Unit-2

Fourier Series

Periodic function:-

A function $f(x)$ is said to be periodic, if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest value of p for which this equation is true for every value of x be called the period of the function.

Example:

$\sin x$ is a periodic function with the period 2π

$\cos x$ " " " " 2π

$\tan x$ " " " " π

Dirichlet's Conditions :-

If a function $f(x)$ is defined in $c \le x \le c+2\pi$, it can be expanded as a Fourier Series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, provided the following DIRICHLET'S Conditions are satisfied.

(i) $f(x)$ is single valued and finite in $(c, c+2\pi)$

(ii) $f(x)$ is continuous (or) piece-wise continuous with finite number of finite discontinuities in $(c, c+2\pi)$.

(iii) $f(x)$ has a finite number of maxima or minima in $(c, c+2\pi)$

Note:

These conditions are not necessary but only sufficient for the existence of Fourier series.

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$ is called Bernoulli's extended formula of integration by parts.

If $f(x)$ is defined in $(c, c+2\pi)$ with the period 2π, then Fourier series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{where}$$

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx.$$

Problems :-

1) If $f(x) = \frac{\pi - x}{2}$, Find the Fourier series of period 2π in the interval $(0, 2\pi)$. Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Soln :- Let the required Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right) dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\,dx = \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \frac{4\pi^2}{2}\right] = 0.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\cos nx\,dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx.$$

$u = \pi - x$ $\qquad$ $v = \cos nx$

$u' = -1$ $\qquad$ $v_1 = \frac{\sin nx}{n}$

$u'' = 0$ $\qquad$ $v_2 = -\frac{\cos nx}{n^2}$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[0 - \frac{1}{n^2} - \left[0 - \frac{1}{n^2}\right]\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right] = 0.$$

Note:-

$\cos n\pi = (-1)^n$; $\quad \cos 2n\pi = 1$;

$\sin n\pi = 0$; $\quad \sin 2n\pi = 0$;

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$\Rightarrow$ $u = \pi - x$ $\qquad$ $v = \sin nx$.

$u' = -1$ $\qquad$ $v_1 = \dfrac{-\cos nx}{n}$

$u'' = 0$ $\qquad$ $v_2 = -\dfrac{\sin nx}{n^2}$

$$\Rightarrow \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(-\pi)\left(\frac{-1}{n}\right) - 0 - \left[\pi\left(-\frac{1}{n}\right) - 0\right]\right]$$

$$= \frac{1}{2\pi}\left[\pi/n + \pi/n\right] = 1/n$$

$$\therefore \quad f(x) = 0 + 0 + \sum_{n=1}^{\infty} 1/n \sin nx.$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

$$= \frac{1}{1} \cdot \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots$$

Put $x = \pi/2$, which is a point of continuity, the value of the Fourier series at $x = \pi/2$ is $f(\pi/2)$.

$$f(\pi/2) = \frac{\pi - \pi/2}{2} = \pi/4.$$

$$\therefore \quad \pi/4 = \sin \pi/2 + 1/2 \sin 2\pi/2 + 1/3 \sin 3\pi/2 + \cdots$$

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$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \cdots$$

$$\therefore \quad 1 - 1/3 + 1/5 - 1/7 + \cdots = \pi/4.$$

2) Obtain the Fourier series for the function

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi . \end{cases}$$

Soln:

Let the required Fourier series be:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx .$

$$= \frac{1}{\pi} \left[\int_0^{\pi} f(x) dx + \int_{\pi}^{2\pi} f(x) dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} (1) dx + \int_{\pi}^{2\pi} 0 \, dx \right]$$

$$= \frac{1}{\pi} [x]_0^{\pi} = \frac{1}{\pi} (\pi) = 1$$

$\therefore a_0 = 1$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} f(x) \cos nx \, dx + \int_{\pi}^{2\pi} f(x) \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} (1) \cos nx \, dx + \int_{\pi}^{2\pi} (0) \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{+\sin nx}{n} \right]_0^{\pi}$$

$$= \frac{1}{\pi} [0] = 0 .$$

$\therefore \boxed{a_n = 0}$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} (1)\sin nx\,dx + \int_{\pi}^{2\pi} 0\,\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^n}{n} - \left(-\frac{1}{n}\right)\right] = \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n} + \frac{1}{n}\right]$$

$$= \frac{1}{\pi n}\left(1 + (-1)^{n+1}\right) \qquad \underbrace{(-1)(-1)^n = (-1)^{n+1}}$$

$$\therefore\; f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\left(1+(-1)^{n+1}\right)\sin nx.$$

HW.

① Determine the Fourier series for the function $f(x) = x^2$ of period 2π in $0 < x < 2\pi$.

Ans :- $a_0 = \frac{8\pi^2}{3}$ $a_n = 4/n^2$ $b_n = \frac{-4\pi}{n}$

If $f(x)$ is defined on $(0,2l)$ or $(-l,l)$ with period $2l$, then Fourier series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \quad \text{where}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx \qquad \text{(or)} \qquad a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Problems in $(-l,l)$ and $(0,2l)$:

① If $f(x) = \begin{cases} x/l, & 0<x<l \\ \frac{2l-x}{l}, & l<x<2l \end{cases}$ express $f(x)$ as a Fourier series of periodicity $2l$.

Soln:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$ where

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\left[\int_0^{l} f(x)\,dx + \int_l^{2l} f(x)\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} x/l\,dx + \int_l^{2l} \frac{2l-x}{l}\,dx\right]$$

$$= \frac{1}{l^2}\left[\int_0^{l} x\,dx + \int_l^{2l} (2l-x)\,dx\right]$$

$$= \frac{1}{l^2}\left[\left(\frac{x^2}{2}\right)_0^{l} + \left(2lx - \frac{x^2}{2}\right)_l^{2l}\right]$$

$$= \frac{1}{l^2}\left[\frac{l^2}{2} + 4l^2 - \frac{\cancel{4}^{2}l^2}{\cancel{2}} - 2l^2 + \frac{l^2}{2}\right]$$

$$= \frac{1}{l^2}\left[\frac{2l^2}{2}\right] = 1$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\left[\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} f(x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} \frac{x}{l}\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} \cancel{f(x)}\ \frac{2l-x}{l}\cos\frac{n\pi x}{l}\,dx\right)$$

$$= \frac{1}{l^2}\left[\int_0^{l} x\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} (2l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$u = x \qquad v = \cos\frac{n\pi x}{l} \qquad\qquad u = 2l - x$

$u' = 1 \qquad v_1 = \dfrac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} \qquad\qquad u' = -1$

$u'' = 0 \qquad\qquad\qquad\qquad\qquad u'' = 0.$

$v_2 = -\dfrac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

$$= \frac{1}{l^2}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{l} +$$

$$(2l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\Bigg|_l^{2l}$$

$\cos\frac{n\pi 2k}{k} = 1$

$\cos\frac{n\pi k}{k}$

$$= \frac{1}{l^2}\left[+\frac{l^2}{n^2\pi^2}(-1)^n - \left(\frac{l^2}{n^2\pi^2}(1)\right) - \frac{l^2}{n^2\pi^2}(1) - \left(-\frac{l^2}{n^2\pi^2}(-1)^n\right)\right]$$

$$= \frac{1}{l^2}\left[\frac{l^2}{n^2\pi^2}(-1)^n - \frac{l^2}{n^2\pi^2} - \frac{l^2}{n^2\pi^2} + \frac{l^2}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{l^2}\left[\frac{2l^2}{n^2\pi^2}(-1)^n - \frac{2l^2}{n^2\pi^2}\right)$$

$$\Rightarrow \frac{2l^2}{l^2(n^2\pi^2)}\left((-1)^n - 1\right) = \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{l}\left[\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} f(x) \sin\frac{n\pi x}{l}\, dx + \int_l^{2l} f(x) \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} x/l \,\sin\frac{n\pi x}{l}\, dx + \int_l^{2l} \frac{2l-x}{l} \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{1}{l^2}\left[\int_0^{l} x \sin\frac{n\pi x}{l}\, dx + \int_l^{2l} (2l-x) \sin\frac{n\pi x}{l}\, dx\right]$$

$u = x \qquad v = \sin\frac{n\pi x}{l} \qquad u = 2l - x$

$u' = 1 \qquad u' = -1$

$u'' = 0 \qquad v_1 = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} \qquad u'' = 0.$

$$v_2 = \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}.$$

$$= \frac{1}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l +$$

$$(2l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\Bigg]_l^{2l}$$

$$= \frac{1}{l^2}\left[-l \cdot \frac{l}{n\pi}\left((-1)^n\right) - l\left(\frac{l}{n\pi}\right)\left(-(-1)^n\right)\right.$$

$$= \frac{1}{l^2}\left[\frac{-l^2}{n\pi}(-1)^n + \frac{l^2}{n\pi}(-1)^n\right] = 0.$$

$$\therefore f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos\frac{n\pi x}{l}.$$

2) Find Fourier series of periodicity 2 for $f(x)$ given

$$f(x) = \begin{cases} 0 & \text{in } -1 < x < 0 \\ 1 & \text{in } 0 < x < 1. \end{cases}$$

$(-1, 1)$
$(-l, l)$ here $l = 1$

Soln :- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{1} + b_n \sin \frac{n\pi x}{1}$.

where $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \frac{1}{1}\left[\int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx\right]$

$$= \int_0^1 1\,dx = x\Big|_0^1 = 1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0} f(x)\cos n\pi x\,dx + \int_0^1 f(x)\cos n\pi x\,dx$$

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$$= \int_0^1 \cos n\pi x\,dx = \frac{\sin n\pi x}{n\pi}\Big|_0^1 = 0.$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} f(x)\sin n\pi x\,dx + \int_0^1 f(x)\sin n\pi x\,dx$$

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$$= \int_0^1 \sin n\pi x\,dx = -\frac{\cos n\pi x}{n\pi}\Big|_0^1$$

$$= -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi} = \frac{1}{n\pi}(1-(-1)^n)$$

$$\therefore \quad f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-(-1)^n)\sin n\pi x.$$

3) Find Fourier series of periodicity 2 for

$f(x) = \begin{cases} x & \text{in } -1 < x \le 0 \\ x+2 & \text{in } 0 < x \le 1 \end{cases}$ and hence deduce that sum of

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ to ∞. $(-1, 1)$ $(-l, l)$ $l = 1$

Soln:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{1} + b_n \sin\frac{n\pi x}{1}$

where $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} x\,dx + \int_{0}^{1} (x+2)\,dx$

$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{n+1}$

$=$

$= -\frac{1}{2} + \left[\frac{(x+2)^2}{2}\right]_0^1 = \frac{-1}{2} + \frac{9}{2} - \frac{4}{2}$

