Assignment - 1

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Problem

1.Show that the area of the triangle whose vertices are:

$$(a\tan\theta, b\cot\theta), (a\tan\phi, b\cot\phi), (a\tan\gamma, b\cot\gamma)$$

$$is \ \ \frac{4ab\ \sin(\theta-\phi)\sin(\phi-\gamma)\sin(\gamma-\theta)}{\sin 2\theta \sin 2\phi \sin 2\gamma}$$

Solution

$$\mathbf{x_1} = \left(a\tan\theta\right), \mathbf{x_2} = \left(a\tan\phi\right) \mathbf{x_3} = \left(a\tan\gamma\right),$$

$$\mathbf{y_1} = \left(b\cot\theta\right), \mathbf{y_2} = \left(b\cot\phi\right), \mathbf{y_3} = \left(b\cot\gamma\right)$$

$$area \quad of \quad triangle \quad = \quad \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad (1)$$

$$= \quad \frac{1}{2} \begin{vmatrix} a\tan\theta & b\cot\theta & 1 \\ a\tan\phi & b\cot\phi & 1 \\ a\tan\gamma & b\cot\gamma & 1 \end{vmatrix} \quad (2)$$

$$= \quad \frac{ab}{2} \begin{vmatrix} \tan\theta & \cot\theta & 1 \\ \tan\phi & \cot\phi & 1 \\ \tan\gamma & \cot\gamma & 1 \end{vmatrix} \quad (3)$$

$$\because taking\ a\ for\ C1\ and\ b\ from\ C2\ common$$

$$= \quad \frac{ab}{2} \begin{vmatrix} \tan\theta & \cot\theta & 1 \\ \tan\phi - \tan\theta & \cot\phi - \cot\theta & 0 \\ \tan\gamma - \tan\theta & \cot\gamma - \cot\theta & 0 \end{vmatrix} \quad (4)$$

$$\because R2 = R2 - R1\ and\ R3 = R3 - R1$$

$$= \quad \frac{ab}{2} \begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin\phi}{\cos\phi} - \frac{\sin\theta}{\cos\theta} & \frac{\cos\phi}{\sin\phi} - \frac{\cos\theta}{\sin\theta} & 0 \\ \frac{\sin\gamma}{\cos\gamma} - \frac{\sin\theta}{\cos\theta} & \frac{\cos\gamma}{\sin\gamma} - \frac{\cos\theta}{\sin\theta} & 0 \end{vmatrix} \quad (5)$$

$$\because \tan x = \frac{\sin x}{\cos x}\ and\ \cot x = \frac{\cos x}{\sin x}$$

$$= \frac{ab}{2} \begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin\phi\cos\theta - \sin\theta\cos\phi}{\cos\phi\cos\theta} & \frac{\sin\theta\cos\phi - \sin\phi\cos\theta}{\sin\phi\sin\theta} & 0 \\ \frac{\sin\gamma\cos\theta - \sin\theta\cos\gamma}{\cos\gamma\cos\theta} & \frac{\sin\theta\cos\gamma - \sin\gamma\cos\theta}{\sin\gamma\sin\theta} & 0 \end{vmatrix} \quad (6)$$

$$\because \ rearranging\ terms$$

$$= \quad \frac{ab}{2} \begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin(\phi-\theta)}{\cos\phi\cos\theta} & \frac{\sin(\theta-\phi)}{\sin\phi\sin\theta} & 0 \\ \frac{\sin(\gamma-\theta)}{\cos\phi\cos\theta} & \frac{\sin(\theta-\gamma)}{\sin\phi\sin\theta} & 0 \end{vmatrix} \quad (7)$$

$$\because \sin A\cos B - \sin B\cos A = \sin(A-B)$$

$$= \frac{ab}{2}[\left(\frac{\sin(\phi-\theta)\sin(\theta-\gamma)}{\sin\theta\cos\theta\cos\phi\sin\gamma}\right) - \left(\frac{\sin(\gamma-\theta)\sin(\theta-\phi)}{\sin\theta\cos\theta\sin\phi\cos\gamma}\right)] \quad (8)$$

$$\because expanding\ determinant$$

$$= \frac{ab}{2\sin\theta\cos\theta\cos\phi\sin\phi\cos\gamma\sin\gamma}[$$
$$\sin\phi\cos\gamma\,(\sin(\phi-\theta)\sin(\theta-\gamma)) -$$
$$\sin\gamma\cos\phi\,(\sin(\gamma-\theta)\sin(\theta-\phi))\ \ ] \quad (9)$$

$$\because solving\ above\ equation$$

$$= \frac{ab*\ 8}{2*\ 8\sin\theta\cos\theta\cos\phi\sin\phi\cos\gamma\sin\gamma}[$$
$$\sin\phi\cos\gamma\,(\sin(\phi-\theta)\sin(\theta-\gamma)) -$$
$$\sin\gamma\cos\phi\,(\ (-sin(\theta-\gamma))(-\sin(\phi-\theta)))\ ] \quad (10)$$

$$\because multiplying\ 8\ and\ dividing\ 8\ and$$
$$\sin(b-a) = -\sin(a-b)$$

$$= \frac{4ab}{2\sin\theta\cos\theta\ 2\cos\phi\sin\phi\ 2\cos\gamma\sin\gamma}[$$
$$\sin\phi\cos\gamma\,(\sin(\phi-\theta)\sin(\theta-\gamma)) -$$
$$\sin\gamma\cos\phi\,(\sin(\theta-\gamma)sin(\phi-\theta))\ ] \quad (11)$$

$$\because\ rearranging\ terms$$

$$= \frac{4ab\ \sin(\phi-\theta)\ \sin(\theta-\gamma)}{\sin 2\theta\ \sin 2\phi\ \sin 2\gamma}[$$
$$\sin\phi\cos\gamma - \sin\gamma\cos\phi] \quad (12)$$

$$\because \sin X\cos X = \sin 2X\ and\ taking$$

$$= \frac{4ab\ \sin(\phi-\theta)\ \sin(\theta-\gamma)\ \sin(\phi-\gamma)}{\sin 2\theta\ \sin 2\phi\ \sin 2\gamma\ \sin(\phi-\gamma)} \quad (13)$$

$$\because \sin A\cos B - \sin B\cos A = \sin(A-B)$$

$$= \frac{4ab\ (-\sin(\theta-\phi))\ \sin(\phi-\gamma)\ (-\sin(\gamma-\theta))}{\sin 2\theta\ \sin 2\phi\ \sin 2\gamma\ \sin(\phi-\gamma)} \quad (14)$$

$$\sin(b-a) = -\sin(a-b)$$

$$= \frac{4ab\ \sin(\theta-\phi)\ \sin(\phi-\gamma)\ \sin(\gamma-\theta)}{\sin 2\theta\ \sin 2\phi\ \sin 2\gamma\ \sin(\phi-\gamma)} = R.H.S$$