Unit 4

RARE { $f(x) = 1$ ✓
$f(x) = x$ ✓

15M { $f(x) = a^2 - x^2$ ✓ } with deduction
$f(x) = a - |x|$ ✓

} 1 15M, 3 8M

8M $f(x) = 1 - |x|$

Using Parseval's identity, $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$ FST ①, $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ ② FCT

Unit 5

8M {
Inverse Z-transform
Partial fraction → (3Q's)
Residue method
Convolution

15M Solve differential eq using z transfer (2 CW sums)

Unit 4

- $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s)$ } Fourier transform
  (parameter)
- $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} dx$ } Inverse Fourier transform

1. Linear: $F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)]$
2. Shifting: When $F[f(x)] = F(s)$, $F[f(x-a)] = e^{isa} F(s)$
3. Shifting: When $F[f(x)] = F(s)$, $F[e^{isa} f(x)] = F(s+a)$
4. Modulation: When $F[f(x)] = F(s)$, $F[f(x)\cos x] = \frac{1}{2}[F(s+a) + F(s-a)]$
5. Change of scale: When $F[f(x)] = F(s)$, $F[f(ax)] = \frac{1}{a} F(\frac{s}{a})$; $a > 0$
6. Derivatives: When $F[f(x)] = F(s)$, $F[f'(x)] = -isF(s)$ $\forall f(x) \to 0, x \to \pm\infty$
7. When $F[f(x)] = F(s)$, $\overline{F[f(x)]} = \overline{F(s)}$

Convolution thm: $f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) g(x-u) du$

Parseval's identity: For $F[f(x)] = F(s)$, $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$

- $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x) \cos sx \, dx = F_c(s)$ } Fourier Cos Transform
- $f(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} F_c(s) \cos sx \, ds$ } Inverse Fourier Cos Transform
- $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x) \sin sx \, dx = F_s(s)$

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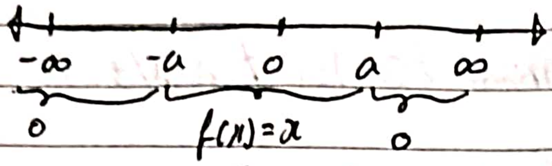
-Q 1. Fourier's Transform of $f(x) = \begin{cases} x & \forall |x| < a \\ 0 & \forall |x| > a \end{cases}$

(RARE)



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} \underbrace{f(x)}_{0} e^{isx}\, dx + \int_{-a}^{a} \underbrace{f(x)}_{x} e^{isx}\, dx + \int_{a}^{\infty} \underbrace{f(x)}_{0} e^{isx}\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\, e^{isx}\, dx \qquad \left(e^{i\theta} = \cos\theta + i\sin\theta \rightarrow e^{isx} = \cos sx + i \sin sx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} x\cos sx\, dx + i \int_{-a}^{a} x \sin sx\, dx \right\}$$

$\boxed{f(-x) = -f(x) \rightarrow \text{odd}}$ $\boxed{f(-x) = f(x) \rightarrow \text{even}}$

$f(x) = x\cos sx \rightarrow f(-x) = -x\cos sx \rightarrow \text{odd} \Rightarrow 0$

$f(x) = x\sin sx \rightarrow f(-x) = +x(+\sin x) \rightarrow \text{even} \rightarrow 2i\int_0^a x \sin sx\, dx$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left\{ 0 + 2i \int_0^a x \sin sx\, dx \right\}$$

$\int uv = uv_1 - u'v_2 + \cdots$

$u = x \quad v = \sin sx$

$u' = (1) \quad v_1 = -\frac{\cos sx}{s}$

$v_2 = -\frac{\sin sx}{s^2}$

$$= \frac{2i}{\sqrt{2\pi}} \int_0^a x \sin sx\, dx$$

$$= \frac{2i}{\sqrt{2\pi}} \left[(x)\left(-\frac{\cos sx}{s}\right) + (1)\left(+\frac{\sin sx}{s^2}\right) \right]_0^a$$

$$= \frac{2i}{\sqrt{2\pi}} \left[-\frac{x\cos sx}{s} + \frac{\sin sx}{s^2} \right]_0^a$$

$$F[f(x)] = \frac{2i}{\sqrt{2\pi}} \left\{ \frac{-a\cos sa}{s} + \frac{\sin sa}{s^2} \right\} //$$

