Dimensional Analysis

What is Dimensional Analysis?

- Analytical Techniques like:
  - Control Volume Analysis
  - Bernoulli's equation
  - Potential Flows

These straight *analytical* techniques are limited to simple geometries and uniform boundary conditions. Only a fraction of engineering flow problems can be solved by direct analytical formulas.

- Most practical fluid flow problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by EXPERIMENTS (or) approximated by computational fluid dynamics (CFD).

- These results are typically reported as experimental data points and curves. Such data have much more generality if they are expressed in compact, economic form. Such data are best presented in *dimensionless* form.

- Experiments that might result in tables of output, or even multiple volumes of tables, might be reduced to a single set of curves—or even a single curve—when suitably nondimensionalized. The technique for doing this is *dimensional analysis*.

Fundamental Dimensions (or) Basic Dimensions (or) Reference Dimensions

Mass — $[M]$

Length — $[L]$

Time — $[T]$

Temperature — $[\Theta]$

Volume flow rate $\frac{m^3}{s}$ $[L^3 T^{-1}]$

Velocity $= \frac{m}{s} = [L T^{-1}]$

Acceleration $\frac{m}{s^2}$ $[L T^{-2}]$

Force : $N - \frac{kg\,m}{s^2}$ $[M L T^{-2}]$

Dimensions of fluid mechanics properties

Quantity	Symbol	Dimensions $MLT\Theta$
Length	L	L
Area	A	L^2
Volume	$\mathcal{V}$	L^3
Velocity	V	LT^{-1}
Acceleration	dV/dt	LT^{-2}
Speed of sound	a	LT^{-1}
Volume flow	Q	L^3T^{-1}
Mass flow	$\dot{m}$	MT^{-1}
Pressure, stress	p, σ, τ	$ML^{-1}T^{-2}$
Strain rate	$\dot{\varepsilon}$	T^{-1}
Angle	θ	None
Angular velocity	ω, Ω	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$
Kinematic viscosity	ν	L^2T^{-1}
Surface tension	Υ	MT^{-2}
Force	F	MLT^{-2}
Moment, torque	M	ML^2T^{-2}
Power	P	ML^2T^{-3}
Work, energy	W, E	ML^2T^{-2}
Density	ρ	ML^{-3}
Temperature	T	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$
Thermal expansion coefficient	β	Θ^{-1}

Buckingham Pi Theorem

If an equation involving *n* variables, it can be reduced to a relationship among *j*=(*n*-*k*) independent dimensionless products, where *k* is the minimum number of fundamental dimensions required to describe the variables.

The dimensionless products are frequently referred to as ***pi terms,*** and the theorem is called the ***Buckingham pi theorem*.**

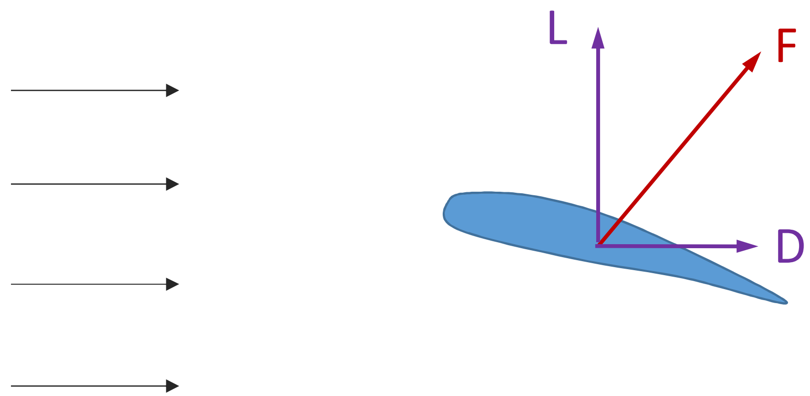
Dimensional form: $$u_1 = f(u_2, u_3, u_4, u_5, \ldots\ldots\ldots, u_n)$$

- If N dimensional variables have k fundamental dimensions

n dimensional variables can be reduced to (n-k) dimensionless variables or π-terms

Dimensionless form: $$\pi_1 = f(\pi_2, \pi_3, \pi_4, \ \ldots\ldots\ldots, \pi_{N-k})$$

Example:
Derive the Lift and Drag relations for an airfoil using dimensional analysis. Prove that their coefficients are functions of Reynolds number and Mach number



