MODULE-5 Z-Transform:-

Introduction of z-transform - Elementary properties - Change of scale property - shifting property - Z transform of a^n, $1/n$, $1/(n+1)$, $1/n^2$, $1/(n+1)^2$, $r^n \cos n\theta$ and $r^n \sin n\theta$ - Initial value theorem, Final value theorem, Inverse z-transform - long division method - related problems - partial fraction method - Residue method - Problems - convolution theorem (without proof) - convolution theorem applications - solution of linear difference equations with constant coefficients using z-transform.

DEFINITION:-

The z-transform of a sequence $x(n)$ defined for discrete values $n=0,1,\ldots$ is denoted by $z[x(n)]$ and is defined by

$$z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n} = X(z)$$

where X is a function of z.

PROPERTIES OF Z-TRANSFORM

1. LINEAR PROPERTY

If $z[x(n)] = X(z)$ and $z[y(n)] = Y(z)$ then

$$z[a x(n) + b y(n)] = a X(z) + b Y(z)$$

Proof:-

$$z[a x(n) + b y(n)]$$

$$= \sum_{n=0}^{\infty} [a x(n) + b y(n)] z^{-n}$$

$$= \sum_{n=0}^{\infty} a x(n) z^{-n} + \sum_{n=0}^{\infty} b y(n) z^{-n}$$

$$= a \sum_{n=0}^{\infty} x(n) z^{-n} + b \sum_{n=0}^{\infty} y(n) z^{-n}$$

$= a X(z) + b Y(z)$

$= Z[a x(n) + b y(n)] = a X(z) + b Y(z)$

2. CHANGE OF SCALE PROPERTY

If $Z[x(n)] = X(z)$ then

i) $Z[a^n x(n)] = X(z/a)$

ii) $Z[a^{-n} x(n)] = X(az)$

Proof: Given $Z[x(n)] = X(z)$

$$\sum_{n=0}^{\infty} x(n) z^{-n} = X(z)$$

i) $Z[a^n x(n)]$

$$Z[a^n x(n)] = \sum_{n=0}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) \frac{z^{-n}}{a^{-n}}$$

$$= \sum_{n=0}^{\infty} x(n) (z/a)^{-n}$$

$$Z[a^n x(n)] = X(z/a)$$

ii) $Z[a^{-n} x(n)]$

$$Z[a^{-n} x(n)] = \sum_{n=0}^{\infty} a^{-n} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) (az)^{-n}$$

$$\boxed{Z[a^{-n} x(n)] = X(az)}$$

3. LEFT SHIFTING THEOREM

If $Z[x(n)] = X(z)$, then

$$Z[x(n-k)] = z^{-k} X(z)$$

Proof:

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n)\, z^{-n} = X(z)$$

$$Z[x(n-k)] = \sum_{n-k=0}^{\infty} x(n-k)\, z^{-n}$$

put $n-k = m \Rightarrow n = m+k$

$n = k \Rightarrow m = 0$

$n = \infty \Rightarrow m = \infty$

$$\Rightarrow \sum_{n=k}^{\infty} x(n-k)\, z^{-n}$$

$$= \sum_{m=0}^{\infty} x(m)\, z^{-m-k}$$

$$= \sum_{m=0}^{\infty} x(m)\, z^{-m}\, z^{-k}$$

$$= z^{-k} \sum_{m=0}^{\infty} x(m)\, z^{-m}$$

put $m = n$,

$$= z^{-k} \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= z^{-k} X(z)$$

$$\boxed{Z[x(n-k)] = z^{-k} X(z)}$$

4. FIRST SHIFTING THEOREM

If $Z[x(n)] = X(z)$, Then,

$$Z[x(n+1)] = zX(z) - z\,x(0)$$

Proof:

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n)\, z^{-n} = X(z)$$

$$Z[x(n+1)] = \sum_{n=-1}^{\infty} x(n+1)\, z^{-n}$$

put $n+1=m \Rightarrow n=m-1$

when $n=-1$, $m=0$

$n=\infty$, $m=\infty$

$$\Rightarrow \sum_{m-1=0}^{\infty} x(m)\, z^{-(m-1)}$$

$$\Rightarrow \sum_{m=1}^{\infty} x(m)\, z^{-m} \cdot z$$

$$\Rightarrow z \sum_{m=1}^{\infty} x(m)\, z^{-m}$$

put $m=n$

$$\Rightarrow z \sum_{n=1}^{\infty} x(n)\, z^{-n}$$

$$\Rightarrow z \left\{ \sum_{n=0}^{\infty} x(n)\, z^{-n} - x(0) \right\}$$

$$\Rightarrow z \left\{ X(z) - x(0) \right\}$$

$$\boxed{Z[x(n+1)] = z[X(z) - x(0)]}$$

5. SECOND SHIFTING THEOREM

If $Z[x(n)] = X(z)$, then prove that

$$Z[x(n+2)] = z^2 X(z) - z^2 x(0) - z\, x(1)$$

proof:-

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n)\, z^{-n} = X(z)$$

$$Z[x(n+2)] = \sum_{n=-2}^{\infty} x(n+2)\, z^{-n}$$

put $n+2=m$ $\quad n=m-2$

when $n=0$ $\quad m=2$

$n=\infty$ $\quad m=\infty$

$$\Rightarrow \sum_{m=2}^{\infty} x(m) z^{-(m-2)}$$

$$= \sum_{m=2}^{\infty} x(m) z^{-m} z^{2}$$

$$\Rightarrow z^2 \sum_{m=2}^{\infty} x(m) z^{-m}$$

put $m=n$

$$\Rightarrow z^2 \sum_{n=2}^{\infty} x(n) z^{-n}$$

$$\Rightarrow z^2 \left[\sum_{n=0}^{\infty} x(n) z^{-n} - x(1) z^{-1} - x(0) \right]$$

$$\Rightarrow z^2 \sum_{n=0}^{\infty} x(n) z^{-n} - x(1) z - z^2 x(0)$$

$$\Rightarrow z^2 X(z) - z^2 x(0) - z x(1)$$

$$\therefore \boxed{Z[x(n+2)] = z^2 X(z) - z^2 x(0) - z x(1)}$$

Binomial Expansion:-

i) $(1-z)^{-1} = 1 + z + z^2 + \dots \quad ; \ |z| < 1$

ii) $(1+z)^{-1} = 1 - z + z^2 - z^3 + \dots \quad ; \ |z| < 1$

iii) $(1-z)^{-2} = 1 + 2z + 3z^2 + \dots \quad |z| < 1$

iv) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 - \dots$

v) $1 + 4z + 9z^2 + \dots = (1+z)(1-z)^{-3} \quad ; \ |z| < 1$

vi) $e^z = 1 + z/1! + z^2/2! + z^3/3! + \dots$

vii) $e^{-z} = 1 - z/1! + z^2/2! + -z^3/3! + \dots$

Z-Transform for standard functions:-

i) $Z(1)$

Soln

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n} = X(z)$$

$$Z[1] = \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} (1/z)^n$$

$$= 1 + 1/z + (1/z)^2 + (1/z)^3 + \dots$$

$$= (1 - 1/z)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1} = z/z-1$$

$$\boxed{Z[1] = z/z-1} \qquad |1/z| < 1$$

ii) $Z[(-1)^n]$

$$Z[(-1)^n] = \sum_{n=0}^{\infty} (-1)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (-1/z)^n$$

$$= 1 + (-1/z)^{+1} + (-1/z)^2 + (-1/z)^3 + \dots$$

$$\Rightarrow 1 - 1/z + (1/z)^2 - (1/z)^3 + \dots$$

$$= (1 + 1/z)^{-1}$$

$$= \left(\frac{z+1}{z}\right)^{-1}$$

$$= \frac{z}{z+1} \qquad \boxed{Z[(-1)^n] = \frac{z}{z+1}} \qquad |1/z| < 1 \ ; |z| > 1$$

3) $z[a^n]$

$$z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a/z)^n$$

$$= 1 + a/z + (a/z)^2 + (a/z)^3 + \dots$$

$$= (1 - a/z)^{-1}$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$$\boxed{z[a^n] = \frac{z}{z-a}} \qquad |a/z| < 1$$

iv) $z[(-1)^n a^n]$

$$z[(-1)^n a^n] = \sum_{n=0}^{\infty} (-1)^n a^n . z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(-a)^n}{z^n} = \sum_{n=0}^{\infty} (-a/z)^n$$