$= 2$

$-\frac{1}{2} + \frac{9}{2} - 2$

$-\frac{1}{2} + \frac{7}{2}$

$\rightarrow \frac{4}{2}$

$\rightarrow 2$

$a_n = \frac{1}{1}\int_{-1}^{1} f(x) \cos\frac{n\pi x}{1}\,dx$

$= \int_{-1}^{0} f(x) \cos n\pi x\,dx + \int_{0}^{1} f(x) \cos n\pi x\,dx$

$= \int_{-1}^{0} x \cos n\pi x\,dx + \int_{0}^{1} (x+2) \cos n\pi x\,dx$

$u = x$ $\quad v = \cos n\pi x$ $\quad u = x+2$

$u' = 1$ $\quad v_1 = \frac{\sin n\pi x}{n\pi}$ $\quad u' = 1$

$u'' = 0$ $\quad u'' = 0$

$v_2 = -\frac{\cos n\pi x}{n^2\pi^2}$

$= \left[x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0} +$

$\left[(x+2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$

$= +\cancel{\frac{1}{n^2\pi^2}} - \left(\cancel{\frac{(-1)^n}{n^2\pi^2}}\right) + \frac{(-1)^n}{n^2\pi^2} - \cancel{\frac{1}{n^2\pi^2}} = 0$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx = \int_{-1}^{0} f(x)\sin n\pi x\,dx + \int_{0}^{1} f(x)\sin n\pi x\,dx$$

$$= \int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1}(x+2)\sin n\pi x\,dx.$$

$u = x$, $v = \sin n\pi x$, $u = x+2$

$u' = 1$, $v_1 = -\frac{\cos n\pi x}{n\pi}$, $u' = 1$

$u'' = 0$, $u'' = 0$

$v_2 = -\frac{\sin n\pi x}{n^2\pi^2}$

$$= \left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0} +$$

$$\left[(x+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{0}^{1}$$

$$= 0 - \left((-1)\left(\frac{-(-1)^n}{n\pi}\right)\right) + \left(\frac{-3\cos n\pi}{n\pi} + 2\cos\left(\frac{1}{n\pi}\right)\right).$$

$$= \frac{-(-1)^n}{n\pi} - \frac{3(-1)^n}{n\pi} + \frac{2}{n\pi} = \frac{-4(-1)^n}{n\pi} + \frac{2}{n\pi}.$$

$$\therefore \qquad = \frac{2}{n\pi}\left[1 - 2(-1)^n\right]$$

$$f(x) = \frac{\cancel{2}^{1}}{\cancel{2}} + \sum_{n=1}^{\infty}\frac{2}{n\pi}(1-2(-1)^n)\sin n\pi x.$$

Put $x = \frac{1}{2}$ which is a point of continuity, the value of the Fourier series at $x = 1/2$ is $f(1/2)$.

$$= 1 + \frac{2}{\pi}\left[3\cdot\frac{1}{1} - 3\cdot\frac{1}{3} + 3\cdot\frac{1}{5} - \cdots\right] = \frac{1}{2} + 2 \qquad \frac{\sin n\pi}{2}$$

$n=1, +1$

$n=2, 0$

$n=3, -1$

$n=4, 0$

$n=5, +1$

$$1 + \frac{6}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) = \frac{5}{2}$$

$$\frac{6}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) = \frac{5}{2} - 1 = \frac{3}{2}$$

$$\therefore 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{3}{2} \times \frac{\pi}{6} = \frac{\pi}{4}.$$

$$\therefore 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

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4) Obtain the Fourier series for the function given by

$$f(x) = \begin{cases} 1 + \frac{2x}{l} & \text{in } -l \le x \le 0 \\ 1 - \frac{2x}{l} & \text{in } 0 \le x \le l \end{cases}$$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Soln:- ~~Let~~ Given $f(x) = \begin{cases} 1 + \frac{2x}{l} & \text{in } -l \le x \le 0 \\ 1 - \frac{2x}{l} & \text{in } 0 \le x \le l. \end{cases}$

$$f(-x) = \begin{cases} 1 + \frac{2(-x)}{l} & \text{in } -l \le -x \le 0 \\ 1 - \frac{2(-x)}{l} & \text{in } 0 \le -x \le l. \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{l} & \text{in } l \ge x \ge 0 \\ 1 + \frac{2x}{l} & \text{in } 0 \ge x \ge -l. \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{l} & \text{in } 0 \le x \le l \\ 1 + \frac{2x}{l} & \text{in } -l \le x \le 0. \end{cases}$$

$$= f(x).$$

$\therefore$ $f(x)$ is an even function. Hence $b_n = 0$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l}\right)$.

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l \left(1 - \frac{2x}{l}\right) dx$.

$$= \frac{2}{l}\left[x - \frac{2x^2}{2l}\right)_0^l$$

$$= \frac{2}{l}\left(l - \frac{2l^{\cancel{2}}}{2\cancel{l}}\right) = 0$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \left(1 - \frac{2x}{l}\right)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\left[\left(1 - \frac{2x}{l}\right)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(-\frac{2}{l}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-\frac{2}{l}\left(\frac{l^2}{n^2\pi^2}\right)(-1)^n - \left(-\frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}(1)\right)\right]$$

$$= \frac{2}{l}\left[\frac{-2}{\cancel{l}}\left[\frac{l^{\cancel{2}}}{n^2\pi^2}\right)(-1)^n + \frac{2}{\cancel{l}}\cdot\frac{l^{\cancel{2}}}{n^2\pi^2}\right]$$

$$= \frac{2}{\cancel{l}}\left(\frac{2\cancel{l}}{n^2\pi^2}\right)\left[1 - (-1)^n\right] = \frac{4}{n^2\pi^2}(1 - (-1)^n).$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(1 - (-1)^n).$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}(1 - (-1)^n).$$

$n = 1$

$\frac{1}{1^2}(1 - (-1)^1) = 2/1^2$

$n = 2$ $\quad 0$

$n = 3$ $\quad 2/3^2$

Put $x=0$, here $x=0$ is a point of continuity of $f(x)$. The value of the Fourier series at $x=0$ is $f(0)$.

either $f(x) = 1+\frac{2x}{l}$ or $f(x) = 1-\frac{2x}{l}$.

$$f(0) = 1+\frac{2(0)}{l} = 1.$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right).$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

Exercise problems :-

① Find the Fourier series expansion of period $2l$ for the function $f(x) = (l-x)^2$ in the range $(0,2l)$. Deduce that $\sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6}$.

Ans: $a_0 = \frac{2l^2}{3}$ $\quad a_n = \frac{4l^2}{n^2\pi^2}$ $\quad b_n = 0$.

② Find the Fourier series for the function

$$f(x) = \begin{cases} x & \text{in } 0<x<1 \\ 1-x & \text{in } 1<x<2 \end{cases}$$

($(0,2l)$ – Full range series; $(0,2)$ given; here $2l=2$, $l=1$)

deduce that $1/1^2 + 1/3^2 + 1/5^2 + \cdots = \frac{\pi^2}{8}$.

ans: $$f(x) = \sum_{n=1}^{\infty} \frac{2((-1)^n-1)}{n^2\pi^2}\cos n\pi x + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\sin n\pi x.$$

③ If $f(x)=x$ is defined in $-l<x<l$ with period $2l$, find the Fourier expansion of $f(x)$.

ans: $a_0=0$ $\quad a_n=0$ $\quad b_n = \frac{-2l(-1)^n}{n\pi}$

problems in $(0, 2l)$ and $(-l, l)$.

① Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in ~~$0<x<2l$~~ $0<x<3$.

It is a full range series. $(0,2l)$ $(0,3)$.

$\therefore 2l = 3$ $l = 3/2$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{3/2} + b_n \sin \frac{n\pi x}{3/2} \right)$.

ie, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{3} + b_n \sin \frac{2n\pi x}{3} \right)$

where $a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx = \frac{1}{3/2} \int_0^3 (2x - x^2)\, dx$.

$= \frac{2}{3} \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3} \right]_0^3 \Rightarrow \frac{2}{3} \left[9 - 27/3 \right] = 0$.

$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx$.

$= \frac{1}{3/2} \int_0^3 (2x - x^2) \cos \frac{n\pi x}{3/2}\, dx$.

$= 2/3 \int_0^3 (2x - x^2) \cos \frac{2n\pi x}{3}\, dx$

$u = 2x - x^2$ $v = \cos \frac{2n\pi x}{3}$

$u' = 2 - 2x$ $v_1 = \frac{\sin \frac{2n\pi x}{3}}{\frac{2n\pi}{3}}$

$u'' = -2$

$u''' = 0$.

$v_2 = -\frac{\cos \frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}$

$v_3 = -\sin \frac{2n\pi x}{3} \Big/ \left(\frac{2n\pi}{3}\right)^3$.

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) - (2-2x)\left(\frac{-\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right) + (-2)\left(\frac{-\sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[0 + (\overset{2-6}{2-2\times 3})\left(\frac{1}{\frac{4n^2\pi^2}{9}}\right) + 0 - \left[0 + (2)\frac{1}{\frac{4n^2\pi^2}{9}} + 0\right]\right]$$

$$= \frac{2}{3}\left[\frac{-\cancel{4}\times 9}{\cancel{4}n^2\pi^2} - \frac{\cancel{2}\times 9}{\underset{2}{\cancel{4}}n^2\pi^2}\right]$$

$$= \frac{2}{3}\left[\frac{-9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right] = \frac{2}{3}\left[\frac{9}{n^2\pi^2}\right]\left[-1-\frac{1}{2}\right]$$

$$= \frac{\cancel{2}}{\cancel{3}} \times \frac{9}{n^2\pi^2} \times \frac{-\cancel{3}}{\cancel{2}} = \frac{-9}{n^2\pi^2}.$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{3/2}\int_0^3 (2x-x^2)\sin\frac{n\pi x}{3/2}\,dx.$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\frac{2n\pi x}{3}\,dx.$$

$u = 2x-x^2$ $\quad$ $v = \sin\frac{2n\pi x}{3}$

$u' = 2-2x$ $\quad$ $v_1 = \dfrac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}$

$u'' = -2$ $\quad$ $v_2 = \dfrac{-\sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}$

$u''' = 0.$ $\quad$ $v_3 = \dfrac{\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}$

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$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}} \right) - (2-2x)\left(\frac{-\sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2} \right) + (-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3} \right) \right]_0^3$$

$$= \frac{2}{3}\left[-\left(2(3)-9\right)\frac{3}{2n\pi}(1) + 0 - 2\left(\frac{27}{4n^2\pi^2}\right)(1) - \left[0 + 0 - 2\times\frac{27}{4n^2\pi^2}(1) \right] \right].$$

$$= \frac{2}{3}\left[9/2n\pi - \cancel{\frac{27}{2n^2\pi^2}} + \cancel{\frac{27}{2n^2\pi^2}} \right]$$

$$= 2/3\left(\frac{9}{2n\pi}\right) = 3/n\pi.$$

$$\therefore f(x) = 0 + \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \frac{3}{n\pi}\sin\frac{2n\pi x}{3} \right)$$

---

2) $f(x)$ is defined in $(-2,2)$ as follows. Express $f(x)$ in a Fourier series of periodicity 4.

$$f(x) = \begin{cases} 0, & -2<x<-1 \\ 1+x, & -1<x<0 \\ 1-x, & 0<x<1 \\ 0, & 1<x<2. \end{cases}$$

$(-2,2)$

$(-l,l)$

$\boxed{\therefore l=2}$.

