Assignment - 1

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Problem

1.Show that the area of the triangle whose vertices are:

$(a\tan\theta, b\cot\theta), (a\tan\phi, b\cot\phi), (a\tan\gamma, b\cot\gamma)$

$$is \ \ \frac{4ab\ \sin(\theta-\phi)\sin(\phi-\gamma)\sin(\gamma-\theta)}{\sin 2\theta\sin 2\phi\sin 2\gamma}$$

Solution

$$\mathbf{x_1} = \left(a\tan\theta\right), \mathbf{x_2} = \left(a\tan\phi\right)\mathbf{x_3} = \left(a\tan\gamma\right),$$

$$\mathbf{y_1} = \left(b\cot\theta\right), \mathbf{y_2} = \left(b\cot\phi\right), \mathbf{y_3} = \left(b\cot\gamma\right)$$

area of triangle= $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} a\tan\theta & b\cot\theta & 1 \\ a\tan\phi & b\cot\phi & 1 \\ a\tan\gamma & b\cot\gamma & 1 \end{vmatrix}$

$= \frac{ab}{2}\begin{vmatrix} \tan\theta & \cot\theta & 1 \\ \tan\phi & \cot\phi & 1 \\ \tan\gamma & \cot\gamma & 1 \end{vmatrix}$

$$\because taking\ a\ for\ C1\ and\ b\ from\ C2\ common$$

$= \frac{ab}{2}\begin{vmatrix} \tan\theta & \cot\theta & 1 \\ \tan\phi-\tan\theta & \cot\phi-\cot\theta & 0 \\ \tan\gamma-\tan\theta & \cot\gamma-\cot\theta & 0 \end{vmatrix}$

$$\because R2 = R2 - R1\ and\ R3 = R3 - R1$$

$= \frac{ab}{2}\begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin\phi}{\cos\phi} - \frac{\sin\theta}{\cos\theta} & \frac{\cos\phi}{\sin\phi} - \frac{\cos\theta}{\sin\theta} & 0 \\ \frac{\sin\gamma}{\cos\gamma} - \frac{\sin\theta}{\cos\theta} & \frac{\cos\gamma}{\sin\gamma} - \frac{\cos\theta}{\sin\theta} & 0 \end{vmatrix}$

$$\because \tan x = \frac{\sin x}{\cos x}\ and\ \cot x = \frac{\cos x}{\sin x}$$

$= \frac{ab}{2}\begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin\phi\cos\theta-\sin\theta\cos\phi}{\cos\phi\cos\theta} & \frac{\sin\theta\cos\phi-\sin\phi\cos\theta}{\sin\phi\sin\theta} & 0 \\ \frac{\sin\gamma\cos\theta-\sin\theta\cos\gamma}{\cos\gamma\cos\theta} & \frac{\sin\theta\cos\gamma-\sin\gamma\cos\theta}{\sin\gamma\sin\theta} & 0 \end{vmatrix}$

$$\because\ rearranging\ terms$$

$= \frac{ab}{2}\begin{vmatrix} \frac{\sin\theta}{\cos\theta} & \frac{\cos\theta}{\sin\theta} & 1 \\ \frac{\sin(\phi-\theta)}{\cos\phi\cos\theta} & \frac{\sin(\theta-\phi)}{\sin\phi\sin\theta} & 0 \\ \frac{\sin(\gamma-\theta)}{\cos\phi\cos\theta} & \frac{\sin(\theta-\gamma)}{\sin\phi\sin\theta} & 0 \end{vmatrix}$

$$\because \sin A\cos B - \sin B\cos A = \sin(A-B)$$

$$= \frac{ab}{2}[\left(\frac{\sin(\phi-\theta)\sin(\theta-\gamma)}{\sin\theta\cos\theta\cos\phi\sin\gamma}\right) - \left(\frac{\sin(\gamma-\theta)\sin(\theta-\phi)}{\sin\theta\cos\theta\sin\phi\cos\gamma}\right)]$$

$$\because expanding\ determinant$$

$$= \frac{ab}{2\sin\theta\cos\theta\cos\phi\sin\phi\cos\gamma\sin\gamma}[\sin\phi\cos\gamma\left(\sin(\phi-\theta)\sin(\theta-\gamma)\right) - \sin\gamma\cos\phi\left(\sin(\gamma-\theta)\sin(\theta-\phi)\right)\ ]$$

$$\because solving\ above\ equation$$

$$= \frac{ab*\ 8}{2*\ 8\sin\theta\cos\theta\cos\phi\sin\phi\cos\gamma\sin\gamma}[\sin\phi\cos\gamma\left(\sin(\phi-\theta)\sin(\theta-\gamma)\right) - \sin\gamma\cos\phi\left(\ (-sin(\theta-\gamma))(-\sin(\phi-\theta))\right)\ ]$$

$$\because multiplying\ 8\ and\ dividing\ 8\ and$$
$$\sin(b-a) = -\sin(a-b)$$

$$= \frac{4ab}{2\sin\theta\cos\theta\ \ 2\cos\phi\sin\phi\ \ 2\cos\gamma\sin\gamma}[$$

$$\sin\phi\cos\gamma\,(\sin(\phi-\theta)\sin(\theta-\gamma)) -$$

$$\sin\gamma\cos\phi\,(\sin(\theta-\gamma)sin(\phi-\theta))\ \ ]$$

$$\because\ \textit{rearranging terms}$$

$$= \frac{4ab\ \ \sin(\phi-\theta)\ \ \sin(\theta-\gamma)}{\sin 2\theta\ \ \sin 2\phi\ \ \sin 2\gamma}[$$

$$\sin\phi\cos\gamma - \sin\gamma\cos\phi]$$

$$\because \sin X\cos X = \sin 2X\ \ \textit{and taking}$$

$$= \quad \frac{4ab\ \ \sin(\phi-\theta)\ \ \sin(\theta-\gamma)\ \ \sin(\phi-\gamma)}{\sin 2\theta\ \ \sin 2\phi\ \ \sin 2\gamma\ \ \sin(\phi-\gamma)}$$

$$\because \sin A\cos B - \sin B\cos A = \sin(A-B)$$

$$= \frac{4ab\ \ (-\sin(\theta-\phi))\ \ \sin(\phi-\gamma)\ \ (-\sin(\gamma-\theta))}{\sin 2\theta\ \ \sin 2\phi\ \ \sin 2\gamma\ \ \sin(\phi-\gamma)}$$

$$\sin(b-a) = -\sin(a-b)$$

$$= \frac{4ab\ \ \sin(\theta-\phi)\ \ \sin(\phi-\gamma)\ \ \sin(\gamma-\theta)}{\sin 2\theta\ \ \sin 2\phi\ \ \sin 2\gamma\ \ \sin(\phi-\gamma)} = R.H.S$$