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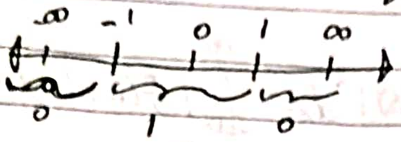
(RARE)

2. Fourier's transform of $f(x) = \begin{cases} 1 & \forall |x| \le 1 \\ 0 & \text{else} \end{cases}$ | Deduce that i) $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ ii) $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-1} \cancel{f(x)} e^{isx} dx + \int_{-1}^{1} \cancel{f(x)} e^{isx} dx + \int_{1}^{\infty} \cancel{f(x)} e^{isx} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx \, dx$$

$f(x) = \cos sx \to f(-x) = \cos sx \to f(-x) = f(x) \Rightarrow$ even
$f(x) = \sin sx \to f(-x) = -\sin sx \to f(-x) = -f(x) \to$ odd

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left[2 \int_0^1 \cos sx \, dx \right] + 0$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sx}{s} \right]_0^1 = \sqrt{\frac{2}{\pi}} \frac{\sin s}{s} \to \boxed{F(s) = \sqrt{\frac{2}{\pi}} \frac{\sin s}{s}}$$

i) To prove $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

By Inverse Fourier Transform, $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

$$1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin s}{s} e^{-isx} ds$$

take $x = 0$

$$1 = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} ds$$

puts $s = x \to ds = dx \Rightarrow 1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

$f(x) = \frac{\sin x}{x} \to f(-x) = \frac{+\sin x}{+x} \Rightarrow f(-x) = f(x) \to$ even

$$\Rightarrow \pi = 2 \int_{0}^{\infty} \frac{\sin x}{x} dx$$

$$\Rightarrow \frac{\pi}{2} = \int_0^\infty \frac{\sin x}{x} dx \quad \text{Hence Proved}$$

11) To prove $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$

By Parseval's Identity; $\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

$$\Rightarrow \int_{-\infty}^{\infty} \left[\sqrt{\tfrac{2}{\pi}} \frac{\sin s}{s}\right]^2 ds = \int_{-1}^{1} 1^2 dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = [x]_{-1}^{1}$$

$$\frac{\cancel{2}}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \cancel{2}$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \pi$$

put $s = x \rightarrow ds = dx$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi$$

$f(x) = \frac{\sin x}{x} \rightarrow f(-x) = \frac{-\sin x}{-x} \Rightarrow$ even

$$\Rightarrow 2 \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$ Hence proved,,

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta \rightarrow 1-\cos a = 2\sin^2\theta/2$

$\cos 2\theta = 2\cos^2\theta - 1$

Q. 3 15M Find F.T for $f(x) = \begin{cases} a-|x| & \forall\ |x| \le a \\ 0 & \text{else} \end{cases}$ and deduce

i) $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$ (ii) $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$

$-\infty \quad -a \quad 0 \quad a \quad \infty$

$0 \quad a-|x| \quad 0$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} \underbrace{f(x)}_{0} e^{isx} dx + \int_{-a}^{a} \underbrace{f(x)}_{a-|x|} e^{isx} dx + \int_{a}^{\infty} \underbrace{f(x)}_{0} e^{isx} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \underbrace{(a-|x|)\cos sx\, dx}_{\text{even}} + \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \underbrace{(a-|x|)\sin sx\, dx}_{\text{odd}}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \underset{u}{(a-|x|)}\, \underset{v}{\cos sx}\, dx + 0$$

