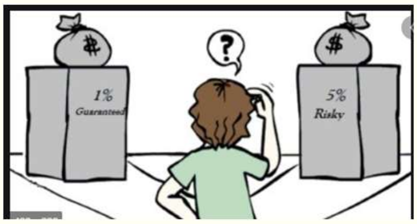
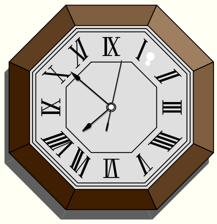
CHAPTER 2– INTEREST AND TIME VALUE OF MONEY



References

- I would like to acknowledge all the authors of Engineering economics for their contribution. I this session of Engineering Economic study I have referred them directly or indirectly and its only for the teaching learning methodology.
- Specially I have followed
  - Contemporary Engineering Economics six edition by Chan S. Park
  - Engineering Economy 16th Edition by William G. Sullivan and Elin M. Wicks
  - Principles Of Economics by N. Gregory Mankiw
  - Engineering economic analysis by Eschenbach, 11th Ted G. Newnan, Donald G. Lavelle, Jerome P
  - Engineering economics by Yates, J. K
  - Engineering Economy by Leland Blank, Anthony Tarquin
  - Principles of Engineering Economics with Applications by Zahid A. Khan, Arshad N. Siddiquee, Brajesh Kumar, Mustufa H. Abidi
  - Fundamentals of Engineering Economic Analysis by John A. White et al.
  - Nanda Shakya sir MANUAL-OF-ENGINEERING-ECONOMY
  - Engineering Economics by Santosh Kumar Shrestha
  - Fundamentals of Engineering Economics and Decision Analysis by David L. Whitman, Ronald E. Terry

Time Value of Money



- Money has value
  - Money can be leased or rented
  - The payment is called interest
  - If you put $100 in a bank at 9% interest for one time period you will receive back your original $100 plus $9.

Time Value of Money

- The "time value of money" seems like a sophisticated concept, yet it is a concept that you grapple with every day.
- Should you buy something today or save your money and buy later?
- Suppose you have two options;
  (1) receiving $1,000 cash now, or
  (2) receiving $1,090 one year from now. Which option would you prefer?

Assume that there is no risk of not receiving the money a year from now. Ask yourself what you would do if you decide to receive $1,000 now. If you don't need money now for any other purpose, you may deposit the money in a bank. If the bank's interest rate is 10%, then your balance would grow to $1,100. Certainly waiting a year to receive $1,090 does not make sense.

Again, Suppose that you could expect a 4% inflation during your deposit period. Then what is the case?

Contd…

- The receipt of $1,090 one year from now is only worth $1,048 (= $1,090/1.04) if you expect a 4% inflation.
- What the example above illustrates is that we must connect the "earning power" and the "purchasing power" to the concept of time.

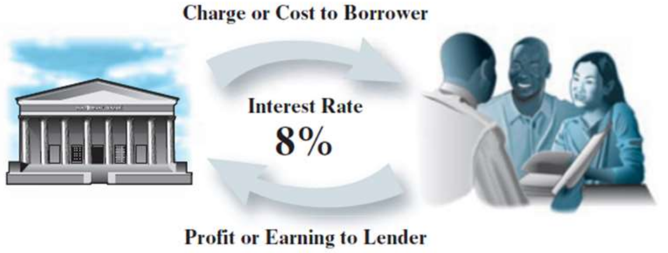


Figure 3.1 The meaning of *interest rate* to the lender (bank) and to the borrower.

[Contemporary Engineering Economics; Park C.S. (2016)]

Earning and Purchasing Power

- Money has both **earning power** as well as **purchasing power** over time, as shown in Figure 3.2 (it can be put to work, earning more money for its owner), a dollar received today has a greater value than a dollar received at some future time.

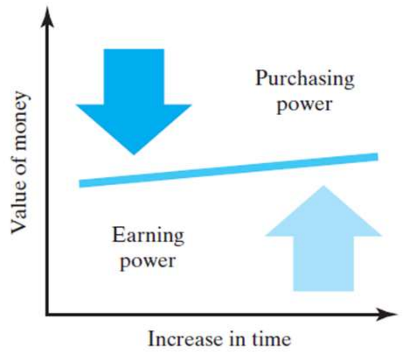


Figure 3.2 The time value of money. This is a two-edged sword whereby earning grows, but purchasing power decreases as time goes by due to inflation.

[Contemporary Engineering Economics; Park C.S. (2016)]

Elements for Transactions Involving Interest

Transactions

- borrowing or investing money, or purchasing machinery on credit

- **Principal (P)**
- **Interest rate (i%)**
- **Interest period (Daily, Monthly, Weekly , Quarterly, Semi-annually, Annually)**
- **Number of interest periods (N: 1,2,3,…………., n,………….. N.)**
- **Plan for receipts or disbursements**
- **Future amount of money (F)**

End-of-period Convention

In practice, cash flows can occur
- at the beginning or
- in the middle
- End of an interest period
- Indeed, at practically any point in time.

One of the simplifying assumptions we make in engineering economic analysis is the **end-of-period convention, which is the practice of** placing all cash flow transactions at the end of an interest period.

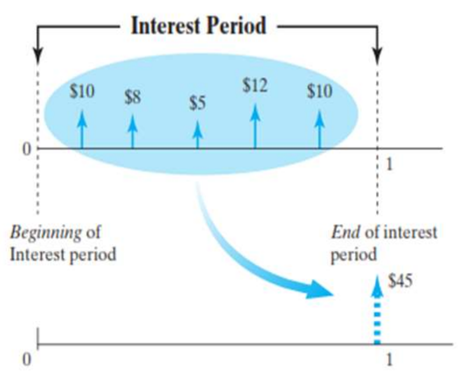


Figure 3.4 Any cash flows occurring during the interest period are summed to a single amount and placed at the end of the interest period.

[Contemporary Engineering Economics; Park C.S. (2016)]

Method of Calculating Interest

- Simple Interest
- Compound Interest
- Note: Engineering Economics use compound interest in most of the case if not specified
- **Simple Interest**
- Simple interest is interest earned on only the principal amount during each interest period. In other words, with simple interest, the interest earned during each interest period does not earn additional interest in the remaining periods, *even though you do not withdraw it.*

$$I = (iP)N$$

The total amount available at the end of N periods thus would be

$$F = P + I = P(1 + iN)$$

Compound Interest

- Under a compound-interest scheme, the interest earned in each period is calculated on the basis of the total amount at the end of the previous period. This total amount includes the original principal plus the accumulated interest that has been left in the account. In this case, you are, in effect, increasing the deposit amount by the amount of interest earned.
- In general, if you deposited (invested) P dollars at interest rate i, you would have P + iP = P(1 + i) dollars at the end of one period. If the entire amount (principal and interest) is reinvested at the same rate i for another period, at the end of the second period you would have

$$P(1 + i) + i[P(1 + i)] = P(1 + i)(1 + i)$$
$$= P(1 + i)^2$$

Continuing, we see that the balance after the third period is

$$P(1 + i)^2 + i[P(1 + i)^2] = P(1 + i)^3$$

This interest-earning process repeats, and after N periods, the total accumulated value (balance) F will grow to

$$F = P(1 + i)^N \qquad (3.3)$$

F=P (F/P,i%,N)

single payment compound-amount factor

End of Period	(A) Amount Owed	(B) Interest for Next Period	(C) = (A) + (B) Amount Owed for Next Period*
0	P	Pi	$P + Pi = P(1+i)$
1	$P(1+i)$	$P(1+i)i$	$P(1+i) + P(1+i)i = P(1+i)^2$
2	$P(1+i)^2$	$P(1+i)^2 i$	$P(1+i)^2 + P(1+i)^2 i = P(1+i)^3$
3	$P(1+i)^3$	$P(1+i)^3 i$	$P(1+i)^3 + P(1+i)^3 i = P(1+i)^4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$P(1+i)^{n-1}$	$P(1+i)^{n-1} i$	$P(1+i)^{n-1} + P(1+i)^{n-1} i = P(1+i)^n$
n	$P(1+i)^n$		

*Notice, the value in column (C) for the end of period $(n-1)$ provides the value in column (A) for the end of period n.

[Principles Of Engineering Economic Analysis, White (2012)]

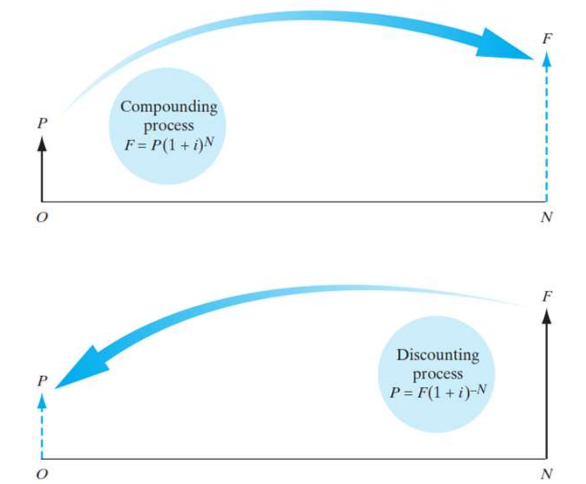


Figure 3.10 Equivalence relation between P and F.

[Contemporary Engineering Economics; Park C.S. (2016)]

Discounting Process

$$P = F\left[\frac{1}{(1+i)^N}\right] = F(P/F, i, N)$$

The factor $(1+i)^{-N}$ is known as the **single-payment present-worth factor** and is designated $(P/F, i, N)$. Tables have been constructed for P/F factors and for various values of i and N. The interest rate i and the P/F factor are also referred to as the **discount rate** and **discounting factor**, respectively.

- The total interest earn ed over *N periods is*

$$I = F - P = P[(1+i)^N - 1]$$

- Compared with the simple-interest scheme, the additional interest earned with compound interest is

$$\begin{aligned}\Delta I &= P[(1+i)^N - 1] - (iP)N \\ &= P[(1+i)^N - (1+iN)]\end{aligned}$$

Comparison of Simple and Compound Interest Over Time

- If you loaned a friend money for short period of time the difference between simple and compound interest is negligible.
- If you loaned a friend money for a long period of time the difference between simple and compound interest may amount to a considerable difference.

Check the table to see the difference over time.

Short or long? When is the $ difference significant?
You pick the time period.

Simple and compound interest
Single payment

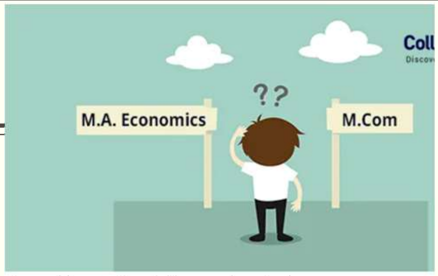
Principal = 100.00
Interest = 9.00%

Period	Simple amount factor	Compound amount factor
	Find Fs Given P	Find F Given P
n	Fs/P	F/P
0	100.000	100.000
1	109.000	109.000
2	118.000	118.810
3	127.000	129.503
4	136.000	141.158
5	145.000	153.862
6	154.000	167.710
7	163.000	182.804
8	172.000	199.256
9	181.000	217.189
10	190.000	236.736
11	199.000	258.043
12	208.000	281.266
13	217.000	306.580
14	226.000	334.173
15	235.000	364.248
16	244.000	397.031
17	253.000	432.763
18	262.000	471.712
19	271.000	514.166
20	280.000	560.441

Four Discounted Cash Flow Rules

1. Money has a time value;
2. Money cannot be added or subtracted unless it occurs at the same point(s) in time;
3. To move money forward one time unit, multiply by one plus the discount or interest rate;
4. To move money backward one time unit, divide by one plus the discount or interest rate.

Economic equivalence



https://www.collegedekho.com/articles/ma-economics-vs-mcom-comparison/

- Economic equivalence exists between cash flows that have the same economic effect and could therefore be traded for one another in the financial marketplace, which we assume to exist.

