MODULE - 4

Introduction to Fourier Transforms - Problems and properties - standard Results - Fourier sine and cosine Transforms - properties - Convolution of two functions - Convolution theorem - parseval's Identity for fourier transforms, sine and cosine Transforms - FT using differentiation property - Solving Integral equation - self-reciprocal using FT - sine and cosine Transforms.

DEFINITION - FOURIER TRANSFORM

Let $f(x)$ be a function defined in the interval $(-\infty,\infty)$. Then the Fourier transform of $f(x)$ is defined as

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s), \text{ s is a parameter.}$$

INVERSE FOURIER TRANSFORM

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} dx$$

PROPERITIES OF FOURIER TRANSFORMS

1. Linear property

$$F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$$

Proof:-

$$F[a f(x) + bg(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{isx} dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af(x) e^{isx} + bg(x)e^{isx}) dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a f(x) e^{isx} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b\, g(x) e^{isx} dx$$

$$\Rightarrow a \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \right\} + b \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx \right\}$$

$$= aF[f(x)] + b[g(x)]$$

$$= F[a f(x) + b\, g(x)]$$

$$= aF[f(x)] + bF[g(x)]$$

2. shifting property:-

If $F[f(x)] = F(s)$ then $F[f(x-a)] = e^{isa} \cdot F(s)$

Proof:-

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx} dx$$

put $x - a = t$ $\quad x = a+t \quad dx = dt$

when $x = \infty$; $t = \infty$; $x = -\infty$; $t = -\infty$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(a+t)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{isa} \cdot e^{ist} dt$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist} dt$$

put $t = x$

$$\Rightarrow \frac{e^{isa}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isa} dx \Rightarrow e^{isa} F[f(x)]$$

$$\boxed{\therefore F[f(x-a)] = e^{isa} F[f(x)]}$$

3. shifting property:-

If $F[f(x)] = F(s)$, then $F[e^{isa} f(x)] = F(s+a)$

proof:-

$$F[e^{isa}(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$$

$$= F(s+a)$$

4. Modulation property:-

If $F[f(x)] = F(s)$ then

$$F[f(x) \cdot \cos ax] = \tfrac{1}{2}[F(s+a) + F(s-a)]$$

proof:

$$F[f(x) \cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax\, e^{isx}\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right) e^{isx}\, dx$$

$$\Rightarrow \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (e^{iax} + e^{-iax})\, e^{isx}\, dx$$

$$\Rightarrow \frac{1}{2}\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx \right\}$$

$$= \tfrac{1}{2}\{F(s+a) + F(s-a)\}$$

$$\boxed{\therefore F[f(x) \cos ax] = \tfrac{1}{2}[F(s+a) + F(s-a)]}$$

5. Change of scale property :-

If $F[f(x)] = F(s)$ then $F[f(ax)] = 1/a \, F(s/a)$; $a>0$

Proof :

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $ax = t$ $x = t/a$; $dx = dt/a$

where $x = -\infty$; $t = -\infty$; $x = \infty$; $t = \infty$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist/a}\, dt/a$$

$$= 1/a \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)}\, dt$$

put $t = x$

$$\Rightarrow 1/a \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{is(x/a)}\, dx$$

$$= 1/a \, F(s/a)$$

$$\boxed{F[f(ax)] = 1/a \, F(s/a)}$$

6. Fourier transform for derivative

If $F[f(x)] = F(s)$, then

$F[f'(x)] = -is\, F(s)$ Provided $f(x) \to 0$

for $x \to \pm\infty$

Proof

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{isx}\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, d[f(x)]$$

$u = e^{isx}$ $\qquad d[f(x)] = dv$

$du = e^{isx} is\,dx$ $\qquad v = f(x)$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\left[e^{isx} f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) is\, e^{isx}\,dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\left[0 - \int_{-\infty}^{\infty} is\, f(x)\, e^{isx}\,dx\right]$$

$$\Rightarrow \frac{(-is)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx = (-is) F[f(x)] = -is\, F(s)$$

$$\boxed{\therefore F[f'(x)] = -is\,F(s)}$$

7. If $F[f(x)] = F(s)$, then $F[\overline{f(x)}] = \overline{F(s)}$

Proof: $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)\, e^{isx}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, \overline{e^{isx}}\,dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\,dx$$

but $x = -t$ $\quad t = -x$; $dx = -dt$

If $x = \infty$, $t = -\infty$; $x = -\infty$; $t = \infty$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-t)}\, e^{ist}\,(-dt)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\left[-\int_{\infty}^{-\infty} \overline{f(-t)}\, e^{ist}\,(+dt)\right]$$

Result: $\int_{-a}^{a} f(x)\,dx = -\int_{a}^{-a} f(x)\,dx$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\, e^{-ist}\,dt$$

$\because$ t is the dummy variable we can replace it by 'x'

$$\boxed{\overline{F(s)} = \overline{F[f(-x)]}}$$

CONVOLUTION THEOREM OF TWO FUNCTIONS

Let $f(x)$ and $g(x)$ be the two functions defined in the region $(\infty, -\infty)$. Then convolution of $f(x)$ and $g(x)$ is

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

CONVOLUTION THEOREM IN FOURIER TRANSFORM

Statement:-

The fourier transform of convolution of two functions is equal to the product of their fourier transform.

$$F[f(x) * g(x)] = F(s)\, G(s)$$

where $F[f(x)] = F(s)$; $F[g(x)] = G(s)$

Proof:

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$$F[f(x) * g(x)] = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} f(x) * g(x)\, e^{isx}\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \right] e^{isx}\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-u)\, e^{isx}\, dx \right] f(u)\, du$$

(on changing the order of integration)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) F[g(x-u)]\, du \quad [\because \text{definition of F.T}]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{ius} G(s)\, du$$

$\left[\text{using } F(x-a) = e^{ias} F(s)\right.$
$\left. \therefore F[g(x-u)] = e^{ius} G(s)\right]$

$$= G(s) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{isu}\, du$$

$$= G(s) \cdot F(s)$$

$$\therefore F[(f * g)x] = F(s) \cdot G(s)$$

1. Find the FT for $f(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & \text{else} \end{cases}$

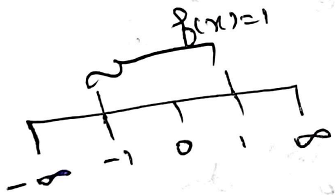
Deduce that i) $\int_0^{\infty} \frac{\sin x}{x}\, dx = \pi/2$

ii) $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$

Soln

WKT $F[f(x)] = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$

Given $f(x) = \begin{cases} 1 & ; |x| \le 1 \\ 0 & ; \text{else} \end{cases}$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-1} f(x) e^{isx}\, dx + \int_{-1}^{1} f(x) e^{isx}\, dx + \int_{1}^{\infty} f(x) e^{isx}\, dx \right\}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx}\, dx \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos sx + i \sin sx)\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos sx\, dx + i \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx\, dx$$

Since $\cos sx$ is an even function.

