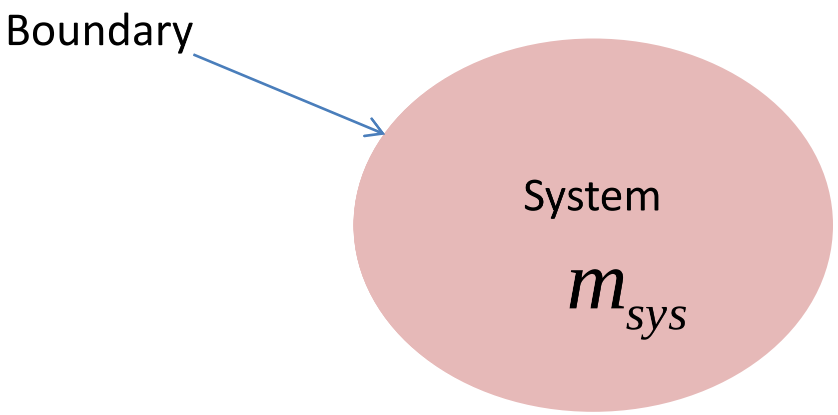
System and Control volume concept

System concept (or Control Mass concept)

- It has a FIXED MASS with a FIXED IDENTITY that move in the flow
- Always contains same fluid particles
- Only energy interaction with the surroundings
- It may continually change size and shape, but it always contains the same mass.



$$m_{sys} = \text{constant}$$

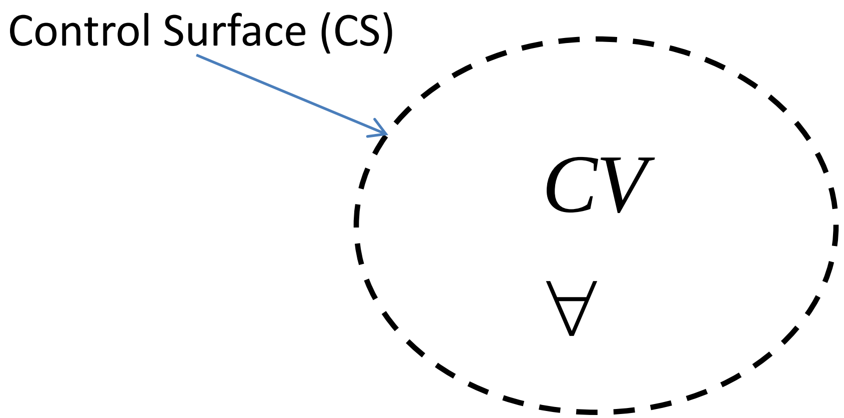
- In fluid mechanics, it is often quite difficult to identify and keep track of a specific quantity of matter.

- A finite portion of a fluid contains an uncountable number of fluid particles that move about quite freely, unlike a solid that may deform but usually remains relatively easy to identify.

- For example, we cannot as easily follow a specific portion of water flowing in a river as we can follow a branch floating on its surface.

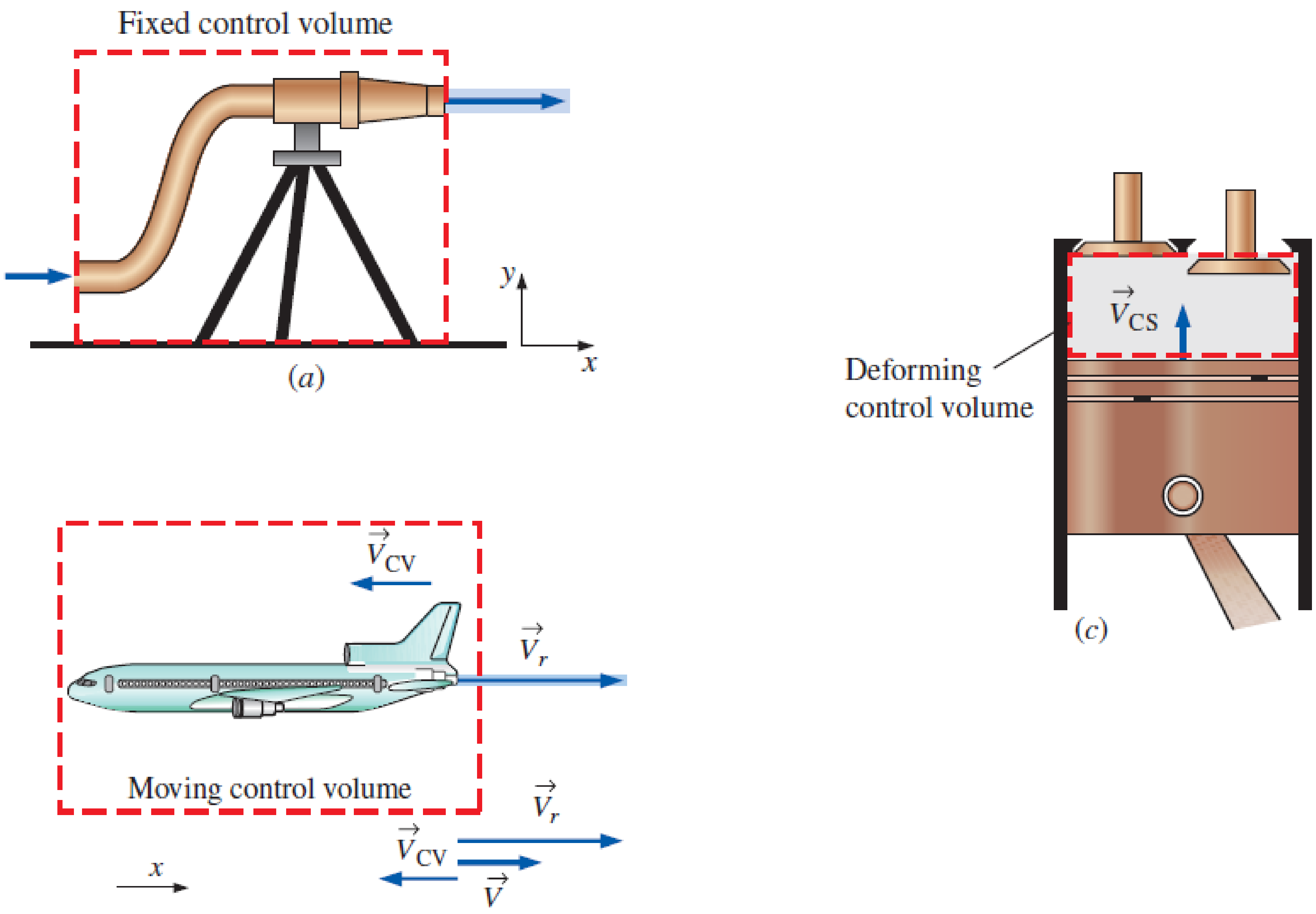
Control Volume concept

- It is a volume in space through which fluid flows
- The fluid particles flow through the control volume
- It is a geometric entity, has no fixed identity of mass
- Matter within a control volume may change with time as the fluid flows through it.
- The amount of mass within the volume may change with time.



We apply control volume concept for analysing fluid flow

Types of control volume



All the laws of mechanics are written for a *system*

Conservation of mass:

$$m_{sys} = \text{constant}$$

$$\frac{dm_{sys}}{dt} = 0$$

Conservation of momentum:

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d(m\vec{V})}{dt}$$

Conservation of energy:

$$\dot{Q} - \dot{W} = \frac{dE}{dt}$$

We need an analytical tool to shift from System concept to Control Volume concept $\longrightarrow$ **Reynolds Transport Theorem**

The Reynolds Transport Theorem

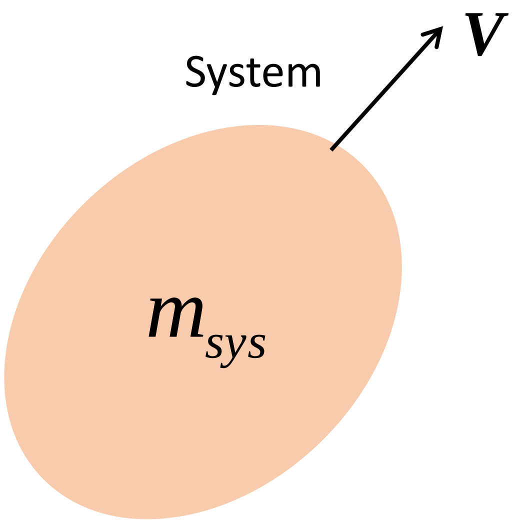
Conservation parameters: mass (m), momentum (mV) and energy (E)

	B	b=B/m
Mass	m	1
Momentum	$m\vec{V}$	$\vec{V}$
Energy	E	e

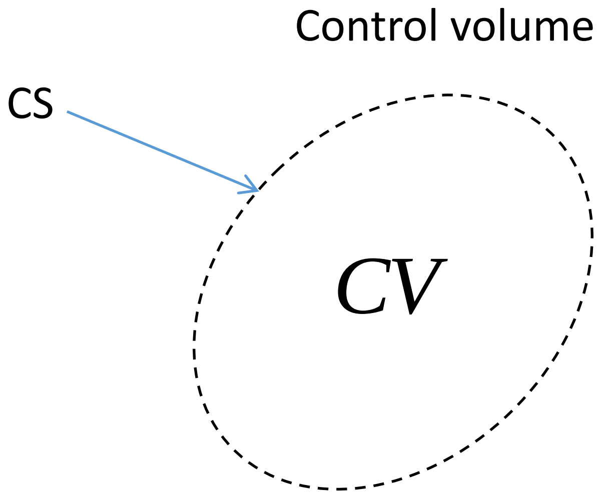
B - Any flow parameter

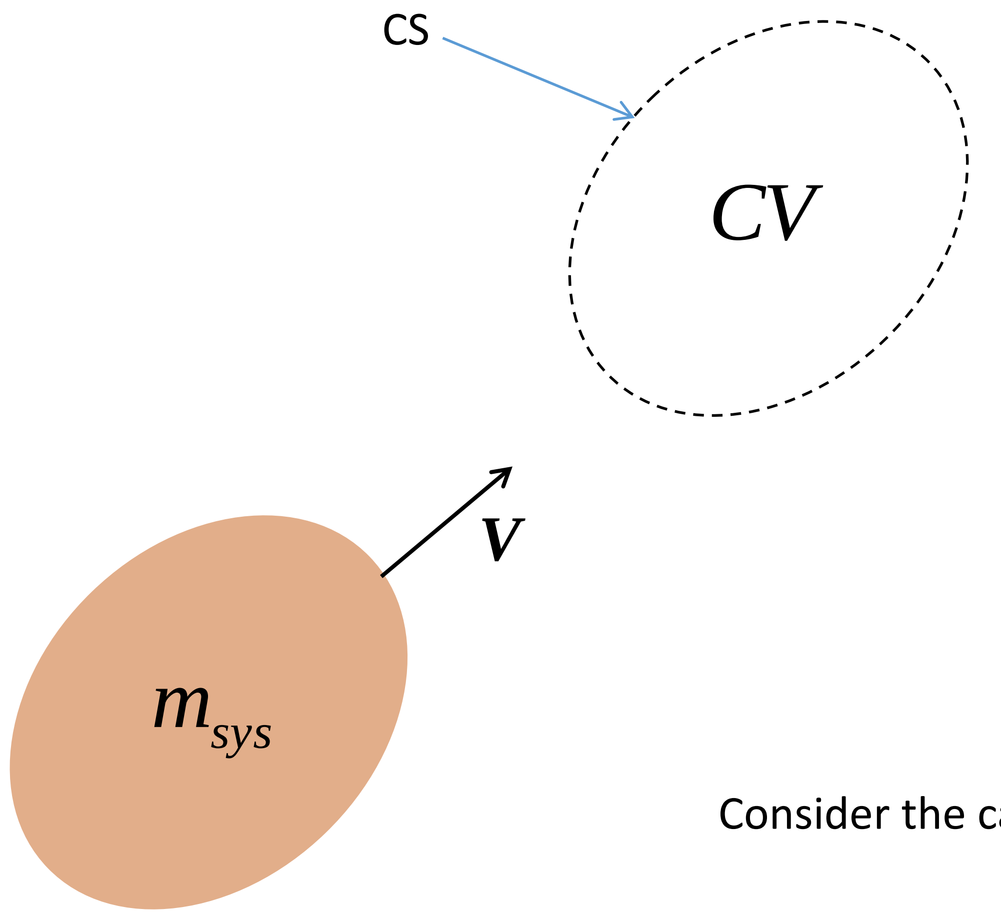
b - Amount of that parameter per unit mass = B/m

Consider a



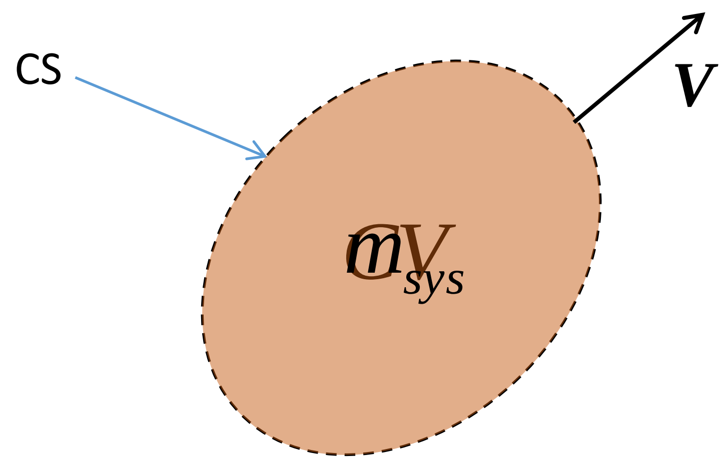
and





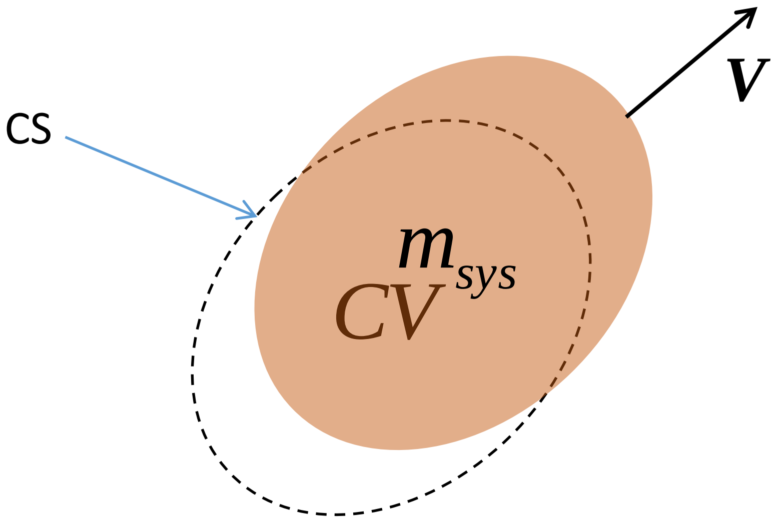
Consider the case of System flowing through the Control Volume