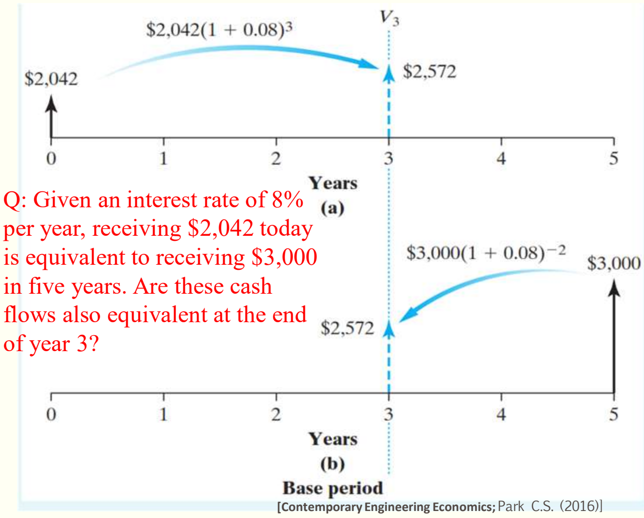
- Economic equivalence refers to the fact that a cash flow—whether a single payment or a series of payments—can be converted to an *equivalent cash flow at any point in time. For* example, we could find the equivalent future value F *of a present amount P at interest rate i at period n; or we could determine the equivalent present value P of N equal payments A.*

Equivalence Calculation: General Principles

➢ **Principle 1: Equivalence calculations made to compare alternatives require a common time basis**

❑One aspect of this basis is the choice of a single point in time at which to make our calculations.

❑If we had been given the magnitude of each cash flow and had been asked to determine whether they were equivalent, we could have chosen any reference point and used the compound interest formula to find the value of each cash flow at that point.



Q: Given an interest rate of 8% per year, receiving $2,042 today is equivalent to receiving $3,000 in five years. Are these cash flows also equivalent at the end of year 3?

[Contemporary Engineering Economics; Park C.S. (2016)]

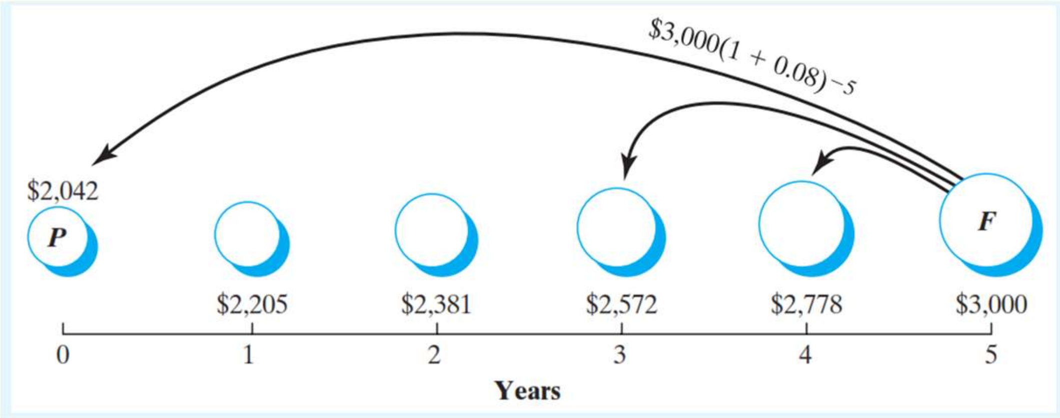


Fig: Various dollar amounts that will be economically equivalent to $3,000 in five years, given an interest rate of 8%

[Contemporary Engineering Economics; Park C.S. (2016)]

Principle 2:

Equivalence Depends on Interest Rate

- The equivalence between two cash flows is a function of the amount and timing of individual cash flows and the interest rate or rates that operate on those flows.

Changing the interest rate Destroys equivalence

Q: Given an interest rate of 8% per year, receiving $2,042 today is equivalent to receiving $3,000 in five years. Are these cash flows also equivalent at an interest rate of 10%? If not, which option is more economical?

SOLUTION

Given: $P = \$2{,}042$, $i = 10\%$ per year, and $N = 5$ years.

Find: F and is it equal to \$3,000?

We first determine the base period under which an equivalence value is computed. Since we can select any period as the base period, let's select $N = 5$. We then need to calculate the equivalent value of \$2,042 today five years from now.

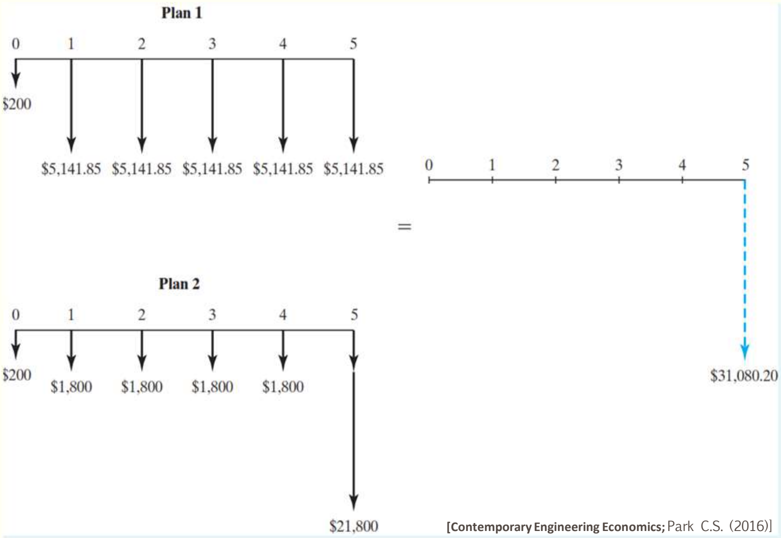
$$F = \$2{,}042(1 + 0.10)^5 = \$3{,}289.$$

Since this amount is greater than \$3,000, the change in interest rate breaks the equivalence between the two cash flows. As you may have already guessed, at a lower interest rate, P must be higher to be equivalent to the future amount. For example, at $i = 4\%$, $P = \$2{,}466$.

Principle 3:

❖ **Equivalence Calculations May Require the Conversion of Multiple Payment Cash Flows to a Single Cash Flow**

- Cash flow series involves moving each individual cash flow in the series to the same single point in time and summing these values to yield a single equivalent cash flow

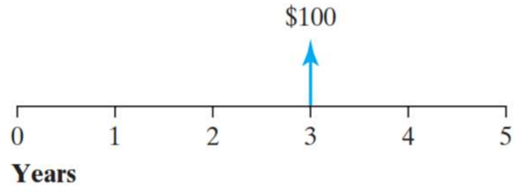


[Contemporary Engineering Economics; Park C.S. (2016)]

- Principle 4: Equivalence is maintained regardless of point of view

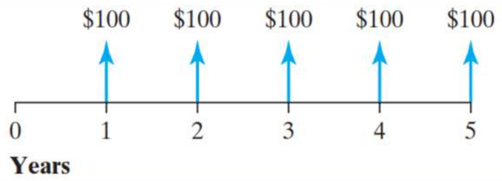
Categories of cash flow transactions

- There are five major categories of cash flow transactions :
    1) a single cash flow,
    2) a uniform series,
    3) a linear-gradient series,
    4) a geometric-gradient series, and
    5) an irregular series.



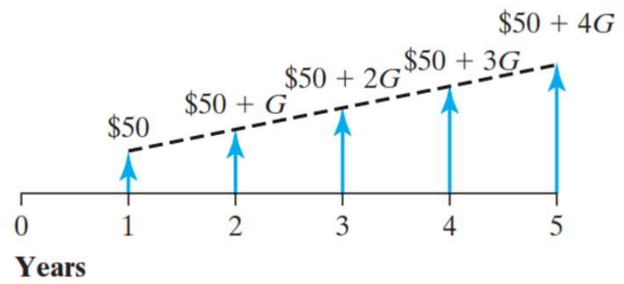
(a) Single cash flow

[Contemporary Engineering Economics; Park C.S. (2016)]



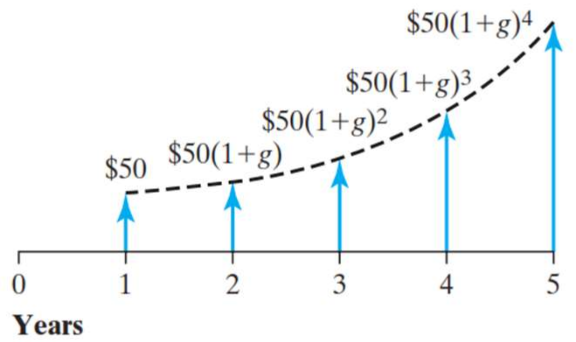
(b) Equal (uniform) payment series at regular intervals

[Contemporary Engineering Economics; Park C.S. (2016)]



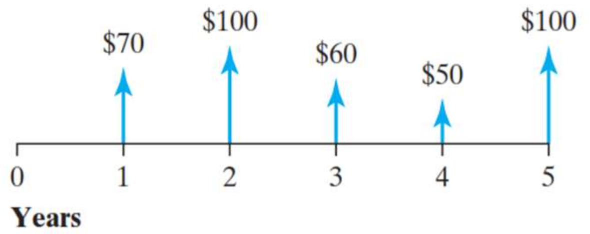
(c) Linear-gradient series, where each cash flow in the series increases or decreases by a fixed amount *G*

[Contemporary Engineering Economics; Park C.S. (2016)]



(d) Geometric-gradient series, where each cash flow in the series increases or decreases by a fixed rate (percentage) *g*

[Contemporary Engineering Economics; Park C.S. (2016)]



(e) Irregular payment series, which exhibits no regular overall pattern

[Contemporary Engineering Economics; Park C.S. (2016)]

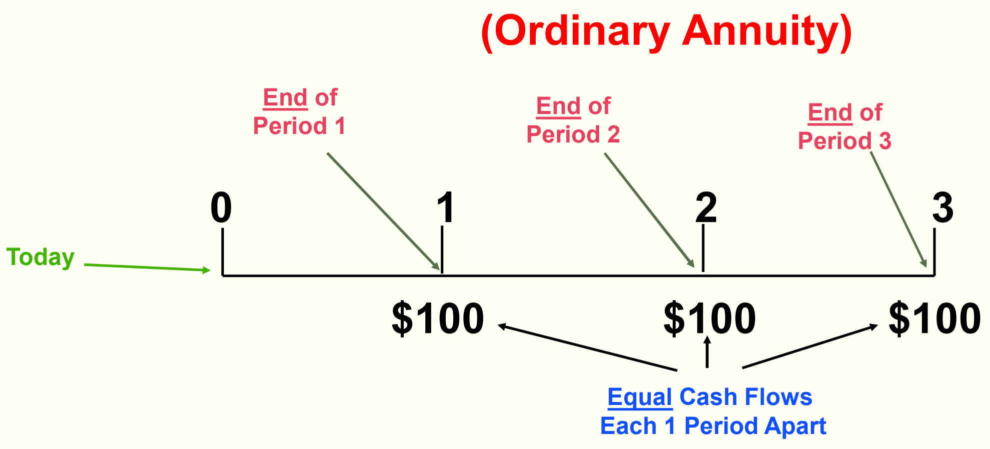
Types of Annuities

- ***An Annuity*** **represents a series of equal payments (or receipts) occurring over a specified number of equidistant periods.**
- **Ordinary Annuity: Payments or receipts occur at the end of each period.**
- **Annuity Due: Payments or receipts occur at the beginning of each period.**

