CS105 (DIC on Discrete Structures)
Problem set 5

- Attempt *all* questions.
- Apart from things proved in lecture, you cannot assume anything as "obvious". Either quote previously proved results or provide clear justification for each statement.

Posets, chains and anti-chains

Basic

1. Consider the set $S = \{1, 2, 3, 4\}$ with the subseteq relation $\subseteq$. We know that $(\mathcal{P}(S), \subseteq)$ is a poset.
   (a) Draw its Hasse diagram.
   (b) What is the length of the longest chain in this poset?
   (c) What is the size of the largest anti-chain in this poset?
2. Consider the poset $(\mathbb{Z}^+, |)$, i.e., positive integers with divisbility ordering.
   (a) Give an example of a chain of length 5 in this poset.
   (b) Give an example of an anti-chain of length 5 in this poset.
   (c) Does this poset have:
      i. a minimal element
      ii. a maximal element
      iii. a minimum or least element
      iv. a maximum or greatest element
      v. an infinite chain
      vi. an infinite anti-chain

   For each of the above, if you claim there exists one, give an example, otherwise explain why there can't be any.
3. Prove carefully that each finite poset has a topological sort (i.e., a linearization).

Advanced

4. Show that every maximal chain in a finite poset $(S, \preceq)$ contains a minimal element of S. (A maximal chain is a chain that is not a subset of a larger chain.)

5. For all $t > 0$, prove that any poset with n elements must have either a chain of length greater than t or an antichain with at least $\frac{n}{t}$ elements.

6. *Consider a permutation of the numbers from 1 to n arranged as a sequence from left to right on a line. Using Mirsky's theorem done in class, prove that there exists a $\sqrt{n}$-length subsequence of these numbers that is completely increasing or completely decreasing as you move from right to left.

   For example, the sequence 2, 3, 4, 7, 9, 5, 6, 1, 8 has an increasing subsequence of length 3, for example: 2, 3, 4, and a decreasing subsequence of length 3, for example: 9, 6, 1. (Hint: Use the previous question!)

Lattices

7. Give an example of a poset with five elements that is a lattice and an example of another poset with five elements which is not a lattice.

8. Prove or disprove: Every totally ordered set is a lattice.

9. Let L be a lattice. For any two elements $x, y \in L$ we use $x \vee y$ to denote the least upper bound of $\{x, y\}$ and $x \wedge y$ to denote the greatest lower bound of $\{x, y\}$ (note that both these elements exist by the definition of a lattice).

   (a) Show the following properties for all $x, y, z \in L$,

      i. (commutative laws) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$
      ii. (associative laws) $((x \vee y) \vee z) = (x \vee (y \vee z))$ and $((x \wedge y) \wedge z) = (x \wedge (y \wedge z))$.
      iii. (absorption laws) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$
      iv. (idempotency laws) $x \vee x = x$ and $x \wedge x = x$.

   (b) Use the above to prove that every finite nonempty subset of a lattice must have a greatest lower bound and a least upper bound.