$$= 1 + (-a/z) + (-a/z)^2 + (-a/z)^3 + \dots$$

$$= 1 - a/z + (a/z)^2 - (a/z)^3 + \dots$$

$$= (1 + a/z)^{-1}$$

$$\Rightarrow \left(\frac{z+a}{z}\right)^{-1}$$

$$\Rightarrow z/z+a$$

$$\boxed{z[(-1)^n a^n] = z/z+a} \qquad |a/z| < 1$$

5. Unit-Step function's z-transform.

$$u(n) = \begin{cases} 1 & n > 0 \\ 0 & n < 0 \end{cases}$$

$$Z[u(n)] = \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} (1/z)^n$$

$$= 1 + 1/z + (1/z)^2 + (1/z)^3 + \cdots$$

$$= (1 - 1/z)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1} = z/z-1$$

$$\boxed{Z[u(n)] = \frac{z}{z-1}} \quad |1/z| < 1$$

6. Z-transform of unit impluse function.

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n) z^{-n}$$

$$= \delta(0) z^{-0} + \delta(1) z^{-1} + \cdots$$

$$= 1 + 0 + 0$$

$$\boxed{Z[\delta(n)] = 1}$$

7. $Z[n]$

Soln

$$z[n] = \sum_{n=0}^{\infty} n z^{-n}$$

$$= \sum_{n=0}^{\infty} (n/z^n)$$

$$= 0 + 1/z + 2/z^2 + 3/z^3 + \cdots$$

$$= 1/z + 2(1/z)^2 + 3(1/z)^3 + \cdots$$

$$= 1/z\left[1 + 2(1/z)^1 + 3(1/z)^2 + \cdots\right]$$

$$= 1/z\left[1 - 1/z\right]^{-2}$$

$$= 1/z\left[\frac{z-1}{z}\right]^{-2} = 1/z\left[\frac{z^2}{(z-1)^2}\right]$$

$$\boxed{z[n] = z/(z-1)^2}$$

8. $z[n^2]$ method 1

Soln

$$z[n^2] = \sum_{n=0}^{\infty} n^2 z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{n^2}{z^n}\right)$$

$$= 0 + 1/z + 4/z^2 + 9/z^3 + \cdots$$

$$= 1/z\left[1 + 4/z + 9/z^2 + \cdots\right]$$

$$= 1/z\left[(1 + 1/z)(1 - 1/z)^{-3}\right]$$

$$= 1/z\left[\left(\frac{z+1}{z}\right)\left(\frac{z-1}{z}\right)^{-3}\right]$$

$$= 1/z\left[\frac{z+1}{z} \cdot \frac{z^3}{(z-1)^3}\right]$$

$$\boxed{z[n^2] = \frac{z(z+1)}{(z-1)^3}}$$

9. $z[1/n!]$

Soln $z[1/n!] = \sum_{n=0}^{\infty} (1/n!) z^{-n}$

$= \sum_{n=0}^{\infty} 1/n! (1/z)^n$

$= 1 + 1/z + 1/1! + 1/2! (1/z)^2 + \dots$

$$\boxed{z(1/n!) = e^{1/z}}$$

10. $z\left[\frac{a^n}{n!}\right]$

$z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$

$= \sum_{n=0}^{\infty} 1/n! (a^n/z^n)$

$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n$

$= 1 + 1/1! (a/z) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \dots$

$$\boxed{z\left[\frac{a^n}{n!}\right] = e^{a/z}}$$

11) $z[1/n]$

$z[1/n] = \sum_{n=1}^{\infty} 1/n \cdot z^{-n} = \sum_{n=1}^{\infty} \frac{1}{n} (1/z)^n$

$\Rightarrow 1/z + \frac{1}{2}\left(\frac{1}{z}\right)^2 + 1/3 (1/z)^3 + \dots$

$$\boxed{z[1/n] = \log\left(\frac{z}{z-1}\right)}$$

2

12. $Z\left[1/n^2\right]$

$$Z\left[1/n^2\right] = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{-n} = \sum_{n=1}^{\infty} 1/n^2 \,(1/z)^n$$

$$= \frac{1}{1}(1/z) + 1/2^2\left(\frac{1}{z}\right)^2 + \frac{1}{3^2}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= \frac{1}{1}\left(\frac{1}{z}\right) + \left(1/2z\right)^2 + \left(1/3z\right)$$

13. $Z\left[1/n+1\right]$

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(1/n+1\right) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)(1/z)^n$$

$$= 1 + 1/2\,(1/z) + 1/3\,(1/z)^2 + \cdots$$

$$= z\left[\frac{1}{z} + \frac{1}{2}(1/z)^2 + 1/3\,(1/z)^3 + \cdots\right]$$

$$\boxed{Z[x(n)] = z \log\left(\frac{z}{z-1}\right)}$$

14. $Z\left[\frac{1}{(n+1)^2}\right]$

15. $Z[\cos n\theta]$ and $z[\sin n\theta]$

Soln

WKT $\quad z[a^n] = \frac{z}{z-a}$

$a = e^{i\theta}$; $a^n = (e^{i\theta})^n$

$$z\left[e^{i\theta n}\right] = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$z[\cos n\theta + i\sin n\theta] = \frac{z}{z-\cos\theta - i\sin\theta} \times \frac{z-\cos\theta + i\sin\theta}{z-\cos\theta+i\sin\theta}$$

$$\Rightarrow \frac{z(z-\cos\theta + i\sin\theta)}{(z-\cos\theta)^2 + \sin^2\theta} \Rightarrow \frac{z(z-\cos\theta)+iz\sin\theta}{z^2+\cos^2\theta - 2z\cos\theta + \sin^2\theta}$$

$$\Rightarrow \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+1}$$

$$z[\cos n\theta + i\sin n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\,\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Equating real & imaginary parts,

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

4

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

problems:-

1. Find $z[n(n-1)]$

soln

$$z[n^2-n] = z[n^2] - z[n]$$

$$= \frac{z^2+z}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$\Rightarrow \frac{z^2+z-z(z-1)}{(z-1)^3} \Rightarrow \frac{z^2+z-z^2+z}{(z-1)^3} = \frac{2z}{(z-1)^3}$$

$$\boxed{Z[n^2-n] = \frac{2z}{(z-1)^3}}$$

2. Find $Z\left[\frac{(n+1)(n+2)}{2}\right]$

Soln

$$Z\left[\frac{(n+1)(n+2)}{2}\right] = Z\left[\frac{n^3+3n+2}{2}\right]$$

$$\Rightarrow \frac{1}{2}\left\{Z[n^2] + 3Z[n] + 2Z[1]\right\}$$

$$\Rightarrow \frac{1}{2}\left\{\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}\right\}$$

$$\Rightarrow \frac{1}{2}\left\{\frac{z^2+z+3z(z-1)+2z(z-1)^2}{(z-1)^3}\right\}$$

$$\Rightarrow \frac{1}{2}\left\{\frac{z^2+z+3z^2-3z+2z^3-4z^2+2z}{(z-1)^3}\right\}$$

$$= \frac{1}{2}\left\{\frac{2z^3}{(z-1)^3}\right\} \Rightarrow \boxed{Z\left[\frac{(n+1)(n+2)}{2}\right] = \frac{z^3}{(z-1)^3}}$$

3. Find $Z\left[\sin\left(\frac{n\pi}{2}+\pi/4\right)\right]$

Soln

$$\Rightarrow Z\left[\sin\frac{n\pi}{2}\cdot\cos\frac{\pi}{4} + \cos\frac{n\pi}{2}\sin\pi/4\right]$$

$$\Rightarrow Z\left[\sin\frac{n\pi}{2}\cdot\frac{1}{\sqrt{2}} + \cos\frac{n\pi}{2}\cdot 1/\sqrt{2}\right]$$

$$\Rightarrow \frac{1}{\sqrt{2}} Z\left[\sin\frac{n\pi}{2}\right] + \frac{1}{\sqrt{2}} Z\left[\cos\frac{n\pi}{2}\right]$$

$$\Rightarrow \frac{1}{\sqrt{2}} \frac{z}{z^2+1} + \frac{1}{\sqrt{2}} \frac{z^2}{z^2+1} \Rightarrow \frac{z^2+z}{\sqrt{2}(z^2+1)}$$

$$\boxed{Z\left[\sin\left(\frac{n\pi}{2} + \pi/4\right)\right] = \frac{z(z+1)}{(z^2+1)\sqrt{2}}}$$

4. Find $Z\left[\frac{n}{(n-1)(n+2)}\right]$

Soln

Consider $\frac{n}{(n-1)(n+2)} = \frac{A}{n-1} + \frac{B}{n+2}$

$$\frac{n}{(n-1)\cancel{(n+2)}} = \frac{A(n+2) + B(n-1)}{\cancel{(n-1)}(n+2)}$$

$n = A(n+2) + B(n-1)$ —— 1

put $n=1$ in eqn 1

$1 = 3A$

$\boxed{A = 1/3}$

put $n=-2$ in eqn 1

$-2 = -3B$

$\boxed{B = 2/3}$

$$\frac{n}{(n-1)(n+2)} = \frac{1/3}{n-1} + \frac{2/3}{n+2}$$

$$\frac{n}{(n-1)(n+2)} = \frac{1}{3(n-1)} + \frac{2}{3(n+2)}$$

$$Z\left[\frac{n}{(n-1)(n+2)}\right] = 1/3\, Z\left[1/n-1\right] + 2/3\, Z\left[\frac{1}{n+2}\right]$$

$$\boxed{Z\left[\frac{n}{(n-1)(n+2)}\right] = 1/3 \log\left(\frac{z}{z-1}\right) + 2/3\left[z^2 \log\left(\frac{z}{z-1}\right) - z\right]}$$

$$Z[a^n x(n)] = Z[x(n)]_{z \to z/a}$$

$$Z[a^{-n} x(n)] = Z[x(n)]_{z \to az}$$

5. $Z[na^n]$.