Examples of Annuities

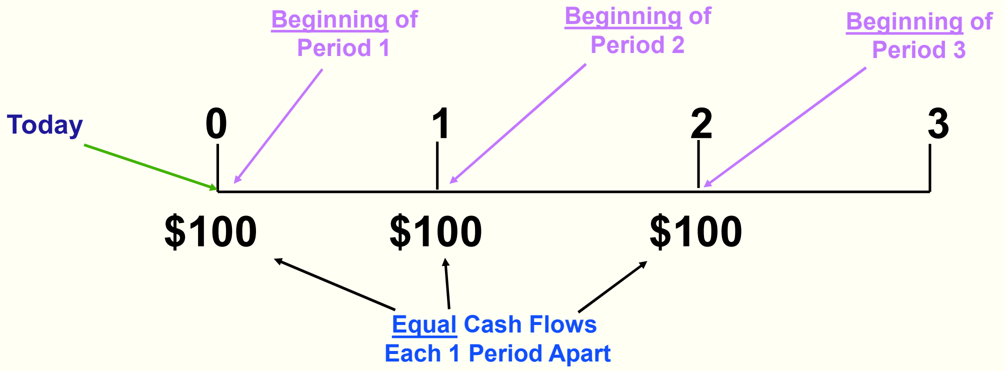
- Student Loan Payments
- Car Loan Payments
- Insurance Premiums
- Mortgage Payments
- Retirement Savings

Parts of an Annuity

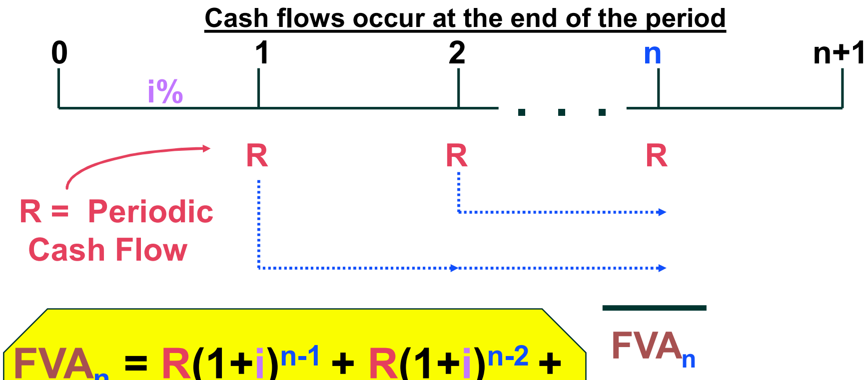


Parts of an Annuity

(Annuity Due)

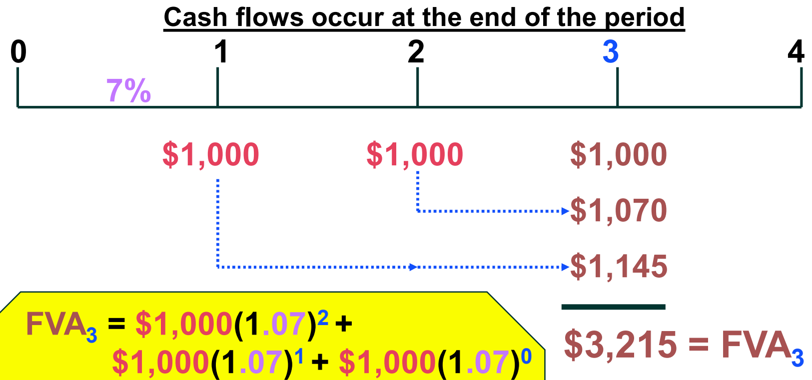


Overview of an Ordinary Annuity -- FVA



$$FVA_n = R(1+i)^{n-1} + R(1+i)^{n-2} + \dots + R(1+i)^{1} + R(1+i)^{0}$$

Example of an Ordinary Annuity -- FVA



FVA_3 = \$1,000(1.07)^2 + \$1,000(1.07)^1 + \$1,000(1.07)^0

= \$1,145 + \$1,070 + \$1,000

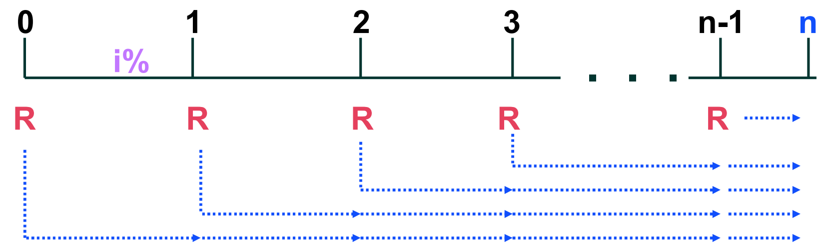
= \$3,215

Hint on Annuity Valuation

The future value of an ordinary annuity can be viewed as occurring at the *end* of the last cash flow period, whereas the future value of an annuity due can be viewed as occurring at the *beginning* of the last cash flow period.

Overview View of an Annuity Due -- FVAD

Cash flows occur at the beginning of the period

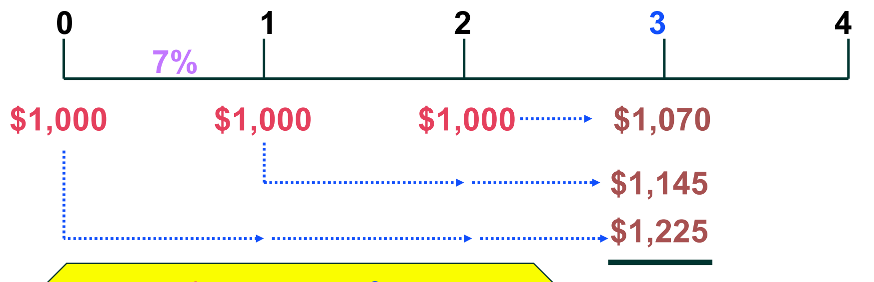


$$\text{FVAD}_n = R(1+i)^n + R(1+i)^{n-1} + \ldots + R(1+i)^2 + R(1+i)^1 = \text{FVA}_n(1+i)$$

FVAD_n

Example of an Annuity Due -- FVAD

Cash flows occur at the beginning of the period



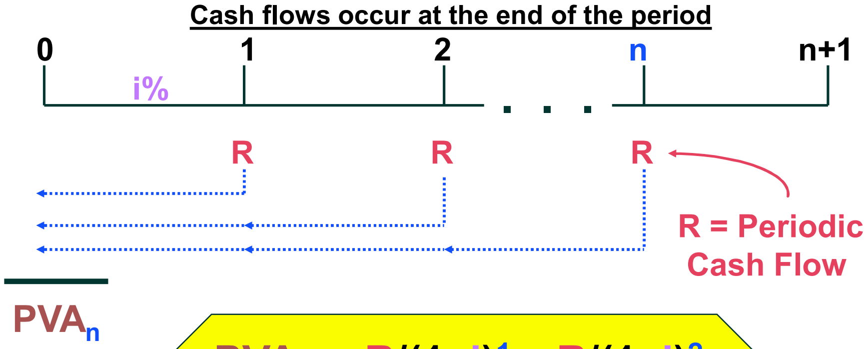
$$\text{FVAD}_3 = \$1{,}000(1.07)^3 + \$1{,}000(1.07)^2 + \$1{,}000(1.07)^1$$

$$= \$1{,}225 + \$1{,}145 + \$1{,}070$$

$$= \$3{,}440$$

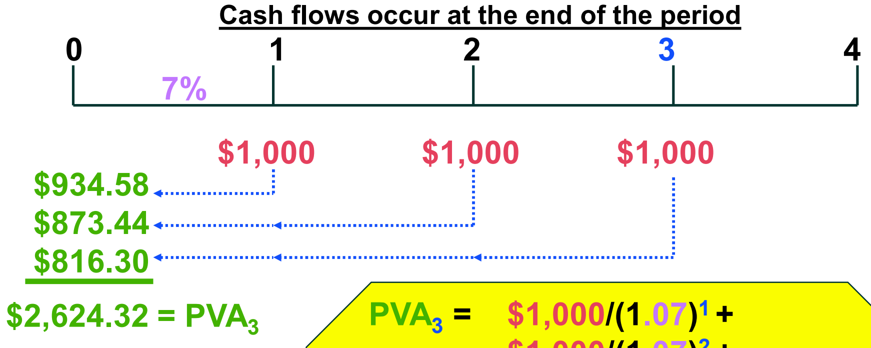
$\$3{,}440 = \text{FVAD}_3$

Overview of an Ordinary Annuity -- PVA



$$PVA_n = R/(1+i)^1 + R/(1+i)^2 + \ldots + R/(1+i)^n$$

Example of an Ordinary Annuity -- PVA



$$PVA_3 = \$1{,}000/(1.07)^1 + \$1{,}000/(1.07)^2 + \$1{,}000/(1.07)^3$$

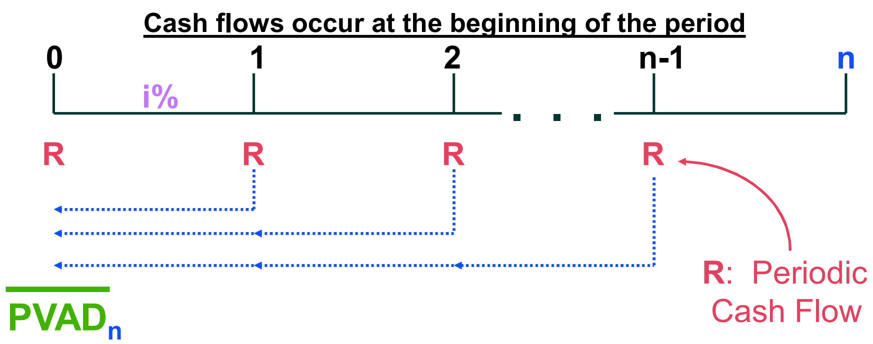
$$= \$934.58 + \$873.44 + \$816.30$$

$$= \$2{,}624.32$$

Hint on Annuity Valuation

The present value of an ordinary annuity can be viewed as occurring at the *beginning* of the first cash flow period, whereas the future value of an annuity due can be viewed as occurring at the *end* of the first cash flow period.

Overview of an Annuity Due -- PVAD



$$PVAD_n = R/(1+i)^0 + R/(1+i)^1 + \ldots + R/(1+i)^{n-1}$$
$$= PVA_n\,(1+i)$$

Example of an Annuity Due -- PVAD

Cash flows occur at the beginning of the period

$1,000.00 $1,000 $1,000

$ 934.58

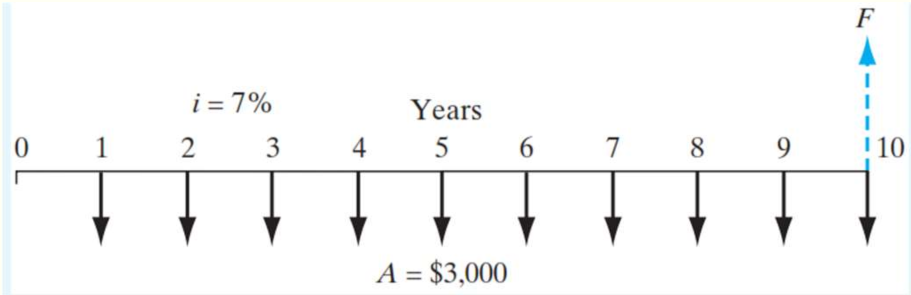
$ 873.44

$2,808.02 = $PVAD_n$

$$PVAD_n = \$1{,}000/(1.07)^0 + \$1{,}000/(1.07)^1 + \$1{,}000/(1.07)^2 = \$2{,}808.02$$

Question

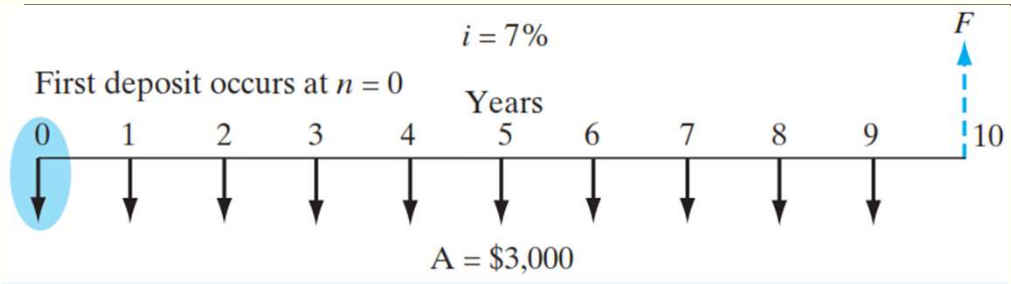
Suppose you make an annual contribution of $3,000 to your savings account at the end of each year for 10 years. If the account earns 7% interest annually, how much can be withdrawn at the end of 10 years.