At a time instant t



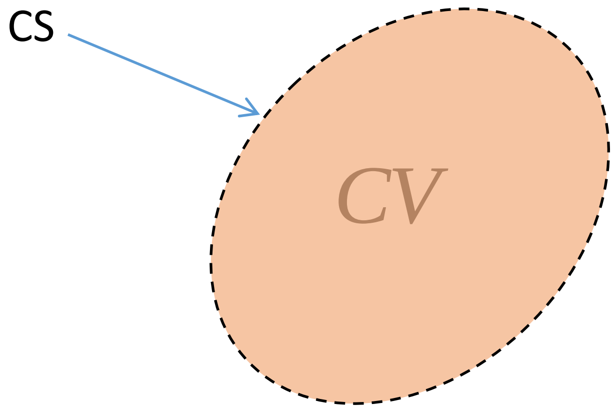
the system occupies the control volume

At a time instant t+δt



the system moves out of CV

At time t



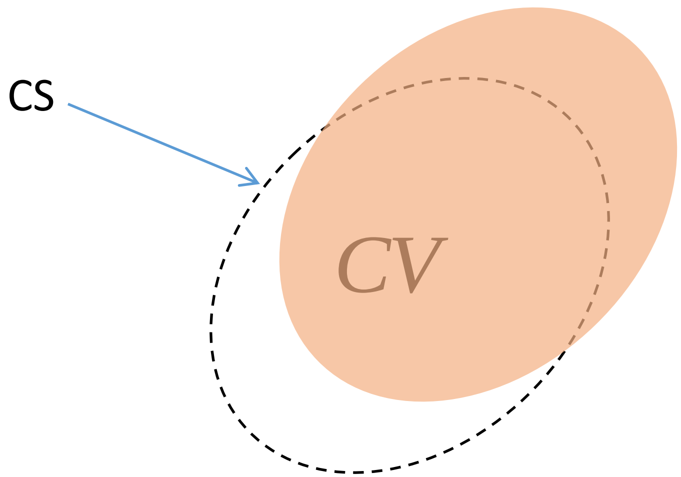
At time t, the system occupies the control volume

Sys = CV

At time t,

$$B_{sys}(t) = B_{CV}(t)$$

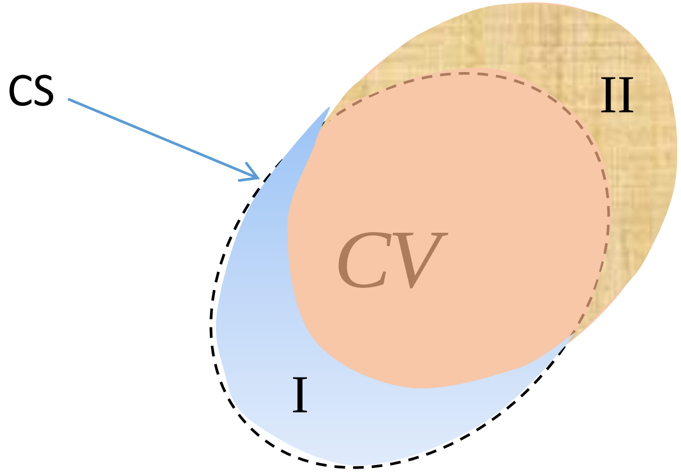
At time $t+\delta t$



At time $t+\delta t$, the system moving out of CV

Sys =

At time $t+\delta t$



At time $t+\delta t$, the system moving out of CV

Sys = CV – I + II

At time $t+\delta t$,

$$B_{sys}(t+\delta t) = B_{CV}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t)$$

Change in the amount of B in the system in the time interval δt is:

$$\frac{\delta B_{sys}}{\delta t} = \frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t}$$

$$= \frac{B_{CV}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t) - B_{sys}(t)}{\delta t}$$

$$= \frac{B_{CV}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t) - B_{CV}(t)}{\delta t}$$

$$\Rightarrow \frac{\delta B_{sys}}{\delta t} = \frac{B_{CV}(t+\delta t) - B_{CV}(t)}{\delta t} + \frac{B_{II}(t+\delta t)}{\delta t} - \frac{B_{I}(t+\delta t)}{\delta t}$$

Simplifying the above equation in the limit of $\delta t \rightarrow 0$

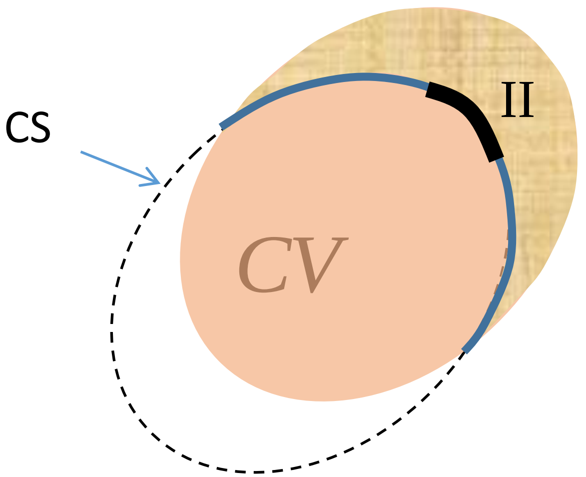
LHS:

$$\lim_{\delta t \to 0} \frac{\delta B_{sys}}{\delta t} = \frac{dB_{sys}}{dt} = \frac{DB_{sys}}{Dt}$$

First term on RHS:

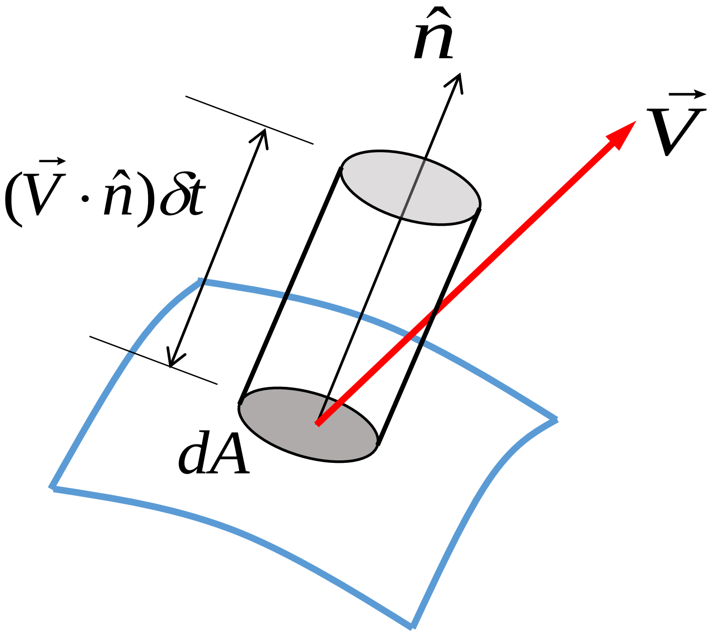
$$\lim_{\delta t \to 0} \frac{B_{CV}(t+\delta t) - B_{CV}(t)}{\delta t} = \frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t} \iiint_{CV} \rho \, b \, d\forall$$

Second term on RHS:



$$\lim_{\delta t \to 0} \frac{B_{II}(t+\delta t)}{\delta t} = \dot{B}_{out}$$

the rate at which the extensive parameter *B* flows from the control volume across the control surface

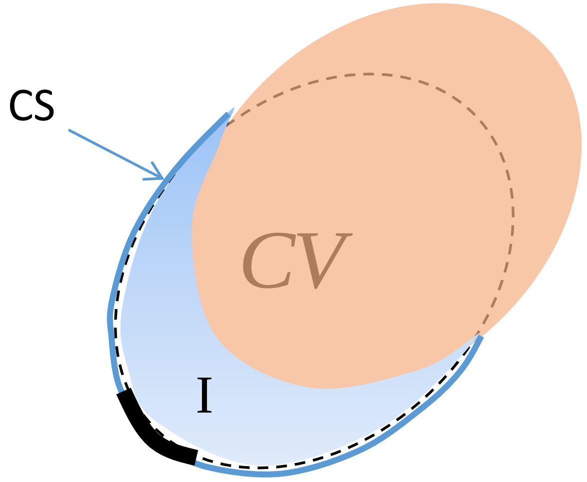


$$B_{II}(t+\delta t) = \iint_{CS_{out}} \rho\, b\,(\vec{V}\cdot\hat{n})\delta t\, dA$$

$$\frac{B_{II}(t+dt)}{\delta t} = \iint_{CS_{out}} \rho\, b\,(\vec{V}\cdot\hat{n})\, dA$$

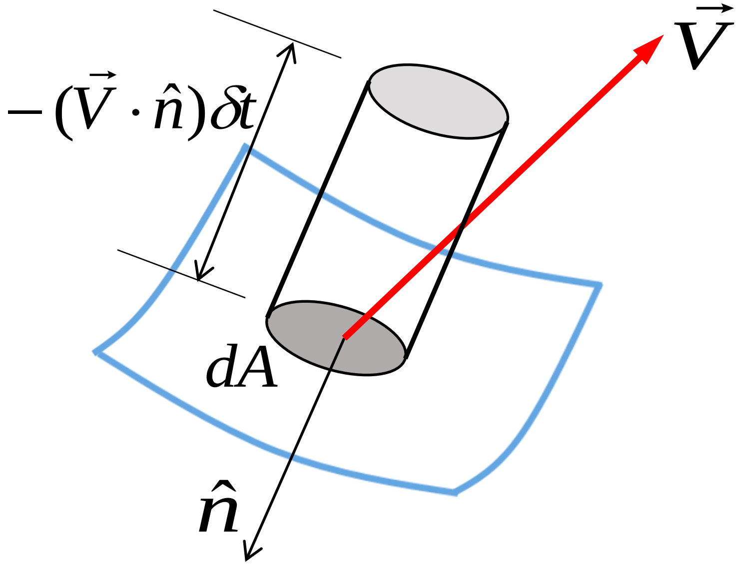
$$\Rightarrow \dot{B}_{out} = \iint_{CS_{out}} \rho\, b\,(\vec{V}\cdot\hat{n})\, dA$$

Third term on RHS:



$$\lim_{\delta t \to 0} \frac{B_I(t+\delta t)}{\delta t} = \dot{B}_{in}$$

the rate at which the extensive parameter *B* flows into the control volume across the control surface



$$B_I(t+\delta t) = -\iint_{CS_{in}} \rho\, b\, (\vec{V}\cdot\hat{n})\delta t\, dA$$

$$\Rightarrow \dot{B}_{in} = -\iint_{CS_{in}} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA$$

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \dot{B}_{out} - \dot{B}_{in}$$

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \iint_{CS_{out}} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA - \left(- \iint_{CS_{in}} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA \right)$$

$$\boxed{\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \oiint_{CS} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA}$$

Reynolds transport theorem – Relates System to Control Volume

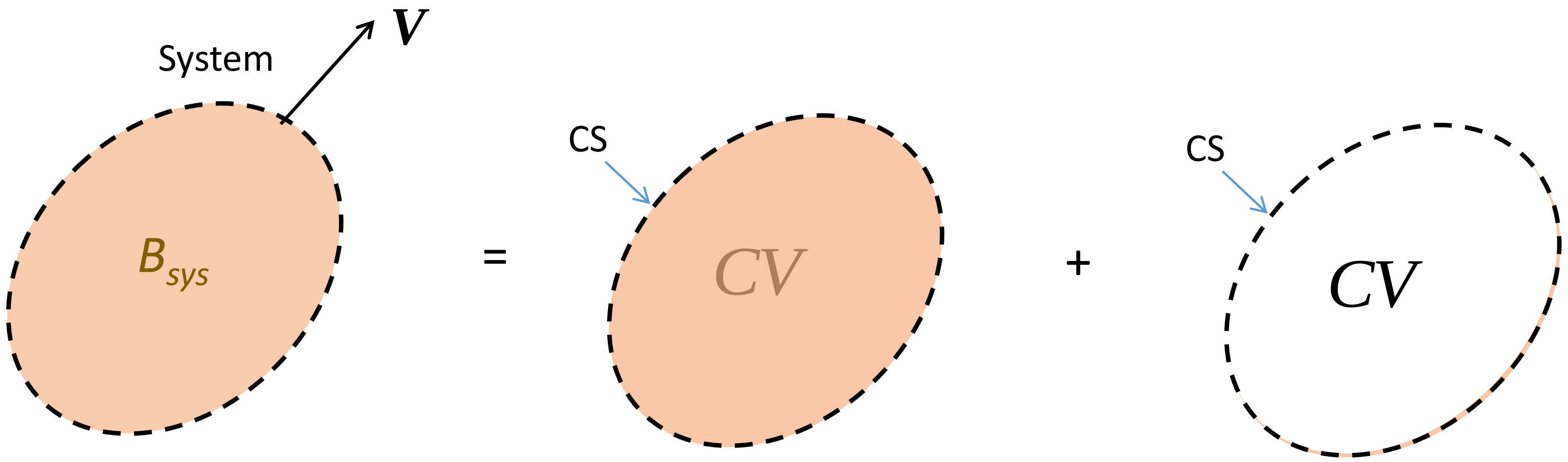
$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \oiint_{CS} \rho\, b\, (\vec{V} \cdot \hat{n})\, dA$$

Time rate of change of a parameter B of a system	=	Time rate of change of parameter B within the control volume as the fluid flows through it	+	Net flux of the parameter B across the control surface

End of Reynolds Transport Theorem Derivation

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \oiint_{CS} \rho\, b\,(\vec{V}\cdot\hat{n})\, dA$$

Time rate of change of a parameter B of a system	=	Time rate of change of parameter B within the control volume as the fluid flows through it	+	Net flux of the parameter B across the control surface



Governing equations expressed in Control Volume Concept:

RTT: $$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, b\, d\forall + \oiint_{CS} \rho\, b\, (\vec{V}\cdot\hat{n})\, dA$$ $$b = \frac{B}{m}$$

Mass conservation/Continuity:

Substituting $B = m \quad \Rightarrow \quad b = 1$ in *RTT*

$$\frac{Dm}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, d\forall + \oiint_{CS} \rho\, (\vec{V}\cdot\hat{n})\, dA$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}\iiint_{CV} \rho\, d\forall + \oiint_{CS} \rho\, (\vec{V}\cdot\hat{n})\, dA = 0}$$

Momentum conservation:

Substituting $B = m\vec{V} \;\;\Rightarrow\;\; b = \vec{V}$ in *RTT*

$$\frac{D(m\vec{V})}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\,\vec{V}\,d\forall + \oiint_{CS} \vec{V}\,\rho\,(\vec{V}\cdot\hat{n})\,dA$$

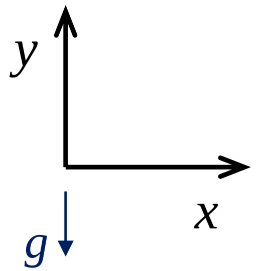
Newton's second law: $\dfrac{D(m\vec{V})}{Dt} = \sum \vec{F}$

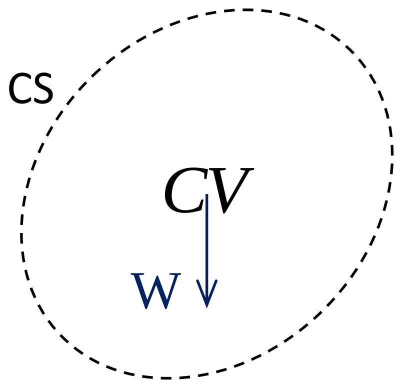
$$\Rightarrow \quad \frac{\partial}{\partial t}\iiint_{CV} \rho\,\vec{V}\,d\forall + \oiint_{CS} \vec{V}\,\rho\,(\vec{V}\cdot\hat{n})\,dA = \sum \vec{F}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\iiint_{CV} \rho\,\vec{V}\,d\forall + \oiint_{CS} \vec{V}\,\rho\,(\vec{V}\cdot\hat{n})\,dA = \vec{F}_{body} + \vec{F}_{surface} + \vec{F}_{external}$$

$\vec{F}_{body}$:

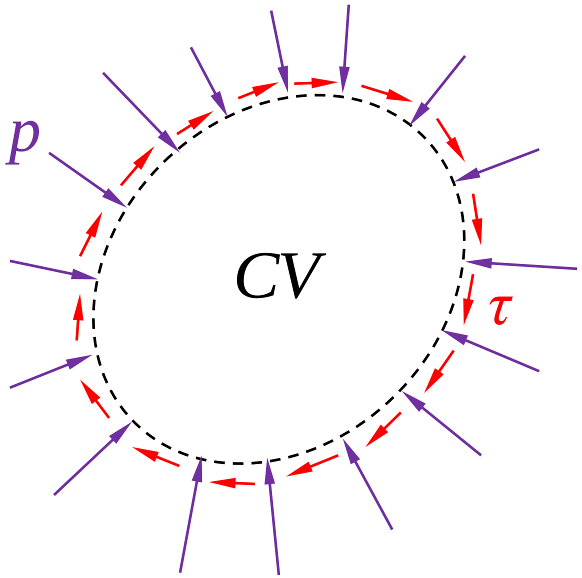
Body force – Weight of the control volume





$$\vec{F}_{body} = \vec{W} = \iiint_{CV} \rho \, \vec{g} \, d\forall$$

$\vec{F}_{surface}$:

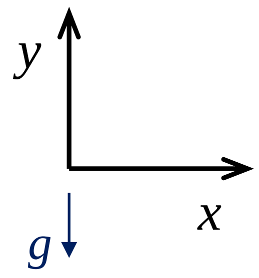


$$\vec{F}_{surface} = \vec{F}_{pressure} + \vec{F}_{shear}$$

$$\vec{F}_{surface} = -\oint\!\!\oint_{CS} p \, dA + \oint\!\!\oint_{CS} \tau \, dA$$

Governing equations for control volume analysis

$$\vec{V} = u\hat{i} + v\hat{j}$$



$$F_{x,body} = 0$$

$$F_{y,body} = -W$$

Continuity equation:

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, d\forall + \oiint_{CS} \rho \, (\vec{V}\cdot\hat{n}) \, dA = 0$$

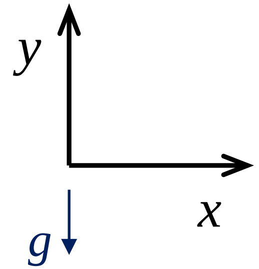
x-momentum equation:

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, u \, d\forall + \oiint_{CS} u \, \rho \, (\vec{V}\cdot\hat{n}) \, dA = F_{x,body} + F_{x,press} + F_{x,shear} + F_{x,ext}$$

y-momentum equation:

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, v \, d\forall + \oiint_{CS} v \, \rho \, (\vec{V}\cdot\hat{n}) \, dA = F_{y,body} + F_{y,press} + F_{y,shear} + F_{y,ext}$$

If the flow properties are uniform over the area
The equations can be expressed in simplified form as….



