Unit - 1

Partial Differential equations.

- A partial differential equation which involves partial derivatives.
- The order of a partial differential equation is the order of the highest derivative occuring in it.

$$z = f(x, y)$$

x, y are independent Variable

z dependent variable.

Notations :-

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r,$$

$$\frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

Formation of differential equations.

- elimination of arbitrary Constants
- elimination of arbitrary functions.

By elimination of arbitrary Constants.

let the function $f(x, y, z, a, b) = 0 \longrightarrow (1)$

where 'a' and 'b' are arbitrary constants.

Diffentiating the eqn ① partially with respect ⓐ to independent variables x, y,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0.$$

ie, $$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \longrightarrow ②$$

IIIly $$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0.$$

ie, $$\frac{\partial f}{\partial y} + q \cdot \frac{\partial f}{\partial z} = 0 \longrightarrow ③$$

eliminating the 2 arbitrary constants 'a' and 'b' from ①, ② & ③, we get a partial differential eqn. of the first order of the form

$$\phi(x, y, z, p, q) = 0.$$

Problems:-

1) Form the partial differential equation by eliminating the arbitrary constants 'a' and 'b' from

$z = (x^2+a)(y^2+b)$.

a, b - arbitrary constants

x, y - independent variables

Soln:-

$$z = (x^2+a)(y^2+b)$$

differentiating partially w.r to 'x' & 'y',

$$\frac{\partial z}{\partial x} = 2x(y^2+b) \quad \text{ie, } p = 2x(y^2+b) \longrightarrow ②$$

$$\frac{\partial z}{\partial y} = 2y(x^2+a) \quad \text{ie, } q = 2y(x^2+a) \longrightarrow ③$$

From ② & ③, we get

$$\left.\begin{aligned} \frac{p}{2x} &= y^2+b. \\ \frac{q}{2y} &= x^2+a \end{aligned}\right\} \quad (*)$$

Sub (*) in eqn ①, we get

$$z = p/2x \cdot q/2y$$

$$\Rightarrow \boxed{4xyz = pq}$$ is the required equation.

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2) Form the partial differential equation by elimination 'a' & 'b' from $\log(az-1) = x+ay+b$.

Soln:-

$$\log(az-1) = x+ay+b \longrightarrow ①$$

diff. partially w.r.to 'x' & 'y', we get

$$\frac{1}{az-1} \cdot a\frac{\partial z}{\partial x} = 1$$

ie, $$\frac{1}{az-1} ap = 1 \longrightarrow ②.$$

$$\frac{1}{az-1} \cdot \cancel{a}\frac{\partial z}{\partial y} = \cancel{a}$$

ie, $$\frac{1}{az-1} \cdot q = 1 \longrightarrow ③.$$

divide ② by ③, we get

$$\frac{\frac{1}{\cancel{az-1}} \cdot ap}{\frac{1}{\cancel{az-1}} \cdot q} = \frac{1}{1} \Rightarrow \frac{ap}{q} = 1$$

$$\Rightarrow a = \frac{q}{p}$$

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Sub 'a' in eqn ③,

$$\frac{1}{(q/p)z-1} \cdot q = 1$$

$$\frac{pq}{qz-p} = 1 \Rightarrow pq = qz - p$$

$$pq + p = qz$$

$\boxed{p(q+1) = qz}$ is the required equation.

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3) Obtain the partial differential equation of all spheres whose centres lie on $z=0$ and whose radius is constant and equal to r.

Soln:- The centre of sphere is $(a,b,0)$

The equation of sphere is $(x-a)^2 + (y-b)^2 + z^2 = r^2$ → ①

diff. par. w.r.to 'x' & 'y', we get

$$2(x-a) + 2z \cdot \frac{\partial z}{\partial x} = 0.$$

ie, $2(x-a) + 2zp = 0$.

$$\cancel{2}(x-a) = -\cancel{2}zp \Rightarrow \boxed{-zp = (x-a)} \rightarrow ②$$

$$2(y-b) + 2z \cdot \frac{\partial z}{\partial y} = 0.$$

ie, $2(y-b) + 2zq = 0$.

$$\cancel{2}(y-b) = -\cancel{2}zq \Rightarrow \boxed{y-b = -zq} \rightarrow ③$$

Sub ② & ③ in eqn ①, we get

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

$$\Rightarrow z^2p^2 + z^2q^2 + z^2 = r^2 \Rightarrow \underline{z^2(p^2+q^2+1) = r^2}$$

which is the req. eqn.

4) Find the partial differential equation of all planes cutting equal intercepts from the 'x' and 'y' axes.

Soln:

Let a, c be the intercepts on x and z axes respectively.

The equation of the plane is $\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \rightarrow (1)$

diff. part. w.r to x & y,

$$\frac{1}{a} + \frac{1}{c} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{1}{a} + \frac{1}{c} \cdot p = 0 . \rightarrow (2)$$

$$\frac{1}{a} + \frac{1}{c} \cdot \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{1}{a} + \frac{1}{c} q = 0 \longrightarrow (3) .$$

From (2) & (3), we get

$$\frac{1}{a} + p/c - \left(\frac{1}{a} + \frac{q}{c}\right) = 0 .$$

$$\cancel{\frac{1}{a}} + p/c - \cancel{1/a} - q/c = 0$$

$$\frac{p-q}{c} = 0 \Rightarrow p - q = 0$$

$$\Rightarrow \boxed{p = q}$$ which is required equation.

Note:

(1) The number of arbitrary constants is equal to the number of independent variables, we get a partial differential equation will be of the first order.

Solution of Partial differential equations.

A solution or integral of a partial differential equation is a relation between the independent and the dependent variables which satisfy the given partial differential equation.

Consider the equations $z = ax + by$ → ① , $z = x f(x/y)$ → ②

where 'a' and 'b' are arbitrary constants & f be an arbitrary function.

① → $z = ax + by$

diff eqn ① par. w.r to 'x' & 'y',

$\frac{\partial z}{\partial x} = a$ ie, $p = a$

$\frac{\partial z}{\partial y} = b$, ie, $q = b$.

$\boxed{\therefore z = px + qy}$

② → $z = x f(x/y)$

diff. eqn ② par. w.r to 'x' & 'y'.

$\frac{\partial z}{\partial x} = x f'(x/y) \cdot 1/y + f(x/y)(1)$

ie, $p = \frac{x}{y} f'(x/y) + f(x/y) \rightarrow$ ③

$\frac{\partial z}{\partial y} = x f'(x/y)\left(-x/y^2\right)$.

ie, $q = f'(x/y)\left(-x^2/y^2\right) \rightarrow$ ④

From ③,

$p = \frac{x}{y} f'(x/y) + z/x$

(∵ from ②).

$p - z/x = x/y\, f'(x/y) \rightarrow$ ⑤

$\frac{⑤}{④} \Rightarrow \frac{p - z/x}{q} = \frac{\cancel{f'(x/y)}\, x/y}{\cancel{f'(x/y)}\left(-x^2/y^2\right)}$

$\frac{px - z}{qx} = \frac{\cancel{x}}{y}\left(\frac{-y^{\cancel{2}}}{x^{\cancel{2}}}\right)$

$\boxed{\text{ie, } px + qy = z}$

We can see that eqn ① & ② are the solutions of the same partial differential equation

$z = px + qy$.

Types of Solutions:-

Complete Solution:-

A solution which contains the number of arbitrary constant is equal to the number of independent variables is called Complete solution or Complete integral.

Particular integral:-

If we give particular values to 'a' and 'b' in the complete solution, then it is a particular integral.

To find the Singular integral:

Suppose that $f(x,y,z,p,q) = 0 \longrightarrow ①$

is the PDE whose Complete integral is

$$\phi(x,y,z,a,b) = 0. \longrightarrow ②$$

where 'a' and 'b' are arbitrary constants.

diff. ② part. w.r to 'a' and 'b', we obtain

$$\frac{\partial \phi}{\partial a} = 0 \longrightarrow ③$$

and $$\frac{\partial \phi}{\partial b} = 0 \longrightarrow ④$$

The eliminant of 'a' and 'b' from the equations ②, ③ and ④, when it exists, is called the Singular integral of eqn ①.

To find the general integral :

In the complete integral, $\phi(x,y,z,a,b)=0 \longrightarrow (1)$

assume that one of the constant is a function of the other. Ie, $b=f(a)$.

then, $\phi(x,y,z,a,f(a)) = 0. \longrightarrow (2)$

diff (1) par. w. r to 'a', then we have

$$\frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial b} \cdot f'(a) = 0 \longrightarrow (3)$$

Eliminating 'a' between (2) and (3) we get the general integral.

Methods to solve the first order partial differential equations.

Type : 1 $F(p,q)=0$.

Suppose that $z=ax+by+c \longrightarrow (1)$ is a solution of the equation $F(p,q)=0$.

Then diff. (1) part. w. r to 'x' & 'y', we get

$$\frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = b.$$

Ie, $p=a$, $q=b$.

Substituting these in the given equation, we get $F(a,b)=0$.

Hence the complete solution of the given equation is

$z=ax+by+c$ where $F(a,b)=0$

Solving for 'b' from $F(a,b)=0$ we get $b=\phi(a)$ (say)

Then $z=ax+\phi(a)y+c$ is the complete integral of the given equation since it contains two arbitrary constants.

Singular integral is got by eliminating 'a' & 'c'

from $z = ax + \phi(a)y + c$

diff. partially w.r to 'a', we get

$\frac{\partial z}{\partial a} = 0 = x + \phi'(a)y$

& diff. partially w.r to 'c', we get

$\frac{\partial z}{\partial c} = 0 = 1.$

which is absurd, there is no singular integral for the given partial differential eqn.

To find the general integral:

put $c = f(a)$ in eqn ① f being arbitrary

Then $z = ax + \phi(a)y + f(a) \longrightarrow ②$

diff. partially w.r to 'a',

$\frac{\partial z}{\partial a} = 0 = x + \phi'(a)y + f'(a) \longrightarrow ③$

Eliminating 'a' between eqn ② & ③, we get the general solution.

