Hydrodynamics in General Relativistic Setup

9th-Semester Project Report

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Abstract

This work conceptually divides into three sections. First we present a brief review of the general relativistic Boltzmann equation, the relaxation time approximation and motivate the idea of deriving fluid dynamics from kinetic theory. The second part reports a failed attempt at calculating the second order transport coefficients from the Boltzmann equation using the Chapman-Enskog like expansion. Finally, in the third part we report an attempt at making a general relativistic extension of the anisotropic hydrodynamics (aHydro) formalism. In the end, we show that in the limit of trivial metric, the aHydro equations obtained here match with those given by Strickland in his work on aHydro in flat spacetime.

1 The General Relativistic Boltzmann Equation

The Boltzmann equation relates the phase-space derivatives of a single-particle distribution function to the net effect of non-reactive collisions occurring between the component particles. It derives from the Liouville equation which describes the phase-space evolution of a given ensemble of particles. The single particle distribution function is, in general, a function of phase-space coordinates. $f \equiv f(x^\mu, p^\nu)$. If λ is some affine parameter in the phase-space, the Boltzmann equation reads:

$$\frac{df}{d\lambda} = \mathcal{C}[f]$$

Where $\mathcal{C}[f]$ represents the phase-space density of collisions. Using the chain rule of derivatives, it can be rewritten as:

$$\frac{dx^\mu}{d\lambda}\frac{\partial f}{x^\mu} + \frac{dp^\mu}{d\lambda}\frac{\partial f}{p^\mu} = \mathcal{C}[f] \tag{1.1}$$

For neutral particles, the trajectories are described by the following geodesic equation:

$$\frac{dp^\mu}{d\lambda} + \Gamma_\rho \sigma^\mu p^\rho p^\sigma = 0$$

Incorporating the above equation in (1.1), one obtains:

$$p^\mu \partial_\mu f - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = \mathcal{C}[f] \tag{1.2}$$

The exact form of $C[f]$ depends on the type of interactions between the particles. When there is a global equilibrium in the system, the function f has to be stationary; that is $df/d\lambda = 0$ when $f = f_{eq}$, implying that the collision term becomes zero when $f = f_{eq}$. This equilibrium distribution function f_{eq} is called the Maxwellian and it is the only known exact solution of the Boltzmann equation used in practice (there is another one by Ikenberry and Truesdell but its use is limited to textbooks). The equilibrium distribution for a relativistic fluid at temperature T, chemical potential μ and fluid four-velocity u^μ is given by the Maxwell-Jüttner distribution:

$$f_{eq}(x,p) \sim exp\left(\frac{\mu}{k_B T} - \frac{u^\mu p_\mu}{k_B T}\right) \tag{1.3}$$

Where k_B is the Boltzmann constant. Throughout our exposition, we will be dealing with fluids with zero chemical potential and we will be working in natural units. Hence the equilibrium distribution function will look like

$$f_{eq} = exp(-\beta(u \cdot p)) \tag{1.4}$$

Where β is inverse temperature. Note that the temperature may be a function of coordinates, that is $T \equiv T(x)$. Used in this form, the Maxwellian is no longer an exact solution of the Boltzmann equation, but when imposed as a solution it gives constraint equations relating the gradients of the temperature and chemical potential fields with fluid velocity and acceleration. Therefore, only the global equilibrium distribution function is an exact solution which makes it uninteresting for practical applications.

If we require a description of more realistic non-equilibrium processes, we have to make use of approximation techniques. One such approximation technique is to linearize the collision kernel and then solve the Boltzmann equation for perturbations around the f_{eq}. One specific prescription for linearising the collision integral is the relaxation time approximation, etymology of which will be clear in the next section.

1.1 Relaxation Time Approximation

The relaxation time approximation was introduced by Anderson and Witting in 1974 following the non-relativistic version developed by Bhatnagar, Gross and Krook. If the system under consideration is close to equilibrium, it is natural to linearize the collision kernel in the neighbourhood of a local equilibrium state. The relaxation time approximation specifies this linear approximation with one parameter τ_R, the relaxation time of the system which is interpreted as the time scale within which the system reaches equilibrium. With collision kernel given by the relaxation time approximation, the Boltzmann equation (1.2) becomes:

$$p^\mu \partial_\mu f - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\frac{u \cdot p}{\tau_R}(f - f_{eq}) \tag{1.5}$$

where $\tau_{eq} = 5\bar{\eta}/T$ is the relaxation time [], and $\bar{\eta} = \eta/s$ is the ratio of shear viscosity to entropy density. For the rest of our work, we will be using the Boltzmann equation in the form of (1.5).

Deriving hydrodynamics from kinetic theory The fluid dynamics equations often contain parameters of microscopic origin, for example the transport coefficients associated to various dissipative processes in a system that is out of equilibrium. With this picture in mind, one would like to start with a theory containing microscopic parameters and then perform a coarse graining to get to the macroscopic picture. In the transition from kinetic theory to fluid dynamics this coarse graining corresponds to integrating out the momentum dependencies from analog physical quantities. This paradigm will be best apparent with some examples which are in order.

- The energy momentum tensor for an ideal fluids is of the form:

$$T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu - p(g^{\mu\nu} - u^\mu u^\nu).$$

  From kinetic theory, it is constructed as:

$$\int_p \frac{d^4p}{(2\pi)^3} p^\mu p^\nu \delta(p^\alpha p_\alpha - m^2) 2\theta(p^0) f(p,x) \equiv T^{\mu\nu}.$$

- The particle current is similarly constructed with the first moment of the distribution function.

$$\int_p \frac{d^4p}{(2\pi)^3} p^\mu \delta(p^\alpha p_\alpha - m^2) 2\theta(p^0) f(p,x) \equiv N^\mu. \tag{1.6}$$

While moments of the distribution function can generate coarse-grained hydrodynamical quantities, the moments of Boltzmann equation itself can generate their evolution. It is well known that the zeroth and first moments of the Boltzmann equation lead to particle conservation and energy-momentum conservation equations respectively. The higher moments, being sensitive to the high-momentum end of the distribution function also give independent equations in terms of higher order tensors. In the next section we present our attempts at calculating the second-order transport coefficients for the evolution of the shear stress tensor starting from the general relativistic Boltzmann equation.

2 Attempts at calculating second order transport coefficients

2.1 From derivatives of the first order

We wish to calculate hydrodynamical quantities from distribution function upto first order (spacetime derivative) expansion around the equilibrium distribution function, $f \equiv f(u^\mu, T)$. We introduce the notation, $\delta f = (f - f_{eq})$ for the Chapman-Enskog (CE) like expansion around the equilibrium distribution function. In the CE expansion, one expands the distribution function in powers of spacetime derivatives around the equilibrium value. When this expansion contains terms which are first order in spacetime derivatives, we will call it δf_1, and δf_2 when the terms are second order in derivatives with respect to the spacetime coordinates. Since δf_1 is a scalar, the most general expression for δf_1 can be written with the scalars constructed out of first order derivatives of T and U^μ, which are listed as follows:

Table 2.1: First order derivatives of fluid degrees of freedom.

