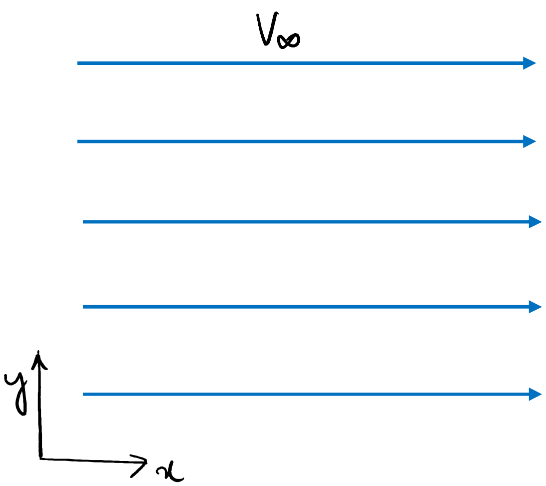
Elementary potential flows

Uniform flow



Uniform flow with velocity V_∞ oriented in positive x-direction

$$u = V_\infty \qquad v = 0$$

Check: This flow satisfies $\bar{\nabla} \cdot \bar{V} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ ✓✓ satisfy

$\bar{\nabla} \times \bar{V} = 0 \Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ ✓✓ satisfy

This flow satisfies both incompressible flow and irrotational flow condition $\Rightarrow$ so, it is a potential flow.

ϕ:

$$\frac{\partial \phi}{\partial x} = u \qquad \Big| \qquad \frac{\partial \phi}{\partial y} = v$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = V_\infty \qquad \Big| \qquad \frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow \phi = V_\infty x + f(y) \qquad \Big| \qquad \phi = f(x)$$

on comparison, $f(y) = 0 \Rightarrow \boxed{\phi = V_\infty x}$

$$x = \frac{\phi}{V_\infty} = \text{Const}$$

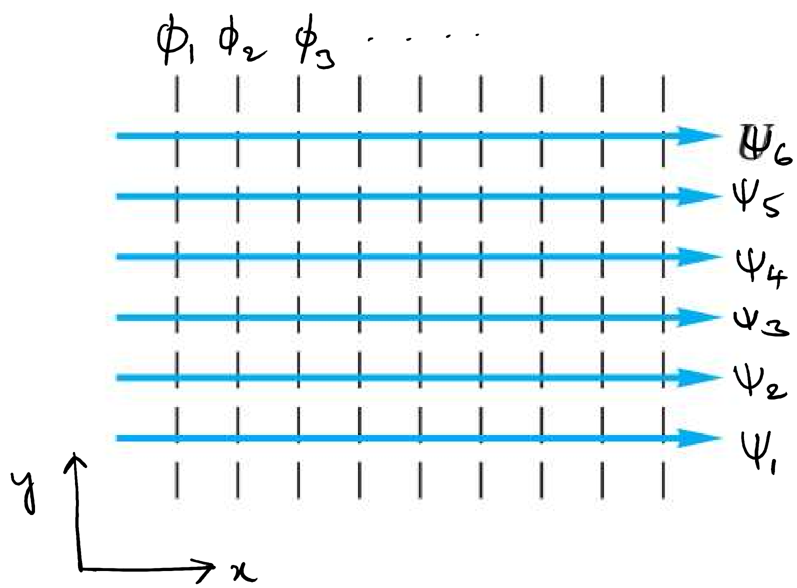
$\underline{\underline{\psi:}}$

$$\frac{\partial \psi}{\partial y} = u \qquad \frac{\partial \psi}{\partial x} = -v$$

$$\frac{\partial \psi}{\partial y} = V_\infty \qquad \frac{\partial \psi}{\partial x} = 0$$