$$Z[a^n n] \Rightarrow Z[n]_{z \to z/a}$$

$$\Rightarrow \left[\frac{z}{(z-1)^2}\right]_{z \to z/a}$$

$$\Rightarrow \frac{z/a}{(z/a-1)^2} \Rightarrow \frac{z/a}{\frac{(z-a)^2}{a^2}} \Rightarrow \frac{z}{a} \cdot \frac{a^2}{(z-a)^2} = \frac{az}{(z-a)^2}$$

$$\boxed{Z[a^n n] = \frac{az}{(z-a)^2}}$$

6. Find $Z[a^{-n} n^2]$

$$Z[a^{-n} n^2] = Z[n^2]_{z \to az}$$

$$= \left[\frac{z(z+1)}{(z-1)^3}\right]_{z \to az} \Rightarrow \frac{az(az+1)}{(az-1)^3}$$

$$\boxed{Z[a^{-n} n^2] = \frac{az(az+1)}{(az-1)^3}}$$

7. Find $Z\left[\frac{1}{(n+2)!}\right]$ using Shifting property.

Soln

$x(n+2) = \frac{1}{(n+2)!}$; $x(n) = 1/n!$ $x(0) = 1/0! = 1$

$$X(z) = Z[x(n)] = Z[1/n!] = e^{1/z}$$

$$\boxed{Z[x(n+2)] = z^2 e^{1/z} - z^2 - z}$$

INITIAL VALUE THEOREM (IVT)

If $Z[x(n)] = X(z)$, then

$$x(0) = \lim_{z\to\infty} X(z)$$

FINAL VALUE THEOREM (FVT)

If $Z[x(n)] = X(z)$, then

$$\lim_{z\to\infty} x(n) = \lim_{z\to 1} (z-1)X(z).$$

1. If $X(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$, evaluate $x(0)$

Soln

$$x(0) = \lim_{z\to\infty} X(z)$$

$$= \lim_{z\to\infty}\left(\frac{2z^2+5z+14}{(z-1)^4}\right)$$

$$\Rightarrow \lim_{z\to\infty} \frac{z^2[2+5/z+14/z^2]}{z^4[1-1/z]^4}$$

$$\Rightarrow \lim_{z\to\infty} \frac{(2+5/z+14/z^2)}{z^2(1-1/z)^4} = 0$$

$$\boxed{x(0)=0}$$

2. If $F(z) = \dfrac{5z}{(z-2)(z-3)}$ find $f(0)$ and $\lim_{t\to\infty} f(t)$

$$\boxed{\frac{1}{\infty}=0}$$

Soln

$$f(0) = \lim_{z\to\infty} \frac{5z}{(z-2)(z-3)} \quad \left(\frac{\infty}{\infty}\right)$$

$$\Rightarrow \lim_{z\to\infty} \frac{\frac{d}{dz}(5z)}{d/dz[z^2-5z+6]} \quad \text{[L'Hospital rule]} = \lim_{z\to\infty} \frac{5}{2z-5} = 0$$

$$\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) \frac{5z}{(z-2)(z-3)} = 0$$

Inverse Z-transform

PARTIAL FRACTION METHOD:- I

procedure:-

i) Assume $X(z)$, find $\frac{X(z)}{z}$

ii) Find partial fraction for $\frac{X(z)}{z}$

iii) Find $X(z)$

iv) find $z^{-1}[X(z)]$

1. Find $z^{-1}\left[\frac{z^2-3z}{(z-5)(z+2)}\right]$ using partial fraction.

Soln

$$X(z) = \frac{z^2-3z}{(z-5)(z+2)} = \frac{z(z-3)}{(z-5)(z+2)} \Rightarrow \frac{X(z)}{z} = \frac{z-3}{(z-5)(z+2)}$$

$$\frac{z-3}{(z-5)(z+2)} = \frac{A}{z-5} + \frac{B}{z+2}$$

$$\frac{z-3}{(z-5)(z+2)} = \frac{A(z+2)+B(z-5)}{(z-5)(z+2)}$$

$$z-3 = A(z+2) + B(z-5) \quad \text{—(1)}$$

Put $z=5$ in eqn (1)

$2 = 7A$

$\boxed{A = 2/7}$

put $z=-2$ in eqn (1)

$-5 = -7B$

$\boxed{B = 5/7}$

$$\frac{X(z)}{z} = \frac{2/7}{z-5} + \frac{5/7}{z+2}$$

$$X(z) = \frac{2}{7}\frac{z}{z-5} + \frac{5}{7}\frac{z}{z+2}$$

$$z^{-1}[X(z)] = \frac{2}{7} z^{-1}\left[\frac{z}{z-5}\right] + \frac{5}{7} z^{-1}\left[\frac{z}{z+2}\right]$$

$$\boxed{z^{-1}[X(z)] = \frac{2}{7} 5^n + \frac{5}{7}(-2)^n}$$

2. Find $z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$

$$X(z) = \frac{z^2}{(z+2)(z^2+4)}$$

$$\frac{X(z)}{z} = \frac{z}{(z+2)(z^2+4)}$$

$$\frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$$

$$\frac{z}{(z+2)(\cancel{z^2+4})} = \frac{A(z^2+4) + (Bz+C)(z+2)}{(z+2)(\cancel{z^2+4})}$$

$$z = A(z^2+4) + (Bz+C)(z+2) \quad \text{— (1)}$$

put $z=-2$ in eqn (1)

$-2 = 8A$

$\boxed{A = -1/4}$

put $z=0$ in eqn (1)

$0 = 4A + 2C$

$0 = 4(-1/4) + 2C$

$2C = 1$

$\boxed{C = 1/2}$

put $z=1$ in eqn (1)

$1 = 5A + (B+C)(3)$

$1 = 5A + 3B + 3C$

$= 5(-1/4) + 3B + 3/2$

$1 = \frac{-5}{4} + \frac{3}{2} + 3B$

$1 = \frac{-5+6}{4} + 3B$

$= 1/4 + 3B$

$1 - 1/4 = 3B$

$3/4 = 3B \rightarrow \boxed{B = 1/4}$

$$\frac{X(z)}{z} = \frac{-1/4}{z+2} + \frac{1/4 z + 1/2}{z^2+4}$$

$$X(z) = -1/4 \frac{z}{z+2} + 1/4 \frac{z^2}{z^2+4} + 1/2 \frac{z}{z^2+4}$$

$$z^{-1}[X(z)] = -1/4\, z^{-1}\left[\frac{z}{z+2}\right] + 1/4\, z^{-1}\left[\frac{z^2}{z^2+4}\right] + 1/2\, z^{-1}\left[\frac{z}{z^2+4}\right]$$

$$z^{-1}[X(z)] = -1/4\,(-2)^n + 1/4\, 2^n \cos n\frac{\pi}{2} + 1/4\, 2^n \sin n\pi/2$$

Z- Transform using Convolution theorem

Definition of Convolution:-

Let $x(n)$ and $y(n)$ be two sequences defines for discrete values $n = 0,1,2 \ldots$ Then the Convolution for $x(n)$ and $y(n)$ is defined as

$$x(n) * y(n) = \sum_{k=0}^{\infty} x(k)\, y(n-k)$$

Convolution Theorem:-

If $z[x(n)] = X(z)$ and $z[y(n)] = Y(z)$ then

$$z[x(n) * y(n)] = X(z)\,Y(z)$$

1. Find $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ by using Convolution theorem.

Soln

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{k=0}^{n} a^k b^{n-k}$$

$$\Rightarrow \sum_{k=0}^{n} a^k b^n . b^{-k}$$

$$= b^n \sum_{k=0}^{n} a^k b^{-k}$$

$$= b^n \sum_{k=0}^{n} (a/b)^k$$

Formula

$a + ar + ar^2 + \dots + ar^n$

$\Rightarrow \frac{a[1-r^{n+1}]}{1-r}$ if $r<1$

Here $a=1$

$$= b^n\left[1+(a/b)+(a/b)^2+\dots\right]$$

$$= b^n\left[\frac{1-(a/b)^{n+1}}{1-(a/b)}\right]$$

$$= b^n\left[\frac{1-\frac{a^{n+1}}{b^{n+1}}}{\frac{b-a}{b}}\right]$$

$$= b^n\left[\frac{[b^{n+1}-a^{n+1}]}{b^{n+1}} \cdot \frac{b}{b-a}\right]$$

$$= \cancel{b^n}\left(\frac{b^{n+1}-a^{n+1}}{\cancel{b^n}.\cancel{b}} \cdot \frac{\cancel{b}}{b-a}\right)$$

$$= \frac{b^{n+1} - a^{n+1}}{b-a}$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{b^{n+1} - a^{n+1}}{b-a}$$

2. Find $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right]$ using C.T

Soln

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] = Z^{-1}\left[\frac{8z^2}{2(z-\frac{1}{2})\,4(z-\frac{1}{4})}\right]$$

$$= Z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}\right]$$

$$\Rightarrow Z^{-1}\left[\frac{z}{z-\frac{1}{2}} \cdot \frac{z}{z-\frac{1}{4}}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$\Rightarrow (\tfrac{1}{2})^n * (\tfrac{1}{4})^n \Rightarrow \sum_{k=0}^{n} (\tfrac{1}{2})^k (\tfrac{1}{4})^{n-k}$$

$$\Rightarrow \sum_{k=0}^{n} (\tfrac{1}{2})^k (\tfrac{1}{4})^n (\tfrac{1}{4})^{-k} \Rightarrow (\tfrac{1}{4})^n \sum_{k=0}^{n} (\tfrac{1}{2})^k \cdot (\tfrac{1}{4})^{-k}$$

$$= (\tfrac{1}{4})^n \sum_{k=0}^{n} 2^k$$

$$\Rightarrow (\tfrac{1}{4})^n \sum_{k=0}^{n} \left(\frac{\frac{1}{2}}{\frac{1}{4}}\right)^k$$

$$= (\tfrac{1}{4})^n \left\{1 + 2 + 2^2 + 2^3 + \cdots\right\}$$

$$\Rightarrow (\tfrac{1}{4})^n \left\{\frac{1-2^{n+1}}{1-2}\right\} = (\tfrac{1}{4})^n \left(\frac{1-2^{n+1}}{-1}\right)$$

$$= (\tfrac{1}{4})^n \left(2^{n+1} - 1\right)$$

$$\boxed{Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] = (\tfrac{1}{4})^n \left(2^{n+1}-1\right)}$$