$\sin sx$ is an odd function.

$$\int_{-1}^{1} \cos sx\,dx = 2\int_{0}^{1} \cos sx\,dx$$

$$\Rightarrow \int_{-1}^{1} \sin sx\,dx = 0$$

$$\Rightarrow \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos sx\,dx$$

$$\Rightarrow \frac{2}{\sqrt{2\pi}} \left(\frac{\sin sx}{s}\right)_{0}^{1} = \sqrt{2/\pi}\, \frac{\sin s}{s}$$

$$F(s) = \sqrt{2/\pi}\, \frac{\sin s}{s}$$

i) To prove $\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi/2$

WKT $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\,ds$

$$\Rightarrow 1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2/\pi}\, \frac{\sin s}{s}\, e^{isx}\,ds$$

Put $x = 0$

$$1 = \frac{1}{\sqrt{2\pi}} \sqrt{2/\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s}\,ds$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s}\,ds$$

Put $s = x$; $ds = dx$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

$\frac{\sin x}{x}$ is an even function

$$2\int_0^\infty \frac{\sin x}{x} = \pi$$

$$\boxed{\int_0^\infty \frac{\sin x}{x} = \pi/2}$$

ii, To prove $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$

WKT

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \left[\sqrt{2/\pi}\,\frac{\sin s}{s}\right]^2 ds = \int_{-1}^{1} 1^2 dx$$

$$2/\pi \int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = [x]_{-1}^{1} = 2$$

$$1/\pi \int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \pi$$

Put $s = x$; $ds = dx$ and $\frac{\sin s}{s}$ is an even function

$$\boxed{\int_0^\infty \left(\frac{\sin x}{x}\right)^2 ds = \pi/2}$$

Parseval's Identity:-

Let $F(s)$ be the Fourier transform of $f(x)$. Then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

2. Find the FT for $f(x) = \begin{cases} a-|x| & ; |x| \le a \\ 0 & ; \text{else} \end{cases}$

and hence deduce that

i) $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$

ii) $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \pi/3$.

Soln

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} f(x)\, e^{isx}\, dx + \int_{-a}^{a} f(x)\, e^{isx}\, dx + \int_{a}^{\infty} f(x)\, e^{isx}\, dx \right\}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)\, e^{isx}\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) \sin sx)\, dx$$

Since $(a-|x|)\cos sx$ is an even function and $(a-|x|)\sin sx$ is an odd function.

$$\Rightarrow \frac{2}{\sqrt{2\pi}} \int_0^a (a-|x|) \cos sx\, dx$$

$$\Rightarrow \frac{2}{\sqrt{2\pi}} \int_0^a (a-x) \cos sx\, dx$$

$dv = \cos sx\, dx$

$u = a-x$ $\quad v = \frac{\sin sx}{s}$; $v_1 = \frac{-\cos x}{s^2}$

$u' = -1$

$$\Rightarrow \sqrt{2/\pi}\left[(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^a$$

$$\Rightarrow \sqrt{2/\pi}\left[\left(0-\frac{\cos sa}{s^2}\right)-\left(0-1/s^2\right)\right]$$

$$\Rightarrow \sqrt{2/\pi}\left[\frac{1-\cos sa}{s^2}\right] \qquad \left[1-\cos\theta = 2\sin^2\theta/2\right]$$

$$\Rightarrow \sqrt{2/\pi}\left[\frac{2\sin^2 sa/2}{s^2}\right] \Rightarrow \sqrt{2/\pi}\, 2\, \frac{\sin^2(sa/2)}{s^2}$$

$$F(s) \Rightarrow 2\sqrt{2/\pi}\left(\frac{\sin(sa/2)}{s}\right)^2$$

i, To prove that $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$

Since the power of $F(s)$ and power of integrant are same, we ~~h~~ use inversion formula

WKT $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{2/\pi}\left(\frac{\sin sa/2}{s}\right)^2 e^{-isx}\, ds$$

$$a-|x| = 2/\pi \int_{-\infty}^{\infty}\left(\frac{\sin sa/2}{s}\right) e^{-isx}\, ds \qquad |x|\le a$$

Put $x=0$

$$a = 2/\pi \int_{-\infty}^{\infty}\left(\frac{\sin(sa/2)}{s}\right)^2 ds$$

$$\frac{\pi a}{2} = \int_{-\infty}^{\infty}\left(\frac{\sin(sa/2)}{s}\right)^2 ds$$

put $\frac{sa}{2} = x \Rightarrow s = \frac{2x}{a}$

$ds = \frac{2dx}{a}$; $s \to \infty$; $x \to \infty$
$s \to -\infty$; $x \to -\infty$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{2x/a}\right)^2 \cdot \frac{2dx}{a} = \pi a/2$$

$$\int_{-\infty}^{\infty} \frac{(\sin x)^2}{4x^2/a^2} \cdot \frac{2\,dx}{a} = \pi a/2$$

$$\int_{-\infty}^{\infty} (\sin x)^2 \left(\frac{a^2}{4x^2}\right) \frac{2dx}{a} = \frac{\pi a}{2}$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi$$

since $\left(\frac{\sin x}{x}\right)^2$ is an even function

$$2\int_{0}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi$$

$$\boxed{\therefore \int_{0}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi/2}$$

ii, To prove that $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \pi/3$

Since the power of integrant is square of $F(s)$,

we use Parseval's identity

WKT $\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

$$\Rightarrow \int_{-\infty}^{\infty} \left[2\sqrt{2/\pi}\left(\frac{\sin(sa/2)}{s}\right)^2\right]^2 ds = \int_{-a}^{a}(a-|x|)^2\,dx$$

$$\Rightarrow 8/\pi \int_{-\infty}^{\infty} \left(\frac{\sin(sa/2)}{s}\right)^4 ds = \int_{-a}^{a} (a-x)^2\,dx$$

$$8/\pi \int_{-\infty}^{\infty} \left(\frac{\sin sa/2}{s}\right)^4 ds = \left[\frac{(a-x)^3}{-3}\right]_{-a}^{a}$$

$$8/\pi \int_{-\infty}^{\infty} \left(\frac{\sin(sa/2)}{s}\right)^4 ds = \left[0 + \frac{8a^3}{3}\right]$$

$$8/\pi \int_{-\infty}^{\infty} \left(\frac{\sin(sa/2)}{s}\right)^4 ds = \frac{2a^3}{3}$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin(sa/2)}{s}\right)^4 ds = \frac{a^3\pi}{4\cdot 3}$$

put $\frac{sa}{2} = x$; $s = \frac{2x}{a}$ $\quad ds = \frac{2\,dx}{a}$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{2x/a}\right)^4 \frac{2\,dx}{a} = \frac{a^3\pi}{4\cdot 3}$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 \frac{a^4}{16}\cdot\frac{2\,dx}{a} = \frac{a^3\pi}{4\times 3}$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 = 2\pi/3$$

Since $\left(\frac{\sin x}{x}\right)^4$ is an even

$$2\int_0^\infty \left(\frac{\sin x}{x}\right)^4 = \frac{2\pi}{3}$$

$$\boxed{\int_0^\infty \left(\frac{\sin x}{x}\right)^4 = \pi/3}$$

3. Find FT of $f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & \text{else} \end{cases}$ and hence deduce $\int_0^\infty \left(\frac{\sin x - \cos x}{x^3}\right) \cos\left(\frac{x}{2}\right) dx = \frac{3\pi}{16}$.