Problems:-

① Solve $\sqrt{p} + \sqrt{q} = 1 \rightarrow ①$

This is of the form $F(p,q) = 0$.

Suppose that $z = ax + by + c \rightarrow ②$ be the solution of the equation ①.

diff ② par. w.r to 'x' & 'y',

$\frac{\partial z}{\partial x} = a$ ie, $p = a$ & $\frac{\partial z}{\partial y} = b$ ie, $q = b$

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$\therefore$ Sub. in the eqn ①, we get $\sqrt{a}+\sqrt{b}=1$.

Hence the complete solution of the given equation is $z = ax+by+c$ where $\sqrt{a}+\sqrt{b}=1$

ie, $\sqrt{b} = 1-\sqrt{a}$

$b = (1-\sqrt{a})^2$

$\therefore$

Then $z = ax + (1-\sqrt{a})^2 y + c \longrightarrow ③$ is the complete integral of the given eqn. $\because$ since it contains two arbitrary constants.

To find Singular integral :

Diff ③ partially w.r to 'c' we get

$$\frac{\partial z}{\partial c} = 0 = 1.$$

which is absurd, there is no S.I for the given PDE.

To find general integral :-

Put $c = f(a)$ in eqn ③, where f is arbitrary.

ie, $z = ax + (1-\sqrt{a})^2 y + f(a) \longrightarrow ④$

diff ④ par. w.r to 'a', we get

$$\frac{\partial z}{\partial a} = 0 = x + 2y(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right) + f'(a) \longrightarrow ⑤$$

Eliminating 'a' bet. (4) & (5), we get the general integral.

2) Solve $p^2+q^2 = npq \longrightarrow (1)$

Soln:. This is of the form $F(p,q)=0$.

Suppose that $z = ax+by+c \longrightarrow (2)$ be the soln. of eqn (1).

diff. (2) par. w. r to 'x' & 'y', we get

$p=a$ and $q=b$

Sub in given eqn (1), we get $a^2+b^2=nab$.

Hence the complete soln. of the given eqn is

$z=ax+by+c$ where $a^2+b^2-nab=0$.

Solving for 'b', $b = \dfrac{na \pm \sqrt{a^2n^2-4a^2}}{2}$

ie, $b = \dfrac{a}{2}\left[n \pm \sqrt{n^2-4}\right]$

$\therefore$ The complete integral is

$$z = ax + \frac{a}{2}\left(n \pm \sqrt{n^2-4}\right)y + c \longrightarrow (3)$$

($\because$ it contains two arbitrary constants.)

To find the Singular integral :

differentiating eqn ③ par. w.r to 'c', we get

$$0 = 1.$$

which is absurd, there is no singular integral. for the given partial differential eqn.

To find the general integral :

Put $c = f(a)$ in eqn ③, where f is arbitrary.

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2-4}\right] y + f(a) \longrightarrow ④$$

diff. ④ partially w.r to 'a',

$$0 = x + \frac{1}{2}\left[n \pm \sqrt{n^2-4}\right] y + f'(a) \longrightarrow ⑤.$$

Eliminating 'a' between ④ & ⑤, we get the general solution of the given equation.

Exercise problems

① Solve : $p + q = pq$

② Solve : $p^2 + q^2 = 4pq$

③ Solve : $p + q = 1$.

Type: 2 clairaut's form.

$$z = px + qy + f(p,q). \rightarrow (1)$$

Suppose that $z = ax + by + c$ $\rightarrow (2)$ be the soln. of eqn (1).

diff (2) par. w.r to 'x' & 'y', we get

$\frac{\partial z}{\partial x} = a$ ie, $p = a$ $\quad \frac{\partial z}{\partial y} = b$ ie, $q = b$.

sub in eqn (1), we get

$z = ax + by + f(a,b)$ $\rightarrow (3)$ is the Complete integral of equation (1), where a, b are arbitrary constants.

To find S.I:-

Differentiating eqn (3) par. w. r to 'a' & 'b', we get

$$x + \frac{\partial f}{\partial a} = 0 \longrightarrow (4)$$

& $$y + \frac{\partial f}{\partial b} = 0 \longrightarrow (5)$$

By eliminating a and b from (3), (4) & (5), we get the Singular integral of (1).

To find G.I:-

Taking $b = \phi(a)$ in (3),

$$z = ax + \phi(a)y + f(a, \phi(a)) \rightarrow (6)$$

diff. (6) partially w.r to a, we get

$$0 = x + \phi'(a)y + f'(a) \longrightarrow (7)$$

Eliminating 'a' between (6) & (7), we get the general integral of eqn (1).

Problems :-

① Solve : $z = px + qy + \sqrt{1+p^2+q^2}$

This is of the form $z = px + qy + f(p,q)$

Hence the complete integral is $z = ax + by + \sqrt{1+a^2+b^2}$ → ①

where a and b are arbitrary constants.

To find S.I :-

diff ① par. w. r to 'a' and 'b', we get

$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}} \longrightarrow ②$$

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}} \longrightarrow ③$$

From ② & ③,

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \rightarrow ④ , \quad y = \frac{-b}{\sqrt{1+a^2+b^2}} \rightarrow ⑤$$

$$x^2 = \frac{a^2}{1+a^2+b^2} \qquad y^2 = \frac{b^2}{1+a^2+b^2}$$

$$x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1 - \left(\frac{a^2+b^2}{1+a^2+b^2}\right) = \frac{1+\cancel{a^2}+\cancel{b^2}-\cancel{a^2}-\cancel{b^2}}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}}$$

$$\therefore \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

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$\therefore$ From ④, $a = -x\sqrt{1+a^2+b^2}$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}}$$

From ⑤, $\cancel{y} \neq$ $b = -y\sqrt{1+a^2+b^2}$

$$b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$\therefore$ Sub in eqn ①, we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} = \frac{\sqrt{1-x^2-y^2}\,\cancel{\sqrt{1-x^2-y^2}}}{\cancel{\sqrt{1-x^2-y^2}}}$$

$$z = \sqrt{1-x^2-y^2} \Rightarrow z^2 = 1-x^2-y^2$$

$z^2+x^2+y^2=1$ which is required S.I.

To find G.I :-

Put $b=\phi(a)$ in eqn ①

$(1+a^2+(\phi(a))^2$

$$z = ax + \phi(a)y + \sqrt{1+a^2+(\phi(a))^2} \longrightarrow ⑥$$

diff. ⑥ w.r to a,

$$0 = x + \phi'(a)y + \frac{1}{2}\left(1+a^2+(\phi(a))^2\right)^{-1/2}\cdot\left(2a+2\phi(a).\phi'(a)\right)$$

$$0 = x + \phi'(a)y + \frac{1}{\cancel{2}\sqrt{1+a^2+(\phi(a))^2}}\cdot\cancel{2}\left(a+\phi(a).\phi'(a)\right) \longrightarrow ⑦$$

Eliminate 'a' between ⑥ & ⑦ we get the general solution.

(2) Solve: $z = px + qy + p^2 - q^2$

This is of the form $z = px + qy + f(p,q)$.

Hence the complete integral is $z = ax + by + a^2 - b^2 \longrightarrow (1)$

To get singular integral:

diff (1) w.r to 'a' & 'b',

$$0 = x + 2a \longrightarrow (2)$$

$$0 = y - 2b \longrightarrow (3)$$

from (2), $a = -x/2$; from (3) $b = y/2$.

Sub in (1), we get $z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$

$$\Rightarrow z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4}$$

$$z = \frac{y^2 - x^2}{4}.$$

$\Rightarrow$ $4z = y^2 - x^2$ which is S.I.

To get general integral:-

Put $b = \phi(a)$ in eqn (1).

$$z = ax + \phi(a)y + a^2 - (\phi(a))^2 \longrightarrow (4)$$

diff (4). w.r to 'a',

$$0 = x + y\,\phi'(a) + 2a - 2\phi(a)\cdot\phi'(a) \longrightarrow (5)$$

Eliminating 'a' between (4) & (5), we get the general solution.

③ $Z = px + qy + 2\sqrt{pq}$

This is the form of $Z = px + qy + f(p,q)$

The Complete integral is $Z = ax + by + 2\sqrt{ab}$. $\to$ ①

where a & b are arbitrary Constants.

TO get S.I :-

diff ① w.r to 'a' & 'b', $(ab)^{1/2}$

$$0 = x + 2 b^{1/2} \cdot \tfrac{1}{2} a^{-1/2}$$

$$0 = x + \sqrt{\frac{b}{a}}$$

$$x = -\sqrt{b/a} \longrightarrow ②$$

$$0 = y + \cancel{2} a^{1/2} \tfrac{1}{2} b^{-1/2}$$

$$0 = y + \sqrt{a/b}$$

$$y = -\sqrt{a/b} \longrightarrow ③$$

From ③ & ④, we get

$$xy = -\sqrt{b/a} \cdot -\sqrt{a/b} = 1$$

$\therefore$ $xy = 1$ which is Singular integral.

TO get G.I :-

Put $b = \phi(a)$ in ① $(a\,\phi(a))^{1/2}$

$\frac{1}{2}(a\,\phi(a))^{-1/2} \cdot a\phi'(a) + \phi(a)(1)$

$$z = ax + \phi(a) y + 2\sqrt{a\,\phi(a)} \longrightarrow ④$$

$\Rightarrow \frac{1}{2\sqrt{a\phi(a)}}$

diff. ④ w.r to 'a', we get

$$0 = x + \phi'(a) y + \cancel{2}\left(\frac{1}{\cancel{2}\sqrt{a\phi(a)}}(a\phi'(a) + \phi(a))\right) \to ⑤$$

Eliminating 'a' between (4) & (5), we get general integral.

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Exercise problems

① $z = px + qy + p^2q^2$

② $z = px + qy + p/q - p$

③ $z = px + qy + 3pq$

Type: III.

1) $F(z, p, q) = 0$.

Assume that z is a function of $u = x + ay$, where a is arbitrary constant

$$z = f(u) = f(x+ay)$$

$u = x + ay$

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial u}{\partial y} = a$

$$\therefore \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$$

ie, $p = \frac{dz}{du} \cdot 1$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

ie, $q = a\frac{dz}{du}$

Sub. these values of p & q in $F(z,p,q) = 0$ we get

$F\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$ which is an ordinary differential equation of the First order.