Type	**Independent terms**
Scalars	$\theta = \nabla_\mu u^\mu, \quad \dot{\beta} = (u_\mu \nabla^\mu)\beta = \beta\theta/3$
Vectors	$\nabla^\perp_\mu \beta = \triangle^\alpha_\mu \nabla_\alpha \beta = -\beta \dot{u}_\mu, \quad a_\nu \equiv \dot{u}_\mu$
Rank-two Tensors	$\sigma_{\mu\nu} = \triangle^{\mu\alpha}\triangle^{\nu\beta}\left[\frac{1}{2}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{1}{3}\theta g_{\alpha\beta}\right], \qquad \omega_{\mu\nu} = \triangle^{\mu\alpha}\triangle^{\nu\beta}\left(\frac{1}{2}(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)\right)$

Notation

Throughout the text, 'dot' over a quantity represents its derivative with respect to time, or the action of the operator $(u^\mu \nabla_\mu)$ both of which are same (∇_μ is covariant derivative). The two index projection operator $\triangle^{\mu\nu}$ is defined as

$$\triangle^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu$$

$$\triangle^{\alpha\beta}_{\mu\nu} := \frac{1}{2}\left(\triangle^\mu_\alpha \triangle^\nu_\beta + \triangle^\nu_\alpha \triangle^\mu_\beta - \frac{2}{3}\triangle_{\mu\nu}\triangle^{\alpha\beta}\right)$$

Out of these five terms, only three are independent owing to the equation of motion (EoM) $\nabla_\mu T^{\mu\nu} = 0$ that allows us to express the β derivatives in terms of first order derivatives of u^μ. Hence the most general first order deviation from the equilibrium distribution function is:

$$\delta f_1 = f_{eq}\left(a\theta + b p^\mu \dot{u}_\mu + c p^\mu p^\nu \sigma_{\mu\nu}\right) \tag{2.1}$$

Where a, b, c are scalar functions of $(u \cdot p)$ and β. Substituting the above form into the following Boltzmann equation (with the derivatives w.r.t. spacetime retained upto first order)

$$\underbrace{p^\mu \partial_\mu f_{eq}}_{\text{A}} - \underbrace{\Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f_{eq}}{\partial p^\mu}}_{\text{B}} = \frac{u \cdot p}{\tau_R}\delta f_1 \tag{2.2}$$

Note that the right hand side is entirely dependent on the equilibrium distribution function f_{eq} only. This is because first order spacetime derivative enter through the $\partial_\mu f_{eq}$ and Christoffel terms.

$$\textbf{Term A} \qquad p^\mu \partial_\mu f_{eq} = -p^\mu f_{eq}\left((u\cdot p)\partial_\mu \beta + \beta p^\nu \partial_\mu u_\nu\right)$$

Converting the partial derivatives ∂_μ to covariant derivatives ∇_μ and using (1.4), we obtain:

$$\begin{aligned} p^\mu \partial_\mu f_{eq} &= -p^\mu f_{eq}\left[(u\cdot p)(u_\mu \dot{\beta} + \nabla^\perp_\mu \beta) + \beta p^\nu \left(\triangle_\mu u_\nu + \Gamma^\rho_{\mu\nu} u_\rho\right)\right] \\ &= -f_{eq}\left[(u\cdot p)^2 \frac{\beta\theta}{3} + \beta p^\mu p^\nu \nabla^\perp_\mu u_\nu + \beta p^\mu p^\nu \Gamma^\rho_{\mu\nu} u_\rho\right] \end{aligned}$$

$$\textbf{Term B} \qquad \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f_{eq}}{\partial p^\mu} = \beta u_\mu \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma$$

Combining terms A and B with (2.1),

$$\delta f_1 = \frac{\tau_R f_{eq}}{u\cdot p}\left[(u\cdot p)^2 \frac{\beta\theta}{3} + \beta p^\mu p^\nu \nabla^\perp_\mu u_\nu\right]$$

With the identity

$$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{\theta}{3}\triangle_{\mu\nu} + u_\mu (u\cdot\nabla) u_\nu \qquad \text{and the fact that} \qquad p_\mu p^\mu = 0$$

, we get:

$$\delta f_1 = \frac{\tau_R f_{eq}}{u\cdot p}\left[\beta p^\mu p^\nu \sigma_{\mu\nu}\right] \tag{2.3}$$

We calculate the time evolutoin of the shear stress tensor in the following manner:

$$\dot{\Pi}^{<\mu\nu>} = \triangle^{\mu\nu}_{\alpha\beta} \int dP p^\alpha p^\beta (u\cdot\nabla)(\delta f_1) \tag{2.4}$$

This approach fails to give any curvature terms (like Ricci scalar, Ricci tensor, Riemann tensor or their contractions with fluid four velocity).

2.2 From Second Order

Adopting an approach similar to that in the previous section, we list all possible second order terms, scalars or tensors that can be constructed from derivatives of T and U^μ, and that from the curvature of spacetime. Here we closely follow the development by Sayantani Bhattacharyya in [3]

Table 2.2: Second order derivatives of fluid degrees of freedom.

Type	**All terms, before imposing EoMs**	**Independent terms**
Scalars	$(u^\mu \nabla_\mu)\theta$, $\nabla^2 \beta$, $u^\mu u^\nu \nabla_\mu \nabla_\nu \beta$	$(u^\mu \nabla_\mu)\theta$

Vectors	$\triangle^{\mu\nu}\ (u^\mu\nabla_\mu)a_\nu,\quad \triangle^{\mu\nu}\ \nabla^2 u_\nu,$ $\triangle^{\mu\nu}\ \nabla_\nu\theta,\quad \triangle^{\mu\nu}\ \nabla_\nu\dot{\beta}$	$\triangle^{\mu\nu}\ \nabla_\nu\theta,\quad \triangle^{\mu}_{\alpha}\ \nabla_\beta\sigma^{\alpha\beta}$
Rank-two Tensors	$\triangle^{\mu\alpha}\triangle^{\nu\beta}\ (u^\mu\nabla_\mu)\sigma_{\alpha\beta},$ $\nabla_{<\mu}\nabla_{\nu>}\beta$	$\triangle^{\mu\alpha}\triangle^{\nu\beta}\ (u^\mu\nabla_\mu)\sigma^{\alpha\beta}$

