

16. $n(n-1)\,2^n$ [Ans. $\dfrac{8z}{(z-2)^3}$]

17. nC_3 [Ans. $\dfrac{z}{(z-1)^4}$]

18. $\dfrac{1}{n(n+1)(n+2)}$ [Ans. $\dfrac{7}{2}\left[(1-z)^2\log\left(\dfrac{z}{z-1}\right)-z\right]$]

19. $\dfrac{1}{n!}(a^n + a^{-n})$ [Ans. $e^{a/z} + e^{1/az}$]

20. $n\sin n\theta$ [Ans. $\dfrac{(z^3-z)\sin\theta}{(z^2-2z\cos\theta+1)^2}$]

21. $na^n \cos n\theta$ [Ans. $\dfrac{a(z^3\cos\theta-2az^2+a^2z\cos\theta)}{(z^2-2az\cos\theta+a^2)^2}$]

22. $na^n \sin n\theta$ [Ans. $\dfrac{az(z^2-a^2)\sin\theta}{(z^2-2az\cos\theta+a^2)^2}$]

23. $r^n\cos(n\theta+\phi)$ [Ans. $\dfrac{z^2\cos\phi-zr\cos(\theta-\phi)}{z^2-2zr\cos\theta+r^2}$]

24. $n(n-1)(n-2)$ [Ans. $\dfrac{6z}{(z-1)^4}$]

25. $\dfrac{n-3}{n(n-1)}$ [Ans. $\dfrac{4-3z}{z}\log\left(\dfrac{z}{z-1}\right)$]

26. $\left(\dfrac{1}{5}\right)^n u(n)$ [Ans. $\dfrac{5z}{5z-1}$]

27. $5^n u(n)$ [Ans. $\frac{z}{z-5}$]

28. $n^2 u(n)$ [Ans. $\frac{z(z+1)}{(z-1)^3}$]

29. $a + bn + cn^2$ [Ans. $\frac{az}{z-1} + \frac{bz}{(z-1)^2} + \frac{cz(z+1)}{(z-1)^3}$]

30. $2n + 5\sin\frac{n\pi}{4} - 3a^4$ [Ans. $\frac{2z}{(z-1)^2} + \frac{z/\sqrt{2}}{z^2 - \sqrt{2}z + 1} + \frac{z}{z-1}$]

31. $(n-1)^2$ [Ans. $\frac{z^3 - 3z^2 + 4z}{(z-1)^3}$]

32. $(n+1)^2$ [Ans. $\frac{z^3 + z^2}{(z-1)^3}$]

33. $\sin(n+1)\theta$ [Ans. $\frac{z^2\sin\theta}{z^2 - 2z\cos\theta + 1}$]

34. $\cos(n+1)\theta$ [Ans. $\frac{z^2\cos\theta - z}{z^2 - 2z\cos\theta + 1}$]

35. $\sin(3n+5)$ [Ans. $\frac{z^2\sin 5 - z\sin 2}{z^2 - 2z\cos 3 + 1}$]

36. $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$ [Ans. $\frac{z(z-1)}{\sqrt{2}(z^2+1)}$]

37. $e^{-an}\cosh\theta$ [Ans. $\frac{ze^a(ze^a - \cos\theta)}{z^2e^{2a} - 2ze^a\cos\theta + 1}$]

38. $e^{-an}\sin n\theta$ [Ans. $\frac{ze^a\sin\theta}{z^2e^{2a} - 2ze^a\cos\theta + 1}$]

39. $\cos\left(\frac{n\pi}{8} + \theta\right)$ [Ans. $\frac{z^2\cos\theta - z\cos\left(\frac{\pi}{8} - \theta\right)}{z^2 - 2z\cos\frac{\pi}{8} + 1}$]

40. $\cosh\left[\frac{n\pi}{2} + \theta\right]$ [Ans. $\frac{z^2\cosh\theta - z\cosh\left(\frac{\pi}{2} - \theta\right)}{z^2 - 2z\cosh\frac{\pi}{2} + 1}$]

41. $\left(\frac{1}{2}\right)^n u(n)$ [Ans. $\frac{2z}{2z-1}$]

EXERCISE - 5.2

I. **Find the inverse Z-transform of the following sequence by any method.**

1. $\frac{z}{(z-1)(z-2)}$ [Ans. $(2^n - 1)u(n)$]