Solving for $\frac{dz}{du}$, we ~~obtain~~ obtain $\frac{dz}{du} = \phi(z, a)$ (say)

ie, $\frac{dz}{\phi(z,a)} = du$

integrating, $\int \frac{dz}{\phi(z,a)} = \int du$

$$f(z,a) = u + c$$

ie, $f(z,a) = x + ay + c$

which is the C.I of given equation.

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2) Suppose that the given equation is of the form

$F(x, p, q) = 0 \longrightarrow (1)$

Since z is a function of 'x' & 'y',

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \Rightarrow dz = p\,dx + q\,dy$$

Assume $q = a$, Then the equation becomes $F(x, p, a) = 0$

Solving for p, we obtain $p = \phi(x, a)$.

$$\therefore dz = \phi(x,a)dx + a\,dy$$

int. on. b.s, we get

$$z = \int \phi(x,a)dx + a\int dy$$

$$z = f(x,a) + ay + c.$$ which is the C.I of given eqn (1).

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3) Suppose that the given equation is of the form

$F(y, p, q) = 0 \longrightarrow (1)$

Since z is a function of 'x' & 'y',

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \Rightarrow dz = p\,dx + q\,dy$$

Assume $p = a$, Then the eqn. becomes $F(y, a, q) = 0$

Solving for q, we obtain $q = \phi(y, a)$

$$\therefore dz = a\,dx + \phi(y,a)\,dy$$

Int. on b.s we get

$$z = ~~ay~~ \; ax + \int \phi(y,a)\,dy + c$$

$$z = ax + f(y,a) + c$$ which is the C.I of given eqn (1).

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For Type : III, Singular integrals and General integrals are found out as usual.

Problems.

① Solve : $p(1+q) = qz$.

Assume that $z = f(u)$ where $u = x+ay$, & a is an arb. Constant.

Then $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

Sub. these values in the given eqn, we get

$$\cancel{\frac{dz}{du}}\left(1 + a\frac{dz}{du}\right) = a\cancel{\frac{dz}{du}} \cdot z$$

$$1 + a\frac{dz}{du} = az.$$

$$\Rightarrow a\frac{dz}{du} = az-1$$

$$\frac{dz}{du} = \frac{az-1}{a}$$

$$\therefore \frac{a\,dz}{az-1} = du$$

int. on b.s, $\int \frac{a\,dz}{az-1} = \int du$

$$\log(az-1) = u + c.$$

ie, $\log(az-1) = x+ay+c$ which is Complete integral of given equation.

2) Solve: $9(p^2 z + q^2) = 4.$

This is of the form $F(z, p, q) = 0$.

Assume that $z = f(u)$ where $u = x + ay$ & a is an arb. Constant

Then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$.

Sub. these values in the given eqn, we get

$$9\left(\left(\frac{dz}{du}\right)^2 z + \left(a\frac{dz}{du}\right)^2\right) = 4$$

$$9\left(\frac{dz}{du}\right)^2 (z + a^2) = 4.$$

$$\left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)}$$

$$\frac{dz}{du} = \frac{2}{3\sqrt{z+a^2}}$$

$$\sqrt{z+a^2}\, dz = 2/3\, du.$$

Int. on b.s, $\int \sqrt{z+a^2}\, dz = 2/3\, du.$

$$\frac{(z+a^2)^{3/2}}{3/2} = 2/3\, u + c.$$

$$\cancel{3/2}\; 2/3\, (z+a^2)^{3/2} = 2/3\, u + c$$

ie, $(z+a^2)^{3/2} = u + b.$

ie, $(z+a^2)^{3/2} = x + ay + b$

$\Rightarrow (z+a^2)^3 = (x+ay+b)^2$ which is the Complete integral of given eqn.

3) $z^2 = 1+p^2+q^2$

This is of the form $F(z,p,q)=0$.

Assume that $z=f(u)$ where $u=x+ay$, & a is an arb. Constant.

Then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$.

Sub. these values in the given eqn,

$$z^2 = 1+\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2$$

$$z^2-1 = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$z^2-1 = \left(\frac{dz}{du}\right)^2\left(1+a^2\right)$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{1+a^2}$$

$$\frac{dz}{du} = \frac{\sqrt{z^2-1}}{\sqrt{1+a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z^2-1}} = \frac{du}{\sqrt{1+a^2}}$$

Int. on. b. s, we get

$$\int\frac{dz}{\sqrt{z^2-1}} = \int\frac{du}{\sqrt{1+a^2}}$$

$$\log\left(z+\sqrt{z^2-1}\right) = \frac{u}{\sqrt{1+a^2}}+c$$

ie, $$\log\left(z+\sqrt{z^2-1}\right) = \frac{x+ay}{\sqrt{1+a^2}}+c$$

which is Complete integral of the given eqn.

arb.
mt.

④ Solve: $p = 2qx$.

This is of the form $F(x, p, q) = 0$.

Assume $q = a$.

sub. in given eqn, we get $p = 2ax$.

WKT $dz = pdx + qdy$

$dz = 2ax + ady$

Int. on b.s, $\int dz = 2a \int x + a \int dy$

$$z = \frac{\cancel{2}ax^2}{\cancel{2}} + ay + c$$

which is the CI of the given eqn.

---

⑤ Solve: $q = px + p^2$

This is of the form $F(x, p, q) = 0$.

Assume $q = a$,

sub in given eqn, we get $a = px + p^2$.

solving for p, $p^2 + px - a = 0$.

$a = 1 \quad b = x$

$c = -a$.

$$p = \frac{-x \pm \sqrt{x^2 + 4a}}{2(1)}$$

we know that, $dz = pdx + qdy$.

$$dz = \left(\frac{-x \pm \sqrt{x^2 + 4a}}{2}\right) dx + ady$$

integrating on b.s,

$$z = \int \left(\frac{-x \pm \sqrt{x^2 + 4a}}{2}\right) dx + a \int dy$$

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left\{\frac{x}{2}\sqrt{x^2+4a} + 2a\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right)\right\} + ay + b.$$

which is C.I of given eqn.

---

6) Solve: $pq = y$

This is of the form $f(y, p, q) = 0$.

Assume, $p = a$.

sub in given eqn, $aq = y$

solving for q, $q = y/a$.

WKT $dz = p\,dx + q\,dy$

$dz = a\,dx + y/a\,dy$

Int. on b.s,

$z = ax + \frac{y^2}{2a} + c$ is the complete integral

since it contains 2 arb. constants.

---

7) $\sqrt{p} + \sqrt{q} = \sqrt{y}$

This is of the form $f(y, p, q) = 0$

Assume $p = a$,

sub in given eqn; $\sqrt{a} + \sqrt{q} = \sqrt{y}$

solving for q, $\sqrt{q} = \sqrt{y} - \sqrt{a}$

$q = (\sqrt{y} - \sqrt{a})^2$

$q = y + a - 2\sqrt{y}\sqrt{a}$

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WKT, $dz = p\,dx + q\,dy$

$y^{1/2}$

$\dfrac{y^{1/2+1}}{3/2}$

$dz = a\,dx + (y + a - 2\sqrt{y}\sqrt{a})\,dy$

Int. on. b.s, we get

$$z = ax + \frac{y^2}{2} + ay - 2\sqrt{a} \cdot \frac{2}{3} y^{3/2} + c.$$

$z = ax + \dfrac{y^2}{2} + ay - \dfrac{4}{3}\sqrt{a}\,y^{3/2} + c$ which is

Complete soln. of the given eqn.

Exercise problems

1. $z = p^2 + q^2$.
2. $p^2z^2 + q^2 = 1$.
3. $\sqrt{p} + \sqrt{q} = \sqrt{x}$
4. $pq = z$.
5. $z^2(p^2 + q^2 + 1) = 1$.
6. $z^2 = pq$.
7. $q = 2px$

(i) Lagrange's linear equation:-

A linear partial differential equation of the first order known as Lagrange's linear equation is of the form $\underline{Pp + Qq = R}$ where P, Q and R are functions of x, y, z.

We have already seen that by eliminating the arbitrary function ϕ from the relation

$$\phi(u,v) = 0$$

where u, v are functions of x, y, z, we get a partial differential equation of the form $Pp + Qq = R$.

$\therefore$ $\phi(u,v) = 0$ is the general solution of $Pp + Qq = R$, where ϕ is arbitrary.

To solve the equation $Pp + Qq = R$.

(i) form the auxiliary simultaneous equations or subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

(ii) Solve these auxiliary simultaneous equations giving two independent solutions $u = a$ and $v = b$.

(iii) then write down the solution as $\phi(u,v) = 0$ or $u = f(v)$ (or) $v = g(u)$, where the function is arbitrary.

Solution of the Subsidiary equation by the method of multipliers.

The Subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

then by properties of ratio and proportion

each ratio $= \dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \dfrac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$

where the two set of multipliers l, m, n ; l', m', n' be a constants or variables in x, y, z.

Choosing l, m, n such that $lP + mQ + nR = 0$, we have $l\,dx + m\,dy + n\,dz = 0 \longrightarrow (1)$

If $l\,dx + m\,dy + n\,dz$ is a perfect differential of some function, say, $u(x,y,z)$ then $du = 0$ by (1).

Hence integrating (1), we get

$u = a$, as one solution.

Illly, the other set of multipliers l', m', n' can be found out so that $l'P + m'Q + n'R = 0$.

Hence $l'dx + m'dy + n'dz = 0$.

This gives another solution $v = b$.

$\therefore$ The general solution is $\phi(u,v) = 0$. (or) $u = f(v)$ (or) $v = g(u)$

$\therefore$ the set of multipliers l, m, n and l', m', n' are called Lagrangian multipliers.

① Find the general integral of $px + qy = z$.

Soln :-

This is Lagrange's equation $Pp + Qq = R$.

here, $P = x$, $Q = y$ & $R = z$.