<u>Step 1:</u> List all the variables present in the problem

Aerodynamic force : F

Flow velocity : V_∞

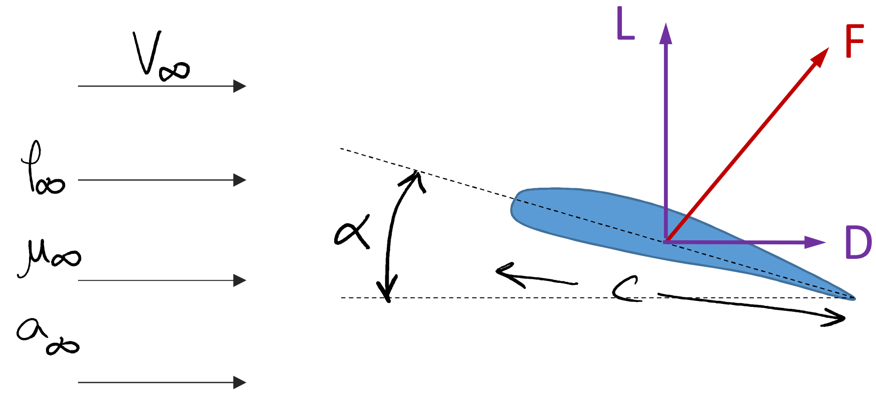
Density : ρ_∞

Viscosity : μ_∞

Speed of sound : a_∞

Chord : c

Angle of attack : α



$$F = f^n(V_\infty, \rho_\infty, \mu_\infty, a_\infty, c, \alpha)$$

no: of dimensional variables $\underline{\underline{n = 7}}$

Step 2: Express each of the variables in terms of fundamental dimensions.

$$[F] = MLT^{-2}$$

$$[V_\infty] = LT^{-1}$$

$$[\rho_\infty] = ML^{-3}$$

$$[\mu_\infty] = ML^{-1}T^{-1}$$

$$[a_\infty] = LT^{-1}$$

$$[c] = L$$

$$[\alpha] = \text{None}$$

$\frac{kg\,m}{s^2}$ MLT^{-2}

$\frac{kg}{m^3}$

$\mu - \frac{kg}{ms}$

M, L, T

no: of fundemental dimensions present

$K = 3$

Step 3: Determine the required number of pi terms.

$n = 7$

$K = 3$

So No. of pi terms (dimensionless variables) we can get $j = n - K$

$= 7 - 3$

$\underline{\underline{j = 4}}$

Step 4: Select a number of repeating variables, where the number required is equal to the number of fundamental dimensions

No: of repeating variables = No of fundamental dimensions = 3

— Select Repeating Variables

No: of repeating Variables = 3

— Among 7 variables, we have to select 3 repeating variables

— How to select repeating variables ??

* The repeating variables should not form a π-term among themselves

* Each repeating variable should contain a unique fundamental dimension

* Thumb rule

(*) Thumb rule

generally, we select

one geometric variable — Eg: Length, Area, volume etc......

one flow variable — Eg: Velocity, Acceleration etc.....

one fluid property — Eg: Density, viscosity etc.....

$$F = f(\checkmark c, \checkmark V_\infty, \checkmark \rho_\infty, \mu_\infty, a_\infty, \alpha)$$

Repeating Variables (3) — c, V_∞, ρ_∞

Non-Repeating Variables (4) — $F, \mu_\infty, a_\infty, \alpha$

Step 5: Form a pi term by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.

$$\Pi_1 = [F][c]^{a_1}[V_\infty]^{b_1}[\rho_\infty]^{c_1}$$

$$M^0 L^0 T^0 = [MLT^{-2}][L]^{a_1}[LT^{-1}]^{b_1}[ML^{-3}]^{c_1}$$

$$M^0 L^0 T^0 = M^{1+c_1} \; L^{1+a_1+b_1-3c_1} \; T^{-2-b_1}$$

Equating powers will give a_1, b_1, c_1

$$\left.\begin{aligned} c_1+1=0 &\Rightarrow c_1=-1 \\ -2-b_1=0 &\Rightarrow b_1=-2 \\ 1+a_1+b_1-3c_1=0 &\Rightarrow a_1 = 3c_1-b_1-1 \\ &\qquad = -3+2-1 = -2 \end{aligned}\right\}$$