$$\psi = V_\infty y + f(x) \qquad \psi = f(y)$$

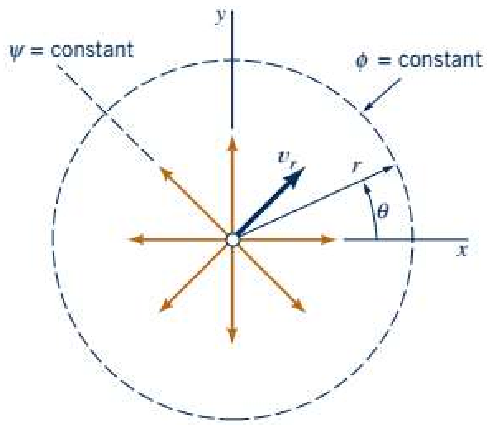
on Comparison $f(x) = 0 \Rightarrow \boxed{\psi = V_\infty y}$ $\quad y = \frac{\psi}{V_\infty}$ $\quad$ $y = \underline{\underline{\text{Constant}}}$

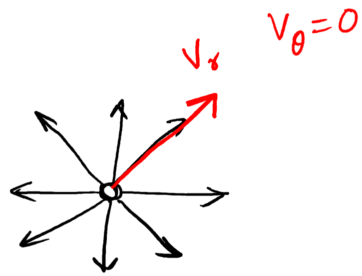


y = Constant lines are the Streamlines

x = Constant lines are the equipotential lines.

Source flow and Sink flow





Source

V_r — Radial velocity

V_θ — Tangential velocity

<u>Source flow:</u>

$$V_r \propto \frac{1}{r} \quad , \quad V_\theta = 0$$

$$\Rightarrow \quad V_r = \frac{C}{r} \quad , \quad V_\theta = 0 \qquad C - \text{Constant}$$

To find C, consider volume flow rate at any distance r.

Volume flow rate Q = Area × Velocity

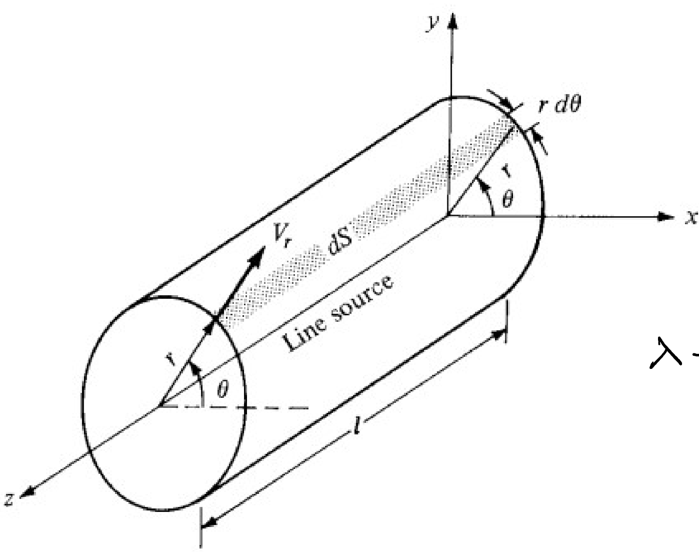
$$Q = 2\pi r l \times V_r$$

$$\Rightarrow \quad \frac{Q}{l} = 2\pi r V_r$$

$$\Rightarrow \quad \lambda = 2\pi r V_r$$

$$\Rightarrow \quad \boxed{V_r = \frac{\lambda}{2\pi r}} \quad \Rightarrow \quad C = \frac{\lambda}{2\pi}$$

λ — Volume flow rate at any radius r per unit length.



λ is called the SOURCE STRENGTH

$\underline{\underline{\phi}}$:

$$\frac{\partial \phi}{\partial r} = V_r$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial r} = \frac{\lambda}{2\pi r}$$

$$\Rightarrow \quad \phi = \frac{\lambda}{2\pi} \ln r + f(\theta)$$

$$\Rightarrow \quad \boxed{\phi = \frac{\lambda}{2\pi} \ln r}$$

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = V_\theta$$

$$\frac{\partial \phi}{\partial \theta} = 0$$

$$\phi = f(r)$$

Equipotential lines correspond to lines of

$\ln r = \text{constant}$

$\Rightarrow r = C_1$

$\underline{\psi}$:

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_r$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\lambda}{2\pi r}$$

$$\Rightarrow \quad \psi = \frac{\lambda}{2\pi}\theta + f(r)$$

$$\Rightarrow \quad \boxed{\psi = \frac{\lambda}{2\pi}\theta}$$

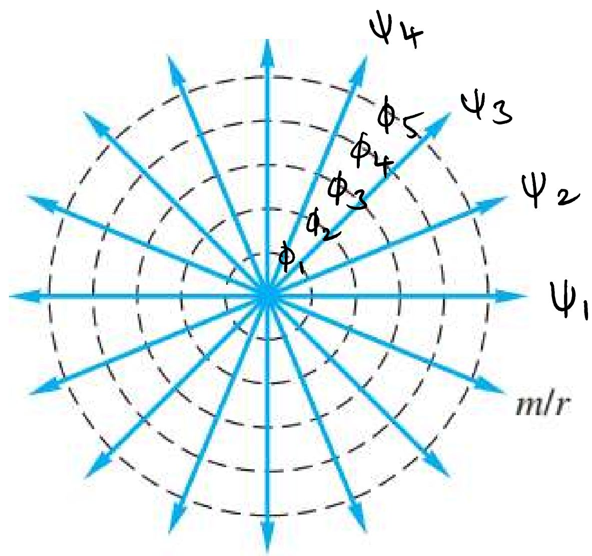
$$\frac{\partial \psi}{\partial r} = -V_\theta$$

$$\frac{\partial \psi}{\partial r} = 0$$

$$\psi = f(\theta)$$

Streamlines corresponds to the lines of

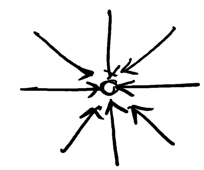
$\theta = \text{constant}$



Sink strength $= -\lambda$

<u>Sink:</u>

opposite of source.



$$V_r = \frac{-\lambda}{2\pi r}$$

$$V_\theta = 0$$

$$\phi = \frac{-\lambda}{2\pi} \ln r$$

$$\psi = \frac{-\lambda}{2\pi} \theta$$

Free Vortex flow

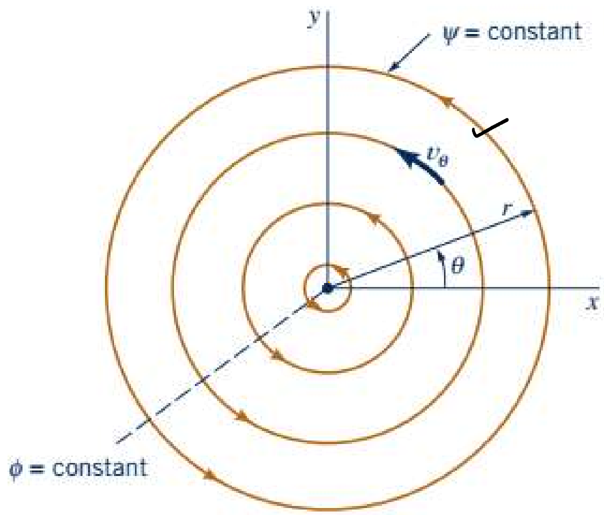
— Flow in closed circular streamlines is called a vortex flow

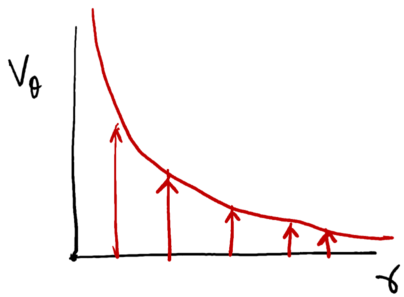
$$V_r = 0 \qquad V_\theta \propto \frac{1}{r}$$

$$V_r = 0 \text{ and } V_\theta = \frac{c}{r}$$

c – constant

satisfies
$\bar{\nabla} \cdot \bar{V} = 0$
$\bar{\nabla} \times \bar{V} = 0$
$\Rightarrow$ potential flow