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Considering the first two ratios, $\frac{dx}{x} = \frac{dy}{y}$

integrating on b.s, we get

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log a$$

$$\log x - \log y = \log a$$

$$\log (x/y) = \log a$$

$$\boxed{a = x/y}$$

Considering the second and third ratios,

$$\frac{dy}{y} = \frac{dz}{z}.$$

integrating, $\int dy/y = \int dz/z$.

$$\log y = \log z + \log b$$

$$\log y - \log z = \log b.$$

$$\log (y/z) = \log b$$

$$\boxed{b = y/z}$$

$\therefore$ the general solution is $\phi(x/y, y/z) = 0$.

2) Solve : $(mz-ny)p+(nx-lz)q = ly-mx$.

Soln:-

This is of the form $Pp+Qq=R$.

$P=(mz-ny)$ $Q=(nx-lz)$ $R=ly-mx$.

The subsidiary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx} \quad (1)$$

Using the two set of multipliers x,y,z ; l,m,n each of the ratio in (1)

$$=\frac{xdx+ydy+zdz}{\cancel{xmz}-\cancel{xny}+\cancel{ynx}-\cancel{ylz}+\cancel{lyz}-\cancel{mxz}} = \frac{ldx+mdy+ndz}{\cancel{lmz}-\cancel{lny}+\cancel{nmx}-\cancel{mlz}+\cancel{nly}-\cancel{nmx}}$$

(0) (0)

hence $xdx+ydy+zdz=0$ & $ldx+mdy+ndz=0$.

int. on b.s, we get

$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=a$ & $lx+my+nz=b$.

Hence the general integral is $\phi\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ lx+my+nz\right)=0$.

---

(3) Find the general integral of $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$.

Soln:- This is of the form $Pp+Qq=R$.

$P=x(z^2-y^2)$ $Q=y(x^2-z^2)$ $R=z(y^2-x^2)$

The subsidiary eqns are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)} \longrightarrow (1)$$

Taking the two set of multipliers as x, y, z and $1/x, 1/y, 1/z$ each of ratio in (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2z^2} - \cancel{x^2y^2} + \cancel{y^2x^2} - \cancel{y^2z^2} + \cancel{z^2y^2} - \cancel{z^2x^2}} =$$

$$\frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{\cancel{z^2} - \cancel{y^2} + \cancel{x^2} - \cancel{z^2} + \cancel{y^2} - \cancel{x^2}}$$

(0)

hence $x\,dx + y\,dy + z\,dz = 0$ & $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$.

Int. $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a$ & $\log x + \log y + \log z = \log b$

$\log(xyz) = \log b$

$b = xyz$.

Hence the general integral is $\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2},\ xyz\right) = 0$.

---

(4) Solve :-

$$\frac{y^2 z}{x} p + xzq = y^2$$

This is of the form $Pp + Qq = R$.

$P = \frac{y^2 z}{x}$ $Q = xz$ $R = y^2$.

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

From the first two ratios, we get $\frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x \cancel{z}}$

$$x^2 dx = y^2 dy$$

Int. $\frac{x^3}{3} - \frac{y^3}{3} = a$

From the first and last ratios, we get

$$\frac{x\,dx}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2}}$$

$$x\,dx = z\,dz$$

$$\frac{x^2}{2} - \frac{z^2}{2} = b.$$

$\therefore$ The general integral is $\phi\left(x^3/_3 - y^3/_3,\ x^2/_2 - z^2/_2\right) = 0.$

---

(5) Find the general solution of $(y+z)p + (z+x)q = x+y$.

Soln: This is of the form $Pp + Qq = R$.

$$P = y+z \qquad Q = z+x \qquad R = x+y$$

The Subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

each is equal to $\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y+\cancel{z}-\cancel{z}-x} = \frac{dy-dz}{z+\cancel{x}-\cancel{x}-y}$

Taking the first two ratios,

$$\frac{d(x+y+z)}{2(x+y+z)} = \frac{dx-dy}{-(x-y)} = \frac{d(x-y)}{-(x-y)}$$

integrating, $\frac{1}{2}\log(x+y+z) = -\log(x-y) + \log a.$

$$\log(x+y+z)^{1/2} + \log(x-y) = \log a$$

$$\log\left((x+y+z)^{1/2}.(x-y)\right) = \log a$$

$$\boxed{a = (x+y+z)^{1/2}(x-y)}$$

Taking the last two ratios of eqn (1)

$$\frac{d(x-y)}{+(x-y)} = \frac{d(y-z)}{+(y-z)}$$

integrating, $\log(x-y) = \log(y-z) + \log b$.

$$\log\left(\frac{x-y}{y-z}\right) = \log b$$

$$\boxed{b = \frac{x-y}{y-z}}$$

$\therefore$ The general solution is $\phi\left((x+y+z)^{1/2}(x-y), \frac{x-y}{y-z}\right) = 0.$

---

6) Find the general solution of $p\tan x + q\tan y = \tan z$.

Soln:- This is of the form $Pp + Qq = R$.

$P = \tan x \quad Q = \tan y \quad R = \tan z$.

The auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}.$$

$$\cot x dx = \cot y dy = \cot z dz.$$

Taking the first two ratios we get

$$\cot x dx = \cot y dy$$

Integrating on b.s, we get

$$\log(\sin x) = \log(\sin y) + \log a$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log a$$

$$\boxed{a = \frac{\sin x}{\sin y}}$$

Taking the last two ratios, we get

$$\cot y dx = \cot z dz.$$

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Integrating on b.s, we get

$$\log(\sin y) = \log(\sin z) + \log b$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log b$$

$$\boxed{b = \frac{\sin y}{\sin z}}$$

$\therefore$ The general soln. is $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$.

---

7) Solve: $z(x-y) = px^2 - qy^2$.

This is of the form $Pp + Qq = R$.

$P = x^2 \quad Q = -y^2 \quad R = z(x-y)$.

The auxiliary eqns are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)} = \frac{dx+dy}{x^2-y^2}$$

Taking the first two ratios, we get

$$\frac{dx}{x^2} = -\frac{dy}{y^2}$$

x^{-2}

$\frac{x^{-2+1}}{-2+1}$

$\frac{x^{-1}}{-1}$

Integrating on b.s, we get

$$-\frac{1}{x} = \frac{1}{y} + k.$$

$$+\frac{1}{y} + \frac{1}{x} = a.$$

Taking the last two ratios, we get

$$\frac{dz}{z(\cancel{x-y})} = \frac{d(x+y)}{(x+y)(\cancel{x-y})}$$

Integrating, $\log z = \log(x+y) + \log b$

$$\log z - \log(x+y) = \log b$$

$$\log\left(z/x+y\right) = \log b$$

$$\boxed{b = z/x+y}$$

$\therefore$ The general integral is $\phi\left(\frac{1}{x} + \frac{1}{y}, z/x+y\right) = 0$.

8) Solve: $(3z-4y)p + (4x-2z)q = 2y-3x$.

Soln:- This is form of $Pp + Qq = R$.

$P = 3z - 4y \quad Q = 4x - 2z \quad R = 2y - 3x$.

The auxiliary eqns are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x} \longrightarrow (1)$$

Taking the two set of multipliers as x, y, z and $2, 3, 4$ each of ratio in (1).

$$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{3xz} - \cancel{4xy} + \cancel{4xy} - \cancel{2yz} + \cancel{2yz} - \cancel{3xz}} =$$

$$\frac{2dx + 3dy + 4dz}{\cancel{6z} - \cancel{8y} + \cancel{12x} - \cancel{6z} + \cancel{8y} - \cancel{12x}}$$

hence $x\,dx + y\,dy + z\,dz = 0$ & $2dx + 3dy + 4dz = 0$.

integrating, $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = k$ $\quad \boxed{2x + 3y + 4z = b}$

$\boxed{x^2 + y^2 + z^2 = a}$

$\therefore$ The general solution is $\phi(x^2+y^2+z^2, 2x+3y+4z) = 0$.

Exercise problems

① $(y-z)p + (z-x)q = x-y$

② $x(y-z)p + y(z-x)q = z(x-y)$

③ $(y-xz)p + (yz-x)q = (x+y)(x-y)$

④ $pz + qy = x$

⑤ $p-q = \log(x+y)$

⑥ $(2z-y)p + (x+z)q + 2x+y = 0$.

⑦ $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$.

⑧ $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.

⑨ $px^2 - qy^2 = z^2$

⑩ $py^2z + qx^2z = xy^2$.

Homogeneous linear equation :-

A homogeneous linear partial differential equation of n^{th} order with constant coefficients is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \longrightarrow (1)$$

where a's are constants and F is known function of x, y.

[Note: $\partial/\partial x = D$, $\partial/\partial y = D'$.]

eqn (1) symbolically can be written as

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n) z = F(x,y) \longrightarrow (2)$$

(or) $$f(D, D') z = F(x,y) \longrightarrow (3)$$

The solution of $f(D, D') z = 0$ is called the Complementary function (CF) of (3).

We find a particular integral (P.I.) of (3) which is given by $\frac{1}{f(D, D')} F(x,y)$. Then

$z = CF + PI$ is the complete solution of (3) (or) general solution.

Method of finding C.F.

(1) To get the auxiliary equation of $f(D,D')z = F(x,y)$

Put $D=m$ and $D'=1$

(2) The auxiliary equation is $f(D,D') = 0$

ie, $f(m,1) = 0$

ie, $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0 . \longrightarrow (*)$

Let $m_1, m_2, \dots, m_n$ be the roots of $(*)$

Case (i). The roots $m_1, m_2, \dots, m_n$ are distinct.

Then $C.F. = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_n(y+m_nx)$

where $\phi_1, \phi_2, \dots \phi_n$ are arbitrary functions.

Case (ii) The auxiliary equation has repeated roots.

Suppose $m_1 = m_2 = \dots = m_r = m$.

Then $CF = \phi_1(y+mx) + x\,\phi_2(y+mx) + \dots + x^{r-1}\phi_r(y+mx)$

where $\phi_1, \phi_2, \dots \phi_r$ are arbitrary functions.

The Particular Integral.

