Second order p.d.e in the function u of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Here, $A = 1,\ B = 2,\ C = 4$

$$B^2 - 4AC = (2)^2 - 4(1)(4)$$
$$= 4 - 16 = -12 < 0$$

$\therefore$ The given equation is an elliptic equation.

23. Classify the partial differential equation
$u_{xx} + u_{xy} + u_{yy} = 0$ **[A.U A/M 2019 R-13]**

Solution :

Given : $u_{xx} + u_{xy} + u_{yy} = 0$

Second order p.d.e in the function u of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Here, $A = 1,\ B = 1,\ C = 1$

$$B^2 - 4AC = (1)^2 - 4(1)(1)$$
$$= 1 - 4 = -3 < 0$$

$\therefore$ The given equation is an elliptic equation.

Unit-IV

FOURIER TRANSFORMS

Statement of Fourier integral theorem - Fourier transform pair - Fourier sine and cosine transforms - Properties - Transforms of simple functions - Convolution theorem - Parseval's identity.

4.0 INTRODUCTION

Integral transforms are used in the solution of partial differential equations. The choice of a particular transform to be used for the solution of a differential equation depends upon the nature of the boundary conditions of the equation and the facility with which the transform F(s) can be converted to give $f(x)$.

4.1 STATEMENT OF FOURIER INTEGRAL THEOREM

4.1a Fourier integral theorem (without proof)

[A.U N/D 2018-A R-17]

If $f(x)$ is piece-wise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(t-x)}\, dt\, ds \qquad \dots (1)$$

Equation (1) can be re-written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left[\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right] ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right] ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\, F(s)\, ds, \text{ where}$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

i.e., $F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$ [$\because$ t is a dummy variable]

or equivalently,

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda(t-x)\, dt\, d\lambda$$

This is known as Fourier integral theorem or Fourier integral formula.

(OR)

Let us assume the following conditions on a function $f(x)$

1. $f(x)$ is piece-wise continuous in any finite interval (a, b)
2. $\int_{-\infty}^{\infty} |f(x)|\, dx$ is convergent.

Then the Fourier integral theorem states that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda(t-x)\, dt\, d\lambda$$

The double integral in the right hand side is known as a Fourier integral expansion of $f(x)$

(OR)

If $f(x)$ is a function defined in $(-l, l)$ satisfying Dirichlet conditions, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda(t-x)\, dt\, d\lambda$$

The double integral in the right hand side is known as Fourier integral to represent $f(x)$

Note : We assume the following conditions on $f(x)$

(i) $f(x)$ is defined as single-valued except at finite points in $(-l, l)$

(ii) $f(x)$ is periodic outside $(-l, l)$ with period $2l$.

(iii) $f(x)$ and $f'(x)$ are sectionally continuous in $(-l, l)$

(iv) $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges

i.e., $f(x)$ is absolutely integrable in $(-\infty, \infty)$

I.(a) Problems based on Fourier integral theorem :

Example 4.1.a(1) : Show that $f(x) = 1$, $0 < x < \infty$ cannot be represented by a Fourier integral.

Solution : $\int_{0}^{\infty} |f(x)|\, dx = \int_{0}^{\infty} 1\, dx = \left[x\right]_0^{\infty} = \infty - 0 = \infty$

i.e., $\int_{0}^{\infty} |f(x)|\, dx$ is not convergent.

Hence, $f(x) = 1$ cannot be represented by a Fourier integral.

Example 4.1.a(2) : If $f(x) = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$,

show that $f(x) = \int_{0}^{\infty} \frac{\cos\lambda x \sin\lambda}{\lambda}\, d\lambda$

Hence show that $\int_{0}^{\infty} \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$

Solution : We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{is(t-x)}\, dt\, ds$$

$$\boxed{\begin{aligned} \text{i.e., } f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds, \text{ where} \quad \ldots (1) \\ F(s) = F[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \quad \ldots (2) \end{aligned}}$$

Given : $f(x) = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$ i.e., $f(x) = \begin{cases} \frac{\pi}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$(2) \Rightarrow F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{\pi}{2} e^{isx}\, dx$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left(\frac{1}{2}\right) \int_{-1}^{1} e^{isx}\, dx$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left(\frac{1}{2}\right) \int_{-1}^{1} [\cos sx + i \sin sx]\, dx$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left(\frac{1}{2}\right) \left[\int_{-1}^{1} \cos sx\, dx + i \int_{-1}^{1} \sin sx\, dx\right]$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left(\frac{1}{2}\right) \left[2 \int_{0}^{1} \cos sx\, dx + i\,(0)\right]$$

$\because$ $\cos sx$ is an even function in $(-1, 1)$
$\sin sx$ is an odd function in $(-1, 1)$

$$= \left(\sqrt{\frac{\pi}{2}}\right) \left[\frac{\sin sx}{s}\right]_{x=0}^{x=1}$$

$$= \left(\sqrt{\frac{\pi}{2}}\right) \left[\frac{\sin s}{s} - 0\right]$$

$$= \sqrt{\frac{\pi}{2}}\, \frac{\sin s}{s} \quad \text{... (3)}$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}}\, \frac{\sin s}{s}\, e^{-isx}\, ds$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin s}{s} [\cos sx - i \sin sx]\, ds$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin s}{s} \cos sx\, ds - \frac{i}{2} \int_{-\infty}^{\infty} \frac{\sin s}{s} \sin sx\, ds$$

$$= \left(\frac{1}{2}\right)\left[(2) \int_{0}^{\infty} \frac{\sin s}{s} \cos sx\, ds\right] - \frac{i}{2}[0]$$

$\because$ $\frac{\sin s}{s} \cos sx$ is an even function in $(-\infty, \infty)$
$\frac{\sin s}{s} \sin sx$ is an odd function in $(-\infty, \infty)$

$$= \int_{0}^{\infty} \frac{\sin s}{s} \cos sx\, ds$$

$$= \int_{0}^{\infty} \frac{\sin \lambda}{\lambda} \cos \lambda x\, d\lambda \qquad [\because s \text{ is a dummy variable}]$$

$$\text{i.e., } f(x) = \int_{0}^{\infty} \frac{\cos \lambda x \sin \lambda}{\lambda}\, d\lambda \quad \text{... (4)}$$

Put $x = 0$, (we cannot assign any value to the dummy variable 'λ')

$$\left.\begin{matrix} f(x) \\ \text{at } x = 0 \end{matrix}\right\} = \frac{\pi}{2}, \qquad [\because x = 0 \text{ is a point of continuity in } -1 < x < 1]$$

$$\therefore (4) \Rightarrow \frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin \lambda}{\lambda}\, d\lambda \qquad \text{i.e., } \int_{0}^{\infty} \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$$

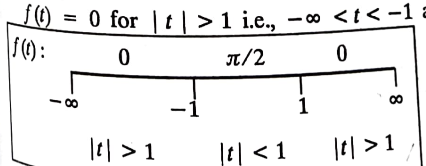
Aliter :

We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t - x)\, dt\, d\lambda \quad \text{... (1)}$$

Here, $f(t) = \frac{\pi}{2}$ for $|t| < 1$ is $-1 < t < 1$

$f(t) = 0$ for $|t| > 1$ i.e., $-\infty < t < -1$ and $1 < t < \infty$



$$(1) \Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \int_{-1}^{1} \frac{\pi}{2} \cos\lambda(t-x)\,dt\,d\lambda$$

$$= \frac{1}{\pi}\frac{\pi}{2}\int_0^\infty \int_{-1}^{1} \cos\lambda(t-x)\,dt\,d\lambda$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{-1}^{1} d\lambda$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{\sin\lambda(1-x)}{\lambda} - \frac{\sin\lambda(-1-x)}{\lambda}\right] d\lambda$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{\sin\lambda(1-x)}{\lambda} + \frac{\sin\lambda(1+x)}{\lambda}\right] d\lambda$$

$[\because \sin(-\theta) = -\sin\theta]$

$$= \frac{1}{2}\int_0^\infty \frac{\sin\lambda(1-x)+\sin\lambda(1+x)}{\lambda}\,d\lambda$$

$$= \frac{1}{2}\int_0^\infty \frac{\sin(\lambda-\lambda x)+\sin(\lambda+\lambda x)}{\lambda}\,d\lambda$$

$$= \frac{1}{2}\int_0^\infty \frac{2\sin\lambda\cos\lambda x}{\lambda}\,d\lambda \quad [\because \sin(A+B)+\sin(A-B) = 2\sin A\cos B]$$

$$= \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$$

i.e., $\displaystyle\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = f(x)$

$$= \frac{\pi}{2} \text{ for } |x| < 1$$

$$= 0 \text{ for } |x| > 1$$

Putting $x = 0$, we get $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{\pi}{2}$

Example 4.1.a(3) : Express the function $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ as a Fourier integral. Hence evaluate $\displaystyle\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ and find the value of $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$. **[A.U. April, 2001] [A.U N/D 2015 R-8] [A.U N/D 2018 R-8]**

Solution : We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(t-x)}\,dt\,ds$$

i.e., $\displaystyle f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\,ds$, where ... (1)

$\displaystyle F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$ ... (2)

Given : $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ i.e., $f(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$$(2) \Rightarrow F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1)\, e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} [\cos sx + i\sin sx]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \cos sx\,dx + \frac{i}{\sqrt{2\pi}}\int_{-1}^{1} \sin sx\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^1 \cos sx\,dx\right] + \frac{i}{\sqrt{2\pi}}[0]$$

$\because$ $\cos sx$ is an even function in $(-1, 1)$
$\sin sx$ is an odd function in $(-1, 1)$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_{x=0}^{x=1}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} - 0\right] \qquad [\because \sin 0 = 0]$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \frac{\sin s}{s}$$

$$(1) \Rightarrow \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin s}{s} e^{-isx} \, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} [\cos sx - i \sin sx] \, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} \cos sx \, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} \sin sx \, ds$$

$$= \frac{1}{\pi} \left[2 \int_0^{\infty} \frac{\sin s}{s} \cos sx \, ds\right] - \frac{i}{\pi} [0]$$

$\because$ $\frac{\sin s}{s} \cos sx$ is an even function in $(-\infty, \infty)$

$\frac{\sin s}{s} \sin sx$ is an odd function in $(-\infty, \infty)$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin s}{s} \cos sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda}{\lambda} \cos \lambda x \, d\lambda \qquad [\because s \text{ is a dummy vari}$$

$$\text{i.e., } \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x)$$

$$\text{Hence, } \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \begin{cases} \frac{\pi}{2}, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Put $x = 0$, (we cannot assign any value to the dummy variable 'λ')

$$\left.\begin{matrix} f(x) \\ at\ x = 0 \end{matrix}\right\} = 1 \qquad [\because x = 0 \text{ is a point of continuity in } -1 \le x \le 1]$$

$$\therefore (4) \Rightarrow \quad \int_0^{\infty} \frac{\sin \lambda}{\lambda} d\lambda = \frac{\pi}{2}$$

Example 4.1.a(4) : Find the Fourier integral of the function

$$\mathbf{f(t)} = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}$$

Solution : We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(t-x)} \, dt \, ds$$

$$\text{i.e., } f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds, \text{ where} \quad \dots (1)$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx \quad \dots (2)$$

$$\text{Given : } f(t) = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases} \quad \text{i.e., } f(t) = \begin{cases} e^{at}, & -\infty < t < 0 \\ e^{-at}, & 0 < t < \infty \end{cases}$$

$$\text{i.e., } f(x) = \begin{cases} e^{ax}, & -\infty < x < 0 \\ e^{-ax}, & 0 < x < \infty \end{cases}$$

$$(2) \Rightarrow F(s) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{ax} e^{isx} \, dx + \int_0^{\infty} e^{-ax} e^{isx} \, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(a+is)x} \, dx + \int_0^{\infty} e^{-(a-is)x} \, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{e^{(a+is)x}}{a+is}\right)_{-\infty}^{0} + \left(\frac{e^{-(a-is)x}}{-(a-is)}\right)_0^{\infty}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is} - 0 + 0 - \left(\frac{1}{-(a-is)}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is} + \frac{1}{a-is}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a-is+a+is}{a^2+s^2}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2a}{a^2+s^2}\right]$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+s^2}\,e^{-isx}\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+s^2}[\cos sx - i\sin sx]\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+s^2}\cos sx\,ds - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+s^2}\sin sx\,ds$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\frac{a}{a^2+s^2}\cos sx\,ds - \frac{i}{\pi}(0)$$

$\because \frac{a}{a^2+s^2}\cos sx$ is an even function in $(-\infty,\infty)$ &

$\frac{a}{a^2+s^2}\sin sx$ is an odd function in $(-\infty,\infty)$

$$= \frac{2a}{\pi}\int_{0}^{\infty}\frac{\cos sx}{a^2+s^2}\,ds$$

$$f(x) = \frac{2a}{\pi}\int_{0}^{\infty}\frac{\cos \lambda x}{a^2+\lambda^2}\,d\lambda \qquad [\because s \text{ is a dummy variable}]$$

$$\text{i.e., } \int_{0}^{\infty}\frac{\cos \lambda x}{a^2+\lambda^2}\,d\lambda = \frac{\pi}{2a}f(x)$$

$$= \begin{cases}\frac{\pi}{2a}e^{at}, & t<0\\ \frac{\pi}{2a}e^{-at}, & t>0\end{cases}$$

Example 4.1.a(5) : Find the Fourier integral of the function

$$f(x) = \begin{cases}0, & x<0\\ 1/2, & x=0\\ e^{-x}, & x>0\end{cases}$$

Verify the representation directly at the point x = 0.

[A.U N/D 2010, M/J 2012]

Solution : We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{is(t-x)}\,dt\,ds$$

$$\text{i.e., } f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)\,e^{-isx}\,ds, \text{ where} \quad \ldots (1)$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{isx}\,dx \quad \ldots (2)$$

$$\text{Given : } f(x) = \begin{cases}0, & x<0\\ ½, & x=0\\ e^{-x}, & x>0\end{cases} \quad \text{i.e., } f(x) = \begin{cases}0, & -\infty<x<0\\ ½, & x=0\\ e^{-x}, & 0<x<\infty\end{cases}$$

$$(2) \Rightarrow F(s) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-x}\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-(1-is)x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(1-is)x}}{-(1-is)}\right]_{x=0}^{x=\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - \left(\frac{1}{-(1-is)}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-is}\right] = \frac{1}{\sqrt{2\pi}}\,\frac{1+is}{1+s^2} \quad \ldots (3)$$

$$(1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,\frac{1+is}{1+s^2}\,e^{-isx}\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{1+s^2}e^{-isx} + i\,\frac{s}{1+s^2}e^{-isx}\right]ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+s^2}(\cos sx - i\sin sx)\,ds + \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{s}{1+s^2}(\cos sx - i\sin sx)\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+s^2}\cos sx\,ds - \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+s^2}\sin sx\,ds + \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{s}{1+s^2}\cos sx\,ds + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{s}{1+s^2}\sin sx\,ds$$

$$= \frac{1}{2\pi}\left[2\int_{0}^{\infty}\frac{1}{1+s^2}\cos sx\,ds\right] - \frac{i}{2\pi}[0] + \frac{i}{2\pi}[0] + \frac{1}{2\pi}\left[2\int_{0}^{\infty}\frac{s}{1+s^2}\sin sx\,ds\right]$$

$\because$ $\frac{1}{1+s^2}\cos sx$ is an even function in $(-\infty, \infty)$,
$\frac{s}{1+s^2}\sin sx$ is an even function in $(-\infty, \infty)$,
$\frac{1}{1+s^2}\sin sx$ is an odd function in $(-\infty, \infty)$ &
$\frac{s}{1+s^2}\cos sx$ is an odd function in $(-\infty, \infty)$

$$= \frac{1}{\pi}\left[\int_{0}^{\infty}\frac{1}{1+s^2}\cos sx\,ds + \int_{0}^{\infty}\frac{s}{1+s^2}\sin sx\,ds\right]$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos sx + s\sin sx}{1+s^2}\,ds$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\,d\lambda \qquad \dots (4)$$

[$\because$ s is a dummy variable]

Verification :

Put $x = 0$ in (4), we get

$$f(0) = \frac{1}{\pi}\int_{0}^{\infty}\frac{1}{1+\lambda^2}\,d\lambda = \frac{1}{\pi}\left[\tan^{-1}\lambda\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}$$

The value of the given function at $x = 0$ is $\frac{1}{2}$.

Hence, verified.

4.1.b Complex form of the Fourier integrals.

The Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos[\lambda(t-x)]\,dt\,d\lambda \qquad \dots (1)$$

because $\cos[\lambda(t-x)]$ is an even function of λ.

Also since $\sin[\lambda(t-x)]$ is an odd function of λ.

$$\text{We have, } \int_{-\infty}^{\infty} f(t)\sin[\lambda(t-x)]\,d\lambda = 0$$

$$\text{i.e., } \int_{-\infty}^{\infty} f(t)\,i\,\sin[\lambda(t-x)]\,d\lambda = 0$$

$$(1) \Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,[\cos\lambda(t-x) + i\sin\lambda(t-x)]\,dt\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\lambda(t-x)}\,dt\,d\lambda$$

which is the complex form of the Fourier integral.