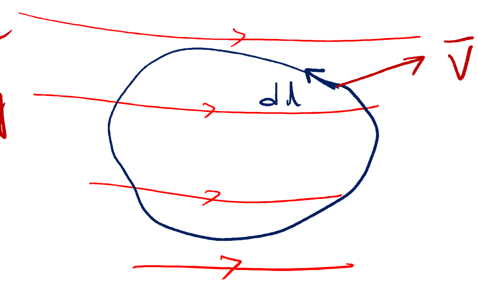


* Irrotational Vortex

Circulation:

$$\Gamma = -\oint \bar{V} \cdot d\bar{l}$$

– It is defined as the negative of line integral of velocity along the closed curve



$$\Gamma = -\oint \bar{V} \cdot d\bar{l} = -V_\theta \oint dl = -V_\theta \, 2\pi r$$

Γ – circulation $\Rightarrow$ $\boxed{V_\theta = \frac{-\Gamma}{2\pi r}}$ $\Rightarrow c = \frac{-\Gamma}{2\pi}$

Γ is called VORTEX STRENGTH

ϕ:

$$\frac{\partial \phi}{\partial r} = V_r \qquad \Bigg| \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = V_\theta$$

$$\frac{\partial \phi}{\partial r} = 0 \qquad \Bigg| \qquad \frac{1}{\cancel{r}}\frac{\partial \phi}{\partial \theta} = \frac{-\Gamma}{2\pi \cancel{r}}$$

$$\phi = f(\theta) \qquad \Bigg| \qquad \phi = -\frac{\Gamma}{2\pi}\theta + f(r)$$

on comparison $\boxed{\phi = \frac{-\Gamma}{2\pi}\theta}$ $\Rightarrow$ θ = constant are the equipotential lines

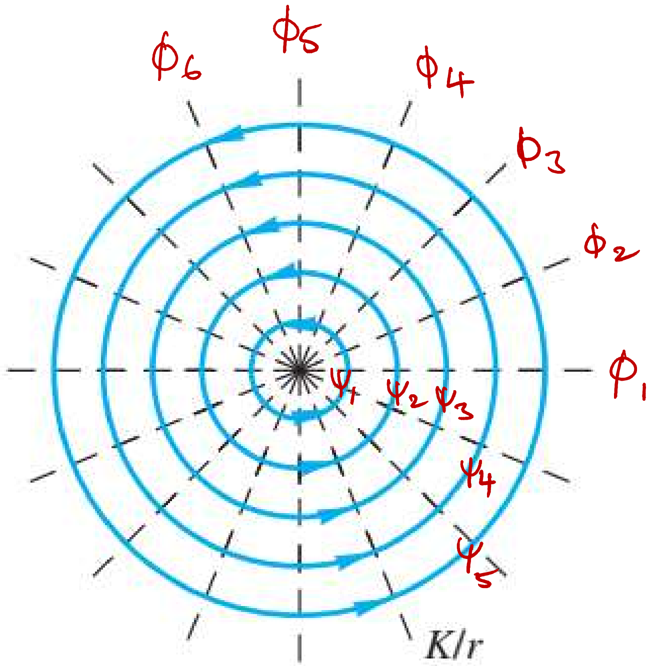
ψ:

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_r \qquad \Bigg| \qquad -\frac{\partial \psi}{\partial r} = V_\theta$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \qquad \Bigg| \qquad -\frac{\partial \psi}{\partial r} = \frac{-\Gamma}{2\pi r}$$

$$\psi = f(r) \qquad \Bigg| \qquad \psi = \frac{\Gamma}{2\pi}\ln r + f(\theta)$$

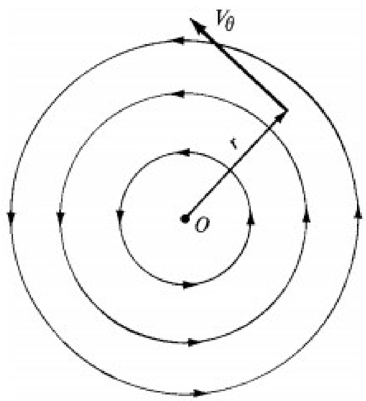
on comparison, $\boxed{\psi = \frac{\Gamma}{2\pi}\ln r}$ $\Rightarrow$ $\underbrace{r = \text{constant}}_{\text{circles}}$ are the streamlines

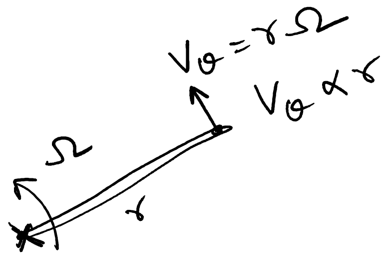


Note: $\Gamma = -\oint \bar{V} \cdot d\bar{l}$

why there is a -ve sign? θ and V_θ are positive in Counter clock-wise direction, whereas Γ is taken to be +ve in clock-wise direction.

