Problem: *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$.

Solution. Given :

Velocity distribution $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

(*i*) Displacement thickness δ^* is given by

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$, we have

$$\delta^* = \int_0^\delta \left\{1 - \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right\} dy$$

$$= \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right\} dy = \left[y - \frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta$$

$$= \delta - \frac{\delta^2}{\delta} + \frac{\delta^3}{3\delta^2} = \delta - \delta + \frac{\delta}{3} = \frac{\delta}{3}. \quad \textbf{Ans.}$$

(*ii*) Momentum thickness θ, is given by

$$\theta = \int_0^\delta \frac{u}{U}\left\{1-\frac{u}{U}\right\} dy = \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left[1-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^4}{\delta^4}\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right] dy = \left[\frac{2y^2}{2\delta}-\frac{5y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{\delta^2}{\delta}-\frac{5\delta^3}{3\delta^2}+\frac{\delta^4}{\delta^3}-\frac{\delta^5}{5\delta^4}\right] = \delta-\frac{5\delta}{3}+\delta-\frac{\delta}{5}$$

$$= \frac{15\delta-25\delta+15\delta-3\delta}{15} = \frac{30\delta-28\delta}{15} = \mathbf{\frac{2\delta}{15}}.\quad \textbf{Ans.}$$

(*iii*) Energy thickness δ^{**} is given by

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1-\frac{u^2}{U^2}\right]dy = \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]^2\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\left[\frac{4y^2}{\delta^2}+\frac{y^4}{\delta^4}-\frac{4y^3}{\delta^3}\right]\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{4y^2}{\delta^2}-\frac{y^4}{\delta^4}+\frac{4y^3}{\delta^3}\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{8y^3}{\delta^3}-\frac{2y^5}{\delta^5}+\frac{8y^4}{\delta^4}-\frac{y^2}{\delta^2}+\frac{4y^4}{\delta^4}+\frac{y^6}{\delta^6}-\frac{4y^5}{\delta^5}\right)dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}-\frac{8y^3}{\delta^3}+\frac{12y^4}{\delta^4}-\frac{6y^5}{\delta^5}+\frac{y^6}{\delta^6}\right]dy$$

$$= \left[\frac{2y^2}{2\delta}-\frac{y^3}{3\delta^2}-\frac{8y^4}{4\delta^3}+\frac{12y^5}{5\delta^4}-\frac{6y^6}{6\delta^5}+\frac{y^7}{7\delta^6}\right]_0^\delta$$

$$= \frac{\delta^2}{\delta}-\frac{\delta^3}{3\delta^2}-\frac{2\delta^4}{\delta^3}+\frac{12\delta^5}{5\delta^4}-\frac{\delta^6}{\delta^5}+\frac{\delta^7}{7\delta^6} = \delta-\frac{\delta}{3}-2\delta+\frac{12}{5}\delta-\delta+\frac{\delta}{7}$$

$$= -2\delta-\frac{\delta}{3}+\frac{12}{5}\delta+\frac{\delta}{7} = \frac{-210\delta-35\delta+252\delta+15\delta}{105}$$

$$= \frac{-245\delta+267\delta}{105} = \mathbf{\frac{22\delta}{105}}.\quad \textbf{Ans.}$$