2. $\frac{z^2}{(z-a)(z-b)}$ [Ans. $\frac{1}{a-b}[a^{n+1} - b^{n+1}]$]

3. $\frac{5z}{(2-z)(3z-1)}$ [Ans. $\left(\frac{1}{3}\right)^n - 2^n$] **[A.U A/M 2019 R-8]**

4. $\frac{z}{z^2 + 11z + 30}$ [Ans. $(-5)^n - (-6)^n$]

5. $\frac{z^2 + z}{(z-1)^3}$ [Ans. $\frac{t^2}{T^2}$]

6. $\frac{z^2 + 2z}{(z-1)(z-2)(z-3)}$ [Ans. $\frac{3}{2} - 4(2^n) + \frac{5}{2}3^n$]

7. $\frac{z^2 - 3}{(z+2)(z^2+1)}$ [Ans. $-\frac{1}{10}(-2)^n + \frac{1}{10}\cos\frac{n\pi}{2} - \frac{1}{20}\sin\frac{n\pi}{2}$]

8. $\frac{2z^2 + 4z}{(z-2)^3}$ [Ans. $n^2 2^n$]

9. $\frac{z^{-2}}{(1+z^{-1})^2(1-z^{-1})}$ [Ans. $\frac{1}{4} - \frac{1}{4}(-1)^n + \frac{n}{2}(-1)^n$]

10. $\dfrac{z^2 - 3z}{z^2 - 3z + 10}$ [Ans. $\frac{2}{7}5^n + \frac{5}{7}(-2)^n$]

11. $\dfrac{4z^2 - 2z}{(z-1)(z-2)^2}$ [Ans. $2 + (3n-2)\,2^n$]

12. $\dfrac{z(z^2-1)}{(z^2+1)^2}$ [Ans. $\frac{n}{2}[i^{n-1} - (-1)^{n-1}]$]

13. $\dfrac{z(3z^2 - 6z + 4)}{(z-1)^2(z-2)}$ [Ans. $4\ 2^n - n\,(1)^n - (1)^n$]

14. $\dfrac{z^2}{z^2+4}$ [Ans. $2^n \cos\frac{n\pi}{2}$]

EXERCISE - 5.3

Solve the following equations, using Z-transform.

1. $y(n+2) - y(n) = 2^n$ given that $y(0) = 0$ and $y(1) = 1$

   [Ans. $y(n) = \frac{1}{3}[2^n - (-1)^n]$]

2. $y(n+2) - 4y(n) = 2^n$ given that $y(0) = 0$ and $y(1) = 0$

   [Ans. $y(n) = \frac{1}{16}[(-2)^n - 2^n + n\,2^{n+1}]$]

3. $y(n+2) + 2y(n+1) + y(n) = n$ given that $y(0) = 0$ and $y(1) = 0$

   [Ans. $y(n) = \frac{1}{4}(n-1)[1 + (-1)^{n-1}]$]

4. $y(n+2) - 4y(n+1) + 3y(n) = 2^n n^2$ given that $y(0) = 0$ and $y(1) = 0$

   [Ans. $y(n) = 3 + 5\ 3^n - 2^n(n^2 + 8)$]

5. $y(n+2) - 4y(n+1) + 4y(n) = \pi$ given that $y(0) = 0$ and $y(1) = 0$

   [Ans. $y(n) = \pi[1 + (n-2)\,2^{n-1}]$]

6. $x(n+2) - y(n) = 1$ and $y(n+2) - x(n) = 1$ given that $x(0) = 0$ and $y(0) = -1$

   [Ans. $x(n) = \frac{1}{2}\left[1 - \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$,

   $y(n) = \frac{1}{2}\left[\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2} - 1\right]$]

7. $y(n+2) - 5y(n+1) + 6y(n) = n(n-1)$ given that $y(0) = 0$ and $y(1) = 0$ [Ans. $y(n) = \frac{1}{2}3^n - 2^{n+1} + \frac{3}{2} + n$]

8. $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$ [Ans. $y_n = c_1 + c_2\,3^n + \frac{5^n}{8}$]

9. $u_{n+2} - 5u_{n+1} - 6u_n = 2^n$ [Ans. $u_n = c_1(-1)^n + c_2\,6^n - \frac{2^n}{12}$]

10. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$

    [Ans. $y_n = (c_1 + c_2 n)\,3^n + \frac{1}{2}n(n-1)\,3^{n-2}$]

11. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \ (n \geq 0),\ y_0 = 0$

    [Ans. $y_n = 2\left[\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right]$]

12. $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$

    [Ans. $y_n = c_1\,4^n + \left(c_2 - \frac{n}{4}\right)^{2n} + 2n - \frac{8}{3}$]

13. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $y_0 = 2,\ y_1 = 1$

    [Ans. $u_k = 1 - 2k + 2^k$] **[A.U A/M 2019 R-8]**

14. $u_{n+2} + 4u_{n+1} - 3u_n = 3^n$ with $u_n = 0,\ u_1 = 1$

    [Ans. $u_n = \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n$]

15. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$

[Ans. $y_n = \frac{1}{25}\left[2^n - (-3)^n + \frac{5}{3}n(-3)^n\right]$]

16. Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$ with $y_0 = 0, y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3...$ by Z-transform method.

[Ans. $y_n = \frac{1}{2} - 2\,2^n + \frac{3}{2}3^n$]

17. Solve the difference equation $y_n + \frac{1}{4}y_{n-1} = u_n + \frac{1}{3}u_{n-1}$ where u_n is a unit step sequence.

[Ans. $y_n = \frac{1}{12}\left[-\frac{1}{4}\right]^{n-1}$]

18. $u_{n+2} - 2u_{n+1} + u_n = 3^n + 5$ **[A.U A/M 2019 R-17]**

[Ans. $u_n = \frac{1}{2}n(n-1)(n+3) + c_0 + c_1 n$]

19. Find the response of the system given by $y_n + 3y_{(n-1)} = u_n$ where u_n is a unit step sequence and $y(-1) = 1$.

[Ans. $y_n = \left[\frac{1}{4} - \frac{9}{4}(-3)^n\right]\mu(n)$]

20. Find the impulse response of a system described by $y_{n+1} + 2y_n = 8n$; $y_0 = 0$.

[Ans. $y_n = (-2)^{n-1}\ (n \geq 1)$]

21. $y_{n+2} - 2\cos\alpha\, y_{n+1} + 2y_n = 0$ given $y_0 = 1, y_1 = \cos\alpha$.

[Ans. $y_n = \cos n\alpha$]

22. $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = 0$ if $y_0 = 0, y_1 = 1$

[Ans. $y_n = \frac{\sin n\alpha}{\sin\alpha}$]

23. $y_{n+2} + y_n = na^n$

[Ans. $y_n = A\cos\frac{\pi n}{2} + B\sin\frac{n\pi}{2} + \frac{a^n}{1+a^2}\left[n - \frac{2a^2}{1+a^2}\right]$]

24. $f(n) + 3f(n-1) - 4f(n-2) = 0, n \geq 2$ given that $f(0) = 3$ and $f(1) = -2$.

[Ans. $f(n) = \frac{1}{5}4^n + \frac{14}{5}(-1)^n$]

25. $y_{n+2} + 2y_{n+1} + y_n = n$ if $y_0 = y_1 = 0$

[Ans. $y_n = \frac{n-1}{4}[1 - (-1)^n]$] **[A.U N/D 2013]**

26. $a_{n+1} = a_n + \alpha b_n, b_{n+1} = b_n + \alpha a_n$.

Find $\frac{a_n}{b_n}$ as a function of α and n.

[Ans. $\tan(n\tan^{-1}\alpha)$]

27. $x_{n+1} = 7x_n + 10y_n$

$y_{n+1} = x_n + 4y_n$ given $x_0 = 3, y_0 = 2$

[Ans. $x_n = 5(9^n) - 2(2^n)$; $y_n = 9^n + 2^n$]

28. $x(n+2) - 3x(n+1) - 10x(n) = 0$, given $x(0) = 1, x(1) = 0$

[Ans. $x(n) = \frac{2}{7}5^n + \frac{5}{7}(-2)^n, n = 0, 1, 2, ...$]

[A.U M/J 2013] [A.U M/J 2014]

29. $x(k+2) + 2x(k+1) + x(c) = u(k)$ where $x(0) = 0, x(1) = 0$ and $u(k) = k$ for $k = 0, 1, 2, ...$