Type : 1 $\dfrac{1}{f(D,D')} e^{ax+by} = \dfrac{1}{f(a,b)} e^{ax+by}$ if $f(a,b) \neq 0$

Replace D by a
D' by b.

Type : 2 $\dfrac{1}{f(D,D')} x^r y^s = [f(D,D')^{-1}] \cdot x^r y^s$ where

$[f(D,D')]^{-1}$ is to be expanded in powers of D, D'

Type : 3 $\dfrac{1}{f(D^2, DD', D'^2)}$ $\sin(ax+by)$ (or) $\cos(ax+by)$

Replace D^2 by $-a^2$
DD' by $-ab$
D'^2 by $-b^2$

$= \dfrac{1}{f(-a^2, -ab, -b^2)}$ $\sin(ax+by)$ (or) $\cos(ax+by)$

Type : 4 $\dfrac{1}{f(D,D')} e^{ax+by} \phi(x,y) = e^{ax+by} \dfrac{1}{f(D+a, D'+b)} \phi(x,y)$

Type : 5 $\dfrac{\sin ax \sin by}{f(D^2, D'^2)} = \dfrac{\sin ax \sin by}{f(-a^2, -b^2)}$ if denominator $\neq 0$

$\dfrac{\cos ax \cos by}{f(D^2, D'^2)} = \dfrac{\cos ax \cos by}{f(-a^2, -b^2)}$ if denominator $\neq 0$.

General rule to find $\dfrac{1}{D-mD'} F(x,y)$ [Replace y by $a-mx$]

Integrate $F(x, a-mx)$ with respect to x and after integration replace 'a' by $y+mx$.

Problems :

1. $(D^2 - 4DD' + 4D'^2)z = 0$.

The auxiliary eqn is $m^2 - 4m + 4 = 0$. (Replace D by m and D' by 1 in $f(D,D')$ and equate to zero).

$m = 2, 2$ (equal roots).

Since RHS is zero, there is no particular integral.

hence $z = CF$ alone.

ie, $z = \phi_1(y+2x) + x\phi_2(y+2x)$.

---

2. $(D^3 - 3D^2D' + 2DD'^2)z = 0$.

The aux. eqn is $m^3 - 3m^2 + 2m = 0$.

$m(m^2 - m + 2) = 0$.

$m(m-1)(m-2) = 0$.

$m = 0, 1, 2$.

$\therefore$ General soln is $z = \phi_1(y+0x) + \phi_2(y+x) + \phi_3(y+2x)$.

---

3. Solve: $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$

The eqn is $(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$.

The aux. eqn is $m^3 - 3m^2 + 4 = 0$.

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$m = -1$ is one of the root.

$m^2 - 4m + 4 = 0$.

$m = 2, 2$.

$\therefore$ CF $= \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x)$

$$PI = \frac{1}{f(D,D')} \cdot F(x,y)$$

$$= \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y} \qquad a=1, \quad b=2.$$

Replace D by a
D' by b.

$$= \frac{1}{1 - 3(1)^2(2) + 4(8)} e^{x+2y} = \frac{1}{1-6+32} e^{x+2y}$$

$$= \frac{e^{x+2y}}{27}$$

∴ The complete solution is

$$z = CF + PI.$$

$$z = \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{e^{x+2y}}{27}.$$

---

(4) Solve:

$$(D^3 + D^2D' - DD'^2 - D'^3) z = e^x \cos 2y.$$

The auxiliary eqn. is $m^3 + m^2 - m - 1 = 0$.

$$\begin{array}{c|cccc} 1 & 1 & 1 & -1 & -1 \\ & 0 & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$m=1$ is one of the root.

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1. \qquad \therefore m = 1, -1, -1$$

CF: $\phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x)$

$$PI = \frac{1}{f(D,D')} F(x,y)$$

$$= \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^x \cos 2y \quad (\text{type: } 4)$$

e^{ax+by} Replace D by $D+a$ & D' by $D'+b$.

$a=1$ $b=0$.

$$= e^x \frac{1}{(D+1)^3 + (D+1)^2 D' - (D+1)D'^2 - D'^3} \cos 2y$$

$e^{i\theta} = \cos\theta + i\sin\theta.$

$e^{2iy} = \cos 2y + i\sin 2y.$

Real part of $e^{2iy} = \cos 2y.$

$$= e^{x}.\ \text{Real part of}\ \frac{e^{i2y}}{(D+1)^3 + (D+1)^2 D' - (D+1) D'^2 - D'^3}$$

(Type: 1)

Replace D by a $a=0$

D' by b $b=2i$.

$4i^2$

$(2i)^3 = -8i$

$$= e^{x}.\ R.P\ \text{of}\ \frac{e^{i2y}}{1 + 2i - (2i)^2 - (2i)^3}$$

$1+2i+4+8i$

$$= e^{x}\ RP\ \text{of}\ \frac{e^{i2y}}{1+2i+4+8i}$$

$$= e^{x}\ RP\ \text{of}\ \frac{e^{i2y}}{5+10i}$$

$$= \frac{e^{x}}{5}\ R.P.\ \text{of}\ \frac{e^{i2y}}{1+2i}$$

$$= \frac{e^{x}}{5}\ RP\ \text{of}\ \frac{1}{1+2i} \times \frac{1-2i}{1-2i}\ e^{i2y}$$

$$= \frac{e^{x}}{5}\ RP\ \text{of}\ \frac{1-2i}{1-4i^2}\ e^{i2y} \qquad i^2 = -1.$$

$$= \frac{e^{x}}{25}\ R.P\ \text{of}\ \left[(1-2i)(\cos 2y + i\sin 2y)\right]$$

$$= \frac{e^{x}}{25}\ R.P\ \text{of}\ \left[\cos 2y + i\sin 2y - 2i\cos 2y + 2\sin 2y\right]$$

$$= \frac{e^{x}}{25}\left[\cos 2y + 2\sin 2y\right]$$

$\therefore$ The complete soln is

$$z = \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x) + \frac{e^{x}}{25}\left[\cos 2y + 2\sin 2y\right]$$

(5) $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = e^{x+2y}$

$(D^3 - 2D^2D')z = e^{x+2y}$

Replace D by m
D' by 1

The auxiliary eqn is $m^3 - 2m^2 = 0$.

$m^2(m-2) = 0$.

$m = 0, 0, 2$.

CF : $\phi_1(y) + x\,\phi_2(y) + \phi_3(y+2x)$

$PI = \frac{1}{f(D,D')} \cdot F(x,y)$

$= \frac{1}{D^3 - 2D^2D'} e^{x+2y}$ $a=1$ $b=2$. Replace D by a, D' by b

$= \frac{1}{1 - 2(1)^2(2)} e^{x+2y} = \frac{1}{1-4} e^{x+2y} = \frac{-1}{3} e^{x+2y}$

$\therefore$ The Complete soln. is

$z = CF + PI$

$z = \phi_1(y) + x\,\phi_2(y) + \phi_3(y+2x) - \frac{1}{3} e^{x+2y}$

---

(6) Solve :- $(D^2 - 4DD' + 4D'^2)y = e^{2x+y}$

The aux. eqn is $m^2 - 4m + 4 = 0$. Replace D by m, D' by 1.

$m = 2, 2$.

CF : $\phi_1(y+2x) + x\,\phi_2(y+2x)$.

$PI = \frac{1}{f(D,D')} F(x,y)$

$= \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$ here, $a=2$ $b=1$

Replace D by a
D' by b

$$= \frac{1}{4-4(1)(2)+4(1)^2} \cdot e^{2x+y}$$

$$= \frac{1}{4-8+4} e^{2x+y}$$

$$= \frac{1}{0} e^{2x+y}$$ [multiply by 'x' and differentiate denominator w.r to 'D'].

$$= x \cdot \frac{1}{2D-4D'} e^{2x+y}$$

$$= x \cdot \frac{1}{2(2)-4(1)} e^{2x+y}$$

$$= x \cdot \frac{1}{0} e^{2x+y}$$ [Again multiply by 'x' & diff. denominator w.r to 'D']

$$= x^2 \cdot \frac{1}{2} e^{2x+y}$$

$$= \frac{x^2}{2} e^{2x+y}.$$

$\therefore$ The complete soln. is $z = CF + PI$

$$z = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{x^2}{2} e^{2x+y}.$$

---

(7) Solve :-

$$(D^2+2DD'+D'^2)z = \sinh(x+y).$$

The aux. eqn is $m^2+2m+1=0$.

$m = -1, -1$.

CF: $\phi_1(y-x) + x\phi_2(y-x)$.

$$PI = \frac{1}{f(D,D')} \cdot F(x,y)$$

$$= \frac{1}{D^2+2DD'+D'^2} \sinh(x+y)$$ $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

$$= \frac{1}{D^2+2DD'+D'^2}\left[\frac{e^{x+y} - e^{-(x+y)}}{2}\right]$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{D^2+2DD'+D'^2} e^{x+y}}_{PI_1} - \underbrace{\frac{1}{D^2+2DD'+D'^2} e^{-(x+y)}}_{PI_2}\right]$$

$$PI_1 = \frac{1}{D^2+2DD'+D'^2} e^{x+y} \qquad a=1 \quad b=1.$$

Replace D by a
D' by b

$$\Rightarrow \frac{1}{1+2+1} e^{x+y} = \frac{e^{x+y}}{4}$$

$$PI_2 = \frac{1}{D^2+2DD'+D'^2} e^{-(x+y)} \qquad a=-1 \quad b=-1.$$

$$= \frac{1}{(-1)^2+2(-1)\cdot(-1)+(-1)^2} e^{-(x+y)}$$

$$= \frac{1}{1+2+1} e^{-(x+y)} = \frac{e^{-(x+y)}}{4}.$$

$$PI = \frac{1}{2}\left[\frac{e^{x+y}}{4} + \frac{e^{-(x+y)}}{4}\right]$$

$$\Rightarrow \frac{1}{4}\left[\frac{e^{x+y}+e^{-(x+y)}}{2}\right]$$

$$= \frac{1}{4}\sinh(x+y)$$

$$\therefore \; z = CF + PI = \phi_1(y-x) + x\phi_2(y-x) + \frac{1}{4}\sinh(x+y).$$

which is complete soln.