Sol:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}$.

ie, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2} + b_n\sin\frac{n\pi x}{2}\right)$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx.$

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$$= \frac{1}{2}\left[\cancel{\int_{-2}^{-1} 0\,dx} + \int_{-1}^{0}(1+x)\,dx + \int_{0}^{1}(1-x)\,dx + \cancel{\int_{1}^{2} 0\,dx}\right]$$

$$= \frac{1}{2}\left[\left(x+\frac{x^2}{2}\right)_{-1}^{0} + \left(x-\frac{x^2}{2}\right)_{0}^{1}\right]$$

$$= \frac{1}{2}\left[-\left(-1+\tfrac{1}{2}\right) + 1-\tfrac{1}{2}\right] = \frac{1}{2}\left(1-\tfrac{1}{2}+1-\tfrac{1}{2}\right)$$

$$= \frac{1}{2}\left[2-1\right] = \tfrac{1}{2}.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx.$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(1+x)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{1}(1-x)\cos\frac{n\pi x}{2}\,dx\right]$$

(since the other integrals vanish)

$$= \frac{1}{2}\left[(1+x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_{-1}^{0} +$$

$$(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\Bigg]_{0}^{1}$$

$$\frac{\cos n\pi(-1)}{2} = \cos\frac{n\pi}{2}.$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}(1) - \left(\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \left(-\frac{4}{n^2\pi^2}(1)\right)\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\right]$$

$$= \frac{1}{2}\times\frac{4}{n^2\pi^2}\left[1 - \cos\frac{n\pi}{2} - \cos\frac{n\pi}{2} + 1\right]$$

$$= \frac{2}{n^2\pi^2}\left[2 - 2\cos\frac{n\pi}{2}\right] = \frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(1+x)\sin\frac{n\pi x}{2}\,dx + \int_{0}^{1}(1-x)\sin\frac{n\pi x}{2}\,dx\right]$$

the other integrals being zero.

$$= \frac{1}{2}\left[(1+x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_{-1}^{0} +$$

$$(1-x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\Bigg]_{0}^{1}$$

$\sin(-\theta) = -\sin\theta$.

$$= \frac{1}{2}\left((1)\frac{2}{n\pi}(-1) + 0 - \left(0 + \frac{4}{n^2\pi^2}\sin\frac{n\pi(-1)}{2}\right) + \right.$$

$$\left. 0 - \frac{4}{n^2\pi^2}\sin\frac{n\pi(1)}{2} - \left((1)\frac{2}{n\pi}(-1) - 0\right)\right]$$

$$\Rightarrow \frac{1}{2}\left[\cancel{\frac{-2}{n\pi}} + \cancel{\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}} - \cancel{\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}} + \cancel{\frac{2}{n\pi}}\right]$$

$$= 0.$$

$$\therefore f(x) \triangleq \frac{\frac{1}{2}}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

$$\therefore f(x) = 1/4 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\cos\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}.$$

Exercise problems :-

① If $f(x) = x$ is defined in $-l < x < l$ with period $2l$, find the Fourier expansion of $f(x)$.

ans : $a_0 = 0$, $a_n = 0$, $b_n = \frac{-2l(-1)^n}{n\pi}$.

② Find the Fourier series expansion of the periodic function $f(x)$ of period $2l$ defined by

$$f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l. \end{cases}$$

Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

ans : $a_0 = l$; $a_n = \frac{2l}{n^2\pi^2}(1-(-1)^n)$; $b_n = 0$

③ Obtain Fourier series of $f(x)$ of period $2l$ and defined as follows.

$$f(x) = \begin{cases} l-x & 0 \le x \le l \\ 0 & l \le x \le 2l. \end{cases}$$

Hence deduce that $1 - 1/3 + 1/5 - 1/7 + \cdots = \pi/4$ &

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

ans : $a_0 = l/2$; $a_n = \frac{1}{n^2\pi^2}(1-(-1)^n)$; $b_n = 1/n\pi$

④ Expand $f(x) = x - x^2$ as Fourier series in $-1 < x < 1$.

ans : $a_0 = -2/3$; $a_n = \frac{4(-1)^{n+1}}{n^2\pi^2}$; $b_n = \frac{-2}{n\pi}(-1)^n$

Harmonic Analysis

The process of finding the Fourier series for a function given by numerical values is known as harmonic analysis.

The Fourier constants are evaluated by the following formulae:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{2l-0}\int_0^{2l} f(x)\,dx \qquad \left[\because \text{Mean} = \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

(or)

$$a_0 = 2\left[\text{mean value of } f(x) \text{ in } (0,2l)\right] = 2\left[\frac{\Sigma f(x)}{n}\right]$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_n = 2\left[\text{mean value of } f(x)\cos\frac{n\pi x}{l} \text{ in } (0,2l)\right] = 2\left[\frac{\Sigma f(x)\cos\frac{n\pi x}{l}}{n}\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = 2\left[\text{mean value of } f(x)\sin\frac{n\pi x}{l} \text{ in } (0,2l)\right] = 2\left[\frac{\Sigma f(x)\sin\frac{n\pi x}{l}}{n}\right]$$

Note:

① In a Fourier expansion, the term $(a_1\cos x + b_1\sin x)$ is called the fundamental (or) first harmic, the term $(a_2\cos 2x + b_2\sin 2x)$, the second harmonic and so on...

① Find the Fourier Series upto the third harmonic for $y = f(x)$ in $(0, 2\pi)$ defined by the table of values given below

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1.0

Soln: Since the last value of $f(x)$ is a repetition of the first only the 1st six values will be used.

we know that the Fourier series is given by

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

x		y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$\cos 3x$	$\sin 3x$
0	0°	1	1	0	1	0	1	0
$\frac{\pi}{3}$	60°	1.4	0.5	0.866	−0.5	0.866	−1	0
$\frac{2\pi}{3}$	120°	1.9	−0.5	0.866	−0.5	−0.866	1	0
π	180°	1.7	−1	0	1	0	−1	0
$\frac{4\pi}{3}$	240°	1.5	−0.5	−0.866	−0.5	0.866	1	0
$\frac{5\pi}{3}$	300°	1.2	0.5	−0.866	−0.5	−0.866	−1	0
		8.7						

$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$	$y\cos 3x$	$y\sin 3x$
1	0	1	0	1	0
0.7	1.2124	−0.7	1.2124	−1.4	0
−0.95	1.6454	−0.95	−1.6454	1.9	0
−1.7	0	1.7	0	−1.7	0
−0.75	−1.299	−0.75	1.299	1.5	0
0.6	−1.0392	−0.6	−1.0392	−1.2	0
1.1	0.5196	−0.3	−0.1732	0.1	0

$a_0 = 2\left[\text{mean value of } y \text{ in } (0, 2\pi)\right]$ In formula, $(0,2l)$ $(0,2\pi)$ where $l=\pi$

$= 2\left[\frac{\Sigma y}{n}\right] = 2\left[\frac{8.7}{6}\right] = 2.9.$

$a_1 = 2\left[\text{mean value of } y\cos x \text{ in } (0,2\pi)\right]$

$= 2\left[\frac{\Sigma y\cos x}{n}\right] = 2\left[\frac{-1.1}{6}\right] = -0.37.$

$a_2 = 2\left[\frac{\Sigma y\cos 2x}{n}\right] = 2\left[\frac{-0.3}{6}\right] = -0.1$

$a_3 = 2\left[\frac{\Sigma y\cos 3x}{n}\right] = 2\left[\frac{0.1}{6}\right] = 0.03$

$b_1 = 2\left[\frac{\Sigma y\sin x}{n}\right] = 2\left[\frac{0.5196}{6}\right] = 0.17$

$b_2 = 2\left[\frac{\Sigma y\sin 2x}{n}\right] = 2\left[\frac{-0.1732}{6}\right] = -0.06$

$b_3 = 2\left[\frac{\Sigma y\sin 3x}{n}\right] = 2\left[0/6\right] = 0.$

$\therefore\ y = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) +$
$(a_3\cos 3x + b_3\sin 3x)$

$= \frac{8.7}{2} + (-0.37\cos x + 0.17\sin x) + (-0.1\cos 2x - 0.06\sin 2x) +$
$0.03\cos 3x$

$= 1.45 - 0.37\cos x + 0.17\sin x - 0.1\cos 2x - 0.06\sin 2x$
$+ 0.03\cos 3x.$

2) Find an empirical formula of the form

$f(x) = a_0 + a_1 \cos x + b_1 \sin x$ for the following data given

that $f(x)$ is periodic with period 2π.

x in degrees	0	60	120	180	240	300	360
y = f(x)	40.00	31.0	-13.7	20.0	3.7	-21.0	40.0

Soln: Since the last value of y is a repetition of the first, only the first six values will be used.

x	y	cos x	sin x	y cos x	y sin x
0	40.0	1	0	40.0	0
60	31.0	0.5	0.866	15.50	26.846
120	-13.7	-0.5	0.866	6.85	-11.864
180	20.0	1	0	-20.0	0
240	3.7	-0.5	-0.866	-1.85	-3.204
300	-21.0	0.5	-0.866	-10.50	18.186
	60			30	29.964

$$a_0 = 2\left[\frac{\Sigma y}{n}\right] = 2\left[\frac{60}{6}\right] = 20$$

$$a_1 = 2\left[\frac{\Sigma y \cos x}{n}\right] = 2\left[\frac{30}{6}\right] = 10$$

$$b_1 = 2\left[\frac{\Sigma y \sin x}{n}\right] = 2\left[\frac{29.964}{6}\right] = 9.988$$

$$\therefore f(x) = 20 + 10\cos x + 9.988 \sin x$$

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3) The values of x and the corresponding values of $f(x)$ over a period T are given below, show that

$f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$ where $\theta = \frac{2\pi x}{T}$.

x	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$y=f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Solution:- First and last values are same. Hence we omit the last value.

When x varies from 0 to T, θ varies from 0 to 2π.

Let $f(x) = F(\theta) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$.

x	$\theta = \frac{2\pi x}{T}$		y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	0	0	1.98	1	0	1.98	0
$\frac{T}{6}$	$\frac{\pi}{3}$	60°	1.30	0.5	0.866	0.65	1.1258
$\frac{T}{3}$	$\frac{2\pi}{3}$	120°	1.05	-0.5	0.866	-0.525	0.9093
$\frac{T}{2}$	π	180°	1.30	-1	0	-1.3	0
$\frac{2T}{3}$	$\frac{4\pi}{3}$	240°	-0.88	-0.5	-0.866	0.44	0.762
$\frac{5T}{6}$	$\frac{5\pi}{3}$	300°	-0.25	0.5	-0.866	-0.125	0.2165
						1.12	3.0136

Note:-

When $x=0$, $\theta = 0$

When $x = \frac{T}{6}$, $\theta = \frac{2\pi \frac{T}{6}}{T} = \frac{2\pi T}{6T} = \frac{\pi}{3}$

When $x = T/3$, $\theta = \frac{2\pi T}{3T} = \frac{2\pi}{3}$.