F = $3,000(F/A, 7%, 10)
= $3,000(13.8164)
= $41,449.20

[Contemporary Engineering Economics; Park C.S. (2016)]

Suppose that all deposits were made at the *beginning of each period instead.* How would you compute the balance at the end of period 10?



Each payment has been shifted to one year earlier; thus, each payment would be compounded for one extra year. Note that with the end-of-year deposit, the ending balance (F) *was $41,449.20. With the beginningof-year* deposit, the same balance accumulates by the end of period 9. This balance can earn interest for one additional year. Therefore, we can easily calculate the resulting balance as

F = $41,449.20(1.07) = $44,350.64

[Contemporary Engineering Economics; Park C.S. (2016)]

Equal Payment Series (Or Annuity Series)

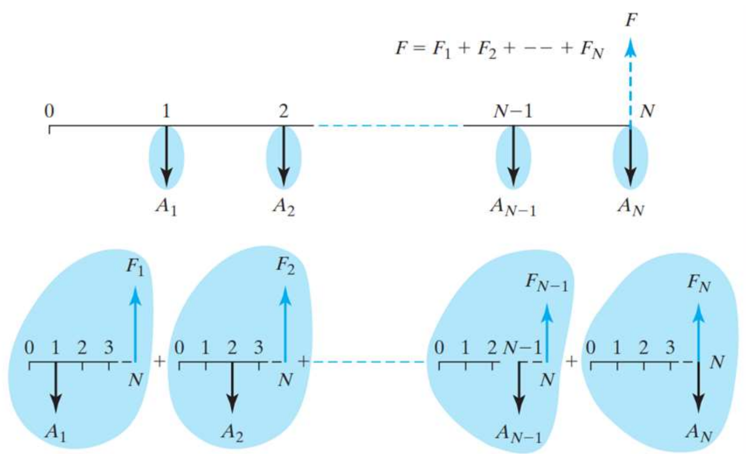


Figure 3.19 The future worth of a cash flow series obtained by summing the future-worth figures of each of the individual flows.

[Contemporary Engineering Economics; Park C.S. (2016)]

$$F = A(1 + i)^{N-1} + A(1 + i)^{N-2} + \cdots + A(1 + i) + A$$

or expressed alternatively,

$$F = A + A(1 + i) + A(1 + i)^2 + \cdots + A(1 + i)^{N-1} \quad (3.8)$$

To find a closed form expression to the problem, we may multiply Eq. (3.8) by $(1 + i)$

$$(1 + i)F = A(1 + i) + A(1 + i)^2 + \cdots + A(1 + i)^N \quad (3.9)$$

Subtracting Eq. (3.8) from Eq. (3.9) to eliminate common terms gives us

$$F(1 + i) - F = -A + A(1 + i)^N$$

Solving for F yields

$$F = A\left[\frac{(1 + i)^N - 1}{i}\right] = A(F/A, i, N) \quad (3.10)$$

The bracketed term in Eq. (3.10) is called the **equal-payment series compound-amount factor**, or the **uniform series compound-amount factor**; its factor notation is $(F/A, i, N)$. This

[Contemporary Engineering Economics; Park C.S. (2016)]

Sinking-Fund Factor:

$$A = F\left[\frac{i}{(1 + i)^N - 1}\right] = F(A/F, i, N)$$

The term within the brackets is called the **equal-payment series sinking-fund factor, or sinking-fund factor, and is referred to by the notation (*A/F, i, N*). *A sinking*** fund is an interest-bearing account into which a fixed sum is deposited each interest period; it is commonly established for the purpose of replacing fixed assets or retiring corporate bonds.

[Contemporary Engineering Economics; Park C.S. (2016)]

Consider three investment plans for an individual who just celebrated her 24th birthday at an annual interest rate of 8% (Figure 3.24):

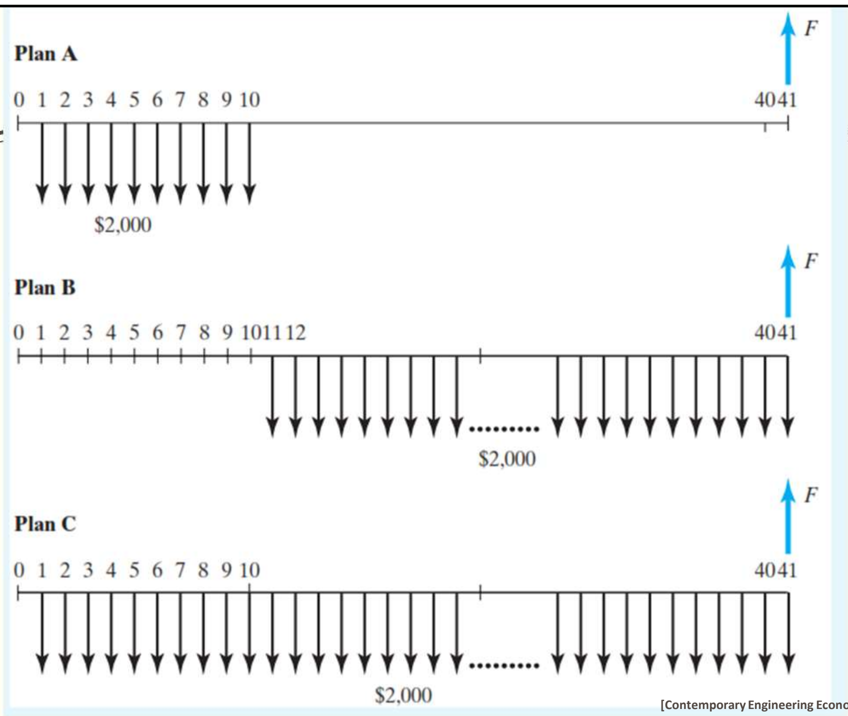
Plan A. Invest $2,000 per year for the first 10 years of your career. At the end of 10 years, make no further investments, but reinvest the amount accumulated at the end of 10 years for the next 31 years.

Plan B. Do nothing for the first 10 years. Then start investing $2,000 per year for the next 31 years.

Plan C. Invest $2,000 per year for the entire 41 years.

Note that all investments are made at the birthday of each year; the first deposit will be made on the 25th birthday ($n = 1$), and you want to calculate the balance on the 65th birthday ($n = 41$).

[Contemporary Engineering Economics; Park C.S. (2016)]



[Contemporary Engineering Economics; Park C.S. (2016)]

SOLUTION

Given: Three different deposit scenarios with $i = 8\%$ and $N = 41$ years.

Find: Balance at the end of 41 years (or on the 65th birthday).

Plan A:

$$F_{65} = \overbrace{\underbrace{\$2{,}000(F/A, 8\%, 10)}_{\$28{,}973.12}}^{\text{Balance at the end of 10 years}}(F/P, 8\%, 31)$$

$$= \$314{,}870.34$$

Plan B:

$$F_{65} = \underbrace{\$2{,}000(F/A, 8\%, 31)}_{\$246{,}691.74}$$

$$= \$246{,}691.74$$

Plan C:

$$F_{65} = \underbrace{\$2{,}000(F/A, 8\%, 41)}_{\$561{,}562.08}$$

$$= \$561{,}562.08$$

[Contemporary Engineering Economics; Park C.S. (2016)]

Capital Recovery

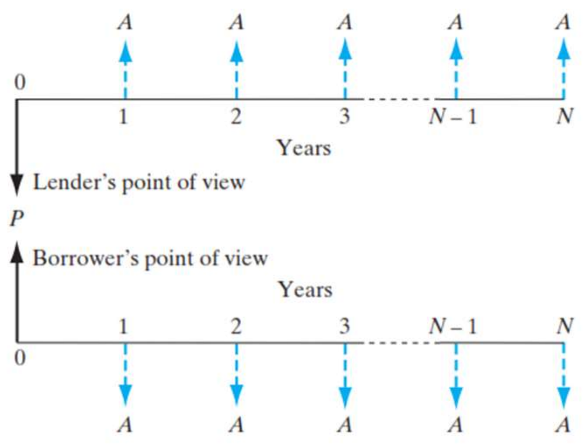


Figure 3.25 A repayment series for a loan by two different viewpoints.

[Contemporary Engineering Economics; Park C.S. (2016)]

Capital Recovery Factor (Annuity Factor): Find *A*, Given *P*, *i*, and *N*

We can determine the amount of a periodic payment A if we know P, i, and N. Figure 3.25 illustrates this situation. To relate P to A, recall the relationship between P and F in Eq. (3.3): $F = P(1 + i)^N$. Replacing F in Eq. (3.11) by $P(1 + i)^N$, we get

$$A = P(1 + i)^N\left[\frac{i}{(1 + i)^N - 1}\right]$$

or

$$A = P\left[\frac{i(1 + i)^N}{(1 + i)^N - 1}\right] = P(A/P, i, N) \qquad (3.12)$$

Now we have an equation for determining the value of the series of end-of-period payments A when the present sum P is known. The portion within the brackets is called the **equal-payment series capital-recovery factor**, or simply **capital-recovery factor**, which is designated $(A/P, i, N)$. In finance, this A/P factor is referred to as the **annuity factor** and indicates a series of payments of a fixed, or constant, amount for a specified number of periods.

[Contemporary Engineering Economics; Park C.S. (2016)]

Question

BioGen Company, a small biotechnology firm, has borrowed $250,000 to purchase laboratory equipment for gene splicing. The loan carries an interest rate of 8% per year and is to be repaid in equal installments over the next six years. Compute the amount of the annual installments.

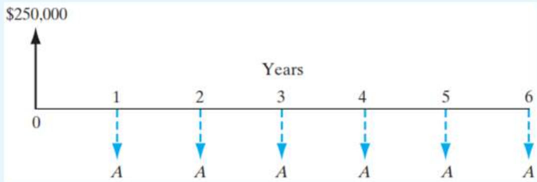


Figure 3.26 A loan cash flow diagram from BioGen's point of view.

SOLUTION

Given: $P = \$250{,}000$, $i = 8\%$ per year, and $N = 6$ years.

Find: A.

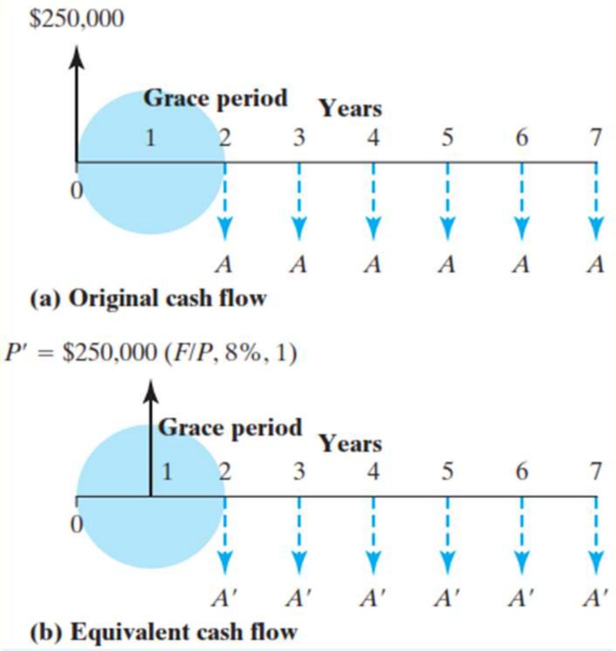
$$\begin{aligned} A &= \$250{,}000(A/P, 8\%, 6) \\ &= \$250{,}000(0.2163) \\ &= \$54{,}075 \end{aligned}$$

[Contemporary Engineering Economics; Park C.S. (2016)]

Question

BioGen Company, a small biotechnology firm, has borrowed $250,000 to purchase laboratory equipment for gene splicing. The loan carries an interest rate of 8% per year and is to be repaid in equal installments over the next six years. Compute the amount of the annual installments.