FOURIER SINE AND COSINE INTEGRALS

Fourier sine and cosine integrals

(i) $f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx \, dt \, ds$

[Fourier sine integral]

(ii) $f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, dt \, ds$

[Fourier cosine integral]

Proof : We know that, the Fourier integral theorem is

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos s(t-x) \, dt \, ds$$

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos (st - sx) \, dt \, ds$$

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) [\cos st \cos sx + \sin st \sin sx] \, dt \, ds$$

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos st \cos sx \, dt \, ds$$

$$+ \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \sin st \sin sx \, dt \, ds \quad \ldots (1)$$

Case (i) : When $f(t)$ is odd,

$f(t) \cos st$ is an odd function in $(-\infty, \infty)$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{\pi}(0) + \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx \, dt \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx \, dt \, ds \quad \ldots (2)$$

Case (ii) : When $f(t)$ is even,

$f(t) \sin st$ is an odd function in $(-\infty, \infty)$

$$\therefore (1) \Rightarrow f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, dt \, ds + \frac{1}{\pi}(0)$$

$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, dt \, ds \quad \ldots (3)$$

Note I :

Equation (2) can be re-written as

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin sx \left[\int_0^\infty f(t) \sin st \, dt \right] ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st \, dt \right] ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \; F_s(s) \, ds, \text{ where}$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st \, dt$$

i.e., $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

[$\because$ t is a dummy variable]

Note II :

Equation (3) can be re-written as

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos sx \left[\int_0^\infty f(t) \cos st \, dt \right] ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \, dt \right] ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \; F_c[s] \, ds, \text{ where}$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \, dt$$

i.e., $$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

[$\because$ t is a dummy variable]

I.(b) Problems based on Fourier cosine and Fourier sine integrals

Example 4.1.b(1) : Find Fourier cosine integral of the function e^{-ax}. Hence deduce the value of the integral $\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda$

Solution : We know that, the Fourier Cosine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, dt \, ds$$

i.e., $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds, \text{ where} \quad \ldots (1)$$

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \quad \ldots (2)$$

Given : $f(x) = e^{-ax}$

$$(2) \Rightarrow F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

[Formula : $\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$]

$$F_c[s] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

i.e.,

$$(1) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \cos sx \, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2+a^2} ds$$

$$\Rightarrow \int_0^\infty \frac{\cos sx}{s^2+a^2} ds = \frac{\pi}{2a} f(x)$$

i.e., $$\int_0^\infty \frac{\cos \lambda x}{\lambda^2+a^2} d\lambda = \frac{\pi}{2a} f(x)$$ [$\because$ s is a dummy variable]

$$= \frac{\pi}{2a} e^{-ax}$$

Put $a = 1$, we get

$$\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$$

Example 4.1.b(2) : Using Fourier integral formula, show that

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{\lambda^2+2}{\lambda^4+4} \cos \lambda x \, d\lambda$$

Solution : We know that, the Fourier Cosine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, dt \, ds$$

i.e., $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds, \text{ where} \quad \ldots (1)$$

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \quad \ldots (2)$$

Here, $f(x) = e^{-x} \cos x$

$$(2) \Rightarrow F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos x \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \left[\frac{\cos(s+1)x + \cos(s-1)x}{2}\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-x}\cos(s+1)x}{2} dx + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-x}\cos(s-1)x}{2} dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{1+(s+1)^2}\right] + \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{1+(s-1)^2}\right]$$

$$\left[\because \text{Formula : } \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{1+(s+1)^2} + \frac{1}{1+(s-1)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{1+s^2+1+2s} + \frac{1}{1+s^2+1-2s}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{s^2+2+2s} + \frac{1}{s^2+2-2s}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{s^2+2-2s+s^2+2+2s}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{2(s^2+2)}{(s^2+2)^2-4s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s^2+2}{s^4+4+4s^2-4s^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{s^2+2}{s^4+4}\right]$$

$$(1) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s^2+2}{s^4+4}\right) \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s^2+2}{s^4+4} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\lambda^2+2}{\lambda^4+4} \cos \lambda x \, d\lambda \qquad [\because s \text{ is a dummy variable}]$$

$$\Rightarrow e^{-x}\cos x = \frac{2}{\pi} \int_0^\infty \frac{\lambda^2+2}{\lambda^4+4} \cos \lambda x \, d\lambda$$

Example 4.1.b(3) : **Express** $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$ **as a Fourier sine integral and hence evaluate** $\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin(x\lambda) \, d\lambda$

Solution :

We know that, the Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx \, dt \, ds$$

i.e., $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$, where ... (1)

$$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx \quad \text{... (2)}$$

Given : $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

$$(2) \Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s}\right]_{x=0}^{x=\pi}$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{\cos sx}{s}\right]_{x=0}^{x=\pi}$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\cos \pi s}{s} - \frac{1}{s}\right]$$

i.e., $$F_s(s) = \sqrt{\frac{2}{\pi}}\,\frac{1}{s}\,[1 - \cos \pi s] \quad \dots (3)$$

$$(1) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\,\frac{1}{s}(1 - \cos \pi s)\sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{1 - \cos \pi s}{s}\sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{1 - \cos \pi \lambda}{\lambda}\sin \lambda x\, d\lambda$$

[$\because$ s is a dummy variable]

$$\Rightarrow \int_0^{\infty}\frac{1 - \cos \pi \lambda}{\lambda}\sin \lambda x\, d\lambda = \frac{\pi}{2}f(x) \quad \dots (4)$$

$$= \begin{cases}\left(\frac{\pi}{2}\right)(1), & 0 < x < \pi \\ \left(\frac{\pi}{2}\right)(0), & x > \pi\end{cases} = \begin{cases}\frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi\end{cases}$$

Note : If $x = \pi$ is a finite point of discontinuity of $f(x)$, then

$$\left.\begin{array}{l} f(x) \\ at\ x = \pi\end{array}\right\} = \frac{1+0}{2} = \frac{1}{2}$$

$$\therefore (4) \Rightarrow \int_0^{\infty}\frac{1 - \cos \pi \lambda}{\lambda}\sin \pi \lambda\, d\lambda = \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

Example 4.1.b(4) : Using the Fourier integral representation show that $\displaystyle\int_0^{\infty}\frac{\omega \sin x\omega}{1+\omega^2}\,d\omega = \frac{\pi}{2}e^{-x}\ (x > 0)$

Solution :

We know that, the Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\sin st\ \sin sx\, dt\, ds$$

i.e., $$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}F_s(s)\sin sx\, ds, \text{ where} \quad \dots (1)$$

$$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin sx\, dx \quad \dots (2)$$

Here, $f(x) = e^{-x}$, $x > 0$

$$(2) \Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-x}\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+1}\right]$$

[$\because$ Formula : $\displaystyle\int_0^{\infty}e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}$]

$$(1) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+1}\sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\omega}{\omega^2+1}\sin \omega x\, d\omega$$ [$\because$ s is a dummy variable]

$$\Rightarrow \int_0^{\infty}\frac{\omega}{1+\omega^2}\sin \omega x\, d\omega = \frac{\pi}{2}f(x)$$

$$= \frac{\pi}{2}e^{-x}, \quad x > 0$$

Example 4.1.b(5) : Using Fourier integral formula, prove that

$$e^{-ax} - e^{-bx} = \frac{2(b^2 - a^2)}{\pi}\int_0^{\infty}\frac{\lambda \sin \lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)}\,d\lambda,\ a, b > 0$$

Solution: We know that, the Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\sin st\ \sin sx\, dt\, ds$$

i.e., $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$, where ... (1)

$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$ ... (2)

Here, $f(x) = e^{-ax} - e^{-bx}, \quad x > 0$, and $a, b > 0$

$$(2) \Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty (e^{-ax} - e^{-bx}) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2}\right] - \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + b^2}\right]$$

$$\left[\because \text{Formula} : \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \, s \left[\frac{1}{s^2 + a^2} - \frac{1}{s^2 + b^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \, s \left[\frac{s^2 + b^2 - s^2 - a^2}{(s^2 + a^2)(s^2 + b^2)}\right]$$

$$= \sqrt{\frac{2}{\pi}} \, s \left[\frac{b^2 - a^2}{(s^2 + a^2)(s^2 + b^2)}\right]$$

$$(1) \Rightarrow \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \, s \left[\frac{b^2 - a^2}{(s^2 + a^2)(s^2 + b^2)}\right] \sin sx \, ds$$

$$= \frac{2}{\pi} (b^2 - a^2) \int_0^\infty \frac{s}{(s^2 + a^2)(s^2 + b^2)} \sin sx \, ds$$

$$= \frac{2(b^2 - a^2)}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)} d\lambda,$$

[$\because$ s is a dummy variable]

$$\Rightarrow \frac{2(b^2 - a^2)}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)} d\lambda = f(x)$$

$$= e^{-ax} - e^{-bx}, \; x > 0 \text{ and } a, b > 0$$

Example 4.1.b(6) : Applying the Fourier sine integral formula to the function $f(t) = \sin x$ when $0 < x \le \pi$

$= 0$ when $x > \pi$

show that $\int_0^\infty \frac{\sin \lambda x \;\; \sin \pi \lambda}{1 - \lambda^2} d\lambda = \frac{\pi}{2} \sin x$ if $0 < x < \pi$

$= 0$ if $x > \pi$

Solution: We know that, the Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \; \sin sx \, dt \, ds$$

i.e., $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$, where ... (1)

$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$ ... (2)

Given : $f(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x > \pi \end{cases}$

$$(2) \Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin sx \sin x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \frac{\cos(s-1)x - \cos(s+1)x}{2} dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \int_0^\pi \left[\cos(s-1)x - \cos(s+1)x\right] dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_{x=0}^{x=\pi}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\left[\frac{\sin(s-1)\pi}{(s-1)} - \frac{\sin(s+1)\pi}{s+1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\left[\frac{\sin s\pi\,.\cos\pi - \cos s\pi\sin\pi}{s-1} - \frac{\sin s\pi\cos\pi + \cos s\pi\sin\pi}{s+1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\left[\frac{-\sin s\pi}{s-1} + \frac{\sin s\pi}{s+1}\right] \qquad [\because \sin\pi = 0]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\sin s\pi\left[\frac{-1}{s-1} + \frac{1}{s+1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\sin s\pi\left[\frac{-s-1+s-1}{s^2-1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\sin s\pi\left[\frac{-2}{s^2-1}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{\sin s\pi}{s^2-1}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin s\pi}{1-s^2}$$

$$(1) \Rightarrow \quad f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{\sin s\pi}{1-s^2}\right)\sin sx\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin s\pi}{1-s^2}\sin sx\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\pi}{1-\lambda^2}\sin\lambda x\,d\lambda \qquad [\because s \text{ is a dummy variable}]$$

$$\Rightarrow \int_0^\infty \frac{\sin\lambda\pi\sin\lambda x}{1-\lambda^2}\,d\lambda = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2}\sin x, & 0 < x \le \pi \\ 0, & x > \pi \end{cases}$$

4.2 FOURIER TRANSFORM PAIR :

4.2.a. Fourier Transform : [Complex Fourier Transform]

Definition : The complex (or infinite) Fourier Transform

[A.U N/D 2016 R-8]

The complex (or infinite) Fourier Transform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx \qquad \text{... (1)}$$

Then the function $f(x)$ is the inverse Fourier Transform of $F(s)$ and is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds \qquad \text{... (2)}$$

The above (1) & (2) are jointly called Fourier transform pair.

OTHER FORMATS OF FOURIER TRANSFORM PAIR

(1)	$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-isx}\,dx$	$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\,e^{isx}\,ds$
(2)	$F(s) = \int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds$
(3)	$F(s) = \int_{-\infty}^{\infty} f(x)\,e^{-isx}\,dx$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{isx}\,ds$
(4)	$F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$	$f(x) = \int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds$
(5)	$F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-isx}\,dx$	$f(x) = \int_{-\infty}^{\infty} F(s)\,e^{isx}\,ds$

Note : Whatever definitions or format we use, there will be a difference in constant factor while finding $F(s) = F[f(x)]$. But this will be adjusted while expressing $f(x)$ as a Fourier integral.

For example, $\int_0^\infty \frac{\sin t}{t}\, dt$ or $\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda$ is equal to $\frac{\pi}{2}$, whatever definitions or format we use.

4.2.b. INVERSION FORMULA FOR FOURIER TRANSFORM

Let $f(x)$ be a function satisfying Dirichlet's conditions in every finite interval $(-l, l)$. Let $F(s)$ denote the Fourier transform of $f(x)$. Then at every point of continuity of $f(x)$, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

Proof : By Fourier integral theorem,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i(x-t)s}\, dt\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixs} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-its}\, dt\right] ds$$

put $s = -\omega$,

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-ix\omega} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{it\omega}\, dt\right] (-d\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega}\, F(s)\, d\omega \text{ (by definition of F.T)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-ixs}\, ds \qquad [\because \omega \text{ is a dummy variable}]$$

PROPERTIES - TRANSFORMS OF SIMPLE FUNCTIONS

4.2.c. PROPERTIES OF FOURIER TRANSFORMS :

1. Linear property

$F[a f(x) + b g(x)] = a F[f(x)] + b F[g(x)]$

where a and b are real numbers.

[A.U N/D 2015 R-8] [A.U N/D 2018 R-18]

Proof : $F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a f(x) + b g(x)]\, e^{isx}\, dx$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx$$

$$= a\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right] + b\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx\right]$$

$$= a\, F[f(x)] + b\, F[g(x)]$$

2. Change of scale property **[A.U Tvli. N/D 2010]**

For any **non-zero real a, $F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right]$.**

[A.U M/J 2013] [A.U N/D 2015 R-13] [A.U N/D 2018 R-13]

Proof : We know that, $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $t = ax$, $x \to -\infty \Rightarrow t \to -\infty$, if $a > 0$

$dt = a\, dx$, $x \to \infty \Rightarrow t \to \infty$, if $a > 0$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist/a} \frac{dt}{a} = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)}\, dt$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s/a)x}\, dx \quad [\because t \text{ is a dummy variable}]$$

$$= \frac{1}{a} F\left[\frac{s}{a}\right] \qquad \dots (1)$$

Similarly if $a < 0$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $t = ax$, $x \to -\infty \Rightarrow t \to \infty$

$dt = a\,dx$, $x \to \infty \Rightarrow t \to -\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{(is/a)t}\, \frac{dt}{a}$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{(is/a)t}\, \frac{dt}{a} = \frac{-1}{a} F\left[\frac{s}{a}\right] \quad \dots (2)$$

Combining (1) & (2), we get

$$F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right], \quad a \neq 0$$

3. Shifting property [A.U. May, 2000, April, 1999, April/May 2001]

(i) $F[f(x-a)] = e^{ias} F(s)$ (ii) $F[e^{iax} f(x)] = F[s+a]$

[A.U. N/D 2006, A/M 2006, M/J 2007, CBT Dec. 2008]
[A.U CBT N/D 2010][A.U N/D 2013][A.U N/D 2014 R-08, 13]
[A.U A/M 2015 R-13, A.U A/M 2017 R-13]

Proof : (i) We know that, **[A.U N/D 2018-A, R-17]**

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

put $t = x - a$ | $x \to -\infty \Rightarrow t \to -\infty$

$dt = dx$ | $x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, e^{isa}\, dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = e^{ias} F[f(t)]$$

$$= e^{isa} F[s]$$

(ii) We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$$

$$= F[s+a]$$

4. Modulation Property :

Modulation theorem :

If F (s) is the Fourier transform of f (x), then

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

[A.U CBT Dec. 2008][A.U N/D 2014 R-2013]

Proof : We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\frac{e^{iax} + e^{-iax}}{2}\right] e^{isx}\, dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{i(s+a)x} + e^{i(s-a)x}\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx\right]$$

$$= \frac{1}{2}\,[F(s+a) + F(s-a)]$$

5. $F[x^n f(x)] = (-i)^n \dfrac{d^n F(s)}{ds^n}$. **[A.U Tvli M/J 2011]**

Proof : We know that,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Differentiating both sides n times w.r.to s, we get

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}}\, \frac{d^n}{ds^n}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^n}{\partial s^n}\,[f(x)\, e^{isx}]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,(ix)^n\, e^{isx}\, dx$$

$$= i^n\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, x^n\, e^{isx}\, dx$$

$$= (i)^n\, F[x^n f(x)]$$

Hence, $F[x^n f(x)] = \dfrac{1}{(i)^n}\, \dfrac{d^n}{ds^n} F(s) = (-i)^n \dfrac{d^n}{ds^n} F(s)$

6. (i) $F[f'(x)] = -is\, F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

(ii) $F[f^{(n)}(x)] = (-i)^n s^n F(s)$ if $f(x), f'(x), \ldots f^{(n-1)}(x) \to 0$ as $x \to \pm\infty$.

Proof : We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f'(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\, d[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left[e^{isx} f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, e^{isx}\,(is)\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[(0-0) - is\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right]$$

$$= (-is)\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \qquad [\because f(x) \to 0 \text{ as } x \to \pm\infty]$$

$$= -is\, F(s)$$

Similarly, $F[f^{(n)}(x)] = (-is)^n F(s)$ if $f, f', f'', \ldots f^{(n-1)} \to 0$ as $x \to \pm\infty$.

7. $F\left[\int_a^x f(x)\, dx\right] = \dfrac{F(s)}{(-is)}$

Proof : Let $\phi(x) = \int_a^x f(x)\, dx$

then $\phi'(x) = f(x)$

$F[\phi'(x)] = (-is)\, F[\phi(x)]$ by property (6)

$$= (-is)\, F\left[\int_a^x f(x)\, dx\right]$$

$$F\left[\int_a^x f(x)\, dx\right] = \frac{1}{-is}\, F[\phi'(x)] = \frac{1}{-is}\, F[f(x)]$$

8. $F[\overline{f(x)}] = \overline{F(-s)}$

Proof : We know that,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\therefore F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

Taking complex conjugate on both sides

$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)} \quad e^{isx}\, dx = F[\overline{f(x)}]$$

9. $F[f(-x)] = F(-s)$

Proof : We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x)\, e^{isx}\, dx$$

put $-x = t$ $\quad x \to -\infty \Rightarrow t \to \infty$

$-dx = dt$ $\quad x \to \infty \Rightarrow t \to -\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{-ist}\, (-dt)$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{\infty}^{-\infty} f(t)\, e^{-ist}\, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$$

$$= F(-s)$$

Note : $F[\overline{f(-x)}] = \overline{F(s)}$

4.2.d. CONVOLUTION THEOREM - PARSEVAL'S IDENTITY

Definition : Convolution

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

Convolution Theorem :

[A.U Trichy N/D 2009, CBT A/M 2011, N/D 2011, A/M 2012]
[A.U N/D 2018 R-13][A.U N/D 2018-A, R-17]

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

(i.e.,) $F[f(x) * g(x)] = F(s)\, G(s) = F[f(x)]\, F[g(x)]$

Proof : We know that, $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(x)*g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)*g(x)]\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt \right] e^{isx}\, dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right) \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{isx}\, dt\, dx$$

By changing the order of integration, we get

$$= \left(\frac{1}{\sqrt{2\pi}}\right) \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(x-t)\; e^{isx}\, dx\, dt$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx \right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx \right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\;\; F[g(x-t)]\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, G(s)\, dt$$

by shifting property $F[f(x-a)] = e^{ias} F(s)$

$$= G(s)\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right]$$

$$= G(s)\, F(s) = F(s)\, G(s)$$

Note : $F^{-1}[F(s)\, G(s)] = f(x) * g(x)$

$$= F^{-1}[F(s)] * F^{-1}[G(s)]$$

PARSEVAL'S IDENTITY :

If F (s) is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

[A.U. CBT Dec. 2008, A.U N/D 2010, A.U CBT N/D 2010, M/J 2012]

Proof : By convolution theorem,

$$F[f(x) * g(x)] = F(s).\, G(s)$$

$$f(x) * g(x) = F^{-1}[F(s).\, G(s)]$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, e^{-isx}\, ds \quad \text{...(1)}$$

Put $x = 0$, we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, ds \quad \text{...(2)}$$

$g(-t) = \overline{f(t)}$ there it follows that

$G(s) = \overline{F(s)}$

$$(2) \Rightarrow \int_{-\infty}^{\infty} f(t)\, \overline{f(t)}\, dt = \int_{-\infty}^{\infty} F(s)\, \overline{F(s)}\, ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Note : In the same way, we can prove Parseval's identity for Fourier sine and cosine transforms.