$u' = -1 \quad v_1 = \frac{\sin sx}{s} \quad v_2 = \frac{-\cos sx}{s^2}$ $\quad \int uv = uv_1 - u'v_2 + \dots$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(a-|x|)\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(a-|x|)\sin sx}{s} - \frac{\cos sx}{s^2} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left(0 - \frac{\cos sa}{s^2}\right) - \left(0 - \frac{1}{s^2}\right) \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ -\frac{\cos sa}{s^2} + \frac{1}{s^2} \right\} = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos sa}{s^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2(\frac{sa}{2})}{s^2} \right]$$

$F(s) \rightarrow 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin(\frac{sa}{2})}{s} \right)^2$, to prove $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$, use

IFT $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$ (IFT used because power of integrand is same as F(s))

$a - |x| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin(\frac{sa}{2})}{s}\right)^2 e^{-isx} ds$, put $x=0$

$$\frac{\pi a}{2} = \int_{-\infty}^{\infty} \left(\frac{\sin(sa/2)}{s}\right)^2 ds$$

put $\frac{sa}{2} = x \rightarrow s = \frac{2x}{a} \rightarrow ds = \frac{2dx}{a}$

$$\int_{-\infty}^{\infty} \frac{(\sin x)^2}{4x^2/a^2} \cdot \frac{2dx}{a} = \frac{\pi a}{2}$$

$$\int_{-\infty}^{\infty} \sin^2 x \cdot \frac{\cancel{a^2}}{\cancel{4}x^2} \cdot \frac{\cancel{2}dx}{\cancel{a}} = \frac{\pi \cancel{a}}{\cancel{2}}$$

$$\underbrace{\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2}_{\text{even}} dx = \pi$$

$$\Rightarrow \frac{\pi}{2} = \int_{0}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx \quad \text{hence proved}$$

ii) To prove $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$

As power of integrand is square of $F(s)$, use Parseval's identity:

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \left(\frac{\sin\left(\frac{sa}{2}\right)}{s}\right)^2\right]^2 ds = \int_{-a}^{a} (a-|x|)^2 dx$$

$$\frac{8}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(sa/2)}{s}\right)^4 ds = \int_{-a}^{a} (a-|x|)^2 dx \quad *$$

$$* \quad \frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(sa/2)}{s}\right)^4 ds = \int_{-a}^{a}(a-|x|)^2 dx$$

$$\text{even} \Rightarrow 2\int_0^a (a-|x|)^2 dx$$

$$= 2\left[a^2x + \frac{x^3}{3} - \frac{\cancel{2}ax^2}{\cancel{2}}\right]_0^a$$

$$= 2\left\{\left(\cancel{a^3} + \frac{a^3}{3} - \cancel{a^3}\right) - (0+0-0)\right\}$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(sa/2)}{s}\right)^4 ds = \frac{2a^3}{3}$$

take $x = \frac{sa}{2}$ & $s = \frac{2x}{a}$ & $ds = \frac{2dx}{a}$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{\sin x}{2x/a}\right)^4 \frac{\cancel{2}dx}{a} = \frac{\cancel{2}a^3}{3}\times\frac{\pi}{8}$$

$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^4 \frac{\cancel{a^4}}{16}\frac{dx}{\cancel{a}} = \frac{\cancel{a^3}\pi}{3\times 8}$$

$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^4 dx = \frac{2\pi}{3}$$

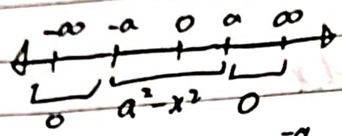
$$\text{even} \Rightarrow \cancel{2}\int_0^{\infty}\left(\frac{\sin x}{x}\right)^4 dx = \frac{\cancel{2}\pi}{3}$$

$$\int_0^{\infty}\left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$$ hence proved