Suppose that BioGen wants to negotiate with the bank to defer the first loan repayment until the end of year 2 (but still desires to pay in six equal installments at 8% interest). If the bank wishes to earn the same profit, what should be the annual installment, also known as **deferred annuity**



[Contemporary Engineering Economics; Park C.S. (2016)]

SOLUTION

Given: $P = \$250{,}000$, $i = 8\%$ per year, and $N = 6$ years, but the first payment occurs at the end of year 2.

Find: A.

By deferring the loan for a year, the bank will add the interest accrued during the first year to the principal. In other words, we need to find the equivalent worth P' of $250,000 at the end of year 1:

$$\begin{aligned} P' &= \$250{,}000(F/P, 8\%, 1) \\ &= \$270{,}000 \end{aligned}$$

In fact, BioGen is borrowing $270,000 for six years. To retire the loan with six equal installments, the deferred equal annual payment on P' will be

$$\begin{aligned} A' &= \$270{,}000(A/P, 8\%, 6) \\ &= \$58{,}401 \end{aligned}$$

By deferring the first payment for one year, BioGen needs to make additional payments of $4,326 in each year.

[Contemporary Engineering Economics; Park C.S. (2016)]

Linear-Gradient Series

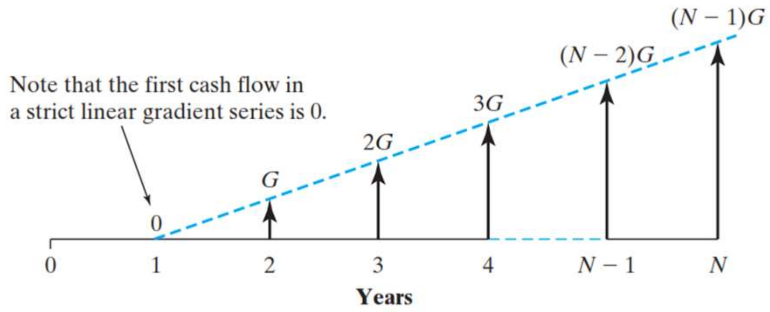


Figure 3.30 A cash flow diagram for a strict gradient series.

[Contemporary Engineering Economics; Park C.S. (2016)]

Contd…………..

- Situations involving periodic payments that increase or decrease by a constant amount (G) *from period to period*

Figure above illustrates a **strict gradient series,** $A_n = (n - 1)G$.

- *Note:* that the origin of the series is at the end of the first period with a zero value.
- The gradient *G can be either positive or negative.*
  - **If** *$G > 0$, the series is referred to as an increasing gradient series.*
  - **If** *$G < 0$, it is a decreasing gradient series.*

Gradient Series as Composite Series

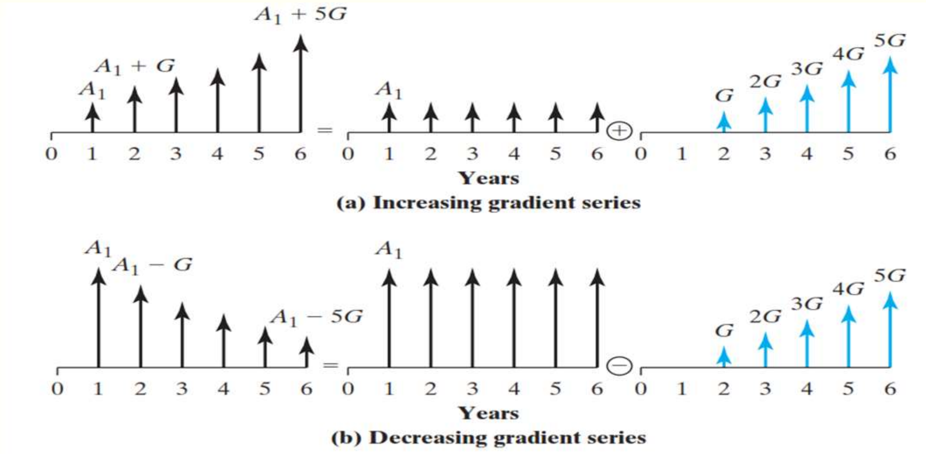


Figure 3.31 Two types of linear-gradient series as composites of a uniform series of N payments of A_1 and the gradient series of increments of constant amount G.

[Contemporary Engineering Economics; Park C.S. (2016)]

Present-Worth Factor: Linear Gradient: Find P, Given G, N, and i

How much would you have to deposit now to withdraw the gradient amounts specified in Figure 3.30? To find an expression for the present amount P, we apply the single-payment present-worth factor to each term of the series and obtain

$$P = 0 + \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \cdots + \frac{(N-1)G}{(1+i)^N}$$

or

$$P = \sum_{n=1}^{N} (n-1)G(1+i)^{-n} \qquad (3.14)$$

Letting $G = a$ and $1/(1 + i) = x$ yields

$$\begin{aligned} P &= 0 + ax^2 + 2ax^3 + \cdots + (N-1)ax^N \\ &= ax[0 + x + 2x^2 + \cdots + (N-1)x^{N-1}] \qquad (3.15) \end{aligned}$$

Since an arithmetic–geometric series $\{0, x, 2x^2, \ldots, (N-1)x^{N-1}\}$ has the finite sum

$$0 + x + 2x^2 + \cdots + (N-1)x^{N-1} = x\left[\frac{1 - Nx^{N-1} + (N-1)x^N}{(1-x)^2}\right]$$

we can rewrite Eq. (3.15) as

$$P = ax^2\left[\frac{1 - Nx^{N-1} + (N-1)x^N}{(1-x)^2}\right]$$

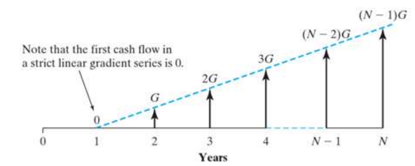


Figure 3.30 A cash flow diagram for a strict gradient series.

Replacing the original values for A and x, we obtain

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right] = G(P/G, i, N) \qquad (3.17)$$

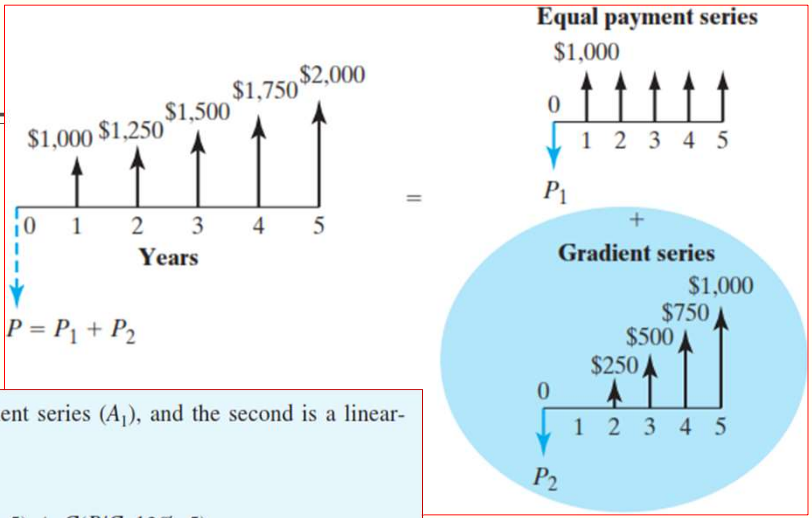
The resulting factor in brackets is called the **gradient-series present-worth factor**, which we denote as $(P/G, i, N)$.

[Contemporary Engineering Economics; Park C.S. (2016)]

Question

A textile mill has just purchased a lift truck that has a useful life of five years. The engineer estimates that maintenance costs for the truck during the first year will be $1,000. As the truck ages, maintenance costs are expected to increase at a rate of $250 per year over the remaining life. Assume that the maintenance costs occur at the end of each year. The firm wants to set up a maintenance account that earns 12% annual interest. All future maintenance expenses will be paid out of this account. How much does the firm have to deposit in the account now?

Solution



The first component is an equal-payment series (A_1), and the second is a linear-gradient series (G). We have

$$P = P_1 + P_2$$

$$P = A_1(P/A, 12\%, 5) + G(P/G, 12\%, 5)$$

$$= \$1{,}000(3.6048) + \$250(6.397)$$

$$= \$5{,}204$$

Note that the value of N in the gradient factor is 5, not 4. This is because, by definition of the series, the first gradient value begins at period 2.

[Contemporary Engineering Economics; Park C.S. (2016)]

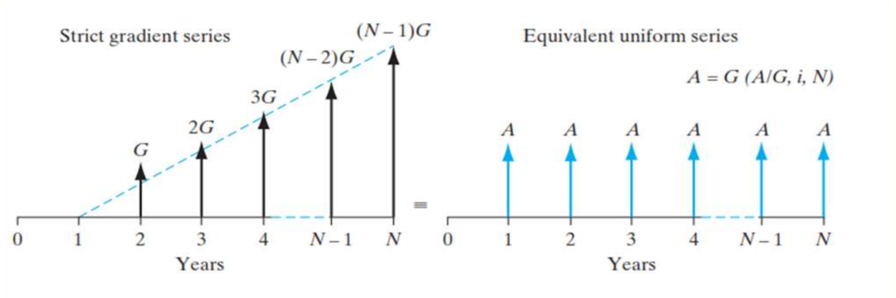


Figure 3.34 Conversion of a gradient series to its equivalent uniform series.

Gradient-to-Equal-Payment Series Conversion Factor: Find *A*, Given *G*, *i*, and *N*

We can obtain an equal payment series equivalent to the gradient series, as depicted in Figure 3.34, by substituting Eq. (3.17) for P into Eq. (3.12) to obtain

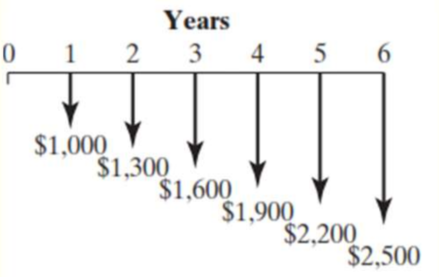
$$A = G\left[\frac{(1+i)^N - iN - 1}{i[(1+i)^N - 1]}\right] = G(A/G, i, N) \qquad (3.18)$$

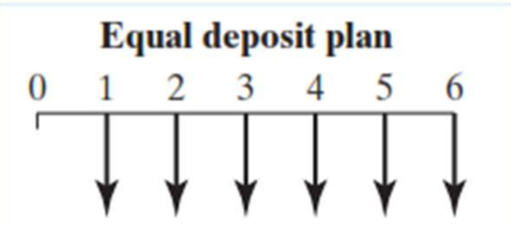
where the resulting factor in brackets is referred to as the **gradient-to-equal-payment series conversion factor** and is designated $(A/G, i, N)$.

[Contemporary Engineering Economics; Park C.S. (2016)]

Question

John and Barbara have just opened two savings accounts at their credit union. The accounts earn 10% annual interest. John wants to deposit $1,000 in his account at the end of the first year and increase this amount by $300 for each of the next five years. Barbara wants to deposit an equal amount each year for the next six years. What should be the size of Barbara's annual deposit so that the two accounts will have equal balances at the end of six years?





