CS105 (DIC on Discrete Structures)
Exercise Problem Sheet 4

Instructions:

- Attempt *all* questions.
- If you have any doubts or you find any typos in the questions, post them on piazza at once!

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1. Give an example for each of the following, if such an example exists. Else prove why it cannot exist.
   (a) A relation that is irreflexive, antisymmetric and not transitive.
   (b) An antisymmetric relation which has a symmetric relation as its subset.
   (c) Relations R_1 and R_2 on set S such that both are symmetric but $R_1 \cap R_2$ is not symmetric.

2. Suppose R_1 and R_2 are two equivalence relations on set S.
   (a) Is $R_1 \cap R_2$ an equivalence relation?
   (b) Is $R_1 \cup R_2$ an equivalence relation?
   (c) Let $f : S \to S$ be a function. Then is the relation R_3, defined by aR_3b if $f(a)R_1f(b)$, an equivalence relation?

   For each of the above, if your answer is "yes", you must prove it, and if your answer is "no", you must provide a counterexample.

3. Let $(S, \preceq)$ be a (non-empty) poset. We write $a \prec b$ if we have $a \prec b$ and $a \neq b$. An element $a \in S$ is called *maximal* if $\nexists b \in S$ s.t. $a \prec b$. Similarly, an element $a \in S$ is called *minimal* if $\nexists b \in S$ s.t. $b \prec a$.
   (a) Consider the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |\})$. What are its maximal and minimal elements?
   (b) Consider poset $(\mathcal{P}(S), \subseteq)$. What are its maximal and minimal elements?

4. Consider a necklace made of 3 beads, each of which can be either red, white or blue. Let S be the set of all such necklaces. Define the following relation R on S as: $N_1 \; R \; N_2$ iff necklace N_2 can be obtained from necklace N_1 by rotating it (and *not* allowing to flip the necklace).
   (a) Show that R is an equivalence relation.
   (b) What are the equivalence classes of R?
   (c) Is the number of elements in each equivalence class the same? Is there a relationship between the number of elements in an equivalence class of R and the total number of elements in S?
   (d) If in the definition of the relation, we allow flipping of the necklace as well: that is, $N_1 \; R' \; N_2$ iff necklace N_2 can be obtained from necklace N_1 by rotating or flipping it. Is R' an equivalence relation? Why or why not?