If $F_s[f(s)] = F_s[s]$ and $F_c[g(x)] = F_c(s)$ then

(i) $\int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_s(s)|^2 ds$ and

(ii) $\int_0^{\infty} |g(x)|^2 dx = \int_0^{\infty} |F_c(s)|^2 ds.$

II. (a) Problems based on Fourier Transform [Complex Fourier Transform]

Formula : $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

Example 4.2.a(1) : Find the Fourier Transform of

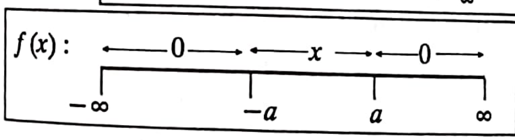
$$f(x) = \begin{cases} x & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$$

[A.U. Oct/Nov. 1996, Tvli M/J 2011] [A.U A/M 2007 R-8]

Solution : The given function can be written as

$$f(x) = \begin{cases} x & \text{if } -a \le x \le a \\ 0 & \text{if } -\infty < x < -a \text{ and } a < x < \infty \end{cases}$$

We know that, $\boxed{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\, dx + \int_{-a}^{a} x e^{isx}\, dx + \int_{a}^{\infty} 0\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x[\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} x \sin sx\, dx$$

$x \cos sx$ is an odd function in $(-a, a)$ $\therefore \int_{-a}^{a} x \cos sx \, dx = 0$

$x \sin sx$ is an even function in $(-a, a)$ $\therefore \int_{-a}^{a} x \sin sx \, dx = 2\int_{0}^{a} x \sin sx \, dx$

$$= \frac{1}{\sqrt{2\pi}}(0) + \frac{i}{\sqrt{2\pi}} \, 2\int_0^a x \sin sx \, dx$$

$$= \frac{2i}{\sqrt{2\pi}} \left[(x)\left[\frac{-\cos sx}{s}\right] - (1)\left[\frac{-\sin sx}{s^2}\right] \right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}} \left[-x\frac{\cos sx}{s} + \frac{\sin sx}{s^2} \right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}} \left[\left(\frac{-a \cos sa}{s} + \frac{\sin sa}{s^2} \right) - (-0 + 0) \right]$$

$$= i\sqrt{\frac{2}{\pi}} \left[\frac{\sin sa - as \cos sa}{s^2} \right]$$

Example 4.2.a(2) : Find the Fourier Transform of

$$f(x) = \begin{cases} 1 & \text{in } |x| < a \\ 0 & \text{in } |x| > a \end{cases}$$

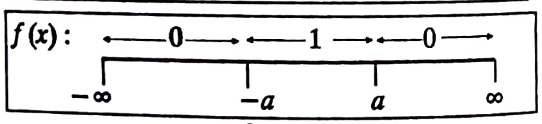
[A.U. Dec. 2005, April, 2004, April 2003] [A.U A/M 2017 R-8]

Solution : The given function can be written as

$$f(x) = \begin{cases} 1 & \text{in } -a < x < a \\ 0 & \text{in } -\infty < x < -a \text{ and } a < x < \infty \end{cases}$$

We know that,

$$\boxed{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx}$$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \sin sx \, dx$$

$\cos sx$ is an even function in $(-a, a)$ $\therefore \int_{-a}^{a} \cos sx \, dx = 2\int_0^a \cos sx$

$\sin sx$ is an odd function in $(-a, a)$ $\therefore \int_{-a}^{a} \sin sx \, dx = 0$

$$= \frac{1}{\sqrt{2\pi}} \, 2\int_0^a \cos sx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0 \right] = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} \right]$$

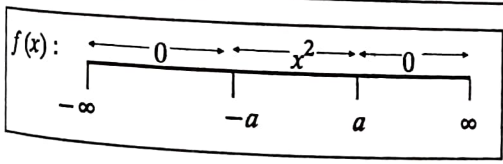
Example 4.2.a(3) : Find the Fourier transform of f(x) given by

$$f(x) = \begin{cases} x^2 & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$$

Solution : The given function can be written as

$$f(x) = \begin{cases} x^2 & \text{if } -a \le x \le a \\ 0 & \text{if } -\infty < x < -a \text{ and } a < x < \infty \end{cases}$$

We know that, $\boxed{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx}$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 [\cos sx + i \sin sx] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} x^2 \sin sx \, dx$$

$x^2 \cos sx$ is an even function in $(-a, a)$

$$\therefore \int_{-a}^{a} x^2 \cos sx \, dx = 2 \int_{0}^{a} x^2 \cos sx \, dx$$

$x^2 \sin sx$ is an odd function in $(-a, a)$

$$\therefore \int_{-a}^{a} x^2 \sin sx \, dx = 0$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} x^2 \cos sx \, dx + \frac{i}{\sqrt{2\pi}} (0)$$

$$= \sqrt{\frac{2}{\pi}} \left[(x^2) \left[\frac{\sin sx}{s}\right] - (2x) \left[\frac{-\cos sx}{s^2}\right] + (2) \left[\frac{-\sin sx}{s^3}\right] \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[x^2 \frac{\sin sx}{s} + \frac{2x}{s^2} \cos sx - \frac{2}{s^3} \sin sx \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{a^2 \sin sa}{s} + \frac{2a \cos sa}{s^2} - \frac{2}{s^3} \sin sa \right) - (0 + 0 - 0) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a^2 s^2 \sin sa + 2as \cos sa - 2 \sin sa}{s^3} \right]$$

Definition : Self reciprocal : [A.U N/D 2013]

If a transformation of a function $f(x)$ is equal to $f(s)$ then the function $f(x)$ is called self reciprocal.

Example 4.2.a(4) : Show that the Fourier Transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$.

(OR)

[A.U. Dec-1996, May-2000]
[A.U N/D 2011, A.U CBT N/D 2011][A.U M/J 2013]
[A.U M/J 2016 R-13][A.U N/D 2018-A, R-17]

Show that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transform.

Solution :

Given : $f(x) = e^{-x^2/2}$, $\quad F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx} dx$$

$$F[e^{-x^2/2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2/2) + isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}}\right)^2 - isx\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2 - \left(\frac{is}{\sqrt{2}}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-\left[\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right]^2} dx$$

We know that,

$$a^2 - 2ab = (a-b)^2 - b^2$$

Here, $a = \frac{x}{\sqrt{2}}$

$$2ab = isx$$

$$2 \frac{x}{\sqrt{2}} b = isx$$

$$b = \frac{is}{\sqrt{2}}$$

Put $y = \frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}$

$$dy = \frac{1}{\sqrt{2}} dx$$

$x \to -\infty \Rightarrow y \to -\infty$

$x \to \infty \Rightarrow y \to \infty$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$$

$$\text{i.e., } F[e^{-x^2/2}] = e^{-s^2/2} \qquad \dots (1)$$

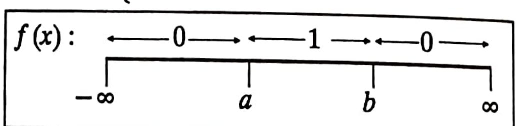
Hence, $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier transform.

Example 4.2.a(5) : Find the Fourier transform of f(x) defined by

$$f(x) = \begin{cases} 0, & x < a \\ 1, & a < x < b \\ 0, & x > b \end{cases}$$

Solution : We know that, $\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx}$

$$\text{Given : } f(x) = \begin{cases} 0, & x < a \\ 1, & a < x < b \\ 0, & x > b \end{cases}$$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{isx} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isb}}{is} - \frac{e^{isa}}{is}\right] = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isb} - e^{isa}}{is}\right]$$

Example 4.2.a(6) : Show that the Fourier transform of

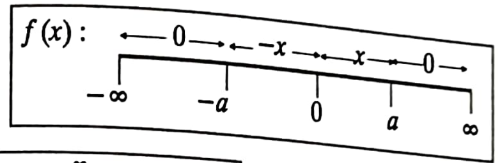
$$f(x) = |x| \text{ for } |x| < a$$
$$= 0 \text{ for } |x| > a,\ a > 0$$

$$\text{is } \sqrt{\frac{2}{\pi}} \left[\frac{sa \sin sa + \cos sa - 1}{s^2}\right]$$

Solution : Given function can be written as

$$f(x) = |x| \text{ for } -a < x < a$$
$$= 0 \text{ for } |x| > a,\ a > 0$$

$$\text{(i.e.,) } f(x) = \begin{cases} -x & \text{for } -a < x < 0 \\ +x & \text{for } 0 < x < a \\ 0 & \text{for } |x| > a \end{cases}$$



We know that,

$$\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} |x| \,[\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} |x| \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} |x| \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^a x \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}} [0]$$

$\because |x| \cos sx$ is an even function in $(-a, a)$,

$|x| \sin sx$ is an odd function in $(-a, a)$ &

$|x| = x$ in $(0, a)$

$$= \sqrt{\frac{2}{\pi}} \int_0^a x \cos sx\, dx = \sqrt{\frac{2}{\pi}} \left[(x)\left[\frac{\sin sx}{s}\right] - (1)\left[\frac{-\cos sx}{s^2}\right]\right]_{x=0}^{x=a}$$

$$= \sqrt{\frac{2}{\pi}} \left[x \frac{\sin sx}{s} + \frac{\cos sx}{s^2}\right]_{x=0}^{x=a}$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{a \sin sa}{s} + \frac{\cos sa}{s^2}\right) - \left(0 + \frac{1}{s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s\, a \sin sa + \cos sa - 1}{s^2}\right]$$

Example 4.2.a(7) : Find the (complex) Fourier transform of

$$f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a, x > b \end{cases}$$

[A.U N/D 2009]

Solution : We know that, $\boxed{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx}$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx} e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(k+s)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)} \right]_a^b = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{i(k+s)} \right] \left[e^{i(k+s)x} \right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{i(k+s)} [e^{i(k+s)b} - e^{i(k+s)a}]$$

$$= \frac{-i}{\sqrt{2\pi}(k+s)} [e^{i(k+s)b} - e^{i(k+s)a}]$$

$$= \frac{i}{\sqrt{2\pi}(k+s)} [e^{i(k+s)a} - e^{i(k+s)b}]$$

Example 4.2.a(8) : Find the Fourier transform of

$$f(x) = \begin{cases} \cos x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution : We know that, **[A.U N/D 2018, R-17]**

$$\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx}$$

Given : $f(x) = \begin{cases} \cos x & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_0^1 \cos x \, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{(is)^2 + 1} [is \cos x + \sin x] \right]_0^1$$

$$\boxed{\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] \quad \text{Here, } a = is, \ b = 1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{1 - s^2} [is \cos x + \sin x] \right]_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{e^{is}}{1 - s^2} (is \cos 1 + \sin 1) - \left(\frac{1}{1 - s^2} (is) \right) \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{is}}{1 - s^2} is \cos 1 + \frac{e^{is}}{1 - s^2} \sin 1 - \frac{is}{1 - s^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1 - s^2} [e^{is} is \cos 1 + e^{is} \sin 1 - is]$$

Example 4.2.a(9) : Find the Fourier transform of $\frac{1}{\sqrt{|x|}}$

Solution : We know that, **[A.U M/J 2014]**

$$\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} e^{isx} dx$$

$$\text{Let } I = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \frac{e^{isx}}{\sqrt{-x}} dx + \int_0^{\infty} \frac{e^{isx}}{\sqrt{x}} dx \right] \quad \dots (1)$$

Since, $|x| = \begin{cases} -x, & x < 0 \\ x, & x > 0 \end{cases}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{isx}}{\sqrt{-x}} dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{isx}}{\sqrt{x}} dx$$

$$= I_1 + I_2$$

In I_1, put $-x = y$; $-dx = dy$

when $x \to -\infty \Rightarrow y \to \infty$

$x \to 0 \Rightarrow y \to 0$

$$\therefore I_1 = \int_{\infty}^{0} \frac{e^{-isy}}{\sqrt{y}}(-dy) = \int_{0}^{\infty} \frac{e^{-isy}}{\sqrt{y}} dy = \int_{0}^{\infty} \frac{e^{-isx}}{\sqrt{x}} dx$$

[$\because$ y is a dummy variable]

There is no change in I_2

$$\therefore I = I_1 + I_2$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-isx}}{\sqrt{x}} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{isx}}{\sqrt{x}} dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{isx} + e^{-isx}}{\sqrt{x}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{2\cos sx}{\sqrt{x}} dx \quad \left[\because \cos sx = \frac{e^{isx} + e^{-isx}}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \text{R.P.} \frac{e^{-isx}}{\sqrt{x}} dx$$

$$= \sqrt{\frac{2}{\pi}} \text{ R.P.} \int_{0}^{\infty} \frac{e^{-isx}}{\sqrt{isx}\sqrt{is}} d(isx)$$

$$= \sqrt{\frac{2}{\pi}} \text{ R.P.} \frac{1}{\sqrt{is}} \int_{0}^{\infty} \frac{e^{-T} dT}{\sqrt{T}} \text{ where } T = isx$$

$$= \sqrt{\frac{2}{\pi}} \text{ R.P.} \frac{(i)^{-1/2}}{\sqrt{s}} \int_{0}^{\infty} e^{-T} T^{-1/2} dT$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{s}} \text{ R.P.} \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{-1/2} \Gamma_{1/2}$$

$$\left[\because i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$= \frac{1}{\sqrt{s}} \sqrt{\frac{2}{\pi}} \text{ R.P.} \left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] \sqrt{\pi} \quad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{1}{\sqrt{s}} \sqrt{\frac{2}{\pi}} \cos\frac{\pi}{4} \sqrt{\pi}$$

$$= \sqrt{\frac{2}{s}} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

$$\therefore F\left[\frac{1}{\sqrt{|x|}}\right] = \frac{1}{\sqrt{s}}$$

Example 4.2.a(10) : Find the Fourier transform of $e^{-a^2x^2}$, $a > 0$, Hence, show that $e^{-x^2/2}$ is self reciprocal under Fourier transform.

[A.U N/D 2014 R-08, 13] [A.U A/M 2015] [A.U N/D 2016 R-13] [A.U N/D 2018, R-8] [A.U A/N 2019 R-17] [A.U A/M 2019, R-13]

Solution : Given : $f(x) = e^{-a^2x^2}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2 + isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax)^2 - isx]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left[(ax) - \frac{is}{2a}\right]^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$

We know that,

$A^2 - 2AB = (A - B)^2 - B^2$

Here, $A = ax$

$2AB = isx$

$2ax\,B = isx$

$B = \dfrac{is}{2a}$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Put $y = ax - \dfrac{is}{2a}$ | $x \to -\infty \Rightarrow y \to -\infty$

$dy = a\,dx$ | $x \to \infty \Rightarrow y \to \infty$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-y^2} \frac{1}{a} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a}\right) e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a}\right) e^{-s^2/4a^2} \sqrt{\pi} \qquad [\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}]$$

$$F[e^{-a^2x^2}] = \frac{1}{\sqrt{2}\,a} e^{-s^2/4a^2} \qquad \dots (1)$$

Put $a = \dfrac{1}{\sqrt{2}}$, we get

$$F[f(x)] = e^{-s^2/2}$$

i.e., $$F[e^{-x^2/2}] = e^{-s^2/2}$$

Hence, $e^{-x^2/2}$ is self reciprocal under Fourier transform.

> **Note :** e^{-x^2} is not self reciprocal, because, by putting $a = 1$ in (4),
> we get $F[e^{-x^2}] = \dfrac{1}{\sqrt{2}} e^{-s^2/4} \neq e^{-s^2}$

Example 4.2.a(11) : Find the Fourier transform of Dirac delta function $\delta(t - a)$.

Solution : The Dirac delta function is defined as

$$\delta(t-a) = \lim_{h \to 0} I(h, t-a), \text{ where}$$

$$I(h, t-a) = \begin{cases} \frac{1}{h} & \text{for } a < t < a+h \\ 0 & \text{for } t < a \text{ and } t > a+h \end{cases}$$

The Fourier transform of $\delta(t - a)$ is

$$F[\delta(t-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \delta(t-a)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \int_{a}^{a+h} \frac{1}{h} e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \frac{1}{h} \left[\frac{e^{ist}}{is}\right]_a^{a+h}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \frac{1}{h} \left[\frac{e^{is(a+h)} - e^{ias}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \left[\frac{e^{ias}(e^{ish} - 1)}{ish}\right]$$

$$= \frac{e^{ias}}{\sqrt{2\pi}} \lim_{h \to 0} \left[\frac{e^{ish} - 1}{ish}\right]$$

$$= \frac{e^{ias}}{\sqrt{2\pi}} \qquad \left[\text{since } \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta}\right) = 1\right]$$

II.(b) Problems based on Fourier transform and its inversion formula

Formula :

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\; e^{-isx}\, ds$$

Example 4.2.b(1) : Find the Fourier transform of the function f (x) defined by $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$

Hence, prove that $\displaystyle\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$

[A.U. April, 1996, 2000, 2001, M/J 2006] [A.U. CBT Dec. 2008] [A.U. CBT Dec. 2009][A.U. M/J 2016 R-8] [A.U N/D 2016 R-8] [A.U N/D 2018, R-17]

Solution : The given function can be written as

$$f(x) = \begin{cases} 1 - x^2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

We know that,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, [\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^1 (1 - x^2) \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}} [0]$$

$\because$ $(1 - x^2) \cos sx$ is an even function in $(-1, 1)$

$(1 - x^2) \sin sx$ is an odd function in $(-1, 1)$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x^2) \cos sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1 - x^2)\left[\frac{\sin sx}{s}\right] - (-2x)\left[\frac{-\cos sx}{s^2}\right] + (-2)\left[\frac{-\sin sx}{s^3}\right]\right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1 - x^2) \frac{\sin sx}{s} - 2x \frac{\cos sx}{s^2} + \frac{2 \sin sx}{s^3}\right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(0 - \frac{2\cos s}{s^2} + \frac{2 \sin s}{s^3}\right) - (0 - 0 + 0)\right]$$

$$= \frac{4}{s^3\sqrt{2\pi}} [\sin s - s \cos s] \quad \ldots (1)$$

By Fourier inversion formula, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{s^3\sqrt{2\pi}} (\sin s - s \cos s)\; e^{-isx}\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) (\cos sx - i \sin sx)\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds - i \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx\, ds \quad \ldots (2)$$

$$= \frac{2}{\pi} \left[2 \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds\right] - i \frac{2}{\pi} [0]$$

II.(b) Problems based on Fourier transform and its inversion formula

Formula :

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\; e^{-isx}\, ds$$

Example 4.2.b(1) : Find the Fourier transform of the function f (x) defined by $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$

Hence, prove that $\int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$

[A.U. April, 1996, 2000, 2001, M/J 2006] [A.U. CBT Dec. 2008] [A.U. CBT Dec. 2009][A.U. M/J 2016 R-8] [A.U N/D 2016 R-8] [A.U N/D 2018, R-17]

Solution : The given function can be written as

$$f(x) = \begin{cases} 1 - x^2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

We know that,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\; [\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^1 (1 - x^2) \cos sx\, dx \right] + \frac{i}{\sqrt{2\pi}} [0]$$

$\because$ $(1 - x^2) \cos sx$ is an even function in $(-1, 1)$

$(1 - x^2) \sin sx$ is an odd function in $(-1, 1)$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x^2) \cos sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1 - x^2) \left[\frac{\sin sx}{s}\right] - (-2x) \left[\frac{-\cos sx}{s^2}\right] + (-2) \left[\frac{-\sin sx}{s^3}\right] \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1 - x^2) \frac{\sin sx}{s} - 2x \frac{\cos sx}{s^2} + \frac{2 \sin sx}{s^3} \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(0 - \frac{2 \cos s}{s^2} + \frac{2 \sin s}{s^3}\right) - (0 - 0 + 0) \right]$$

$$= \frac{4}{s^3 \sqrt{2\pi}} [\sin s - s \cos s] \qquad \text{... (1)}$$

By Fourier inversion formula, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{s^3 \sqrt{2\pi}} (\sin s - s \cos s)\; e^{-isx}\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3}\right) (\cos sx - i \sin sx)\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3}\right) \cos sx\, ds - i \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3}\right) \sin sx\, ds \qquad \text{... (2)}$$

$$= \frac{2}{\pi} \left[2 \int_0^{\infty} \left(\frac{\sin s - s \cos s}{s^3}\right) \cos sx\, ds \right] - i \frac{2}{\pi} [0]$$

$\because \dfrac{\sin s - s\cos s}{s^3}\cos sx$ is an even function in $(-\infty, \infty)$

$\dfrac{\sin s - s\cos s}{s^3}\sin sx$ is an odd function in $(-\infty, \infty)$

i.e., $f(x) = \dfrac{4}{\pi}\displaystyle\int_0^\infty \dfrac{\sin s - s\cos s}{s^3}\cos sx\, ds$

$$\Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos sx\, ds = \frac{\pi}{4} f(x) \quad \ldots (3)$$

Put $x = \dfrac{1}{2}$, we get

$$\left.\begin{matrix} f(x) \\ at\, x = \frac{1}{2}\end{matrix}\right\} = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$[\because x = \frac{1}{2}$ is a point of continuity is $-1 < x < 1]$

$$\therefore (3) \Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\, ds = \frac{\pi}{4}\left(\frac{3}{4}\right) = \frac{3\pi}{16}$$

Example 4.2.b(2) : Find the Fourier Transform of

$$\mathbf{f(x) = \begin{cases} a - |x|, & \text{for } |x| < a \\ 0, & \text{for } |x| > a > 0 \end{cases}}$$ **hence deduce that**

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

[A.U Nov/Dec, 1996] [A.U. CBT Dec. 2008]
[A.U Tvli N/D 2011] [A.U A/M 2015 R-08]

Solution : We know that,

$$\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

Here, $f(x) = \begin{cases} a - |x|, & \text{for } |x| < a \text{ (i.e.,) } -a < x < a \\ 0, & \text{otherwise}\end{cases}$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}[a - |x|]\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}[a - |x|](\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}[a - |x|]\cos sx\, dx + i\int_{-a}^{a}[a - |x|]\sin sx\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a - x)\cos sx\, dx + 0\right]$$

Since, $[a - |x|]\cos sx$ is an even function in $(-a, a)$
$[a - |x|]\sin sx$ is an odd function in $(-a, a)$ &
$a - |x| = a - x$ in $(0, a)$

$$= \sqrt{\frac{2}{\pi}}\int_0^a [a - x]\cos sx\, dx \qquad [\because |x| = x \text{ in } (0, a)]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a - x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[(a - x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(0 - \frac{\cos as}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos as}{s^2} + \frac{1}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{s^2}[1 - \cos as]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\frac{1}{s^2}\left[2\sin^2\frac{as}{2}\right] \quad \ldots (1)$$

By inversion formula,

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\, \frac{1}{s^2} \sin^2 \frac{as}{2}\, e^{-isx}\, ds$$

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{s^2} \sin^2 \frac{as}{2}\, e^{-isx}\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left[\frac{\sin \frac{as}{2}}{s}\right]^2 e^{-isx}\, ds$$

We have to deduce that $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$ in the above integrand put

$x = 0$ and $a = 2$, we get

$$(1) \Rightarrow f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2}\, ds$$

$$= \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2 s}{s^2}\, ds \quad \left[\because \frac{\sin^2 s}{s^2} \text{ is an even function}\right]$$

$$\int_0^{\infty} \frac{\sin^2 s}{s^2}\, ds = \frac{\pi}{4} f(0)$$

$$= \frac{\pi}{4} [2] \qquad \left[\because f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & \text{otherwise} \end{cases}\right.$$

$f(0) = a$ but Here $a = 2$

$$\text{(i.e.,)} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2} \qquad [\because s \text{ is a dummy variable}]$$

Note :

Even function :

If $f(-x) = f(x)$ in $(-l, l)$ then $f(x)$ is an even function.