[Ans. $x(k) = \frac{k+1}{4} + \frac{(-1)^k}{4}(1-k), k = 0, 1, 2, ...$]

30. $y_{n+2} - 4y_{n+1} + 4y_n = \pi$ if $y_0 = y_1 = 0$

[Ans. $y_n = \pi\left[1 + 2^n\left(\frac{n}{2} - 1\right)\right]$]

Miscellaneous Problems

Find the bilateral Z - transform of the following :

1. $a^{|n|}$

2. $f(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

3. $f(n) = \begin{cases} \frac{a^n}{n!}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$

4. $\left(\frac{1}{2}\right)^{|k|}$

5. $x(n) = \{1, 2, 3, 4, 0, 6, 7\}$

1. Find $Z\{a^{|n|}\}$

Solution : We know that, $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \; z^{-n}$

$$Z\{a^{|n|}\} = \sum_{n=-\infty}^{\infty} a^{|n|} \; z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= [\ldots + a^3 z^3 + a^2 z^2 + az] + Z[a^n]$$

$$= (az)[\ldots + a^2 z^2 + az + 1] + Z[a^n]$$

$$= (az)[1 - az]^{-1} + Z[a^n]$$

$$= \frac{az}{1 - az} + \frac{z}{z - a}$$

$$= \frac{z - a^2 z}{(1 - az)(z - a)}$$

Formula : (1) $(1-x)^{-1} = 1 + x + x^2 + \ldots$

(2) $Z[a^n] = \frac{z}{z - a}$

2. Find $Z\{f(n)\}$ if $f(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Solution :

We know that $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} n z^{-n}$$

$$= Z[n]$$

$$= \frac{z}{(z - 1)^2}$$

Formula :

$Z(n) = \frac{z}{(z - 1)^2}$

3. Find $Z[f(n)]$ if $f(n) = \begin{cases} \frac{a^n}{n!}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Solution :

We know that, $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \; z^{-n}$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!}$$

$$= 1 + \frac{az^{-1}}{\lfloor 1} + \frac{(az^{-1})^2}{\lfloor 2} + \ldots$$

$$= e^{az^{-1}}$$

$$= e^{a/z}$$

Formula :

$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots$

Here, $x = az^{-1}$

4. Find the Z-transform of $\left\{\left(\frac{1}{2}\right)^{|k|}\right\}$

Solution : We know that, $Z\{a^{|k|}\} = \frac{z - a^2 z}{(1 - az)(z - a)}$

Here, $a = \frac{1}{2}$

$$\therefore Z\left[\left(\frac{1}{2}\right)^{|k|}\right] = \frac{z - \frac{1}{4}z}{\left(1 - \frac{1}{2}z\right)\left(z - \frac{1}{2}\right)} = \frac{\frac{3z}{4}}{\left(\frac{2 - z}{2}\right)\left(\frac{2z - 1}{2}\right)}$$

$$= \frac{3z}{(2 - z)(2z - 1)} = \frac{3z}{5z - 2 - 2z^2}$$

5. Find the Z-transform of the sequence $x(n) = \{1, 2, 3, 4, 0, 6, 7\}$

Solution : Given : $x(n) = \{1, 2, 3, 4, 0, 6, 7\}$

i.e., $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, $x(3) = 4$

$x(4) = 0$, $x(5) = 6$, $x(6) = 7$

The Z-transform is

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

As $x(n)$ has values from $n = 0$ to $n = 6$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 0z^{-4} + 6z^{-5} + 7z^{-6}$$

$$X(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{6}{z^5} + \frac{7}{z^6}$$

$X(z)$ has finite values except at $z = 0$.

At $z = 0$, $X(z)$ is infinite.

Hence, $X(z)$ is convergent for all values of z, except $z = 0$

$\therefore$ The region of convergence (ROC) is the entire z plane except $z=0$.

