

Assignment - 1

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PROBLEM

1. Show that the area of the triangle whose vertices are:

$$(a \tan \theta, b \cot \theta), (a \tan \phi, b \cot \phi), (a \tan \gamma, b \cot \gamma)$$

is $\frac{4ab \sin(\theta - \phi) \sin(\phi - \gamma) \sin(\gamma - \theta)}{\sin 2\theta \sin 2\phi \sin 2\gamma}$

$$= \frac{ab}{2} \begin{vmatrix} \frac{\sin \theta}{\cos \theta} & \frac{\cos \theta}{\sin \theta} & 1 \\ \frac{\sin \phi}{\cos \phi} - \frac{\sin \theta}{\cos \theta} & \frac{\cos \phi}{\sin \phi} - \frac{\cos \theta}{\sin \theta} & 0 \\ \frac{\sin \gamma}{\cos \gamma} - \frac{\sin \theta}{\cos \theta} & \frac{\cos \gamma}{\sin \gamma} - \frac{\cos \theta}{\sin \theta} & 0 \end{vmatrix} \quad (5)$$

$$\because \tan x = \frac{\sin x}{\cos x} \text{ and } \cot x = \frac{\cos x}{\sin x}$$

SOLUTION

$$\mathbf{x}_1 = (a \tan \theta), \mathbf{x}_2 = (a \tan \phi), \mathbf{x}_3 = (a \tan \gamma),$$

$$\mathbf{y}_1 = (b \cot \theta), \mathbf{y}_2 = (b \cot \phi), \mathbf{y}_3 = (b \cot \gamma)$$

$$= \frac{ab}{2} \begin{vmatrix} \frac{\sin \theta}{\cos \theta} & \frac{\cos \theta}{\sin \theta} & 1 \\ \frac{\sin \phi \cos \theta - \sin \theta \cos \phi}{\cos \phi \cos \theta} & \frac{\sin \theta \cos \phi - \sin \phi \cos \theta}{\sin \phi \sin \theta} & 0 \\ \frac{\sin \gamma \cos \theta - \sin \theta \cos \gamma}{\cos \gamma \cos \theta} & \frac{\sin \theta \cos \gamma - \sin \gamma \cos \theta}{\sin \gamma \sin \theta} & 0 \end{vmatrix} \quad (6)$$

$$\text{area of triangle} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad (1)$$

\because rearranging terms

$$= \frac{1}{2} \begin{vmatrix} a \tan \theta & b \cot \theta & 1 \\ a \tan \phi & b \cot \phi & 1 \\ a \tan \gamma & b \cot \gamma & 1 \end{vmatrix} \quad (2)$$

$$= \frac{ab}{2} \begin{vmatrix} \frac{\sin \theta}{\cos \theta} & \frac{\cos \theta}{\sin \theta} & 1 \\ \frac{\sin(\theta - \phi)}{\cos \phi \cos \theta} & \frac{\sin(\theta - \phi)}{\sin \phi \sin \theta} & 0 \\ \frac{\sin(\gamma - \theta)}{\cos \gamma \cos \theta} & \frac{\sin(\theta - \gamma)}{\sin \gamma \sin \theta} & 0 \end{vmatrix} \quad (7)$$

$$= \frac{ab}{2} \begin{vmatrix} \tan \theta & \cot \theta & 1 \\ \tan \phi & \cot \phi & 1 \\ \tan \gamma & \cot \gamma & 1 \end{vmatrix} \quad (3)$$

$$\because \sin A \cos B - \sin B \cos A = \sin(A - B)$$

\because taking a for $C1$ and b from $C2$ common

$$= \frac{ab}{2} \begin{vmatrix} \tan \theta & \cot \theta & 1 \\ \tan \phi - \tan \theta & \cot \phi - \cot \theta & 0 \\ \tan \gamma - \tan \theta & \cot \gamma - \cot \theta & 0 \end{vmatrix} \quad (4)$$

$$= \frac{ab}{2 \sin \theta \cos \theta} \begin{vmatrix} \sin \theta & \cos \theta & 1 \\ -\frac{\sin(\theta - \phi)}{\cos \phi} & \frac{\sin(\theta - \phi)}{\sin \phi} & 0 \\ \frac{\sin(\gamma - \theta)}{\cos \gamma} & -\frac{\sin(\gamma - \theta)}{\sin \gamma} & 0 \end{vmatrix} \quad (8)$$

$\because R2 = R2 - R1$ and $R3 = R3 - R1 \quad \because$ taking $\frac{1}{\cos \theta}, \frac{1}{\sin \theta}$ common from column 1 and 2

$$= \frac{ab (-\sin(\theta - \phi)) (-\sin(\gamma - \theta))}{2 \sin \theta \cos \theta} \begin{vmatrix} \sin \theta & \cos \theta & 1 \\ \frac{1}{\cos \phi} & \frac{1}{\sin \phi} & 0 \\ \frac{1}{\cos \gamma} & \frac{1}{\sin \gamma} & 0 \end{vmatrix} \quad (9)$$

\therefore taking $\sin(\theta - \phi)$ common from row 2
and $\sin(\gamma - \theta)$ common from row 3

$$= \frac{ab \sin(\theta - \phi) \sin(\gamma - \theta)}{\sin 2\theta} \left[\frac{1}{\cos \phi \sin \gamma} - \frac{1}{\sin \phi \cos \gamma} \right] \quad (10)$$

\therefore solving determinate

$$= \frac{ab \sin(\theta - \phi) \sin(\gamma - \theta)}{\sin 2\theta} \left[\frac{\sin \phi \cos \gamma - \cos \phi \sin \gamma}{\sin \phi \cos \phi \sin \gamma \cos \gamma} \right] \quad (11)$$

\therefore solving above equation

$$= \frac{ab \sin(\theta - \phi) \sin(\gamma - \theta)}{\sin 2\theta} \left[\frac{4 \sin(\phi - \gamma)}{4 \sin \phi \cos \phi \sin \gamma \cos \gamma} \right] \quad (12)$$

$\therefore \sin A \cos B - \sin B \cos A = \sin(A - B)$
multiply and divide by 4

$$= \frac{4ab \sin(\theta - \phi) \sin(\gamma - \theta) \sin(\phi - \gamma)}{\sin 2\theta \cdot 2 \sin \phi \cos \phi \cdot 2 \sin \gamma \cos \gamma} \quad (13)$$

$$= \frac{4ab \sin(\theta - \phi) \sin(\gamma - \theta) \sin(\phi - \gamma)}{\sin 2\theta \sin 2\phi \sin 2\gamma} = R.H.S$$