Forced Vortex flow

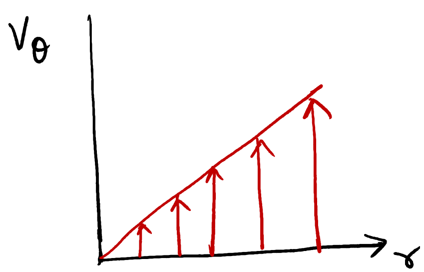




$V_r = 0 \qquad V_\theta \propto r$

$V_\theta = \Omega r \qquad \Omega$ - Angular velocity in the flow

- Not a potential flow
- Does not satisfy the irrotational flow condition



$$\bar{\nabla} \times \bar{V} \neq 0$$

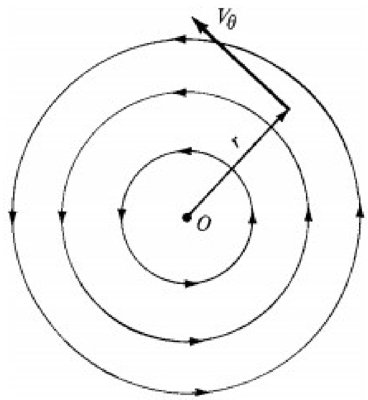
$$\nabla \times \bar{V} = \frac{1}{r}\left[\frac{\partial (r V_\theta)}{\partial r} - \frac{\partial V_r}{\partial \theta}\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r^2 \Omega) = \frac{1}{r} 2r\Omega = 2\Omega \neq 0$$

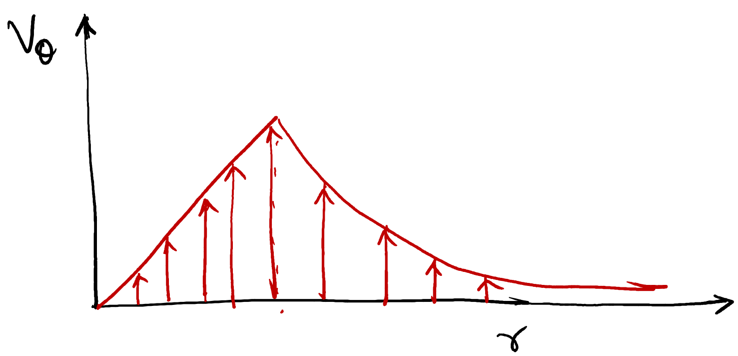
$\Rightarrow$ Forced vortex is rotational

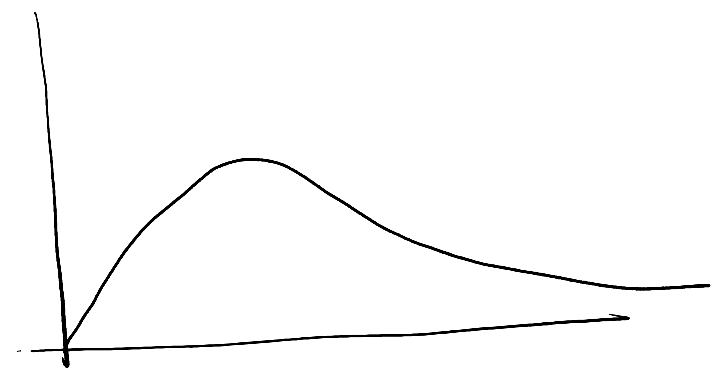
* Rotational vortex

Real Vortex:



Rankine Vortex





* All real vortices will be a combination of both forced vortex and a free vortex.