Q. Find FT of $f(x) = \begin{cases} a^2 - x^2 & \forall\ |x| \le a \\ 0 & \forall\ |x| > a \end{cases}$

Deduce

i) $\displaystyle\int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}$ (ii) $\displaystyle\int_0^\infty \left[\frac{\sin x - x\cos x}{x^3}\right]^2 dx = \frac{\pi}{15}$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\,dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} \underbrace{f(x)}_{0} e^{isx}\,dx + \int_{-a}^{a} f(x) e^{isx}\,dx + \int_{a}^{\infty} \underbrace{f(x)}_{0} e^{isx}\,dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \underbrace{(a^2 - x^2)\cos sx\,dx}_{\text{even}} + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \underbrace{\sin sx\,dx}_{\text{odd}}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a^2 - x^2)\cos sx\,dx$$

~~$f(x) = (a^2 - x^2)\sin sx$~~

$\int uv = uv_1 - u'v_2 + u''v_3 \cdots$

$u = a^2 - x^2$ $\quad v = \cos sx$

$u' = -2x$ $\quad v_1 = \dfrac{\sin sx}{s}$

$u'' = -2$ $\quad v_2 = \dfrac{-\cos sx}{s^2}$

$v_3 = \dfrac{-\sin sx}{s^3}$

$$F[f(x)] = \left[(a^2 - x^2)\frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right)\right]_0^a \frac{2}{\sqrt{2\pi}}$$

$$= \left\{\left[0 - 2a\frac{\cos as}{s^2} + 2\frac{\sin as}{s^3}\right] - [0 - 0 + 0]\right\} \frac{2}{\sqrt{2\pi}}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{2\sin as - 2as\cos as}{s^3}\right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin as - as\cos as}{s^3}\right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin as - as\cos as}{s^3}\right)$$

put $a=1$

$$\boxed{F(s) = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)}$$

i) To prove: $\int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}$.

As power of integrand is same as $F(s)$, use IFT (Inverse Fourier Transform)

$\rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\underbrace{\cos sx}_{\text{even}}\,dx - \frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\underbrace{\sin sx}_{\text{odd}}\,dx$$

$$a^2-x^2 = \frac{4}{\pi}\int_0^\infty\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,dx$$

take $a=1$, $x=0$, $s=x \rightarrow ds=dx$

$$\boxed{\int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}}$$, Hence proved

ii) To prove $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}$

$F(s) = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)$, here, power of integrand is square of $F(s)$, so use Parseval's identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \left| 2\sqrt{\frac{2}{\pi}} \frac{\sin s - s\cos s}{s^3}\right|^2 ds = \int_{-a}^{a} |a^2 - x^2|^2 dx$$

$$4 \times \frac{2}{\pi} \int_{-\infty}^{\infty} \underbrace{\left(\frac{\sin s - s\cos s}{s^3}\right)^2}_{\text{even}} ds = \int_{-a}^{a} \underbrace{(a^2 - x^2)^2}_{\text{even}} dx$$

$$\frac{8}{\pi} \times \cancel{2} \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \cancel{2} \times \int_0^a (a^2 - x^2) dx$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 dx = \frac{\pi}{8} \times \int_0^a (a^2 - x^2)^2 dx$$

take $s = x \Rightarrow dx = ds$, $a = 1$

$$\Rightarrow \int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{8} \int_0^1 (1 - x^2)^2 dx$$

$$= \frac{\pi}{8} \int_0^1 (1 + x^4 - 2x^2) dx$$

$$= \frac{\pi}{8} \left[x + \frac{x^5}{5} - \frac{2x^3}{3} \right]_0^1 \cancel{dx}$$

$$= \frac{\pi}{8} \left\{ \left(1 + \frac{1}{5} - \frac{2}{3}\right) - 0 \right\}$$

$$= \frac{\pi}{\cancel{8}} \left(\frac{\cancel{15 + 3 - 10}}{15} \right)$$

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}$$, Hence proved.

