Lagrangian and Eulerian Flow Descriptions

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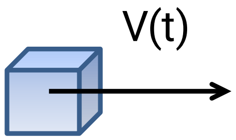
The fluid motion is described by two methods

1. Lagrangian method
2. Eulerian method

Lagrangian method

- Involves following individual fluid particles as they move about in the fluid
- All fluid properties (velocity, acceleration, density etc) described as a function of time
- Particle description

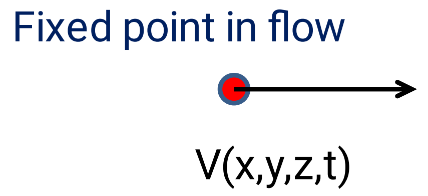


Fluid particle

Lagrangian description is*Same as solid body dynamics*

Eulerian method

- We obtain information about the flow in terms of what happens at fixed points in space as the fluid flows through those points
- Remaining fixed in space and observing different particles as they pass by.
- All fluid properties (velocity, acceleration, density etc) described as a function of space and time
- Field description



***Eulerian method is commonly used in fluid mechanics. It avoids complexity of following individual fluid particles in fluid flow.

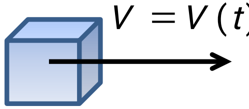
Acceleration field

Velocity field $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

Lagrangian method

Fluid particle

$V = \vec{V}(t)$



$$\vec{a} = \frac{d\vec{V}(t)}{dt} = \frac{d\vec{V}}{dt}$$

Eulerian method

Fixed point

$V = \vec{V}(x, y, z, t)$

$$\vec{a} = \frac{d\vec{V}(x, y, z, t)}{dt}$$

$$\Rightarrow \quad \vec{a} = \frac{\partial \vec{V}}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial \vec{V}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \vec{V}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \vec{V}}{\partial z}\frac{\partial z}{\partial t}$$

$$\Rightarrow \quad \vec{a} = \frac{\partial \vec{V}}{\partial t} + \frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz}{dt}$$

We know $\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$

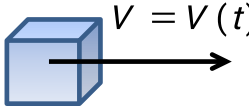
Acceleration field

Velocity field $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

Lagrangian method

Fluid particle

$V = \vec{V}(t)$



$$\vec{a} = \frac{d\vec{V}(t)}{dt} = \frac{D\vec{V}}{Dt}$$

Eulerian method

Fixed point

$V = \vec{V}(x, y, z, t)$

$$\Rightarrow \quad \vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Largangian to Eulerian conversion

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

This equation is called PARTICLE DERIVATIVE / MATERIAL DERIVATIVE / TOTAL DERIVATIVE

$$\frac{D\vec{V}}{Dt} = \left[\frac{\partial}{\partial t} + (\vec{V}.\vec{\nabla})\right]\vec{V}$$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\vec{a} = \frac{D\vec{V}}{Dt} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{Local acceleration}} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{Convective acceleration}}$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

PROBLEM: Given a velocity field $\vec{V} = (4 + xy + 2t)\,\vec{i} + 6x^3\,\vec{j} + (3xt^2 + z)\,\vec{k}$. Find the acceleration of a fluid particle at (2, 4, – 4) and time $t = 3$.

Velocity field: $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$u = 4 + xy + 2t$$

$$v = 6x^3$$

$$w = 3xt^2 + z$$

Acceleration field: $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 2 + 4y + xy^2 + 2ty + 6x^4$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = 72x^2 + 18x^3y + 36tx^2$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 6xt + 12t^2 + 3xyt^2 + 6t^3 + z + 3xt^2$$

The acceleration vector at the point (2, 4, –4) and at time $t = 3$ can be found out by substituting the values of x, y, z and t in the Eq. (3.36) as

$$\vec{a} = 170\vec{i} + 1296\vec{j} + 572\vec{k}$$

Magnitude of resultant acceleration

$$|\vec{a}| = [(170)^2 + (1296)^2 + (572)^2]^{1/2}$$
$$= 1375.39 \text{ units}$$