When $x = T/2$, $\theta = \frac{2\pi T}{2T} = \pi$

When $x = 2T/3$, $\theta = \frac{2\pi . 2T}{3T} = \frac{4\pi}{3}$.

When $x = \frac{5T}{6}$, $\theta = \frac{2\pi 5T}{6T} = \frac{5\pi}{3}$

$$a_0 = \frac{2}{}\left[\frac{\Sigma y}{n}\right] = 2\left[\frac{4.5}{6}\right] = 1.5$$

$$a_1 = 2\left[\frac{\Sigma y \cos\theta}{n}\right] = 2\left[\frac{1.12}{6}\right] = 0.3733$$

$$b_1 = 2\left[\frac{\Sigma y \sin\theta}{n}\right] = 2\left[\frac{3.0136}{6}\right] = 1.0045$$

$$\therefore f(x) = \frac{1.5}{2} + 0.3733\cos\theta + 1.0045\sin\theta$$

$$f(x) = 0.75 + 0.3733\cos\theta + 1.0045\sin\theta$$

---

4) Find the constant term and the coefficient of the first sine and cosine terms in the Fourier expansion of y as given in the following table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Soln:-

Here, the length of the interval is $2l = 6$ ie, $l = 3$

$\therefore$ The Fourier series can be represented by

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}\right) + \left(a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l}\right) + \dots$$

here $$y = \frac{a_0}{2} + a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3} \longrightarrow (1)$$

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$
0	9	1	0	9	0
1	18	0.5	0.866	9	15.588
2	24	-0.5	0.866	-12	20.785
3	28	-1	0	-28	0
4	26	-0.5	-0.866	-13	-22.517
5	20	0.5	-0.866	10	-17.321
	125			-25	-3.465

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}(125) = 41.67$$

$$a_1 = \frac{2}{n}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}(-25) = -8.33$$

$$b_1 = \frac{2}{n}\sum y\sin\frac{\pi x}{3} = \frac{2}{6}(-3.465) = -1.16$$

Substituting in eqn ①, we get

$$y = \frac{41.67}{2} - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3}$$

$$y = 20.84 - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3}.$$

---

5) The turning moment T is given for a series of values of the crank angle $\theta = 75^\circ$

θ°	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

obtain the 1st four terms in a series of sines to represent T and Calculate T for $\theta = 75^\circ$.

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Let the Fourier sine series to represent T in (0,180) be

$T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + \cdots$

$\theta°$	T	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$
0	0	0	0	0	0
30	5224	0.5	0.866	1	0.866
60	8097	0.866	0.866	0	−0.866
90	7850	1	0	−1	0
120	5499	0.866	−0.866	0	0.866
150	2626	0.5	−0.866	1	−0.866

$T\sin\theta$	$T\sin 2\theta$	$T\sin 3\theta$	$T\sin 4\theta$
0	0	0	0
2,612	4523.984	5224	4523.984
7012.002	7012.002	0	−7012.002
7850	0	−7850	0
4762.134	−4762.134	0	4762.134
1,313	−2274.116	2626	−2274.116
23549.136	4499.736	0	0

$b_1 = \frac{2}{6} \sum T\sin\theta = \frac{2}{6}(23549.136) = 7849.712 \simeq 7850$

$b_2 = \frac{2}{6} \sum T\sin 2\theta = \frac{2}{6}(4499.736) = 1499.912 \simeq 1500$

$b_3 = \frac{2}{6} \sum T\sin 3\theta = \frac{2}{6}(0) = 0$

$b_4 = \frac{2}{6} \sum T\sin 4\theta = \frac{2}{6}(0) = 0$

$\therefore \quad T = 7850\sin\theta + 1500\sin 2\theta$.

At $\theta = 75°$, $T = 8332.5177$

$\simeq 8333$

Exercise problems.

① Compute the first three harmonics of the Fourier series for f(x) from the following data:

x	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$f(x)$	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Ans:- $a_0 = 4.202$; $a_1 = -0.280$; $b_1 = 1.618$; $a_2 = -0.178$;

$b_2 = -0.495$; $a_3 = 0$; $b_3 = 0.202$.

② Using six ordinates analyse harmonically the following data to two harmonics.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	π
y	10	12	15	20	17	11	10

③ The table of values of the function $y = f(x)$ is given below:

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Find a Fourier series upto the third harmonic to represent f(x) interms of x.

① Find Fourier Cosine series of $f(x) = \begin{cases} x^2, & 0<x<1 \\ 2-x, & 1<x<2 \end{cases}$

$(0, l)$
$(0, 2)$ here $l=2$

let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx \Rightarrow \frac{\cancel{2}}{\cancel{2}}\int_0^2 f(x)\,dx$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx$$

$$= \left.\frac{x^3}{3}\right]_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \frac{1}{3} + 4 - \cancel{\frac{4}{2}}^{2} - 2 + \frac{1}{2} = \frac{5}{6}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 x^2\cos\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\cos\frac{n\pi x}{2}\,dx.$$

$$= \left[x^2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (2x)\left(\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right) + 2\left(\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^3}\right)\right]_0^1 +$$

$$\left[(2-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_1^2 \qquad (1)\,\frac{2}{n\pi}$$

=

$$= \frac{2}{n\pi}\cancel{\sin\frac{n\pi}{2}} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos n\pi$$

$$- \frac{2}{n\pi}\cancel{\sin\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}.$$

$$= \frac{12}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(-1)^n$$

$$\therefore f(x) = \frac{5}{12} + \sum_{n=1}^{\infty}\left(\frac{12}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(-1)^n\right)\cos\frac{n\pi x}{2}$$

---

2) Find the Sine series of $f(x) = \begin{cases} x & \text{in } 0 \le x \le \frac{\pi}{2} \\ \pi - x & \text{in } \pi/2 \le x \le \pi. \end{cases}$

Soln:- Let the half-range sine series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\sin nx\,dx + \int_{\pi/2}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + (\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\Bigg]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{2}\cdot\frac{1}{n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} - (0)\right] +$$

$$\left[0 - \left(\frac{-\pi}{2}\cdot\frac{1}{n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{2n}\cancel{\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cancel{\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}.$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{4}{\pi n^2}\sin\frac{n\pi}{2}\cdot\sin nx$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx.$$

Exercise problems.

① If $f(x) = \begin{cases} \frac{\pi x}{4}, & 0 < x < \pi/2 \\ \pi/4(\pi - x), & \pi/2 < x < \pi. \end{cases}$

Express $f(x)$ in a series of cosines only (of periocity 2π)

Ans :- $a_0 = \frac{\pi^2}{8}$; $a_n = \frac{1}{2}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{1}{n^2}(-1)^n\right]$

② obtain the sine series for the function

$$f(x) = \begin{cases} x, & 0 \le x \le l/2 \\ l - x, & l/2 \le x \le l. \end{cases}$$

Ans :- ~~$b_n = \frac{4}{}$~~ $b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}.$

③ Expand the function $f(x) = \sin x$, $0 < x < \pi$ in Fourier Cosine series.

Root Mean Square Value of a function :-

The root mean square value of a function $y=f(x)$ over a given interval (a,b) is defined as

$$\bar{y} = \sqrt{\frac{\int_a^b (f(x))^2\,dx}{b-a}}$$

① Find the R.M.S Value of $f(x) = x - x^2$ in the interval $-1 < x < 1$.

$-1<x<1$
$a<x<b$.

Soln:-

$$\text{RMS Value of } f(x) = \sqrt{\frac{\int_{-1}^{1} (x-x^2)^2\,dx}{(1-(-1))}}$$

$$= \sqrt{\frac{\int_{-1}^{1} (x^2+x^4-2x^3)\,dx}{2}}$$

$\int_{-1}^{1} 2(x^3)$ - odd function

$\therefore$ the integral will become (0).

$$= \sqrt{\frac{\cancel{2}\int_0^1 (x^2+x^4)\,dx}{\cancel{2}}}$$

$$= \sqrt{\left(\frac{x^3}{3} + \frac{x^5}{5}\right)_0^1} = \sqrt{1/3 + 1/5} = \sqrt{8/15}$$

(2) Find the RMS value of $f(x) = x^2$ in the interval $(0 < x < l)$.

Sol:-

RMS value of $f(x) = \sqrt{\dfrac{\int_0^l (x^2)^2 dx}{l}}$

$= \sqrt{\dfrac{\int_0^l x^4 dx}{l}} = \sqrt{\dfrac{\frac{x^5}{5}\Big)_0^l}{l}}$

$= \sqrt{\dfrac{\frac{l^5}{5}}{l}} = \sqrt{\dfrac{l^{\cancel{5}}}{5\cancel{l}}} = \sqrt{\dfrac{l^4}{5}}$

$= \dfrac{l^2}{\sqrt{5}}.$

---

(3) Find the RMS value of $f(x) = x$ in the interval $-\pi < x < \pi$.

Sol:- RMS value of $f(x) = \sqrt{\dfrac{\int_{-\pi}^{\pi} x^2 dx}{\pi - (-\pi)}}$

$= \sqrt{\dfrac{\cancel{2}\int_0^{\pi} x^2 dx}{\cancel{2}\pi}} = \sqrt{\dfrac{\frac{x^3}{3}\Big)_0^{\pi}}{\pi}}$

$= \sqrt{\dfrac{\pi^{\cancel{3}}}{3\cancel{\pi}}} = \sqrt{\dfrac{\pi^2}{3}} = \dfrac{\pi}{\sqrt{3}}$

Parseval's identity :-

* If $f(x)$ is defined on $(0, 2l)$, then parseval's identity is

$$\frac{1}{l}\int_0^{2l} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

* If $f(x)$ is defined on $(-l, l)$, then parseval's identity is

$$\frac{1}{l}\int_{-l}^{l} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

* If $f(x)$ is defined on $(0, 2\pi)$, then parseval's identity is

$$\frac{1}{\pi}\int_0^{2\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

* If $f(x)$ is defined on $(-\pi, \pi)$, then parseval's identity is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Half-range sine series :-

If $f(x)$ is defined on $(0, l)$, then parseval's identity is

$$\frac{2}{l}\int_0^{l} (f(x))^2 dx = \sum_{n=1}^{\infty} b_n^2$$

If $f(x)$ is defined on $(0,\pi)$, then parseval's identity is

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2.$$

Half-range cosine series.

If $f(x)$ is defined on $(0,l)$, then parseval's identity is

$$\frac{2}{l}\int_0^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

If $f(x)$ is defined on $(0,\pi)$, then parseval's identity is

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

① Find the Fourier series of periodicity 2π for $f(x)=x^2$ in $-\pi < x < \pi$. Hence show that

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}$$

Soln: $f(x)=x^2$ in $-\pi < x < \pi$.