Soln

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \left\{ \int_{\infty}^{-1} f(x)\, e^{isx}\, dx + \int_{-1}^{1} f(x)\, e^{isx}\, dx + \int_1^\infty f(x)\, e^{isx}\, dx \right\}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)(\cos sx + i \sin sx)\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \sin sx\, dx$$

Since $(1-x^2)\sin sx$ is an odd function.

$$\Rightarrow \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x^2) \cos sx\, dx$$

$u = 1-x^2$ $\quad v = \sin sx/s$

$u' = -2x$ $\quad v_1 = -\cos sx/s$

$u'' = -2$ $\quad v_2 = -\sin sx/s^3$

$$\Rightarrow \frac{2}{\sqrt{2\pi}} \left[(1-x^2)\frac{\sin sx}{s} - \frac{2x \cos sx}{s^2} + \frac{2 \sin sx}{s^3} \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}}\left[\left(0 - \frac{2\cos s}{s^2} + \frac{2\sin s}{s^3}\right) - (0)\right]$$

$$\Rightarrow \frac{2}{\sqrt{2\pi}}\left[\frac{2\sin s}{s^3} - \frac{2\cos s}{s^2}\right]$$

$$\Rightarrow \frac{2}{2\sqrt{\pi}}\left[\frac{2\sin s - 2s\cos s}{s^3}\right] = \frac{4}{\sqrt{2\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$$

$$F(s) = 2\sqrt{2/\pi}\left[\frac{\sin s - s\cos s}{s^3}\right]$$

To deduce that $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)\cos(x/2)\,dx = 3\pi/16$

Since the power of integrand and the power of $F(s)$ are same, we use inversion formula.

WKT $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

$$1 - x^2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{2/\pi}\left[\frac{\sin s - s\cos s}{s^3}\right] e^{-isx}\, ds \quad |x| \le 1$$

$$1 - x^2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{2/\pi}\left[\frac{\sin s - s\cos s}{s^3}\right](\cos sx - i\sin sx)\,ds$$

$$\Rightarrow \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

$$\Rightarrow \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,dx$$

$$- \frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sx\,ds$$

Since $\left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx$ is an odd function

and $\frac{\sin s - s\cos s}{s^3} \cos sx$ is an even function.

$$1-x^2 = 4/\pi \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds$$

put $x = 1/2$

$$1 - 1/4 = 4/\pi \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos(s/2)\, ds$$

$$3/4 = 4/\pi \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos(s/2)\, ds$$

$$\frac{3\pi}{16} = \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos(s/2)\, ds$$

put $s = x$ $\quad ds = dx$; $\quad s \to \infty \quad x \to \infty$

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) \cos(x/2)\, dx = 3\pi/16.$$

SELF-RECIPROCAL

Fourier Transform for $f(x)$ is said to be self-reciprocal if $F[f(x)] = f(s)$

1. verify $f(x) = e^{-x^2/2}$ is self reciprocal.

soln

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(-x^2/2 + isx)} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2/2 - isx)} \, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2 - 2isx}{2}\right)} \, dx$$

WKT $(A-B)^2 = A^2 + B^2 - 2AB$

$A^2 - 2AB = (A-B)^2 - B^2$

$A = x$; $B = is$

$x^2 - 2isx = (x-is)^2 - (is)^2$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{(x-is)^2 - (is)^2}{2}\right)} \, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} e^{(is)^2/2} \, dx$$

$$= 1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} e^{-s^2/2} \, dx$$

$$\Rightarrow \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x-is)^2} \, dx \Rightarrow \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} \, dx$$

Let $t = \frac{x-is}{\sqrt{2}}$; $x - is = t\sqrt{2} \Rightarrow dx = \sqrt{2}\, dt$

$$\Rightarrow \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt \; ; \Rightarrow \frac{e^{-s^2/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 \, dt$$

W.K.T $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ [Gamma function]

$$\Rightarrow \frac{e^{-s^2/2}}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}} \Rightarrow e^{-s^2/2} = f(s)$$

$\boxed{F(f(x)) = f(s)}$

FOURIER COSINE TRANSFORM

$$F_c[f(x)] = F_c(s) = \sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx$$

INVERSE FOURIER COSINE TRANSFORM

$$f(x) = \sqrt{2/\pi} \int_0^\infty F_c(s) \cos sx \, ds$$

FOURIER SINE TRANSFORMS

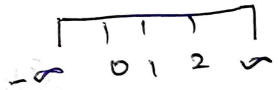
$$F_s[f(x)] = F_s(s) = \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx$$

INVERSE FOURIER SINE TRANSFORMS

$$f(x) = \sqrt{2/\pi} \int_0^\infty F_s(s) \sin sx \, dx$$

1. Find the Fourier sine and Cosine transform for

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$$



Soln

To find Fourier sine Transform:

$$F_s[f(x)] = \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx$$

$$\Rightarrow \sqrt{2/\pi} \left\{ \int_0^1 f(x) \sin sx \, dx + \int_1^2 f(x) \sin sx \, dx + \int_2^\infty f(x) \sin sx \, dx \right\}$$

$$\Rightarrow \sqrt{2/\pi} \left\{ \int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx + 0 \right\}$$

$$\Rightarrow \sqrt{2/\pi} \left\{ \int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx \right\}$$

$dv = \sin s x \, dx$

$u = x \qquad v = \frac{-\cos s x}{s} \qquad u = 2-x$

$u' = 1 \qquad\qquad\qquad\qquad u' = -1$

$v_1 = \frac{-\sin s x}{s^2}$

$$\Rightarrow \sqrt{2/\pi} \left\{ \left[\frac{-x\cos s x}{s} + \frac{\sin s x}{s^2} \right]_0^1 + \left[\frac{(2-x)(-\cos s x)}{s} - \frac{\sin s x}{s^2} \right]_1^2 \right\}$$

$$\Rightarrow \sqrt{2/\pi} \left\{ \left[\frac{-\cos s}{s} + \frac{\sin s}{s^2} \right] - \left[0+0 \right] + \left[0 - \sin 2s - \left(\frac{-\cos s}{s} - \frac{\sin s}{s^2} \right) \right] \right\}$$

$$\Rightarrow \sqrt{2/\pi} \left\{ \frac{-\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2} \right\}$$

$$\Rightarrow \sqrt{2/\pi} \left\{ \frac{2\sin s - \sin 2s}{s^2} \right\}$$

$$\boxed{F_s[f(x)] = \sqrt{2/\pi} \left\{ \frac{2\sin s - \sin 2s}{s^2} \right\}}$$

