Hydrodynamics in General Relativistic Setup

10th-Semester Project Report

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Aknowledgement

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Contents

Abstract

Extending the previous semester's work, we explore the anisotropic hydrodynamics evolution equations arising from different moments. The choice of second moment admits an attractor type solution which we present only numerically in this report. Analytical confirmation is attempted and future prospects of the research undertaken are discussed. The hitherto existing reports of attractors in anisotropic hydrodynamics rely on smallness of the anisotropy parameter. We have found that the attractor exists for any amount of anisotropy in the system. Apart from the attractive business, we also report some failed attempts at kinetic theory derivation of a relaxation type equation used by Misner [1].

1 The General Relativistic Boltzmann Equation

The Boltzmann equation relates the phase-space derivatives of a single-particle distribution function to the net effect of non-reactive collisions occurring between the component particles. It derives from the Liouville equation which describes the phase-space evolution of a given ensemble of particles. The single particle distribution function is, in general, a function of phase-space coordinates. $f \equiv f(x^\mu, p^\nu)$. If λ is some affine parameter in the phase-space, the Boltzmann equation reads:

$$\frac{df}{d\lambda} = \mathcal{C}[f]$$

Where $\mathcal{C}[f]$ represents the phase-space density of collisions. Using the chain rule of derivatives, it can be rewritten as:

$$\frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dp^\mu}{d\lambda}\frac{\partial f}{\partial p^\mu} = \mathcal{C}[f] \tag{1.1}$$

For neutral particles, the trajectories are described by the following geodesic equation:

$$\frac{dp^\mu}{d\lambda} + \Gamma_\rho \sigma^\mu p^\rho p^\sigma = 0$$

Incorporating the above equation in (1.1), one obtains:

$$p^\mu \partial_\mu f - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = \mathcal{C}[f] \tag{1.2}$$

The exact form of $C[f]$ depends on the type of interactions between the particles. When there is a global equilibrium in the system, the function f has to be stationary; that is $df/d\lambda = 0$ when $f = f_{eq}$, implying that the collision term becomes zero when $f = f_{eq}$. This equilibrium distribution function f_{eq} is called the Maxwellian and it is the only known exact solution of the Boltzmann equation used in practice (there is another one by Ikenberry and Truesdell but its use is limited to textbooks). The equilibrium distribution for a relativistic fluid at temperature T, chemical potential μ and fluid four-velocity u^μ is given by the Maxwell-Jüttner distribution:

$$f_{eq}(x, p) \sim exp\left(\frac{\mu}{k_B T} - \frac{u^\mu p_\mu}{k_B T}\right) \tag{1.3}$$

Where k_B is the Boltzmann constant. Throughout our exposition, we will be dealing with fluids with zero chemical potential and we will be working in natural units. Hence the equilibrium distribution function will look like

$$f_{eq} = exp(-\beta(u \cdot p)) \tag{1.4}$$

Where β is inverse temperature. Note that the temperature may be a function of coordinates, that is $T \equiv T(x)$. Used in this form, the Maxwellian is no longer an exact solution of the Boltzmann equation, but when imposed as a solution it gives constraint equations relating the gradients of the temperature and chemical potential fields with fluid velocity and acceleration. Therefore, only the global equilibrium distribution function is an exact solution which makes it uninteresting for practical applications.

If we require a description of more realistic non-equilibrium processes, we have to make use of approximation techniques. One such approximation technique is to linearize the collision kernel and then solve the Boltzmann equation for perturbations around the f_{eq}. One specific prescription for linearising the collision integral is the relaxation time approximation, etymology of which will be clear in the next section.

1.1 Relaxation Time Approximation

The relaxation time approximation was introduced by Anderson and Witting in 1974 following the non-relativistic version developed by Bhatnagar, Gross and Krook. If the system under consideration is close to equilibrium, it is natural to linearize the collision kernel in the neighbourhood of a local equilibrium state. The relaxation time approximation specifies this linear approximation with one parameter τ_R, the relaxation time of the system which is interpreted as the time scale within which the system reaches equilibrium. With collision kernel given by the relaxation time approximation, the Boltzmann equation (1.2) becomes:

$$p^\mu \partial_\mu f - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\frac{u \cdot p}{\tau_R}(f - f_{eq}) \tag{1.5}$$

where $\tau_{eq} = 5\bar{\eta}/T$ is the relaxation time, and $\bar{\eta} = \eta/s$ is the ratio of shear viscosity to entropy density.
For the rest of our work, we will be using the Boltzmann equation in the form of (1.5).

