CS105 (Discrete Structures)
Exercise Problem Set 3

August 25, 2023

Instructions:

- Attempt *all* questions.
- *Some* of the answers will be discussed during the help sessions, but again you are expected to have attempted *all* the questions.
- If you have any doubts or you find any typos in the questions, post them on piazza at once!
- In the following, "disprove" means you have to give a counterexample.

Part 1

1. Give an example of infinite sets A, B and functions f, g, h all from A to B such that f is an injection but not a surjection, g is a surjection but not an injection, and h is an injection and surjection. Is h a bijection?

2. Prove or disprove the following (with complete justifications): Let A, B, C be non-empty sets.
   (a) If there is an injection from A to B then there is a surjection from B to A.
   (b) If there is an injection from A to B then there is a surjection from A to B.
   (c) If there is an bijection from A to B and an injection from B to C, then there is an injection from A to C.
   (d) If there is a bijection from A to B then there is a bijection from $A \times A$ to $B \times B$.
   (e) If $A \subseteq B$ but $A \neq B$, then there must exist an injection from A to B but there can exist no surjection from A to B.
   (f) There is a bijection between $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Q} \times \mathbb{N} \times (\mathbb{N} \cup \{\sqrt{2}\})$.

3. Construct a bijection
   (a) from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$
   (b) from the open interval $(0, 1)$ on the real line to the closed interval $[0, 1]$.

   Prove that the function you constructed is indeed a bijection.

4. Let A be any infinite set. Prove carefully that there is a surjection from A to $\mathbb{N}$. We said in class that this implies that the natural numbers are the "smallest" infinite set! Do you agree? What about the set of even numbers or set of all primes? Discuss.

Part 2

5. Prove that there does not exist an input-free C-program which will always determine whether an arbitrary input-free C-program will halt.

6. In class we showed that there is no bijection from $\mathbb{N}$ to the set of subsets of $\mathbb{N}$. Prove that for *any* non-empty set S, there is no bijection from S to the set of all subsets of S.

7. Prove that there exists a bijection from $\mathbb{R}$ to set of all subsets of $\mathbb{N}$. Can you construct it explicitly? Also, can you conclude whether $\mathbb{R}$ is countable or uncountable from this?

8. Which of the following sets are countable? Justify your answer. If uncountable, use Cantor's diagonalization to show that this is the case.

   (a) The set of all functions from $\{0, 1\}$ to $\mathbb{N}$.

   (b) The set of all functions from $\mathbb{N}$ to $\{0, 1\}$.