The equations of motion (EoM) used to get rid of the inter-dependencies among the terms of column-two, are:

$$u_\nu(u\cdot\nabla)\nabla_\mu T^{\mu\nu}=0 \Longrightarrow u^\nu u^\mu(\nabla_\nu\nabla_\mu\beta)=\frac{\theta^2\beta}{9}+\frac{\dot{\theta}\beta}{3}+a^2\beta$$

$$\nabla_\mu\nabla_\nu T^{\mu\nu}=0 \Longrightarrow \nabla^2\beta=\frac{\theta^2\beta}{9}-\frac{2}{3}\beta\dot{\theta}+\frac{7}{3}\beta a^2-\beta(\sigma^2+\omega^2)$$

Similarly,

$$\triangle^{\mu}_{\alpha}\ (u\cdot\nabla)\nabla_\nu T^{\alpha\nu}=0 \qquad u_\alpha\ \triangle^{\beta\mu}\ \nabla_\beta\nabla_\nu T^{\alpha\nu}=0$$

Table 2.3: Second order curvature terms

Type	**Independent terms**
Scalars	$R\equiv R^{\mu\nu}g_{\mu\nu},\quad R_{00}\equiv u_\mu u_\nu R^{\mu\nu}$
Vectors	$\triangle^{\mu\alpha}\ R^{\alpha\beta}u_\beta$
Rank-two Tensors	$R_{<\mu\nu>},\quad F_{<\mu\nu>}\equiv u^\mu u^\nu R_{\alpha\mu\beta\nu}$

Table 2.4: Second order terms constructed out of combinations of two first order terms

Type	**Independent terms**
Scalars	$\theta^2,\ a^2,\ \omega^2,\ \sigma^2$
Vectors	$\theta a^\nu,\ a_\nu\omega^{\mu\nu},\ a_\nu\sigma^{\mu\nu}$
Rank-two Tensors	$\theta\sigma_{\mu\nu},\quad \sigma^{\alpha}_{<\mu}\sigma_{\alpha\nu>},\quad \omega^{\alpha}_{<\mu}\sigma_{\alpha\nu>},\quad \omega^{\alpha}_{<\mu}\omega_{\alpha\nu>},\quad a_{<\mu}a_{\nu>}$

Combining the independent terms from table (2.2), (2.3) and (2.4), one can construct the most general second order derivative expansion around the equilibrium distribution function as follows:

$$\begin{aligned}\delta f_2 = f_{eq}[&a_2\theta^2+b_2a^2+c_2\omega^2+d_2\sigma^2+e\theta p^\nu\dot{u}_\nu+gp^\mu\dot{u}^\nu\omega_{\mu\nu}+hp^\mu\dot{u}^\nu\sigma_{\mu\nu}+jp^\mu p^\nu\theta\sigma_{\mu\nu}+kp^\mu p^\nu\sigma^{\alpha}_{\mu}\sigma_{\alpha\nu}\\&+lp^\mu p^\nu\omega^{\alpha}_{\mu}\sigma_{\alpha\nu}+mp^\mu p^\nu\omega^{\alpha}_{\mu}\omega_{\alpha\nu}+np^\mu p^\nu a_\mu a_\nu+q(u\cdot\nabla)\theta+rp^\mu\ \triangle^{\alpha}_{\mu}\ \nabla_\alpha\theta+sp^\mu\ \triangle_{\alpha\mu}\ \nabla_\beta\sigma^{\alpha\beta}\\&+tp^\mu p^\nu\ \triangle^{\alpha}_{\mu}\triangle^{\beta}_{\nu}\ (u\cdot\nabla)\sigma_{\alpha\beta}+vR+wu^\mu u^\nu R_{\mu\nu}+xp^\mu\ \triangle^{\alpha}_{\mu}\ R_{\alpha\beta}u^\beta+yp^\mu p^\nu R_{\mu\nu}+zp^\mu p^\nu F_{\mu\nu}].\end{aligned} \tag{2.5}$$

Where the coefficients a to z are scalars yet to be determined. Up to second order, the Boltzmann equation (1.5) looks like:

$$p^\mu \partial_\mu (f_{eq} + \delta f_1) - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial (f_{eq} + \delta f_1)}{\partial p^\mu} = \frac{u \cdot p}{\tau_R} (\delta f_1 + \delta f_2) \tag{2.6}$$

The first order terms 'A' and 'B' were calculated in the previous section, we calculate the second order terms on the left hand side of (2.6).

$$p^\mu \partial_\mu (\delta f_1) = (p^\mu \partial_\mu f_{eq}) \left[a_1 \theta + b_1 p^\nu \dot{u}_\nu + c_1 p^\mu p^\nu \sigma_{\mu\nu}\right] + f_{eq} p^\mu \partial_\mu \left(a_1 \theta + b_1 p^\nu \dot{u}_\nu + c_1 p^\mu p^\nu \sigma_{\mu\nu}\right)$$

Converting partial derivatives to covariant derivatives wherever needed, and using the form of the equilibrium distribution function f_{eq} from (1.4), we obtain:

$$\begin{aligned} p^\mu \partial_\mu (\delta f_1) = & -f_{eq} \beta \left[a_1 \theta p^\mu p^\nu \sigma_{\mu\nu} + b_1 p^\nu \dot{u}_\nu p^\mu p^\nu \sigma_{\mu\nu} + c_1 p^\mu p^\nu \sigma_{\mu\nu} p^\sigma p^\rho \sigma_{\sigma\rho} + \gamma a_1 \theta + \gamma b_1 p^\nu \dot{u}_\nu + \gamma c_1 p^\mu p^\nu \sigma_{\mu\nu}\right] \\ & + f_{eq} \theta p^\mu (\partial_\mu a_1) + f_{eq} p^\mu p^\nu \dot{u}_\nu (\partial \mu b_1) + f_{eq} p^\mu p^\nu \sigma_{\mu\nu} (\partial_\mu c_1) + f_{eq} p^\mu \left[a_1 u_\mu (u \cdot \triangledown) \theta + a_1 \triangle^\alpha_\mu \triangledown_\alpha \theta\right] \\ & + b_1 f_{eq} p^\mu p^\nu \left[\sigma^\alpha_\mu \sigma_{\alpha\nu} + 2 \sigma^\alpha_\mu \omega_{\alpha\nu} + \frac{2}{3} \theta \sigma_{\mu\nu} + \omega^\alpha_\mu \omega_{\alpha\nu} + \frac{\theta^2}{9} \triangle_{\mu\nu}\right] \\ & + b_1 f_{eq} p^\mu p^\nu \left[u_\mu \dot{u}^\nu \sigma_{\mu\nu} + u_\mu \dot{u}^\nu \omega_{\mu\nu} + \frac{\theta}{3} u_\mu \dot{u}^\nu + R_{\theta\nu\beta\mu} u^\theta u^\beta + (u \cdot \triangledown) \sigma_{\mu\nu}\right] \\ & + b_1 f_{eq} p^\mu p^\nu \left[\frac{\triangle_{\mu\nu}}{3} (u \cdot \triangledown) \theta + a_\mu a_\nu + u_\mu (u \cdot \triangledown) u_\nu + \Gamma^\epsilon_{\nu\mu} \dot{u}_\epsilon\right] \\ & + c_1 f_{eq} p^\mu p^\sigma p^\rho (\triangledown_\mu \sigma_{\rho\sigma} + 2 \Gamma^\epsilon_{\rho\nu} \sigma_{\epsilon\sigma} \end{aligned} \quad . \tag{2.7}$$