Exercise Problems :-

① $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

② $(D^3 - 7DD'^2 - 6D'^3)z = e^{2x+y}$

③ $(D^2 - 2DD' + D'^2)z = e^{x+2y}$

④ $(D^4 - D'^4)z = e^{x+y}$

⑤ $(D^2 + 2DD' + D'^2)z = e^{x+2y}.$

(8) **Solve:** $(D^3 - 7DD'^2 - 6D'^3)z = x^2y + \sin(x+2y)$.

Soln: The auxiliary eqn is $m^3 - 7m - 6 = 0$.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & +1 & +6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$m = -1$ is one of the root

$m^2 - m - 6 = 0$.

$m = -2, 3$.

$\therefore \quad m = -1, -2, 3$.

$\therefore$ CF : $\phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$.

$$PI = \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2y + \sin(x+2y)).$$

$$= \underbrace{\frac{1}{D^3 - 7DD'^2 - 6D'^3}x^2y}_{PI_1} + \underbrace{\frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y)}_{PI_2}$$

$$PI_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}x^2y$$

$$= \frac{1}{D^3\left[1 - \left(\frac{7DD'^2 + 6D'^3}{D^3}\right)\right]}x^2y \qquad (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$= \frac{1}{D^3\left(1 - \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right)} \cdot x^2y$$

$$= \frac{1}{D^3}\left(1 - \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right)^{-1} \cdot x^2y$$

$$= \frac{1}{D^3}\left[1 + \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right) + \cdots\right)x^2y.$$

$\left(D' = \partial/\partial y\right)$

$$= \frac{1}{D^3}\left(x^2y + \underbrace{\frac{7D'^2}{D^2}(x^2y)}_{0} + \underbrace{\frac{6D'^3}{D^3}(x^2y)}_{0} + \cdots \right]$$

$$= \frac{1}{D^3}(x^2y) = \frac{x^5y}{60}$$

$$\left[\begin{array}{lll} \text{First integration} : & \int x^2y\,dx = & \frac{x^3y}{3} \\ \text{second} \quad " : & \int \frac{x^3y}{3}\,dx = & \frac{x^4y}{12} \\ \text{third} \quad " : & \int \frac{x^4y}{12}\,dx = & \frac{x^5y}{60} \end{array} \right]$$

$$PI_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x+2y)$$

$\sin(ax+by)$ $a=1$ $b=2$.

D^2 by $-a^2$ $\quad -1$

DD' by $-ab$ $\quad -2$

D'^2 by $-b^2$ $\quad -4$.

$$= \frac{1}{-D-7(-2)D' - 6(-4)D'} \sin(x+2y)$$

$D^2 D = -D$

$$= \frac{1}{-D+14D'+24D'} \sin(x+2y)$$

$DD'^2 = DD'.D'$
$= -2.D'$

$D'^3 = D'^2.D' = -4D'$

$$= \frac{1}{38D'-D} \sin(x+2y).$$

$$= \frac{1}{38D'-D} \times \frac{38D'+D}{38D'+D} \sin(x+2y)$$

38
24
304
76
————
$\cancel{924}$ 6

$$= \frac{38D'+D}{(38D')^2 - D^2} \sin(x+2y)$$

$\frac{38\times 4}{152}$ 3

$$= \frac{38D'+D}{1444D'^2 - D^2} \sin(x+2y)$$

$$= \frac{38D'+D}{1444(-4)-(-1)} \sin(x+2y) \qquad \begin{array}{r}1444\\4\\\hline 5776\end{array}$$

$$= \frac{38D'+D}{-5775} \sin(x+2y)$$

$$= \frac{1}{-5775}\left[38D'(\sin(x+2y)) + D(\sin(x+2y))\right]$$

$$= \frac{1}{-5775}\left[38(\cos(x+2y)\cdot 2) + \cos(x+2y)(1)\right].$$

$$= -\frac{1}{5775}(77\cos(x+2y))$$

$$= -\frac{1}{75}\cos(x+2y).$$

$\therefore$ The complete soln is

$$z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{x^5 y}{60}.$$

---

9) Solve:- $(D^3 - 2D^2D')z = \sin(x+2y) + 3x^2y.$

The aux. eqn is $m^3 - 2m^2 = 0$

$m^2(m-2) = 0$

$m = 0, 0, 2.$

$$CF = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x).$$

$$PI = \frac{1}{f(D,D')} F(x,y).$$

$$= \frac{1}{D^3 - 2D^2D'}(\sin(x+2y) + 3x^2y)$$

$$= \underbrace{\frac{1}{D^3 - 2D^2D'}\sin(x+2y)}_{PI_1} + \underbrace{\frac{1}{D^2 - 2D^2D'} 3x^2y}_{PI_2}$$

$$PI_1 = \frac{1}{D^3-2D^2D'} \sin(x+2y)$$

$a=1 \quad b=2$

D^2 by $-a^2$ $\quad -1$

DD' by $-ab$ $\quad -2$

D'^2 by $-b^2$ $\quad -4.$

$$= \frac{1}{-D-2(-2D)} \sin(x+2y)$$

$$= \frac{1}{-D+4D} \sin(x+2y)$$

$D^2D' = D.DD'$
$= D.(-2)$
$= -2D.$

$$= \frac{1}{3D} \sin(x+2y)$$

$$= \frac{D}{3D^2} \sin(x+2y)$$

$$= \frac{D}{-3} \sin(x+2y) = \frac{-1}{3}\left(\cos(x+2y).1\right)$$

$$= \frac{-\cos(x+2y)}{3}$$

$$PI_2 = \frac{3x^2y}{D^3-2D^2D'}$$

$$= \frac{1}{D^3\left(1-\frac{2D^2D'}{D^3}\right)} \cdot 3x^2y$$

$$= \frac{1}{D^3\left(1-\frac{2D'}{D}\right)} 3x^2y = \frac{1}{D^3}\left(1-\frac{2D'}{D}\right)^{-1} \cdot 3x^2y$$

$$= \frac{1}{D^3}\left(1+\left(\frac{2D'}{D}\right)+\left(\frac{2D'}{D}\right)^2+\cdots\right)(3x^2y)$$

$$= \frac{1}{D^3}\left(3x^2y+\frac{2D'}{D}(3x^2y)+\frac{4}{D^2}D'^2(3x^2y)+\cdots\right)$$

(higher terms are negligable)

$D' = \partial/\partial y$

$\partial^2/\partial y^2(3x^2y) = 0.$

$$= \frac{1}{D^3}\left(3x^2y+\frac{2(3x^2)}{D}\right)$$

$$= \frac{1}{D^3}\left(3x^2y+\frac{6x^2}{D}\right)$$

$$= \frac{3x^3y}{D^3} + \frac{6x^2}{D^4}.$$

$$= 3\left[\frac{x^5y}{60}\right] + 6\left[\frac{x^6}{360}\right]$$

$$= \frac{x^5y}{20} + \frac{x^6}{60}$$

$\int x^2y\,dx = \left[\frac{x^3y}{3}\right]$

$\int \frac{x^3y}{3}\,dx = \frac{x^4y}{12}$

$\int \frac{x^4y}{12}\,dx = \frac{x^5y}{60}$

— x —

$\int x^2\,dx = x^3/3$

$\int x^3/3\,dx = x^4/12$

$\int x^4/12\,dx = x^5/60$

$\int x^5/60\,dx = x^6/360.$

The complete soln is

$z = CF + PI$

$$z = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) - \frac{1}{3}\cos(x+2y) + \frac{x^5y}{20} + \frac{x^6}{60}.$$

Exercise problems :-

① Solve : $(D^2 - 2DD')z = x^3y + e^{2x}$

Ans : $z = \phi_1(y) + \phi_2(y+2x) + \frac{e^{2x}}{4} + \frac{x^5y}{20} + \frac{x^6}{60}.$

② Solve : $(D^2 + 3DD' + 2D'^2)z = x+y.$

Ans : $z = \phi_1(y-x) + \phi_2(y-2x) + \frac{x^2y}{2} - \frac{x^3}{3}.$

③ Solve : $(D^2 + 4DD' - 5D'^2)z = x + y^2 + \pi.$

Ans : $z = \phi_1(x+y) + \phi_2(y-5x) + \frac{x^3}{6} + \frac{x^2}{2}(y^2+\pi) - \frac{4}{3}x^3y + \frac{7}{4}x^4.$

10) Solve: $(D^2-3DD'+2D'^2)z = \sin x \cos y$.

Soln: The aux. eqn is

$m^2-3m+2=0$.

$m=1,2$.

CF: $\phi_1(y+x)+\phi_2(y+2x)$.

$PI = \frac{1}{f(D,D')} F(x,y)$

$\sin A \cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$.

$= \frac{1}{D^2-3DD'+2D'^2} \sin x \cos y$

$= \frac{1}{D^2-3DD'+2D'^2}\left[\frac{1}{2}\sin(x+y)+\sin(x-y)\right]$.

$= \frac{1}{2}\left[\underbrace{\frac{1}{D^2-3DD'+2D'^2}\sin(x+y)}_{PI_1} + \underbrace{\frac{1}{D^2-3DD'+2D'^2}\sin(x-y)}_{PI_2}\right]$

$PI_1 = \frac{1}{D^2-3DD'+2D'^2}\sin(x+y)$ $\quad a=1 \quad b=1$

D^2 by $-a^2$ $\quad -1$

DD' by $-ab$ $\quad -1$

D'^2 by $-b^2$ $\quad -1$

$= \frac{1}{-1-3(-1)+2(-1)}\sin(x+y)$

$-1+3-2$

$= \frac{1}{0}\sin(x+y)$. (∴ multiply by 'x' & differentiate denominator w.r.to 'D')

$= x \cdot \frac{1}{2D-3D'}\sin(x+y)$.