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Using Parsevel's Identity, evaluate $\int_0 \frac{dx}{(x^2+a^2)(x^2+b^2)}$; $a,b>0$

Since numerator is the Integrant, i.e dx, use Fourier cosine transform

Take $\boxed{f(x)=e^{-ax}}$ & $\boxed{g(x)=e^{-bx}}$

$$F_c[f(x)] \Rightarrow F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+s^2}$$

$$F_c[s] = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+s^2}$$

$$G_c[g(x)] = G_c[e^{-bx}] . \sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]$$

$$G_c[s] = \sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]$$

Parsevel's Identity

$$\int_0^\infty F_c(s)\,G_c(s)\,ds = \int_0^\infty f(x)\,g(x)\,dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+b^2}\,\sqrt{\frac{2}{\pi}}\,\frac{b}{b^2+s^2}\,ds = \int_0^\infty e^{-ax}\,e^{-bx}\,dx$$

$$\frac{2}{\pi}\int_0^\pi \frac{ab}{(a^2+s^2)(b^2+s^2)}\,ds = \int_0^\infty e^{-(a+b)x}\,dx$$

$$\cancel{\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)}\left[0-\left[\frac{1}{-(a+b)}\right]\right]}$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)} = \left[0-\frac{1}{-(a+b)}\right]$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)} \Rightarrow \frac{1}{a+b}$$

$$\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2(a+b)} \qquad s \to x \Rightarrow ds = dx$$

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)} \,//$$

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Q. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$; $a,b>0$

Since numerator of integrant is dx, we use Fourier Cosine Transform. Take $f(x)=e^{-ax}$, $g(x)=e^{-bx}$

$$F_c[s] = F_c[f(x)]$$
$$= \sqrt{\tfrac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$
$$= \sqrt{2/\pi} \int_0^\infty e^{-ax} \cos sx \, dx$$
$$F_c(s) = \sqrt{2/\pi}\ \boxed{\frac{a}{a^2+s^2}} \qquad \left(\because \int_0^\infty e^{-ax} \cos sx = \frac{a}{a^2+s^2}\right)$$

Similarly, $\cancel{G_c(s) = \sqrt{2/\pi} \int_0^\infty}$

$$G_c(s) = \sqrt{2/\pi}\ \frac{b}{b^2+s^2} = \sqrt{\tfrac{2}{\pi}} \int_0^\infty e^{-bx} \cos sx \, dx$$

WKT
$$\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$
$$\Rightarrow \int_0^\infty \sqrt{2/\pi}\ \frac{a}{a^2+s^2} \sqrt{2/\pi}\ \frac{b}{b^2+s^2}\, ds = \int_0^\infty e^{-ax} \cdot e^{-bx}\, dx$$
$$\Rightarrow \frac{2ab}{\pi} \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \int_0^\infty e^{-(a+b)x}\, dx$$
$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$
$$= 0 + \left(+1/a+b\right) \qquad (\because e^{-\infty}=0)$$
$$\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

put $s=x$, $ds=dx$

$$\Rightarrow \boxed{\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}}$$

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Q. Evaluate $\int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$; $a, b > 0$

Since numerator is $x^2 dx$, we use FST

Take $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$F_s(s) = F_s(f(x))$

$= \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx$

$= \sqrt{2/\pi} \int_0^\infty e^{-ax} \sin sx \, dx$

$= \sqrt{2/\pi} \cancel{\,s\,} \quad = \sqrt{\frac{2}{\pi}} \boxed{\frac{s}{s^2+a^2}}$

$F_s[s] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$, similarly, $G_s[s] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+b^2}$

WKT $\int_0^\infty F_s(s) G_s(s) ds = \int_0^\infty f(x) g(x) dx$

$\Rightarrow \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+b^2} ds = \int_0^\infty e^{-ax} e^{-bx} dx$

$\frac{2}{\pi} \int_0^\infty \frac{s^2 ds}{(s^2+a^2)(s^2+b^2)} = \int_0^\infty e^{-(a+b)x} dx$

$= \boxed{\frac{1}{a+b}} \quad \left(\hookrightarrow \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty = 0 + \frac{1}{+(a+b)} \right)$

$\therefore \boxed{\int_0^\infty \frac{s^2 ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2(a+b)}}$

$\rightarrow$ put $s = x \rightarrow ds = dx$

$\Rightarrow \boxed{\int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}}$