Deriving hydrodynamics from kinetic theory The fluid dynamics equations often contain parameters of microscopic origin, for example the transport coefficients associated to various dissipative processes in a system that is out of equilibrium. With this picture in mind, one would like to start with a theory containing microscopic parameters and then perform a coarse graining to get to the macroscopic picture. In the transition from kinetic theory to fluid dynamics this coarse graining corresponds to integrating out the momentum dependencies from analog physical quantities. This paradigm will be best apparent with some examples which are in order.

- The energy momentum tensor for an ideal fluids is of the form:

  $$T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu - p(g^{\mu\nu} - u^\mu u^\nu).$$

  From kinetic theory, it is constructed as:

  $$\int_p \frac{d^4p}{(2\pi)^3} p^\mu p^\nu \delta(p^\alpha p_\alpha - m^2) 2\theta(p^0) f(p,x) \equiv T^{\mu\nu}.$$

- The particle current is similarly constructed with the first moment of the distribution function.

  $$\int_p \frac{d^4p}{(2\pi)^3} p^\mu \delta(p^\alpha p_\alpha - m^2) 2\theta(p^0) f(p,x) \equiv N^\mu. \tag{1.6}$$

While moments of the distribution function can generate coarse-grained hydrodynamical quantities, the moments of Boltzmann equation itself can generate their evolution. It is well known that the zeroth and first moments of the Boltzmann equation lead to particle conservation and energy-momentum conservation equations respectively. The higher moments, being sensitive to the high-momentum end of the distribution function also give independent equations in terms of higher order tensors. In the next section we deploy these techniques to analyse the case of an-isotropic hydrodynamics by introducing anisotropy at the level of distribution function.

2 Anisotropic Hydrodynamics

The fireball produced in the relativistic heavy-ion collision experiments has been analysed with scrutiny and it is now established that a very hot-dense plasma of quarks and gluons is indeed formed. The initial strongly interacting phase enjoys free dynamics due to asymptotic freedom and is described by relativistic ideal hydrodynamics. Since the framework of hydrodynamics is based on the assumption of local thermalization, and it successfully describes the evolution of quark-gluon plasma (QGP), it was believed that QGP achieves thermalization very quickly (on a timescale of $1 fm/c$). However, dissipative hydrodynamic descriptions and QCD calculations showed that it is not possible to achieve thermalization in sub-fm/c scale. It was then suggested that thermalization is not needed and anisotropic hydrodynamics formalism can explain the QGP evolution by taking into account the large momentum-space anisotropy initially present in the QGP.
The success of anisotropic hydrodynamics (henceforth referred as aHydro) at explaining the evolution of QGP without resorting to thermalization piques our interest in this formalism and motivates further exploration.

2.1 Brief Introduction to aHydro

In the derivation of aHydro equations, one starts with momentum space anisotropy incorporated into the one-particle distribution function. For example, for the simple pedagogical case of axis-symmetric longitudinal expansion of fluid, one assumes $f(x,p)$ as prescribed by Romatschke and Strickland [?]:

$$f(x,p) = f_{\mathrm{eq}}\left(\frac{\sqrt{\mathbf{p}^2+\xi(x)p_z^2}}{\Lambda(x)}, \frac{\mu(x)}{\Lambda(x)}\right) \tag{2.1}$$

The moments of this distribution function substituted into the Boltzmann equation, provide the evolution equation for the anisotropy parameter $\xi(x)$ and the scaled temperature $\Lambda(x)$.

It was recognised by Jaiswal and Dash [2] that collisionless evolution of a conformal fluid in an anisotropic metric background, can lead to the same aHydro equations of the axis-symmetric case. This happens because the free-streaming particles adopt the anisotropies introduced by the metric. We wish to extend this work in presence of a non-zero collision kernel. The subsequent sections contain our attempts in this direction.