3. Find $z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$ by using C.T

Sol^n

$$z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = z^{-1}\left[\frac{z}{z+a}\cdot\frac{z}{z+a}\right]$$

$$= z^{-1}\left[\frac{z}{z+a}\right] * z^{-1}\left[\frac{z}{z+a}\right]$$

$$= (-a)^n * (-a)^n$$

$$= \sum_{k=0}^{n} (-a)^k \cdot (-a)^{n-k}$$

$$\Rightarrow (-a)^n \cdot \sum_{k=0}^{n} 1^k$$

$$= (-a)^n (1+1+\dots+1)$$

$$= (-1)^n a^n (n+1)$$

$$\boxed{z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = (-1)^n a^n (n+1)}$$

Inverse z-transforms using method of Residues.

CRT

$x(n)$ = sum of the residues of the singular pts of $X(z)z^{n-1}$

1. Find $z^{-1}\left[\frac{z}{z^2+7z+10}\right]$

Sol^n

Let $X(z) = \frac{z}{z^2+7z+10}$

$$X(z)z^{n-1} = \frac{z}{(z+2)(z+5)}z^{n-1}$$

$$X(z)\,z^{n-1} = \frac{z^n}{(z+2)(z+5)}$$

To find poles:-

$(z+2)(z+3)=0$

$z = -2, 5$ are poles of order 1.

$$R_1 = [\text{Res } f(z)]_{z=-2}$$

$$= \lim_{z \to -2} (z+2) \cdot \frac{z^n}{(z+2)(z+5)} = \frac{(-2)^n}{3}$$

$$\boxed{R_1 = \frac{(-2)^n}{3}}$$

$$R_2 = [\text{Res } f(z)]_{z=-3}$$

$$= \lim_{z \to -5} (z+5) \cdot \frac{z^n}{(z+2)(z+5)} = \frac{(-5)^n}{-3}$$

$$z^{-1}[X(z)] = \frac{(-2)^n}{3} - \frac{(-5)^n}{3}$$

$$\boxed{\therefore\ z^{-1}\left[\frac{z}{(z+2)(z+5)}\right] = \frac{(-2)^n - (-5)^n}{3}}$$

2. Find $z^{-1}[X(z)]$, where $X(z) = \dfrac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$

Soln

$$X(z)\,z^{n-1} = \frac{(4z-2)z \cdot z^{n-1}}{z^3 - 5z^2 + 8z - 4} = \frac{4z^{2+n-1} - 2z^n}{(z-1)(z-2)^n}$$

$$X(z)\,z^{n-1} = \frac{4z^{n+1} - 2z^n}{(z-1)(z-2)^2}$$

To find poles:-

$(z-1)(z-2)^2 = 0$

$z=1$ is a poles of order 1

$z=2$ is a poles of order 2.

To find Residues:-

$R_1 = \left[\text{Res } X(z) z^{n-1}\right]_{z=1}$

$$\lim_{z\to 1} (z-1) \frac{4z^{n+1} - 2z^n}{(z-1)(z-2)^2} = \frac{4-2}{(1-2)^n} = 2/_1$$

$\boxed{R_1 = 2}$

$R_2 = \left[\text{Res } X(z) z^{n-1}\right]_{z=2}$

$$= \lim_{z\to 2} \frac{d}{dz} (z-2)^2 \frac{z^{n+1} - 2z^n}{(z-2)^2 (z-1)}$$

$$= \lim_{z\to 2} \left[\frac{(z-1)4(n+1) \cdot z^n - 2nz^{n-1} \cdot (4z^{n+1} - 2z^n)}{(z-1)^2}\right]$$

$$= \frac{4^{(n+1)} \cdot 2^n - 2^n 2^{n-1} - 4 \cdot 2^{n+1} - 2 \cdot 2^n}{(2-1)^2}$$

$$\Rightarrow \frac{4(n+1)2^n - n2^n - 2^{n+3} - 2^{n+1}}{1}$$

$$= 4n2^n + 42^n - n2^n - 82^n + 2(2^n)$$

$$= 2^n[4n + 4 - 4 - 8 + 2] = 2^n[3n-2]$$

$$\boxed{z^{-1}[X(z)] = 2 + 2^n(3n-2)}$$

FORMATION OF DIFFERENCE EQUATION:-

1. From the difference equation from $y_n = a + b3^n$, where a and b are arbitrary constants.

soln

$$y_n = a + b3^n$$

$$y_{n+1} = a + b \cdot 3^n \cdot 3$$

$$y_{n+2} = a + b \cdot 3^n \cdot 9$$

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n(9-3) - 1(9y_{n+1} - 3y_{n+2}) + (y_{n+1} - y_{n+2}) = 0$$

$$6y_n - 9y_{n+1} + 3y_{n+2} + y_{n+1} - y_{n+2} = 0$$

$$6y_n - 8y_{n+1} + 2y_{n+2} = 0$$

$$\boxed{y_{n+2} - 4y_{n+1} + 3y_n = 0}$$

SOLVING DIFFERENCE EQUATION USING Z-TRANSFORM

procedure:-

Step:1 Take z-transform on both sides of the given difference equation.

Step:2 use given condition and solve for $X(z)$

Step:3 Apply partial fractions method.

Step:4 Take inverse z-transform on both sides which is the given sequence

Notations:-

$$Z[y(n+2)] = z^2 y(z) - z^2 y(0) - z y(1)$$

$$Z[y(n+1)] = z Y(z) - z y(0)$$

$$Z[y(n)] = Y(z)$$

1. Solve $y_{n+1} - 3y_n = 0$ given $y(0) = 2$;

Soln

Given $y_{n+1} - 3y_n = 0$; $y_0 = 2$

$$Z[y_{n+1}] - 3Z[y_n] = Z[0]$$

$$z Y[z] - z y(0) - Y(z) = 0$$

$$z Y(z) - 2z - 3Y(z) = 0$$

$$Y(z)(z-3) = 2z$$

$$Y(z) = \frac{2z}{z-3}$$

$$z^{-1}[Y(z)] = 2z^{-1}\left[\frac{z}{z-3}\right] = 2 \cdot 3^n$$

$$\boxed{y_n = 2 \cdot 3^n}$$

2. Solve $4y_n - y_{n+2} = 0$ given $y(0) = 0$; $y(1) = 2$

Soln

$4y_n - y_{n+2} = 0$; $y(0) = 0$; $y(1) = 2$

$$Z[4y_n] - Z[y_{n+2}] = Z[0]$$

$$4Z[y_n] - Z[y_{n+2}] = Z[0]$$

$$4Y(z) - \{z^2 Y(z) - z^2 y(0) - z y(1)\} = 0$$

$$4Y(z) - \{z^2 y(z) - 0 - 2z\} = 0$$

$$Y(z)\,(-4+z^2) = 2z$$

$$Y(z) = \frac{2z}{z^2-4}$$

$$\frac{Y(z)}{z} = \frac{2}{z^2-4}$$

$$\frac{2}{z^2-4} = \frac{2}{(z-2)(z+2)}$$

$$\frac{2}{z^2-4} = \frac{A}{z-2} + \frac{B}{z+2}$$

$$\frac{2}{\cancel{z^2-4}} = \frac{A(z+2)+B(z-2)}{\cancel{z^2-4}}$$

$$2 = A(z+2) + B(z-2) \quad \text{——— } (1)$$

put $z=2$ in eqn (1)

$2 = 4A$

$\boxed{A = 1/2}$

put $z=-2$ in eqn (1)

$2 = -4B$

$\boxed{B = -1/2}$

$$\frac{Y(z)}{z} = \frac{1/2}{z-2} + \frac{-1/2}{z+2}$$

$$Y(z) = \frac{1}{2}\frac{z}{z-2} - \frac{1}{2}\frac{z}{z+2}$$

$$Z^{-1}[Y(z)] = \frac{1}{2} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{2} Z^{-1}\left[\frac{z}{z+2}\right]$$

$$\boxed{y_n = \frac{1}{2}\left[2^n - (-2)^n\right]}$$

3. using z-transform solve $u(n+2)+3u(n+1)+2u(n)=0$ given that $u(0)=1$; $u(1)=2$.

