CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Eulerian walks

Lecture 25
Oct 16 2023

Topic 3: Graph theory

Last topic of this course

Graphs and their properties!

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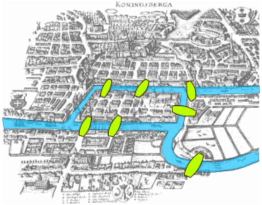
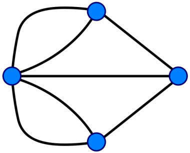
Textbook Reference

- Introduction to Graph Theory, 2^{nd} Ed., by Douglas West.
- Low cost Indian edition available, published by PHI Learning Private Ltd.

Defining graphs





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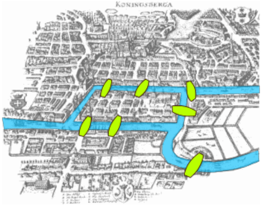

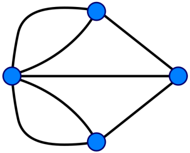

Definition

A simple graph G is a pair (V, E) of a set of vertices/nodes V and edges E between unordered pairs of vertices called end-points: $e = vu$ means e is an edge between v and u $(u \neq v)$.

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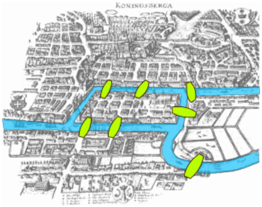

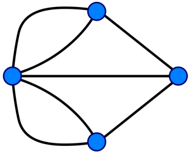

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A graph G is a triple V, E, R where V is a set of vertices, E is a set of edges and $R \subseteq E \times V \times V$ is a relation.

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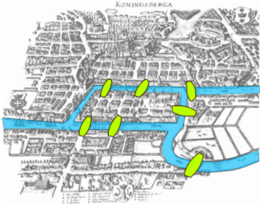

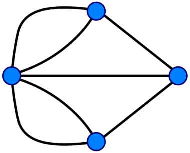

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We will consider only finite graphs (i.e., $|V|, |E|$ are finite) and often simple graphs. **Also, we assume** $|V| > 0$.

Basic terminology

- The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.
- A walk is a sequence of vertices $v_1, \ldots v_k$ such that $\forall i \in \{1, \ldots k-1\}$, $(v_i, v_{i+1}) \in E$. **The vertices v_1 and v_k are called the end-points** and others are called internal vertices.

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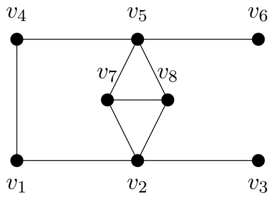
- **The length of a walk is the number of edges in it.**

Basic terminology: Monday Morning Quiz Part 1



1. Give examples for each of the following in the above graph:
   1.1 All vertices of degree 3
   1.2 A walk of length 5
   1.3 A closed walk of length 6

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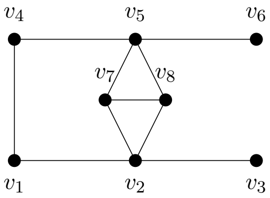

1. Give examples for each of the following in the above graph:
   1.1 All vertices of degree 3
   1.2 A walk of length 5
   1.3 A closed walk of length 6
2. True or False: (a) If a graph is not connected, it must have an isolated vertex.
   (b) A graph is connected iff some vertex has an edge to every other vertex.

Monday morning Quiz Part 2

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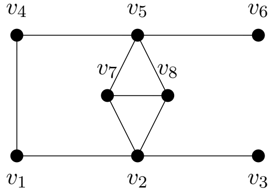

1. Give examples for each of the following in the above graph:
   1.1 A path of length 5
   1.2 A cycle of length 6
2. True or False: Every path is a walk, every cycle is a closed

A final exercise!

Prove or disprove

If every vertex of a graph G has degree at least 2, then G contains a cycle.

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A path P is said to be maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.

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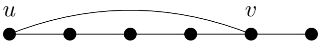

- Let P be a maximal path in G. It starts from some u.

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- Since P cannot be extended, every nbr of u is already in P.

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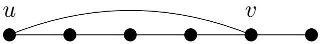

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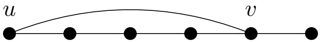

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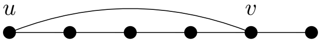

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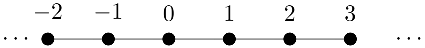
No! Consider $V = \mathbb{Z}$, $E = \{ij : |i - j| = 1\}$.



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- Any two edges are in the same walk implies graph is connected (unless it has isolated vertices).

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Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Base case: $m = 1$.

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Proof ($\Longleftarrow$): By induction on number of edges m in G.

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    - Traverse along cycle C in G and when some G_i is entered for first time, detour along an Eulerian walk of G_i.
    - This walk ends at vertex where we started detour.
    - When we complete traversal of C in this way, we have completed an Eulerian walk on G. $\square$