Continuity equation:

$$\frac{\partial m_{CV}}{\partial t} + \dot{m}_{out} - \dot{m}_{in} = 0$$

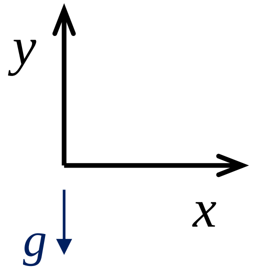
x-momentum equation:

$$\frac{\partial (mu)_{CV}}{\partial t} + (\dot{m}u)_{out} - (\dot{m}u)_{in} = F_{x,press} + F_{x,shear} + F_{x,ext}$$

y-momentum equation:

$$\frac{\partial (mv)_{CV}}{\partial t} + (\dot{m}v)_{out} - (\dot{m}v)_{in} = -W + F_{y,press} + F_{y,shear} + F_{y,ext}$$

If the flow properties are uniform over the area and If the flow is Steady
The equations can be expressed in simplified form as….



Continuity equation:

$$\dot{m}_{out} = \dot{m}_{in}$$

x-momentum equation:

$$(\dot{m}u)_{out} - (\dot{m}u)_{in} = F_{x,press} + F_{x,shear} + F_{x,ext}$$

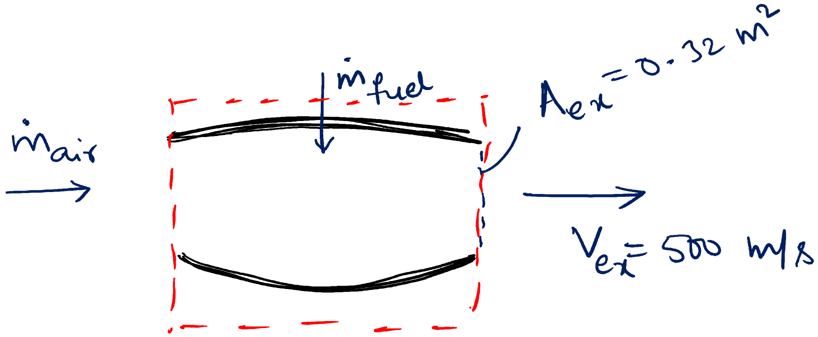
y-momentum equation:

$$(\dot{m}v)_{out} - (\dot{m}v)_{in} = -W + F_{y,press} + F_{y,shear} + F_{y,ext}$$

Problems on Control Volume Analysis

Problem:

At cruise conditions, air flows into a jet engine at a steady rate of 30 kg/s. Fuel enters the engine at a steady rate of 0.27 kg/s. The average velocity of the exhaust gases is 500 m/s relative to the engine. If the engine exhaust effective cross-sectional area is 0.32 m². Estimate the density of the exhaust gases.



$\dot{m}_{air} = 30\ kg/s$

$\dot{m}_{fuel} = 0.27\ kg/s$

$$\dot{m}_{in} = \dot{m}_{out}$$

$$\dot{m}_{air} + \dot{m}_{fuel} = \dot{m}_{ex}$$

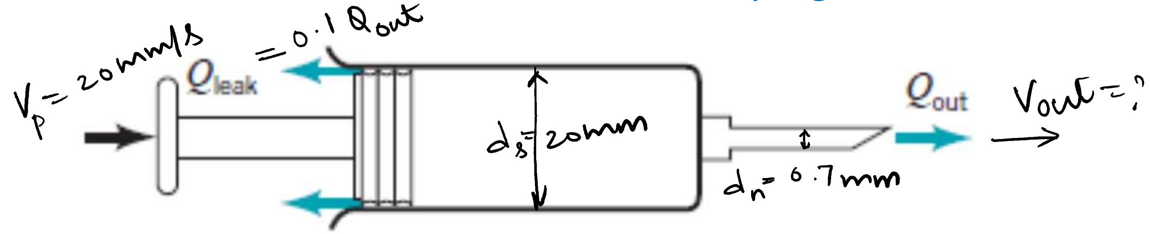
$$30 + 0.27 = \rho_{ex} A_{ex} V_{ex}$$

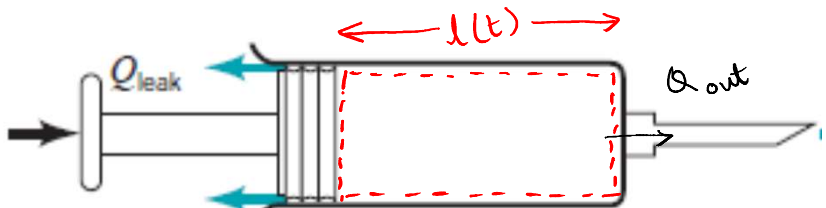
$$\rho_{ex} = \frac{30.27}{0.32 \times 500}$$

$$\rho_{ex} = \underline{\underline{0.189}}\ kg/m^3$$

Problem:

A hypodermic syringe is used to apply a vaccine. If the plunger is moved forward at the steady rate of 20 mm/s and if vaccine leaks past the plunger at 0.1 of the volume flow rate out the needle opening, calculate the average velocity of the needle exit flow. The inside diameters of the syringe and the needle are 20 mm and 0.7 mm.





Continuity:

$$\frac{\partial m_{cv}}{\partial t} + \dot m_{out} - \cancel{\dot m_{in}}^{\,0} = 0 \quad ✓$$

$$m_{cv} = \rho \forall_{cv} = \rho\left[\frac{\pi}{4} d_s^2\, l\right].$$

$$\dot m_{out} = \rho Q_{out} + \rho Q_{leak} = \rho \times 1.1\, Q_{out}.$$

$$\frac{\partial}{\partial t}\left[\rho \frac{\pi}{4} d_s^2\, l\right] + \rho \times 1.1 \times Q_{out} = 0$$

$$\cancel{\rho}\frac{\pi}{4} d_s^2 \frac{dl}{dt} + \cancel{\rho} \times 1.1 \times Q_{out} = 0 \qquad (\rho = \text{const})$$

$$\frac{dl}{dt} = -V_p = -0.02 \text{ m/s}$$

$$\frac{\pi}{4} \times (0.02)^2 \times (-0.02) + 1.1 \times Q_{out} = 0$$

$$= \frac{\pi}{4} \times (0.02)^3 = 1.1 \times Q_{out}$$

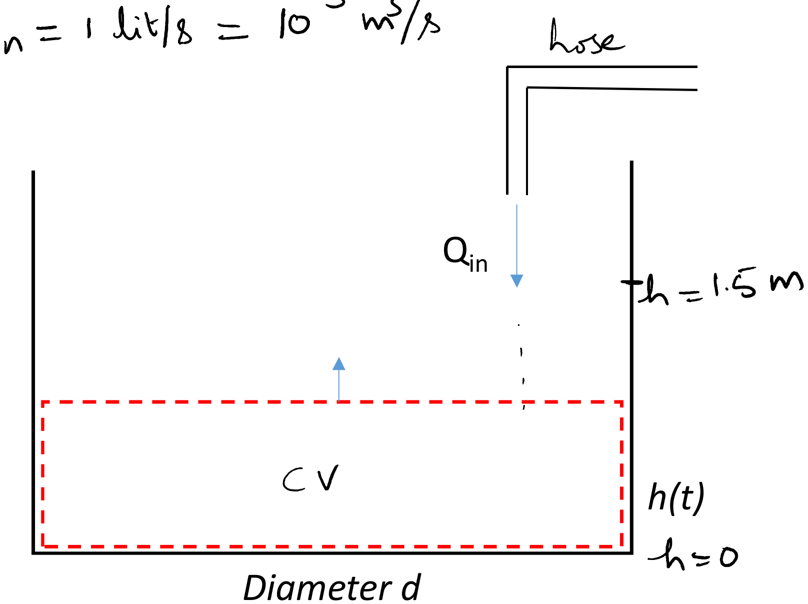
$Q_{out} = 1.$ $\qquad m^3/s$

$$Q_{out} = A_{out} V_{out}$$

$$V_{out} = \frac{Q_{out}}{A_{out}} = \frac{Q_{out}}{\left(\frac{\pi}{4} \times d_n^2\right)} = \qquad m/s$$

5.28 How long would it take to fill a cylindrical-shaped swimming pool having a diameter of 8 m to a depth of 1.5 m with water from a garden hose if the flowrate is 1.0 liter/s?

$Q_{in} = 1 \text{ lit/s} = 10^{-3} \text{ m}^3/\text{s}$



Continuity:

$$\frac{\partial m_{CV}}{\partial t} + \underbrace{\cancel{\dot{m}_{out}}}_{0} - \dot{m}_{in} = 0$$

$$m_{cv} = \rho \forall_{cv} = \rho\left[\frac{\pi}{4} d^2 h(t)\right]$$

$$\dot{m}_{in} = \rho Q_{in}$$

$$\frac{\partial}{\partial t}\left[\rho \frac{\pi}{4} d^2 h(t)\right] = \rho Q_{in}$$

$\rho = \text{Constant}$

$$\cancel{\rho} \frac{\pi}{4} d^2 \frac{dh}{dt} = \cancel{\rho} Q_{in}$$

$$\Rightarrow \frac{\pi}{4} d^2 \cdot \frac{1}{Q_{in}} dh = dt$$

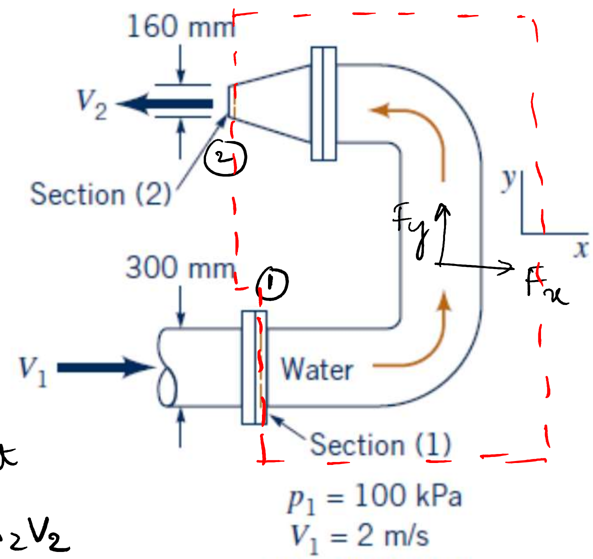
$$\Rightarrow \frac{\pi}{4} d^2 \cdot \frac{1}{Q_{in}} \int_0^{1.5} dh = \int_0^t dt$$

$$\Rightarrow \frac{\pi}{4} \times 8^2 \times \frac{1}{10^{-3}} \times [h]_0^{1.5} = [t]_0^t$$

$$\Rightarrow t = \frac{\pi}{4} \times 8^2 \times \frac{1}{10^{-3}} \times 1.5$$

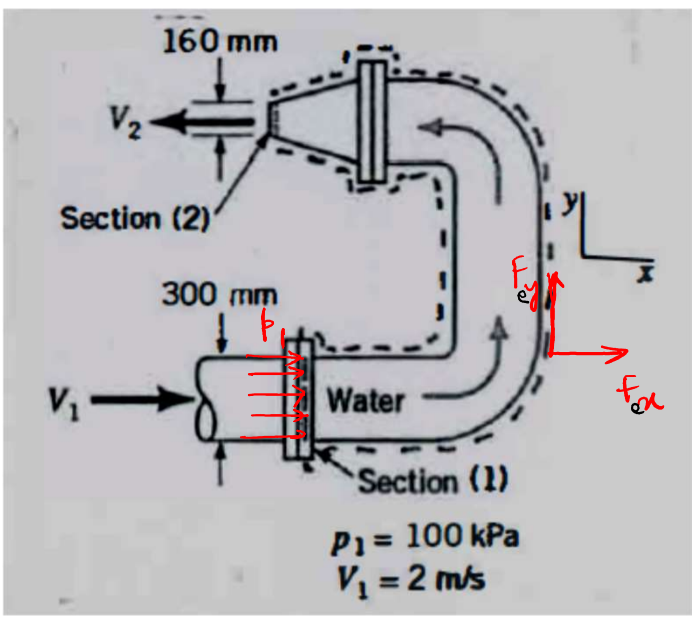
$$= 21 \text{ hours}$$

5.47 Determine the magnitude and direction of the anchoring force needed to hold the horizontal elbow and nozzle combination shown in Fig. P5.47 in place. Atmospheric pressure is 100 kPa(abs). The gage pressure at section (1) is 100 kPa. At section (2), the water exits to the atmosphere.