Already we have solved, where

$$a_0=\frac{2\pi^2}{3}, \quad a_n=\frac{4(-1)^n}{n^2}, \quad b_n=0.$$

By Parseval's identity,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(x^2)^2\,dx=\frac{\left(\frac{2\pi^2}{3}\right)^2}{2}+\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2$$

$$\frac{1}{\pi}\left[\int_{-\pi}^{\pi}x^4\,dx\right]=\frac{\frac{4\pi^4}{9}}{2}+16\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^4} \qquad \left[(-1)^{2n}=1\right.$$

$$\Rightarrow \frac{1}{\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi}=\frac{4\pi^4}{9\times 2}+16\sum_{n=1}^{\infty}1/n^4$$

$$\Rightarrow \frac{1}{\pi}\left[\frac{\pi^5}{5}-\frac{(-\pi)^5}{5}\right]=\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}1/n^4$$

$$=\frac{\frac{1}{\cancel{\pi}}\left[\frac{2\pi^{\cancel{5}\,4}}{5}\right]-\frac{2\pi^4}{9}}{16}=\sum_{n=1}^{\infty}1/n^4$$

$$\Rightarrow \frac{\frac{2\pi^4}{5}-\frac{2\pi^4}{9}}{16}=\sum_{n=1}^{\infty}1/n^4$$

$$\Rightarrow \frac{8\pi^4}{45\times\cancel{16}\,2}=\sum_{n=1}^{\infty}1/n^4$$

$$\therefore \frac{\pi^4}{90}=1/1^4+1/2^4+1/3^4+\cdots$$

2) Express $f(x)=x$ in half range Cosine series and Sine series of periodicity $2l$ in the range $0<x<l$ and deduce the value of

(i) $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$ to ∞.

(ii) $1/1^2+1/2^2+1/3^2+\cdots$ to ∞.

soln:-

Half-range Cosine series :-

$f(x)=x$ in $0<x<l$.

(Refer Previous problem), we know that

$a_0=l$; $a_n=\frac{2l}{n^2\pi^2}((-1)^n-1)$

By Parseval's identity,

$$\frac{2}{l}\int_0^l (f(x))^2dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2$$

$$\frac{2}{l}\int_0^l (x)^2dx = \frac{l^2}{2}+\sum_{n=1}^{\infty}\left(\frac{2l}{n^2\pi^2}((-1)^n-1)\right)^2$$

$$\frac{2}{l}\int_0^l x^2dx = \frac{l^2}{2}+\sum_{n=1}^{\infty}\frac{4l^2}{n^4\pi^4}((-1)^n-1)^2$$

$$\frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{2}+\frac{4l^2}{\pi^4}\sum_{n=1}^{\infty}\frac{((-1)^n-1)^2}{n^4}$$

$((-1)^n-1)^2$

$n=1$,

$((-1)-1)^2$

$=4$

$$\frac{2l^{\cancel{3}2}}{3\cancel{l}} = \frac{l^2}{2}+\frac{4l^2}{\pi^4}\sum_{n=1}^{\infty}\frac{((-1)^n-1)^2}{n^4}$$

$n=2$

$((-1)^2-1)^2$

$(+1-1)^2=0$

$$\frac{2l^2}{3}-\frac{l^2}{2} = \frac{4l^2}{\pi^4}\left[\frac{4}{1^4}+\frac{4}{3^4}+\frac{4}{5^4}+\cdots\right]$$

$n=3$

$((-1)^3-1)^2$

$=(-2)^2=4$

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$$\frac{l^2}{6} = \frac{16l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\frac{l^2}{6} \times \frac{\pi^4}{16l^2} = 1/1^4 + 1/3^4 + 1/5^4 + \cdots$$

∞.

$$\frac{\pi^4}{96} = 1/1^4 + 1/3^4 + 1/5^4 + \cdots$$

(ii) **Half-range sine series:**

$f(x) = x$ in $0 < x < l$.

we know that,

$$b_n = \frac{2l}{n\pi}(-1)^{n+1}.$$

By Parseval's identity,

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$$

$(a^m)^n = a^{mn}$

$$\frac{2}{l}\int_0^l (x)^2\,dx = \sum_{n=1}^{\infty}\left[\frac{2l}{n\pi}(-1)^{n+1}\right]^2$$

$((-1)^{n+1})^2$

$\Rightarrow (-1)^{2(n+1)}$

$$\frac{2}{l}\int_0^l x^2\,dx = \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{2(n+1)}}{n^2}$$

$(-1)^{2(n+1)} = 1$

$$\frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty} 1/n^2$$

$$\frac{\frac{2}{l}\left[\frac{l^3}{3}\right]}{\frac{4l^2}{\pi^2}} = \sum_{n=1}^{\infty} 1/n^2$$

$$\frac{2l^2}{3} \times \frac{\pi^2}{4l^2} = \sum_{n=1}^{\infty} 1/n^2$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} 1/n^2$$

$$\therefore\ 1/1^2 + 1/2^2 + 1/3^2 + \cdots \infty = \frac{\pi^2}{6}.$$

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2) Find the half-range cosine series for the function $f(x) = x(\pi - x)$ in $0<x<\pi$. Deduce that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}.$$

Soln:-

$$f(x) = x(\pi - x) \quad \text{in} \quad 0<x<\pi.$$

Refer previous problems, we know that

$$a_0 = \frac{\pi^2}{3}\ ; \quad a_n = \frac{-2}{n^2}\left((-1)^n+1\right)$$

By Parseval's Identity,

$$\frac{2}{\pi}\int_0^{\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)^2\, dx = \frac{\left(\frac{\pi^2}{3}\right)^2}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{n^2}\left((-1)^n+1\right)\right)^2$$

$$\frac{2}{\pi}\left[\int_0^{\pi}(\pi^2x^2 + x^4 - 2\pi x^3)\, dx\right] = \frac{\pi^4}{9\times 2} + \sum_{n=1}^{\infty}\frac{4}{n^4}\left((-1)^n+1\right)^2$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 x^3}{3} + \frac{x^5}{5} - \frac{2\pi x^4}{4}\right]_0^{\pi} = \frac{\pi^4}{18} + 4\sum_{n=1}^{\infty}\frac{\left((-1)^n+1\right)^2}{n^4}$$

$$= \frac{2}{\pi}\left[\frac{\pi^5}{3} + \frac{\pi^5}{5} - \frac{2\pi^5}{4}\right] = \frac{\pi^4}{18} + 4\sum_{n=1}^{\infty}\frac{\left((-1)^n+1\right)^2}{n^4}$$

$$= \frac{2}{\pi}\left[\frac{2\pi^5}{60}\right] - \frac{\pi^4}{18} = 4\left[\frac{4}{2^4} + \frac{4}{4^4} + \frac{4}{6^4} + \cdots\right]$$

$$2^4.1^4 \quad 2^4.2^4 \quad 2^4.3^4$$

$$\Rightarrow \frac{4\pi^4}{60} - \frac{\pi^4}{18} = \frac{4\times 4}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\Rightarrow \frac{18\pi^4 - 15\pi^4}{15\times 18} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$\Rightarrow \frac{3\pi^4}{15\times 18} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$\left((-1)^n+1\right)^2$

$n=1,\ 0$

$n=2,\ \left((-1)^2+1\right)^2 \Rightarrow 4$

$n=3\ \ (-1+1)^2 = 0$

$n=4\ \ \left((-1)^4+1\right)^2 = 4$

$\frac{20\pi^5 + 12\pi^5 - 30\pi^5}{60}$

$\frac{32\pi^5 - 30\pi^5}{60}$

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Exercise Problems.

① obtain the Fourier series expansion of $f(x)=x^2$ in $(-l,l)$. Find the sum of $1/1^4 + 1/2^4 + 1/3^4 + \cdots$

ans :- $a_0 = \frac{2}{3}l^2$; $a_n = \frac{4l^2(-1)^n}{n^2\pi^2}$; $b_n = 0$.

② Find the Sine series for $f(x)=x$ in $0<x<\pi$. deduce that $\frac{\pi^2}{6} = 1/1^2 + 1/2^2 + 1/3^2 + \cdots$

ans : $b_n = \frac{-2}{n}(-1)^n$.

③ Find the Cosine series for $f(x)=x$ in $(0,\pi)$ and then using Parseval's theorem. Show that $1/1^4 + 1/3^4 + 1/5^4 + \cdots = \frac{\pi^4}{96}$.

ans: $a_0 = \pi$; $a_n = \frac{2}{n^2\pi}((-1)^n - 1)$

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3) Obtain the Fourier Series of Periodicity 2π for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$. Hence deduce the value of $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$. Further, derive a series for $\operatorname{Cosech}\pi$.

Soln:-

Let the required Fourier Series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{where}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{-1}{\pi}\left[e^{-2\pi} - 1\right]$$

$$= \frac{1}{\pi}\left[1 - e^{-2\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$\left[\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]\right]$$

here $a = -1$, $b = n$.

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2+n^2}\left[-\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}[-1] - \left[\frac{e^0}{1+n^2}(-1)\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{-e^{-2\pi}}{1+n^2} + \frac{1}{1+n^2}\right]$$

$$= \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\, dx$$

$$\left[\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

here
$a=-1$
$b=n$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2+n^2}\left[-\sin nx - n\cos nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}(-n) - \left[\frac{e^0}{1+n^2}(-n)\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{-ne^{-2\pi}}{1+n^2} + \frac{n}{1+n^2}\right]$$

$$\Rightarrow \frac{n}{\pi(1+n^2)}(1-e^{-2\pi}) = \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}$$

$$\therefore e^{-x} = \frac{\frac{1-e^{-2\pi}}{\pi}}{2} + \sum_{n=1}^{\infty}\left(\frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx\right)$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{1+n^2} + \frac{n\sin nx}{1+n^2}\right)\right]$$

In $(0, 2\pi)$, e^{-x} is continuous.

Put $x=\pi$ which is a point of continuity, the value of the Fourier series at $x=\pi$ is $f(\pi)$.