Odd function :

If $f(-x) = -f(x)$ in $(-l, l)$ then $f(x)$ is an odd function

In the above problem, $a \cos sx$ is an even function,

$a \sin sx$ is an odd function

$|x|$ is an even function,

$|x| \cos sx$ is an even function

$|x| \sin sx$ is an odd function.

Example 4.2.b(3): Find Fourier transform of $e^{-a|x|}$ **[AU N/D 2012]** **and hence deduce that**

(i) $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2 + a^2)^2}$ **[A.U N/D 2016 R-8]**

(ii) $\int_0^{\infty} \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$ **[A.U A/M 2015 R-08]**

[A.U. Oct.2001, M/J 2007 M/J 2014][A.U N/D 2014 R-08]

Solution : We know that,

$$\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$F[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_0^{\infty} e^{-ax} \cos sx\, dx \qquad [\because |x| = x \text{ in } (0, \infty)]$$

Since, $e^{-a|x|}\cos sx$ is an even function in $(-\infty, \infty)$

$$\therefore \int_{-\infty}^{\infty} e^{-a|x|}\cos sx\,dx = 2\int_0^{\infty} e^{-ax}\cos sx\,dx$$

$e^{-a|x|}\sin sx$ is an odd function, $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} e^{-a|x|}\sin sx\,dx = 0$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right] \qquad \because \text{Formula: } \int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

Using inversion formula, we get

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\,e^{-isx}\,ds}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}\,e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\,a\int_{-\infty}^{\infty}\frac{e^{-isx}}{s^2+a^2}\,ds$$

$$= \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\cos sx - i\sin sx}{s^2+a^2}\,ds$$

$$= \frac{a}{\pi}\left[2\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds\right]$$

Since, $\frac{\cos sx}{s^2+a^2}$ is an even function in $(-\infty, \infty)$

$$\therefore \int_{-\infty}^{\infty}\frac{\cos sx}{s^2+a^2}\,ds = 2\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds$$

$\frac{\sin sx}{s^2+a^2}$ is an odd function in $(-\infty, \infty)$

$$\therefore \int_{-\infty}^{\infty}\frac{\sin sx}{s^2+a^2}\,ds = 0$$

$$f(x) = \frac{2a}{\pi}\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds$$

$$\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds = \frac{\pi}{2a}f(x)$$

$$\int_0^{\infty}\frac{\cos xt}{a^2+t^2}\,dt = \frac{\pi}{2a}e^{-a|x|} \qquad [\because s \text{ is a dummy variable}]$$

$$F[xe^{-a|x|}] = -i\frac{d}{ds}F(s)$$

$$= -i\frac{d}{ds}F[e^{-a|x|}]$$

$$= -i\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}\right]$$

$$= -i\sqrt{\frac{2}{\pi}}\,a\,\frac{d}{ds}\left[\frac{1}{s^2+a^2}\right]$$

$$= -i\,a\sqrt{\frac{2}{\pi}}\left[\frac{0-2s}{(s^2+a^2)^2}\right]$$

$$= i\sqrt{\frac{2}{\pi}}\,\frac{2as}{(s^2+a^2)^2}$$

Example 4.2.b(4): Find the Fourier transform of $e^{-|x|}$ and hence find the Fourier transform of $e^{-|x|}\cos 2x$.

[A.U CBT A/M 2011] [A.U A/M 2015 R-2008]

Solution : We know that, $\boxed{F\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx}$

$$F\,[e^{-|x|}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\,(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cos sx\,dx + \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\sin sx\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}\cos sx\,dx + 0$$

> Since, $e^{-|x|}\cos sx$ is an even function in $(-\infty, \infty)$
>
> $e^{-|x|}\sin sx$ is an odd function in $(-\infty, \infty)$ and
>
> $|x| = x$ in $(0, \infty)$

$$\therefore F(s) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{1+s^2}\right] \quad \left[\because \int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}\right]$$

To find : $F\,[e^{-|x|}\cos 2x]$

By Modulation theorem,

$$\boxed{F\,[f(x)\cos ax] = \frac{1}{2}\,[F(s-a) + F(s+a)]}$$

$$F\,[e^{-|x|}\cos 2x] = \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\left[\frac{1}{(s-2)^2+1}\right] + \sqrt{\frac{2}{\pi}}\left[\frac{1}{(s+2)^2+1}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{s^2-4s+5} + \frac{1}{s^2+4s+5}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s^2+4s+5+s^2-4s+5}{(s^2+5)^2-(4s)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{2(s^2+5)}{s^4+10s^2+25-16s^2}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+5}{s^4-6s^2+25}\right]$$

II.(c) Problems based on inversion formula, Parseval's identity and Convolution theorem

Formula :

$$F(s) = F\,[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

Example 4.2.c(1) : **Find the Fourier transform of**

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| \geq a \end{cases}$$ **where a is a positive real number.**

Hence deduce that (i) $\int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$ **and (ii)** $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

[A.U. March, 1996] [A.U N/D 2007, A/M 2008, M/J 2013] [A.U A/M 2015 R-13] [A.U M/J 2016 R-8] [A.U N/D 2018-A, R-17] [A.U N/D 2019, R-17]

Solution :

The given function can be written as $f(x) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^a \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}} [0]$$

$\because$ $\cos sx$ is an even function in $(-a, a)$ & $\sin sx$ is an odd function in $(-a, a)$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos sx\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_{x=0}^{x=a}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) \quad \dots (1)$$

(i) Now, by **Fourier inversion formula,** we have

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) (\cos sx - i \sin sx)\, ds \quad \text{[using (1)]}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \sin sx\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - \frac{i}{\pi}(0)$$

$\left[\because \left(\frac{\sin as}{s}\right) \sin sx \text{ is an odd function in } (-\infty, \infty)\right]$

$$= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds$$

$\left[\because \left(\frac{\sin as}{s}\right) \cos sx \text{ is an even function in } (-\infty, \infty)\right]$

$$\int_0^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds = \frac{\pi}{2} f(x) \quad \dots (2)$$

Put $x = 0 \Rightarrow \left.\begin{matrix} f(x) \\ at\ x = 0 \end{matrix}\right\} = 1$

$$(2) \Rightarrow \int_0^{\infty} \frac{\sin as}{s}\, ds = \frac{\pi}{2} \quad \dots (3)$$

put $as = t$	$s \to 0 \Rightarrow t \to 0$
$a\, ds = dt$	$s \to \infty \Rightarrow t \to \infty$
$\Rightarrow ds = \frac{1}{a} dt$	

$$(3) \Rightarrow \int_0^{\infty} \frac{\sin t}{(t/a)} \frac{1}{a} dt = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

(ii) Using **Parseval's identity**

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds\ ,}$$

We get, $\displaystyle\int_{-a}^{a} 1\, dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\ \frac{\sin as}{s}\right)^2 ds$

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = a\pi$$

put $t = as$ $dt = a\, ds$ $ds = \frac{1}{a}dt$	$s \to -\infty \Rightarrow t \to -\infty$ $s \to \infty \Rightarrow t \to \infty$

$$\int_{-\infty}^{\infty} \left(\frac{\sin t}{\frac{t}{a}}\right)^2 \frac{dt}{a} = a\pi$$

$$2\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi \quad \left[\because \left(\frac{\sin t}{t}\right)^2 \text{ is an even function}\right]$$

$$\therefore \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

Example 4.2.c(2) : Find the Fourier Transform of

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$ **Hence deduce that**

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2},\ \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$

[A.U. April, 2001, May, 2001]
[A.U. A/M 2005, N/D 2005, M/J 2006, N/D 2007, CBT Dec. 2008]
[A.U N/D 2009, A.U Tvli. N/D 2010, A.U CBT N/D 2010]
[A.U N/D 2011, N/D 2012] [CBT N/D 2011][A.U N/D 2014 R-13]
[A.U N/D 2015 R-13] [A.U N/D 2016 R-13]
[A.U N/D 2018 R-13][A.U A/M 2019 R-13]

Solution : Given $f(x) = \begin{cases} 1 - |x|, & -1 < x < 1 \\ 0 & , x < -1 \text{ and } x > 1 \end{cases}$

The Fourier transform of $f(x)$ is

$$\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1 - |x|)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1 - |x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1 - |x|)\cos sx\, dx + \frac{i}{\sqrt{2\pi}}\int_{-1}^{1} (1 - |x|)\sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{1} (1 - x)\cos sx\, dx + 0$$

$[\because (1 - |x|)\sin sx$ is an odd function in $(-1, 1)]$

$[\because (1 - |x|)\cos sx$ is an even function in $(-1, 1)]$

$[\because |x| = x$ in $(0, 1)]$

$$= \sqrt{\frac{2}{\pi}}\left[(1 - x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[(1 - x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(0 - \frac{\cos s}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right] = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s^2} + \frac{1}{s^2}\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right]$$

(i) Using inverse transform $\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds}$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right][\cos sx - i\sin sx]\, ds \quad \text{by (1)}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1-\cos s}{s^2}\right]\cos sx\, ds \quad [\because \text{ second integrand is odd}]$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{1-\cos s}{s^2}\cos sx\, ds \quad [\because \text{ First integrand is even}]$$

$$\int_0^{\infty}\left[\frac{1-\cos s}{s^2}\right]\cos sx\, ds = \frac{\pi}{2}f(x)$$

$$= \frac{\pi}{2}[1-|x|]$$

put $x = 0$, we get $\int_0^{\infty}\frac{1-\cos s}{s^2}\, ds = \frac{\pi}{2} \Rightarrow \int_0^{\infty}\frac{2\sin^2\frac{s}{2}}{s^2}\, ds = \frac{\pi}{2}$

put	$t = \frac{s}{2}$	$\Rightarrow 2t = s \Rightarrow 2\,dt = ds$	
	$s \to 0$	$\Rightarrow t \to 0,\ s \to \infty \Rightarrow t \to \infty$	

$$\int_0^{\infty}\frac{2\sin^2 t}{(2t)^2}\, 2\,dt = \frac{\pi}{2}$$

$$\int_0^{\infty}\frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

(ii) Using Parseval's identity, $\boxed{\int_{-\infty}^{\infty}|f(x)|^2\, dx = \int_{-\infty}^{\infty}|F(s)|^2\, ds}$ ... (1)

$$\int_{-\infty}^{\infty}|f(x)|^2\, dx = \int_{-1}^{1}[1-|x|]^2\, dx = 2\int_0^1 [1-|x|]^2\, dx$$

$$= 2\int_0^1 (1-x)^2\, dx \quad [\because |x| = x \text{ in } (0, 1)]$$

$$= 2\left[\frac{(1-x)^3}{3(-1)}\right]_0^1 = -\frac{2}{3}\left[(1-x)^3\right]_0^1$$

$$= -\frac{2}{3}[0-1] = \frac{2}{3} \quad \text{... (2)}$$

$$|F(s)|^2 = \frac{2}{\pi}\left[\frac{1-\cos s}{s^2}\right]^2 = \frac{2}{\pi}\left[\frac{2\sin^2(s/2)}{s^2}\right]^2 = \frac{8}{\pi}\frac{\sin^4(s/2)}{s^4}$$

$$\int_{-\infty}^{\infty}|F(s)|^2\, ds = \frac{8}{\pi}\int_{-\infty}^{\infty}\frac{\sin^4(s/2)}{s^4}\, ds = \frac{16}{\pi}\int_0^{\infty}\frac{\sin^4(s/2)}{s^4}\, ds$$

put	$t = s/2$	$\Rightarrow 2t = s$	$\Rightarrow 2\,dt = ds$
	$s \to 0$	$\Rightarrow t \to 0,$	$s \to \infty \Rightarrow t \to \infty$

$$= \frac{16}{\pi}\int_0^{\infty}\frac{\sin^4 t}{(2t)^4}\, 2\,dt = \frac{2}{\pi}\int_0^{\infty}\frac{\sin^4 t}{t^4}\, dt \quad \text{... (3)}$$

$$\therefore (1) \Rightarrow \frac{2}{\pi}\int_0^{\infty}\frac{\sin^4 t}{t^4}\, dt = \frac{2}{3} \quad \text{by (2) \& (3)}$$

$$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$

Example 4.2.c(3) : **Show that the Fourier transform of**

$$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases} \text{ is } 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin as - as\cos as}{s^3}\right)$$

Hence, deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$. **Using Parseval's identity show that** $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$.

[A.U. March 1996, Nov. 2001, A/M 2004] [AU. N/D 2008, A/M 2009] [A.U CBT N/D 2010] [A.U Chennai N/D 2011] [A.U N/D 2012] [A.U N/D 2015 R-8]

Solution :

We know that,

$$\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\,[\cos sx + i\sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a^2 - x^2)\cos sx\, dx + 0$$

$\because$ $(a^2 - x^2)\cos sx$ is an even function in $(-a, a)$ &

$(a^2 - x^2)\sin sx$ is an odd function in $(-a, a)$

$$= \frac{2}{\sqrt{2\pi}} \left[(a^2 - x^2)\left[\frac{\sin sx}{s}\right] - (-2x)\left[\frac{-\cos sx}{s^2}\right] + (-2)\left[\frac{-\sin sx}{s^3}\right]\right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(0 - \frac{2a\cos as}{s^2} + \frac{2\sin as}{s^3}\right) - (0 - 0 + 0)\right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin as - as\cos as}{s^3}\right]$$

$$\text{i.e., } F(s) = 2\sqrt{\frac{2}{\pi}} \left\{\frac{\sin as - as\cos as}{s^3}\right\}$$

(i) Using **inverse Fourier Transform,** we get

$$f(x) = \frac{1}{\sqrt{2\pi}}\, 2\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{1}{s^3} [\sin as - as\cos as]\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\, 2\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{1}{s^3} [\sin as - as\cos as]\,[\cos sx - i\sin sx]\, ds$$

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin as - as\cos as}{s^3} \cos sx\, ds - \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{\sin as - as\cos as}{s^3} \sin sx\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin as - as\cos as}{s^3} \cos sx\, ds + 0$$

The second integrand is odd and hence, its integral is zero.

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin as - as\cos as}{s^3} \cos sx\, ds$$

[$\because$ the first integrand is even]

Put $a = 1$, we get

$$(2) \Rightarrow f(x) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos sx\, ds$$

$$= \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin t - t\cos t}{t^3} \cos tx\, dt \quad [\because s \text{ is a dummy variable}]$$

Put $x = 0$, we get

$$(2) \Rightarrow f(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = 1, \text{ when } a = 1$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}}$$

(ii) Using **Parseval's identity,**

$$\boxed{\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx} \quad \text{... (1)}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,ds = \int_{-1}^{1} (1-x^2)^2\,dx \qquad [\text{Here, } a = 1]$$

$$= 2\int_0^1 \left[1 + x^4 - 2x^2\right] dx = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[\left(1 + \frac{1}{5} - \frac{2}{3}\right) - (0 + 0 - 0)\right] = 2\left[\frac{8}{15}\right] = \frac{16}{15} \quad \text{... (2)}$$

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds \qquad [\text{Here, } a = 1]$$

$$= \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds \quad \text{... (3)}$$

$$(1) \Rightarrow \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

i.e., $\displaystyle\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt = \frac{\pi}{15}$ [$\because$ s is a dummy variable]

Example 4.2.c(4) : Find the Fourier transform of $e^{-a|x|}$ if $a > 0$

Deduce that $\displaystyle\int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a^3}$ if $a > 0$.

Solution : Given : $f(x) = e^{-a|x|}$

See Example 4.2.b(3) in page no. 4.53

$$F(s) = F[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$$

By Parseval's identity,

If F(S) is the Fourier transform of $f(x)$, then

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds} \quad \text{... (1)}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty}\left[e^{-ax}\right]^2 dx = 2\int_0^\infty \left[e^{-ax}\right]^2 dx$$

$$= 2\int_0^\infty e^{-2ax}\,dx = 2\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= -\frac{1}{a}\left[e^{-2ax}\right]_0^\infty = \frac{1}{a}[0-1] = \frac{1}{a} \quad \text{... (2)}$$

$$|F(s)|^2 = \frac{2}{\pi}\,\frac{a^2}{(s^2+a^2)^2} \quad \text{... (3)}$$

$$(1) \Rightarrow \frac{1}{a} = \int_{-\infty}^{\infty} \frac{2}{\pi}\,\frac{a^2}{(s^2+a^2)^2}\,ds \qquad \text{by (2) \& (3)}$$

$$= \frac{4a^2}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds$$

$$\int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds = \frac{\pi}{4a^3}$$

$\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$ if $a > 0$ [$\because$ s is a dummy variable]

Example 4.2.c(5) : Find the Fourier transform of

$$f(x) = 1 - x^2 \text{ if } |x| < 1$$
$$= 0 \quad \text{if } |x| \geq 1$$

Hence, show that $\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\frac{s}{2}\, ds = \frac{3\pi}{16}$

Also show that $\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6}\, dx = \frac{\pi}{15}$.

[A.U N/D 2013][A.U A/M 2019, R-8]

Solution : See Example 4.2.b(1) in page no. 4.48 for problems based on Fourier transform and its inversion formula

$$F(s) = F[f(x)] = \frac{4}{s^3\sqrt{2\pi}}[\sin s - s\cos s]$$

Now, if $F[f(x)] = F(s)$ by **Parseval's identity**

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds \quad \text{... (1)}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-1}^{1} (1-x^2)^2 dx = 2\int_0^1 (1-x^2)^2 dx$$

$$= 2\int_0^1 \left[1 + x^4 - 2x^2\right] dx = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right] = 2\left(\frac{8}{15}\right) = \frac{16}{15}$$

$$|F(s)|^2 = \frac{16}{s^6(2\pi)}[\sin s - s\cos s]^2 = \frac{8}{s^6\pi}[\sin s - s\cos s]^2$$

$$(1) \Rightarrow \quad \frac{16}{15} = \int_{-\infty}^{\infty} \frac{8}{\pi s^6}[\sin s - s\cos s]^2 ds$$

$$= \frac{16}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6} ds$$

$$\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6} ds = \frac{\pi}{15}$$

$$\text{i.e., } \int_0^\infty \frac{(\sin x - x\cos x)^2}{x^6} dx = \frac{\pi}{15} \quad [\because s \text{ is a dummy variable}]$$

$$\text{i.e., } \int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6} dx = \frac{\pi}{15} \quad [\because (a-b)^2 = (b-a)^2]$$

Example 4.2.c(6) : Find the Fourier transform of f(x) given by

$$f(x) = 1 \text{ for } |x| < 2$$
$$= 0 \text{ for } |x| > 2$$

and hence, evaluate $\int_0^\infty \frac{\sin x}{x} dx$ **and** $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$.

