Assignment - 1

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SM21MTECH14002

Problem

1.Show that the area of the triangle whose vertices are:

$$(a\tan\theta, b\cot\theta), (a\tan\phi, b\cot\phi), (a\tan\gamma, b\cot\gamma)$$

$$is\ \ \frac{4ab\ \sin(\theta-\phi)\sin(\phi-\gamma)\sin(\gamma-\theta)}{\sin 2\theta\sin 2\phi\sin 2\gamma}$$

Solution

$$\mathbf{x_1} = \begin{vmatrix} a\tan\theta \end{vmatrix}, \mathbf{x_2} = \begin{pmatrix} a\tan\phi \end{pmatrix} \mathbf{x_3} = \begin{pmatrix} a\tan\gamma \end{pmatrix}$$
$$\mathbf{y_1} = \begin{pmatrix} b\cot\theta \end{pmatrix}, \mathbf{y_2} = \begin{pmatrix} b\cot\phi \end{pmatrix}, \mathbf{y_3} = \begin{pmatrix} b\cot\gamma \end{pmatrix}$$

$$area\quad of\quad triangle \quad = \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{1}$$

$$= \frac{1}{2}\begin{vmatrix} a\tan\theta & b\cot\theta & 1 \\ a\tan\phi & b\cot\phi & 1 \\ a\tan\gamma & b\cot\gamma & 1 \end{vmatrix} \tag{2}$$

$$= \frac{ab}{2}\begin{vmatrix} \tan\theta & \cot\theta & 1 \\ \tan\phi & \cot\phi & 1 \\ \tan\gamma & \cot\gamma & 1 \end{vmatrix} \tag{3}$$

$\because$ taking common $C1 \xleftrightarrow{a} C1$ and $C2 \xleftrightarrow{b} C2$

$$= \frac{ab}{2}\begin{vmatrix} \tan\theta & \cot\theta & 1 \\ \tan\phi-\tan\theta & \cot\phi-\cot\theta & 0 \\ \tan\gamma-\tan\theta & \cot\gamma-\cot\theta & 0 \end{vmatrix} \tag{4}$$

$\because R2 \xleftrightarrow{R2-R1} R2$ and $R3 \xleftrightarrow{R3-R1} R3$

$$= \frac{ab}{2}\begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin\phi}{\cos\phi}-\frac{\sin\theta}{\cos\theta} & \frac{\cos\phi}{\sin\phi}-\frac{\cos\theta}{\sin\theta} & 0 \\ \frac{\sin\gamma}{\cos\gamma}-\frac{\sin\theta}{\cos\theta} & \frac{\cos\gamma}{\sin\gamma}-\frac{\cos\theta}{\sin\theta} & 0 \end{vmatrix} \tag{5}$$

$$\because \tan x = \frac{\sin x}{\cos x}\ and\ \cot x = \frac{\cos x}{\sin x}$$

$$= \frac{ab}{2}\begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin\phi\cos\theta-\sin\theta\cos\phi}{\cos\phi\cos\theta} & \frac{\sin\theta\cos\phi-\sin\phi\cos\theta}{\sin\phi\sin\theta} & 0 \\ \frac{\sin\gamma\cos\theta-\sin\theta\cos\gamma}{\cos\gamma\cos\theta} & \frac{\sin\theta\cos\gamma-\sin\gamma\cos\theta}{\sin\gamma\sin\theta} & 0 \end{vmatrix} \tag{6}$$

$\because$ rearranging terms

$$= \frac{ab}{2}\begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin(\phi-\theta)}{\cos\phi\cos\theta} & \frac{\sin(\theta-\phi)}{\sin\phi\sin\theta} & 0 \\ \frac{\sin(\gamma-\theta)}{\cos\gamma\cos\theta} & \frac{\sin(\theta-\gamma)}{\sin\gamma\sin\theta} & 0 \end{vmatrix} \tag{7}$$

$$\because \sin A\cos B - \sin B\cos A = \sin(A-B)$$

$$= \frac{ab}{2\sin\theta\cos\theta}\begin{vmatrix} \sin\theta & \cos\theta & 1 \\ -\frac{\sin(\theta-\phi)}{\cos\phi} & \frac{\sin(\theta-\phi)}{\sin\phi} & 0 \\ \frac{\sin(\gamma-\theta)}{\cos\gamma} & -\frac{\sin(\gamma-\theta)}{\sin\gamma} & 0 \end{vmatrix} \tag{8}$$

$\because$ taking common $C1 \xleftrightarrow{\frac{1}{\cos\theta}} C1$, $C2 \xleftrightarrow{\frac{1}{\sin\theta}} C2$

$$= \frac{ab\ (-\sin(\theta-\phi))\ (-\sin(\gamma-\theta))}{2\sin\theta\cos\theta}\begin{vmatrix} \sin\theta & \cos\theta & 1 \\ \frac{1}{\cos\phi} & \frac{1}{\sin\phi} & 0 \\ \frac{1}{\cos\gamma} & \frac{1}{\sin\gamma} & 0 \end{vmatrix} \tag{9}$$

$\because$ taking common $R2 \xleftrightarrow{\sin(\theta-\phi)} R2$, $R3 \xleftrightarrow{\sin(\gamma-\theta)} R3$

$$= \frac{ab\ \sin(\theta-\phi)\ \sin(\gamma-\theta)}{\sin 2\theta}\left[\frac{1}{\cos\phi\sin\gamma} - \frac{1}{\sin\phi\cos\gamma}\right] \tag{10}$$

$\because$ solving determinate

$$= \frac{ab\ \sin(\theta-\phi)\ \sin(\gamma-\theta)}{\sin 2\theta} \left[\frac{\sin\phi\cos\gamma-\cos\phi\sin\gamma}{\sin\phi\cos\phi\sin\gamma\cos\gamma}\right] \tag{11}$$

$\because$ solving above equation

$$= \frac{ab\ \sin(\theta-\phi)\ \sin(\gamma-\theta)}{\sin 2\theta} \left[\frac{4\ \sin(\phi-\gamma)}{4\ \sin\phi\cos\phi\sin\gamma\cos\gamma}\right] \tag{12}$$

$\because \sin A\cos B-\sin B\cos A=\sin(A-B)$

$$multiply\ and\ divide\ by\ 4$$

$$= \frac{4ab\ \sin(\theta-\phi)\ \sin(\gamma-\theta)\ \sin(\phi-\gamma)}{\sin 2\theta\ 2\sin\phi\cos\phi\ 2\sin\gamma\cos\gamma} \tag{13}$$

$$= \frac{4ab\ \sin(\theta-\phi)\ \sin(\gamma-\theta)\ \sin(\phi-\gamma)}{\sin 2\theta\ \ \sin 2\phi\ \ \sin 2\gamma} = R.H.S \tag{14}$$

$Putting\ Some\ Numerical\ Values$

$$\frac{4\ (1\ 1)\ \sin(60-45)\ \sin(30-60)\ \sin(45-30)}{\sin 2(60)\ \sin 2(45)\ \sin 2(30)} \tag{15}$$

putting value of a=1, b=1 $\theta=60$ $\phi=45$ $\gamma=30$

$$= \frac{-0.133974}{0.75} = -0.178632 \tag{16}$$