Soln

$$Z[u(n+2)] + 3Z[u(n+1)] + 2Z[y(n)] = Z(0)$$

$$z^2 U(z) - z^2 u(0) - z u(1) + 3\{z U(z) - z u(0)\} + 2U(z) = 0$$

$$U(z) \cdot \{z^2 + 3z + 2\} - z^2 - 3z - z(2) = 0$$

$$U(z) \{z^2 + 3z + 2\} - z^2 - 5z = 0$$

$$(z^2 + 3z + 2)\, U(z) = z(z+5)$$

$$U(z) = \frac{z(z+5)}{z^2+3z+2}$$

$$U(z)\, z^{n-1} = \frac{(z^2+5z)\, z^{n-1}}{(z+1)(z+2)}$$

$$U(z)\, z^{n-1} = \frac{z^{n+1} + 5z^n}{(z+1)(z+2)}$$

To find poles:-

$(z+1)(z+2) = 0$

$z+1 = 0 \;|\; z+2 = 0$

$z = -1$ is a poles of order 1

$z = -2$ is a poles of order 1

To find Residues:-

$$R_1 = \left[\operatorname{Res} U(z)\, z^{n-1}\right]_{z=-2}$$

$$\Rightarrow \lim_{z \to -2} (z+2)\, \frac{z^{n+1} + 5z^n}{(z+2)(z+3)}$$

$$\boxed{R_1 = (-2)^{n+1} + 5(-2)^n}$$

$$R_2 = \left[\operatorname{Res} U(z) z^{n-1}\right]_{z=-1}$$

$$\Rightarrow \lim_{z \to -1} \cancel{(z+1)} \frac{z^{n+1} + 5z^n}{(z+2)\cancel{(z+1)}}$$

$$\Rightarrow \frac{(-1)^{n+1} + 5(-1)^n}{1}$$

$$R_2 = \left[(-1)^{n+1} + (5)(-1)^n\right]$$

$$z\left[U(z)\right] = R_1 + R_2$$

$$= (-2)^n[5-2] + (-1)^n[5-1]$$

$$\boxed{z^{-1}\left[U(z)\right] = 3(-2)^n + 4(-1)^n}$$

Differentiation in Z-Domain [Extra question]

i, $Z[n f(t)] = -z \frac{d}{dz} F(z)$

ii, $Z[n f(n)] = -z \frac{d}{dz} F(z)$

1. Find $Z(n^2)$

W.K.T $Z[n f(n)] = -z \frac{d}{dz} F(z)$ (method 2)

$Z[n^2] = Z[n \cdot n] = -z \frac{d}{dz} [Z(n)]$

$= -z \frac{d}{dz} \left[\frac{z}{(z-1)^2} \right]$

$= -z \left[\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4} \right]$

$\Rightarrow -z \left[\frac{-z+1 \cancel{\ldots}}{(z-1)^3} \right] = -z \left[\frac{1-z}{(z-1)^3} \right]$

$= \frac{z(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$

2. Find $Z(n^3)$

WKT $Z[n f(n)] = -z \frac{d}{dz} F(z)$

$Z[n^3] = Z[n \cdot n^2] = -z \frac{d}{dz} Z[n^2]$

$\Rightarrow -z \frac{d}{dz} \left[\frac{z^2+z}{(z-1)^3} \right]$

Differentiation in Z-Domain [Extra Question]

i) $Z[n\,f(t)] = -z\frac{d}{dz}F(z)$

ii) $Z[n\,f(n)] = -z\frac{d}{dz}F(z)$

1. Find $Z(n^2)$

W.K.T $Z[n\,f(n)] = -z\frac{d}{dz}F(z)$ (method 2)

$$Z[n^2] = Z[n.n] = -z\frac{d}{dz}[Z(n)]$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4}\right]$$

$$\Rightarrow -z\left[\frac{z-1-2z}{(z-1)^3}\right] = -z\left[\frac{1-z}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$$

2. Find $Z(n^3)$

WKT $Z[n\,f(n)] = -z\frac{d}{dz}F(z)$

$$Z[n^3] = Z[n.n^2] = -z\frac{d}{dz}Z[n^2]$$

$$\Rightarrow -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$\Rightarrow -z\left[\frac{(z-1)^3(2z+1)-(z^2+z)3(z-1)^2}{(z-1)^6}\right]$$

$$\Rightarrow -z\left[\frac{(z-1)(2z+1)-3(z^2+z)}{(z-1)^4}\right]$$

$$\Rightarrow -z\left[\frac{2z^2-2z+z-1-3z^2-3z}{(z-1)^4}\right]$$

$$\Rightarrow -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right] = \frac{z(z^2+4z+1)}{(z-1)^4}$$

3. Find $z[n(n-1)(n-2)]$

Soln

$$n(n-1)(n-2) = n[n^2-2n-n+2]$$

$$= n[n^2-3n+2]$$

$$\Rightarrow n^3-3n^2+2n$$

$$z[n(n-1)(n-2)] = z[n^3-3n^2+2n]$$

$$\Rightarrow z[n^3]-3z[n^2]+2z[n]$$

$$= \frac{z[z^2+4z+1]}{(z-1)^4} - 3\frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2}$$

$$\Rightarrow \frac{z^3+4z^2+z-3z(z+1)(z-1)+2z(z-1)^2}{(z-1)^4}$$

$$\Rightarrow \frac{z^3+4z^2+z-3z(z^2-1)+2z(z^2-2z+1)}{(z-1)^4}$$

$$= \frac{z^3+4z^2+z-3z^3+3z+2z^3-4z^2+2z}{(z-1)^4}$$

$$= \frac{6z}{(z-1)^4} //.$$

Find $z[r^n \cos n\theta]$ and $z[r^n \sin n\theta]$

Soln let $a = re^{i\theta}$

$$a^n = (re^{i\theta})^n = r^n . e^{in\theta}$$

$$\Rightarrow r^n[\cos n\theta + i\sin n\theta]$$

$$= r^n \cos n\theta + ir^n \sin n\theta$$

WKT $z[a^n] = \frac{z}{z-a}$

$$z[(re^{i\theta})^n] = \frac{z}{z-re^{i\theta}}$$

$$z[r^n e^{in\theta}] = \frac{z}{z-r(\cos\theta + i\sin\theta)}$$

$$z[r^n \cos n\theta + ir^n \sin n\theta] = \frac{z}{z-r\cos\theta - ir\sin\theta}$$

$$\Rightarrow \frac{z[(z-r\cos\theta)+ir\sin\theta]}{[(z-r\cos\theta-ir\sin\theta][(z-r\cos\theta)+ir\sin\theta]}$$

$$= \frac{z[z-r\cos\theta]+irz\sin\theta}{(z-r\cos\theta)^2+r^2\sin^2\theta}$$

$$= \frac{z(z-r\cos\theta)+izr\sin\theta}{z^2-2zr\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$= \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} + i\,\frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Equating real and imaginary parts we get

$$Z[r^n \cos\theta] = \frac{z[z - r\cos\theta]}{z^2 - 2zr\cos\theta + r^2}$$

$$Z[r^n \sin\theta] = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Problems based on Inverse Z-Transform

i) $Z^{-1}\left[\log\left(\frac{z}{z+1}\right)\right]$

X: Let $F(z) = \log\left(\frac{z}{z+1}\right)$

put $z = 1/y$

$$\Rightarrow \log\left[\frac{1/y}{1/y + 1}\right]$$

$$= \log\left[\frac{1/y}{\frac{1+y}{y}}\right] = \log\left[\frac{1}{1+y}\right]$$

$$= \log(1+y)^{-1}(1) = -\log(1+y)$$

$$= -y + 1/2\, y^2 - 1/3\, y^3 + \cdots$$

$$= -1/z + 1/2z^2 - \frac{1}{3z^3} + \cdots + \frac{(-1)^n}{n} z^{-n}$$

$$f(n) = z^{-1}[f(z)] = \begin{cases} 0 & \text{for } n = 0 \\ \frac{(-1)^n}{n} & \text{otherwise} \end{cases}$$

Method of Residues

To find inverse z-transform using residue theorem.

If $Z[f(n)] = F(z)$, then

$f(n)$ which gives the inverse z-transform of $F(z)$ is obtained from the following result

$$f(n) = \frac{1}{2\pi i} \int_C z^{n-1} F(z)\, dz \quad \text{—} (1)$$

where C is the closed contour which encloses all the poles of the integrand

By residue theorem,

$$\int_C z^{n-1} F(z)\, dz = 2\pi i \left[\begin{array}{c} \text{sum of the residues of} \\ z^{n-1} F(z) \text{ as its poles} \end{array} \right] \quad \text{—} (2)$$

sub (2) in (1)

$$f(n) = \left\{ \begin{array}{l} \text{Sum of the residues of } z^{n-1} F(z) \\ \text{as its poles.} \end{array} \right.$$

Find $z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

X. <u>soln</u>

Let $\frac{z}{(z-1)(z-2)} = F(z)$, $\quad f(n) = z^{-1}[F(z)]$

$$\therefore \quad z^{n-1} F(z) = \frac{z^{n-1} z}{(z-1)(z-2)}$$

$$= \frac{z^n}{(z-1)(z-2)}$$

The poles are $z=1$, $z=2$ (simple poles).