[A.U. Nov/Dec.2003][A.U A/M 2017 R-13]

Solution : The given equation can be written as

$$f(x) = 1 \text{ if } -2 < x < 2$$
$$= 0 \text{ otherwise}$$

$$F(s) = F[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-2}^{2} (\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-2}^{2} \cos sx\, dx + \frac{i}{\sqrt{2\pi}}\int_{-2}^{2} \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^2 \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}}[0]$$

[$\because$ cos sx is an even function in $(-2, 2)$ & sin sx is an odd function in $(-2, 2)$]

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_{x=0}^{x=2}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin 2s}{s} - 0\right]$$

$$F(s) = F[f(x)] = \sqrt{\frac{2}{\pi}}\frac{\sin 2s}{s}$$

(i) Now, by Fourier inversion formula, we have

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin 2s}{s}(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2s}{s}\cos sx\, ds - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2s}{s}\sin sx\, dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2s}{s}\cos sx\, dx - \frac{i}{\pi}(0) \quad \left[\because \frac{\sin 2s}{s}\sin sx \text{ is an odd function in } (-\infty,\infty)\right]$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin 2s}{s}\cos sx\, dx \quad \left[\because \frac{\sin 2s}{s}\cos sx \text{ in an even function in } (-\infty,\infty)\right]$$

$$\text{(i.e.,)} \int_{0}^{\infty}\left(\frac{\sin 2s}{s}\right)\cos sx\, dx = \frac{\pi}{2} f(x)$$

put $x = 0$, we get $\int_{0}^{\infty}\frac{\sin 2s}{s}\, ds = \frac{\pi}{2} f(0) = \frac{\pi}{2}$

$$\therefore \int_{0}^{\infty}\frac{\sin 2s}{s}\, ds = \frac{\pi}{2} \quad [\because f(0) = 1 \text{ in } |x| < 2]$$

Now, put $t = 2s$ $\quad s \to 0 \Rightarrow t \to 0$

$dt = 2\, ds$ $\quad s \to \infty \Rightarrow t \to \infty$

$$\therefore \int_{0}^{\infty}\frac{\sin t}{(t/2)}\frac{dt}{2} = \frac{\pi}{2}$$

Hence, $\int_{0}^{\infty}\frac{\sin t}{t}\, dt = \frac{\pi}{2}$. (i.e.,) $\int_{0}^{\infty}\frac{\sin x}{x}\, dx = \frac{\pi}{2}$ $\quad$ [$\because$ t is a dummy variable]

(ii) Using Parseval's identity, $\boxed{\int_{-\infty}^{\infty}|f(x)|^2\, dx = \int_{-\infty}^{\infty}|F(s)|^2\, ds}$ ... (1)

$$\int_{-\infty}^{\infty}|f(x)|^2\, dx = \int_{-2}^{2}1^2\, dx = \int_{-2}^{2}dx = [x]_{-2}^{2} = 2-(-2) = 4 \quad \text{... (2)}$$

$$|F(s)|^2 = \frac{2}{\pi}\frac{\sin^2 2s}{s^2}$$

$$\int_{-\infty}^{\infty}|F(s)|^2\, ds = \int_{-\infty}^{\infty}\frac{2}{\pi}\frac{\sin^2 2s}{s^2}\, ds = \frac{4}{\pi}\int_{0}^{\infty}\frac{\sin^2 2s}{s^2}\, ds$$

put	$t = 2s$	$\Rightarrow dt = 2\, ds$
	$s \to 0$	$\Rightarrow t \to 0,\ s \to \infty \Rightarrow t \to \infty$

$$= \frac{4}{\pi}\int_{0}^{\infty}\frac{\sin^2 t}{(t/2)^2}\frac{1}{2}dt = \frac{8}{\pi}\int_{0}^{\infty}\frac{\sin^2 t}{t^2}\, dt \quad \text{... (3)}$$

$$\text{(1)} \Rightarrow 4 = \frac{8}{\pi}\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt \quad \text{by (2) \& (3)}$$

$$\Rightarrow \int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2} \quad [\because t \text{ is a dummy variable}]$$

Example 4.2.c(7) : Find the Fourier transform of $e^{-|x|}$, using Parseval's identity show that $\int_{0}^{\infty}\frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$.

Solution : $F[e^{-|x|}] = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+s^2}\right)$ already proved.

By Parseval's identity,

$$\boxed{\int_{-\infty}^{\infty}|F(s)|^2\, ds = \int_{-\infty}^{\infty}|f(x)|^2\, dx} \quad \text{... (1)}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \left[e^{-|x|}\right]^2 dx = 2\int_0^{\infty} \left[e^{-|x|}\right]^2 dx$$

$$= 2\int_0^{\infty} \left[e^{-x}\right]^2 dx \qquad [\because |x| = x \text{ in } (0, \infty)]$$

$$= 2\int_0^{\infty} e^{-2x} dx = 2\left[\frac{e^{-2x}}{-2}\right]_0^{\infty}$$

$$= -\left[e^{-2x}\right]_0^{\infty} = -[(0)-(1)] = 1 \qquad \dots (2)$$

$$|F(s)|^2 = \frac{2}{\pi}\frac{1}{(1+s^2)^2}$$

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1}{(1+s^2)^2} ds = \frac{4}{\pi}\int_0^{\infty} \frac{1}{(1+s^2)^2} ds \qquad \dots (3)$$

$$(1) \Rightarrow \frac{4}{\pi}\int_0^{\infty} \frac{1}{(1+s^2)^2} ds = 1 \qquad \text{by (2) \& (3)}$$

$$\int_0^{\infty} \frac{1}{(1+s^2)^2} ds = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\infty} \frac{1}{(1+x^2)^2} dx = \frac{\pi}{4} \qquad [\because s \text{ is a dummy variable}]$$

Example 4.2.c(8) : If $f(x) = \begin{cases} \cos x, & |x| < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$, **then find the Fourier transform of f(x) and hence, evaluate** $\int_0^{\infty} \frac{\cos^2 \pi x/2}{(1-x^2)^2} dx$ **using Parseval's identity.**

Solution : We know that,

$$\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \cos x\, e^{isx} dx \quad \left[\because f(x) = \begin{cases} \cos x, & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \cos x [\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \cos sx \cos x\, dx + \frac{i}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \sin sx \cos x\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^{\pi/2} \cos sx \cos x\, dx + \frac{i}{\sqrt{2\pi}}(0)$$

> Since, $\cos sx \cos x$ is an even function in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
>
> $\sin sx \cos x$ is an odd function in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$= \frac{2}{\sqrt{2\pi}}\int_0^{\pi/2} \left[\frac{\cos(s+1)x + \cos(s-1)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\pi/2} [\cos(s+1)x + \cos(s-1)x]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{\sin(s+1)\pi/2}{s+1} + \frac{\sin(s-1)\pi/2}{s-1}\right) - (0+0)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin\left(s\frac{\pi}{2}+\frac{\pi}{2}\right)}{s+1} + \frac{\sin\left(\frac{s\pi}{2}-\frac{\pi}{2}\right)}{s-1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin\left(\frac{\pi}{2}+\frac{s\pi}{2}\right)}{s+1} - \frac{\sin\left(\frac{\pi}{2}-\frac{s\pi}{2}\right)}{s-1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\cos\frac{s\pi}{2}}{s+1} - \frac{\cos\frac{s\pi}{2}}{s-1}\right] = \frac{\cos\frac{s\pi}{2}}{\sqrt{2\pi}}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{\cos\frac{s\pi}{2}}{\sqrt{2\pi}}\left[\frac{s-1-s-1}{s^2-1}\right]$$

$$= \frac{\cos\frac{s\pi}{2}}{\sqrt{2\pi}}\left[\frac{-2}{s^2-1}\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{\cos\frac{s\pi}{2}}{1-s^2}\right]$$

By Parseval's identity,

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx \qquad \text{... (1)}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\pi/2}^{\pi/2} \cos^2 x\, dx = 2\int_0^{\pi/2} \cos^2 x\, dx$$

$$= 2\int_0^{\pi/2} \frac{1+\cos 2x}{2}\, dx = \int_0^{\pi/2} (1+\cos 2x)\, dx$$

$$= \left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \left(\frac{\pi}{2}+0\right) - (0+0)$$

$$= \frac{\pi}{2} \qquad \text{... (2)}$$

$$|F(s)|^2 = \frac{2}{\pi}\frac{\cos^2\frac{s\pi}{2}}{(1-s^2)^2} \qquad \text{... (3)}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{2}{\pi}\frac{\cos^2\frac{s\pi}{2}}{(1-s^2)^2}\, ds = \frac{\pi}{2} \quad \text{by (2) \& (3)}$$

$$\Rightarrow \frac{4}{\pi}\int_0^{\infty} \frac{\cos^2\frac{s\pi}{2}}{(1-s^2)^2}\, ds = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos^2\frac{s\pi}{2}}{(1-s^2)^2}\, ds = \frac{\pi^2}{8}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos^2\frac{\pi x}{2}}{(1-x^2)^2}\, dx = \frac{\pi^2}{8} \qquad [\because s \text{ is a dummy variable}]$$

Example 4.2.c(9) : Verify convolution theorem for $f(x) = g(x) = e^{-x^2}$.

[A.U M/J 2013] [A.U N/D 2015 R-13]

Definition : Convolution theorem for Fourier transforms. The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

$$F[f(x) * g(x)] = F\{f(x)\}\, F\{g(x)\}$$

Given : $f(x) = g(x) = e^{-x^2}$

R.H.S. = $F[f(x)]\, F[g(x)]$

We know that, $\boxed{F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx}$

$$F[e^{-x^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2} e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2+isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(x+\frac{is}{2}\right)^2+\frac{s^2}{4}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(x+\frac{is}{2}\right)^2} e^{-s^2/4}\, dx$$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{-\left(x+\frac{is}{2}\right)^2} dx$$

put $t = x + \frac{is}{2}$ | $x \to -\infty \Rightarrow t \to -\infty$

$dt = dx$ | $x \to \infty \Rightarrow t \to \infty$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{-t^2} dt$$

Standard integral formula
$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}} e^{-s^2/4}$$

$$F[f(x)] = \frac{1}{\sqrt{2}} e^{-s^2/4}$$

$$F[g(x)] = F[f(x)] = \frac{1}{\sqrt{2}} e^{-s^2/4}$$

$$\therefore F[f(x)] . F[g(x)] = \left(\frac{1}{\sqrt{2}} e^{-s^2/4}\right)\left(\frac{1}{\sqrt{2}} e^{-s^2/4}\right)$$

$$= \frac{1}{2} e^{-s^2/2} \quad \ldots (1)$$

L.H.S :

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \quad \text{by convolution definition}$$

$$e^{-x^2} * e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2} e^{-(x-u)^2} du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-u)^2 - u^2} du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left[\left(u-\frac{x}{2}\right)^2 + \frac{x^2}{4}\right]} du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2} e^{\frac{-x^2}{2}} du$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2} du$$

put $t = u - \frac{x}{2}$ | $u \to -\infty \Rightarrow t \to -\infty$

$dt = du$ | $u \to \infty \Rightarrow t \to \infty$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2t^2} dt$$

Put $y = \sqrt{2}\, t$

$dy = \sqrt{2}\, dt$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2} \frac{dy}{\sqrt{2}}$$

$$= \frac{e^{-x^2/2}}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{e^{-x^2/2}}{2\sqrt{\pi}}\sqrt{\pi} \quad \text{by standard integral formula.}$$

$$= \frac{1}{2} e^{-x^2/2}$$

$$F[f(x) * g(x)] = F\left[\frac{1}{2}e^{-x^2/2}\right]$$

$$= \frac{1}{2} F[e^{-x^2/2}]$$

We know that, $e^{-x^2/2}$ is self reciprocal under Fourier transform.

$$\therefore \quad F[e^{-x^2/2}] = e^{-s^2/2}$$

$$\therefore F[f(x) * g(x)] = \frac{1}{2}e^{-s^2/2} \quad \text{... (2)}$$

$$\therefore (1) = (2)$$

Hence, convolution theorem is verified.

4.3 FOURIER SINE & COSINE TRANSFORMS :

4.3.a. FOURIER COSINE TRANSFORM :

The infinite Fourier cosine transform of $f(x)$ is defined by

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

The inverse Fourier cosine transform $F_c[f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds$$

4.3.b INVERSION FORMULA FOR FOURIER COSINE TRANSFORM

Let $F_c(s)$ denote the F.C.T of $f(x)$. Then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

Proof : By the definition of F.C.T,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

Here, $f(x)$ is defined for all $x \geq 0$

Now define $g(x)$ by $g(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ f(-x), & \text{if } x < 0 \end{cases}$

Clearly,

$g(-x) = g(x)$ for all x and hence g is an even function.

To prove that, the Fourier transform of $g(x)$ is the F.C.T of $f(x)$.

$$F[g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty g(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty g(x) [\cos sx + i \sin sx] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty g(x) \cos sx \, dx + \frac{1}{\sqrt{2\pi}} i \int_{-\infty}^\infty g(x) \sin sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty g(x) \cos sx \, dx$$

[$\because$ $g(x) \cos sx$ is an even function

$g(x) \sin sx$ is an odd function]

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos sx \, dx$$

$$= F_c[g(x)]$$

$$= F_c[f(x)] \quad [\because g(x) = f(x) \text{ for all } x \geq 0]$$

Hence, by inversion formula for F.T, we have

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(s)\, [\cos sx - i \sin sx]\, ds$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} F_c(s) \cos sx\, ds \quad [\because F_c(s) \text{ is an even function}]$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s) \cos sx\, ds$$

Since, $g(x) = f(x)$ for all $x \geq 0$, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s) \cos sx\, ds$$

4.3.c FOURIER SINE TRANSFORM :

The infinite Fourier sine transform of $f(x)$ is defined by

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

The inverse Fourier sine transform of $F_s[f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s[f(x)] \sin sx\, ds$$

4.3.d INVERSION FORMULA FOR FOURIER SINE TRANSFORM

Let $F_s(s)$ denote the F.S.T of $f(x)$. Then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s) \sin sx\, ds$$

Proof : By the definition of F.S.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

Here, $f(x)$ is defined for all $x \geq 0$

Now, we define $g(x)$ by

$$g(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ -f(x), & \text{if } x < 0 \end{cases}$$

Clearly, $g(x)$ is an odd function

$$F[g(x)] = F_s[f(x)] \text{ [by the above 5.8]}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s) \sin sx\, ds$$

4.3.e Properties of Fourier sine transform and Fourier cosine transform

1. Linear property

(i) $F_s[af(x) + bg(x)] = a\, F_s[f(x)] + b\, F_s[g(x)]$

(ii) $F_c[af(x) + bg(x)] = a\, F_c[f(x)] + b\, F_c[g(x)]$

Proof : (i) We know that,

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx}$$

$$F_s[af(x) + bg(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} [af(x) + bg(x)] \sin sx\, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx + b\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(x) \sin sx\, dx$$

$$= a\, F_s[f(x)] + b\, F_s[g(x)]$$

(ii) We know that,

$$\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx}$$

$$F_c[af(x)+bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty [af(x)+bg(x)]\cos sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos sx\,dx$$

$$= a\,F_c[f(x)] + b\,F_c[g(x)]$$

2. Modulation property :

(i) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

(ii) $F_s[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(iii) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$

(iv) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

Proof : **[A.U. N/D 2008] [A.U Tvli N/D 2009]**

$$F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\sin ax\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\cos(s-a)x - \cos(s+a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx\right]$$

$$= \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$

(ii) $$F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\cos ax\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\cos(s+a)x + \cos(s-a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx\right]$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

(iii) $$F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\sin(a+s)x + \sin(a-s)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(a+s)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(a-s)x\,dx\right]$$

$$= \frac{1}{2}\left[F_s(a+s) + F_s(a-s)\right]$$

(iv) $$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\cos ax\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\cos(s+a)x + \cos(s-a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx\right]$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

3. $F_s\,[f(ax)] = \frac{1}{a} F_s\left[\frac{s}{a}\right]$. **[Change of scale property]**

[A.U N/D 2016 R-13]

Proof : $F_s\,[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \sin sx\, dx$

put $ax = t$, $a\,dx = dt$ | $x \to 0 \Rightarrow t \to 0$, $x \to \infty \Rightarrow t \to \infty$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\left(\frac{st}{a}\right) \frac{dt}{a}$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\left(\frac{s}{a} t\right) dt$$

$$= \frac{1}{a} F_s\left[\frac{s}{a}\right]$$

Similarly, $F_c\,[f(ax)] = \frac{1}{a} F_c\left[\frac{s}{a}\right]$

4. $F_s\,[f'(x)] = -s\, F_c(s)$, **if** $f(x) \to 0$ **as** $x \to \infty$. **[Transform of derivative]**

Proof : $F_s\,[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx\, d[f(x)]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\sin sx\, f(x)\right)_0^\infty - s\int_0^\infty f(x) \cos sx\, dx \right]$$

$$= -s\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx \text{ [Assuming } f(x) \to 0 \text{ as } x \to \infty]$$

$$= -s\, F_c(s)$$

5. $F_c\,[f'(x)] = -\sqrt{\frac{2}{\pi}}\, f(0) + s\, F_s(s)$ **if** $f(x) \to 0$ **as** $x \to \infty$.

[Transform of derivative]

Proof : $F_c\,[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos sx\, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx\, d[f(x)]$$

$$= \sqrt{\frac{2}{\pi}} \left[[\cos sx\, f(x)]_0^\infty + s\int_0^\infty f(x) \sin sx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} [(0 - f(0)] + s\, F_s(s) \text{ Assuming } f(x) \to 0 \text{ as } x \to \infty$$

$$= -\sqrt{\frac{2}{\pi}}\, f(0) + s\, F_s(s)$$

6. $F_s\,[x f(x)] = -\frac{d}{ds}[F_c(s)]$ **[Derivatives of transform]**

[A.U CBT A/M 2011] [A.U A/M 2015 R-08]

Proof : We know that,

$$F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

Differentiating both sides w.r.to 's', we get

$$\frac{d}{ds} F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s}(\cos sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)(-x \sin sx)\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, x \sin sx\, dx$$

$$= -F_s\,[x f(x)]$$

$$\text{(i.e.,) } F_s\,[x f(x)] = -\frac{d}{ds} F_c\,[f(x)]$$

7. $F_c\,[x f(x)] = \frac{d}{ds} F_s(s)$

[A.U A/M 2015 R-08]

Proof : We know that,

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, \sin s x\, dx$$

Differentiating both sides w.r.to 's', we get

$$\frac{d}{ds} F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x)\, \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s} (\sin s x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, \cos sx\, (x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, x \cos sx\, dx$$

$$= F_c\,[x f(x)]$$

$$\text{i.e., } F_c\,[x f(x)] = \frac{d}{ds} F_s\,[f(x)]$$

III(a). Problems based on Fourier Cosine Transform

Formula :

$$F_c(s) = F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

Example 4.3.a(1) : Find the Fourier cosine transform of

$$f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x \geq a \end{cases}$$

Solution :

We know that,

$$\boxed{F_c(s) = F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx\, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos sx \cos x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{1}{2} \int_0^a [\cos (s+1)x + \cos (s-1)x]\, dx$$

$$\left[\because \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin (s+1)x}{s+1} + \frac{\sin (s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{\sin (s+1)a}{s+1} + \frac{\sin (s-1)a}{s-1}\right) - (0+0)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin (s+1)a}{s+1} + \frac{\sin (s-1)a}{s-1}\right]$$

provided $s \neq 1$; $s \neq -1$

Example 4.3.a(2) : Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$ and hence, find $F_c\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$.