Eg: whirlpools

Tornados

Cyclones

Combination of elementary flows

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

2nd order *linear equations*

ψ

ϕ

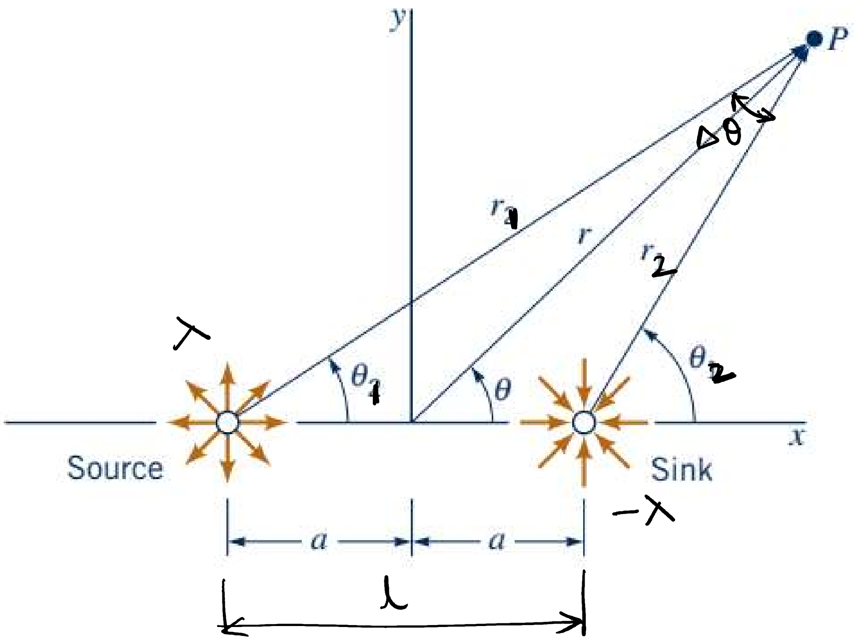
ψ_1 is a solution

ψ_2 is a solution

then $\underbrace{\psi_1 + \psi_2}_{\text{Physically possible potential flow}}$ is also a solution of this eqn

Doublet flow

Source and Sink pair



$$\psi|_{source} = \frac{\lambda}{2\pi}\theta$$

$$\psi_{sink} = \frac{-\lambda}{2\pi}\theta$$

At a point P in the flow,

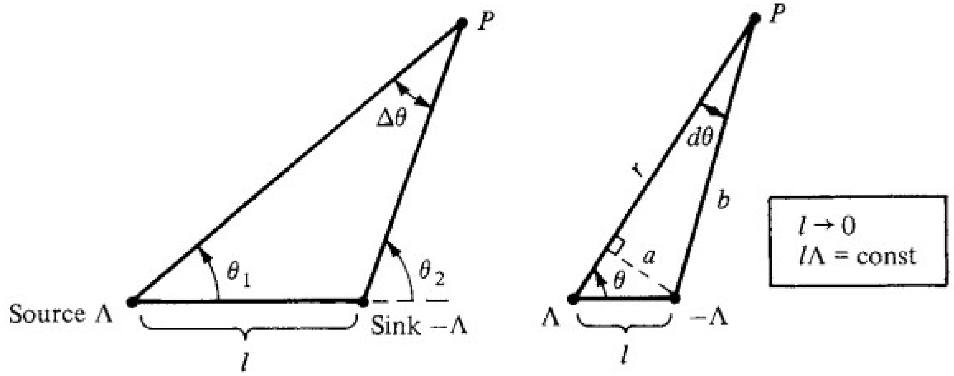
$$\psi = \psi_{source} + \psi_{sink}$$

$$= \frac{\lambda}{2\pi}\theta_1 - \frac{\lambda}{2\pi}\theta_2$$

$$\psi = \frac{\lambda}{2\pi}(\theta_1 - \theta_2) = \frac{-\lambda}{2\pi}\Delta\theta$$