Where γ is shorthand for $u_\mu \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma$. The momentum derivative term in (2.6) contributes the following:

$$\frac{\partial (\delta f_1)}{\partial p^\mu} = -f_{eq} \beta u_\mu \left(a_1 \theta + b_1 p^\nu \dot{u}_\nu + c_1 p^\mu p^\nu \sigma_{\mu\nu}\right) + f_{eq} (b_1 \dot{u}_\mu + 2 c_1 p^\nu \sigma_{\mu\nu}) + f_{eq} (\theta \frac{\partial a_1}{\partial p^\mu} + p^\nu \dot{u}_\nu \frac{\partial b_1}{\partial p^\mu} + p^\rho p^\sigma \sigma_{\rho\sigma} \frac{\partial c_1}{\partial p^\mu}) \tag{2.8}$$

If we plug the expressions (2.7) and (2.8) into (2.6) and compare coefficients of independent terms on the left hand side (of 2.6) to the unknown coefficients on the right hand side (from the δf_1 and δf_2 expressions), we obtain:

LHS	RHS
$-\beta p^\mu p^\nu \sigma_{\mu\nu} f_{eq} p^\mu p^\nu \sigma_{\mu\nu} (\partial_\mu c_1) + p^\rho p^\sigma \sigma_{\rho\sigma} \frac{\partial c_1}{\partial p^\mu}$	$-\frac{(u \cdot p)}{\tau_R} c_1$
$p^\mu \partial_\mu a_1 - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial a_1}{\partial p^\mu}$	$\frac{u \cdot p}{\tau_R} a_1$
$p^\mu \partial_\mu b_1 - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial b_1}{\partial p^\mu}$	$\frac{u \cdot p}{\tau_R} b_1$
.	
.	
.	

Table 2.5: Equations obtained by comparing both sides of (2.6)

We note from equation (2.7) that the coefficient of curvature term viz. the Riemann tensor is b_1, which is the coefficient of the acceleration term $\dot{u}_\mu \equiv a_\mu$ in the general expression for δf_1. Since all terms are mutually independent, we conclude that the only way to obtain curvature data is to have double derivatives of velocity like terms. The above table highlights three relations obtained by comparing coefficients of identical terms on either side of (2.6). There are as many equations as there are independent terms and they solve the system of equations with $b_1 = 0$. This implies that the presented calculations give a zero coefficient for the curvature terms.

Spotting the problem We came to know of the article by Romatschke et al [5] where they discuss the properties of the RTA collision kernel. They point out that RTA collision kernel has symmetries which do not allow any anti-symmetric tensor to be calculated from the Boltzmann equation. It is worthwhile to note that just like the anti-symmetric $\omega_{\mu\nu}$, the Riemann tensor is also anti-symmetric in its pair of indices. We therefore conclude this attempt by noting that a different choice of the collision kernel which doesn't have such symmetries might allow us to calculate the terms sought here.

3 Anisotropic Hydrodynamics

The fireball produced in the relativistic heavy-ion collision experiments has been analysed with scrutiny and it is now established that a very hot-dense plasma of quarks and gluons is indeed formed. The initial strongly interacting phase enjoys free dynamics due to asymptotic freedom and is described by relativistic ideal hydrodynamics. Since the framework of hydrodynamics is based on the assumption of local thermalization, and it successfully describes the evolution of quark-gluon plasma (QGP), it was believed that QGP achieves thermalization very quickly (on a timescale of $1 fm/c$). However, dissipative hydrodynamic descriptions and QCD calculations showed that it is not possible to achieve thermalization in sub-fm/c scale. It was then suggested that thermalization is not needed and anisotropic hydrodynamics formalism can explain the QGP evolution by taking into account the large momentum-space anisotropy initially present in the QGP.
The success of anisotropic hydrodynamics (henceforth referred as aHydro) at explaining the evolution of QGP without resorting to thermalization piques our interest in this formalism and motivates further exploration.

3.1 Brief Introduction to aHydro

In the derivation of aHydro equations, one starts with momentum space anisotropy incorporated into the one-particle distribution function. For example, for the simple pedagogical case of axis-symmetric longitudinal expansion of fluid, one assumes $f(x,p)$ as prescribed by Romatschke and Strickland [6]:

$$f(x,p) = f_{\mathrm{eq}}\left(\frac{\sqrt{\mathbf{p}^2 + \xi(x)p_z^2}}{\Lambda(x)}, \frac{\mu(x)}{\Lambda(x)}\right) \tag{3.1}$$

The moments of this distribution function substituted into the Boltzmann equation, provide the evolution equation for the anisotropy parameter $\xi(x)$ and the scaled temperature $\Lambda(x)$.

It was recognised by Jaiswal and Dash [1] that collisionless evolution of a conformal fluid in an anisotropic metric background, can lead to the same aHydro equations of the axis-symmetric case. This happens because the free-streaming particles adopt the anisotropies introduced by the metric. We wish to extend this work in presence of a non-zero collision kernel. The subsequent sections contain our attempts in this direction.