$\cancel{f(\pi) = e^{-x}}$ $\quad f(\pi) = e^{-\pi}$

$$e^{-\pi} = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\right].$$

$$\frac{\pi e^{-\pi}}{1-e^{-2\pi}} = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}.$$

$$\Rightarrow \frac{\pi e^{-\pi}}{1-e^{-\pi}e^{-\pi}} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\frac{\pi \cancel{e^{-\pi}}}{\cancel{e^{-\pi}}(e^{\pi}-e^{-\pi})} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\left(\because 1-e^{-\pi}e^{-\pi} = e^{-\pi}\left(\frac{1}{e^{-\pi}} - e^{-\pi}\right) = e^{-\pi}(e^{\pi}-e^{-\pi}). \quad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} \right).$$

$$\frac{\pi}{2\sinh\pi} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\frac{\pi}{2}\operatorname{cosech}\pi = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi \operatorname{cosech}\pi}{2}$$

$$\therefore \operatorname{cosech}\pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}.$$

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4) Find the Fourier series of periodicity 2π for $f(x) = \begin{cases} x & \text{in } (0,\pi) \\ 2\pi - x & \text{in } (\pi, 2\pi) \end{cases}$ and hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

<u>soln:</u>

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx \right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx \right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right].$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi^2 - \left(\frac{3\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right] = \underline{\underline{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\,dx\right]$$

$u = x \quad v = \cos nx$

$u' = 1 \quad v_1 = \frac{\sin nx}{n}$

$u'' = 0 \quad v_2 = -\frac{\cos nx}{n^2}$

$u = 2\pi - x \quad v = \cos nx$

$u' = -1 \quad v_1 = \frac{\sin nx}{n}$

$u'' = 0. \quad v_2 = -\frac{\cos nx}{n^2}$

$$= \frac{1}{\pi}\left[\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right].$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \left(\frac{1}{n^2}\right) + \left(-\frac{1}{n^2} - \left(-\frac{(-1)^n}{n^2}\right)\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right] \Rightarrow \frac{2}{n^2\pi}\left((-1)^n - 1\right).$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx\,dx\right]$$

$u = x$	$v = \sin nx$	$u = 2\pi - x$	$v = \sin nx$
$u' = 1$	$v_1 = -\frac{\cos nx}{n}$	$u' = -1$	$v_1 = -\frac{\cos nx}{n}$
$u'' = 0$	$v_2 = -\frac{\sin nx}{n^2}$	$u'' = 0$	$v_2 = -\frac{\sin nx}{n^2}$

$$= \frac{1}{\pi}\left[\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-(-1)^n}{n}\right) + 0 - \left((\pi)\left(-\frac{(-1)^n}{n}\right)\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right] = 0.$$

$$\therefore\ f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

$$= \frac{\pi}{2} + \left[\frac{+2}{\pi}\left(\frac{-2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right)\right]$$

	$(-1)^n - 1$
$n=1$,	-2
$n=2$,	0
$n=3$,	-2
$n=4$,	0
$n=5$,	-2

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

Deduction:- $x=0$ is an end point of the range.

Therefore the value of Fourier series at $x=0$ is the average of the values of $f(x)$ at $x=0$ & $x=2\pi$.

$\therefore$ At $x=0$,

$$\frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{1}{2}\left[f(0) + f(2\pi)\right]$$

$$= \frac{1}{2}\left[0+0\right] = 0.$$

$$\therefore \cancel{-}\frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \cancel{-}\pi/2$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

---

(4) If $f(x) = \begin{cases} \sin x & \text{in } 0 \le x \le \pi \\ 0 & \text{in } \pi \le x \le 2\pi \end{cases}$

find a Fourier series of periodicity 2π and hence evaluate

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots \infty.$$

Soln:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx + \int_{\pi}^{2\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} 0\,dx\right]$$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\, dx + \int_{\pi}^{2\pi} f(x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x \cos nx\, dx + \int_{\pi}^{2\pi} 0 \cdot \cos nx\, dx\right]$$

$A = nx,\ B = x$

$$\cos A \sin B = \left[\frac{\sin(A+B) - \sin(A-B)}{2}\right]$$

$$\Rightarrow \frac{1}{\pi}\left[\int_0^{\pi} \frac{\sin(nx+x) - \sin(nx-x)}{2}\, dx\right.$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} \left(\sin(n+1)x - \sin(n-1)x\right) dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \quad \text{if } n \neq$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \left[\frac{-1}{n+1} + \frac{1}{n-1}\right]\right]$$

$\cos(n+1)\pi = (-1)^{n+1}$

$\cos(n-1)\pi = (-1)^{n-1}$

$$= \frac{1}{2\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right)$$

$$= \frac{1}{2\pi}\left((-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\left((-1)^n + 1\right)$$

$$= \frac{1}{2\pi}\left(\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right)\left((-1)^n+1\right)$$

$$\Rightarrow \frac{1}{\cancel{2}\pi}\left(\frac{-\cancel{2}}{n^2-1}\right)\left((-1)^n+1\right)$$

$$\Rightarrow -\frac{(1+(-1)^n)}{(n^2-1)\pi}, \quad \text{if } n \neq 1.$$

n=1

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos x\,dx + \int_{\pi}^{2\pi} f(x)\cos x\,dx\right].$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x\cos x\,dx + \int_{\pi}^{2\pi} 0\cos x\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx \qquad \underline{\sin 2\theta = \sin\theta\cos\theta}$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{1}{2} + \frac{1}{2}\right] = 0$$

$$\boxed{a_1 = 0}$$

$$b_n = \frac{1}{\pi}\left[\int_0^{2\pi} f(x) \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x) \sin nx \, dx + \int_{\pi}^{2\pi} f(x) \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x \sin nx \, dx + \cancel{\int_{\pi}^{2\pi} 0 \sin nx \, dx}\right]$$

$$\left[\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}\right].$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \frac{\cos(n-1)x - \cos(n+1)x}{2} \, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} \left(\cos(n-1)x - \cos(n+1)x\right) dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1} - (0-0)\right]$$

$$= \frac{1}{2\pi}(0-0) = 0 \quad \text{if } n \neq 1$$

$$\because \sin(n+1)\pi = 0$$
$$\sin(n-1)\pi = 0.$$

$$\boxed{\therefore \quad b_n = 0.}$$

$$b_1 = \frac{1}{\pi}\left[\int_0^{2\pi} f(x) \sin x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x) \sin x \, dx + \int_{\pi}^{2\pi} f(x) \sin x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x \sin x \, dx + \cancel{\int_{\pi}^{2\pi} 0 \sin x \, dx}\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin^2 x\, dx = \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi} = \frac{1}{2\pi}(\pi) = 1/2.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= \frac{\frac{2}{\pi}}{2} + 0 + \sum_{n=2}^{\infty} \frac{-(1+(-1)^n)}{(n^2-1)\pi} \cos nx + \frac{1}{2}\sin x + 0.$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{-(1+(-1)^n)}{(n^2-1)\pi} \cos nx.$$

Deduction: $x=0$ and $x=2\pi$ are end points of the range.

Therefore the value of Fourier series at $x=0$ is the average of the values of $f(x)$ at $x=0$ & $x=2\pi$.

Put $x=0$ in Fourier series.

$$\frac{f(0)+f(2\pi)}{2} = \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{-((-1)^n+1)}{(n^2-1)\pi}(1)$$

$$\frac{0+0}{2} = \frac{1}{\pi} + \left[\frac{-1}{\pi}\left(\frac{2}{3} + \frac{2}{15} + \frac{2}{35} + \cdots\right)\right]$$

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots\right]$$

$n=2$ $\frac{(-1)^2+1}{4-1} = \frac{2}{3}$.

$n=3$ 0

$n=4$ $\frac{(-1)^4+1}{16-1} = \frac{2}{15}$

$n=5$ 0

$n=6$ $\frac{(-1)^6+1}{36-1} = \frac{2}{35}$

$$\frac{+\ 1/\pi}{+\ 2/\pi} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots$$

$$\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots$$

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⑤ Expand $f(x) = (\pi - x)^2$ as a Fourier series of Periodicity 2π in $0 < x < 2\pi$ and hence deduce the sum $\sum_{n=1}^{\infty} 1/n^2$.

Soln:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2\,dx \qquad \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a \cdot n+1}$$

$$= \frac{1}{\pi}\left[\frac{(\pi-x)^3}{(-1)3}\right]_0^{2\pi} = \frac{-1}{3\pi}\left[-\pi^3 - \pi^3\right] = \frac{-1}{3\pi}(-2\pi^3)$$

$$= \qquad = \frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\,dx.$$

$u = (\pi - x)^2$ $\qquad$ $v = \cos nx$.

$u' = 2(\pi - x)(-1)$ $\qquad$ $v_1 = \frac{\sin nx}{n}$

$u'' = -2(-1)$ $\qquad$ $v_2 = -\frac{\cos nx}{n^2}$

$= 2$

$\qquad$ $v_3 = -\frac{\sin nx}{n^3}.$

$$= \frac{1}{\pi}\left[(\pi-x)^2 \frac{\sin nx}{n} - (-2)(\pi-x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(-2(-\pi)\left(\frac{1}{n^2}\right) - \left(-2\pi\left(1/n^2\right)\right)\right] .$$

$$\Rightarrow \frac{1}{\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{4\cancel{\pi}}{\cancel{\pi}(n^2)} = 4/n^2$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^2 \sin nx \, dx .$$

$u = (\pi-x)^2$ $\qquad$ $V = \sin nx$

$u' = 2(\pi-x)(-1)$ $\qquad$ $V_1 = -\frac{\cos nx}{n}$

$u'' = -2(-1) = +2$ $\qquad$ $V_2 = -\frac{\sin nx}{n^2}$

$u''' = 0 .$ $\qquad$ $V_3 = +\frac{\cos nx}{n^3}$

$$= \frac{1}{\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - \left((-2)(\pi-x)\left(-\frac{\sin nx}{n^2}\right)\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\pi^2\left(\frac{-1}{n}\right) + \frac{2}{n^3} - \left(\pi^2\left(-1/n\right) + \frac{2}{n^3}\right)\right) .$$

$$= \frac{1}{\pi}\left[\cancel{\frac{-\pi^2}{n}} + \cancel{\frac{2}{n^3}} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}}\right] = 0 .$$

$$f(x) = (\pi - x)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

$x = 0$ is an end point of the range.

$\therefore$ The value of the Fourier series at $x = 0$ is

$$\frac{f(0) + f(2\pi)}{2}$$

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (1) = \frac{\pi^2 + (-\pi)^2}{2}$$

$$\frac{\pi^2}{3} + 4 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right] = \frac{\cancel{2}\pi^2}{\cancel{2}}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2 - \pi^2/3}{4}$$

$$\Rightarrow \frac{2\pi^2}{12} = \frac{\pi^2}{6}$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Exercise problems

① Expand $f(x) = x(2\pi - x)$ as Fourier series in $(0, 2\pi)$ and hence deduce that the sum of

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Ans: $a_0 = \frac{4\pi^2}{3}$, $a_n = \frac{-4}{n^2}$, $b_n = 0$

② Determine the Fourier series for the function $f(x) = x^2$ of period 2π in $0 < x < 2\pi$.

Ans: $a_0 = \frac{8\pi^2}{3}$; $a_n = 4/n^2$; $b_n = \frac{-4\pi}{n}$.

(5) Expand in Fourier series of $f(x) = x \sin x$ for $0 < x < 2\pi$ and deduce the result

$$\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \cdots \infty = \frac{\pi - 2}{4}.$$

$a_0 = -2$; $\quad a_n = \dfrac{2}{n^2-1}, \ n \neq 1 \quad a_1 = -1/2$.

$b_n = 0, \ n \neq 1, \quad b_1 = \pi$.

[**Note:** $\quad \sin 2(n+1)\pi = 0 \qquad \sin 2(n-1)\pi = 0$

$\cos 2(n+1)\pi = 1 \qquad \cos 2(n-1)\pi = 1$]

Even and odd functions :-

$f(x)$ is said to be even if $f(-x) = f(x)$

Example:

x^2, $\cosh x$, $\cos kx$, $x^2 - 3\cos x + 1$, ...