[Contemporary Engineering Economics; Park C.S. (2016)]

To find the equal payment series beginning at the end of year 1 and ending at year 6 that would have the same present worth as that of the gradient series, we may proceed as follows:

$$A = \$1{,}000 + \$300(A/G, 10\%, 6)$$
$$= \$1{,}000 + \$300(2.2236)$$
$$= \$1{,}667.08$$

Barbara's annual contribution should be $1,667.08.

OR

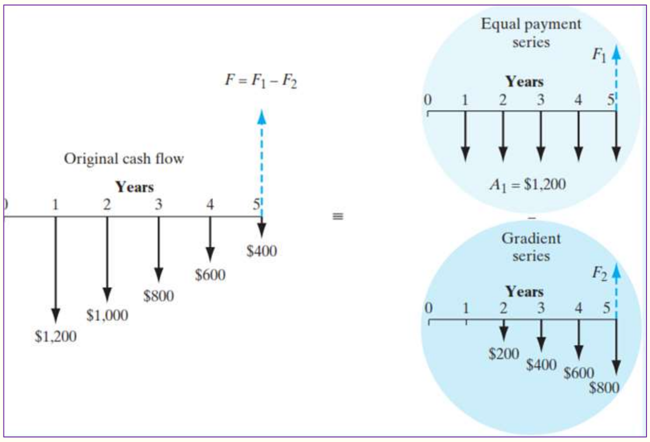
$$P = \$1{,}000(P/A, 10\%, 6) + \$300(P/G, 10\%, 6)$$
$$= \$1{,}000(4.3553) + \$300(9.6842)$$
$$= \$7{,}260.56$$

The equivalent uniform deposit is

$$A = \$7{,}260.56(A/P, 10\%, 6) = \$1{,}667.02$$

Question

Suppose that you make a series of annual deposits into a bank account that pays 10% interest. The initial deposit at the end of the first year is $1,200. The deposit amounts decline by $200 in each of the next four years. How much would you have immediately after the fifth deposit?



$$F = F_1 - F_2$$
$$= A_1(F/A, 10\%, 5) - \$200(P/G, 10\%, 5)(F/P, 10\%, 5)$$
$$= \$1{,}200(6.1051) - \$200(6.8618)(1.6105)$$
$$= \$5{,}115.93$$

[Contemporary Engineering Economics; Park C.S. (2016)]

GEOMETRIC GRADIENT SERIES

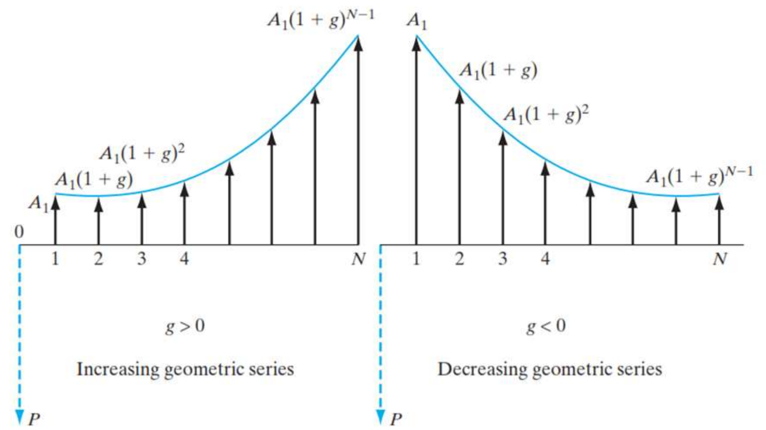


Figure 3.37 A geometrically increasing or decreasing gradient series at a constant rate g.

[Contemporary Engineering Economics; Park C.S. (2016)]

Present-Worth Factor: Find P, Given A_1, g, i, and N

Notice that the present worth of any cash flow A_n at interest rate i is

$$P_n = A_n(1+i)^{-n} = A_1(1+g)^{n-1}(1+i)^{-n}$$

To find an expression for the present amount P for the entire series, we apply the **single-payment present-worth factor** to each term of the series:

$$P = \sum_{n=1}^{N} A_1(1+g)^{n-1}(1+i)^{-n}$$

Bringing the constant term $A_1(1+g)^{-1}$ outside the summation yields

$$P = \frac{A_1}{(1+g)} \sum_{n=1}^{N} \left[\frac{1+g}{1+i}\right]^n$$

Let $a = \dfrac{A_1}{1+g}$ and $x = \dfrac{1+g}{1+i}$. Then rewrite Eq. (3.21) as

$$P = a(x + x^2 + x^3 + \cdots + x^N)$$

Since the summation in Eq. (3.22) represents the first N terms of a geometric series, we may obtain the closed-form expression as follows. First, multiply Eq. (3.22) by x to get

$$xP = a(x^2 + x^3 + x^4 + \cdots + x^{N+1}) \quad (3.23)$$

Then, subtract Eq. (3.23) from Eq. (3.22):

$$P - xP = a(x - x^{N+1})$$

$$P(1-x) = a(x - x^{N+1})$$

$$P = \frac{a(x - x^{N+1})}{1-x}, \quad (x \neq 1). \quad (3.24)$$

If we replace the original values for a and x, we obtain

$$P = \begin{cases} \dfrac{A_1}{i-g}\left[1 - \left(\dfrac{1+g}{1+i}\right)^N\right] & \text{if } i \neq g \\ \dfrac{NA_1}{1+i} & \text{if } i = g \end{cases} \quad (3.25)$$

Or in terms of engineering economy factor notation, we have

$$P = A_1(P/A_1, g, i, N)$$

The factor within brackets is called the **geometric-gradient series present-worth factor** and is designated $(P/A_1, g, i, N)$. In the special case where $i = g$, Eq. (3.21) becomes $P = [NA_1/(1+i)]$.

[Contemporary Engineering Economics; Park C.S. (2016)]

Question

Suppose that you have a savings account with your federal credit union. By looking at the history of the account, you learned the interest rate in each period during the last five years was as shown in Figure 3.43. Show how the credit union calculated your balance at the end of year 5.

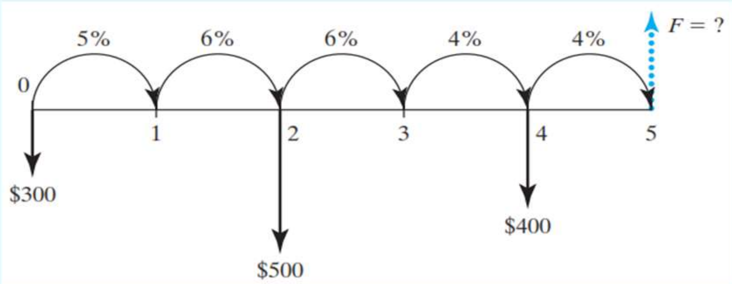


Figure 3.43 A future-value calculation with changing interest rates.

[Contemporary Engineering Economics; Park C.S. (2016)]

SOLUTION

Given: Deposit series shown in Figure 3.43 with varying interest rates; $N = 5$ years.

Find: Balance at the end of year 5 (F_5).

- Contribution of \$300 at $n = 0$ toward F_5:

$$\underbrace{\underbrace{\overbrace{300(F/P, 5\%, 1)}^{\text{Balance at } n=1}(F/P, 6\%, 2)}_{\text{Balance at } n=3}(F/P, 4\%, 2)}_{\text{Balance at } n=5} = \$382.82$$

- Contribution of \$500 at $n = 2$ toward F_5:

$$\underbrace{\overbrace{500(F/P, 6\%, 1)}^{\text{Balance at } n=3}(F/P, 4\%, 2)}_{\text{Balance at } n=5} = \$573.25$$

- Contribution of \$400 at $n = 4$ toward F_5:

$$\$400(F/P, 4\%, 1) = \$416$$

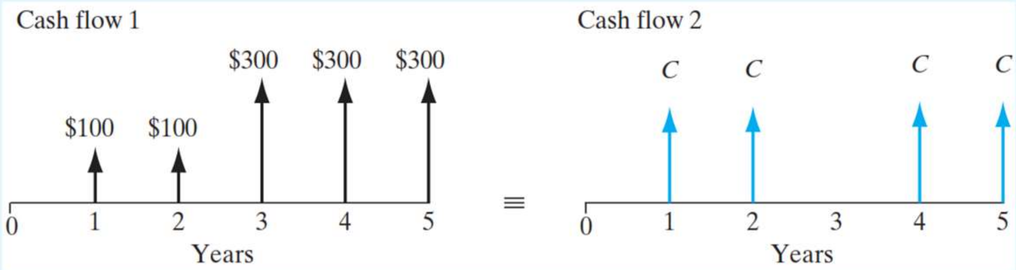
- Total balance at $n = 5$:

$$F_5 = \$382.82 + \$573.25 + \$416.00 = \$1{,}372.06$$

[Contemporary Engineering Economics; Park C.S. (2016)]

Question :

The two cash flows in Figure 3.47 are equivalent at an interest rate of 12% compounded annually. Determine the unknown value C.



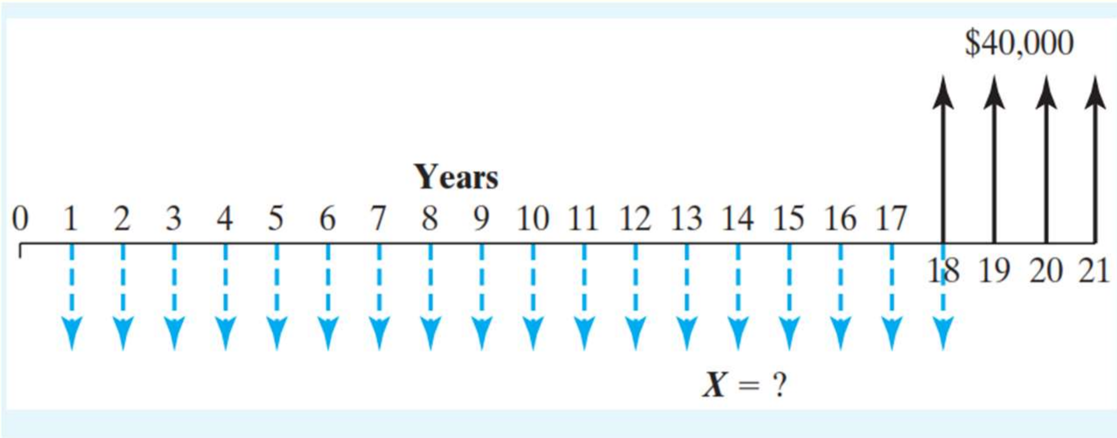
Answer
C = $256.97

[Contemporary Engineering Economics; Park C.S. (2016)]

Question :

A couple with a newborn daughter wants to save for their child's college expenses in advance. The couple can establish a college fund that pays 7% annual interest. Assuming that the child enters college at age 18, the parents estimate that an amount of $40,000 per year (actual dollars) will be required to support the child's college expenses for four years. Determine the equal annual amounts the couple must save until they send their child to college. (Assume that the first deposit will be made on the child's first birthday and the last deposit on the child's 18th birthday. The first withdrawal will be made at the beginning of the freshman year, which also is the child's 18th birthday.)

[Contemporary Engineering Economics; Park C.S. (2016)]



Answer : X = $4,264

[Contemporary Engineering Economics; Park C.S. (2016)]

Summary of Discrete Compounding Interest Factors.