$= x \cdot \frac{1}{2D-3D'} \times \frac{2D+3D'}{2D+3D'}\sin(x+y)$

$= x \cdot \frac{2D+3D'}{4D^2-9D'^2}\sin(x+y)$

$= x \cdot \frac{2D+3D'}{4(-1)-9(-1)}\sin(x+y)$

16

$$= x \cdot \frac{2D+3D'}{-4+9} \sin(x+y).$$

$$= \frac{x}{5}\left[2D(\sin(x+y)) + 3D'(\sin(x+y))\right].$$

$$= \frac{x}{5}\left[2\cos(x+y)(1) + 3\cos(x+y)(1)\right].$$

$$= \frac{x}{5}\left(5\cos(x+y)\right) = \underline{x\cos(x+y)}.$$

$$PI_2 = \frac{1}{D^2-3DD'+2D'^2} \sin(x-y)$$

$a=1 \quad b=-1.$ $\quad D^2$ by $-a^2$ $\;-1$
DD' by $-ab$ $\;+1$
D'^2 by $-b^2$ $\;\cdot -1$
$-(-1)^2 = -1$

$$= \frac{1}{-1-3(+1)+2(-1)} \sin(x-y)$$

$$= \frac{1}{-1-3-2} \sin(x-y) = \frac{-1}{6}\sin(x-y).$$

$$PI = \frac{1}{2}\left[x\cos(x+y) - \frac{1}{6}\sin(x-y)\right].$$

The complete soln is

$z = CF + PI$

$$z = \phi_1(y+x) + \phi_2(y+2x) + \frac{x}{2}\cos(x+y) - \frac{1}{12}\sin(x-y).$$

---

11) Solve: $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$.

Soln:- The aux. eqn. is, $m^2 - 2m + 1 = 0$.
$m = 1, 1$.

$$CF: \phi_1(y+x) + x\phi_2(y+x).$$

$$PI = \frac{1}{f(D,D')} \cdot F(x,y)$$

$$= \frac{1}{D^2-2DD'+D'^2} \cdot \cos(x-3y). \quad a=1, \; b=-3.$$

D^2 by $-a^2$
DD' by $-ab$ $\quad D'^2$ by $-b^2$

$$= \frac{\cos(x-3y)}{-1-2(3)-9}$$

$$= \frac{\cos(x-3y)}{-16}$$

$\therefore$ The complete soln is

$$z = \phi_1(y+x) + x\phi_2(y+x) - \frac{1}{16}\cos(x-3y).$$

12-2) Solve:- $(D^2 - DD')z = \sin x \sin 2y$

The aux. eqn is $m^2 - m = 0$.

$m(m-1) = 0$

$m = 0, 1$.

$$CF: \phi_1(y) + \phi_2(y+x).$$

To find PI:-

$$PI = \frac{1}{f(x,y)} \cdot F(x,y)$$

$$= \frac{1}{D^2 - DD'} \sin x \sin 2y.$$

$$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$\sin x \sin 2y = \frac{\cos(x-2y) - \cos(x+2y)}{2}$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{D^2-DD'}(\cos(x-2y)) - \frac{1}{D^2-DD'}\cos(x+2y)\right]$$

$a=1$, $b=-2$ $a=1$, $b=2$. $\rightarrow$ $-a^2$ -1, $-ab$ -2, b^2 -4

Replace D^2 by $-a^2$ -1

DD' by $-ab$ $+2$

D'^2 by $-b^2$ -4

$$= \frac{1}{2}\left[\frac{1}{-1-2}\cos(x-2y) - \frac{1}{-1+2}\cos(x+2y)\right]$$

$$= \frac{-1}{6}\cos(x-2y) - \frac{1}{2}\cos(x+2y)$$

$\therefore$ The Complete soln is

$z = CF + PI$

$$z = \phi_1(y) + \phi_2(y+x) - \tfrac{1}{6}\cos(x-2y) - \tfrac{1}{2}\cos(x+2y).$$

Exercise problems :-

1. Solve: $(D^2 - 4D'^2)z = \cos 2x \cos 3y$.
2. Solve: $(D^2 - D'^2)z = \sin 2x \sin 3y$
3.

---

13) Solve: $(D^2 + DD' - 6D'^2)z = y\cos x$.

The aux. eqn is $m^2 + m - 6 = 0$. Replace D by m, D' by 1.

$m = 2, -3$

CF : $\phi_1(y+2x) + \phi_2(y-3x)$.

To find PI :

$$PI = \frac{1}{f(D,D')} \cdot F(x,y)$$

$$= \frac{1}{D^2 + DD' - 6D'^2} \cdot y\cos x.$$

Apply : General rule : $\frac{1}{D - mD'} F(x,y)$.

$P = -6$
$S = 1$
$3 \quad -2$

$D^2 + DD' - 6D'^2 =$

$D^2 + 3DD' - 2DD' - 6D'^2$

$D(D + 3D') - 2D'(D + 3D')$

$(D - 2D')(D + 3D')$.

$$= \frac{1}{(D-2D')(D+3D')} y\cos x$$

(A)

$\Rightarrow \frac{1}{D-2D'} \cdot \frac{1}{D+3D'} y\cos x$ $\quad$ Replace y by $a-mx$

$D-mD'$ $\quad$ here, $m=-3$

$= \frac{1}{D-2D'} \int (a+3x)\cos x \, dx.$ $\quad$ $\therefore$ y by $\underline{a+3x}$

$u = a+3x \qquad dv = \cos x \, dx$

$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = \cos x$

$du = 3dx \qquad v = +\sin x.$

$= \frac{1}{D-2D'} \left[(a+3x)\sin x - \int \sin x (3) dx \right].$

$\Rightarrow \frac{1}{D-2D'} \left[(a+3x)\sin x + 3\cos x \right]$ $\quad$ again replace 'a' by $y+mx$.

$= \frac{1}{D-2D'} \left(y\sin x + 3\cos x \right]$ $\quad$ $\therefore$ $a+3x$

$= y - \cancel{3x} + \cancel{3x} = y$

Again general rule,

$\frac{1}{D-2D'}$

$D-mD'$ $\quad$ here $m=2$

Replace y by $a-mx$

$\therefore$ y by $a-2x$

$\Rightarrow \int \left[(a-2x)\sin x + 3\cos x \right] dx$

$\Rightarrow \int (a-2x)\sin x \, dx + 3\int \cos x \, dx.$

$u = a-2x \qquad dv = \sin x \, dx.$

$\frac{du}{dx} = -2 \qquad \frac{dv}{dx} = \sin x$

$du = -2dx \qquad v = -\cos x.$

$= (a-2x)(-\cos x) - \int -\cos x (-2) dx$

$= (a-2x)(-\cos x) - \int \cos x dx$

$= -(a-2x)\cos x - 2\sin x. + 3\sin x$ $\quad$ Replace 'a' by $y+mx$.

$= -(a-2x)\cos x + \sin x.$ $\quad$ $\therefore$ a by $y+2x$.

$= \underline{-y\cos x + \sin x}.$ $\quad$ $\therefore$ $a-2x = y+\cancel{2x}-\cancel{2x}$

$\therefore$ The Complete soln:

$z = CF + PI$

$z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x.$

14) Solve: $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$

The aux. eqn is $m^2 + 2m + 1 = 0$ Replace D by m, D' by 1

$m = -1, -1$

$\therefore$ CF: $\phi_1(y-x) + x\phi_2(y-x).$

$$PI = \frac{1}{D^2+2DD'+D'^2}(2\cos y - x\sin y)$$

$[\because D^2+2DD'+D'^2 = (D+D')(D+D')]$

$$= \frac{1}{(D+D')(D+D')}(2\cos y - x\sin y).$$

$$= \frac{1}{D+D'}\frac{1}{D+D'}(2\cos y - x\sin y)$$

$\frac{1}{D-mD'}$

Apply general rule;

$\boxed{m=-1}$

Replace y by a−mx

$\therefore$ y by a+x.

$$= \frac{1}{D+D'}\left[\int \left(2\cos(a+x) - x\sin(a+x)\right)dx\right].$$

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$$= \frac{1}{D+D'}\left[\int 2\cos(a+x)\,dx - \int x\sin(a+x)\,dx\right].$$

$u = x$ | $dv = \sin(a+x)\,dx$

$\frac{du}{dx} = 1$ | $\frac{dv}{dx} = \sin(a+x)$

$du = dx$ | $\therefore$ $v = -\cos(a+x)$

$$\Rightarrow \quad -x\cos(a+x) - \int -\cos(a+x)\,dx.$$

$$= -x\cos(a+x) + \sin(a+x).$$

$$= \frac{1}{D+D'}\left[2\sin(a+x) + x\cos(a+x) - \sin(a+x)\right]$$

$$= \frac{1}{D+D'}\left[\sin(a+x) + x\cos(a+x)\right]$$

Replace a by $y+mx$

a by $y-x$.

$a+x \Rightarrow y-\cancel{x}+\cancel{x}$

$$= \frac{1}{D+D'}(\sin y + x\cos y).$$

$D - mD'$

Replace y by $a-mx$.

here $m=-1$

Replace. y by $a+x$

$$= \int\left[\sin(a+x) + x\cos(a+x)\right]dx$$

$$= \int \sin(a+x)\,dx + \int x\cos(a+x)\,dx$$

$u = x$ | $dv = \cos(a+x)\,dx$

$\frac{du}{dx} = 1$ | $\frac{dv}{dx} = \cos(a+x)$

$v = \sin(a+x)$

$$\Rightarrow \quad x\sin(a+x) - \int \sin(a+x)\,dx.$$

$$= x\sin(a+x) + \cos(a+x)$$

$$= -\cos(a+x) + x\sin(a+x) + \cos(a+x)$$

Replace by 'a' by $y+mx$

'a' by $y-x$.

$\therefore \quad a+x = y-x+x = y.$

$$= -\cancel{\cos y} + x\sin y + \cancel{\cos y} = x\sin y.$$

$\therefore$ The complete soln is

$z = CF + PI$

$z = \phi_1(y-x) + x\phi_2(y-x) + x \sin y.$

Solve: $(D^2 - 3DD' + 2D'^2) z = (2+4x) e^{x+2y}$.

Replace D by m
D' by 1

$\therefore$ The aux. eqn is $m^2 - 3m + 2 = 0$.
$m = 1, 2$.