6. Find $Z[e^t \sin 2t]$ **[A.U N/D 2015 R-8]**

Sol. $Z[e^t \sin 2t] = Z\left[\sin 2t\right]_{z \to ze^{-T}}$

$$= \frac{ze^{-T} \sin 2T}{z^2 e^{-2T} - 2ze^{-T} \cos 2T + 1}$$

7. If $Z[f(x)] = \dfrac{z^2}{z^2+1}$, then find $f(0)$, using initial value theorem.

[A.U A/M 2019 R-13]

Solution : By initial value theorem

$$f(0) = \lim_{z \to \infty} F(z)$$

$$= \lim_{z \to \infty} \frac{z^2}{z^2+1} = \frac{\infty}{\infty}$$

$$= \lim_{z \to \infty} \frac{2z}{2z} \text{ by L'Hospital's rule.}$$

$$= \lim_{z \to \infty} 1 = 1$$

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5.6 CONVERGENCE OF Z-TRANSFORMS

Def :- Region of Convergence (ROC)

The region of the Z-plane in which

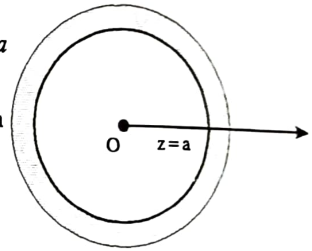
$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \text{ where}$$

u_n represents a number in the sequence for n is an integer converges absolutely is known as the region of convergence (ROC) of $U(z)$

Note : (1) The region of convergence of a one sided Z transform of a right-sided sequence

i.e., $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$ is $|z| > a$

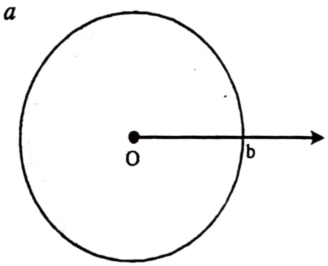
i.e., the exterior of the circle with centre at origin and of radius a say $|z| > |a|$



Note : (2) The region of convergence of a one sided Z transform of a left-sided sequence

i.e., $U(z) = \sum_{n=-\infty}^{0} u_n z^{-n}$ is $|z| < a$

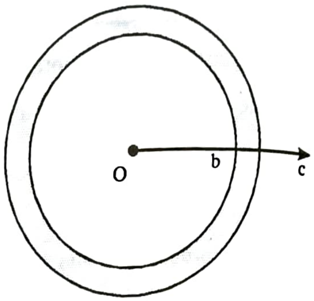
For a left-handed sequence, the ROC is always inside any prescribed contour $|z| < |b|$



Note : (3) The region of convergence of two sided Z transform defined by

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \text{ is the}$$

annulus region $|b| < |z| < |c|$



WORKED EXAMPLES

(1) Find the Z-transform and region of convergence of

$$u(n) = \begin{cases} 4^n, & n < 0 \\ 2^n, & n \geq 0 \end{cases}$$

Solution :

We know that, $Z[u(n)] = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$

$$= \sum_{n=-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$$

Putting $-n = m$ in the first series, we get

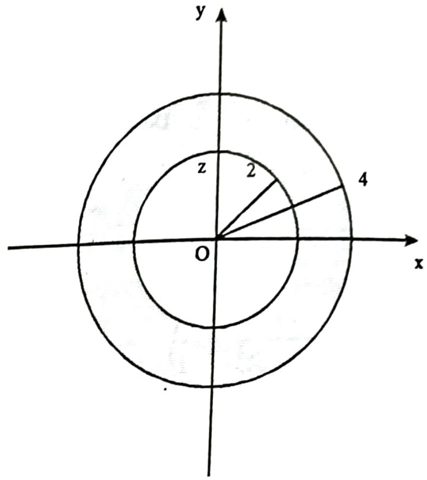
$$Z[u(n)] = \sum_{1}^{\infty} 4^{-m} z^m + \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$= \left\{\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \ldots\right\} + \left\{1 + \frac{2}{z} + \frac{2^2}{z^2} + \ldots\right\}$$

$$= \frac{z}{4}\left\{1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + \ldots\right\} + \left\{1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \ldots\right\} \quad \ldots (1)$$

$$= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(2/z)}$$

$$= \frac{z}{4-z} + \frac{z}{z-2} = \frac{2z}{(4-z)(z-2)}$$



Now the two series in (1) being G.P. will be convergent if $|z/4| < 1$ and $|2/z| < 1$ i.e., if $|z| < 4$ and $2 < |z|$.

i.e., $2 < z < 4$.

Hence $Z[u(n)]$ is convergent if z lies between the annulus as shown shaded in figure. Hence, ROC is $2 < z < 4$.