[A.U CBT A/M 2011]
[A.U N/D 2015 R-13]

Solution : We know that,

$$\boxed{F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx\, dx$$

$$\text{(i.e.,) } F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx\, dx$$

$$\frac{d}{ds} F_c(s) = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s}\left[\frac{e^{-ax}}{x} \cos sx\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} (-x \sin sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty -e^{-ax} \sin sx\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

$$\frac{d}{ds} F_c(s) = -\sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2}\right]$$

Formula :

$$\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}$$

Here, $b = s$

By integrating, we get

$$F(s) = -\sqrt{\frac{2}{\pi}} \int \frac{s}{s^2 + a^2} ds = -\sqrt{\frac{2}{\pi}} \frac{1}{2} \int \frac{2s}{s^2 + a^2} ds$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \log (s^2 + a^2)$$

$$\text{(i.e.,) } F_c\left[\frac{e^{-ax}}{x}\right] = -\frac{1}{\sqrt{2\pi}} \log (s^2 + a^2)$$

Similarly, $F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}} \log (s^2 + b^2)$

Now, $F_c\left[\frac{e^{-ax} - e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$

$$= \frac{-1}{\sqrt{2\pi}} \log (s^2 + a^2) + \frac{1}{\sqrt{2\pi}} \log (s^2 + b^2)$$

$$= \frac{1}{\sqrt{2\pi}} [\log (s^2 + b^2) - \log (s^2 + a^2)]$$

$$= \frac{1}{\sqrt{2\pi}} \log \left[\frac{s^2 + b^2}{s^2 + a^2}\right]$$

Example 4.3.a(3): Find the Fourier cosine transform of e^{-ax}, $a > 0$.

[A.U. April 2001] [A.U N/D 2014 R-08] [A.U N/D 2015 R-8] [A.U. M/J 2016 R-13] [A.U N/D 2016 R-8]

Solution :

We know that, $\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2 + a^2}\right]$$

Formula :

$$\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}$$

Example 4.3.a(4) : Find the Fourier cosine transform of the function $3e^{-5x} + 5e^{-2x}$.

Solution : Let $f(x) = 3e^{-5x} + 5e^{-2x}$

We know that, $\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$

$$F_c\,[3e^{-5x} + 5e^{-2x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty [3e^{-5x} + 5e^{-2x}] \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty 3e^{-5x} \cos sx\, dx + \int_0^\infty 5e^{-2x} \cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\; 3 \int_0^\infty e^{-5x} \cos sx\, dx + \sqrt{\frac{2}{\pi}}\; 5 \int_0^\infty e^{-2x} \cos sx\, dx$$

We know that, $\boxed{\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}}$

$$= 3\sqrt{\frac{2}{\pi}} \left[\frac{5}{5^2+s^2}\right] + \sqrt{\frac{2}{\pi}}\; 5 \left[\frac{2}{2^2+s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{15}{s^2+25} + \frac{10}{s^2+4}\right]$$

Example 4.3.a(5): Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^a 1 \,.\, \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0\right] = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$$

Example 4.3.a(6): Find the Fourier cosine transform of $f(x) = x$.

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\,[x] = \sqrt{\frac{2}{\pi}} \int_0^\infty x \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \int_0^\infty x\, e^{-isx}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \left[x \frac{e^{-isx}}{(-is)} - (1) \frac{e^{-isx}}{(-is)^2}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \left[-x \frac{e^{-isx}}{is} + \frac{e^{-isx}}{s^2}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \left[(0) - \left(0 + \frac{1}{s^2}\right)\right] \qquad [\because e^{-\infty} = 0]$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \left[\frac{-1}{s^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}} \frac{1}{s^2}$$

Example 4.3.a(7): Find the Fourier cosine transform of $e^{-ax} \cos ax$.

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\,[e^{-ax} \cos ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos ax \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \cos ax \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \left[\frac{\cos(s+a)x + \cos(s-a)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty e^{-ax} \cos(s+a)x \, dx + \int_0^\infty e^{-ax} \cos(s-a)x \, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{a}{a^2+(s+a)^2} + \frac{a}{a^2+(s-a)^2}\right]$$

Formula :

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{1}{(s+a)^2+a^2} + \frac{1}{(s-a)^2+a^2}\right]$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{1}{s^2+2as+2a^2} + \frac{1}{s^2-2as+2a^2}\right]$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{1}{(s^2+2a^2)+2as} + \frac{1}{(s^2+2a^2)-2as}\right]$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{s^2+2a^2-2as+(s^2+2a^2)+2as}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{2(s^2+2a^2)}{s^4+4a^4+4s^2a^2-4a^2s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\, a \left[\frac{s^2+2a^2}{s^4+4a^4}\right]$$

Example 4.3.a(8): **Show that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform.**

[A.U M/J 2007] [A.U CBE Dec. 2009] [A.U N/D 2018-A, R-17]

Solution :

We know that,

$$\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx}$$

$$F_c[e^{-x^2/2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^\infty e^{-x^2/2} \cos sx \, dx$$

$[\because e^{-x^2/2} \cos sx$ is an even function$]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \text{R.P} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} e^{isx} \, dx$$

$$= \text{R.P } F[e^{-x^2/2}]$$

$$= \text{R.P } e^{-s^2/2}$$ by (1) See example 4.2a(4)

$$F_c[e^{-x^2/2}] = e^{-s^2/2}$$

Hence, $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier cosine transform.

Example 4.3.a(9): Find the Fourier cosine transform of $e^{-ax} \sin ax$

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\,[e^{-ax} \sin ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin ax \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \sin ax\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{1}{2} \int_0^\infty e^{-ax}\, [\sin (s+a)x - \sin (s-a)x]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty e^{-ax} \sin (s+a)x\, dx - \int_0^\infty e^{-ax} \sin (s-a)x\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s+a}{a^2 + (s+a)^2} - \frac{s-a}{a^2+(s-a)^2}\right]$$

$$\boxed{\because \text{ Formula : } \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s+a}{s^2 + 2as + 2a^2} - \frac{s-a}{s^2 - 2as + 2a^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s^3 - 2as^2 + 2sa^2 + as^2 - 2a^2 s + 2a^3 - s^3 - 2as^2 - 2sa^2 + as^2 + 2a^2 s + 2a^3}{(s^2+2a^2)^2 - (2as)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{4a^3 - 2as^2}{s^4 + 4a^4 + 4s^2a^2 - 4a^2 s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\, \frac{2a\,[2a^2 - s^2]}{s^4 + 4a^4} = \sqrt{\frac{2}{\pi}}\; a \left[\frac{2a^2 - s^2}{s^4 + 4a^4}\right]$$

ample 4.3.a(10): Evaluate $F_c\,[x^{n-1}]$ if $0 < x < 1$.

[A.U A/M 2015 R-13]

educe that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine transform.

[A.U M/J 2012]

lution : $F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \cos sx\, dx$

e know that, $\Gamma(n) = \int_0^\infty e^{-y} y^{n-1}\, dy, \quad n > 0$

t $y = ax$, we get $\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}, \quad a > 0$

Let $a = is$

$$\therefore \int_0^\infty e^{-isx} x^{n-1}\, dx = \frac{\Gamma(n)}{(is)^n}$$

$$\int_0^\infty (\cos sx - i \sin sx)\, x^{n-1}\, dx = \frac{\Gamma(n)\, i^{-n}}{s^n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right]^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right]$$

quating real parts, we get

$$\int_0^\infty x^{n-1} \cos sx\, dx = \frac{\Gamma(n)}{s^n} \cos\left(\frac{n\pi}{2}\right)$$

ing this in (1) we get

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma(n)}{s^n} \cos\left(\frac{n\pi}{2}\right)$$

put $n = \frac{1}{2}$, we get

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\ \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{\sqrt{\pi}}{\sqrt{s}}\ \frac{1}{\sqrt{2}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{s}}$$

Hence, $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine transform.

> **Note :** Equating imaginary part, we get
> $$F_s\,[x^{n-1}] = \sqrt{\frac{2}{\pi}}\ \frac{\Gamma(n)}{s^n} \sin\left(\frac{n\pi}{2}\right)$$

Example 4.3.a(11): Find the Fourier cosine transform of

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$$

[A.U. N/D 2006]

[A.U CBT N/D 2011] [A.U A/M 2019, R-I]

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos sx\, dx + \int_1^2 (2 - x) \cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{\left[x\frac{\sin sx}{s} - (1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1 + \left[(2-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_1^2\right.$$

$$= \sqrt{\frac{2}{\pi}} \left\{\left[x\frac{\sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^1 + \left[(2 - x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_1^2\right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{\left[\left(\frac{1\sin s}{s} + \frac{\cos s}{s^2}\right) - \left(\frac{1}{s^2}\right)\right] + \left[\left(0 - \frac{\cos 2s}{s^2}\right) - \left(\frac{\sin s}{s} - \frac{\cos s}{s^2}\right)\right]\right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\cos s - \cos 2s - 1}{s^2}\right]$$

Example 4.3.a(12): Find the Fourier cosine transform of $\frac{1}{a^2 + x^2}$.

Solution :

We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\left[\frac{1}{1 + x^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} e^{-s}\right]$$

$$F_c\left[\frac{1}{a^2 + x^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2a} e^{-as}\right]$$

$$= \sqrt{\frac{\pi}{2}}\ \frac{1}{a} e^{-as}$$

Example 4.3.a(13): Find the Fourier cosine transform of $e^{-a^2x^2}$.

[A.U M/J 2006, A.U CBT A/M 2011] [A.U Tvli N/D 2011]
[A.U N/D 2012] [A.U A/M 2017 R-13]

Solution : We know that,

$$\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx}$$

$$F_c[e^{-a^2x^2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^\infty e^{-a^2x^2} \cos sx \, dx$$

$[\because e^{-a^2x^2} \cos sx$ is an even function$]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \text{R.P.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-a^2x^2} e^{isx} \, dx$$

$$= \text{R.P } F[e^{-a^2x^2}]$$

$$= \text{R.P } \frac{1}{\sqrt{2}\, a} e^{-s^2/4a^2}$$ by (1) See example 4.2a(10)

$$F_c[e^{-a^2x^2}] = \frac{1}{\sqrt{2}\, a} e^{-s^2/4a^2}$$

III. (b) Problems based on Fourier cosine transform and its inversion formula.

Formula :

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

Example 4.3.b(1): Solve the integral equation

$\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$ **and also show that**

$\int_0^\infty \frac{\cos \lambda x}{1+x^2} dx = \frac{\pi}{2} e^{-\lambda}$ **[A.U. Dec. 1996] [A.U N/D 2015 R-08]**

[A.U N/D 2016 R-13]

Solution : Given : $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$

$$\int_0^\infty f(x) \cos sx \, dx = e^{-s} \qquad [\text{take } \lambda = s]$$

Multiplying by $\sqrt{\frac{2}{\pi}}$ on both sides, we get

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} e^{-s}$$

by formula, $$F_c[f(x)] = \sqrt{\frac{2}{\pi}} e^{-s} \qquad \dots (1)$$

by Fourier cosine inversion formula, we get

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-s} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty e^{-s} \cos sx \, ds$$

$$f(x) = \frac{2}{\pi} \left[\frac{1}{1+x^2}\right]$$

$[\because$ Formula :

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$$

Here $a = 1$, $b = x]$

$$F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{2}{\pi} \left[\frac{1}{1+x^2}\right] \cos sx\, dx$$

$$\sqrt{\frac{2}{\pi}}\, e^{-s} = \frac{2}{\pi} \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\frac{1}{1+x^2}\right] \cos sx\, dx$$

$$e^{-s} = \frac{2}{\pi} \int_0^\infty \left[\frac{1}{1+x^2}\right] \cos sx\, dx$$

$$\int_0^\infty \left[\frac{1}{1+x^2}\right] \cos sx\, dx = \frac{\pi}{2} e^{-s}$$

$$\Rightarrow \int_0^\infty \left[\frac{1}{1+x^2}\right] \cos \lambda x\, dx = \frac{\pi}{2} e^{-\lambda} \qquad \text{[Replace } s \text{ by } \lambda]$$

Example 4.3.b(2): Solve the integral equation

$$\int_0^\infty f(x) \cos \lambda x\, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$$

Hence, evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} dt$.

[A.U.April, 2001]

Solution :

Given that, $\int_0^\infty f(x) \cos \lambda x\, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$

$$\int_0^\infty f(x) \cos sx\, dx = \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases} \qquad [\text{take } \lambda = s]$$

Multiply by $\sqrt{\frac{2}{\pi}}$ on both sides, we have

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx = \sqrt{\frac{2}{\pi}} \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases}$$

by Fourier cosine formula, $F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases}$

$$F_c\,[f(x)] = \begin{cases} \sqrt{\frac{2}{\pi}}\,(1-s), & 0 \le s \le 1 \\ 0, & s > 1 \end{cases} \qquad \dots (1)$$

We know that, Fourier cosine inversion formula is

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c\,[f(x)] \cos sx\, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{2}{\pi}}\,(1-s) \cos sx\, ds$$

$$= \frac{2}{\pi} \int_0^1 (1-s) \cos sx\, ds$$

$$= \frac{2}{\pi} \left[(1-s)\frac{\sin sx}{x} - (-1)\left(\frac{-\cos sx}{x^2}\right)\right]_{s=0}^{s=1}$$

$$= \frac{2}{\pi} \left[(1-s)\frac{\sin sx}{x} - \frac{\cos sx}{x^2}\right]_{s=0}^{s=1}$$

$$= \frac{2}{\pi} \left[\left(0 - \frac{\cos x}{x^2}\right) - \left(0 - \frac{1}{x^2}\right)\right]$$

$$= \frac{2}{\pi} \left[\frac{-\cos x}{x^2} + \frac{1}{x^2}\right]$$

$$f(x) = \frac{2}{\pi} \left[\frac{1-\cos x}{x^2}\right]$$

We know that,

$$F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos s\,x\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{2}{\pi}\left[\frac{1-\cos x}{x^2}\right]\cos s\,x\,dx \qquad \cdots (2)$$

From (1) and (2), we have,

$$\sqrt{\frac{2}{\pi}}\,\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\cos s\,x\,dx = \begin{cases}\sqrt{\frac{2}{\pi}}\,(1-s), & 0\le s\le 1\\ 0, & s>1\end{cases}$$

Now, put $s \to 0$, we get

$$\sqrt{\frac{2}{\pi}}\,\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \sqrt{\frac{2}{\pi}}$$

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\,dx = 1$$

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{2\sin^2 x/2}{x^2}\,dx = \frac{\pi}{2}$$

put $t = \frac{x}{2}$	$x \to 0 \Rightarrow t \to 0$
$dt = \frac{1}{2}dx$	$x \to \infty \Rightarrow t \to \infty$

$$\int_0^\infty \frac{2\sin^2 t}{(2t)^2}\,2\,dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$$

Example 4.3.b(3): Find the Fourier cosine transform of $e^{-|x|}$ and deduce that $\int_0^\infty \frac{\cos xt}{1+t^2}\,dt = \frac{\pi}{2}e^{-|x|}$.

Solution : We know that, $\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx}$

$$F_c\,[e^{-|x|}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\,dx \qquad [\because \text{ in the interval } (0,\infty),\ e^{-|x|} = e^{-x}]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right] \qquad \boxed{\begin{array}{l}\because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}\\ \text{Here, } a = 1,\ b = s\end{array}}$$

Now, using Fourier cosine inversion formula

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c\,[f(x)]\cos sx\,ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right]\cos sx\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos sx}{s^2+1}\,ds$$

(i.e.,) $$\int_0^\infty \frac{\cos sx}{s^2+1}\,ds = \frac{\pi}{2}f(x)$$

$$= \frac{\pi}{2}e^{-|x|}$$

(i.e.,) $$\int_0^\infty \frac{\cos xt}{1+t^2}\,dt = \frac{\pi}{2}e^{-|x|} \qquad (\because s \text{ is a dummy variable})$$

Example 4.3.b(4): Find the Fourier cosine transform of e^{-ax}, $a > 0$ and deduce that $\int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds = \frac{\pi}{2a} e^{-ax}$. **[A.U N/D 2019, R-17]**

Solution : We know that, $\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right] \qquad \because \int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$$

Applying the inversion formula, we have

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[e^{-ax}] \cos sx\, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \cos sx\, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds$$

$$\text{(i.e.,)} \int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds = \frac{\pi}{2a} f(x) = \frac{\pi}{2a} e^{-ax}, \; a > 0$$

III. (c) Problems Based on Fourier Sine Transform. [F.S.T]

Formula :

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

Example 4.3.c(1): Find the Fourier sine transform of

$$f(x) = e^{-x} \cos x.$$

Solution : We know that,

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx}$$

$$F_s[e^{-x} \cos x] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos x \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \cos x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \left[\frac{\sin(s+1)x + \sin(s-1)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty e^{-x} \sin(s+1)x\, dx + \int_0^\infty e^{-x} \sin(s-1)x\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s+1}{(s+1)^2+1} + \frac{s-1}{(s-1)^2+1}\right]$$

$$\left[\because \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s+1}{s^2+2s+2} + \frac{s-1}{s^2-2s+2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s^3-2s^2+2s+s^2-2s+2+s^3+2s^2+2s-s^2-2s-2}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2s^3}{s^4+4+4s^2-4s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2s^3}{s^2+4} = \frac{1}{\sqrt{2\pi}} \frac{2s^3}{s^2+4}$$

Example 4.3.c(2): Find the Fourier sine transform of

$$f(x) = \begin{cases} \sin x, & 0 \le x < a \\ 0, & x > a \end{cases}$$ **[A.U N/D 2018-A, R-17]**

Solution : We know that,

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx}$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^a \sin x \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \sin sx \sin x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{\cos(s-1)x - \cos(s+1)x}{2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^a \cos(s-1)x \, dx - \int_0^a \cos(s+1)x \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{\sin(s-1)x}{s-1} \right)_0^a - \left(\frac{\sin(s+1)x}{s+1} \right)_0^a \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} \right]$$

where $s \neq 1$ and $s \neq -1$

Example 4.3.c(3): Find the Fourier sine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

[A.U N/D 2010]

Solution : We know that, **[A.U N/D 2016 R-8]**

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[x \left(\frac{-\cos sx}{s} \right) - (1) \left(\frac{-\sin sx}{s^2} \right) \right]_0^1 + \left[(2-x) \left(\frac{-\cos sx}{s} \right) - (-1) \left(\frac{-\sin sx}{s^2} \right) \right]_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-x \frac{\cos sx}{s} + \frac{\sin sx}{s^2} \right)_0^1 \right] + \left[\left(-(2-x) \frac{\cos sx}{s} - \frac{\sin sx}{s^2} \right)_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\left(\frac{-\cos s}{s} + \frac{\sin s}{s^2} \right) - (-0+0) \right] + \left[\left(-0 - \frac{\sin 2s}{s^2} \right) - \left(\frac{-\cos s}{s} - \frac{\sin s}{s^2} \right) \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s - \sin 2s}{s^2} \right]$$

Example 4.3.c(4): Find the Fourier sine transform of $\frac{1}{x}$.

[A.U. CBT Dec. 2008] [A.U N/D 2009] [A.U A/M 2015 R-13]

Solution : We know that, **[A.U N/D 2016 R-13, A/M 2017 R-13]**

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$$

[A.U N/D 2018 R-8]

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx \, dx$$

Let $sx = \theta$ | $x \to 0 \Rightarrow \theta \to 0$

$s\,dx = d\theta$ | $x \to \infty \Rightarrow \theta \to \infty$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{s}{\theta} \right) \sin\theta \frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta} d\theta$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} \right] = \sqrt{\frac{\pi}{2}} \qquad \left[\because \int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2} \right]$$

Example 4.3.c(5): Find the Fourier sine transform of $3e^{-5x} + 5e^{-2x}$.

Solution : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$

$$F_s[3e^{-5x} + 5e^{-2x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty (3e^{-5x} + 5e^{-2x}) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty 3e^{-5x} \sin sx \, dx + \int_0^\infty 5e^{-2x} \sin sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[3\int_0^\infty e^{-5x} \sin sx \, dx + 5\int_0^\infty e^{-2x} \sin sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[3\left[\frac{s}{s^2+25}\right] + 5\left[\frac{s}{s^2+4}\right] \right]$$

$$\left[\because \text{Formula} : \int e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \, s \left[\frac{3}{s^2+25} + \frac{5}{s^2+4}\right]$$

Example 4.3.c(6): Find the Fourier sine transforms of $f(x) = e^{-ax}$.