$l \to 0 \qquad \lambda \to \infty$

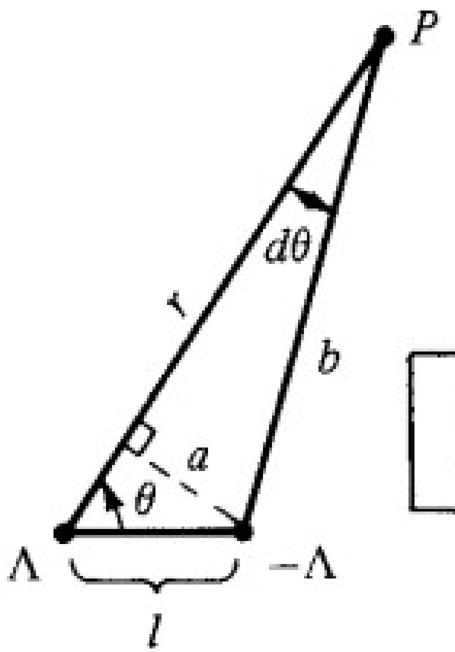
$l\lambda$ = constant



(*a*) Source-sink pair (*b*) Limiting case for a doublet

Figure 3.24 How a source-sink pair approaches a doublet in the limiting case.

- let the distance *l* approach zero while the absolute magnitudes of the strengths of the source and sink increase in such a fashion that the product $l\lambda$ remains constant.
- In the limit, as $l \rightarrow 0$ while $l\lambda$ remains constant, we obtain a special flow pattern defined as a *doublet*.
- The *strength* of the doublet is denoted by κ and is defined as $\kappa \equiv l\,\lambda$.



$l \to 0$
$l\Lambda = \text{const}$

Stream function of a doublet is

$$\psi = \lim_{\substack{l \to 0 \\ K = l\lambda = \text{constant}}} \left(\frac{-\lambda}{2\pi} d\theta \right) \quad \text{— (1)}$$

From the diagram,

$a = l \sin\theta$

$\cos d\theta = \dfrac{r - l\cos\theta}{b} \Rightarrow b = \dfrac{r - l\cos\theta}{\cos d\theta}$

As $l \to 0$, $d\theta \to 0 \Rightarrow \cos d\theta \approx 1$, $\tan d\theta \approx d\theta$

$\Rightarrow b = r - l\cos\theta$

$$\tan d\theta = \frac{a}{b} \Rightarrow d\theta = \frac{a}{b} = \frac{l \sin\theta}{r - l\cos\theta} \quad \text{— (2)}$$

substituting ② in ①

$$\Rightarrow \quad \psi = \lim_{\substack{l \to 0 \\ K = l\lambda = \text{const}}} \left[\frac{-\lambda}{2\pi} \frac{l \sin\theta}{r - l\cos\theta} \right] = \lim_{l \to 0} \left[\frac{-K}{2\pi} \frac{\sin\theta}{r - l\cos\theta} \right]$$

$$\Rightarrow \boxed{\psi = \frac{-K}{2\pi} \frac{\sin\theta}{r}}$$ stream function for a Doublet flow.

$\underline{\underline{\phi:}}$

$$\phi = \phi_{source} + \phi_{sink} = \frac{\lambda}{2\pi} \ln r_1 - \frac{\lambda}{2\pi} \ln r_2$$

$$= \frac{\lambda}{2\pi} \ln\left(\frac{r_1}{r_2}\right) = \frac{-\lambda}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

From the diagram $r_1 = r$ and $r_2 = b = r - l\cos\theta$

$$\Rightarrow \frac{r_2}{r_1} = \frac{r - l\cos\theta}{r} = 1 - \frac{l}{r}\cos\theta$$

Velocity potential ϕ for a doublet is

$$\phi = \lim_{\substack{l \to 0 \\ K = l\lambda = \text{Const}}} \left[-\frac{\lambda}{2\pi} \ln \frac{r_2}{r_1} \right] = \lim_{\substack{l \to 0 \\ K = l\lambda = \text{Const}}} \left[-\frac{\lambda}{2\pi} \underbrace{\ln\left(1 - \frac{l}{r}\cos\theta\right)}_{\text{Use series expansion to simplify}} \right]$$