3.2 Our Setup

We work with a general Bianchi type-I metric where a line element looks like: $ds^2 = dt^2 - g_{ij}dx^i dx^j$. With a change of coordinates, the symmetric spatial part can be diagonalised and without loss of generality, we can take our background metric to be:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -A^2(t) & 0 & 0 \\ 0 & 0 & -B^2(t) & 0 \\ 0 & 0 & 0 & -C^2(t) \end{bmatrix} \tag{3.2}$$

where $A(t)$, $B(t)$, and $C(t)$ are scale factors dependent on time only. The determinant will be useful in defining the integral measure while taking moments of the Boltzmann equation. $\sqrt{-g} := \sqrt{-det(g_{\mu\nu})} = ABC$. The assumed metric has only six non-zero Christoffel symbols:

$$\Gamma^x_{xt} = \Gamma^x_{tx} = \frac{\dot{A}}{A}, \qquad \Gamma^y_{yt} = \Gamma^y_{ty} = \frac{\dot{B}}{B}, \qquad \Gamma^z_{tz} = \Gamma^z_{zt} = \frac{\dot{C}}{C}$$

Our setup consists of a transversely symmetric fluid, hence $A(t) = B(t)$. With the prescribed metric and transverse homogeneity, the Boltzmann equation (1.5) takes the form:

$$p^\mu \partial_\mu f - 2p^0 \left(\frac{\dot{A}}{A} p^x \frac{\partial f}{\partial p^x} + \frac{\dot{A}}{A} p^y \frac{\partial f}{\partial p^y} + \frac{\dot{C}}{C} p^z \frac{\partial f}{\partial p^z} \right) = -\frac{u \cdot p}{\tau_R}(f - f_{eq}) \tag{3.3}$$

Next we define the basis vectors as follows:

$$\begin{aligned} u^\mu &= (1, 0, 0, 0) \\ X^\mu &= (0, 1/A(t), 0, 0) \\ Y^\mu &= (0, 0, 1/A(t), 0) \\ Z^\mu &= (0, 0, 0, 1/C(t)) \end{aligned}$$

Note that the basis vectors scale as metric expands or contracts in the respective directions. This is to be contrasted with the choice of basis where $X^\mu = (0, 1, 0, 0)$, $Y^\mu = (0, 0, 1, 0)$ and so on. The later choice introduces a co-moving coordinate frame where the observer is oblivious to the dynamics of the metric. Our choice of basis vector helps set up a proper frame where the observer is an asymptotic observer (assuming that the metric is asymptotically flat). Any tensor of any rank can be constructed out of these four vectors defined above. For example, it can be checked that the metric tensor can be decomposed as:

$$g^{\mu\nu} = u^\mu u^\nu - \sum_{i=1}^{3} X_i^\mu X_i^\nu$$

The general form of the current and energy-momentum tensor can be given as follows, where we only use the symmetric nature of these tensors.

$$
\begin{aligned}
J^\mu &= nu^\mu + n^i X_i^\mu \\
T^{\mu\nu} &= t_{00} g^{\mu\nu} + \sum_{i=1}^{3} t_{ii} X_i^\mu X_i^\nu + \sum_{i,j\neq 0, i>j}^{3} t_{ij} X_i^\mu X_j^\nu
\end{aligned}
\tag{3.4}
$$

The spatial components of the current vanishes because of the spatial reflection symmetry, and therefore:

$$
J^\mu = nu^\mu \tag{3.5}
$$

Name the z axis as the longitudinal axis, and the xy plane, as the transverse plane then in the frame of a stationary (with respect to the fluid) asymptotic observer, the energy-momentum tensor assumes the following form:

$$
T^{\mu\nu}_{asymp} = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_T/A^2(t) & 0 & 0 \\ 0 & 0 & P_T/A^2(t) & 0 \\ 0 & 0 & 0 & P_L/C^2(t) \end{bmatrix} \tag{3.6}
$$

Where ϵ is the energy density, P_T and P_L are the transverse and longitudinal pressures. This motivates the following tensor decomposition for $T^{\mu\nu}$.

$$
T^{\mu\nu} = (\epsilon + P_T) u^\mu u^\nu - P_T g^{\mu\nu} + (P_L - P_T) Z^\mu Z^\nu \tag{3.7}
$$

To get a physical insight, imagine an expanding metric, that it $A(t)$, $B(t)$ and $C(t)$ are increasing functions of time. Expression (3.7) says that the pressures, as measured by the asymptotic observer, will decrease with time. This is expected because the same amount of radiation is now present in a larger volume. The above form can also be derived by imposing the transverse homogeneity on the general form given in (3.4). In the next few subsections we concern ourselves with taking the zeroth, first and second moments of the Boltzmann equation (3.3). The invariant integral measure is:

$$
\int dP \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\sqrt{-g}}{p^0}
$$

Note that the integral measure now has a coordinate dependent term $\sqrt{-g}$. Before we embark on taking moments of the Boltzmann equation, calculations for a some of the thermodynamical quantities are in order:

Number density, n defined as:

$$
n(\xi, \Lambda) \equiv \int \sqrt{-g} \frac{d^3p}{(2\pi)^3} f_{\rm eq}\left(\sqrt{\mathbf{p}^2 + \xi(x) p_z^2}/\Lambda(x)\right)
$$

After a change in parameter $p_z'^2 = (1+\xi) p_z^2$, the above expression for number density reads:

$$
n(\xi, \Lambda) = \frac{\sqrt{-g}}{\sqrt{1+\xi}} \int \frac{d^3p}{(2\pi)^3} f_{\rm eq}(|\mathbf{p}|/\Lambda(x)) \Longrightarrow n(\xi, \Lambda) = \frac{\sqrt{-g}}{\sqrt{1+\xi}} n_{\rm eq}(\Lambda) \tag{3.8}
$$

Energy density, ϵ defined as

$$\epsilon = \int dPE^2 f_{\rm eq}\left(\sqrt{\mathbf{p}^2+\xi(x)p_z^2}/\Lambda(x)\right)$$
$$= \int \sqrt{-g}\frac{d^3p}{(2\pi)^3}\sqrt{p_x^2+p_y^2+p_z^2}f_{\rm eq}\left(\sqrt{\mathbf{p}^2+\xi(x)p_z^2}/\Lambda(x)\right)$$

After introducing parameter change similar to what was done for n, but in spherical coordinates and we also get another term under a square root.

$$= \frac{\sqrt{-g}}{\sqrt{1+\xi}}\int \frac{d^3p}{(2\pi)^3}|\mathbf{p}|\sqrt{\sin^2\theta+\frac{\cos^2\theta}{1+\xi}}f_{\rm eq}(|\mathbf{p}|/\Lambda(x))$$

The integral over θ does not involve f_{eq}, so we take it out and multiply and divide by another $\int d\theta$ factor (this way integral over d^{3}p remains, and we get another 1/2 factor)

$$\epsilon = \left(\frac{\sqrt{-g}}{2\sqrt{1+\xi}}\int d(\cos\theta)\sqrt{\sin^2\theta+\frac{\cos^2\theta}{1+\xi}}\right)\int\frac{d^3p}{(2\pi)^3}|\mathbf{p}|f_{\rm eq}(|\mathbf{p}|/\Lambda(x)) \tag{3.9}$$

We write the equation (3.9) in a simpler form: $\epsilon = \mathcal{R}(\xi)\epsilon_{\rm eq}(\Lambda)$, where $\mathcal{R}(\xi) = \frac{1}{2}\left[\frac{1}{1+\xi}+\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]$
Following a similar procedure, one can get

$$\begin{aligned} P_T &= \mathcal{R}_T(\xi)P_{\rm eq}(\Lambda) \\ P_L &= \mathcal{R}_L(\xi)P_{\rm eq}(\Lambda) \end{aligned} \tag{3.10}$$

where $\mathcal{R}_T(\xi) = \frac{3}{2\xi}\left[\frac{1+(\xi^2-1)\mathcal{R}(\xi)}{\xi+1}\right]$, $\mathcal{R}_L(\xi) = \frac{3}{\xi}\left[\frac{(\xi+1)\mathcal{R}(\xi)-1}{\xi+1}\right]$, and $P_{eq}(\Lambda) = \epsilon/3$ owing to the equation of state for conformal fluid.