$$\text{Res}\left[z^{n-1} F(z)\right]_{z=1} = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{z^n}{\cancel{(z-1)}(z-2)}$$

$$= (-1)^n$$

$$\text{Res}\left[z^{n-1} F(z)\right]_{z=2} = \lim_{z \to 2} (z-\cancel{2}) \cdot \frac{z^n}{(z-1)(z-\cancel{2})}$$

$$= 2^n$$

$$f(n) = \begin{cases} \text{Sum of the residues of } z^{n-1} F(z) \\ \text{as its poles} \end{cases}$$

$$= 2^n - (-1)^n \; ; \; n \geq 0$$

$$z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] = 2^n - (-1)^n \quad n \geq 0$$

$$\left[\text{Res}_{z=z_0} F(z) = \lim_{z \to z_0} (z - z_0) \, F(z)\right]$$

Find the inverse z-transform of using inversion integral $\frac{z+3}{(z+1)(z-2)}$

$$F(z) = \frac{z+3}{(z+1)(z+2)} \quad \text{where } F(z) = Z[f(n)]$$

$$F(z) \cdot z^{n-1} = \frac{z^{n-1}(z+3)}{(z+1)(z+2)}$$

The poles are given by $(z+1)(z-2) = 0$

$z = -1, 2$ which are simple poles

$$R(-1) = \lim_{z \to 1} \frac{(z+1) z^{n-1} (z+3)}{(z+1)(z-2)}$$

$$= \lim_{z \to 1} \frac{z^{n-1}(z+3)}{(z-2)} = \frac{(-1)^n(-1+3)}{-1-2} = -\tfrac{2}{3}(-1)^n$$

$$R(2) = \lim_{z \to 2} \frac{(z-2) z^{n-1} (z+3)}{(z+1)(z-2)}$$

$$\lim_{z \to 2} \frac{z^{n-1}(z+3)}{z+1} = \frac{2^{n-1}(2+3)}{2+1} = \frac{5}{3} 2^{n-1} = \frac{5}{6} 2^n$$

By inversion method

$$f(n) = \text{Sum of residues}$$

$$= R(-1) + R(+2) = \frac{2}{3}(-1)^n + \frac{5}{6} 2^n$$

$n = 0, 1, 2 \ldots$

3. Find $z^{-1}\left[\frac{2z^2+4z}{(z-2)^3}\right]$ $\left[\because \text{order } m' \quad \operatorname*{Res}_{z=z_0} F(z) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m F(z) z^{n-1}\right]\right]$

$$F(z) = \frac{2z^2+4z}{(z-2)^3} \quad \text{where } F(z) = Z[f(n)]$$

$$F(z) \cdot z^{n-1} = z^{n-1} \frac{(2z^2+4z)}{(z-2)^3}$$

$\therefore$ the poles are given by $(z-2)^3 = 0 \Rightarrow z = 2$

which is a pole of order 3.

$$R(2) = \lim_{z \to 2} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left[F(z) \cdot z^{n-1} (z-2)^3\right]$$

$$= \lim_{z \to 2} \frac{1}{2} \frac{d^2}{dz^2}\left[\frac{z^{n-1}(2z^2+4z)(z-2)^3}{(z-2)^3}\right]$$

$$\Rightarrow \frac{1}{2} \lim_{z \to 2} \frac{d^2}{dz^2}\left[2z^{n+1} + 4z^n\right]$$

$$= \frac{1}{2} \lim_{z \to 2} \frac{d}{dz}\left[2(n+1) z^n + 4 n z^{n-1}\right]$$

$$= \frac{1}{2} \lim_{z \to 2} \left[2(n+1)\, n z^{n-1} + 4n(n-1) z^{n-2} \right]$$

$$= \frac{1}{2} \left[2(n+1)(2^{n-1})(n) + 4n(n-1)\, 2^{n-2} \right]$$

$$= \frac{n}{2} \left[(n+1)2^n + (n-1)2^n \right]$$

$$= \frac{n}{2} 2^n \left[n + \cancel{1} + n - \cancel{1} \right] = \frac{n 2^n \cdot \cancel{2} n}{\cancel{2}} = n^2 \cdot 2^n$$

By inversion integral method or Residue method

$f(n) =$ sum of residues

$= n^2 \cdot 2^n \;; \quad n = 0, 1, 2, 3 \ldots$

Evaluate $z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$ using partial fraction.

Let $X(z) = \frac{z^3}{(z-1)^2(z-2)}$

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$z^2 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

Put $z=1$ we get

$1 = 0 + B(-1)$

$\boxed{B = -1}$

Put $z=2$ we get

$4 = 0 + 0 + C$

$\boxed{C = 4}$

Put $z=0$ we get

$0 = A(-1)(-2) + B(-2) + C(-1)^2$

$0 = 2A - 2B + C$

$0 = 2A + 2 + 4 \Rightarrow A + 1 + 2$

$\boxed{A = -3}$

$$\therefore \frac{X(z)}{z} = \frac{-3}{z-1} - \frac{1}{(z-1)^2} + \frac{4}{(z-2)}$$

$$\therefore X(z) = \frac{-3z}{z-1} - \frac{z}{(z-1)^2} + \frac{4z}{z-2}$$

$$Z[x(n)] = X(z) = -3\frac{z}{z-1} - \frac{z}{(z-1)^2} + 4\frac{z}{z-2}$$

$$x(n) = -3\, z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -3(1)^n - n + 4(2)^n \,//.$$

X. Find $z^{-1}\left[\dfrac{z^2+2z}{z^2+2z+4}\right]$

$$F(z) = \frac{z^2+2z}{z^2+2z+4}$$

$$\frac{F(z)}{z} = \frac{z+2}{z^2+2z+4}$$

$$\frac{F(z)}{z} = \frac{z+2}{[z+(1-i\sqrt{3})][z+(1+i\sqrt{3})]} \quad -(1)$$

$$\frac{z+2}{[z+(1-i\sqrt{3})][z+(1+i\sqrt{3})]} = \frac{A}{z+1-i\sqrt{3}} + \frac{B}{z+1+i\sqrt{3}} \quad -(2)$$

$$z+2 = A(z+1+i\sqrt{3}) + B(z+1-i\sqrt{3})$$

Put $z = -1 - i\sqrt{3}$

$-1 - i\sqrt{3} + 2 = B(-2i\sqrt{3})$

$1 - i\sqrt{3} = -2i\sqrt{3}\,B$

$B = \dfrac{-(1-i\sqrt{3})}{2i\sqrt{3}} \times \dfrac{i}{i}$

$B = \dfrac{1}{2\sqrt{3}}(\sqrt{3}+i)$

Equating coefficient of z

$A + B = 1$

$A = 1 - \dfrac{1}{2\sqrt{3}}(\sqrt{3}+i)$

$= \dfrac{2\sqrt{3} - \sqrt{3} - i}{2\sqrt{3}}$

$A = \dfrac{1}{2\sqrt{3}}(\sqrt{3}-i)$

Sub A & B in ②

$$\frac{F(z)}{z} = \frac{1}{2\sqrt{3}}(\sqrt{3}-i) \cdot \frac{1}{(z+1) - i\sqrt{3}} +$$

$$\frac{1}{2\sqrt{3}}(\sqrt{3}+i) \frac{1}{(z+1) + i\sqrt{3}}$$

$$F(z) = c_1 \frac{z}{z+1-i\sqrt{3}} + c_2 \frac{z}{z+1+i\sqrt{3}}$$

$c_1 = \dfrac{1}{2\sqrt{3}}(\sqrt{3}-i)$ & $c_2 = \dfrac{1}{2\sqrt{3}}(\sqrt{3}+i)$

$$\therefore z^{-1}[F(z)] = c_1 z^{-1}\left[\frac{z}{z+1-i\sqrt{3}}\right] + c_2 z^{-1}\left[\frac{z}{z+1+i\sqrt{3}}\right]$$

$$= c_1[-1+i\sqrt{3}]^n + c_2[-1-i\sqrt{3}]^n$$

$$\Rightarrow \frac{1}{2\sqrt{3}}[\sqrt{3}-i]\left\{2^n\left(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right)\right\}$$

$$+ \frac{1}{2\sqrt{3}}[\sqrt{3}+i]\left\{2^n\left(\cos\frac{2n\pi}{3} - i\sin\frac{2n\pi}{3}\right)\right\}$$

$$\left[\because -1 \pm i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}\right)\right]$$

$$Z^{-1}\left[\frac{z^2+2z}{z^2+2z+4}\right] = 2^n\left\{\cos\frac{2n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{2n\pi}{3}\right\}$$

method of residues

X. Find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

$$F(z) = \frac{z^2}{(z-a)(z-b)}$$

$$\therefore z^{n-1}F(z) = \frac{z^{n-1}\cdot z^2}{(z-a)(z-b)}$$

$$= \frac{z^{n+1}}{(z-a)(z-b)}$$

The poles are $z=a$ & $z=b$ [simple pole]

$$\text{Res}\left[z^{n-1}F(z)\right]_{z\to a} = \lim_{z\to a}(z-a)\frac{z^{n+1}}{(z-a)(z-b)}$$

$$= \frac{a^{n+1}}{a-b}$$

$$\text{Res}\left[z^{n-1}F(z)\right]_{z=b} \Rightarrow \lim_{z\to b}[\cancel{z-b}]\frac{z^{n+1}}{(z-a)(\cancel{z-b})}$$

$$= \frac{b^{n+1}}{b-a}$$

$f(n) = \left\{\begin{array}{l}\text{sum of the residues of } z^{n-1}f(z) \\ \text{as its poles.}\end{array}\right.$

$$= \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right] \quad n \geq 0$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right] ; n \geq 0$$

Ex. Find $z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$ by using convolution theorem.