$Q = AV$

$\dot{m} = \rho A V$



Continuity: $\dot{m}_{in} = \dot{m}_{out}$

Incompressible $Q_{in} = Q_{out}$

$$A_1 V_1 = A_2 V_2$$

$$\frac{\pi}{4} \times d_1^2 \cdot V_1 = \frac{\pi}{4} d_2^2 \times V_2$$

$$V_2 = \left(\frac{d_1}{d_2}\right)^2 \cdot V_1 = \left(\frac{0.3}{0.16}\right)^2 \times 2 = 7.03 \text{ m/s}$$

x-momentum eqn:

$$(\dot{m}u)_{out} - (\dot{m}u)_{in} = \Sigma F_x$$

$$\rho A_2 V_2 (-V_2) - \rho A_1 V_1 (V_1) = p_1 A_1 + F_{ex}$$

$$\Rightarrow \quad F_{ex} = -\rho A_2 V_2^2 - \rho A_1 V_1^2 - p_1 A_1$$

$$= -1000 \times 0.02 \times 7^2 - 1000 \times 0.07 \times 2^2 - 100 \times 10^3 \times 0.07$$

$$= -8260 \text{ N}$$

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.3^2 = 0.07 \, m^2$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.16^2 = 0.02 \, m^2$$

$$\rho_{water} = 1000 \, Kg/m^3$$

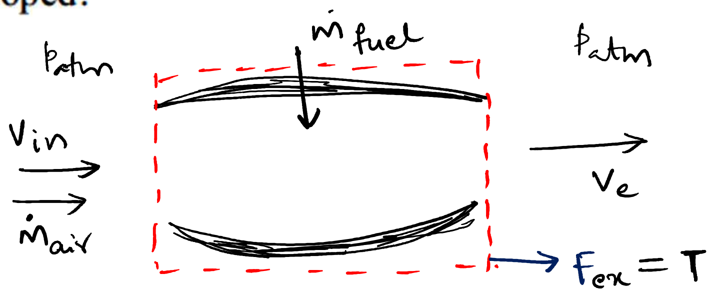
Y-momentum eqn:

$$(\dot{m} v)_{out} - (\dot{m} v)_{in} = \sum F_y = -\underbrace{\cancel{W}}_{\text{Neglect}} + \underbrace{\cancel{F_{y,pres}}}_{0} + \underbrace{\cancel{F_{y,shear}}}_{0} + F_{y,ext}$$

$$0 - 0 = F_{ey}$$

$$F_{ey} = 0$$

Air enters the intake duct of a jet engine at atmospheric pressure and at 152 m/s. Fuel air ratio of the engine is 1/50 by mass. The intake duct area is 0.042 m^2 and the density of air is 1.24 kg/m^3. If the velocity of the exhaust gases relative to the aircraft is 1525 m/s and the exit pressure is atmospheric, what thrust is developed?



$V_{in} = 152$ m/s

$\rho_{air} = 1.24$ kg/m^3

$A_{in} = 0.042$ m^2

$V_e = 1525$ m/s

$\dot{m}_{fuel} = \frac{1}{50}\dot{m}_{air}$

Continuity:

$\dot{m}_{out} = \dot{m}_{in}$

$\dot{m}_{out} = \dot{m}_{air} + \dot{m}_{fuel}$

$\dot{m}_{air} = \rho A_{in} V_{in}$

$= 1.24 \times 0.042 \times 152$

$= 7.9$ Kg/s

x-momentum:-

$$(\dot{m}u)_{out} - (\dot{m}u)_{in} = \Sigma F_x = \cancel{F_{x,pr}}^{\,0} + \cancel{F_{x,sh}}^{\,0} + F_{x,ex}$$

$$(\dot{m}_{air} + \dot{m}_{fuel})V_e - \dot{m}_{air}V_{in} = F_{ex} = T$$

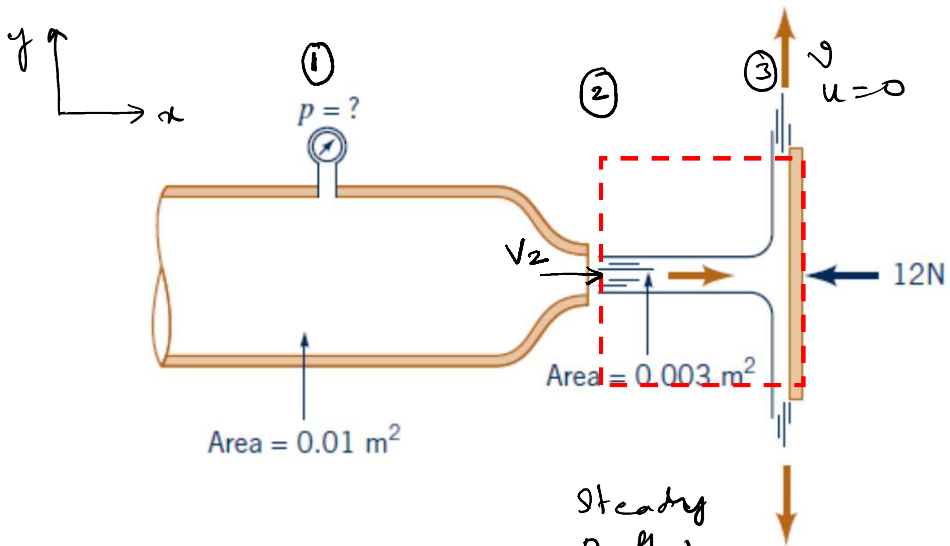
$$T = \left(1 + \frac{1}{50}\right)\dot{m}_{air}V_e - \dot{m}_{air}V_{in}$$

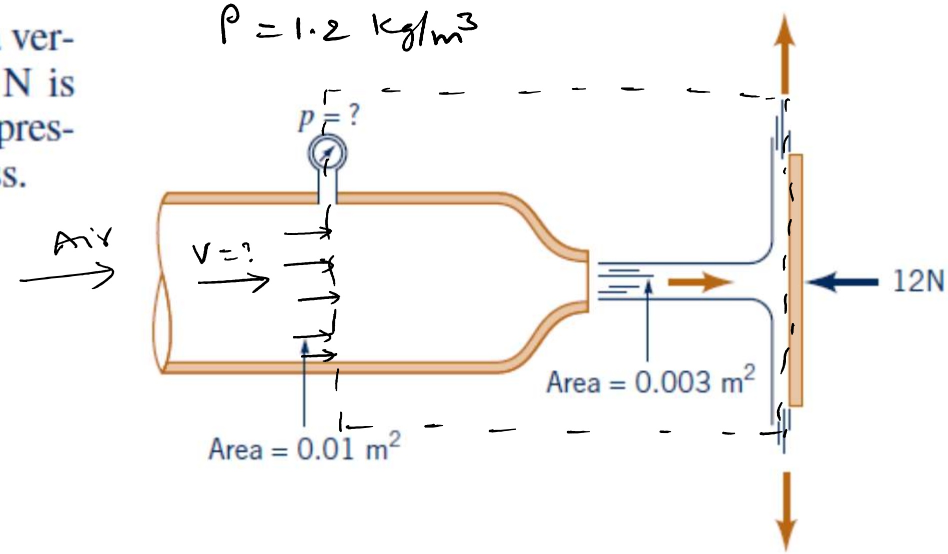
$$= 11{,}087 \text{ N}$$

$$\Rightarrow T = 11 \text{ KN}$$

5.57 Air flows into the atmosphere from a nozzle and strikes a vertical plate as shown in Fig. P5.57. A horizontal force of 12 N is required to hold the plate in place. Determine the reading on the pressure gage. Assume the flow to be incompressible and frictionless.

$\rho = 1.2\ kg/m^3$





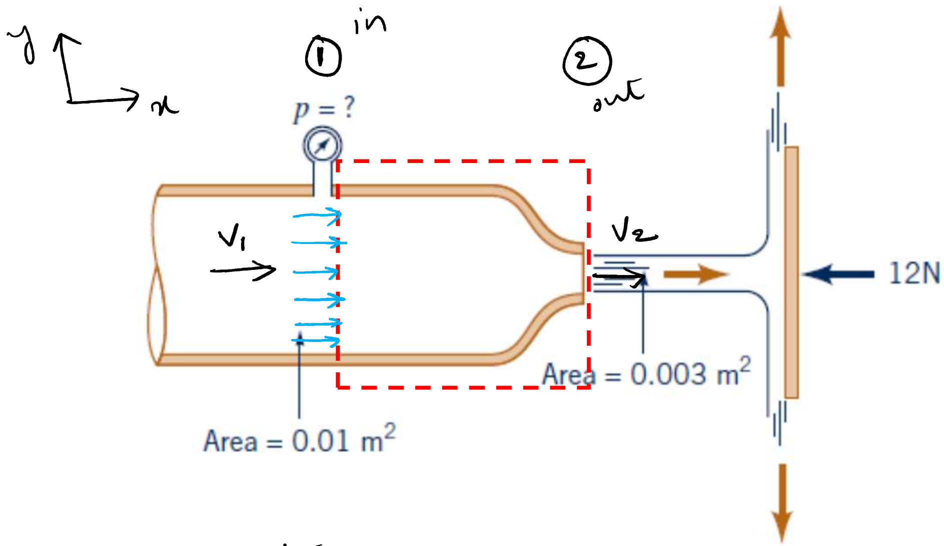
x-momentum:

Steady flow

$$\cancel{\frac{\partial (mu)_{cv}}{\partial t}}^{\,0} + \cancel{(\dot m u)_{out}}_{\,0} - (\dot m u)_{in} = \sum F_x = \cancel{F_{pr,x}}^{\,0} + \cancel{F_{sh,x}}^{\,0} + F_{ex,x}$$

$$-\rho A_2 V_2 V_2 = -12$$

$$\Rightarrow \quad V_2 = \sqrt{\frac{12}{1.2 \times 0.003}} = 57.7\ m/s$$



Continuity:

$$\dot{m}_{in} = \dot{m}_{out} \quad (or) \quad Q_{in} = Q_{out}$$

$$\Rightarrow A_1 V_1 = A_2 V_2$$

$$\Rightarrow 0.01 \times V_1 = 0.003 \times 57.7$$

$$\Rightarrow V_1 = 17.3 \ m/s$$

x-momentum:

$$(\dot{m}u)_{out} - (\dot{m}u)_{in} = \sum F_x = F_{x,pr} + \cancel{F_{x,sh}}^{\,0} + \cancel{F_{x,ext}}^{\,0}$$

frictionless

$$\dot{m} = \rho A V$$

$$(\rho A_2 V_2 V_2) - (\rho A_1 V_1 V_1) = p_1 \times A_1$$

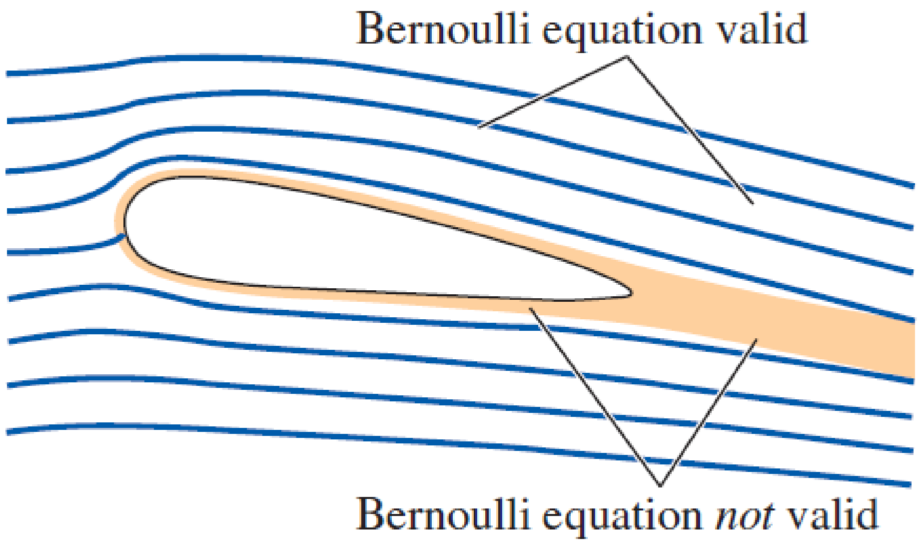
$$p_1 = \frac{\rho A_2 V_2^2 - \rho A_1 V_1^2}{A_1} = \frac{1.2 \times 0.003 \times 57.7^2 - 1.2 \times 0.01 \times 17.3^2}{0.01}$$

$$p_1 = 839.3 \ pa \ (gage)$$

Bernoulli’s equation

- The **Bernoulli equation** is *an approximate relation between pressure, velocity, and elevation,* and is valid in *regions of steady, incompressible flow where net frictional forces are negligible.*

- key approximation in the derivation of the Bernoulli equation - *viscous effects are negligibly small compared to inertial, gravitational, and pressure effects*.



The *Bernoulli equation* is an *approximate* equation that is valid only in *inviscid regions of flow* where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur outside of *boundary layers* and *wakes*.

Euler's equation of motion along a streamline

Consider the motion of a fluid particle along a streamline in steady flow.

Applying Newton's second law in the *s*-direction on a particle moving along a streamline

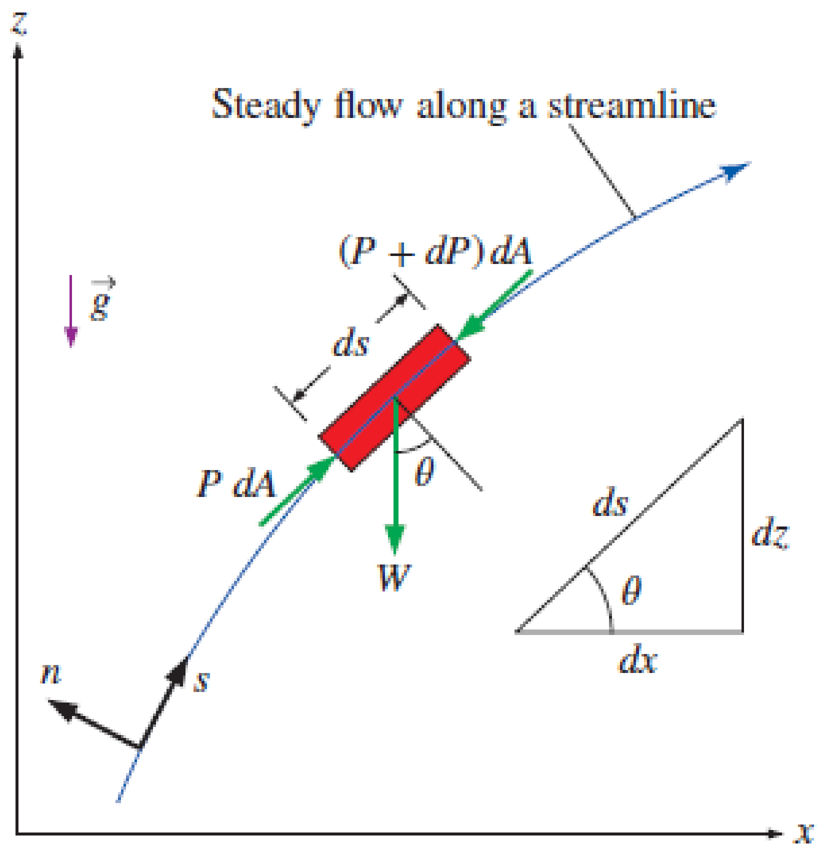
$$\sum F_s = ma_s$$

To find acceleration of the fluid particle a_s

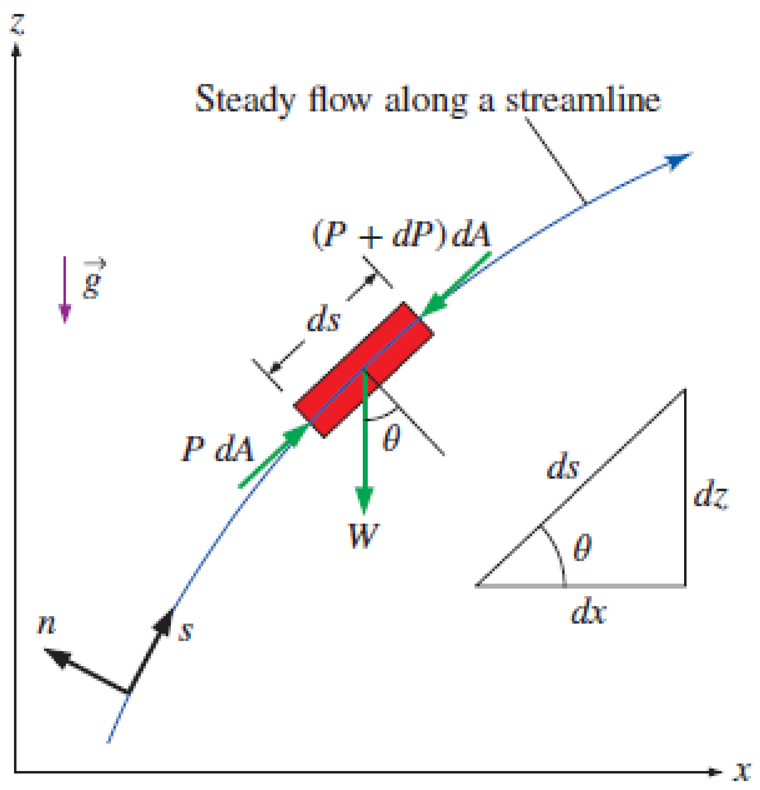
$$dV = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial t} dt$$

$$\Rightarrow \qquad \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t}$$

$$\Rightarrow \quad a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} = \frac{\partial V}{\partial s} V = V\frac{dV}{ds}$$



$$\sum F_s = ma_s \quad \Rightarrow \quad P\,dA - (P + dP)\,dA - W \sin\theta = mV\frac{dV}{ds}$$



Substituting $m = \rho V = \rho\,dA\,ds$

$$W = mg = \rho g\,dA\,ds$$

$$-dP\,dA - \rho g\,dA\,ds\,\frac{dz}{ds} = \rho\,dA\,ds\,V\frac{dV}{ds}$$

$$-dP - \rho g\,dz = \rho V\,dV$$

Noting that $V\,dV = \frac{1}{2}d(V^2)$

$$\boxed{\frac{dP}{\rho} + \frac{1}{2}d(V^2) + g\,dz = 0}$$

On Integrating,

$$\textit{Steady flow:} \qquad \int \frac{dP}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

for an incompressible flow, ρ is constant and can be taken outside the integral

$$\textit{Steady, incompressible flow:} \quad \frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

The Bernoulli equation can also be written between any two points as

$$\textit{Steady, incompressible flow:} \qquad \frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$\overbrace{\frac{P}{\rho}}^{\text{Flow energy}} + \underbrace{\frac{V^2}{2}}_{\text{Kinetic energy}} + \overbrace{gz}^{\text{Potential energy}} = \text{constant}$$

The sum of the kinetic, potential, and flow energies (per unit mass) of a fluid particle is constant along a streamline during steady flow when compressibility and frictional effects are negligible.