$$\Rightarrow \Pi_1 = F c^{-2} V_\infty^{-2} \rho_\infty^{-1}$$

$$\Rightarrow \boxed{\Pi_1 = \frac{F}{\rho_\infty V_\infty^2 c^2}}$$

$$\Pi_2 = \mu_\infty [c]^{a_2}[V_\infty]^{b_2}[\rho_\infty]^{c_2}$$

$$\Pi_3 = a_\infty [c]^{a_3}[V_\infty]^{b_3}[\rho_\infty]^{c_3}$$

$$\Pi_4 = \alpha [c]^{a_4}[V_\infty]^{b_4}[\rho_\infty]^{c_4}$$

Step 6: Repeat Step 5 for each of the remaining nonrepeating variables.

$$\Pi_2 = [\mu_\infty][c]^{a_2}[V_\infty]^{b_2}[\rho_\infty]^{c_2}$$

$$M^0 L^0 T^0 = M L^{-1} T^{-1} [L]^{a_2} [L T^{-1}]^{b_2} [M L^{-3}]^{c_2}$$

$$M^0 L^0 T^0 = M^{1+c_2} \; L^{-1+a_2+b_2-3c_2} \; T^{-1-b_2}$$

Equating powers

$$\left.\begin{array}{l} 1+c_2=0 \Rightarrow c_2=-1 \\ -1-b_2=0 \Rightarrow b_2=-1 \\ -1+a_2+b_2-3c_2=0 \Rightarrow a_2=-1 \end{array}\right\} \quad \Pi_2 = \mu_\infty \, c^{-1} \, V_\infty^{-1} \, \rho_\infty^{-1}$$

$$\Rightarrow \boxed{\Pi_2 = \frac{\rho_\infty V_\infty c}{\mu_\infty}}$$

$$\Pi_3 = [a_\infty]\,[c]^{a_3}\,[V_\infty]^{b_3}\,[\rho_\infty]^{c_3}$$

$$\frac{a_\infty}{V_\infty}$$

$$M^0 L^0 T^0 = [LT^{-1}]\,[L]^{a_3}\,[LT^{-1}]^{b_3}\,[ML^{-3}]^{c_3}$$

$$M^0 L^0 T^0 = L^{1+a_3+b_3-3c_3}\;T^{-1-b_3}\;M^{c_3}$$

Equating powers

$$c_3 = 0$$

$$-1-b_3 = 0 \Rightarrow b_3 = -1$$

$$1+a_3+b_3-3c_3 = 0 \Rightarrow a_3 = 0$$

$$\boxed{\Pi_3 = \frac{a_\infty}{V_\infty}}$$

$$\boxed{\Pi_4 = \alpha}$$

Dimensional form $\quad F = f^n(V_\infty, \rho_\infty, \mu_\infty, c, a_\infty, \alpha)$

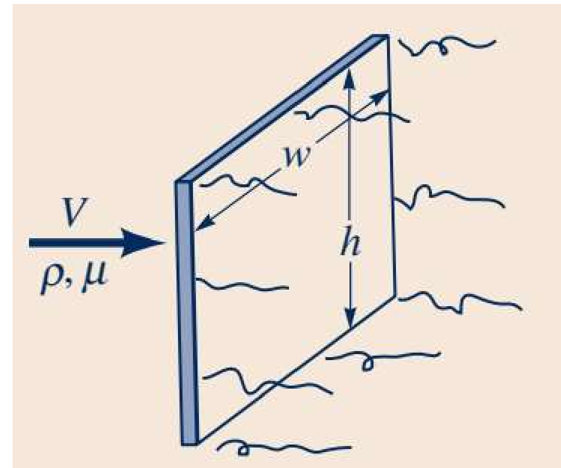
$\downarrow$ Buckingham pi-Theorem

Dimensionless form : (No units)

$$\underbrace{\frac{F}{\rho_\infty V_\infty^2 c^2}}_{\text{Force coefficient } C_F} = f^n\left(\underbrace{\frac{\rho_\infty V_\infty c}{\mu_\infty}}_{\text{Reynolds number}}, \underbrace{\frac{V_\infty}{a_\infty}}_{\text{Mach number}}, \underbrace{\alpha}_{\text{Angle of attack}} \right)$$

- $C_L = f^n(Re, Ma, \alpha)$
- $C_D = f^n(Re, Ma, \alpha)$

GIVEN A thin rectangular plate having a width w and a height h is located so that it is normal to a moving stream of fluid as shown in Fig. E7.1. Assume the drag, $\mathcal{D}$, that the fluid exerts on the plate is a function of w and h, the fluid viscosity and density, μ and ρ, respectively, and the velocity V of the fluid approaching the plate.