Series expansion for $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots$$

$$\Rightarrow \quad \phi = \lim_{\substack{l \to 0 \\ K = l\lambda = \text{Const}}} \left[-\frac{\lambda}{2\pi} \left(-\frac{l}{r}\cos\theta - \frac{l^2}{2r^2}\cos^2\theta - \frac{l^3}{3r^3}\cos^3\theta - \cdots\cdots \right) \right]$$

$$\Rightarrow \quad \boxed{\phi = \frac{K}{2\pi} \frac{\cos\theta}{r}}$$

Velocity potential for a Doublet flow

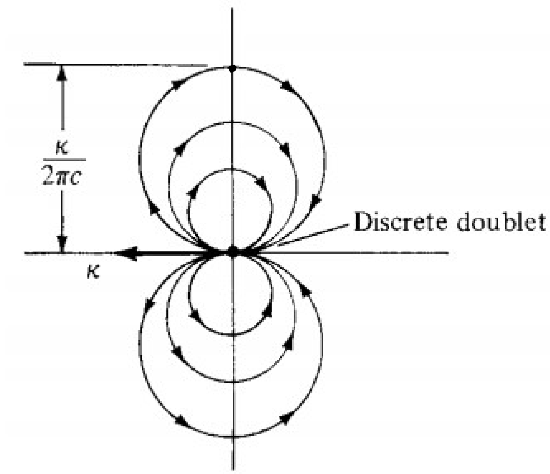
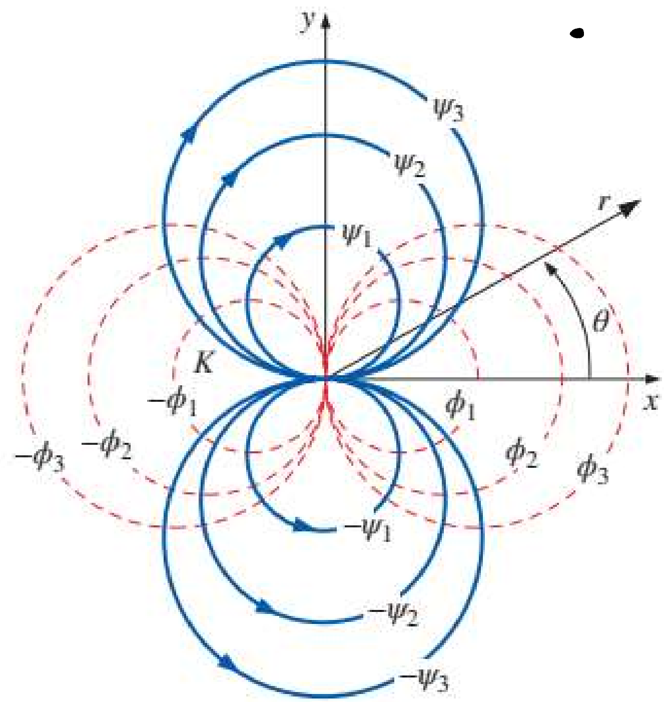


Figure 3.25 Doublet flow with strength κ.



SUMMARY

Type of flow	Velocity	ϕ	ψ
Uniform flow in x direction	$u = V_\infty$	$V_\infty x$	$V_\infty y$
Source	$V_r = \dfrac{\Lambda}{2\pi r}$	$\dfrac{\Lambda}{2\pi} \ln r$	$\dfrac{\Lambda}{2\pi}\theta$
Vortex	$V_\theta = -\dfrac{\Gamma}{2\pi r}$	$-\dfrac{\Gamma}{2\pi}\theta$	$\dfrac{\Gamma}{2\pi} \ln r$
Doublet	$V_r = -\dfrac{\kappa}{2\pi}\dfrac{\cos\theta}{r^2}$	$\dfrac{\kappa}{2\pi}\dfrac{\cos\theta}{r}$	$-\dfrac{\kappa}{2\pi}\dfrac{\sin\theta}{r}$
	$V_\theta = -\dfrac{\kappa}{2\pi}\dfrac{\sin\theta}{r^2}$		