Bernoulli's equation in terms of the heads,

$$\overbrace{\frac{P}{\rho g}}^{\text{Pressure head}} + \underbrace{\frac{V^2}{2g}}_{\text{Velocity head}} + \overbrace{z}^{\text{Elevation head}} = \underbrace{H}_{\text{Total head}} = \text{constant}$$

Between two points,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

▶ 6.6 BERNOULLI'S EQUATION FOR REAL FLUID

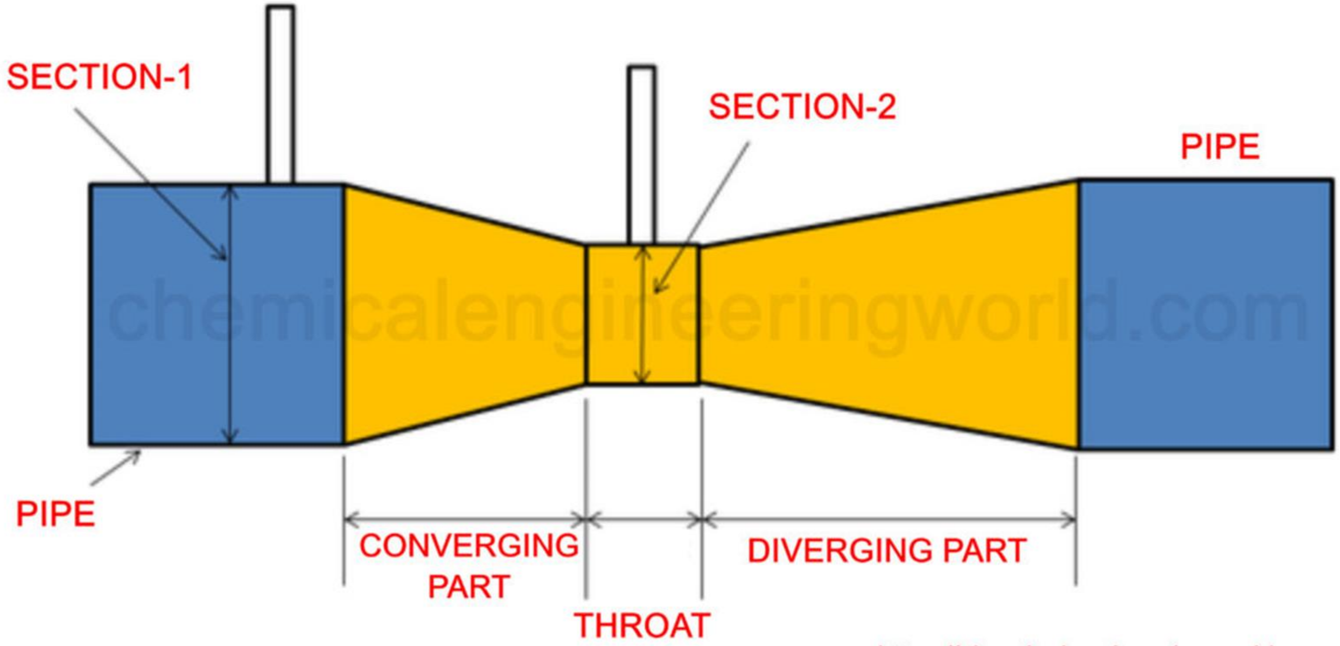
The Bernoulli's equation was derived on the assumption that fluid is inviscid (non-viscous) and therefore frictionless. But all the real fluids are viscous and hence offer resistance to flow. Thus there are always some losses in fluid flows and hence in the application of Bernoulli's equation, these losses have to be taken into consideration. Thus the Bernoulli's equation for real fluids between points 1 and 2 is given as

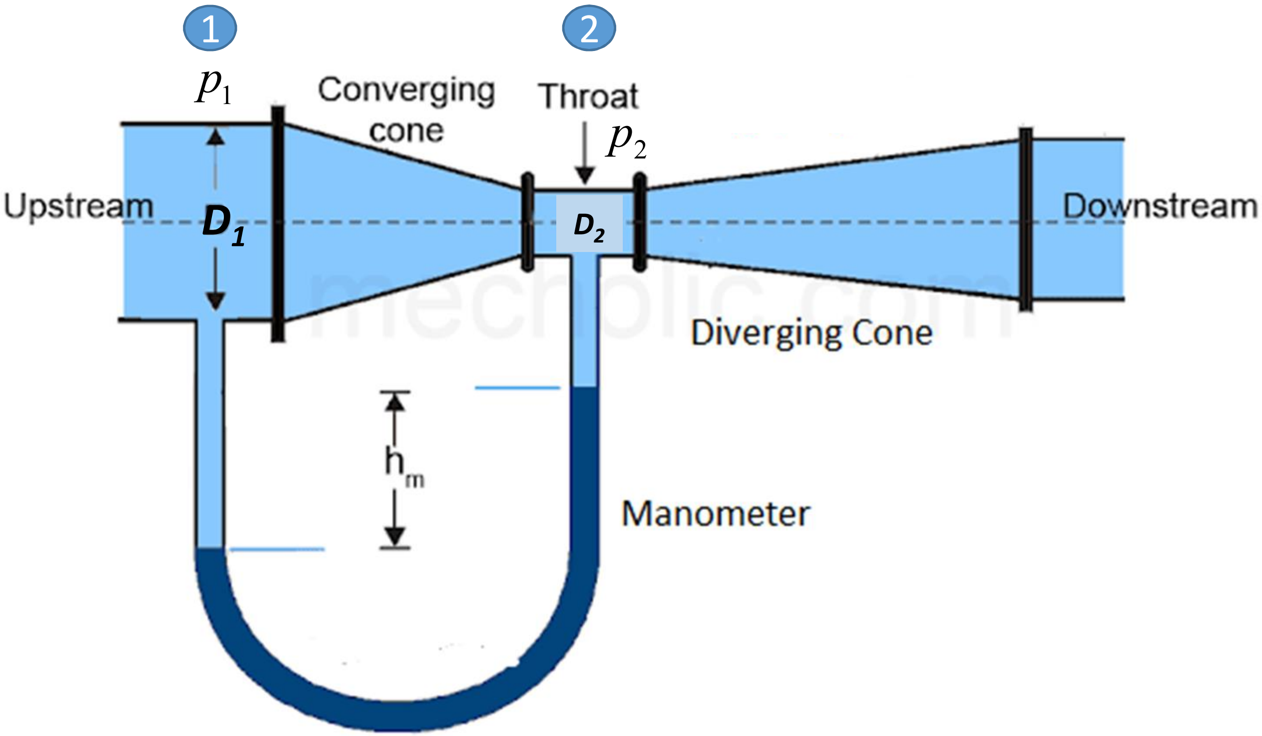
$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_L \qquad \text{...(6.5)}$$

where h_L is loss of energy between points 1 and 2.

Applications of Bernoulli’s equation

Venturimeter





Considering the fluid to be ideal

Applying Bernoulli's equation between Sections 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\Rightarrow \frac{V_2^2}{2g} - \frac{V_1^2}{2g} = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} \qquad \text{------ (1)}$$

From continuity equation between 1 and 2 $\quad V_2 A_2 = V_1 A_1$

$$\Rightarrow V_1 = \frac{A_2}{A_1} V_2 \qquad \text{------ (2)}$$

Substituting (2) in (1)

$$\Rightarrow \frac{V_2^2}{2g}\left(1 - \frac{A_2^2}{A_1^2}\right) = \frac{p_1 - p_2}{\rho g}$$

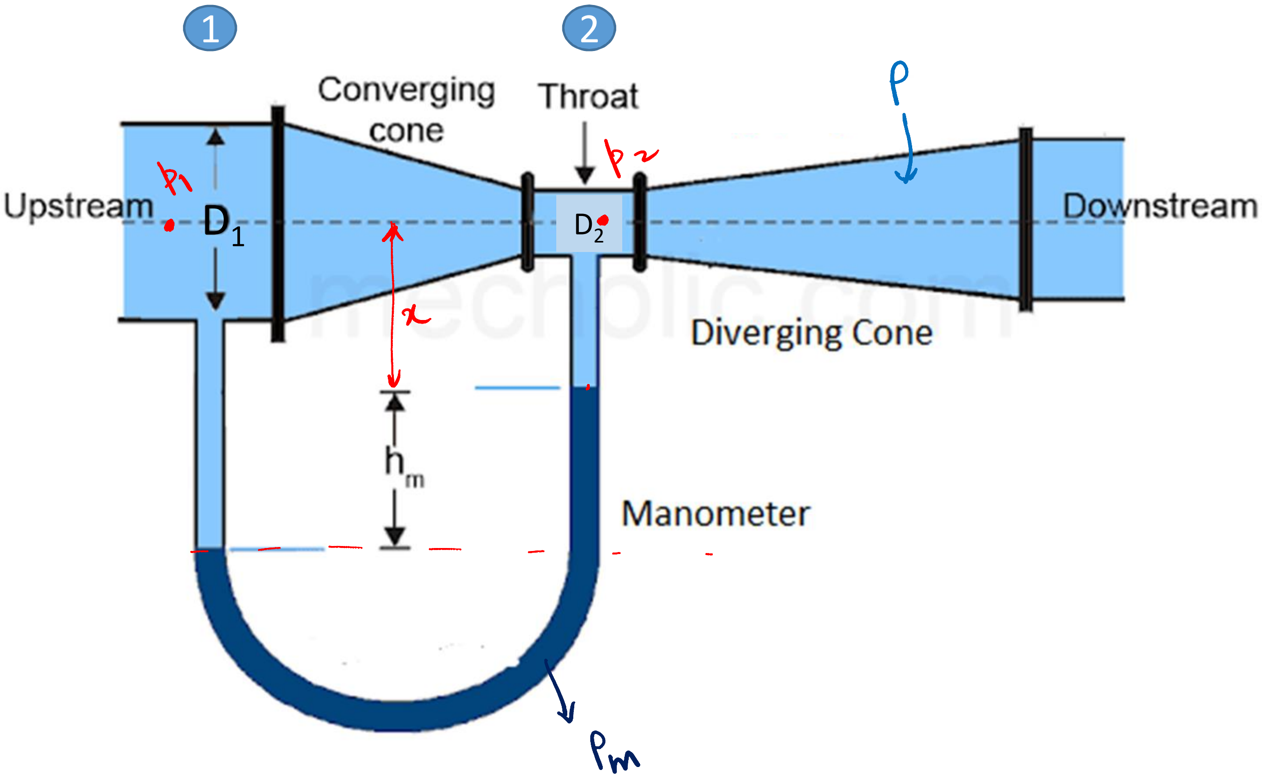
Let $\dfrac{p_1 - p_2}{\rho g} = H$ (Differential Pressure Head)

$$\frac{V_2^2}{2g}\left(1 - \frac{A_2^2}{A_1^2}\right) = H \qquad \Rightarrow V_2 = \sqrt{\frac{2gH}{\left(1 - \frac{A_2^2}{A_1^2}\right)}}$$

$$\Rightarrow V_2 = A_1\sqrt{\frac{2gH}{A_1^2 - A_2^2}}$$

Hence, the volume flow rate through the pipe is given by

$$Q = A_2 V_2 = A_1\, A_2\sqrt{\frac{2gH}{A_1^2 - A_2^2}} \qquad \text{------ } (3)$$



H to be obtained from the manometer

From manometry,

$$p_1 + \rho g(x + h_m) - \rho_m g h_m - \rho g x = p_2$$

$$\Rightarrow p_1 - p_2 = \rho_m g h_m - \rho g h_m$$

$$\Rightarrow \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = \left(\frac{\rho_m}{\rho} - 1\right) h_m$$

$$\Rightarrow H = \left(\frac{\rho_m}{\rho} - 1\right) h_m \qquad \text{------ (4)}$$

Substituting (4) in (3)

Hence, the volume flow rate through the pipe is given by

$$Q = \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2g\left(\frac{\rho_m}{\rho} - 1\right) h_m} \qquad \text{------ eqn (5)}$$

- Measured values of h_m between Sections 1 and 2, for a real fluid will always be greater than that assumed in case of an ideal fluid because of frictional losses in addition to the change in momentum.