To find Fourier Cosine transform

$$F_c[f(x)] = \sqrt{2/\pi} \int_0^\infty f(x) \cos s x \, dx$$

$$= \sqrt{2/\pi} \left\{ \int_0^1 f(x) \cos s x \, dx + \int_1^2 f(x) \cos s x \, dx + \int_2^\infty f(x) \cos s x \, dx \right\}$$

$$\Rightarrow \sqrt{2/\pi} \left\{ \int_0^1 x \cos s x \, dx + \int_1^2 (2-x) \cos s x \, dx \right\}$$

$dv = \cos s x \, dx$

$u = x \qquad v = \frac{\sin s x}{s} \qquad u = 2-x$

$u' = 1 \qquad\qquad\qquad\qquad u' = -1$

$v_1 = \frac{-\cos s x}{s^2}$

$$= \sqrt{2/\pi}\left\{\left[\frac{x\sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^1 + \left[\frac{(2-x)\sin sx}{s} - \frac{\cos sx}{s^2}\right]_1^2\right\}$$

$$\Rightarrow \sqrt{2/\pi}\left\{\left[\left(\frac{\sin s}{s} + \frac{\cos s}{s^2}\right) - \left(0 + 1/s^2\right)\right]\right.$$

$$\left.\left[\left(0 - \frac{\cos 2s}{s^2}\right) - \left(\frac{\sin s}{s} - \frac{\cos s}{s^2}\right)\right]\right\}$$

$$\Rightarrow \sqrt{2/\pi}\left[\frac{\sin s}{s} + \frac{\cos s}{s^2} - 1/s^2 - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right]$$

$$\Rightarrow \sqrt{2/\pi}\left[\frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - 1/s^2\right]$$

$$\Rightarrow \sqrt{2/\pi}\left[\frac{2\cos s - \cos 2s - 1}{s^2}\right]$$

$$\boxed{F_c[f(s)] = \sqrt{2/\pi}\left[\frac{2\cos s - \cos 2s - 1}{s^2}\right]}$$

2. Find the Fourier sine transform for

$$f(x) = \begin{cases} e^x; & 0 < x < 1 \\ -1 & 1 \le x \le 2 \\ 0 & ; x > 2 \end{cases}$$

Soln

$$F_s[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx$$

$$\Rightarrow \sqrt{2/\pi}\left\{\int_0^1 f(x)\sin sx\,dx + \int_1^2 f(x)\sin sx\,dx + \int_2^\infty f(x)\sin sx\,dx\right\}$$

$$\Rightarrow \sqrt{2/\pi}\left\{\int_0^1 e^x \sin sx\,dx + \int_1^2 -\sin sx\,dx + 0\right\}$$

$$\Rightarrow \sqrt{2/\pi}\left\{\int_0^1 e^x \sin sx\,dx\right\} + \sqrt{2/\pi}\left\{\int_0^1 (-2\sin sx\,dx)\right\}$$

WKT

$$\int e^{ax} \sin dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$$

$$\Rightarrow \sqrt{2/\pi}\left\{\left[\frac{e^x}{1+s^2}\left[\sin sx - s\cos sx\right]\right]_0^1\right\} + \sqrt{2/\pi}\left[\frac{-\cos sx}{s}\right]_1^2$$

$$\Rightarrow \sqrt{2/\pi}\left[\left(\frac{e}{1+s^2}(\sin s - s\cos s)\right) - \left(\frac{1}{1+s^2}[0-s]\right)\right]$$

$$+ \sqrt{2/\pi}\left[\frac{-\cos 2s}{s^2} + \frac{\cos s}{s}\right]$$

$$\Rightarrow \sqrt{2/\pi}\left\{\frac{e}{1+s^2}(\sin s - s\cos s) + \frac{s}{1+s^2} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s}\right\}$$

$$F_s[f(x)] = \sqrt{2/\pi}\left\{\frac{e}{1+s^2}(\sin s - s\cos s) + \frac{s}{1+s^2} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s}\right\}$$

3. Find the FST and FCT for $f(x) = e^{-ax}$; $x > 0$ $a > 0$ and deduce that the inversion formula.

Soln

To find FST:

$$F_s[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{2/\pi}\int_0^\infty e^{-ax} \sin sx \, dx$$

WKT
$$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a \sin sx - s\cos sx)\right]_0^\infty$$

$$\Rightarrow \sqrt{2/\pi}\left[0 - 1/a^2+s^2 (-s)\right] = \sqrt{2/\pi}\,\frac{s}{s^2+a^2}$$

$$\boxed{F_s[f(x)] = \sqrt{2/\pi}\,\frac{s}{a^2+s^2}}$$

To find inversion formula :-

The inversion formula is

$$f(x) = \sqrt{2/\pi} \int_0^\infty F_s(s) \sin sx \, dx$$

$$e^{-ax} = \sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi} \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\boxed{\int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds = \pi/2 \, e^{-ax}}$$

To find FCT :

WKT $F_c[f(x)] = \sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx$

$$\Rightarrow \sqrt{2/\pi} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$\Rightarrow \sqrt{2/\pi} \left[\frac{e^{-ax}}{a^2+s^2}(-a \cos sx + s \sin sx)\right]_0^\infty$$

$$\Rightarrow \sqrt{2/\pi} \left[0 - 1/a^2+s^2 \, (-a+0)\right] = \sqrt{2/\pi} \, a/a^2+s^2$$

$$\boxed{F_c[f(x)] = \sqrt{2/\pi} \, a/a^2+s^2}$$

To find inversion formula

WKT $f(x) = \sqrt{2/\pi} \int_0^\infty F_c(s) \cos sx \, dx$

$$e^{-ax} = \sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi} \frac{a}{a^2+s^2} \cos sx \, dx$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{a \cos sx}{a^2+s^2} ds$$

$$\boxed{\int_0^\infty \frac{\cos sx}{a^2+s^2} = \frac{\pi e^{-ax}}{2a}}$$

Result :- FT under derivatives

i, $F_c[x f(x)] = \frac{d}{ds}[F_s(f(x))]$

ii, $F_s[x f(x)] = \frac{-d}{ds}[F_c f(x)]$

Find the sine and cosine Fourier transform for $f(x) = xe^{-ax}$
$x > 0$

soln

WKT $F_s[f(x)] = \frac{d}{ds}[F_c[f(x)]]$

$$F_s[xe^{-ax}] = \frac{d}{ds}\left[\sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx\right]$$

$$= \frac{d}{ds}\left[\sqrt{2/\pi} \int_0^\infty e^{-ax} \cos sx \, dx\right]$$

$$= d/ds\left[\sqrt{2/\pi} \frac{s}{s^2+a^2}\right]$$

$$\Rightarrow \sqrt{2/\pi} \frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \sqrt{2/\pi}\left[\frac{(a^2+s^2)(1) - s(2s)}{(a^2+s^2)^2}\right]$$