3.3 Moments of the General Relativistic Boltzmann Equation

We define the $n^{\rm th}$ moment of the collision kernel, $\mathcal{C}[f]$ for a given r,

$$\mathcal{C}_r^{\mu_1\mu_2\cdots\mu_n} \equiv -\int dP(p.u)^r p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}\mathcal{C}[f] \tag{3.11}$$

As is well known, the first moment of the Boltzmann equation gives energy-momentum conservation. While evaluating the first moment of the Boltzmann equation with collision kernel given by RTA, one has to impose the conservation by hand. This amounts to the condition

$$\mathcal{C}_0^\nu = \int dPp^\nu\frac{u\cdot p}{\tau_R}\left(f_{eq}(\xi,\Lambda)-f_{eq}(T)\right) = 0 \tag{3.12}$$

While, owing to the symmetries of $f(x,p)$, the above integral vanishes for $\mu = 1,2,3$, the vanishing of $\mathcal{C}_0^0$ gives the following condition:

$$T = \mathcal{R}^{1/4}(\xi)\Lambda \tag{3.13}$$

This relation can be obtained as follows:

$$\mathcal{C}_0^0 = \frac{\sqrt{-g}}{\tau_R} \int \frac{d^3\mathbf{p}}{(2\pi)^3} |\mathbf{p}| \left(f_{eq}(\xi, \Lambda) - f_{eq}(T) \right) = 0 \tag{3.14}$$

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} |\mathbf{p}| \frac{1}{\exp\left(\sqrt{\mathbf{p}^2 + \xi p_z^2}/\Lambda\right) + a} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} |\mathbf{p}| \frac{1}{\exp(|\mathbf{p}|/T) + a}$$

The form we get is same as (3.9), hence with appropriate substitutions ($p_z'^2 = (1+\xi)p_z^2)$), one obtains:

$$\mathcal{R}(\xi) \int \frac{d^3\mathbf{p}}{(2\pi)^3} |\mathbf{p}| \frac{1}{\exp\left(|\mathbf{p}|/\Lambda\right) + a} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} |\mathbf{p}| \frac{1}{\exp(|\mathbf{p}|/T) + a} \Longrightarrow \mathcal{R}(\xi)\Lambda^4 = T^4$$

Zeroth Moment

The 0^{th} moment of the Boltzmann equation is,

$$\partial_\mu \underbrace{\int dP (p^\mu f)}_{\mathrm{J}^\mu = n u^\mu} - (\partial_\mu \sqrt{-g}) \int \frac{d^3 p}{(2\pi)^3 E} (p^\mu f) - \Gamma^\mu_{\rho\sigma} \int dP p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = \underbrace{- \int dP \frac{u \cdot p}{\tau_R} (f - f_{eq})}_{\mathcal{C}_0} \tag{3.15}$$

Where the second term on the left hand side enters because the integral measure is now coordinate dependent and hence one cannot just pull the partial derivative out of the integral. We have the identity:

$$\partial_\mu \sqrt{-g} = \frac{1}{2} \sqrt{-g}\, g^{\alpha\beta} \partial_\mu g_{\alpha\beta}$$

We use the above result and rewrite (3.15) as:

$$\partial_\mu (n u^\mu) - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) n(\xi, \Lambda) - \Gamma^\mu_{\rho\sigma} \int dP p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\mathcal{C}_0$$

Evaluating $\mathcal{C}_0$ using equations (3.11) and (3.8), one obtains:

$$\mathcal{C}_0 = \int \frac{p.u}{\tau_{\text{eq}}} [f - f_{eq}(T)] dP = \frac{n_{\text{eq}}}{\tau_{\text{eq}}} \left(\frac{1}{\sqrt{1+\xi}} - \mathcal{R}^{3/4}(\xi) \right)$$

The first term above comes trivially using equation (3.8), but the calculation for the second term involving f_{eq} (writing $T = \mathcal{R}^{1/4}(\xi)\Lambda$ and reparameterising p=p'$\mathcal{R}^{1/4}$, in spherical co-ordinates) is as shown below.

$$- \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} \frac{p.u}{\tau_{eq}} \frac{1}{\exp(|\mathbf{p}|/T) + a} = - \frac{\mathcal{R}^{3/4}(\xi)}{\tau_{eq}} \int \frac{d^3\mathbf{p}' |p'^2|}{(2\pi)^3} \frac{1}{\exp(|\mathbf{p}'|/\Lambda) + a} = -R^{3/4}(\xi) \frac{n_{eq}}{\tau_{eq}}$$

Note that $\partial_\mu J^\mu = Dn + n\partial_\mu u^\mu$, where $D = u^\mu \partial_\mu$ and $\partial_\mu u^\mu = 0$ in the fluid rest frame. With these substitutions, and the definition of number density from (3.8), the zeroth moment of the Boltzmann equation becomes:

$$\partial_t \left(\sqrt{-g} \frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}} \right) + \left(\sqrt{-g} \frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}} \right) \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) = \sqrt{-g} \left[\frac{1}{\sqrt{1+\xi}} - R^{3/4}(\xi) \right] \frac{n_{eq}(\Lambda)}{\tau_R}$$

Using $\partial_t \sqrt{-g} = \sqrt{-g} \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right)$, on the first term on the left

$$\partial_t\left(\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right)+\left(2\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right)\left(2\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)=\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\left[\frac{1}{\tau_R}-R^{3/4}(\xi)\right]$$

Again using equation (3.8), we can expand the left-hand-side of the above equation in terms of derivatives of ξ and Λ, this gives us:

$$\frac{1}{1+\xi}\partial_t\xi-\frac{6}{\Lambda}\partial_t\Lambda-4\left(2\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)=\frac{-2}{\tau_{\text{eq}}}\left(1-R^{3/4}(\xi)\sqrt{1+\xi}\right) \tag{3.16}$$