CF: $\phi_1(y+x) + \phi_2(y+2x)$.

To find PI:-

$$PI = \frac{1}{f(D,D')} \cdot F(x,y).$$

$$= \frac{1}{D^2 - 3DD' + 2D'^2} \cdot (2+4x) e^{x+2y}.$$

$$= \frac{1}{D^2 - 3DD' + 2D'^2} \cdot e^{x+2y} (2+4x).$$

Type: 4.

$$= \frac{e^{x+2y} \times (2+4x)}{(D-2D')(D-D')}$$

$a=1 \quad b=2$

Replace D by $D+1$

D' by $D'+2$

$$= e^{x+2y} \frac{1}{((D+1)-2(D'+2))(D+1-(D'+2))} (2+4x).$$

$$= e^{x+2y} \frac{1}{(D+1-2D'-4)(D+1-D'-2)} (2+4x)$$

$$= e^{x+2y} \frac{1}{(D-2D'-3)(D-D'-1)} (2+4x)$$

$$= e^{x+2y} \frac{1}{(-3)\left(1+\left(\frac{D-2D'}{-3}\right)\right)(-1)\left(1+\left(\frac{D-D'}{-1}\right)\right)} (2+4x)$$

$$= \frac{e^{x+2y}}{3}\left(\frac{1}{\left(1-\left(\frac{D-2D'}{3}\right)\right)(1-(D-D'))} . (2+4x)\right)$$

$$= \frac{e^{x+2y}}{3}\left(\left(1-\left(\frac{D-2D'}{3}\right)\right)^{-1}(1-(D-D'))^{-1} . (2+4x)\right)$$

$$= \frac{e^{x+2y}}{3}\left(1+\left(\frac{D-2D'}{3}\right)+\cdots\right)\left(1+(D-D')+\cdots\right)(2+4x)$$

$$= \frac{e^{x+2y}}{3}\left(1+\left(\frac{D-2D'}{3}\right)+\cdots\right)\left(2+4x+(D-D')(2+4x)\right)$$

$D = \partial/\partial x$

$$= \frac{e^{x+2y}}{3}\left(1+\left(\frac{D-2D'}{3}\right)+\cdots\right)(2+4x+4))$$

$$= \frac{e^{x+2y}}{3}\left(1+\left(\frac{D-2D'}{3}\right)+\cdots\right)(4x+6))$$

$$= \frac{e^{x+2y}}{3}\left(4x+6+\left(\frac{D-2D'}{3}\right)(4x+6)\right).$$

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$$= \frac{e^{x+2y}}{3}\left(4x+6+4/3\right)$$

$$= \frac{e^{x+2y}}{3}\left(\frac{22}{3}+4x\right).$$

$\therefore$ The complete soln is

$$z = CF + PI$$

$$z = \phi_1(y+x) + \phi_2(y+2x) + \frac{e^{x+2y}}{3}\left(\frac{22}{3}+4x\right)$$

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5) Obtain the partial differential equation by eliminating a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Soln:

Since the number of arbitrary Constants is more than the number of independent variables, we will get the partial differential equation of order greater than 1.

a, b, c - Constants

x, y - independent Variables.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \rightarrow (1)$$

diff. partially w.r to x and y,

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0.$$

ie, $$\frac{2x}{a^2} + \frac{2z}{c^2} p = 0. \longrightarrow (2)$$

diff. eqn (2) w.r to 'x' again,

$$\frac{2}{a^2} + \frac{2}{c^2}\left[z \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x}\right] = 0.$$

$$\frac{2}{a^2} + \frac{2}{c^2}\left[z \cdot \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2\right] = 0. \rightarrow (3).$$

From (2),

$$\frac{2x}{a^2} = -\frac{2z}{c^2} p.$$

$$\boxed{\frac{2}{a^2} = -\frac{2z}{c^2 x} p.} \longrightarrow (4).$$

Sub eqn (4) in eqn (3), we get

$$-\frac{2z}{c^2 x}p + \frac{2}{c^2}\left[z\cdot\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2\right] = 0.$$

$$+\frac{2}{c^2}\left[-\frac{zp}{x} + z\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2\right] = 0.$$

$$-\frac{zp}{x} + zr + p^2 = 0.$$

$$-zp + zrx + p^2 x = 0.$$

$\boxed{p^2 x + zrx = zp}$ which is the required eqn.

Note :-

2. If the number of arbitrary constants is more than the number of independent variables, the resulting partial differential equation will be of the second or higher orders.

3. The answer is not unique. We may also get different PDE.

By elimination of arbitrary functions:

Let u and v be any two given functions of x, y and z. Let u and v be connected by an arbitrary function ϕ by the relation

$$\phi(u,v) = 0.$$

Note:

- The relation $\phi(u,v)=0$ is a solution of $Pp + Qq = R$, whatever may the arbitrary function ϕ.
- $Pp + Qq = R$ which is called Lagrange's linear equation.

Form the partial differential equation by eliminating the arbitrary functions:

① $z = f(x^2+y^2) \longrightarrow ①$

differenting ① partially w.r to 'x' and 'y',

$$\frac{\partial z}{\partial x} = f'(x^2+y^2) \cdot 2x$$

ie, $p = f'(x^2+y^2) \cdot 2x \longrightarrow ②$

$$\frac{\partial z}{\partial y} = f'(x^2+y^2) \cdot 2y$$

ie, $q = f'(x^2+y^2) \cdot 2y \longrightarrow ③$

Divide, $\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{\cancel{f'(x^2+y^2)} \cdot \cancel{2}x}{\cancel{f'(x^2+y^2)} \cdot \cancel{2}y}$

$\Rightarrow py = qx$

$\Rightarrow py - qx = 0$

$yp - xq = 0$ which is required eqn.

2) $z = x^2 + 2f\left(\frac{1}{y} + \log x\right) \rightarrow (1)$

differentiating eqn (1) part. w. r to 'x' & 'y',

$$\frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y} + \log x\right) \cdot \frac{1}{x}$$

ie, $p = 2x + 2f'\left(\frac{1}{y} + \log x\right) \cdot \frac{1}{x}$

$$p - 2x = 2f'\left(\frac{1}{y} + \log x\right) \cdot \frac{1}{x} \longrightarrow (2)$$

$$\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right)$$

ie, $q = 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right) \longrightarrow (3)$

From (2) & (3),

$$\frac{(2)}{(3)} = \frac{p-2x}{q} = \frac{\cancel{2f'\left(\frac{1}{y}+\log x\right)} \cdot \frac{1}{x}}{\cancel{2f'\left(\frac{1}{y}+\log x\right)}\left(\frac{-1}{y^2}\right)}$$

$$\frac{p-2x}{q} = \frac{1}{x}\left(\frac{-y^2}{1}\right)$$

$$\frac{p-2x}{q} = \frac{-y^2}{x}.$$

$$\Rightarrow \quad px - 2x^2 = -y^2 q.$$

$$px + y^2 q = 2x^2.$$

$\Rightarrow \quad xp + y^2 q = 2x^2$ which is the req. eqn.

③ Form the pde by eliminating the arbitrary function from the relation $\phi(x^2+y^2+z^2,\ lx+my+nz)=0$.

Soln:- The equation $\phi(x^2+y^2+z^2,\ lx+my+nz)=0$ may be written as $x^2+y^2+z^2 = f(lx+my+nz)$ where f is an arbitrary function. $\longrightarrow$ ①.

diff. eqn ① part. w.r to 'x' & 'y',

$$2x+2z\cdot\frac{\partial z}{\partial x} = f'(lx+my+nz)\cdot\left(l+n\frac{\partial z}{\partial x}\right)$$

ie, $2[x+zp] = f'(lx+my+nz)\cdot(l+np) \longrightarrow$ ②.

$$2y+2z\cdot\frac{\partial z}{\partial y} = f'(lx+my+nz)\cdot\left(m+n\frac{\partial z}{\partial y}\right).$$

ie, $2[y+zq] = f'(lx+my+nz)\cdot(m+nq) \longrightarrow$ ③.

$$\frac{②}{③} \Rightarrow \frac{\cancel{2}(x+zp)}{\cancel{2}(y+zq)} = \frac{\cancel{f'(lx+my+nz)}\cdot(l+np)}{\cancel{f'(lx+my+nz)}\cdot(m+nq)}$$

$$(x+zp)(m+nq) = (l+np)(y+zq)$$

$$xm+nqx+zpm+\cancel{zpnq} = ly+lzq+npy+\cancel{npzq}$$

$$xm+nqx+zpm-ly-lzq-npy = 0.$$

$(mz-ny)p+(nx-lz)q = ly-mx$ which is req. eqn.

④ Form the pde by eliminating f from

$$f(x^2+y^2+z^2,\ x+y+z)=0.$$

Soln:-

The equation $f(x^2+y^2+z^2,\ x+y+z)=0$ can be written as $x^2+y^2+z^2 = \phi(x+y+z)$ $\longrightarrow$ ① where ϕ is an arbitrary functions

diff. eqn ① partially w.r.to 'x' & 'y',

$$2x+2z\cdot\frac{\partial z}{\partial x} = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right)$$

ie, $2[x+zp] = \phi'(x+y+z)(1+p).$ $\longrightarrow$ ②

$$2y+2z\cdot\frac{\partial z}{\partial y} = \phi'(x+y+z)\cdot\left(1+\frac{\partial z}{\partial y}\right)$$

ie, $2[y+zq] = \phi'(x+y+z)(1+q)$ $\longrightarrow$ ③.

$\frac{②}{③} \Rightarrow$. $$\frac{\cancel{2}(x+zp)}{\cancel{2}(y+zq)} = \frac{\cancel{\phi'(x+y+z)}(1+p)}{\cancel{\phi'(x+y+z)}(1+q)}$$

$$(x+zp)(1+q) = (y+zq)(1+p)$$

$$x+xq+zp+\cancel{zpq} = y+yp+zq+\cancel{zqp}$$

$$x+xq+zp-y-yp-zq=0.$$

$(z-y)p+(x-z)q = y-x$. which is req. eqn.

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