To Find	Given	Factor	Symbol	Name
P	F	$(1+i)^{-n}$	$(P\|F\ i\%,n)$	Single sum, present worth factor
F	P	$(1+i)^{n}$	$(F\|P\ i\%,n)$	Single sum, compound amount factor
P	A	$\frac{(1+i)^n-1}{i(1+i)^n}$	$(P\|A\ i\%,n)$	Uniform series, present worth factor
A	P	$\frac{i(1+i)^n}{(1+i)^n-1}$	$(A\|P\ i\%,n)$	Uniform series, capital recovery factor
F	A	$\frac{(1+i)^n-1}{i}$	$(F\|A\ i\%,n)$	Uniform series, compound amount factor
A	F	$\frac{i}{(1+i)^n-1}$	$(A\|F\ i\%,n)$	Uniform series, sinking fund factor
P	G	$\frac{[1-(1+ni)(1+i)^{-n}]}{i^2}$	$(P\|G\ i\%,n)$	Gradient series, present worth factor
A	G	$\frac{(1+i)^n-(1+ni)}{i[(1+i)^n-1]}$	$(A\|G\ i\%,n)$	Gradient to uniform series conversion factor
P	A_1,j	$\frac{1-(1+j)^n(1+i)^{-n}}{i-j}$ for $i \neq j$	$(P\|A_1\ i\%,j\%,n)$	Geometric series, present worth factor
F	A_1,j	$\frac{(1+i)^n-(1+j)^n}{i-j}$ for $i \neq j$	$(F\|A_1\ i\%,j\%,n)$	Geometric series, future worth factor

[Principles Of Engineering Economic Analysis, White (2012)]

Summary of Continuous Compounding Interest Factors for Discrete Flows.

To Find	Given	Factor	Symbol
P	F	e^{-rn}	$(P\|F\ r,n)_\infty$
F	P	e^{rn}	$(F\|P\ r,n)_\infty$
F	A	$\frac{e^{rn}-1}{e^r-1}$	$(F\|A\ r,n)_\infty$
A	F	$\frac{e^r-1}{e^{rn}-1}$	$(A\|F\ r,n)_\infty$
P	A	$\frac{e^{rn}-1}{e^{rn}(e^r-1)}$	$(P\|A\ r,n)_\infty$
A	P	$\frac{e^{rn}(e^r-1)}{e^{rn}-1}$	$(A\|P\ r,n)_\infty$
P	G	$\frac{e^{rn}-1-n(e^r-1)}{e^{rn}(e^r-1)^2}$	$(P\|G\ r,n)_\infty$
A	G	$\frac{1}{e^r-1}-\frac{n}{e^{rn}-1}$	$(A\|G\ r,n)_\infty$
P	A_1, c	$\frac{1-e^{(c-r)n}}{e^r-e^c}$	$(P\|A_1\ r,c,n)_\infty$*
F	A_1, c	$\frac{e^{rn}-e^{cn}}{e^r-e^c}$	$(F\|A_1\ r,c,n)_\infty$*

*$r \neq c$.

[Principles Of Engineering Economic Analysis, White (2012)]

In above all examples and questions the implicit assumption was that payments are received once a year (or annually) or at the end of interest period.

However, some of the most familiar financial transactions involve payments that are not based on one annual payment—for example, monthly mortgage payments and quarterly earnings on savings accounts.

This need has led to the development of the concepts of the **nominal interest rate** and **the effective interest rate.**

Nominal Interest Rate OR annual percentage rate (APR),

Many banks, for example, state the interest arrangement for credit cards in this way:

APR 18% compounded monthly

This statement simply means that each month the bank will charge 1.5% interest on an unpaid balance. We say that 18% is the **nominal interest rate or annual percentage rate (APR), and the compounding frequency is monthly (12). As shown in Figure ,** to obtain the interest rate per compounding period, we divide, for example, 18% by 12 to get 1.5% per month.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Interest rate (%)	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5

Nominal interest rate.
1.5% × 12 = 18%

Figure 4.1 The nominal interest rate is determined by summing the individual interest rates per period.

The APR does not explain precisely the amount of interest that will accumulate in a year. To explain the true effect of more frequent compounding .On annual interest amounts, we will introduce the term **effective interest rate.**

[Contemporary Engineering Economics; Park C.S. (2016)]

Effective Annual Interest Rate

- **Commonly** known as **annual effective yield or annual percentage yield (APY) also.**
- The **effective annual interest rate** is the one rate that truly represents **the interest earned** in one year.
- Suppose you deposit $10,000 in a savings account that pays you at an interest rate of 9% compounded quarterly. Here, 9% represents the nominal interest rate, and the interest rate per quarter is 2.25% (9%/4).

TABLE 4.1 Interest Calculation with Quarterly Compounding

End of Period	Base Amount	Interest Earned 2.25% × (Base Amount)	New Base
First Quarter	$10,000.00	2.25% × $10,000.00 = $225.00	$10,225.00
Second Quarter	10,255.00	2.25% × $10,255.00 = $230.06	10,455.06
Third Quarter	10,455.06	2.25% × $10,455.06 = $235.24	10,690.30
Fourth Quarter	10,690.30	2.25% × $10,690.30 = $240.53	10,930.83

Clearly, you are earning more than 9% on your original deposit. In fact, you are earning = 9.3083% ($930.83/$10,000).

$$F = P(1 + i)^N$$
$$= 10{,}000(1 + 0.0225)^4$$
$$= \$10{,}930.83$$

[Contemporary Engineering Economics; Park C.S. (2016)]

This means for each dollar deposited, you are earning an equivalent annual interest of 9.308 cents.
In terms of an effective annual interest rate (i_{eff}), the interest payment can be rewritten as a percentage of the principal amount.

Example 9.3083%

In other words, earning 2.25% interest per quarter for four quarters is equivalent to earning 9.3083% interest just one time each year

TABLE 4–2 Effective Annual Interest Rates Using Equation [4.3]

r = 18% per year, compounded CP-ly

Compounding Period, CP	Times Compounded per Year, m	Rate per Compound Period, i%	Distribution of i over the Year of Compounding Periods	Effective Annual Rate, $i_a = (1+i)^m - 1$
Year	1	18	18% (1)	$(1.18)^1 - 1 = 18\%$
6 months	2	9	9% 9% (1 2)	$(1.09)^2 - 1 = 18.81\%$
Quarter	4	4.5	4.5% 4.5% 4.5% 4.5% (1 2 3 4)	$(1.045)^4 - 1 = 19.252\%$
Month	12	1.5	1.5% in each (1 2 3 4 5 6 7 8 9 10 11 12)	$(1.015)^{12} - 1 = 19.562\%$
Week	52	0.34615	0.34615% in each (1 2 3 … 24 26 28 … 50 52)	$(1.0034615)^{52} - 1 = 19.684\%$

Engineering economy, eighth edition; Blank et. Al (2018)

TABLE 4.2 Nominal and Effective Interest Rates with Different Compounding Periods

	Effective Interest Rates				
Nominal Rate	Compounding Annually	Compounding Semiannually	Compounding Quarterly	Compounding Monthly	Compounding Daily
4%	4.00%	4.04%	4.06%	4.07%	4.08%
5	5.00	5.06	5.09	5.12	5.13
6	6.00	6.09	6.14	6.17	6.18
7	7.00	7.12	7.19	7.23	7.25
8	8.00	8.16	8.24	8.30	8.33
9	9.00	9.20	9.31	9.38	9.42
10	10.00	10.25	10.38	10.47	10.52
11	11.00	11.30	11.46	11.57	11.62
12	12.00	12.36	12.55	12.68	12.74

[Contemporary Engineering Economics; Park C.S. (2016)]

So, it is required that financial institutions quote both the nominal interest rate and the compounding frequency (i.e., the effective interest) when you deposit or borrow money.

More frequent compounding increases the amount of interest paid over a year at the same nominal interest rate.

Assuming that the nominal interest rate is *r, and M* compounding periods occur during the year, we can calculate the effective annual interest rate or annual percentage yield (APY)

$$i_{eff} = \left(1 + \frac{r}{m}\right)^m - 1$$

When m=1 , $i_{eff} = r$ i.e. the **effective annual interest rate** is equal to the nominal interest.

That is, when compounding takes place once annually, the effective interest is equal to the nominal interest.

Effective Interest Rate Per Payment Periods

- The interest rate and payment periods must have the same time unit for the factors to correctly account for the time value of money.
- to compute the effective interest rate for periods of *any duration*

$$i = \left(1 + \frac{r}{M}\right)^C - 1$$
$$= \left(1 + \frac{r}{CK}\right)^C - 1$$

Where
M = number of interest periods per year
C = the number of interest periods per payment period
K = the number of payment periods per year

$M = CK$
$i = i_{eff}$ with $C = M$ and $K = 1$.

$$i_{eff} = \left(1 + \frac{r}{m}\right)^m - 1$$ The effective annual interest rate(i_{eff})

$$\mathbf{i = \left(1 + \frac{r}{m}\right)^{m/k} - 1}$$ Effective interest rate per payment periods (i)

[Contemporary Engineering Economics; Park C.S. (2016)]

What size monthly payments should occur when \$10,000 is borrowed at 8 percent per year compounded quarterly and the loan is repaid with 36 equal monthly payments?

From Equation 2.47, $r = 0.08$, $m = 4$, and $k = 12$. Therefore,

$$i = (1 + 0.08/4)^{4/12} - 1$$
$$= 0.006623 \text{ or } 0.6623\%/\text{month}$$

Knowing the monthly interest rate, the monthly payment can be determined:

$$A = \$10{,}000(A|P\ 0.6623\%, 36)$$
$$= \$10{,}000[(0.006623)(1.006623)^{36}]/[(1.006623)^{36} - 1]$$
$$= \$313.12$$

[Contemporary Engineering Economics; Park C.S. (2016)]

Continuous Compounding

- As the number of compounding periods (M) becomes very large, the interest rate per compounding period (r/M) becomes very small. As M approaches infinity and r/M approaches zero, we approximate the situation of continuous compounding.

$$
\begin{aligned}
i &= \lim_{CK \to \infty} \left[\left(1 + \frac{r}{CK}\right)^{C} - 1 \right] \\
&= \lim_{CK \to \infty} \left[\left(1 + \frac{r}{CK}\right)^{CK} \right]^{1/K} - 1 \\
&= (e^{r})^{1/K} - 1
\end{aligned}
$$

Therefore, the effective interest rate per payment period is $i = e^{r/K} - 1$

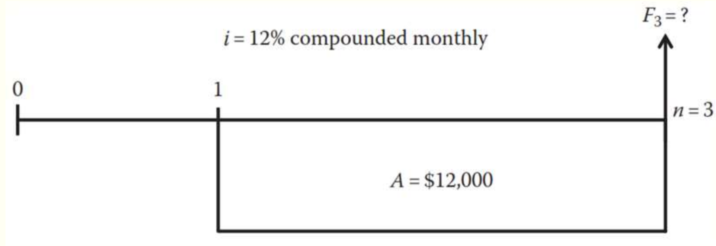
To calculate the effective annual interest rate for continuous compounding, K=1, So $i_a = e^{r} - 1$

TABLE 4.3 Summary of Selected Continuous Compounding Formulas with Discrete Payments

Flow Type	Find	Given	Factor Notation	Formula
Compound-Amount	F	P	$(F/P, r, N)$	$F = P(e^{rN})$
Present-Worth	P	F	$(P/F, r, N)$	$P = F(e^{-rN})$
Compound-Amount	F	A	$(F/A, r, N)$	$F = A\left[\frac{(e^{r})^{N} - 1}{e^{r} - 1}\right]$
Sinking-Fund	A	F	$(A/F, r, N)$	$A = F\left[\frac{e^{r} - 1}{(e^{r})^{N} - 1}\right]$
Present-Worth	P	A	$(P/A, r, N)$	$P = A\left[\frac{1 - e^{-rN}}{e^{r} - 1}\right]$
Capital Recovery	A	P	$(A/P, r, N)$	$A = P\left[\frac{e^{r} - 1}{1 - e^{-rN}}\right]$

[Contemporary Engineering Economics; Park C.S. (2016)]

A systems engineer deposits $12,000.000 into a savings account each year paying 12% per year compounded monthly. How much will be in the account at the end of three years if the interest rate is 12%? Figure 6.13 is the cash flow diagram for the investment plan of the systems engineer.