Note: The graph of an even function will be symmetric about the y-axis.

$f(x)$ is said to be odd if $f(-x) = -f(x)$

Ex: x, $\sin x$, $3x - 4\sin x$, ...

Note: The graph of an odd function will be symmetric about the origin.

There are functions which are neither odd nor even, for example: $1+x$, e^x, ...

Note:

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd.} \end{cases}$$

If $f(x)$ is <u>even function</u> in $(-\pi, \pi)$, then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$b_n = 0.$$

If $f(x)$ is an <u>odd function</u> in $(-\pi, \pi)$, then

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\,dx.$$

<u>Problems in $(-\pi, \pi)$:-</u>

① Expand ~~is~~ $f(x) = x^2$ when $-\pi < x < \pi$, in a Fourier series of periodicity 2π. Hence deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \text{to } \infty = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \text{to } \infty = \frac{\pi^2}{12}$

(iii) $1/1^2 + 1/3^2 + 1/5^2 + \cdots \text{to } \infty = \frac{\pi^2}{8}$

<u>Soln:-</u>

Given $f(x) = x^2$.

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore$ $f(x)$ is an even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \quad \text{where}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx\ ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\ ; \quad b_n = 0.$$

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$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx.$$

$u = x^2 \quad V = \cos nx$

$u' = 2x \quad V_1 = \frac{\sin nx}{n}$

$u'' = 2 \quad V_2 = -\frac{\cos nx}{n^2}$

$u''' = 0 \quad V_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n^2}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left[+2\cancel{\pi}\frac{(-1)^n}{n^2} - [0]\right] = \frac{4(-1)^n}{n^2}.$$

$$\therefore f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

Deduction :-

(i) $x = \pi$ is an end point of the range.

$\therefore$ The value of the Fourier series at $x = \pi$ is equal to $\frac{f(\pi) + f(-\pi)}{2}$

Putting $x = \pi$ in the Fourier series,

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi = \frac{f(\pi) + f(-\pi)}{2}$$

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n = \frac{\pi^2 + (-\pi)^2}{2}$$

$(-1)^n(-1)^n = (-1)^{2n}$

$\frac{\pi^2+\pi^2}{2} = \frac{2\pi^2}{2} = \pi^2$

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$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \pi^2$$

$$\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \pi^2$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2 - \frac{\pi^2}{3}}{4} = \frac{\frac{2\pi^2}{3}}{4} = \frac{2\pi^2}{3 \times 4} = \frac{\pi^2}{6}.$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \longrightarrow (i)$$

(ii) Put $x=0$ which is point of continuity, the value of the Fourier series at $x=0$ is $f(0)$.

$f(x)=x^2$
$\because f(0)=0$

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(1) = f(0).$$

$$\frac{\pi^2}{3} + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right) = 0.$$

$$\frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right) = 0.$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\cancel{-}\frac{\pi^2}{3}}{\cancel{-}4} = \frac{\pi^2}{12}.$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}. \longrightarrow (ii).$$

(iii) Adding (i) & (ii),

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right) = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{3\pi^2}{12} = \frac{\pi^2}{4}.$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

2) In $(-\pi, \pi)$, find the Fourier series of periodicity 2π $f(x) = \begin{cases} 1+x & \text{in } 0<x<\pi \\ -1+x & \text{in } -\pi<x<0. \end{cases}$

Soln:- Given $f(x) = \begin{cases} 1+x & \text{in } 0<x<\pi \\ -1+x & \text{in } -\pi<x<0. \end{cases}$

$$f(-x) = \begin{cases} 1-x & \text{in } 0<-x<\pi \\ -1-x & \text{in } -\pi<-x<0. \end{cases}$$

$$= \begin{cases} -(x-1) & \text{in } 0<-x<\pi. \\ -(1+x) & \text{in } -\pi<-x<0. \end{cases}$$

$$= -\begin{cases} x-1 & \text{in } 0>x>-\pi. \\ 1+x & \text{in } \pi>x>0. \end{cases}$$

$$= -\begin{cases} x-1 & \text{in } -\pi<x<0 \\ 1+x & \text{in } 0<x<\pi. \end{cases}$$

$$= -f(x).$$

$\therefore$ Given $f(x)$ is an odd function

Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} (1+x)\sin nx\, dx.$$

$u = 1+x$ $\qquad v = \sin nx.$

$u' = 1$ $\qquad v_1 = \dfrac{-\cos nx}{n}$; $\quad v_2 = -\dfrac{\sin nx}{n^2}$

$u'' = 0$

$$= \frac{2}{\pi}\left[(1+x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left((1+\pi)\left(-\frac{\cos n\pi}{n}\right) - \left[\frac{-1}{n}\right]\right)$$

$$= \frac{2}{\pi}\left((1+\pi)\left(\frac{-(-1)^n}{n}\right) + \frac{1}{n}\right).$$

$$\Rightarrow \frac{2}{n\pi}\left(1 - (-1)^n(1+\pi)\right).$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n(1+\pi)\right)\sin nx.$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(1-(-1)^n(1+\pi)\right)\sin nx$$

---

③ Find the Fourier series of $f(x) = |\sin x|$ in $(-\pi, \pi)$ of periodicity 2π.

Soln :- Given $f(x) = |\sin x|$

$$f(-x) = |\sin(-x)|$$
$$= |-\sin x|$$
$$= |\sin x| = f(x)$$

$$\therefore f(-x) = f(x).$$

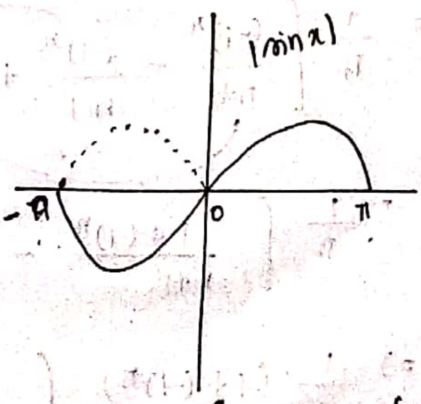
$\therefore f(x)$ is an even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0.$$



$$|\sin x| = \begin{cases} -\sin x, & (-\pi, 0) \\ \sin x, & (0, \pi) \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\sin x|\, dx \quad (\because \text{ even integrand})$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\, dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(-1)+1\right] = \frac{4}{\pi}$$
$$(1+1)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$\cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{\sin(n+1)x - \sin(n-1)x}{\cancel{2}}\, dx$$

$$\Rightarrow \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \left[\frac{-1}{n+1} + \frac{1}{n-1}\right]\right] \text{ if } n \neq 1.$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad \because \cos(n+1)\pi = (-1)^{n+1}, \ \cos(n-1)\pi = (-1)^{n-1}.$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$-(-1)^{n+1}$$
$$\Rightarrow (-1)(-1)^n(-1)^1$$
$$= (-1)^n$$

$$= \frac{1}{\pi}\left[\frac{1+(-1)^n}{n+1} - \frac{1}{n-1}\left(1+(-1)^n\right)\right]$$

$$\Rightarrow \frac{1}{\pi}\left(1+(-1)^n\right)\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{(1+(-1)^n)}{\pi}\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right]$$

$$\Rightarrow \frac{1+(-1)^n}{\pi}\left(\frac{-2}{n^2-1}\right). \quad \text{if } n \neq 1.$$

$n=1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx \qquad 2\sin\theta\cos\theta = \sin 2\theta$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{\sin 2x}{\cancel{2}}\, dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{1}{2\pi}[-1+1] = 0$$

$\therefore a_1 = 0.$

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx.$$

$$= \frac{\frac{4}{\pi}}{2} + 0 + \sum_{n=2}^{\infty} \frac{-2}{\pi(n^2-1)}(1+(-1)^n)\cos nx.$$

$$= \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{(1+(-1)^n)}{n^2-1}\cos nx.$$

---

(4). Find the Fourier series of $f(x) = x+x^2$ in $(-\pi,\pi)$ of periodicity 2π. Hence deduce that $\sum 1/n^2 = \frac{\pi^2}{6}$

Soln:-

Given $f(x) = x+x^2$.

$\therefore$ $f(x)$ is neither odd nor even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \; ; \; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \; ; \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi} x\,dx}_{(0)} + \int_{-\pi}^{\pi} x^2\,dx\right]$$

$\because$ integrand 'x' is odd   integrand 'x^2' is even.

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi} x\cos nx\,dx}_{(0)} + \int_{-\pi}^{\pi} \underbrace{x^2\cos nx}\,dx\right].$$

$= (\because x\cos nx$ is odd$)$ even function

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left[2\cancel{\pi}\left(\frac{(-1)^n}{n^2}\right)\right] = \frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \underbrace{x\sin nx}\,dx + \underbrace{\int_{-\pi}^{\pi} x^2\sin nx\,dx}_{(0)}\right]$$

even function $(\because x^2\sin nx$ is an odd fn$)$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\pi\left(-\frac{(-1)^n}{n}\right)\right) = \frac{-2}{n}(-1)^n$$

$$f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\cos nx + \left(\frac{-2}{n}(-1)^n\right)\sin nx\right)$$

$$\therefore x+x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right].$$

Deduction :-

$x=\pi$ and $x=-\pi$ are end point of the range.

$\therefore$ The value of Fourier series at $x=\pi$ is the average of the values of $f(x)$ at $x=\pi$ & $x=-\pi$.

hence, put $x=\pi$ in Fourier series,

$(-1)^n(-1)^n=(-1)^{2n}$

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos n\pi = \frac{f(\pi)+f(-\pi)}{2}$$

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}(-1)^n = \frac{(\cancel{\pi}+\pi^2)+(\cancel{-\pi}+(-\pi)^2)}{2}$$

$$\frac{\pi^2}{3} + 4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] = \frac{\cancel{2}(\pi^2)}{\cancel{2}}$$

$$4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] = \pi^2-\frac{\pi^2}{3} = \frac{2\pi^2}{3}$$

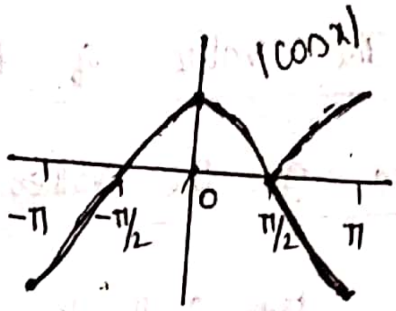
$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{2\pi^2}{3\times4} = \frac{\pi^2}{6}.$$

$$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}.$$

Exercise problems.

① Find the Fourier series for $f(x) = |\cos x|$ in $(-\pi, \pi)$ of periodicity 2π.