$$F(z) = \left[\frac{8z^2}{(2z-1)(4z+1)}\right] = z^{-1}\left[\frac{8z^2}{8\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right]$$

$$= z^{-1}\left[\frac{z^2}{(z-1/2)(z+1/4)}\right] = z^{-1}\left[\frac{z}{z-1/2} \cdot \frac{z}{z-1/4}\right]$$

$$= z^{-1}\left[\frac{z}{z-1/2}\right] * z^{-1}\left[\frac{z}{z+1/4}\right]$$

$$= (1/2)^n * \left(\frac{-1}{4}\right)^n$$

$$= \sum_{r=0}^{\infty} (1/2)^{n-r} \cdot (-1/4)^r$$

$$= \sum_{r=0}^{\infty} (1/2)^n \cdot (1/2)^{-r} (-1/4)^r$$

$$= (1/2)^n \sum_{r=0}^{n} (1/2)^{-r} (-1/4)^r$$

$$= (1/2)^n \sum_{r=0}^{n} 2^r (-1/4)^r$$

$$= (1/2)^n \sum_{r=0}^{n} \left(2 \times \frac{-1}{4}\right)^r$$

$$\Rightarrow \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{-1}{2}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \left[1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)^2 + \cdots + \left(\frac{-1}{2}\right)^n\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1 - (-1/2)^{n+1}}{1 - \left(\frac{-1}{2}\right)}\right]$$

$$\Rightarrow (1/2)^n \left\{\frac{1 - (-1/2)^{n+1}}{1 + 1/2}\right\}$$

$$= \left(\frac{1}{2}\right)^n \left\{\frac{1 - \left(\frac{-1}{2}\right)^{n+1}}{3/2}\right\}$$

$$= \frac{2}{3} \{1/2\}^n \left\{1 - \left(\frac{-1}{2}\right)^n (-1/2)\right\}$$

$$= \frac{2}{3} \{1/2\}^n \left\{1 + \frac{1}{2}(-1/2)^n\right\}$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n \left(\frac{-1}{2}\right)^n$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{2} \times \frac{-1}{2}\right)^n$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{-1}{4}\right)^n$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}(-1/4)^n \,//.$$

X: Find $z^{-1}\left[\frac{3z^2-18z+26}{(z-1)(z-3)(z-4)}\right]$

Given $F(z) = \left[\frac{3z^2-18z+26}{(z-1)(z-3)(z-4)}\right]$

Note that we do not have z in the numerator to rewrite $\frac{F(z)}{z}$. If we divide by z, then one more factor will be introduced in the denominator which will introduce one more term. Instead we put into partial fraction as it is.

Let $\frac{3z^2-18z+26}{(z-1)(z-3)(z-4)} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z-4}$

$3z^2-18z+26 = A(z-3)(z-4) + B(z-1)(z-4) + C(z-1)(z-3)$

put $z=1$

$3(1) - 18(1) + 26 = A(1-3)(1-4) + B(0) + C(0)$

$3-18+26 = A(-2)(-3)$

$29-18 = 6A$

$11 = 6A$

$\boxed{A = 11/6}$

put $z = 3$

$3(3)^2 - 18(3) + 26 = A(0) + B(3-1)(3-4) + C(0)$

$27 - 54 + 26 = B(2)(-1)$

$53-54 = -2B$

$\cancel{-}1 = \cancel{-}2B$

$\boxed{B = 1/2}$

put $z=4$

$$3(4)^2-18(4)+26 = A(0)+B(0)+C(4-2)(4-3)$$

$$48-72+26 = C(2)(1)$$

$$74-72 = C(2)$$

$$\boxed{C=1}$$

$$F(z) = \frac{11}{6(z-1)} + \frac{1}{2(z-3)} + \frac{1}{(z-4)}$$

$$z[f(n)] = \frac{11}{6(z-1)} + \frac{1}{2(z-3)} + \frac{1}{(z-4)}$$

$$f(n) = z^{-1}\left[\frac{1}{z-1}\right]\frac{11}{6} + \frac{1}{2}z^{-1}\left[\frac{1}{z-3}\right] + z^{-1}\left[\frac{1}{z-4}\right]$$

$$= \frac{11}{6}1^{n-1} + \frac{1}{2}3^{n-1} + 4^{n-1}$$

Ex. Find $z^{-1}\left[\frac{2z+z^2}{z^2+2z+4}\right]$ by the long division method.

Let $X(z) = \dfrac{z^2+2z}{(z^2+2z+4)}$

$$\Rightarrow \frac{z^2\left[1+\frac{2}{z}\right]}{z^2\left[1+\frac{2}{z}+\frac{4}{z^2}\right]} = \frac{1+2z^{-1}}{1+2z^{-1}+4z^{-2}}$$

$$
\begin{array}{r|l}
 & 1-4z^{-2}+8z^{-3}-32z^{-5}+\dots \\
\hline
1+2z^{-1}+4z^{-2} & 1+2z^{-1} \\
 & 1+2z^{-1}+4z^{-2} \\
 & (-)\quad(-)\quad(-) \\
\hline
 & \cancel{-4z^{-2}} \\
 & \cancel{-4z^{-2}}-8z^{-3}-16z^{-4} \\
 & (+)\quad(+)\quad(+) \\
\hline
 & \cancel{+8z^{-3}}\ \cancel{+16z^{-4}} \\
 & \cancel{8z^{-3}}+\cancel{16z^{-4}}+32z^{-5} \\
 & (-)\quad(-)\quad(-) \\
\hline
 & -32z^{-5}
\end{array}
$$

$$\therefore X(z) = \frac{z^2+2z}{z^2+2z+4} = 1-4z^{-2}+8z^{-3}-32z^{-5}+\dots$$

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = 1-4z^{-2}+8z^{-3}+(-32)z^{-5}+\dots$$

$$x(0)z^{-0}+x(1)z^{-1}+x(2)z^{-2}+\dots = 1-4z^{-2}+8z^{-3}-32z^{-5}+\dots$$

$\therefore x(0)=1$

$x(1)=0$

$x(2)=-4$

$x(3)=8$

$x(4)=0$

$x(5)=-32$

The sequence is $1, 0, -4, 8, 0, -32, \dots$

method: 1

X. Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$

Given

$$y_{n+2} + 6y_{n-1} + 9y_n = 2^n$$

$$Z[y_{n+2}] + 6Z[y_{n-1}] + 9Z[y_n] = Z[2^n]$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] + 6[zy(z) - zy(0)] + 9Y(z)$$

$$= \frac{z}{z-2}$$

$$z^2 Y(z) + 6zY(z) + 9Y(z) = \frac{z}{z-2}$$

$$[z^2 + 6z + 9] Y(z) = \frac{z}{z-2}$$

$$(z+3)^2 Y(z) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$Y(z) z^{n-1} = \frac{z^n}{(z-2)(z+3)^2}$$

$Y(z) z^{n-1}$ has poles at $z = 2$ & $z = -3$

$z = 2$ is a simple pole

$z = -3$ is a pole of order 2

$$\operatorname*{Res}_{z=-3} Y(z) z^{n-1} = \lim_{z \to -3} \frac{1}{1!} \frac{d}{dz} \left[\frac{(z+3)^2 z^n}{(z-2) + (z+3)^2} \right]$$

$$= \lim_{z \to -3} \frac{d}{dz}\left[\frac{z^n}{z-2}\right]$$

$$= \lim_{z \to -3}\left[\frac{(z-2)nz^{n-1} - z^n}{(z-2)^n}\right]$$

$$= \frac{(-5)n(-3)^{n-1} - (-3)^n}{(-5)^2} \Rightarrow \frac{(-3)^n\left[-5n(-3)^{-1} - 1\right]}{25}$$

$$= \frac{(-3)^n\left[\frac{5}{3}n - 1\right]}{25} = \frac{(-3)^n\left[5n-3\right]}{75}$$

$$\operatorname*{Res}_{z=2} Y(z)z^{n-1} = \lim_{z \to 2} (z-2)\frac{z^n}{(z-2)(z+3)^2}$$

$$= \lim_{z \to 2} \frac{z^n}{(z+3)^2}$$

$$= \frac{2^n}{5^2} = \frac{2^n}{25}$$

$y(n)$ = Sum of the residues

$$= \frac{1}{75}\left[(-3)^n(5n-3)\right] + \frac{2^n}{25}$$

$$= \frac{1}{25}\left[2^n - (-3)^n + \frac{5}{3}n(-3)^n\right]$$

Two sided or bilateral z-transform

Let $\{f(n)\}$ be a sequence defined for $n = 0, \pm 1, \pm 2, \ldots$

then z-transform is defined as

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n} \qquad z \to \text{a complex form}$$

$$= F(Z)$$

one sided z-transform

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} \qquad z \to \text{a complex form.}$$

$$= F(Z)$$

Find $Z(1/n)$ $|z| > 1$, $n > 0$

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z(1/n) = \sum_{n=1}^{\infty} 1/n \, z^{-n} \qquad \because n > 0$$

$$= z^{-1} + \frac{1}{2} z^{-2} + 1/3 \, z^{-3} + \ldots$$

$$= 1/z + 1/2 (1/z^2) + \frac{1}{3} (1/z^3) + \ldots$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \ldots$$

$$= -\log\left[1 - 1/z\right]$$

$$= -\log\left[\frac{z-1}{z}\right]$$

$$= \log\left[\frac{z-1}{z}\right]^{-1} \text{ (or) } \log\left(\frac{z}{z-1}\right)$$

method: 2

Solve the difference equation using z-transform techique $y_{n+2} + 6y_{n+1} + 9y_n = 2^n;\ y_0 = y_1 = 0$