FIND Determine a suitable set of pi terms to study this problem experimentally.

Step 1:

$$D = f^n(w, h, \mu, \rho, V)$$

no: of Variables $n = 6$

Step 2:

$[D] = MLT^{-2}$

$[w] = L$

$[h] = L$

$[\mu] = ML^{-1}T^{-1}$

$[\rho] = ML^{-3}$

$[V] = LT^{-1}$

No: of fundemental dimensions present (M, L, T)

$K = 3$

Step 3: No: of pi terms $j = n - k = 6 - 3 = 3$ $\underline{j = 3}$

Step 4:

Select repeating variables

no: of repeating Variables $= 3$

Repeating variables :— w, V, ρ

Non-repeating Variables — D, h, μ

Step 5:

$$\pi_1 = [D]\,[w]^{a_1}\,[V]^{b_1}\,[\rho]^{c_1}$$

$$M^0 L^0 T^0 = [MLT^{-2}]\,[L]^{a_1}\,[LT^{-1}]^{b_1}\,[ML^{-3}]^{c_1}$$

$$M^0 L^0 T^0 = M^{1+c_1}\,L^{1+a_1+b_1-3c_1}\,T^{-2-b_1}$$

Equating powers :

$1 + c_1 = 0 \Rightarrow c_1 = -1$

$-2 - b_1 = 0 \Rightarrow b_1 = -2$

$1 + a_1 + b_1 - 3c_1 = 0 \Rightarrow a_1 = -2$

$$\boxed{\pi_1 = \frac{D}{w^2 V^2 \rho}}$$

$$\Pi_2 = h\,[W]^{a_2}\,[V]^{b_2}\,[\rho]^{c_2}$$

$$M^0L^0T^0 = [L]\,[L]^{a_2}\,[LT^{-1}]^{b_2}\,[ML^{-3}]^{c_2}$$

$$M^0L^0T^0 = L^{1+a_2+b_2-3c_2}\;T^{-b_2}\;M^{c_2}$$

Equating powers $\Rightarrow$ $b_2 = 0$, $c_2 = 0$, $a_2 = -1$

$$\boxed{\Pi_2 = \frac{h}{W}}$$

$$\Pi_3 = \mu\,[W]^{a_3}\,[V]^{b_3}\,[\rho]^{c_3}$$

$a_3 = -1$, $b_3 - 1$, $c_3 = -1$

$$\boxed{\Pi_3 = \frac{\mu}{\rho V W}}$$

$$D = f^n(w, h, \rho, \mu, V)$$

$$\downarrow$$

$$\frac{D}{\rho V^2 w^2} = f^n\left(\frac{h}{w}, \frac{\mu}{\rho V w}\right)$$

$$\Rightarrow \underbrace{\frac{D}{\rho V^2 A}}_{\text{Drag coefficient } C_D} = f^n\left(\underbrace{\frac{h}{w}}_{\text{Aspect ratio of plate } AR}, \underbrace{\frac{\rho V w}{\mu}}_{\text{Reynolds number}}\right) \qquad A = L \times w$$

$$\Rightarrow \bullet \; C_D = f^n(AR, Re)$$

Some common dimensionless groups in fluid mechanics

Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, ℓ; Density, ρ; Frequency of oscillating flow, ω; Pressure, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\dfrac{\rho V \ell}{\mu}$	Reynolds number, Re	$\dfrac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\dfrac{V}{\sqrt{g\ell}}$	Froude number, Fr	$\dfrac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\dfrac{p}{\rho V^2}$	Euler number, Eu	$\dfrac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\dfrac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\dfrac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\dfrac{V}{c}$	Mach number,[a] Ma	$\dfrac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\dfrac{\omega \ell}{V}$	Strouhal number, St	$\dfrac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\dfrac{\rho V^2 \ell}{\sigma}$	Weber number, We	$\dfrac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important