In equation (3.16), the first and third term on the LHS, and the RHS are trivially obtained, but the second term on the LHS ($\frac{1}{\sqrt{1+\xi}}\partial_t n_{eq}(\Lambda)=\frac{6}{\Lambda}\partial_t\Lambda$) needs some explanation:

$$\frac{1}{\sqrt{1+\xi}}\partial_\tau n_{eq}(\Lambda)=\frac{\partial_\tau}{\sqrt{1+\xi}}\int\frac{d^3p}{(2\pi^3)}|p|^2\frac{1}{e^{|p|/\Lambda}+a}$$

$$\frac{\partial_t}{\sqrt{1+\xi}}\int\Lambda^3\frac{d^3y}{(2\pi^3)}|y|^2\frac{e^{-y}}{1+ae^y}\qquad\text{(Using y = p/}\Lambda\text{)}$$

Since only Λ is dependent on time, the rest of the calculation simple.

$$\begin{aligned}\frac{\partial_t(\Lambda^3)}{\sqrt{1+\xi}}\int\frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x}&=\frac{3\Lambda^2\partial_t\Lambda}{\sqrt{1+\xi}}\int\frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x}\\&=\frac{3\partial_t\Lambda}{\Lambda\sqrt{1+\xi}}\int\Lambda^3\frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x}=\frac{3n_{eq}\partial_t\Lambda}{\Lambda\sqrt{1+\xi}}\end{aligned}$$

First Moment

The 1^{st} moment gives the equation of motion for the energy-momentum tensor,

$$\partial_\mu\int dPp^\mu p^\nu f-\Gamma^\mu_{\rho\sigma}\int dPp^\rho p^\sigma p^\nu\frac{\partial f}{\partial p^\mu}=0 \tag{3.17}$$

$$T^{\mu\nu}:=\int dPp^\mu p^\nu f=(\epsilon+P_T)u^\mu u^\nu-P_Tg^{\mu\nu}+(P_L-P_T)Z^\mu Z^\nu$$

When (3.17) is projected along u_ν, one obtains,

$$u_\nu\partial_\mu T^{\mu\nu}=(\dot{\epsilon}+\dot{P}_T)+\underbrace{(\epsilon+P_T)\partial_\mu u^\mu}_{0}+\underbrace{(\epsilon+P_T)u_\nu u^\mu\partial_\mu u^\nu}_{0}-u^\mu\partial_\mu P_T+P_T(\Gamma^\mu_{\mu\alpha}g^{\nu\alpha}+\Gamma^\nu_{\mu\alpha}g^{\mu\alpha})u_\nu+\underbrace{(P_L-P_T)u_\nu Z^\mu\partial_\mu Z^\nu}_{0}$$

Performing a similar manipulation on the first moment equation of $\mathcal{C}_0$, we get a simplified form of energy conservation in $0+1$d, i.e.

$$\frac{\partial\epsilon(\tau)}{\partial\tau}+P_T\left(2\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)=0$$

$$\frac{d}{dt}\left(\epsilon_{eq}(\Lambda)R(\xi)\right)+\left[\frac{1}{\xi}\left(R(\xi)-\frac{1}{\xi+1}\right)+R(\xi)\right]\epsilon_{eq}(\Lambda)\left(2\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)=0$$

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau\xi+\frac{4}{\Lambda}\partial_\tau\Lambda=\left(2\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)}-\frac{1}{\xi}-1\right] \tag{3.18}$$

Second Moment

The 2^{nd} moment gives the equation of motion for $I^{\mu\delta\gamma}$,

$$\partial_\mu \underbrace{\int dPp^\mu p^\delta p^\gamma f}_{\mathrm{I}^{\mu\delta\gamma}} - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\int dPp^0 p^\delta p^\gamma f - \Gamma^\mu_{\rho\sigma}\int dPp^\rho p^\sigma p^\delta p^\gamma \frac{\partial f}{\partial p^\mu} = -\underbrace{\int dPp^\delta p^\gamma \frac{u\cdot p}{\tau_R}(f - f_{eq})}_{\mathcal{C}^{\delta\gamma}} \quad (3.19)$$

Where the second term is a result of the partial derivative acting on the $\sqrt{-g}$ factor.
A rank three tensor that respects the symmetries of the system can be written in its most general form in terms of the given basis as:

$$\begin{aligned} I^{\mu\delta\gamma} &= I_u(u^\mu u^\delta u^\gamma) + I_z(u^\mu Z^\delta Z^\gamma + Z^\mu u^\delta Z^\gamma + Z^\mu Z^\delta u^\gamma) \\ &+ I_y(u^\mu Y^\delta Y^\gamma + Y^\mu u^\delta Y^\gamma + Y^\mu Y^\delta u^\gamma) + I_x(u^\mu X^\delta X^\gamma + X^\mu u^\delta X^\gamma + X^\mu X^\delta u^\gamma) \end{aligned}$$

Where

$$\left.\begin{aligned} I_u &= \mathcal{S}_u(\xi) I_{\mathrm{eq}}(\Lambda); & \mathcal{S}_u(\xi) &= \frac{3+2\xi}{(1+\xi)^{3/2}} \\ I_x = I_y &= \mathcal{S}_T(\xi) I_{\mathrm{eq}}(\Lambda); & \mathcal{S}_T(\xi) &= \frac{1}{\sqrt{1+\xi}} \\ I_z &= \mathcal{S}_L(\xi) I_{\mathrm{eq}}(\Lambda); & \mathcal{S}_L(\xi) &= \frac{1}{(1+\xi)^{3/2}} \end{aligned}\right\} I_{eq}(\Lambda) = \frac{1}{3}\int dPE^3 f_{eq}$$

zz projection

$$Z_\delta Z_\gamma \partial_\mu I^{\mu\delta\gamma} = \frac{dI_z}{dt}$$

Plugging this into (3.19),

$$\frac{dI_z}{dt} - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\int dPp^0 p^z p^z f - \Gamma^\mu_{\rho\sigma}\int dPp^z p^z p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\int dPp^z p^z \frac{u\cdot p}{\tau_R}(f - f_{eq})$$

$$\frac{d}{dt}\left(\sqrt{-g}\; I_{eq}(\Lambda) S_L(\xi)\right) + \left(\sqrt{-g}\; I_{eq}(\Lambda) S_L(\xi)\right)\left(2\frac{\dot{A}}{A} + 5\frac{\dot{C}}{C}\right) = \frac{\sqrt{-g}}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda) S_L(\xi)\right]$$

$$\frac{d}{dt}\left(I_{eq}(\Lambda) S_L(\xi)\right) + 2\left(I_{eq}(\Lambda) S_L(\xi)\right)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{1}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda) S_L(\xi)\right] \quad (3.20)$$

xx and yy projection

$$\frac{d}{dt}\left(I_{eq}(\Lambda) S_T(\xi)\right) + \left(I_{eq}(\Lambda) S_T(\xi)\right)\left(\frac{4\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{1}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda) S_T(\xi)\right] \quad (3.21)$$