- Therefore, eqn (5) always overestimates the actual flow rate. In order to take this into account, a multiplying factor C_d, called the coefficient of discharge, is incorporated in the eqn (5) as

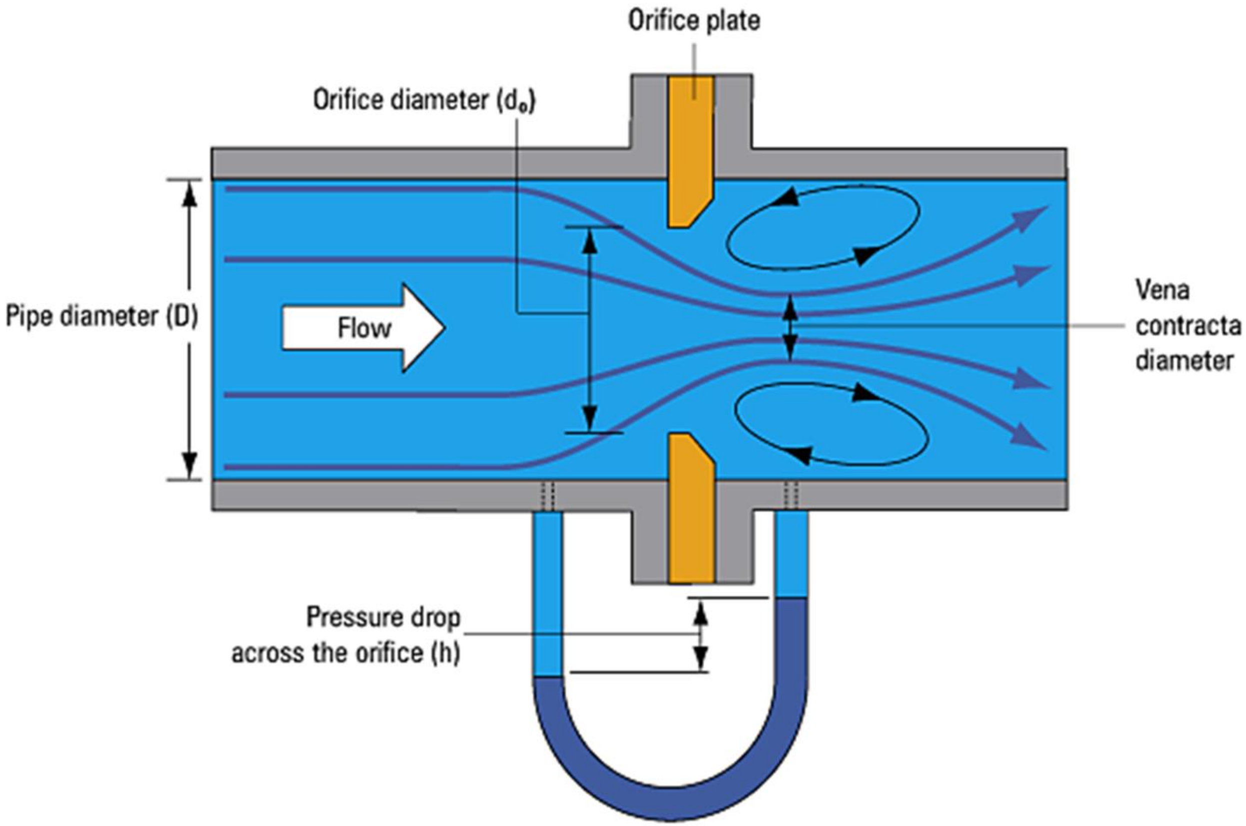
$$Q_{actual} = C_d \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2g\left(\frac{\rho_m}{\rho} - 1\right) h_m}$$

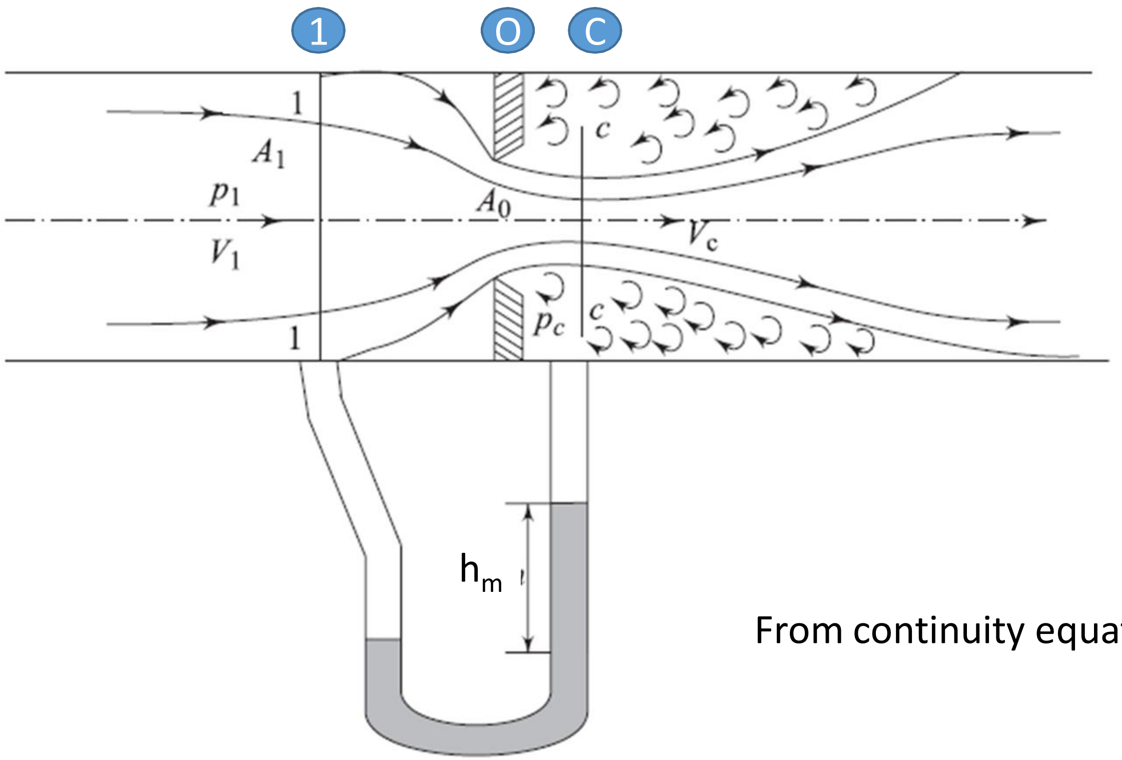
- The **coefficient of discharge** C_d is always less than unity and is defined as $C_d = \dfrac{\text{Actual rate of discharge}}{\text{Theoretical rate of discharge}}$

- Value of C_d for a venturimeter usually lies between 0.95 to 0.98.

Orificemeter

- a simpler and cheaper arrangement for the measurement of flow through a pipe.
- An orificemeter is essentially a thin circular plate with a sharp edged concentric circular hole in it.
- Flow attains minimum cross section at a small distance downstream of the orifice plate called Vena contracta
- Diameter of Vena contracta less than orifice hole diameter





Considering the fluid to be ideal and the downstream pressure taping to be at the vena contracta (Sec. c-c), we can write, by applying Bernoulli's theorem between Sections 1-1 and Sec. c-c,

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_C}{\rho g}+\frac{V_C^2}{2g}+z_C$$

$$\Rightarrow \frac{V_C^2}{2g}-\frac{V_1^2}{2g}=\frac{p_1}{\rho g}-\frac{p_C}{\rho g} \qquad \text{------ } (1)$$

From continuity equation between 1 and c $\quad A_1V_1=A_CV_C$

$$\Rightarrow \quad V_1=\frac{A_C}{A_1}V_C \qquad \text{------ } (2)$$

Substituting (2) in (1) $\quad \Rightarrow \dfrac{V_C^2}{2g}\left(1-\dfrac{A_C^2}{A_1^2}\right)=\dfrac{p_1}{\rho g}-\dfrac{p_C}{\rho g}$

Let $\dfrac{p_1 - p_C}{\rho g} = H$ (Differential Pressure Head)

$$\Rightarrow \frac{V_C^2}{2g}\left(1 - \frac{A_C^2}{A_1^2}\right) = H$$

$$\Rightarrow V_C = \sqrt{\frac{2gH}{\left(1 - \dfrac{A_C^2}{A_1^2}\right)}}$$

V_c has been obtaining assuming the flow to be ideal (frictionless). Therefore, for a real fluid

$$V_C = C_V \sqrt{\frac{2g(h_1 - h_C)}{\left(1 - \dfrac{A_C^2}{A_1^2}\right)}}$$

Where, C_V is the coefficient of velocity and $C_V<1$

From manometry, $H = \left(\frac{\rho_m}{\rho} - 1\right) h_m$

$$\Rightarrow V_C = C_V \sqrt{\frac{2g\left(\frac{\rho_m}{\rho} - 1\right) h_m}{\left(1 - \frac{A_C^2}{A_1^2}\right)}}$$

Hence, the volume flow rate through the pipe is given by $Q_{actual} = A_C V_C$

$$= C_C A_0 V_C$$

where, C_C is the coefficient of contraction defines as $C_C = A_C / A_0$

$$Q_{actual} = C_C A_0 V_C$$

$$= C_C A_0 C_V \sqrt{\frac{2g\left(\frac{\rho_m}{\rho} - 1\right)h_m}{\left(1 - \frac{A_C^2}{A_1^2}\right)}}$$

$$= C_C C_V \frac{A_0 A_1}{\sqrt{A_1^2 - C_C^2 A_0^2}} \sqrt{2g\left(\frac{\rho_m}{\rho} - 1\right)h_m}$$

$$\Rightarrow \; Q_{actual} = C_d \frac{A_1 A_0}{\sqrt{A_1^2 - C_C^2 A_0^2}} \sqrt{2g\left(\frac{\rho_m}{\rho} - 1\right)h_m}$$

$$Q_{actual} = C_d \frac{A_1 A_0}{\sqrt{A_1^2 - C_C^2 A_0^2}} \sqrt{2g\left(\frac{\rho_m}{\rho} - 1\right) h_m}$$

$$= \frac{C_d \sqrt{A_1^2 - A_0^2}}{\sqrt{A_1^2 - C_C^2 A_0^2}} \frac{A_0 A_1}{\sqrt{A_1^2 - A_0^2}} \sqrt{2g\left(\frac{\rho_m}{\rho} - 1\right) h_m}$$

$$Q_{actual} = C_d^{'} \frac{A_0 A_1}{\sqrt{A_1^2 - A_0^2}} \sqrt{2g\left(\frac{\rho_m}{\rho} - 1\right) h_m}$$

Static, Dynamic, and Stagnation Pressures

Static Pressure:

- The thermodynamic or hydrostatic pressure caused by molecular collisions is known as static pressure in a fluid flow and is usually referred to as the pressure *p*.
- If a hole is made at the wall and is connected to any pressure measuring device, it will then sense the static pressure at the wall. This type of hole at the wall is known as a **wall tap.**

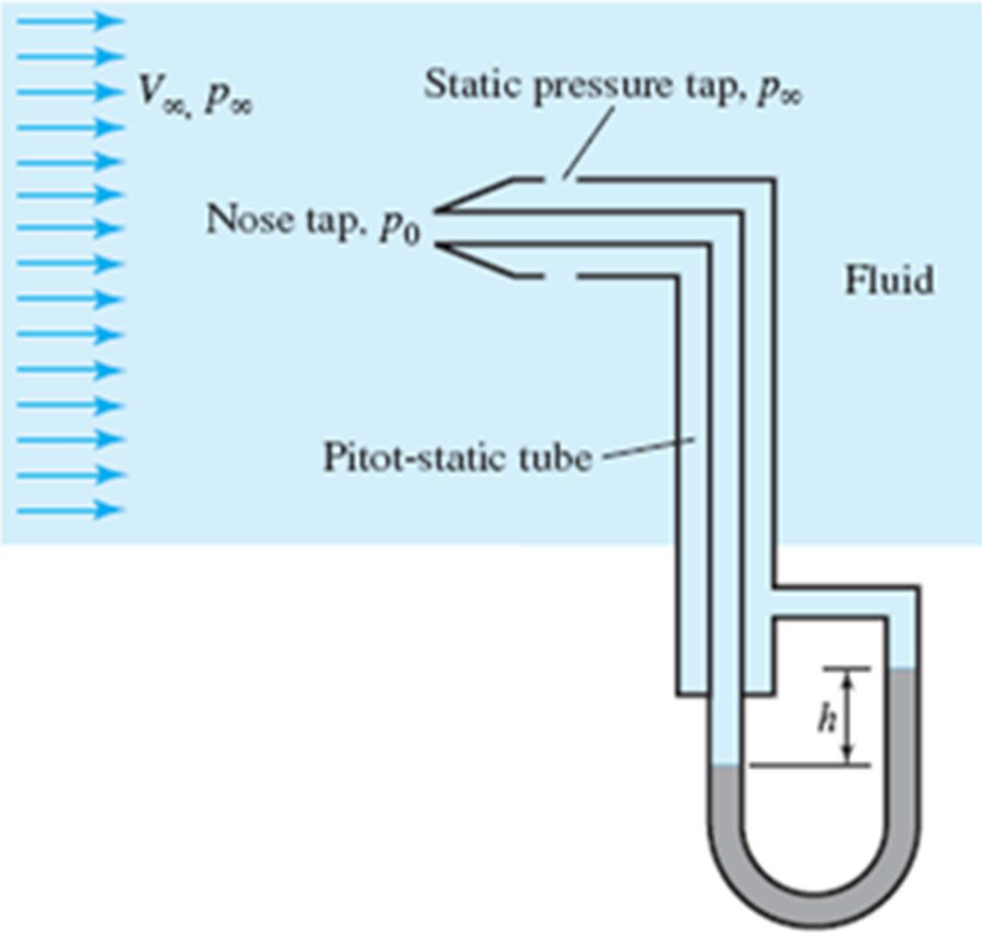
Stagnation Pressure:

- The stagnation pressure at a point in a fluid flow is the pressure which could result if the fluid were brought to rest isentropically.
- It is the sum of static pressure and Dynamic Pressure

$$P + \rho \frac{V^2}{2} = \text{constant}$$ for a negligible change in z,

$$p + \frac{1}{2}\rho V^2 = p_0$$ (or) $$\frac{p}{\rho g} + \frac{V^2}{2g} = \frac{p_0}{\rho g}$$

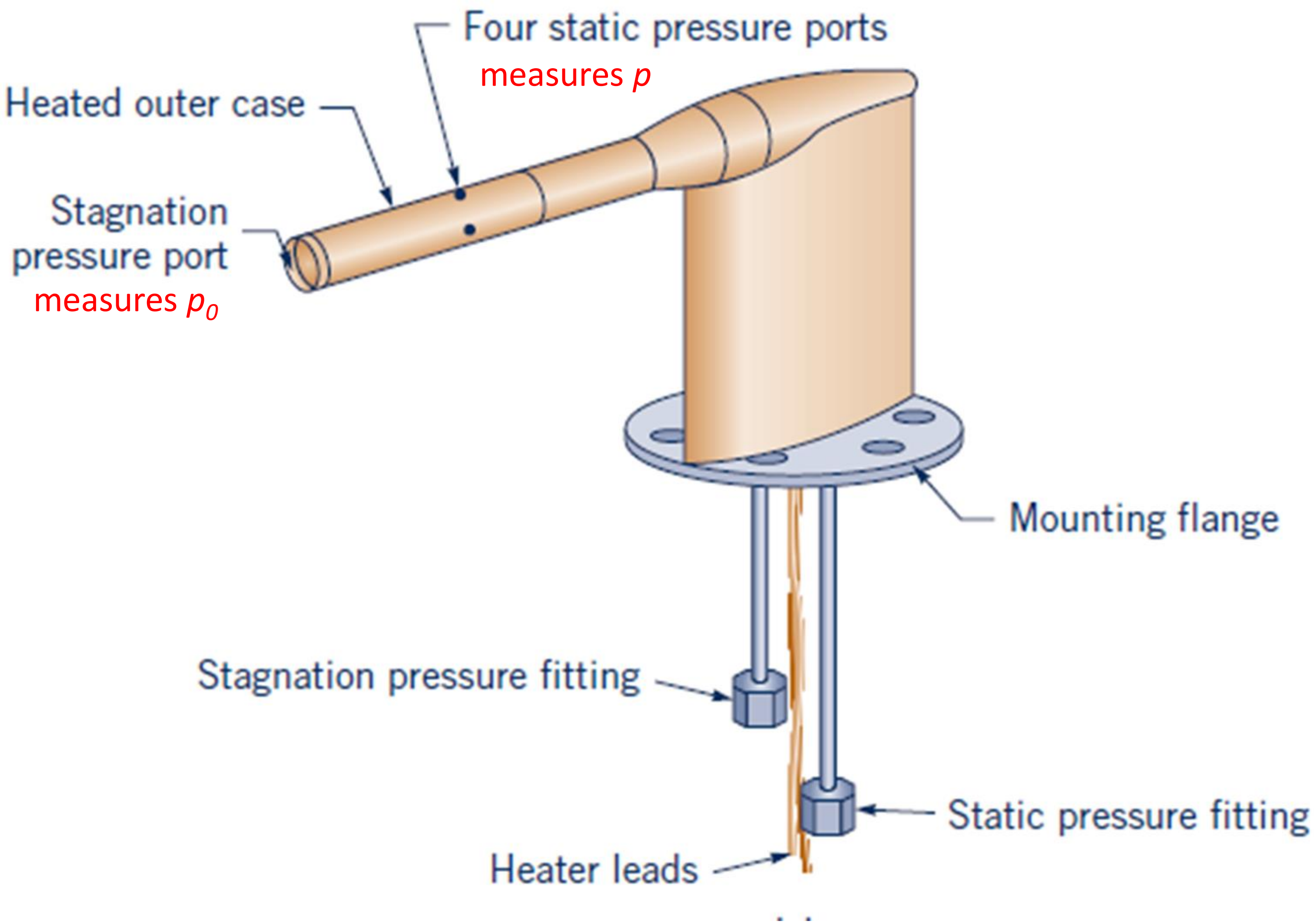
Pitot-static tube



$$p_0 = p_\infty + \frac{1}{2}\rho_\infty V_\infty{}^2$$

$$\Rightarrow V_\infty = \sqrt{\frac{2(p_0 - p_\infty)}{\rho_\infty}}$$

Airplane Pitot-static probe



Problem *An oil of sp. gr. 0.8 is flowing through a venturimeter having inlet diameter 20 cm and throat diameter 10 cm. The oil-mercury differential manometer shows a reading of 25 cm. Calculate the discharge of oil through the horizontal venturimeter. Take $C_d = 0.98$.*

