CS105 2023 Discrete Structures: Propositions, Proofs and Induction
Exercise Problem Set 1

Part 1

1. Give the converse and contrapositive for the following propositions:
   (a) If it rains today, then my hostel room will leak.
   (b) If $|x| = x$, then $x \geq 0$.
   (c) If n is greater than 3, then n^2 is greater than 9.

   Which of the above statements/propositions are true? Are their converses also true?

2. True or False
   (a) A proposition is equivalent to its contrapositive.
   (b) A proposition is equivalent to its converse.
   (c) A proposition is equivalent to the converse of its converse.

3. For each of the following propositions, write its negation. Is the negation true?
   (a) If it rains today, then my hostel room will leak.
   (b) There exists $n \in \mathbb{N}$ such that $n \geq 5$ and $n^2 < 25$.
   (c) For all $n \in \mathbb{N}$, either n is a prime or n^2 is a prime, but n^3 is not a prime.

4. Prove or disprove the following:
   (a) For any real number x, if x^3 is irrational, then so is x.
   (b) For any real number x, if x is irrational, then so is x^3.
   (c) There exists a nonnegative integer $n^2 > 10^{1000}$. Is your proof constructive or non-constructive?
   (d) There is no rational solution to the equation $x^5 + x^4 + x^3 + x^2 + 1 = 0$.

5. Prove (by induction) or disprove: For every positive integer n,
   (a) $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$.
   (b) if $h > -1$, then $1 + nh \leq (1 + h)^n$.
   (c) 12 divides $n^4 - n^2$.

Part 2

6. Use the Well Ordering-Principle to show the following:

   (a) Any two positive integers a, b have a unique greatest common divisor (hint: consider the set of numbers of the form $ax + by$).

   (b) The equation $4a^3 + 2b^3 = c^3$ does not have any solutions over $\mathbb{N}$. What about the equation $a^4 + b^4 + c^4 = d^4$ over $\mathbb{Z}$?

7. In a cricket tournament, every two teams played against each other exactly once. After all games were over, each team wrote down the names of the other teams they defeated, and the names of those teams defeated by some team they defeated. Prove that at least one team listed the names of every other team! (Give two proofs, one by induction and another without using induction.)

8. For any $n \in \mathbb{N}$, $n \geq 2$, prove that

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{(n-1)\sqrt{n}}}}} < 3$$

9. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show, using induction, that there is a car which can complete a lap by collecting gas from the other cars on its way around.