Using the definitions of the terms S_T, I_{eq} and S_L, obtain:

$$\begin{aligned} (\log \mathcal{S}_L)'\,\partial_t\xi + 5\partial_t \log\Lambda + \left(2\frac{\dot{A}}{A} + 3\frac{\dot{C}}{C}\right) &= \frac{1}{\tau_R}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_L} - 1\right] \\ (\log \mathcal{S}_T)'\,\partial_t\xi + 5\partial_t \log\Lambda + \left(\frac{4\dot{A}}{A} + \frac{\dot{C}}{C}\right) &= \frac{1}{\tau_R}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_T} - 1\right] \end{aligned}$$

The equation of motion for the second moment (equation 3.22) is thus obtained from the above equations, after solving further:

$$\frac{1}{1+\xi}\partial_t\xi - 2\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)+\frac{\mathcal{R}^{5/4}(\xi)}{\tau_R}\xi\sqrt{1+\xi}=0 \tag{3.22}$$

3.4 Consistency Checks

Checking for behaviour in free streaming limit

The free streaming case can be obtained by putting the $\tau_R \longrightarrow \infty$ in the equation (3.22). In this limit the equation solves exactly to give:

$$1+\xi = (\frac{C}{A})^2 \tag{3.23}$$

This result matches with the one obtained by Jaiswal and Dash [1].

Consistency with flat-space aHydro

Kasner Metric

Restricting ourselves to a subclass of Bianchi type-I metric, we look at the evolution of anisotropy parameter ξ when

$$A = t^a, \quad B = t^b, \quad C = t^c \qquad \Longrightarrow \qquad \frac{\dot{A}}{A}=\frac{a}{t}, \quad \frac{\dot{B}}{B}=\frac{b}{t}, \quad \frac{\dot{C}}{C}=\frac{c}{t}$$

When this form of the metric is required to satisfy the Einstein's GR field equations, the parameters a, b, c are related to each other via the following equations:

$$a+b+c=1, \qquad a^2+b^2+c^2=1$$

Requiring azimuthal symmetry as present in our setup, the only solution to the above set of equations are:

1. $(a,b,c) \equiv (0,0,1)$
2. $(a,b,c) \equiv (\frac{2}{3},\frac{2}{3},-\frac{1}{3})$

Case I produces the exact aHydro equations as derived in [1] and [2]. The equations (3.22), (3.18) and (3.16) can be seen to reduce to those derived by Strickland (see Appendix I for comparison).

Numerical Solution to the aHydro equations

We solve equation (3.22) numerically and evaluate the pressure anisotropy (P_T/P_L) till $\tau = 100fm/c$. The behaviour of the direct solution of equation (3.22), that is $\xi(\tau)$ is also plotted to show that ξ does eventually converge to zero as expected. The convergence of ξ to zero naturally corresponds to convergence of pressure anisotropy to 1 (that is, the isotropic limit). The time taken for convergence depends on $\overline{\eta}$ among other variables as can be seen in figure (3.1). The plots were generated in python using mid-point method of solving differential equations, with the initial conditions: $T_0 = 0.3GeV$, $\xi_0 \approx 0$, $\tau = 2.5fm/c$. It is worthwhile to note that the pressure anisotropy converges to 1, the isotropic limit, at time scales much larger than $1fm/c$.

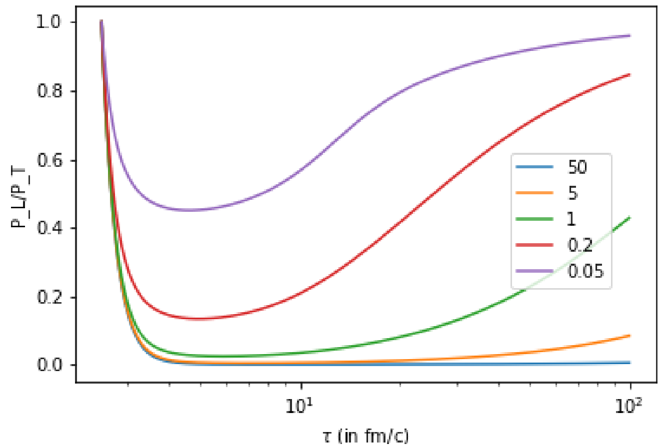

Figure 3.1: The 0+1d pressure anisotropy $\mathrm{P}_L/\mathrm{P}_T$ predicted by the aHydro equation (3.22). The legend contains the $\overline{\eta}$ values used for each curve. Note that the free streaming limit would be a steep vertical line with negative slope, while the ideal hydro limit $\overline{\eta} \longrightarrow 0$ will be a horizontal line at $P_L/P_T = 1$.

4 Conclusion

The attempt at deriving second order transport coefficients from the general relativistic Boltzmann equation taught us to carefully look for symmetries allowed by the collision integrals. Future work in this direction will thus primarily involve a deeper study of the collision kernels approximations available. The attempt at generalising the axis-symmetric aHydro evolution equations can be said to be successful as it meets the requirements put by a number of the consistency checks.

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Appendix I: $(0+1D)$ aHydro

Equations describing the evolution of the anisotropy parameter as obtained from the aHydro formalism are derived in detain in this term paper [Report]

To point out the similarities, we present the final equations here:

Zeroth moment

$$\frac{1}{1+\xi}\partial_\tau\xi - \frac{6}{\Lambda}\partial_\tau\Lambda - \frac{2}{\tau} = \frac{-2}{\tau_{\text{eq}}}\left(1 - R^{3/4}(\xi)\sqrt{1+\xi}\right)$$

First moment, projected along u_μ

$$\frac{1}{1+\xi}\partial_\tau\xi - \frac{2}{\tau} + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_{\text{eq}}}\xi\sqrt{1+\xi} = 0$$

Second moment, zz and xx, yy projections

$$\begin{aligned}
\left(\log\mathcal{S}_L\right)'\partial_\tau\xi + 5\partial_\tau\log\Lambda + \frac{3}{\tau} &= \frac{1}{\tau_{\text{eq}}}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_L} - 1\right] \\
\left(\log\mathcal{S}_T\right)'\partial_\tau\xi + 5\partial_\tau\log\Lambda + \frac{1}{\tau} &= \frac{1}{\tau_{\text{eq}}}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_T} - 1\right]
\end{aligned}$$

The last two equations can be combined to eliminate the Λ term, and one obtains:

$$\frac{1}{1+\xi}\partial_\tau\xi - \frac{2}{\tau} + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_{\text{eq}}}\xi\sqrt{1+\xi} = 0$$