(2) Find the Z-transform and region of convergence of

$$u(n) = {}^nC_k, \ n \geq k$$

Solution :

We know that, $Z[u(n)] = \sum_{-\infty}^{\infty} {}^nC_k z^{-n} = \sum_{n=k}^{\infty} {}^nC_k 2^n z^{-n}$

We replace n by $k + r$

$$\therefore Z[u(n)] = \sum_{r=0}^{\infty} {}^{k+r}C_k z^{-(k+r)} = z^{-k} \sum_{r=0}^{\infty} {}^{k+r}C_r z^{-r}$$

$$= z^{-k}\left[1 + {}^{k+1}C_1 z^{-1} + {}^{k+1}C_2 z^{-2} + \ldots\right]$$

$$= z^{-k}\left(1 - \frac{1}{z}\right)^{-(k+1)}$$

This series is convergence for $|1/z| < 1$ i.e., for $|z| > 1$.

Hence ROC is $|z| > 1$.

(3) **Find $Z^{-1}\{(z-5)^{-3}\}$ when $|z| > 5$. Determine the region of convergence.**

Solution :

Consider

$$(z-5)^{-1} = \frac{1}{(z-5)^3} = \frac{1}{z^3}\,\frac{1}{\left(1-\frac{5}{z}\right)^3}$$

Expanding by Binomial series which is valid when $\left|\frac{5}{z}\right| < 1$ or $|z| > 5$, we have

$$\frac{1}{(z-5)^3} = \frac{1}{z^3}\left[1 + 3\left(\frac{5}{z}\right) + \frac{3.4}{1.2}\cdot\left(\frac{5}{z}\right)^2 + \frac{3.4.5}{1.2.3}\left(\frac{5}{z}\right)^3 + \frac{3.4.5.6}{1.2.3.4}\left(\frac{5}{z}\right)^4 + \ldots\right]$$

$$\frac{1}{(z-5)^3} = \frac{1}{2}\cdot\frac{1}{z^3}\left[1.2\left(\frac{5}{z}\right)^0 + 2.3\left(\frac{5}{3}\right)^3 + 3.4\left(\frac{5}{z}\right)^2 + 4.5\left(\frac{5}{z}\right)^3 + 5.6\left(\frac{5}{z}\right)^4 + \ldots\right]$$

$$= \frac{1}{2}\sum_{m=0}^{\infty}(m+1)(m+2)\,5^m\,.\,z^{-m-3};\ \text{put } m+3 = n$$

$$U(z) = (z-5)^{-3} = \frac{1}{2}\sum_{n=3}^{\infty}(n-1)(n-2)\,5^{n-3}z^{-n}$$

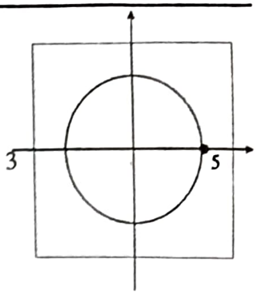
$$= \sum_{n=0}^{\infty} u_n z^{-n}$$

Taking inverse Z-transform

$$Z^{-1}[U(z)] = Z^{-1}\left[(z-5)^{-3}\right] = u_n$$

$$= \frac{1}{2}(n-1)(n-2)\,5^{n-3} \quad \text{for } n \geq 3$$

$$= 0 \quad \text{for } n < 3$$



The region of convergence is the exterior of the circle $|z| = 5$ i.e., with centre at origin and of radius 5.

(4) **Find the Z-transform and the radius of convergence of $u(n) = 2^n,\ n < 0$**

Solution :

Let $u(n) = 0$ for $n \geq 0$ we have

$$Z[u(n)] = \sum_{-\infty}^{\infty} u(n)\,z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

$$= \sum_{m=1}^{\infty} 2^{-m} z^m \quad \text{where } m = -n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots + \infty$$

$$= \frac{z}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \ldots\infty\right]$$

$$= \frac{z}{2}\cdot\frac{1}{1-\left(\frac{z}{2}\right)} = \frac{z}{2-z}$$

This series being a G.P. is convergent if $\left|\frac{z}{2}\right| < 1$ i.e., $|z| < 2$.