[A.U. N/D 2007] [A.U N/D 2009 (Trichy)] [A.U N/D 2014 R-08] [A.U A/M 2019, R-13]

Solution : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

$$\boxed{\text{Formula} : \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}}$$

Example 4.3.c(7): Find the Fourier sine transform of the function

$$f(x) = \frac{e^{-ax}}{x} \text{ and hence find } F_s\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$$

[A.U. N/D 2006] [A.U. T. Chennai N/D 2011] [A.U N/D 2016 R-13] [A.U N/D 2018, R-17]

Solution : We know that,

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$$

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

Diff. w.r.to s on both sides,

$$\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x} \sin sx\right) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x \frac{e^{-ax}}{x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right] \qquad \left[\because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right]$$

Here $b = s$

Integrating w.r.to 's', we get

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int \frac{a}{s^2+a^2} ds$$

$$= \sqrt{\frac{2}{\pi}} \, a \cdot \frac{1}{a} \cdot \tan^{-1}\frac{s}{a} = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{s}{a}$$

Note : Put $a = 0$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

Similarly, $F_s\left[\frac{e^{-bx}}{x}\right] = \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{b}$

$$\therefore F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_s\left[\frac{e^{-ax}}{x}\right] - F_s\left[\frac{e^{-bx}}{x}\right]$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{a} - \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{b}$$

$$= \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\frac{s}{a} - \tan^{-1}\frac{s}{b}\right]$$

Example 4.3.c(8): Find the Fourier sine transform of x^{n-1}. Deduce that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform.

Solution : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx}$

$$F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \qquad \left[\because \int_0^\infty x^{n-1}\sin sx\,dx = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}\right]$$

Taking $n = \frac{1}{2}$, we get

$$F_s[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\frac{\Gamma\frac{1}{2}}{s^{1/2}}\sin\frac{\pi}{4}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

Hence, $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform.

...mple 4.3.c(9): Find the Fourier sine transform of $\frac{x}{a^2+x^2}$.

...tion : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{a^2+x^2}\sin sx\,dx}$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\frac{\pi}{2}e^{-as} = \sqrt{\frac{\pi}{2}}e^{-as}$$

...(d) Problems based on Fourier sine transform and its inversion formula.

Formula :

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds$$

...mple 4.3.d(1): Find Fourier sine transform of e^{-ax}, $a > 0$ and deduce that $\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,dx = \frac{\pi}{2}e^{-ax}$.

[A.U. M/J 2007, A.U Tvli N/D 2009, A.U N/D 2010]
[A.U M/J 2016R-13][A.U N/D 2019, R-17]

...tion : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx}$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right] \qquad \left[\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right]$$

...plying the inversion formula, we get

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2} \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{a^2+s^2} \sin sx\, ds$$

$$\therefore \int_0^\infty \frac{s}{a^2+s^2} \sin sx\, ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-ax},\ a > 0$$

Example 4.3.d(2): Find the Fourier sine transform of e^{-x}, Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m},\ m > 0.$

Solution : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx}$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{1+s^2}\right]$$

Formula : $\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}$

By inversion formula, we get

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[e^{-x}] \sin sx\, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{1+s^2}\right) \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{1+s^2}\, ds$$

$$\int_0^\infty \frac{s \sin sx}{1+s^2}\, ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-x}$$

Changing 'x' to 'm' and 's' to 'x', we get

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}$$

Example 4.3.d(3): Find $f(x)$ if its sine transform is $\frac{e^{-sa}}{s}$. Hence, find $F_s^{-1}\left[\frac{1}{s}\right]$. **[A.U N/D 2016 R-13]**

[A.U Trichy N/D 2010, Tvli M/J 2011][A.U N/D 2013]

Solution : Given $F_s[f(x)] = \frac{e^{-sa}}{s}$

By inversion formula, we get

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx\, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-sa}}{s} \sin sx\, ds$$

$$\frac{d}{dx} f(x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty \frac{e^{-sa}}{s} \sin sx\, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial x} \left[\frac{e^{-sa}}{s} \sin sx\right] ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\frac{e^{-sa}}{s}\, s \cos sx\right] ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-sa} \cos sx\, ds$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2}$$

Formula : $\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$

Integrating w.r.to x, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2+x^2} dx$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)$$

Put $a = 0$, we get

$$F_s^{-1}\left[\frac{1}{s}\right] = \sqrt{\frac{2}{\pi}} \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

Example 4.3.d(4): **Solve the integral equation**

$$\int_0^\infty f(x) \sin sx\, dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$$

[A.U M/J 2014]

[A.U N/D 2015 R-8]

Solution : Let $F_s[f(x)] = \int_0^\infty f(x) \sin sx\, dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$

By inversion formula, we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty F_s[f(x)] \sin sx\, ds}$$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 \sin sx\, ds + \int_1^2 2\sin sx\, ds\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos sx}{x}\right)_0^1 + 2\left(\frac{-\cos sx}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[\left[\left(-\frac{\cos x}{x}\right) - \left(-\frac{1}{x}\right)\right] + 2\left[\left(\frac{-\cos 2x}{x}\right) - \left(\frac{-\cos x}{x}\right)\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

$$= \frac{2}{\pi}\left[\frac{1 - 2\cos 2x + \cos x}{x}\right]$$

$$= \frac{2 + 2\cos x - 4\cos 2x}{\pi x}, \ x > 0$$

4.3.(e) Problems based on properties of F.C.T AND F.S.T.

Example 4.3.e(1): **(i) Find the Fourier cosine transform of** $\frac{1}{1+x^2}$

(ii) Find the Fourier sine transform of $\frac{x}{1+x^2}$.

[A.U N/D 2016 R-8]

Solution : We know that, $\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx}$

$$F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{1+x^2} dx \quad \text{... (A)}$$

$$\text{Let } I = \int_0^\infty \frac{\cos sx}{1+x^2} dx \quad \text{... (1)}$$

$$\frac{dI}{ds} = \frac{d}{ds}\int_0^\infty \frac{\cos sx}{1+x^2} dx$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left(\frac{\cos sx}{1+x^2}\right) dx = \int_0^\infty \frac{1}{1+x^2}\frac{\partial}{\partial s}[\cos sx]\, dx$$

$$\frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{1+x^2} dx \quad \text{... (2)}$$

$$= -\int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx = -\int_0^\infty \frac{[(1+x^2)-1]\sin sx}{x(1+x^2)} dx$$

$$= -\int_0^\infty \frac{\sin sx}{x} dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \quad \text{... (3)}$$

$$\left[\because \int_0^\infty \frac{\sin sx}{x} dx = \frac{\pi}{2}\right]$$

$$\frac{d^2 I}{ds^2} = 0 + \frac{d}{ds}\left[\int_0^\infty \frac{\sin sx}{x(1+x^2)} dx\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[\frac{\sin sx}{x(1+x^2)}\right] dx = \int_0^\infty \frac{1}{x(1+x^2)} \frac{\partial}{\partial s}[\sin sx]\, dx$$

Now, $$\frac{d^2 I}{ds^2} = \int_0^\infty \frac{x \cos sx}{x(1+x^2)} dx = \int_0^\infty \frac{\cos sx}{1+x^2} dx = I$$

$$\frac{d^2 I}{ds^2} - I = 0$$

$$(D^2 - 1) I = 0$$

A.E is $m^2 - 1 = 0$

$$m = 1, -1$$

$$\therefore I = Ae^{-s} + Be^{s} \qquad \dots (4)$$

$$\frac{dI}{ds} = -Ae^{-s} + Be^{s} \qquad \dots (5)$$

$$(4) \Rightarrow Ae^{-s} + Be^{s} = \int_0^\infty \frac{\cos sx}{1+x^2} dx \quad \text{by (1)}$$

put $s = 0$, $$A + B = \int_0^\infty \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^\infty$$

$$A + B = \frac{\pi}{2} \qquad \dots (6)$$

$$(5) \Rightarrow -Ae^{-s} + Be^{s} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \quad \text{by (3)}$$

put $s = 0$ $$-A + B = -\frac{\pi}{2} \qquad \dots (7)$$

Solving (6) & (7), we get

$$A = \frac{\pi}{2}, \; B = 0$$

Substitute in (4), we get

$$I = \frac{\pi}{2} e^{-s} \qquad \dots (8)$$

$$\frac{dI}{ds} = -\frac{\pi}{2} e^{-s} \qquad \dots (9)$$

$$F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2} e^{-s}\right] = \sqrt{\frac{\pi}{2}}\, e^{-s} \quad \text{by (A)}$$

$$F_s\left[\frac{x}{1+x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin sx}{1+x^2} dx$$

$$= -\sqrt{\frac{2}{\pi}} \frac{dI}{ds} \quad \text{by (2)}$$

$$= -\sqrt{\frac{2}{\pi}}\left[-\frac{\pi}{2} e^{-s}\right] \quad \text{by (9)}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-s}$$

Example 4.3.e(2): Find the Fourier sine and cosine transformations of xe^{-ax}.

Solution : (i) We know that, $\boxed{F_s[x f(x)] = \frac{-d}{ds} F_c[f(x)]}$

$$F_s[xe^{-ax}] = -\frac{d}{ds} F_c[e^{-ax}] \qquad [\because f(x) = e^{-ax}]$$

$$= \frac{-d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right] \qquad \left[\because F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\, a \frac{d}{ds}\left[\frac{1}{s^2+a^2}\right]$$

$$= -a\sqrt{\frac{2}{\pi}}\left[\frac{-2s}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$$

(ii) We know that, $\boxed{F_c[x f(x)] = \frac{-d}{ds} F_s[f(x)].}$

Solution : $$F_c[xe^{-ax}] = -\frac{d}{ds} F_s[e^{-ax}]$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}$$

$$= -\sqrt{\frac{2}{\pi}}\frac{a^2-s^2}{(s^2+a^2)^2} = \sqrt{\frac{2}{\pi}}\frac{s^2-a^2}{(s^2+a^2)^2}$$

Example 4.3.e(3): Find Fourier cosine transform of $e^{-a^2x^2}$ and hence find $F_s\,[xe^{-a^2x^2}]$.

[A.U. N/D 2006] [A.U N/D 2018 R-13]

Solution : See Example 4.3 a(13), Page No. 4.97

$$F_c\,[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}}e^{-s^2/4a^2}$$

Now, $F_s\,[xe^{-a^2x^2}] = \frac{-d}{ds}F_c\,[e^{-a^2x^2}]$

$$= \frac{-d}{ds}\left[\frac{1}{a\sqrt{2}}e^{-s^2/4a^2}\right]$$

$$= -\frac{1}{a\sqrt{2}}e^{-s^2/4a^2}\left[\frac{-2s}{4a^2}\right]$$

$$= \frac{s}{2\sqrt{2}\,a^3}e^{-s^2/4a^2}$$

Example 4.3.e(4): Find the Fourier sine transform of e^{-ax} and hence find the Fourier cosine transform of xe^{-ax}.

[A.U N/D 2011]

Solution : We know that,

$$\boxed{F_s\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx}$$

$$F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right] \qquad \left[\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right]$$

To find : $F_c\,[xe^{-ax}]$

$$F_c\,[xe^{-ax}] = \frac{d}{ds}F_s\,[e^{-ax}]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

III (f) Problems based on Parseval's identity in F.S.T and F.C.T

Example 4.3.f(1): Evaluate : $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.

[A.U. April, 2001] [A.U. N/D 2005, M/J 2006, A.U Trichy N/D 2010]

[A.U.T Tvli N/D 2011] [A.U N/D 2014 R-13] [A.U N/D 2015 R-8]

[A.U A/M 2019 R-17, R-8] [A.U N/D 2018 R-17]

[A.U N/D 2018 R-13, R-8] [A.U A/M 2019 R-13]

Solution :

Parseval's identity is

$$\boxed{\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F_c\,[f(x)]\,F_c\,[g(x)]\,ds} \quad \ldots (1)$$

Let $f(x) = e^{-ax}$	Let $g(x) = e^{-bx}$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$ $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$ $= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$	$F_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos sx\, dx$ $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \cos sx\, dx$ $= \sqrt{\frac{2}{\pi}} \left[\frac{b}{s^2+b^2}\right]$

$$f(x)\, g(x) = e^{-ax} e^{-bx} = e^{-(a+b)x}$$

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty e^{-(a+b)x}\, dx = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = 0 - \left[\frac{1}{-(a+b)}\right]$$

$$= \frac{1}{a+b}$$

$$F_c[f(x)]\, F_c[g(x)] = \left[\sqrt{\frac{2}{\pi}}\, \frac{a}{s^2+a^2}\right] \left[\sqrt{\frac{2}{\pi}}\, \frac{b}{s^2+b^2}\right]$$

$$= \frac{2}{\pi}\, \frac{ab}{(s^2+a^2)(s^2+b^2)}$$

$$(1) \Rightarrow \frac{1}{a+b} = \frac{2ab}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}\, ds$$

$$\Rightarrow \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2ab(a+b)}$$

$$\Rightarrow \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2ab(a+b)} \quad [\because s \text{ is a dummy variable}]$$

Note :

(1) **Evaluate :** $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$

[A.U Dec. 2010, A/M 2009, Dec. 2009, M/J 2006, Dec. 2005]

Solution : Step 1 : **[A.U A/M 2017 R-13]**

Prove that : $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

Step 2 : Here, $a = 1, b = 2$

$$\therefore \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{2(1)(2)(1+2)} = \frac{\pi}{12}$$

(2) **Evaluate :** $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+9)}$

Solution :

Step 1 : Prove that : $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

Step 2 : Here, $a = 1,\ b = 3$

$$\therefore \int_0^\infty \frac{dx}{(x^2+1)(x^2+9)} = \frac{\pi}{2(1)(3)(1+3)} = \frac{\pi}{24}$$

Example 4.3.f(2): Using Fourier sine transform, prove that

$$\int_0^\infty \frac{\lambda^2\, d\lambda}{(a^2+\lambda^2)(b^2+\lambda^2)} = \frac{\pi}{2(a+b)}$$

Solution : Parseval's identity is **[A.U N/D 2015 R-13]**

$$\boxed{\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_s[f(x)]\, F_s[g(x)]\, ds} \quad \ldots (1)$$

Let $f(x)=e^{-ax}$	Let $g(x) = e^{-bx}$
$F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$	$F_s[g(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\sin sx\,dx$
$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$	$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin sx\,dx$
$=\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]$	$=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+b^2}$

$$f(x)g(x) = e^{-ax}e^{-bx} = e^{-(a+b)x}$$

$$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty e^{-(a+b)x}\,dx = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= 0-\left[\frac{1}{-(a+b)}\right] = \frac{1}{a+b}$$

$$F_s[f(x)]F_s[g(x)] = \frac{2}{\pi}\frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

$$\therefore (1) \Rightarrow \frac{1}{a+b} = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds$$

$$\Rightarrow \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds = \frac{\pi}{2(a+b)}$$

$$\Rightarrow \int_0^\infty \frac{\lambda^2}{(\lambda^2+a^2)(\lambda^2+b^2)}ds = \frac{\pi}{2(a+b)} \quad [\because s \text{ is a dummy variable}]$$

Note : x^2 in Nr and a & b in Dr. use F.S.T formula

Note : Evaluate : $\int_0^\infty \frac{x^2\,dx}{(x^2+4)(x^2+9)}$ [A.U A/M 2017 R-8]

Solution : Step 1 : Prove that : $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2(a+b)}$

Step 2 : Here, $a=2,\ b=3$

$$\therefore \int_0^\infty \frac{x^2\,dx}{(x^2+4)(x^2+9)} = \frac{\pi}{2(2+3)} = \frac{\pi}{10}$$

Example 4.3.f(3) : Using transform methods, evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

[A.U Tvli M/J 2011] [A.U M/J 2013, N/D 2013] [A.U M/J 2014] [A.U N/D 2019, R-17]

Solution : Parseval's identity is

$$\boxed{\int_0^\infty |f(x)|^2dx = \int_0^\infty |F_c[f(x)]|^2ds} \quad \dots (1)$$

Let $f(x)=e^{-ax}$	$[f(x)]^2 = (e^{-ax})^2 = e^{-2ax}$
$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$	$\lvert F_c[f(x)]\rvert^2 = \frac{2}{\pi}\frac{a^2}{(s^2+a^2)^2}$

$$\int_0^\infty |f(x)|^2dx = \int_0^\infty e^{-2ax}dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = (0)-\left(\frac{1}{-2a}\right) = \frac{1}{2a}$$

$$(1) \Rightarrow \frac{1}{2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)^2}ds$$

$$\Rightarrow \int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3} \quad [\because s \text{ is a dummy variable}]$$

Note : Evaluate : $\int_0^\infty \frac{dx}{(x^2+1)^2}$

Step 1 : Prove that : $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$

Step 2 : Here, $a = 1$

$$\therefore \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4(1)^3} = \frac{\pi}{4}$$

Example 4.3.f(4): Using transform methods, evaluate $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$ where a > 0. [A.U CBT Dec. 2009] [A.U M/J 2014]

[A.U A/M 2019 R-17]

Solution : Parseval's identity is

$$\boxed{\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s[f(x)]|^2 ds} \quad \ldots (1)$$

Let $f(x) = e^{-ax}$	$[f(x)]^2 = \left[e^{-ax}\right]^2 = e^{-2ax}$
$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$ $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$ $= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$	$\|F_s[f(x)]\|^2 = \frac{2}{\pi} \frac{s^2}{(s^2+a^2)^2}$

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty e^{-2ax} dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = (0) - \left(\frac{-1}{2a}\right) = \frac{1}{2a}$$

$$(1) \Rightarrow \frac{1}{2a} = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$$

$$\Rightarrow \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds = \frac{\pi}{4a}$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a} \qquad [\because s \text{ is a dummy variable}]$$

Note :

Evaluate : $\int_0^\infty \frac{x^2}{(x^4+4)^2} dx$

Solution : Step 1 : Prove that : $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$

Step 2 : Here, $a = 2$

$$\therefore \int_0^\infty \frac{x^2}{(x^2+4)^2} dx = \frac{\pi}{4(2)} = \frac{\pi}{8}$$

Example 4.3.f(5): Using Parseval's identity of the Fourier cosine transform, Evaluate $\int_0^\infty \frac{\sin ax}{x(a^2+x^2)} dx$, if

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases} \text{ and}$$

$$g(x) = e^{-a|x|},\ a > 0.$$

Solution :

Parseval's identity is

$$\boxed{\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_c[f(x)]\, F_c[g(x)]\, ds} \quad \ldots (1)$$

$f(x) = 1, \ \lvert x \rvert < a$	$g(x) = e^{-a\lvert x\rvert}, \ a > 0$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$	$F_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos sx \, dx$
$= \sqrt{\frac{2}{\pi}} \int_0^a \cos sx \, dx$	$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$
$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a$	$[\because \lvert x \rvert = x \text{ in } (0, \infty)]$
$= \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$	$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2 + a^2}\right)$

$$f(x)\, g(x) = (1)\, e^{-ax} = e^{-ax} \text{ in } (0, a)$$

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^a e^{-ax}\, dx = \left[\frac{e^{-ax}}{-a}\right]_0^a = \left(\frac{e^{-a^2}}{-a}\right) - \left(\frac{1}{-a}\right)$$

$$= \frac{1}{a}\left[1 - e^{-a^2}\right]$$

$$F_c[f(x)]\, F_c[g(x)] = \frac{2}{\pi} \frac{a \sin sa}{s\,(s^2 + a^2)}$$

$$(1) \Rightarrow \frac{1}{a}\left[1 - e^{-a^2}\right] = \frac{2a}{\pi} \int_0^\infty \frac{\sin sa}{s\,(s^2 + a^2)}\, ds$$

$$\Rightarrow \int_0^\infty \frac{\sin sa}{s\,(s^2 + a^2)}\, ds = \frac{\pi}{2a^2}\left[1 - e^{-a^2}\right]$$

$$\Rightarrow \int_0^\infty \frac{\sin ax}{x\,(x^2 + a^2)}\, dx = \frac{\pi}{2a^2}\left[1 - e^{-a^2}\right] \quad [\because s \text{ is a dummy variable}]$$

EXERCISE 4.1 [Fourier integral theorem]

Using Fourier integral representation, show that

$$\int_0^\infty \frac{\sin \pi\lambda \sin \lambda x}{1 - \lambda^2}\, d\lambda = \begin{cases} \frac{\pi}{2} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

Solve the integral equation $\int_0^\infty f(x) \cos sx\, dx = e^{-s}$. Hence deduce that $\int_0^\infty \frac{\cos sx}{1 + s^2}\, ds = \frac{\pi}{2} e^{-x}$ [**Ans.** $f(x) = \frac{2}{\pi} \frac{1}{1 + x^2}$]

Solve for $f(x)$ from the integral equation

$$\int_0^\infty f(x) \sin sx\, dx = \begin{cases} 1 & 0 \le s < 1 \\ 2 & 1 \le s < 2 \\ 0 & s \ge 2 \end{cases}$$

[A.U M/J 2014]

[Ans. $f(x) = \frac{2}{\pi x}\left[1 + \cos x - 2 \cos 2x\right]$]

Solve the integral equation $\int_0^\infty f(x) \cos \lambda x\, dx = \begin{cases} 1 - \lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$.

Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$ [**Ans.** $f(x) = \frac{2}{\pi}\left[\frac{1 - \cos x}{x^2}\right]$]

EXERCISE 4.2 [Fourier Transform]

Find the Fourier transform of the function.

1. $f(x) = \begin{cases} 1 + \frac{x}{a}, & -a < x < 0 \\ 1 - \frac{x}{a}, & 0 < x < a \\ 0, & \textit{otherwise} \end{cases}$ **Ans.** $\frac{4 \sin^2\left(\frac{as}{2}\right)}{as^2}$

2. $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \textit{otherwise} \end{cases}$ **Ans.** $\frac{1 + e^{is\pi}}{\sqrt{2\pi}\,(1 - s^2)}$

3. $f(x) = x e^{-x},\ 0 \le x \le \infty$ Ans. $\frac{1}{\sqrt{2\pi}}\left[\frac{1}{(1-is)^2}\right]$

4. $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2\in}, & |x| \le \in \\ 0, & |x| > \in \end{cases}$ Ans. $\frac{\sin s\in}{s\in}$

5. $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 0 & \text{Otherwise} \end{cases}$

[Ans. $\left(\frac{\cos s + s\sin s - 1}{\sqrt{2\pi}\, s^2}\right) + i\left(\frac{\sin s - s\cos s}{\sqrt{2\pi}\, s^2}\right)$]

6. $f(x) = e^{-|x|}$ and deduce that $\int_0^\infty \left(\frac{\cos xt}{1+t^2}\right) dt = \frac{\pi e^{-|x|}}{2}$

[Ans. $\sqrt{\frac{2}{4}}\left(\frac{1}{1+s^2}\right)$]

7. $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{l} & \text{if } -l < x < l \\ 0 & \text{if Otherwise} \end{cases}$ [Ans. $\frac{\sqrt{2}\,\pi \sin sl}{sl}$]

8. $f(x) = e^{-a|x|}$ and hence deduce that

(i) $\int_0^\infty \frac{\cos sx}{a^2+t^2} dt = \frac{\pi}{2a} e^{-a|x|}$ (ii) $F[x e^{-a|x|}] = \frac{2as}{(s^2+a^2)^2}$

II. If the Fourier transform of $f(x)$ is $\sqrt{\frac{2}{\pi}}\,\frac{\sin as}{s}$,

find $F\left[F(x)\left(1+\cos\frac{\pi x}{a}\right)\right]$ Ans. $\sqrt{\frac{2}{\pi}}\,\sin as\left[\frac{1}{s}+\frac{2a^2 s}{\pi^2 - a^2 s^2}\right]$

EXERCISE 4.3 [FCT and FST]

I. 1. Find the Fourier cosine transform of e^{-4x}. Deduce that

$\int_0^\infty \frac{\cos 2x}{x^2+16} dx = \frac{\pi}{8} e^{-8}$ and $\int_0^\infty \frac{s\sin 2x}{x^2+16} dx = \frac{\pi}{2} e^{-8}$

[Ans. $F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}}\left[\frac{4}{s^2+16}\right]$]

2. Find the Fourier sine transform of $x e^{-x^2/2}$ [Ans. $s e^{-s^2/2}$]

3. Find $f(x)$ if its cosine transform is

$F_c(s) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - \frac{s}{2}\right) & \text{if } 0 < s < 2a \\ 0 & \text{if } s \ge 2a \end{cases}$ [Ans. $\frac{\sin^2 ax}{\pi x^2}$]

Find the Fourier Cosine Transform of

1. $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & x > 1 \end{cases}$ Ans. $F_c(s) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin s}{s}\right)$

2. $f(x) = \frac{e^{-ax} - e^{-bx}}{x}$ Ans. $\frac{1}{\sqrt{2\pi}} \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

3. $f(x) = e^{-4x}$ and hence deduce that

(i) $\int_0^\infty \frac{\cos 2x}{x^2+16} dx = \frac{\pi}{8} e^{-8}$ (ii) $\int_0^\infty \frac{x\sin 2x}{x^2+16} dx = \frac{\pi}{2} e^{-8}$

Ans. $F(s) = \sqrt{\frac{2}{\pi}}\ \frac{4}{s^2+16}$

Find the Fourier Sine transform of

1. $\frac{x}{x^2+a^2}$ Ans. $\sqrt{\frac{\pi}{2}}\ e^{-as}$

2. $f(x) = \begin{cases} 0 & \text{if } 0 < x < a \\ x & \text{if } a \le x \le b \\ 0 & \text{if } x > b \end{cases}$

Ans. $\sqrt{\frac{2}{\pi}}\left[\frac{a\cos sa - b\cos sb}{s} + \frac{\sin sb - \sin sa}{s^2}\right]$

3. $f(x) = e^{-|x|}$ Ans. $\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+1}\right)$

4. $f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x \ge a \end{cases}$

Ans. $\frac{1}{\sqrt{2\pi}}\left[\frac{2s}{s^2-1} - \left(\frac{\cos(s+1)a}{s+1} + \frac{\cos(s-1)a}{s-1}\right)\right]$

IV. Find the Fourier cosine and sine transforms of

1. $5e^{-2x} + 2e^{-5x}$

Ans. $F_c(s) = 10\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+4} + \frac{1}{s^2+25}\right]$

$F_s(s) = \sqrt{\frac{2}{\pi}}\left[\frac{5s}{s^2+4} + \frac{2s}{s^2+25}\right]$

2. $\cosh x - \sinh x$

Ans. $F_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+1}\right)$

$F_c(s) = \left(\frac{1}{s^2+1}\right)$

******

IMPORTANT QUESTIONS

Find the F.T of $f(x)$ and show that the following by using inversion and parseval's identity

	Function	F.T	F.T and its inversion	Parseval's identity
1.	$f(x)=\begin{cases}1, & \lvert x\rvert < a\\ 0, & \lvert x\rvert \ge a\end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin sa}{s}\right)$	$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$
2.	$f(x)=\begin{cases}1, & \lvert x\rvert < 1\\ 0, & \lvert x\rvert \ge 1\end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin s}{s}\right)$	$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$
3.	$f(x)=\begin{cases}1, & \lvert x\rvert < 2\\ 0, & \lvert x\rvert \ge 2\end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin 2s}{s}\right)$	$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$
4.	$f(x)=\begin{cases}1-x^2, & \lvert x\rvert<1\\ 0, & \lvert x\rvert > 1\end{cases}$	$F(s)=\frac{4}{\sqrt{2\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)$	$\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{16}$	$\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6}\,dx = \frac{\pi}{15}$
5.	$f(x)=\begin{cases}1-\lvert x\rvert, & \lvert x\rvert<1\\ 0, & \lvert x\rvert > 1\end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)$	$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

	Function	F.T	F.T and its inversion	Parseval's identity
6.	$f(x)=\begin{cases} a^2-x^2, & \lvert x\rvert<a \\ 0, & \lvert x\rvert>a>0 \end{cases}$	$F(s)=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - s\cos s}{s^3}\right)$	$\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt=\frac{\pi}{4}$	$\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt=\frac{\pi}{15}$
7.	$f(x)\begin{cases} a-\lvert x\rvert, & \lvert x\rvert<a \\ 0, & \lvert a\rvert>a>0 \end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right]$	$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$
8.	$f(x)=\begin{cases} x, & \lvert x\rvert<a \\ 0, & \lvert x\rvert>a \end{cases}$	$F(s)=i\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\sin sa}{s^2}\right]$		
9.	$f(x)=\begin{cases} x^2, & \lvert x\rvert\le a \\ 0, & \lvert x\rvert>a \end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{a^2s^2\sin as + 2as\cos as - 2\sin as}{s^3}\right]$		
10.	$f(x)=e^{-a\lvert x\rvert},\ a>0$	$F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$		$\int_0^\infty \frac{dx}{(x^2+a^2)^2}=\frac{\pi}{4a^3}$
11.	$f(x)=e^{-\lvert x\rvert}$	$F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right]$		$\int_0^\infty \frac{dx}{(x^2+1)^2}=\frac{\pi}{4}$

	Function	F.T	F.T and its inversion	Parseval's identity
12.	$f(x)=\begin{cases} \cos x, & \lvert x\rvert<\frac{\pi}{2} \\ 0, & \lvert x\rvert>\frac{\pi}{2} \end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{\cos s\frac{\pi}{2}}{1-s^2}\right]$		$\int_0^\infty \frac{\cos^2\frac{\pi x}{2}}{(1-x^2)^2}\,dx=\frac{\pi^2}{8}$
13.	$f(x)=\frac{\sin ax}{x}$	$F(s)=\sqrt{\frac{\pi}{2}}$		$\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2}\,dx=\pi a$
14.	$f(x)=e^{-x^2/2}$	$F(s)=e^{-s^2/2}$		
15.	$f(x)=e^{-a^2x^2},\ a>0$	$F(s)=\frac{e^{-s^2/4a^2}}{a\sqrt{2}}$		
16.	$f(x)=\frac{1}{\sqrt{\lvert x\rvert}}$	$F(s)=\frac{1}{\sqrt{s}}$		

PART-A QUESTIONS AND ANSWERS

1. **State Fourier integral theorem.**

[A.U. Ap. 1996, A/M 2005, A/M. 2008, N/D 2008, AU Tvli. N/D 2010] [A.U Tvli M/J 2011, A.U CBT N/D 2011] [A.U M/J 2016 R-13] [A.U N/D 2018-A, R-17] [A.U N/D 2018 R-13]

See Page 4.1

2. **Show that f(x) = 1, 0 < x < ∞ cannot be represented by a Fourier integral.**

[A.U. April/May 2003] [A.U M/J 2014]

Sol : $\int_0^\infty |f(x)|\,dx = \int_0^\infty 1\,dx = [x]_0^\infty = \infty$

and this value tends to ∞ as $x \to \infty$

i.e., $\int_0^\infty |f(x)|\,dx$ is not convergent.

Hence $f(x) = 1$ cannot be represented by a Fourier integral.

3. **Define Fourier Transform pair. (OR)**

Define Fourier Transform and its inverse transform.

[A.U March 1996] [A.U. A/M. 2001, N/D 2007] [A.U. N/D 2010] [A.U Trichy N/D 2010, CBT N/D 2010,] [A.U. N/D 2011, A.U.T. Tvli. N/D 2011, A.U.T Chennai N/D 2011]

See Page 4.25

4. **What is the Fourier cosine transform of a function ?** **[A.U. O/N 1996]**

See Page 4.78

5. **Find the Fourier cosine transform of** $f(x)=\begin{cases}\cos x & \text{if } 0<x<a\\ 0 & \text{if } x\geq a\end{cases}$

See Page 4.87

6. **Find the Fourier cosine transform of** e^{-ax}, $a > 0$

See Page 4.89

7. **Find Fourier Cosine transform of** e^{-x} **[A.U. Nov./Dec. 2004] [A.U.T CBT N/D 2011]**

Solution : We know that $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\,dx$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{1+s^2}\right]$$

Formula : $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$

Find the Fourier sine transform of e^{-3x}.

[Anna University, Nov/Dec., 1996] [A.U M/J 2013]

Solution : $F_s[e^{-3x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-3x}\sin sx\,dx$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+3^2}\right]$$

Formula : $F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin sx\,dx$

Find the Fourier Sine-transform of $3e^{-2x}$

[Anna University, March, 1996]

Solution : Let $f(x) = 3e^{-2x}$

We know that,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty 3e^{-2x}\sin sx\,dx$$

$$= 3\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x}\sin sx\,dx$$

$$= 3\sqrt{\frac{2}{\pi}} \left[\frac{e^{-2x}}{4+s^2}(-2\sin sx - s\cos sx)\right]_0^\infty$$

$$= 3\sqrt{\frac{2}{\pi}}\left[[0]-\left[\frac{1}{4+s^2}(-s)\right]\right] = 3\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+4}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{3s}{s^2+4}\right]$$

10. Find the Fourier Sine transform of $\frac{1}{x}$

See Page 4.107 **[A.U. A/M 2005, A.U.T Chennai N/D 2011] [A.U M/J 2014] [A.U A/M 2017 R-13]**

11. Define Fourier sine transform and its inversion formula.

See Page 4.80 **[A.U. April/May, 2004 PTMA]**

12. Find the Fourier sine transform of $f(x) = e^{-ax}$, $a > 0$ and hence deduce that $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-a}$.

See Page 4.112 **[A.U, March, 1998, 1999 & 2000]**

13. If Fourier Transform of $f(x) = F(s)$, then what is Fourier Transform of $f(ax)$? **[A.U. N/D 1996] [A.U. M/J 2006]**

See Page 4.27

14. If F denotes the Fourier Transform operator, then show that

$$F\left\{x^k f^{(m)}(x)\right\} = (-1)^{k+m}\frac{d^k}{ds^k}s^m F\left\{f(x)\right\}$$

[Anna University, March, 1996]

Solution :

Property : $F[x^n f(x)] = (-i)^n \frac{d^n F(s)}{ds^n}$

Proof : We have, $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

Differentiating both sides n times w.r.t. 's', we get

$$\frac{d^n}{ds^n}F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,(ix)^n\, e^{isx}\, dx$$

$$= \frac{i^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, x^n\, dx = i^n F[x^n f(x)]$$

Hence, $F[x^n f(x)] = \frac{1}{i^n}\frac{d^n F(s)}{ds^n} = (-i)^n \frac{d^n F(s)}{ds^n}$

...arly, we can prove

(i) $F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$,

(ii) $F_c[x f(x)] = \frac{d}{ds}[F_s(s)]$

...arly, we get

$$F\left\{x^k f^{(m)}(x)\right\} = (-1)^{k+m}\frac{d^k}{ds^k}s^m F\left\{f(x)\right\}$$

... If Fourier transform of $f(x)$ is $F(s)$, prove that the Fourier transform of $f(x) \cos ax$ is $\frac{1}{2}[F(s-a) + F(s+a)]$

See Page 4.29 **[A.U, April, 2001]**

... Prove that $F_c[f(x) \cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$ where F_c denotes the Fourier cosine transform $f(x)$.

See Page 4.82 **[A.U April, May 2001] [A.U N/D 2019, R-17]**

... If $F(s)$ is the Fourier transform of $f(x)$, then show that the Fourier transform of $e^{iax} f(x)$ is $F(s+a)$.

See Page 4.28 **[A.U April, 1996] [A.U CBT N/D 2010]**

18. Given that $e^{-x^2/2}$ is self reciprocal under Fourier cosine transform, find (i) Fourier sine transform of $xe^{-x^2/2}$ and (ii) Fourier cosine transform of $x^2 e^{-x^2/2}$

[Anna University, Dec, 1996]

Solution : Given $F_c\left[e^{-x^2/2}\right] = e^{-s^2/2}$

$$F_s\left[xe^{-x^2/2}\right] = -\frac{d}{ds} F_c\left[xe^{-x^2/2}\right]$$

$$= -\frac{d}{ds}\left[e^{-s^2/2}\right] = -e^{-s^2/2}[-s] = se^{-s^2/2}$$

$$F_c\left[x^2 e^{-x^2/2}\right] = \frac{d}{ds} F_s\left[xe^{-x^2/2}\right]$$

$$= \frac{d}{ds}\left[se^{-s^2/2}\right] = \left[se^{-s^2/2}(-s) + e^{-s^2/2}\right]$$

$$= -s^2 e^{-s^2/2} + e^{-s^2/2} = (1-s^2)\, e^{-s^2/2}$$

19. If F (s) is the Fourier transform of f (x), then find the Fourier transform of f (x – a).

[A.U. N/D 2005, N/D 2006, A.U. CBT Dec. 2008, N/D 2005]

See Page 4.28 **[A.U.T Tvli. N/D 2011]**

20. State the convolution theorem for Fourier transforms.

[A.U. April/May 2003, May 2000 PT] [A.U. A/M 2008]
[A.U N/D 2018, R-17, R-8, A/M 2019 R17, R-8, N/D 2019, R-17]

Solution : Convolution theorem (or) Faltung theorem :

If F(s) and G(s) are the Fourier transform of $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform.

$$F\,[f(x) * g(x)] = F(S)\; G(S)$$

$$= F\,[f(x)]\; F\,[g(x)]$$

...State the Fourier transform of the derivatives of a function.

[A.U. N/D 2005]

See Page 4.30, Q.No. 6. (i) and (ii)

...Find the Fourier sine transform of $f(x) = e^{-x}$. [A.U. M/J 2006]

...tion : We know that $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{1+s^2}\right] \quad \left[\because \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}\right]$$

...Give a function which is self reciprocal under Fourier sine and cosine transforms.

[A.U. CBT Dec. 2008]

...tion : $\frac{1}{\sqrt{x}}$

...State the modulation theorem in Fourier Transform.

See Page 4.29 **[A.U CBT Dec. 2008]**

...State the Parseval's identity on Fourier Transform.

See Page 4.34 **[A.U CBT Dec. 2008] [A.U CBT N/D 2010]**
[A.U N/D 2011]

...Define self reciprocal with respect to Fourier transform.

[A.U N/D 2013] [A.U A/M 2019 R-13]

See Page No. 4.38

...Does Fourier Sine transform of $f(x) = k,\ 0 \le x < \infty$, exist? Justify your answer.

[A.U N/D 2018 R-17]

...tion.: Given : $f(x) = k,\ 0 \le x < \infty$

We know that,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s[k] = \sqrt{\frac{2}{\pi}} \int_0^\infty k \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \, k \int_0^\infty \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \, k \left[\frac{-\cos sx}{s}\right]_0^\infty$$

$\therefore$ $F_s[k]$ does not exist.

Since, $\cos \infty$ is undefined.

28. State the condition for the existence of Fourier cosine and sine transform of derivatives. **[A.U A/M 2019 R-17]**

Solution :

Let $f(x)$ be continuous and absolutely integrable on the x-axis,

Let $f'(x)$ be piecewise continuous on finite interval, and

let $f(x) \to 0$ as $x \to \infty$ then $F_c[f'(x)] = -\sqrt{\frac{2}{\pi}} f(0) + s F_s F(s)$

$$= s F_s(s) - \sqrt{\frac{2}{\pi}} f(0)$$

UNIT - V

Z - TRANSFORMS AND DIFFERENCE EQUATIONS

Z-transforms - Elementary properties - Inverse Z - transform (using partial fraction and residues) - Convolution theorem - Formation of difference equations - Solution of difference equations using Z-transform.

Introduction

The Z - transform plays the same role for discrete systems as the Laplace transform does for continuous systems.

Applications of Z - transform **[A.U N/D 2018 R-17]**

Communication is one of the fields whose development is based on discrete analysis. Difference equations are also based on discrete system and their solutions and analysis are carried out by Z - transform.

In the system analysis area, the Z - transform converts convolutions to a product and difference equations to algebraic equations.

The stability of a discrete linear system can be determined by analyzing the transfer function H (z) given by the Z - transform.

Digital filters can be analyzed and designed using the Z - transform.

Digital control systems can be analyzed and designed using Z- transforms.