$$\Rightarrow \sqrt{2/\pi}\left[\frac{a^2+s^2-2s^2}{(a^2+s^2)^2}\right] = \sqrt{2/\pi} \frac{[-s^2+a^2]}{(s^2+a^2)^2}$$

$$\boxed{F_c[x f(x)] = \sqrt{2/\pi} \frac{(a^2-s^2)}{(s^2+a^2)^2}}$$

WKT $F_c[x f(x)] = -\frac{d}{ds}[F_s(f(x))]$

$$F_c[x f(x)] = -\frac{d}{ds}\left[\sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx\right]$$

$$= -\frac{d}{ds}\left[\sqrt{2/\pi}\int_0^\infty e^{-ax}\sin sx\,dx\right]$$

$$= -\frac{d}{ds}\left[\sqrt{2/\pi}\,\frac{a}{a^2+s^2}\right]$$

$$\Rightarrow -\sqrt{2/\pi}\,a\,\frac{d}{ds}\left[1/s^2+a^2\right]$$

$$\Rightarrow -a\sqrt{2/\pi}\left[\frac{(s^2+a^2)0-1(2s)}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \sqrt{2/\pi}\left[\frac{2as}{(s^2+a^2)^2}\right]$$

$$\boxed{\therefore F_c[x f(x)] = \sqrt{2/\pi}\left(\frac{2as}{(s^2+a^2)^2}\right)}$$

2. Find the Fourier Sine transform for $\frac{x}{x^2+a^2}$ and Fourier cosine transform $\frac{1}{x^2+a^2}$.

$$F_c\left[\frac{1}{x^2+a^2}\right] = \sqrt{2/\pi}\int_0^\infty \frac{1}{x^2+a^2}\cos sx\,dx$$

Let $f(x) = e^{-ax}$

$$F_c[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{2/\pi}\int_0^\infty e^{-ax}\cos sx\,dx.$$

$$F_c[f(x)] = \sqrt{2/\pi}\,\frac{a}{a^2+s^2}$$

WKT $f(x) = \sqrt{2/\pi} \int_0^\infty F_c(s) \cos sx \, dx$

$\Rightarrow \sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi} \cos sx \cdot \frac{a}{a^2+s^2} ds$

$\Rightarrow \frac{2}{\pi} \int_0^\infty \frac{a}{a^2+s^2} \cos sx \, ds \Rightarrow \int_0^\infty \frac{a}{a^2+s^2} \cos sx \, dx = \pi/2 \, e^{-ax}$

$\int_0^\infty \frac{1}{a^2+x^2} \cos sx \, dx = \pi/2a \, e^{-as}$

$F_c\left[\frac{1}{x^2+a^2}\right] = \sqrt{2/\pi} \; \pi/2a \; e^{-as}$

$F_c\left[\frac{1}{x^2+a^2}\right] = \sqrt{\pi/2} \; 1/a \, e^{-as}$

$F_s\left[\frac{x}{x^2+a^2}\right] = F_s\left[x \cdot \frac{1}{x^2+a^2}\right]$

$\Rightarrow \frac{-d}{ds} F_c\left[\frac{1}{x^2+a^2}\right]$

$\Rightarrow \frac{-d}{ds} \sqrt{\pi/2} \, \frac{e^{-ax}}{a}$

$\Rightarrow -\sqrt{\pi/2} \; 1/a \; e^{-ax} (-a) = \sqrt{\pi/2} \, e^{-ax}$

$$\boxed{F_s\left[\frac{x}{x^2+a^2}\right] = \sqrt{\pi/2} \, e^{-ax}}$$

PARSEVAL'S IDENITITY

1. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$; $a, b > 0$

Soln

Since the numerator in the integrant is dx, we use Fourier cosine transform. Take $f(x) = e^{-ax}$

$g(x) = e^{-bx}$

$$F_c[s] = F_c[f(x)]$$

$$= \sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx$$

$$\Rightarrow \sqrt{2/\pi} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$F_c(s) = \sqrt{2/\pi}\ a/a^2+s^2 \, ;$$

Similarly $G_c(s) = \sqrt{2/\pi}\ b/b^2+s^2 \Rightarrow \sqrt{2/\pi} \int_0^\infty e^{-bx} \cos sx \, dx$

WKT

$$\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

$$\Rightarrow \int_0^\infty \sqrt{2/\pi}\ a/a^2+s^2\ \sqrt{2/\pi}\ b/b^2+s^2\ ds = \int_0^\infty e^{-ax} \cdot e^{-bx}\, dx$$

$$\Rightarrow \frac{2ab}{\pi} \int_0^\infty \left(1/a^2+s^2\right)\left(1/b^2+s^2\right) ds = \int_0^\infty e^{-(a+b)x}\, dx$$

$$\frac{2ab}{\pi} \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\frac{2ab}{\pi} \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left[0 - \left(-1/a+b\right)\right]$$

$$\Rightarrow \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

put $s = x$ $\quad ds = dx$

$$\boxed{\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}}$$

2\. Evaluate $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$ $\quad a ; b > 0$

Soln

Since the numerator of the integrant is $x^2 dx$, so we use FST.

Take $f(x) = e^{-ax}$ $\quad f(x) = e^{-bx}$

$F_s[s] = F_s[f(x)]$

$\Rightarrow \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx$

$\Rightarrow \sqrt{2/\pi} \int_0^\infty e^{-ax} \sin sx \, dx$

$\Rightarrow \sqrt{2/\pi} \frac{s}{s^2+a^2}$

$F_s(s) = \sqrt{2/\pi} \frac{s}{s^2+a^2}$; $G_s(s) = \sqrt{2/\pi}\ s/s^2+b^2$

WKT

$$\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

$$\int_0^\infty \sqrt{2/\pi} \frac{s}{s^2+a^2} \sqrt{2/\pi}\ s/s^2+b^2 \, ds = \int_0^\infty e^{-(a+b)x} dx$$

$$\Rightarrow 2/\pi \int_0^\infty \frac{s^2\,ds}{(s^2+a^2)(s^2+b^2)} = 1/a+b$$

$$\boxed{\therefore \int_0^\infty \frac{s^2\,ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2(a+b)}}$$

put $s=x$; $ds=dx$

$$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}$$

3. Evaluate $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$

$$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}$$

4mark

$$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+a^2)} = \frac{\pi}{2(a+a)} \Rightarrow \pi/4a$$

$$\boxed{\therefore \int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+a^2)} = \pi/4a}$$

4. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$

Soln

WKT $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

4mark

$$\int_0^\infty \frac{dx}{(x^2+1^2)(x^2+2^2)} = \frac{\pi}{2\cdot 1\cdot 2\,(1+2)} = \pi/12$$

$$\boxed{\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \pi/12}$$

Integral Equation :-

An equation involving integrating is called integral equation.