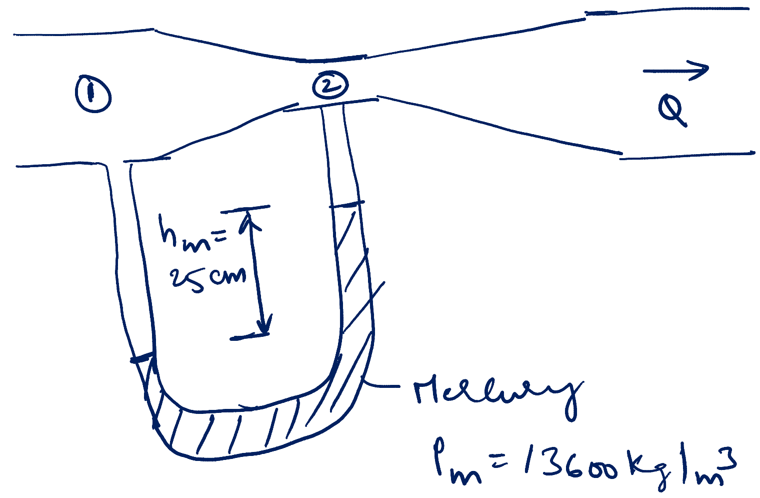
$d_1 = 20\text{ cm}$ $\quad d_2 = 10\text{ cm}$ $\quad \rho = 800\text{ kg/m}^3$

Sol:

* $Q = C_d \dfrac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2g(h_1 - h_2)}$

* $h_1 - h_2 = \left(\dfrac{\rho_m}{\rho} - 1\right) h_m$

$= \left(\dfrac{13600}{800} - 1\right) \times \dfrac{25}{100} = 4 \text{ m}$



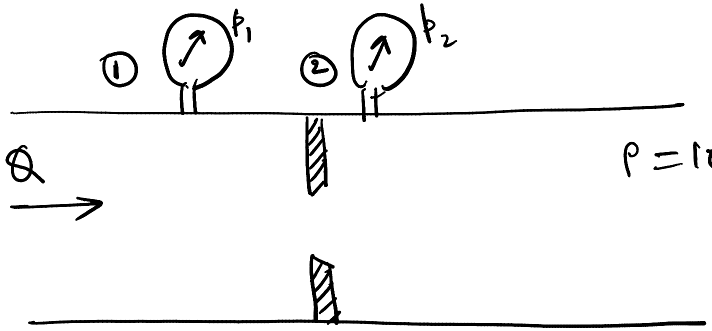
Inlet Area $A_1 = \dfrac{\pi}{4} d_1^2 = \dfrac{\pi}{4} \times 0.2^2 = \qquad \text{m}^2$

Throat Area $A_2 = \dfrac{\pi}{4} d_2^2 = \dfrac{\pi}{4} \times 0.1^2 = \qquad \text{m}^2$

1 - Inlet
2 - Throat

$Q = C_d \dfrac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2g(h_1 - h_2)} = \qquad \text{m}^3/\text{s}$

Problem *An orifice meter with orifice diameter 10 cm is inserted in a pipe of 20 cm diameter. The pressure gauges fitted upstream and downstream of the orifice meter gives readings of 19.62 N/cm^2 and 9.81 N/cm^2 respectively. Co-efficient of discharge for the orifice meter is given as 0.6. Find the discharge of water through pipe.* (Ans: 68.21 lit/s)



$\rho = 1000\, kg/m^3$

$d_1 = 20 cm \quad d_2 = 10 cm$

$p_1 = 19.62 N/cm^2 \quad p_2 = 9.81 N/cm^2$

$C_d = 0.6$

1 – Inlet / pipe

2 – Orifice

$p_1 = 19.62 \frac{N}{cm^2} = \frac{19.62}{(10^{-2})^2} \frac{N}{m^2}$

$p_1 = 19.62 \times 10^4 \, N/m^2$

Sol:

$$Q = C_d \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2g(h_1 - h_2)}$$

$$h_1 = \frac{p_1}{\rho g} = \frac{19.62 \times 10^4 \, N/m^2}{1000 \times 9.81} = 20 \quad m$$

$$h_2 = \frac{p_2}{\rho g} = \frac{9.81 \times 10^4}{1000 \times 9.81} = 10 \quad m$$

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.2^2 = \quad m^2$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.1^2 = \quad m^2$$

$$Q = C_d \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \sqrt{2g(h_1 - h_2)} = \quad m^3/s$$

Problem *A sub-marine moves horizontally in sea and has its axis 15 m below the surface of water. A pitot-tube properly placed just in front of the sub-marine and along its axis is connected to the two limbs of a U-tube containing mercury. The difference of mercury level is found to be 170 mm. Find the speed of the sub-marine knowing that the sp. gr. of mercury is 13.6 and that of sea-water is 1.026 with respect of fresh water.* (Ans: 6.4 m/s)

Sol:

$$p_\infty + \frac{1}{2}\rho_\infty V_\infty^2 = p_0 \Rightarrow V_\infty = \sqrt{\frac{2(p_0 - p_\infty)}{\rho_\infty}}$$

$\rho = SG \times 1000 \ kg/m^3$

$$p_0 - p_\infty = (\rho_m - \rho_\infty) g h$$

$$= (13600 - 1026) \times 9.81 \times \frac{170}{1000}$$

$$= \qquad N/m^2$$

$$V_\infty = \sqrt{\frac{2(p_0 - p_\infty)}{\rho_\infty}} = \sqrt{\frac{2 \times}{1026}} = \qquad m/s$$

Numericals on Bernoullis' equation

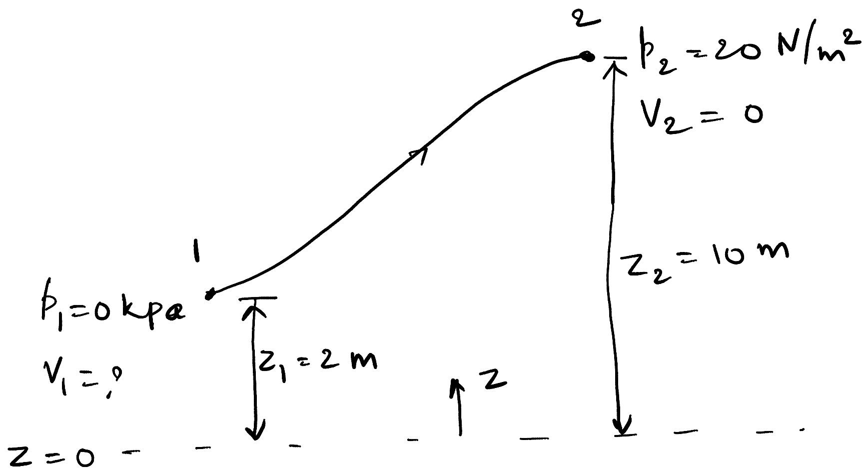
3.2 Air flows steadily along a streamline from point (1) to point (2) with negligible viscous effects. The following conditions are measured: At point (1) $z_1 = 2$ m and $p_1 = 0$ kPa; at point (2) $z_2 = 10$ m, $p_2 = 20\ \text{N/m}^2$, and $V_2 = 0$. Determine the velocity at point (1).

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$0 + \frac{V_1^2}{2\times 9.81} + 2 = \frac{20}{1.2\times 9.81} + 0 + 10$$

$$V_1^2 = 2\times 9.81 \left[\frac{20}{1.2\times 9.81} + 10 - 2\right]$$

$$\Rightarrow V_1 = \qquad \text{m/s}$$

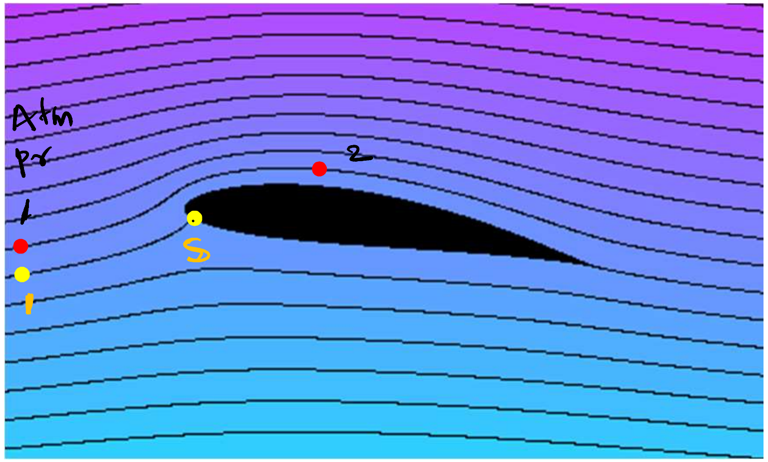


$\rho_{air} = 1.2\ \text{kg/m}^3$

When an airplane is flying 322 kmph at 5000-ft altitude (density=0.74 kg/m3) in a standard atmosphere, the air velocity at a certain point on the wing is 439 kmph relative to the airplane.

(a) What suction pressure is developed on the wing at that point?

(b) What is the pressure at the leading edge (a stagnation point) of the wing?



(a) $V_1 = 322$ km/hr $= 89.4$ m/s

p_1

$z_1 = z_2$

$V_2 = 439$ km/hr $= 121.9$ m/s

p_2

Applying Bernoulli's eqn b/w 1 & 2

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + \cancel{z_2}$$

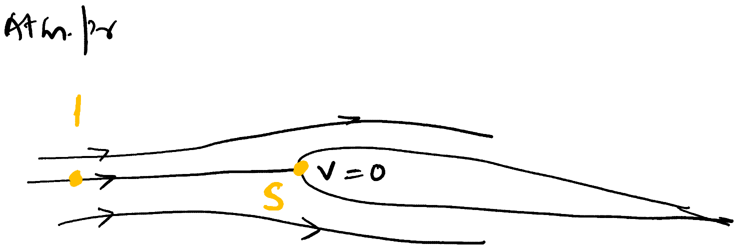
$$\frac{p_2}{\rho g} - \frac{p_1}{\rho g} = \frac{V_1^2}{2g} - \frac{V_2^2}{2g} \Rightarrow p_2 - p_1 = \frac{\rho}{2}\left(V_1^2 - V_2^2\right) = \frac{0.74}{2} \times \left(89.4^2 - 121.9^2\right)$$

$$\Rightarrow p_2 - p_1 = -2540.8 \text{ Pa} \Rightarrow p_2 = p_1 - 2540.8 \text{ Pa}$$

$p_1 = p_{atm} \Rightarrow p_1 = 0$ (gage)

$p_2 = -2540.8$ Pa (gage)

Applying Bernoulli's b/w 1 and S



$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{p_s}{\rho g} + \frac{V_s^2}{2g} + \cancel{z_s}$$

$$\frac{p_1}{\rho g} + \frac{89.4^2}{2\times 9.81} = \frac{p_s}{\rho g} + 0$$

$$\frac{p_s}{\rho g} = \frac{p_1}{\rho g} + \frac{89.4^2}{2\times 9.81}$$

$$p_s = p_1 + \frac{0.74 \times 89.4^2}{2}$$

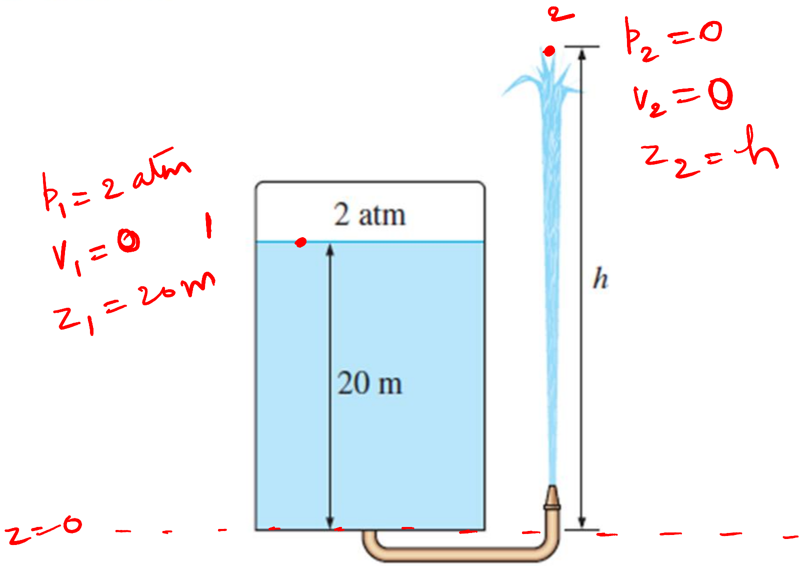
$$\Rightarrow \quad p_s = p_1 + 2957.1 \quad \text{Pa}$$

$$\Rightarrow \quad p_s = 2957.1 \text{ Pa} \quad \text{(gage)}$$

S- Stagnation point
(A point where the velocity is zero)

(p_1 = Atmospheric pressure
$\Rightarrow p_1 = 0$ (gage))

5–58 The water level in a tank is 20 m above the ground. A hose is connected to the bottom of the tank, and the nozzle at the end of the hose is pointed straight up. The tank cover is airtight, and the air pressure above the water surface is 2 atm gage. The system is at sea level. Determine the maximum height to which the water stream could rise. *Answer:* 40.7 m



1 atm = 1.01325×10^5 pa $\approx 1 \times 10^5$ pa

$\rho_{water} = 1000$ Kg/m^3

Apply Bernoulli's b/w 1 and 2

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

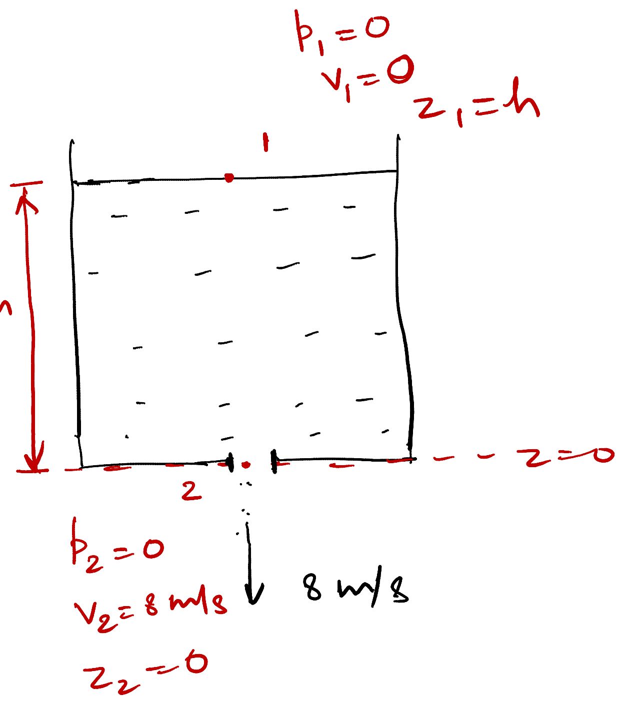
$$\frac{2 \times 101325}{1000 \times 9.81} + 0 + 20 = 0 + 0 + h$$

$$\Rightarrow h = 20.7 + 20$$

$$h = \underline{\underline{40.7 \text{ m}}}$$

3.32 Water flows through a hole in the bottom of a large, open tank with a speed of 8 m/s. Determine the depth of water in the tank. Viscous effects are negligible.