Taking z-transform we get

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$$

$$[z^2 Y(z) - z^2 y(0) - zy(1)] + 6[zY(z) - zy(0)] + 9Y(z) = \frac{z}{z-2}$$

Given $y_0 = y_1 = 0$

$$Y(z)[z^2 + 6z + 9] = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} \quad \text{---} \ (1)$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z+3)(z-2) + C(z-2) \quad \text{---} \ (2)$$

put $z = 2$	put $z = -3$	Equating coeff of z^2
$25A = 1$	$-5C = 1$	$A + B = 0$
$A = 1/25$	$C = -1/5$	$B = -A$
		$B = -1/25$

$$\frac{Y(z)}{z} = \frac{\frac{1}{25}}{(z-2)} - \frac{1/25}{(z+3)} - \frac{1/5}{(z+3)^2}$$

$$Y(z) = \frac{1}{25}\left[\frac{z}{z-2}\right] - \frac{1}{25}\left[\frac{z}{z+3}\right] - \frac{1}{5}\left[\frac{z}{(z+3)^2}\right]$$

$$z^{-1}[Y(z)] = \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$y(n) = 1/25\, 2^n - \frac{1}{25}(-3)^n - 1/5\, n(-3)^{n-1}$$

$$= 1/25\, 2^n - 1/25\,(-3)^n + \frac{1}{15} n(-3)^n \qquad n \geq 0.$$

Solve the difference equation using z-transform

$y_{n+2} - 7y_{n+1} + 12y_n = 2^n$ given $y_0 = y_1 = 0$

$Z[y_{n+2}] - 7Z[y_{n+1}] + 12Z[y_n] = Z[2^n]$

$[z^2 Y(z) - z^2 y(0) - z y(1)] - 7[zY(z) - zy(0)] + 12Y(z)$

$= \frac{z}{z-2}$

Given $y_0 = y_1 = 0$

$z^2 Y(z) - 7zY(z) + 12Y(z) = \frac{z}{z-2}$

$Y(z)[z^2 - 7z + 12] = \frac{z}{z-2}$

$Y(z) = \frac{z}{(z-2)(z^2-7z+12)}$

$\frac{Y(z)}{z} = \frac{1}{(z-2)(z-3)(z-4)}$

$\frac{1}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$

$1 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$

put $z = 2$

$\therefore 1 = A(2-3)(2-4) \Rightarrow 2A = 1 \Rightarrow \boxed{A = 1/2}$

put $z = 3$

$\therefore 1 = B(3-2)(3-4) \Rightarrow -B = 1 \Rightarrow \boxed{B = -1}$

put $z = 4$

$\therefore 1 = C(4-2)(4-3) \Rightarrow 2C = 1 \Rightarrow \boxed{C = 1/2}$

$\frac{F(z)}{z} = \frac{1}{2(z-2)} - \frac{1}{z-3} + \frac{1}{2(z-4)}$

$F(z) = \frac{1}{2}\left[\frac{z}{z-2}\right] - \left[\frac{z}{z-3}\right] + \frac{1}{2}\left[\frac{z}{z-4}\right]$

Taking inverse z-transform

$$y(n) = Z[Y(z)] = \frac{1}{2} z^{-1}\left[\frac{z}{z-2}\right] - z^{-1}\left[\frac{z}{z-3}\right] + \frac{1}{2} z^{-1}\left[\frac{z}{z-4}\right]$$

$$y(n) = \frac{1}{2} 2^n . -3^n + \frac{1}{2} 4^n ; \quad n \geq 0.$$

method: 2

1) $y_{n+2} - 7y_{n+1} + 12y_n = 2^n$ given $y_0 = y_1 = 0$

$$Z[y_{n+2}] - 7Z[y_{n+1}] + 12 Z[y_n] = Z[2^n]$$

$$[z^2 Y(z) - z^2 y(0) - z y(1)] - 7[zY(z) - z y(0)] + 12 Y(z) = \frac{z}{z-2}$$

Given $y_0 = y_1 = 0$

$$z^2 Y(z) - 7zY(z) + 12Y(z) = \frac{z}{z-2}$$

$$Y(z)[z^2 - 7z + 12] = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z^2-7z+12)} = \frac{z}{(z-2)(z-3)(z-4)}$$

$$Y(z) z^{n-1} = \frac{z^{n-1}. z}{(z-2)(z-3)(z-4)}$$

$$Y(z) z^{n-1} = \frac{z^n}{(z-2)(z-3)(z-4)}$$

$z=2$ is a simple pole

$z=3$ is a simple pole

$z=4$ is a simple pole.

$$\left[\text{Res } Y(z)z^{n-1}\right]_{z=2} = \lim_{z\to 2} (\cancel{z-2}) \frac{z^n}{(\cancel{z-2})(z-3)(z-4)}$$

$$\Rightarrow \frac{2^n}{(2-3)(2-4)} = \frac{2^n}{(-1)(-2)} = \frac{2^n}{2}$$

$$\left[\text{Res } Y(z)z^{n-1}\right]_{z=3} = \lim_{z\to 3} (\cancel{z-3}) \frac{z^n}{(z-2)(\cancel{z-3})(z-4)}$$

$$\Rightarrow \frac{3^n}{(3-2)(3-4)} = \frac{3^n}{(1)(-1)} \Rightarrow -(3)^n$$

$$\left[\text{Res } Y(z)z^{n-1}\right]_{z=4} = \lim_{z\to 4} (\cancel{z-4}) \frac{z^n}{(z-2)(z-3)(\cancel{z-4})}$$

$$= \frac{4^n}{(4-2)(4-3)} = \frac{4^n}{(2)(1)} = \frac{4^n}{2}$$

$f(n) =$ ie $y(n) =$ sum of the residues

$= R(2) + R(3) + R(4)$

$= \frac{2^n}{2} - (3)^n + \frac{4^n}{2}$ //.

using power series technique (long division)

find the z-transform of $F(z) = \dfrac{z}{2z^2-3z+1}$ $|z|>1$

$$\begin{array}{r|l} & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots \\ 2z^2-3z+1 & z \\ & z - 3/2 + 1/2\, z^{-1} \\ & -\ \ (+)\ \ (-) \\ \hline & \frac{3}{2} - \frac{1}{2}z^{-1} \\ & \frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2} \\ & (-)\ \ (+)\ \ (-) \\ \hline & \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \cdots \end{array}$$

$$\therefore F(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots \quad -(A)$$

using definition

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \cdots \quad (B)$$

Comparing A & B we get

$f(0) = 0$; $f(1) = 1/2$, $f(2) = 3/4$, $f(3) = 7/8$ ....

$\therefore f(n) = \{0, 1/2, 3/4, 7/8, \ldots\}$

$z^{-1}\left[\dfrac{z^2}{(z+2)(z^2+4)}\right]$ using method of residues.

$$F(z) = \frac{z^2}{(z+2)(z^2+4)} = \frac{z^2}{(z+2)(z+2i)(z-2i)}$$

$$F(z)z^{n-1} = \frac{z^2 \cdot z^{n-1}}{(z+2)(z+2i)(z-2i)}$$

$z=-2$ is the simple pole

$z=-2i$ is the simple pole

$z=2i$ is the simple pole

$$\operatorname*{Res}_{z=-2} F(z)z^{n-1} = \lim_{z\to -2} \cancel{(z+2)} \frac{z^2 \cdot z^{n-1}}{\cancel{(z+2)}(z^2+4)}$$

$$\Rightarrow \frac{(-2)^2(-2)^{n+1}}{(2^2+4)} \Rightarrow \frac{1}{2}(-2)^{n-1}$$

$$\left[\operatorname*{Res}_{z=-2i} F(z)z^{n-1}\right] \lim_{z\to -2i} (z+2i) \frac{z^2 \cdot z^{n-1}}{(z+2)(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i} \frac{z^2}{(z+2)(z-2i)} z^{n-1}$$

$$= \frac{(-2i)^2}{(-2i+2)(-4i)} (-2i)^{n-1}$$

$$= \frac{-4}{2(1-i)(-4i)} (-2i)^{n-1} \Rightarrow \frac{1}{2i(1-i} (-2i)^{n-1}$$

$$\left[\operatorname*{Res}_{z=2i} F(z)z^{n-1}\right] \Rightarrow \lim_{z\to 2i} (z-2i) \frac{z^2}{(z+2)(z+2i)(z-2i)} z^{n-1}$$

$$= \lim_{z\to 2i} \frac{z^2}{(z+2)(z+2i)} z^{n-1} \Rightarrow \frac{(2i)^2}{(2i+2)(4i)} (2i)^{n-1}$$

$$= \frac{-4}{2(1+i)(4i)} (2i)^{n-1} \Rightarrow \frac{-1}{2i(1+i)} (2i)^{n-1}$$

$\therefore$ $f(n) =$ sum of residues

$$= \frac{1}{2}(-2)^{n-1} + \frac{1}{2i(1-i)}(-2i)^{n-1} + \frac{(-1)}{2i(1+i)}(2i)^{n-1}$$