Hint:- $f(x) = |\cos x|$ is even fn.



$$|\cos x| = \begin{cases} \cos x & \text{in } (0, \pi/2) \\ -\cos x & \text{in } (\pi/2, \pi) \end{cases}$$

Ans:-

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx\right]$$

$$= 4/\pi$$

$$a_n = \frac{-4}{\pi(n^2-1)} \cos\frac{n\pi}{2} \quad \text{if } n \neq 1;$$

$$a_1 = 0$$

(2) Find the Fourier expansion of $f(x) = x$ in $-\pi < x < \pi$.

Ans: $a_0 = 0$; $a_n = 0$; $b_n = \frac{2(-1)^{n+1}}{n}$

(3) What is the Fourier expansion of the periodic function whose definition of one period is

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$$? Hence

Evaluate : (i) $\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \frac{1}{7\cdot9} + \cdots$

(ii) $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \frac{1}{7\cdot9} + \cdots$

Ans: $a_0 = 2/\pi$; $a_n = \frac{-1}{\pi}\left[\frac{1}{n^2-1}\right](1+(-1)^n)$, $n \neq 1$

$a_1 = 0$; $b_n = 0$; $b_1 = 1/2$

(4) Obtain the Fourier series to represent the function $f(x) = |x|$ in $-\pi < x < \pi$. and deduce that

$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. Ans: $a_0 = \pi$; $a_n = \frac{2}{\pi n^2}((-1)^n - 1)$

(5) obtain the Fourier series for $f(x) = 1 + x + x^2$ in $(-\pi, \pi)$. Deduce that $1/1^2 + 1/2^2 + 1/3^2 + \cdots \infty = \frac{\pi^2}{6}$.

Ans: $a_0 = \frac{2}{3}(3 + \pi^2)$; $a_n = \frac{4(-1)^n}{n^2}$; $b_n = \frac{-2}{n}(-1)^n$.

Half - range Sine and cosine series :-

In some problems, we are concerned with the interval $(0, \pi)$, (instead of the usual interval of length 2π). ~~Furth.~~ Further, the conditions of the problem may require us to expand the given function in a series of sines alone or a series of cosines alone.

* If we extend the function $f(x)$ by reflecting it in the y-axis (so that $f(-x) = f(x)$), then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms.

* If we extend the function $f(x)$ by reflecting it in the origin, so that $f(-x) = -f(x)$, then the extended function is odd for which $a_0 = 0$, $a_n = 0$. The Fourier expansion of $f(x)$ will contain only sine terms.

The half-range cosine series in $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{where}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \; ; \quad a_n = \frac{2}{l}\int_0^l f(x)\cdot\cos\frac{n\pi x}{l}\,dx$$

The half-range cosine series in $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \; ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

The half-range sine series in $(0, l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

The half-range sine series in $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Problems :-

(1) Find half-range Fourier cosine series and sine series for $f(x)=x$ in $0<x<\pi$.

Half-range cosine series :-

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$

$= \frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left(\frac{1}{n^2}((-1)^n-1)\right).$

$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}((-1)^n-1)\cos nx.$

(ii) Half-range sine series :-

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where

$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$

$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$

$= \frac{2}{\cancel{\pi}}\left[-\frac{\cancel{\pi}}{n}(-1)^n\right] = \frac{-2}{n}(-1)^n.$

$$\therefore f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx.$$

2) Express $f(x) = x(\pi - x)$, $0 < x < \pi$ as a Fourier series of periodicity 2π containing (i) sine terms only (ii) cosine terms only.

Hence, deduce $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

(i) Sine series :-

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$

where $b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx \, dx$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}(-1)^n + \frac{2}{n^3}(1)\right]$$

$$\Rightarrow \frac{4}{\pi n^3}\left(1 - (-1)^n\right).$$

$\sin n\pi/2$	
$n=1,$	1
$n=2$	0
$n=3$	-1

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}(1 - (-1)^n) \sin nx.$$

Put $x = \pi/2$ which is a point of continuity.

The value of Fourier series at $x = \pi/2$ is $f(\pi/2)$.

$$f(\pi/2) = \pi/2(\pi - \pi/2) = \pi/2(\pi/2) = \frac{\pi^2}{4}.$$

$$\frac{\pi^2}{4} = \frac{4}{\pi}\left[\frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \cdots\right]$$

$$= \frac{\pi^2}{4} = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right]$$

$$\frac{\pi^2}{4} \times \frac{\pi}{8} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$

(ii) <u>Cosine series</u>:

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx.$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\left[\frac{\pi^3}{6}\right] = \frac{\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{n^2}(-1)^n - \frac{\pi}{n^2}\right] = \frac{-2\pi}{\pi n^2}\left((-1)^n + 1\right)$$

$$= \frac{-2}{n^2}\left((-1)^n + 1\right).$$

$$\therefore f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} -\frac{2}{n^2}\left((-1)^n + 1\right)\cos nx.$$

<u>Put $x = \pi/2$ which is a point of continuity. The</u>
<u>value of Fourier series at $x = \pi/2$ is $f(\pi/2)$.</u>

$$f(\pi/2) = \pi/2\,(\pi - \pi/2) = \frac{\pi^2}{4}.$$

$$\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi}{2}.$$

$$n=1,\quad \frac{(-1)+1}{1^2}\cos\pi/2 = 0$$

$$n=2,\quad \frac{(-1)^2+1}{2^2}\cos\frac{2\pi}{2} = \frac{2}{2^2}(-1).$$

$$n=3\quad 0$$

$$n=4\quad \frac{(-1)^4+1}{4^2}\cos\frac{\cancel{4}^{2}\pi}{\cancel{2}} = 2/4^2\ (1)$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{6} - 2\left[\frac{-2}{2^2}+\frac{2}{4^2}-\frac{2}{6^2}+\cdots\right]$$

$$\frac{\pi^2}{4}-\frac{\pi^2}{6} = 4\left[\underset{2^2\cdot1^2}{\frac{1}{2^2}}-\underset{2^2\cdot2^2}{\frac{1}{4^2}}+\underset{2^2\cdot3^2}{\frac{1}{6^2}}-\cdots\right]$$

$$\frac{2\pi^2}{24} = \frac{\cancel{4}}{\cancel{2^2}}\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

3) Expand $f(x) = (x-1)^2, \; 0<x<1$ in a Fourier series of sines only.

Soln:- Let $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ $\quad 0<x<l$, $0<x<1$

here $l=1$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{1}\int_0^1 (x-1)^2 \sin n\pi x \, dx$$

$$= 2\left[(x-1)^2 \left(\frac{-\cos n\pi x}{n\pi}\right) - 2(x-1)(1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\frac{2(-1)^n}{n^3\pi^3} + \frac{1}{n\pi} - \frac{(1)2}{n^3\pi^3}\right]$$

$$\Rightarrow \frac{2}{n\pi}\left[1 + \frac{2((-1)^n - 1)}{n^2\pi^2}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + \frac{(2(-1)^n - 1)}{n^2\pi^2}\right] \sin n\pi x.$$

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(4) Express $f(x) = x$ in half-range cosine series and sine series of periodicity $2l$ in the range $0<x<l$.

(i) Cosine series:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left(\frac{x^2}{2}\right)_0^l = \underline{\underline{l}}$$

$$a_n = \frac{2}{l}\int_0^l x \cos\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}(-1)^n - \frac{l^2}{n^2\pi^2}(1)\right]$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}\right]\left((-1)^n - 1\right) = \frac{2l}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$f(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2}\left((-1)^n - 1\right)\cos\frac{n\pi x}{l}$$

<u>Sine series:</u>

let $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

where $b_n = \frac{2}{l}\int_0^l x \sin\frac{n\pi x}{l}\, dx$

$$= \frac{2}{l}\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l^2}{n\pi}(-1)^n\right] = \frac{2l}{n\pi}(-1)^{n+1}.$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{l}.$$

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5) Expand $f(x) = x\sin x$ as a Cosine series in $0<x<\pi$

and Show that $1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots = \frac{\pi}{2}$.

Soln:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\, dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{\pi}$$

$$(1-(-1))$$

$$= \frac{2}{\pi}(\pi) = 2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\, dx.$$

$$\cos A\sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} x\left(\frac{\sin(n+1)x - \sin(n-1)x}{\cancel{2}}\right) dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(n+1)x\, dx - \int_0^{\pi} x\sin(n-1)x\, dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi} -$$

$$\left[x\left(\frac{-\cos(n-1)x}{n-1}\right) - (1)\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi} \text{ if } n \neq 1.$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^{n+1}}{n+1} + \frac{\pi(-1)^{n-1}}{n-1}\right]$$

$$(-1)(-1)^{n+1}$$

$$(-1)(-1)^n(-1)^1$$

$$(-1)^n$$

$$= \frac{(-1)^n(\cancel{\pi})}{\cancel{\pi}}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

case. $(-1)^{n-1}$

$(-1)^n\cdot(-1)^1$

$$= (-1)^n \left[\frac{\cancel{n} - \cancel{n} + \cancel{n} - 1 - \cancel{n} - 1}{n^2-1} \right]$$

$$\Rightarrow \frac{-2(-1)^n}{n^2-1}, \quad \text{if } n \neq 1.$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx \qquad \sin A \cos A = \frac{\sin 2A}{2}.$$

$$= \frac{\cancel{2}}{\pi} \int_0^{\pi} x \cdot \frac{\sin 2x}{\cancel{2}} \, dx$$

$$= \frac{1}{\pi} \left[x \left(\frac{-\cos 2x}{2} \right) - (1) \left(-\frac{\sin 2x}{4} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\pi \left(-\frac{1}{2} \right) \right] = -\frac{1}{2}.$$

$$\therefore f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx.$$

$$f(x) = \frac{2}{2} - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx;$$

Put $x = \pi/2$ which is a point of continuity. The value of the Fourier series at $x = \pi/2$ is $f(\pi/2)$.

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$$f(\pi/2) = \pi/2 \sin \pi/2 = \pi/2.$$

$$\pi/2 = 1 - \frac{1}{2}\cancel{\cos \pi/2}_{(0)} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\pi/2 = 1 + 2\left[\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \cdots\right]$$

$$\frac{\pi}{2} = 1 + 2\left[\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \cdots\right]$$

$\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$

$n=2$, $\frac{-1}{3}(-1) = 1/3$

$n=3$, 0.

$n=4$, $\frac{-1}{15}(1) = -1/15$

$n=5$, 0

$n=6$, $\frac{-1}{35}(-1) = 1/35$

Exercise problems.

① Find the half-range sine series for the function $f(x) = x - x^2$, $0 < x < 1$.

ans: $b_n = \frac{4}{\pi^3 n^3}(1-(-1)^n)$.

② Find the half-range sine series for the function $f(x) = l - x$ in $(0, l)$

ans: $b_n = \frac{2l}{n\pi}$.

③ Expand the function $f(x) = \sin x$, $0 < x < \pi$ in Fourier cosine series.

ans: $a_0 = 4/\pi$; $a_n = \frac{-2}{\pi(n^2-1)}(1+(-1)^n)$; $a_1 = 0$

④ Find the half-range cosine series for $f(x) = (x-1)^2$ in $(0,1)$.