1. Solve the integral equation $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$

Soln $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$

put $\lambda = s$

$$\int_0^\infty f(x) \cos sx \, dx = e^{-s}$$

$$\sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx = \sqrt{2/\pi}\, e^{-s}$$

$$F_c(s) = \sqrt{2/\pi}\, e^{-s}$$

WKT $f(x) = \sqrt{2/\pi} \int_0^\infty F_c(s) \cos sx \, ds$

$$= \sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi}\, e^{-s} \cos sx \, ds$$

$$\Rightarrow 2/\pi \int_0^\infty e^{-s} \cos sx \, dx \Rightarrow 2/\pi \; 1/x^2+1$$

$$\boxed{f(x) = \frac{2}{\pi}\left(1/x^2+1\right)}$$

2. Find the function its Fourier sine transform is $\frac{e^{-as}}{s}$ $a>0$

Soln Given $F_s(s) = \frac{e^{-as}}{s}$

WKT $f(x) = \sqrt{2/\pi} \int_0^\infty F_s(s) \sin sx \, dx$

$$f(x) = \sqrt{2/\pi} \int_0^\infty \frac{e^{-as}}{s} \sin sx \, ds$$

$$\frac{d}{dx}[f(x)] = \sqrt{2/\pi} \int_0^\infty \frac{\partial}{\partial x}(\sin sx) \frac{e^{-as}}{s} \, ds$$

$$\frac{d}{dx}[f(x)] = \sqrt{2/\pi} \int_0^\infty \frac{e^{-as}}{s} \, s \cos sx \, dx$$

$$\frac{d}{dx}[f(x)] = \sqrt{2/\pi}\ \frac{a}{a^2+x^2}$$

$$\int \frac{d}{dx}[f(x)] = \sqrt{2/\pi} \int \frac{a}{a^2+x^2}\,dx$$

$$f(x) = \sqrt{2/\pi}\ \tan^{-1}(x/a) + c$$

As $x \to 0$ $\quad f(x) = 0 \quad c = 0$

$$\boxed{f(x) = \sqrt{2/\pi}\ \tan^{-1}(x/a)}$$

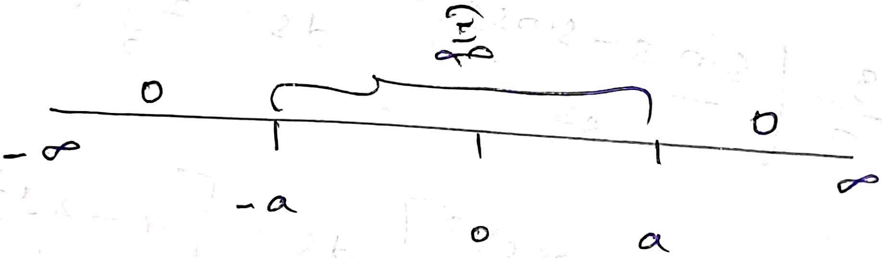
Find the Fourier transform of $f(x) = \begin{cases} x, & |x| < a \\ 0, & |x| > a \end{cases}$

X.

Here $f(x) = x$ in $-a < x < a$

$f(x) = 0$ in $-\infty < x < -a$ & $a < x < \infty$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$



$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x) e^{isx} dx + \int_{-a}^{a} f(x) e^{isx} dx + \int_{a}^{\infty} f(x) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 \cdot e^{isx} dx + \int_{-a}^{a} x e^{isx} dx + \int_{a}^{\infty} 0 \cdot e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x e^{isx} dx \right]$$

Formula

$e^{ix} = \cos x + i \sin x$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x (\cos sx + i \sin sx) \, dx \right.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x \cos sx \, dx + i \int_{-a}^{a} x \sin sx \, dx \right]$$

$$\left[x\cos sx \rightarrow \text{odd} \Rightarrow \int_{-a}^{a} x\cos sx\,dx = 0 \right.$$

$$\left. x\sin sx \rightarrow \text{even} \Rightarrow \int_{-a}^{a} x\sin sx\,dx = 2\int_{0}^{a} x\sin sx\,dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + i2\int_0^a \underset{u}{x}\,\underset{v}{\sin sx}\,dx\right] \qquad \left[\int uv = uv_1 - u'v_2 + u''v_3 - \dots\right]$$

$$= \frac{2i}{\sqrt{2\pi}}\left[x\left(\frac{-\cos sx}{s}\right) - (1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^a$$

$$= \frac{2i}{\sqrt{2\pi}}\left[-\frac{x\cos sx}{s} + \frac{\sin sx}{s^2}\right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}}\left[\left\{-\frac{a\cos as}{s} + \frac{\sin as}{s^2}\right\} - \left\{0+0\right\}\right]$$

$$= i\sqrt{2/\pi}\left[-\frac{a\cos as}{s} + \frac{\sin as}{s^2} - 0 - 0\right]$$

$$= i\sqrt{2/\pi}\left[\frac{\sin as - a\cos as}{s^2}\right] //.$$

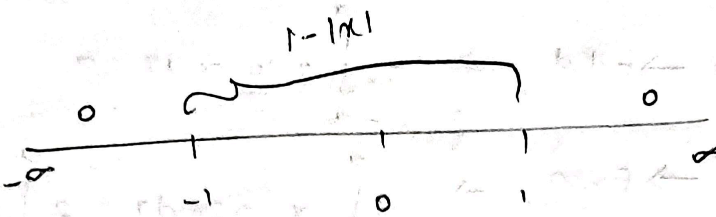
X. Find the Fourier transform of $f(x)$ if

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0 & |x|>1 \end{cases}$$ Hence deduce that

a) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$ b) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \pi/3$ X

Soln

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$



$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} [1-|x|] e^{isx} \, dx \qquad \left[\because f(x)=0 \text{ in } (-\infty,1) \text{ \& } (1,\infty)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left[1-|x| (\cos sx + i \sin sx)\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-|x| \cos sx \, dx) + i \int_{-1}^{1} ((1-|x|) \sin sx \, dx)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-|x|) \cos sx \, dx + i\,(0)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_{0}^{1} (1-x) \cos sx \, dx\right] \qquad \left(\because 1-|x| \cos sx \rightarrow \text{even fn}\right.$$

$$\int_{-1}^{1} 1-|x| \cos sx \, dx = 2 \int_{0}^{1} (1-x) \cos sx \, dx$$

$$\int_{-1}^{1} 1-|x| \sin sx \, dx \rightarrow \text{odd}$$

$$\left.\int_{-1}^{1} 1-|x| \sin sx \, dx = 0\right)$$

$$\because |x| \geq 0 \,;\, |x| = x$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \underbrace{(1-x)}_{u} \underbrace{\cos sx}_{v} \, dx$$

$$\Rightarrow \sqrt{2/\pi} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$= \sqrt{2/\pi} \left[(1-x)\left(\frac{\sin sx}{s}\right) - \frac{\cos sx}{s^2}\right]_0^1$$