Hence, ROC is $|z| < 2$

(5) **Find the Z-transform and the radius of convergence of $u(n) = 5^n/n!,\ n \geq 0$**

Solution :

We know that,

$$Z[u(n)] = \sum_{n=0}^{\infty} \frac{5^n}{n!} . z^{-n} = \sum_{0}^{\infty} \frac{(5/z)^n}{n!}$$

$$= 1 + \left(\frac{5}{z}\right) + \frac{1}{2!}\left(\frac{5}{z}\right)^2 + \frac{1}{3!}\left(\frac{5}{z}\right)^3 + \dots \infty$$

$$= e^{5/z}$$

The above series is convergent for all values of z.

Hence, ROC is the entire z-plane.

EXERCISE

Find the Z-transform and its ROC in each of the following sequences :

1. $u(n) = 2^n/n, n > 1$ Ans. $-\log\left(1 - \frac{3}{z}\right); |z| > 3$
2. $u(n) = e^{an}, n \geq 0$ Ans. $\left(1 - e^a/z\right)^{-1}, |z| > |e^a|$
3. $u(n) = 4^n, n \geq 0$ Ans. $\frac{z}{z-4}, |z| > 4$
4. $u(n) = 3^n/n!, n \geq 0$ Ans. $e^{3/z}$, ROC is Z-plane.
5. $u(n) = n\,5^n, n \geq 0$ Ans. $\frac{5z}{(z-5)^2}, |z| > 5$
6. $u(n) = 2^n, n < 0$ Ans. $\frac{z}{2-z}, |z| < 2$
7. $u(n) = 4^n$, for $n < 0$
   and $= 3^n$ for $n \geq 0$ Ans. $\frac{3z}{(4-z)(z-3)}, 3 < |z| < 4$

PART-A QUESTIONS AND ANSWERS

1. How are Z-transforms related to Laplace transforms? What are the uses of Z-transforms? Give suitable examples.
   [A.U. April/May - 1999]
2. Define one sided Z-transform.
3. Define two sided Z-transform.
4. Define unit step sequence and unit impulse sequence.
5. Find the value of $u(n) - u(n-1) = 0$
   [Ans. $\delta(n)$**]**
6. Define $u(n-k)$
   [Ans. $u(n-k) = \begin{cases} 1, & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$
7. $Z[n f(n)] = \dots\dots\dots\dots$ **[Ans.** $-z \frac{dF(z)}{dz}$**]**
8. $Z[(-1)^n] = \dots\dots\dots\dots$ **[Ans.** $\frac{z}{z+1}$ if $|z| > 1$**]**
9. $Z[a^n u(n)]$ exists only if $\dots\dots\dots\dots$ **[Ans.** $|z| > |a|$**]**
10. State convolution theorem for casual sequences.
    [A.U Trichy N/D 2009]
11. State initial value theorem on Z-transform.
    [A.U N/D 2018 R-13, R8]
12. State final value theorem on Z-transform. **[A.U N/D 2018 R-13]**
    [A.U A/M 2019 R-8]
13. Find $Z[n(n-1)x(n)]$ **[Ans.** $z^{-2} \frac{d^2 X}{d(z^{-1})^2}$**]**
14. Prove that $Z[1] = \frac{z}{z-1}, |z| > 1$

15. Prove that $Z[k] = k\left[\frac{z}{z-1}\right]$

16. Prove that $Z[a^n] = \frac{z}{z-a}, \ |z| > |a|$

17. Prove that $Z[e^{an}] = \frac{z}{z-e^a}$

18. Prove that $Z\left[\frac{1}{n!}\right] = e^{1/z}$

19. Prove that $Z\left[\frac{1}{n}\right] = \log\left[\frac{z}{z-1}\right], \ |z| > 1, n > 0$

20. Prove that $Z[\delta(n)] = 1$

21. Prove that $Z(u(n)) = \frac{z}{z-1}$

22. Prove that $Z[\delta(n-k)] = \frac{1}{z^k}$

23. Prove that $Z[u(n-1)] = \frac{1}{z-1}$

24. Prove that $Z(t) = \frac{Tz}{(z-1)^2}$

25. Prove that $Z[e^{-at}] = \frac{z}{z-e^{-aT}}$

26. Prove that $Z^{-1}\left[\frac{z}{z-1}\right] = 1$

27. Prove that $Z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$

28. Prove that $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

29. Prove that $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$

30. Define second shifting theorem on Z-transform.

31. Define convolution of sequences.