Solution

$$\text{ieff} = \left(1 + \frac{r}{m}\right)^m - 1$$

First, determine (i) for the payment period:

$$i = \frac{i_n}{m} = \frac{12\%/\text{Year}}{12 \text{ compounding periods/Year}} = 1\% \text{ per month}$$

Second, calculate the effective interest for one year:

$$i_e = (1+i)^n - 1 = (1+0.01)^{12} - 1 = 0.1268 = 12.68\% \text{ per year}$$

Third, calculate the future worth at the end of three years:

$$F_3 = A\left[\frac{(1+i)^n - 1}{i}\right] = \$5,000.00\left[\frac{(1+0.01268)^3 - 1}{0.01268}\right] = \$5,000.00\left(\frac{0.038524}{0.01268}\right)$$

$$= \$5,000.00(3.0382)$$

$$= \$15,191.00$$

[Engineering Economics, Yates J.K. (2017)]

Q: (a) For an interest rate of 18% per year, compounded continuously, calculate the effective monthly and annual interest rates.
(b) An investor requires an effective return of at least 15%. What is the minimum annual nominal rate that is acceptable for continuous compounding?

a) The nominal monthly rate is $r = 18\%/12 = 1.5\%$, or 0.015 per month. By Equation [4.11], the effective monthly rate is

$$i\% \text{ per month} = e^r - 1 = e^{0.015} - 1 = 1.511\%$$

Similarly, the effective annual rate using $r = 0.18$ per year is

$$i\% \text{ per year} = e^r - 1 = e^{0.18} - 1 = 19.722\%$$

b) Solve Equation [4.11] for r by taking the natural logarithm.

$$e^r - 1 = 0.15$$

$$e^r = 1.15$$

$$\ln e^r = \ln 1.15$$

$$r = 0.13976$$

Therefore, a rate of 13.976% per year, compounded continuously, will generate an effective 15% per year return. The general formula to find the nominal rate, given the effective continuous rate i, is $r = \ln(1 + i)$.

Engineering economy, eighth edition; blank et. Al (2018)

Q: Engineers Marci and Suzanne both invest \$5000 for 10 years at 10% per year. Compute the future worth for both individuals if Marci receives annual compounding and Suzanne receives continuous compounding.

Solution

Marci: For annual compounding the future worth is

$$F = P(F/P,10\%,10) = 5000(2.5937) = \$12{,}969$$

Suzanne: Using Equation [4.11], first find the effective i per year for use in the F/P factor.

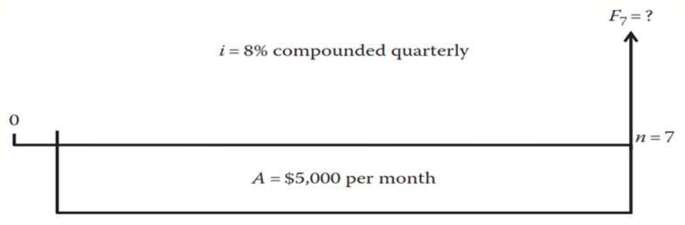
$$\text{Effective } i\% = e^{0.10} - 1 = 10.517\%$$

$$F = P(F/P,10.517\%,10) = 5000(2.7183) = \$13{,}591$$

Continuous compounding causes a \$622 increase in earnings. For comparison, daily compounding yields an effective rate of 10.516% ($F = \$13{,}590$), only slightly less than the 10.517% for continuous compounding.

Engineering economy, eighth edition; blank et. Al (2018)

An agricultural engineer deposits \$5,000.000 into a savings account every month for seven years for his firm. How much would be in the account after the last deposit if the interest rate is 8% compounded quarterly? Figure 6.14 is the cash flow diagram for the savings plan of the agricultural engineer.



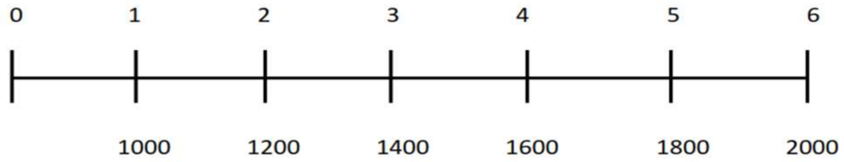
$$\mathbf{i = \left(1 + \frac{r}{m}\right)^{m/k} - 1}$$

m=4
K=12
N=7*12

Effective interest rate per monthly payment periods (i)

[Engineering Economics, Yates J.K. (2017)]

Calculate the future worth of the following 6-year cash diagram if the interest rate is 10% compounded annually.



There are a number of ways to solve this economic problem, which is the case for most cash flow evaluations. One technique might be shorter in terms of the number of formulas to look up or calculate, but all will result in the same answer.

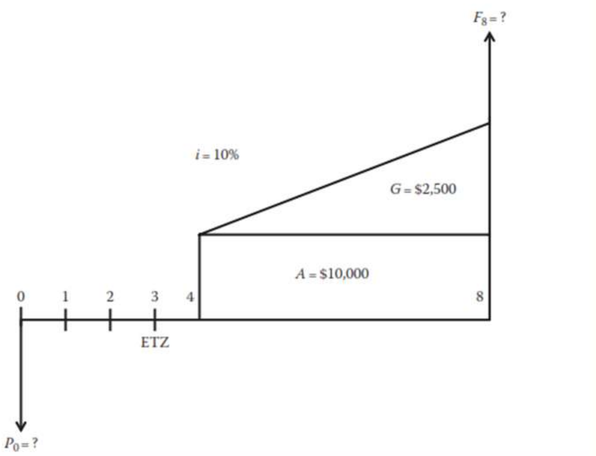
Solution 1:

Note that this series of cash flows can be broken into an annuity of $1,000 per year and a gradient of $200 per year. One can compute the future value of each of these contributions separately and then add to get the final result.

$$F_{\text{Annuity}} = A(F/A)_{i,n} = 1,000(F/A)_{10,6} = 1,000(7.7156) = \$7,715.60$$
$$F_{\text{Gradient}} = G(F/G)_{i,n} = 200(F/G)_{10,6} = 200(17.156) = \$3,431.22$$
$$F = 7,715.60 + 3,431.22 = \$11,147$$

[Engineering Economics, Yates J.K. (2017)]

A process engineer starts investing his money when he graduates from college. He is able to afford investing $10,000.00 a year from the time he graduates in four years until the end of eight years. He also plans to invest an additional $2,500.00 per year increasing by $2,500.00 per year at the end of the year after he graduates until year eight. How much will the process engineer have saved by the end of year eight and what is its present worth if the interest rate is 10%? Figure 6.12 is the cash flow diagram for the investment plan of the process engineer.

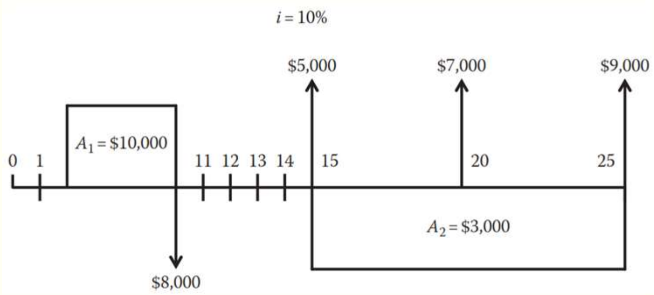


P = $ 41369 .41

F= $88,678.50

[Engineering Economics, Yates J.K. (2017)]

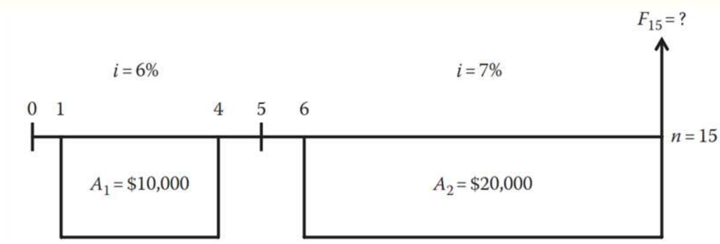
An industrial engineer has determined the amount of money required to be withdrawn from an investment to meet periodic expenses for a project the firm is contemplating undertaking soon. She has determined the firm will be required to withdraw $10,000.00 a year starting at year two through year 10. At year 10, $8,000.00 will be deposited into the investment account paying 10% interest. At year 15, the firm will start depositing $3,000.00 a year until year 25. Also, at year 15, $5,000.00 will be withdrawn from the account. In the 20th year $7,000.00 will be withdrawn, and at year 25, $9,000.00 will be withdrawn. What is the present worth of the future payment and disbursement streams? Figure 6.11 is the cash flow diagram of the industrial engineering project.



$$
\begin{aligned}
P_0 &= \$10{,}000.00(P/A,10,9)(P/F,10,1) - \$8{,}000.00(P/F,10,10) \\
&\quad + \$5{,}000.00(P/F,10,15) + \$7{,}000.00(P/F,10,20) + \$9{,}000.00(P/F,10,25) \\
&\quad - \$3{,}000.00(P/A,10,11)(P/F,10,14) \\
&= \$10{,}000.00(5.7590)(0.90909) - \$8{,}000.00(0.38554) + \$5{,}000.00(0.23939) \\
&\quad + \$7{,}000.00(0.14864) + \$9{,}000.00(0.09230) - \$3{,}000.00(6.4950)(0.26333) \\
&= \$52{,}354.49 - \$3{,}084.32 + \$1{,}196.95 + \$1{,}040.48 + \$830.70 - \$5{,}130.99 \\
&= \$47{,}207.31
\end{aligned}
$$

[Engineering Economics, Yates J.K. (2017)]

A biomedical engineering firm needs to save funds to pay a large tax bill due at the end of 15 years. For the first four years, the firm will be able to save $10,000.00 per year at an interest rate of 6%. Then starting at year six, the firm will save $20,000.00 per year until year 15 at an interest rate of 7%. How much will the firm have in 15 years to pay their taxes? Figure 6.10 is the cash flow diagram for the tax bill savings plan.



Solution

First, calculate the future worth at year four of the first uniform annual series using the interest rate of 6%:

$$
\begin{aligned}
F_4 &= A_1(F/A,i,n) = \$10{,}000.00(F/A,6,4) = \$10{,}000.00(4.3746) \\
&= \$43{,}746.00
\end{aligned}
$$

Second, calculate the future worth of the present value at year four, which is the future worth of the first uniform annual series using the interest rate of 7%:

$$
\begin{aligned}
F_4 &= P_4 \\
F_{15\,of\,A_1} &= P_4(F/P,i,n) = \$43{,}746.00(F/P,7,11) = \$43{,}746.00(2.1048) \\
&= \$92{,}076.58
\end{aligned}
$$

Third, calculate the future worth of the second uniform annual series at year 15 using an interest rate of 7%:

$$
\begin{aligned}
F_{15\,of\,A_2} &= A_2(F/A,i,n) = \$20{,}000.00(F/A,7,10) = \$20{,}000.00(13.816) \\
&= \$276{,}320.00
\end{aligned}
$$

Fourth, solve for the total future worth at year 15 by summing up the future worth of the first and second uniform annual series:

$$
\begin{aligned}
F_{total} &= F_{15\,of\,A_1} + F_{15\,of\,A_2} = \$92{,}076.58 + \$276{,}320.00 \\
&= \$368{,}396.58
\end{aligned}
$$

$$
\begin{aligned}
F_{total} &= F_{15\,of\,A_1} + F_{15\,of\,A_2} = \$92{,}076.58 + \$276{,}320.00 \\
&= \$368{,}396.58
\end{aligned}
$$

[Engineering Economics, Yates J.K. (2017)]