$$= \sqrt{2/\pi}\left[\left\{(1-1)\left(\frac{\sin s}{s}\right) - \frac{\cos s}{s^2}\right\} - \left\{1-0\left(\frac{\sin 0}{s}\right) - \frac{\cos 0}{s^2}\right\}\right]$$

$$= \sqrt{2/\pi}\left[\left(0 - \frac{\cos s}{s^2}\right) - \left(0 - 1/s^2\right)\right]$$

$$= \sqrt{2/\pi}\left[\frac{-\cos s}{s^2} + 1/s^2\right]$$

$$F(s) = \sqrt{2/\pi}\ 1/s^2\ (1-\cos s)$$

i, using parseval's identity

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{(1-\cos s)^2}{s^4}\right) ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$[\because f(x)=0$ in $(-\infty,-1)$ & $(1,\infty)]$

$$= 2\int_0^1 (1-x)^2\, dx \qquad [\because |x| = x,\ x>0\ \&\ 1-|x| \to \text{even}]$$

$$\Rightarrow 2\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{-2}{3}[0-1] \Rightarrow 2/3$$

$$2/\pi \int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = 2/3$$

$$\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = \pi/3$$

put $s = 2t$ | $s=\infty \quad t=\infty$

$ds = 2dt$ | $s=-\infty \quad t=-\infty$

$$\int_{-\infty}^{\infty} \frac{(1-\cos 2t)^2}{16t^4} \cdot 2\,dt = \pi/3$$

$$\int_{-\infty}^{\infty} \frac{(1-\cos 2t)^2}{8t^4}\,dt = \pi/3 \qquad \because \frac{(1-\cos 2t)^2}{8t^4} \rightarrow \text{even}$$

$$2\int_{0}^{\infty} \frac{(1-\cos 2t)^2}{8t^4}\,dt = \pi/3 \qquad \because \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\int_{0}^{\infty} \frac{(2\sin^2 t)^2}{4t^4}\,dt = \pi/3$$

$$\int_{0}^{\infty} \frac{\sin^4 t}{t^4}\,dt = \pi/3$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \pi/3 \;//.$$

ii, To prove that $\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$.

Inversion formula four Fouries transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \sqrt{2/\pi} \left(\frac{1-\cos s}{s^2}\right) (\cos sx - i\sin sx)\, ds \right]$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right) \cos sx\, ds - i \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right) \sin sx\, ds \right]$$

$$\left[\because \left(\frac{1-\cos s}{s^2}\right) \sin sx \rightarrow \text{odd} \right.$$

$$\left. \int_{-\infty}^{\infty} \left(\frac{1-\cos sx}{s^2}\right) \sin sx\, dx = 0 \right.$$

$\left(\frac{1-\cos s}{s}\right)$ $\cos sx \to$ even

$$\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\,dx = 2\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\,dx\Big]$$

$$= \frac{1}{\pi}\left[2\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\,dx - i(0)\right]$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\,ds$$

$$\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\,ds = \pi/_2\, f(x)$$

$$= \pi/_2\left[1-|x|\right]$$

put $x=0$, $\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)ds = \pi/_2$

$\boxed{\because \cos x = 1 - 2\sin^2(x/_2)}$

$$\int_{0}^{\infty}\left(\frac{2\sin^2(s/_2)}{s^2}\right)ds = \pi/_2$$

$$\Rightarrow \int_{0}^{\infty}\frac{4\sin^2(s/_2)}{s^2}\,ds = \pi$$

$$\int_{0}^{\infty}\frac{\sin^2(s/_2)}{(s/_2)^2}\,ds = \pi$$

put $t = s/_2$ | $s=\infty$ $t=\infty$

$dt = ds/_2$ | $s=0$ $t=0$

$$\int_{0}^{\infty}\frac{\sin^2 t}{t^2}\,2dt = \pi \quad \Rightarrow \int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \pi/_2 //.$$

Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} a^2-x^2 & |x| \le a \\ 0 & |x| > a. \end{cases}$$

hence prove that $\int_0^{\infty} \frac{\sin x - x\cos x}{x^3}\, dx = \pi/4.$

Soln

W.K.T FT of $f(x)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} f(x)\, e^{isx} dx + \int_{-a}^{a} f(x) e^{isx} dx + \int_{a}^{\infty} f(x) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} 0 .\ e^{isx} dx + \int_{-a}^{a} (a^2-x^2)\, e^{isx} dx + \int_{a}^{\infty} (a^2-x^2\, e^{isx} dx \right.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2-x^2) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2-x^2) \left(\cos sx + i \sin sx \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} (a^2-x^2) \cos sx\, dx + i \int_{-a}^{a} (a^2-x^2) \sin sx\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_{0}^{a} (a^2-x^2) \cos sx\, dx \right]$$

$(a^2-x^2) \cos sx \rightarrow$ even fn $\rightarrow 2 \int_0^a (a^2-x^2) \cos sx\, dx$

$(a^2-x^2) \sin sx \rightarrow$ odd fn $\rightarrow 0$

$$\frac{2}{\sqrt{2\pi}} \int_0^a (a^2-x^2) \cos sx \, dx$$

$u = a^2 - x^2$ $\qquad$ $v = \cos sx$

$u' = -2x$ $\qquad$ $v_1 = \frac{\sin sx}{s}$

$u'' = -2$ $\qquad$ $v_2 = -\frac{\cos sx}{s^2}$

$v_3 = -\frac{\sin sx}{s^3}$

$\int uv = uv_1 - u'v_2 + u''v_3 - \dots$

$$\frac{2}{\sqrt{2\pi}} \left[(a^2-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(0 - 2a\frac{\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right) - (0+0+0) \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{2\sin sa}{s^3} - \frac{2a\cos sa}{s^2} \right]$$

$$\Rightarrow \frac{4}{\sqrt{2\pi}} \left[\frac{\cancel{2}\sin sa - as\cos sa}{s^3} \right] = F(s)$$

ie $2\sqrt{2/\pi} \left\{ \frac{\sin sa - s\cos s}{s^3} \right\}$ put $a=1$ —— A

using inverse Fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2\sqrt{2/\pi} \int_{-\infty}^{\infty} \frac{1}{s^3} [\sin s - s\cos s] e^{-isx} \, dx \right.$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{s^3} [\sin s - s\cos s] (\cos sx - i\sin sx) \right\} ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \cos sx \, ds -$$

$$\frac{2i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos sx \, ds - 0$$

$$\left[\begin{array}{l} \frac{\sin s - s\cos s}{s^3} \cos sx \rightarrow \text{even} \\ \therefore \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos sx \, ds = 2 \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos sx \, ds \\ \frac{\sin s - s\cos s}{s^3} \sin sx \rightarrow \text{odd} \\ \therefore \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \sin sx \, ds = 0 \end{array} \right]$$

$$= 4/\pi \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos sx \, ds$$

(or) Replacing x by 0

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \, ds \qquad \boxed{\cos 0 = 1}$$