Assume Steady



p_1, p_2 are gage pressures

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

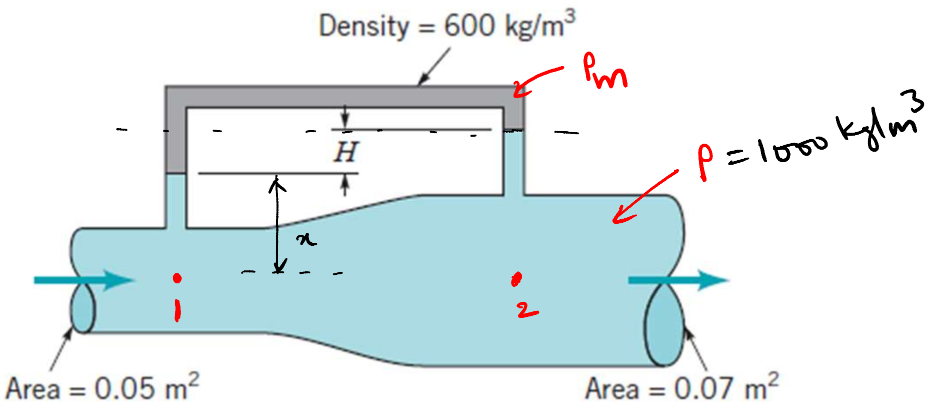
$$0 + 0 + h = 0 + \frac{8^2}{2 \times 9.81} + 0$$

$$\Rightarrow h = 3.26 \text{ m}$$

If you are using Absolute pressures at 1 & 2

$$\cancel{\frac{p_{atm}}{\rho g}} + 0 + h = \cancel{\frac{p_{atm}}{\rho g}} + \frac{8^2}{2 \times 9.81} + 0$$

3.63 Water flows steadily through the variable area pipe shown in Fig. P3.63 with negligible viscous effects. Determine the manometer reading, H, if the flowrate is 0.5 m^3/s and the density of the manometer fluid is 600 kg/m^3.



$$Q = 0.5 \ m^3/s$$

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + \cancel{z_2}$$

Continuity: $Q = A_1V_1 = A_2V_2$

$Q = A_1V_1$

$0.5 = 0.05 \times V_1$

$\Rightarrow V_1 = 10 \ m/s$

$Q = A_2V_2$

$0.5 = 0.07 \times V_2$

$\Rightarrow V_2 = 7.14 \ m/s$

From manometry,

$$p_1 - \rho g x - \rho_m g H + \rho g(H + x) = p_2$$

$$p_1 - p_2 = (\rho_m - \rho) g H$$

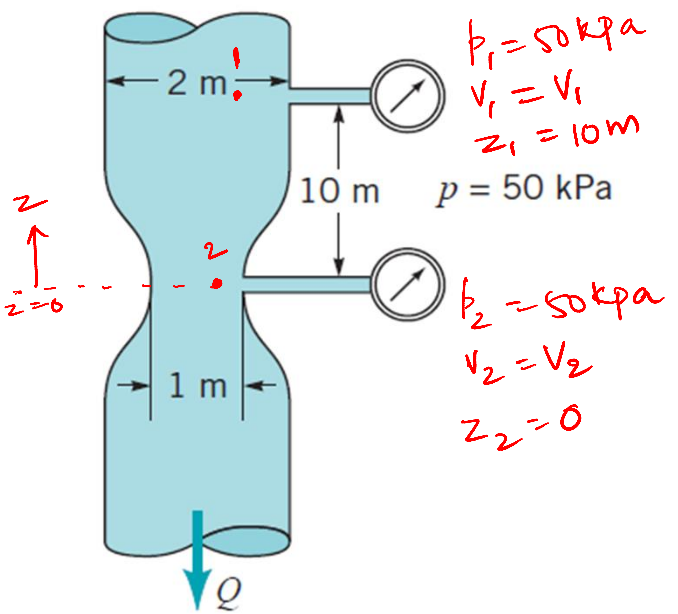
$$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$(\rho_m - \rho) g H = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$(600 - 1000)\, 9.81 \times H = \frac{1000}{2} \times (7.14^2 - 10^2)$$

$$H = \qquad m$$

3.47 Water (assumed inviscid and incompressible) flows steadily in the vertical variable-area pipe shown in Fig. P3.47. Determine the flowrate if the pressure in each of the gages reads 50 kPa.. (Ans: 11.4 m³/s)



1-2: $\frac{\cancel{p_1}}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{\cancel{p_2}}{\rho g} + \frac{V_2^2}{2g} + \cancel{z_2}^{\,0}$ $(p_1 = p_2)$

$V_2^2 - V_1^2 = 2g \times 10 = 2 \times 9.81 \times 10 =$

$V_2^2 - V_1^2 = 196.2$ — ①

Continuity: $Q = A_1 V_1 = A_2 V_2$

$V_1 = \frac{A_2}{A_1} V_2 = \frac{\frac{\pi}{4} d_2^2}{\frac{\pi}{4} d_1^2} \times V_2 = \left(\frac{d_2}{d_1}\right)^2 \times V_2$

$= \frac{1}{4} \times V_2$

$\Rightarrow V_1 = 0.25\, V_2$ — ②

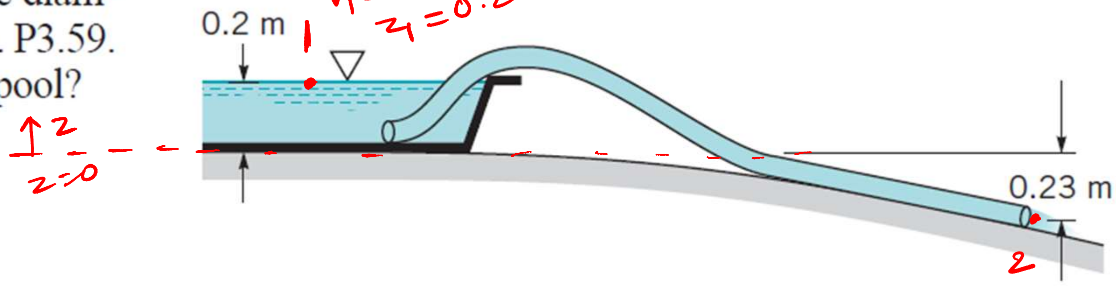
② in ① $\Rightarrow$ $V_2^2 - (0.25 V_2)^2 = 196.2$

$\Rightarrow V_2 = 14.46$ m/s

Flow rate $Q = A_2 V_2 = \frac{\pi}{4} d_2^2 V_2 = \frac{\pi}{4} \times 1^2 \times 14.46 = \underline{\underline{11.35 \text{ m}^3/\text{s}}}$

3.59 A smooth plastic, 10-m-long garden hose with an inside diameter of 20 mm is used to drain a wading pool as is shown in Fig. P3.59. If viscous effects are neglected, what is the flowrate from the pool?
(Ans: $9.11 \times 10^{-4}\ m^3/s$)

$p_1 = 0$
$V_1 = 0$
$z_1 = 0.2\,m$



$p_2 = 0$
$V_2 = V_2$
$z_2 = -0.23\,m$

$$\frac{\cancel{p_1}}{\rho g} + \frac{\cancel{V_1^2}}{2g} + z_1 = \frac{\cancel{p_2}}{\rho g} + \frac{V_2^2}{2g} + z_2$$

0 0 0

$$V_2 = \sqrt{2g(z_1 - z_2)} = \sqrt{2g\,(0.2 - (-0.23))}$$
$$= 2.9\ m/s$$

$$V = \sqrt{2gH}$$

Flow rate $Q = A_2 V_2 = \frac{\pi}{4} \times d_2^2 \times V_2$

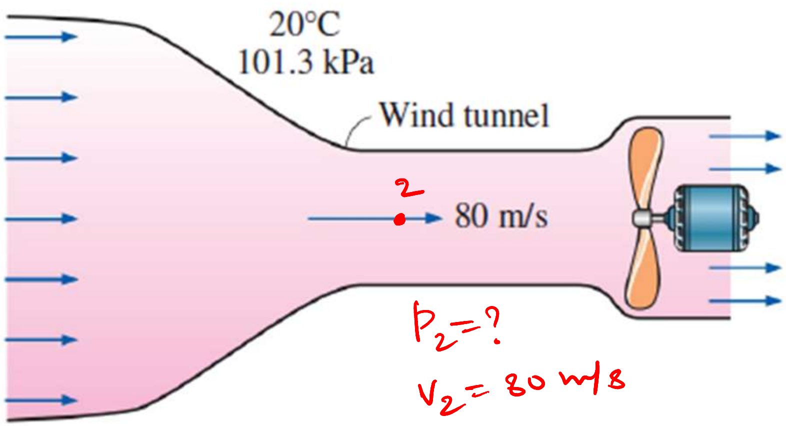
$$= \frac{\pi}{4} \times 0.02^2 \times 2.9$$

$$Q = \underline{\underline{9.11 \times 10^{-4}\ m^3/s}}$$

5–110 A wind tunnel draws atmospheric air at 20°C and 101.3 kPa by a large fan located near the exit of the tunnel. If the air velocity in the tunnel is 80 m/s, determine the pressure in the tunnel.

$p_{atm} = 101325$ Pa

$= 101.325$ kPa



$p_1 = 0$

$V_1 \approx 0$

$z_1 = z_2$

$p_2 = ?$

$V_2 = 80$ m/s

$z_2 = z_1$

$\rho = 1.2$ kg/m^3 (Air)

$$\underbrace{\cancel{\frac{p_1}{\rho g}}}_{0 \text{ atm pr}} + \underbrace{\cancel{\frac{V_1^2}{2g}}}_{0} + \cancel{z_1} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + \cancel{z_2}$$

$$p_2 = -\frac{\rho}{2} V_2^2$$

$$= -\frac{1.2}{2} \times 80^2$$

$$p_2 = -3840 \text{ Pa (gage)}$$

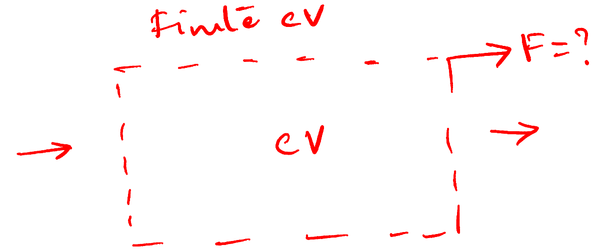
$p_2 <$ <u>Atm. pr.</u>

Differential form of equations of fluid flow

From Reynolds transport theorem,

* INTEGRAL FORM

Conservation of mass / Continuity equation :



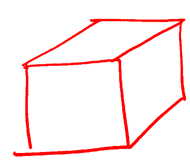
$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, d\forall + \oiint_{CS} \rho \, (\bar{V}\cdot\hat{n})\, dA = 0$$

Conservation of momentum :

$$\frac{d(m\bar{V})}{dt} = \Sigma \bar{F}$$

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \bar{V} \, d\forall + \oiint_{CS} \bar{V} \rho \, (\bar{V}\cdot\hat{n})\, dA = \Sigma \bar{F}$$

$$= \bar{F}_{body} + \bar{F}_{pressure} + \bar{F}_{shear}$$



Differential form of mass conservation

$$\bar{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

The conservation of mass (Continuity equation) equation in integral form is

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \, d\forall + \oiint_{CS} \rho(\bar{V}\cdot\hat{n})\,dA = 0 \quad \text{— (1)}$$

We can convert the surface integral term (2nd term) to a volume integral using the Gauss divergence theorem

Gauss divergence theorem: For a vector field F $\oiint (F\cdot\hat{n})\,dA = \iiint (\nabla\cdot F)\,d\forall$ $\quad F = \bar{V}$

(1) Can be rewritten using Gauss divergence theorem as

$$\iiint_{CV} \frac{\partial\rho}{\partial t}\,d\forall + \iiint_{CV} (\nabla\cdot\rho\bar{V})\,d\forall = 0$$

$$\Rightarrow \quad \iiint \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{V}) \right] d\forall = 0$$

For infinitesimally small volume, $d\forall \to 0$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{V}) = 0}$$

$$\bar{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

This is the differential form of Mass Conservation Equation (or) Continuity equation

For incompressible flow, density ρ = constant $\Rightarrow \dfrac{\partial \rho}{\partial t} = 0$ and $\nabla \cdot (\rho \vec{V}) = \rho(\nabla \cdot \vec{V})$

For incompressible flow, the Continuity equation reduces to,

$$\rho(\nabla \cdot \bar{V}) = 0$$

Since $\rho \neq 0 \Rightarrow (\nabla \cdot \bar{V}) = 0$

$$\nabla \cdot \bar{V} = 0$$

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(u\hat{i} + v\hat{j} + w\hat{k}\right) = 0$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

Mass conservation/Continuity eqn for incompressible flow

Differential form of momentum conservation equation

The momentum conservation equation in integral form is

$$\frac{\partial}{\partial t}\iiint_{CV} \rho \vec{V}\, d\forall + \oiint_{CS} \vec{V}\, \rho\, (\vec{V}\cdot\hat{n})\, dA = \vec{F}_{body} + \vec{F}_{pressure} + \vec{F}_{shear}$$

$$W = mg$$

$$\frac{W}{\forall} = \frac{mg}{\forall} = \rho g$$

Newton's second law :- $\frac{d(m\bar{V})}{dt} = \Sigma F$ or $m\bar{a} = \Sigma F$

$$\rho\bar{a} = \Sigma f \qquad f = \frac{F}{\forall}$$

$$\rho\frac{D\bar{V}}{Dt} = \Sigma f$$

$$\rho\left(\frac{\partial\bar{V}}{\partial t} + (\bar{V}\cdot\nabla)\bar{V}\right) = \Sigma f$$

$$\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

NAVIER-STOKES EQUATION

$$\underbrace{\rho\left(\frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}\right)}_{\text{Acceleration}} = \underbrace{\rho\vec{g}}_{\text{Body force}} \underbrace{-\vec{\nabla}p}_{\text{Pressure force}} + \underbrace{\mu\vec{\nabla}^2\vec{V}}_{\text{Shear force}}$$

Momentum equation in differential form

Per unit volume

Continuity and Navier–Stokes Equations

Governing equations in differential form

Incompressible continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

x-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

to find: Pressure field
Velocity field

Unknowns: u, v, w, p

- Non-linear differential equations
- Analytical solution to these equations not possible

Governing equations for incompressible flow

Continuity equation : $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Navier-Stokes equation : $$\rho\left(\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right) = \rho\vec{g} - \vec{\nabla}p + \mu\vec{\nabla}^2\vec{V}$$

Governing equations for inviscid, incompressible flow

Continuity equation : $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Euler equation : $$\rho\left(\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right) = \rho\vec{g} - \vec{\nabla}p$$

For a fluid element, derive the differential form of momentum equation?