Replacing $s \rightarrow x$

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3} \, dx$$

& $a = 1$

$x \rightarrow 0$

$[\because f(x) = a^2 - x^2$

$f(0) = 1 - 0 = 1$

$f(0) = 1]$

$$\frac{\pi}{4} = \int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3} \, dx \; //$$

ii, $\int_{0}^{\infty} \left[\frac{\sin s - x\cos x}{x^3} \right]^2 dt = \pi/15$

Using Parseval's Identity

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

sub $a=1$

$$\int_{-\infty}^{\infty} \left[\frac{2\sqrt{2/\pi}\,(\sin s - s\cos s)}{s^3}\right]^2 ds = \int_{-1}^{1} \left[(1-x^2)\right]^2 dx$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1-x^2)^2 \, dx$$

$$\frac{16}{\pi}\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 \left[1+x^4-2x^2\right] dx$$

$$\Rightarrow 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1 = 2\left[\left(1+\tfrac{1}{5}-\tfrac{2}{3}\right) - (0+0-0)\right]$$

$$= 2\left[\frac{15+3-10}{15}\right] = 2\left(\frac{8}{15}\right) = \frac{16}{15}$$

ie $$\frac{16}{\pi}\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\Rightarrow \int_0^{\infty} \left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \frac{\pi}{15}$$

put $s = x$

$$\int_0^{\infty} \left[\frac{\sin x - x\cos x}{x^3}\right]^2 dx = \frac{\pi}{15}$$

12 mark

X. Evaluate $\int_0^\infty \frac{dx}{(x^2+4)(x^2+1)}$ using Fourier transform method.

$$F_c\left[e^{-2x}\right] = \sqrt{2/\pi}\ \frac{2}{s^2+4} \quad \text{--- (1)}$$

Illy $$F_c\left[e^{-x}\right] = \sqrt{2/\pi}\ \frac{1}{s^2+1} \quad \text{--- (2)}$$

$$\int_0^\infty F_c[f(x)] \cdot F_c[g(x)]\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

using (1) & (2)

$$\int_0^\infty \sqrt{2/\pi}\ \frac{2}{s^2+4} \cdot \sqrt{2/\pi}\ \frac{1}{s^2+1}\, ds = \int_0^\infty e^{-2x} \cdot e^{-x}\, dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{2}{(s^2+4)(s^2+1)}\, ds = \int_0^\infty e^{-(2+1)x}\, dx$$

$$= \int_0^\infty e^{-3x}\, dx \Rightarrow \left[\frac{e^{-3x}}{-3}\right]_0^\infty$$

$$= -\tfrac{1}{3}\left[e^{-\infty} - e^{-0}\right] \Rightarrow \frac{-1}{3}[0-1] = \tfrac{1}{3}$$

$$\therefore \frac{4}{\pi} \int_0^\infty \frac{ds}{(s^2+4)(s^2+1)} = \tfrac{1}{3}$$

$\boxed{s = x}$

$$\int_0^\infty \frac{dx}{(x^2+4)(x^2+1)} = \frac{\pi}{12}$$

Ex. $F_c[f(ax)] = \frac{1}{a} F_c(s/a)$

Soln $F_c[f(ax)] = \sqrt{2/\pi} \int_0^\infty f(ax) \cos sx \, dx$

put $ax = y$ | $x = 0 \Rightarrow y = 0$
$dx = dy/a$ | $x = \infty \Rightarrow y = \infty$

$$= \sqrt{2/\pi} \int_0^\infty f(y) \cos(sy/a) \, dy/a$$

$$= 1/a \sqrt{2/\pi} \int_0^\infty f(y) \cos(s/a) y \cdot dy$$

$$= 1/a \; F_c[s/a]$$

Evaluate $\int_0^\infty \frac{x^2 \, dx}{(x^2+a^2)^2}$ (12marks)

Since the numerator of the integrant is $x^2 dx$, so we have to use FST,

Take $f(x) = e^{-ax}$ $\quad g(x) = e^{-ax}$

$$F_s[s] = F_s[f(x)] = \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{2/\pi} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{2/\pi} \left[\frac{s}{s^2+a^2}\right]$$

$$G_s[s] = G_s[g(x)] = \sqrt{2/\pi} \int_0^\infty g(x) \sin sx \, dx$$

$$= \sqrt{2/\pi} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{2/\pi} \left[\frac{s}{s^2+a^2}\right]$$

using parsevels identity

$$\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

$$\int_0^\infty \sqrt{2/\pi}\left[\frac{s}{s^2+a^2}\right] \cdot \sqrt{2/\pi}\left[\frac{s}{s^2+a^2}\right] ds =$$

$$\int_0^\infty e^{-ax} \cdot e^{-ax}\, dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = \int_0^\infty e^{-ax-ax}\, dx$$

$$= \int_0^\infty e^{-2ax}\, dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= \left[e^{-\infty} - \left(\frac{-1}{2a}\right)\right]$$

$$2/\pi \int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = 1/2a$$

put $s = x$ $\quad ds = dx$

$$2/\pi \int_0^\infty \frac{x^2\, dx}{(x^2+a^2)^2} = 1/2a$$

$$\int_0^\infty \frac{x^2\, dx}{(x^2+a^2)^2} = \frac{\pi}{4a} \quad //$$

Parseval's Identity (Theorem)

Let $F(s)$ be the Fourier transform of $f(x)$ then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Proof: we know that

$$F[f * g] = F(s) \cdot G(s)$$

$$\therefore F^{-1}[F(s) \cdot G(s)] = f * g$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot G(s) e^{-isx} ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

put $x = 0$ we get

$$\int_{-\infty}^{\infty} F(s) G(s) ds = \int_{-\infty}^{\infty} f(u)\, g(-u)\, du \quad - (1)$$

Let $g(-u) = \overline{f(u)}$ — (2)

$g(u) = \overline{f(-u)}$ — (3)

$$F[g(u)] = F[\overline{g(-u)}]$$

$$G(s) = F\overline{f(-x)} \qquad \text{[By Property (7)]}$$

$$= \overline{F(s)}$$

$$G(s) = \overline{F(s)} \quad - (4)$$

Sub (2) and (4) in (1) we get

$$\int_{-\infty}^{\infty} F(s) \overline{F(s)} ds = \int_{-\infty}^{\infty} f(u) \overline{f(u)} du$$

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\left[\because F(s) \cdot \overline{F(s)} = |F(s)|^2\right]$$

Parseval's identity for Fourier sine transform

$$\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty |F_s(s)|^2 \, ds$$

$$F_s(s) = F_s[f(x)]$$

Parseval's identity for Fourier cosine transform

$$\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty |F_c(s)|^2 \, ds$$

$$F_c(s) = F_c[f(x)]$$

Note:

i) $\int_0^\infty f(x) \cdot g(x) \, dx = \int_0^\infty F_s[f(x)] \cdot F_s[g(x)] \, ds$

ii) $\int_0^\infty f(x) \, g(x) \, dx = \int_0^\infty F_c[f(x)] \cdot F_c[g(x)] \, ds.$