2.2 Our Setup

We work with a general Bianchi type-I metric where a line element looks like: $ds^2 = dt^2 - g_{ij}dx^i dx^j$. With a change of coordinates, the symmetric spatial part can be diagonalised and without loss of generality, we can take our background metric to be:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -A^2(t) & 0 & 0 \\ 0 & 0 & -B^2(t) & 0 \\ 0 & 0 & 0 & -C^2(t) \end{bmatrix} \tag{2.2}$$

where $A(t)$, $B(t)$, and $C(t)$ are scale factors dependent on time only. The determinant will be useful in defining the integral measure while taking moments of the Boltzmann equation. $\sqrt{-g} := \sqrt{-det(g_{\mu\nu})} = ABC$. The assumed metric has only six non-zero Christoffel symbols:

$$\Gamma^x_{xt} = \Gamma^x_{tx} = \frac{\dot{A}}{A}, \qquad \Gamma^y_{yt} = \Gamma^y_{ty} = \frac{\dot{B}}{B}, \qquad \Gamma^z_{tz} = \Gamma^z_{zt} = \frac{\dot{C}}{C}$$

Our setup consists of a transversely symmetric fluid, hence $A(t) = B(t)$. With the prescribed metric and transverse homogeneity, the Boltzmann equation (1.5) takes the form:

$$p^\mu \partial_\mu f - 2p^0\left(\frac{\dot{A}}{A}p^x\frac{\partial f}{\partial p^x} + \frac{\dot{A}}{A}p^y\frac{\partial f}{\partial p^y} + \frac{\dot{C}}{C}p^z\frac{\partial f}{\partial p^z}\right) = -\frac{u\cdot p}{\tau_R}(f - f_{eq}) \tag{2.3}$$

Next we define the basis vectors as follows:

$$\begin{aligned} u^\mu &= (1,0,0,0) \\ X^\mu &= (0, 1/A(t), 0, 0) \\ Y^\mu &= (0,0,1/A(t),0) \\ Z^\mu &= (0,0,0,1/C(t)) \end{aligned}$$

Note that the basis vectors scale as metric expands or contracts in the respective directions. This is to be contrasted with the choice of basis where $X^\mu = (0,1,0,0)$, $Y^\mu = (0,0,1,0)$ and so on. The later choice introduces a co-moving coordinate frame where the observer is oblivious to the dynamics of the metric. Our choice of basis vector helps set up a proper frame where the observer is an asymptotic observer (assuming that the metric is asymptotically flat). Any tensor of any rank can be constructed out of these four vectors defined above. For example, it can be checked that the metric tensor can be decomposed as:

$$g^{\mu\nu} = u^\mu u^\nu - \sum_{i=1}^{3} X_i^\mu X_i^\nu$$

The general form of the current and energy-momentum tensor can be given as follows, where we only use the symmetric nature of these tensors.

$$
\begin{aligned}
J^\mu &= nu^\mu + n^i X_i^\mu \\
T^{\mu\nu} &= t_{00} g^{\mu\nu} + \sum_{i=1}^{3} t_{ii} X_i^\mu X_i^\nu + \sum_{i,j\neq 0, i>j}^{3} t_{ij} X_i^\mu X_j^\nu
\end{aligned}
\tag{2.4}
$$

The spatial components of the current vanishes because of the spatial reflection symmetry, and therefore:

$$
J^\mu = nu^\mu \tag{2.5}
$$

Name the z axis as the longitudinal axis, and the xy plane, as the transverse plane then in the frame of a stationary (with respect to the fluid) asymptotic observer, the energy-momentum tensor assumes the following form:

$$
T^{\mu\nu}_{asymp} = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_T/A^2(t) & 0 & 0 \\ 0 & 0 & P_T/A^2(t) & 0 \\ 0 & 0 & 0 & P_L/C^2(t) \end{bmatrix} \tag{2.6}
$$

Where ϵ is the energy density, P_T and P_L are the transverse and longitudinal pressures. This motivates the following tensor decomposition for $T^{\mu\nu}$.

$$
T^{\mu\nu} = (\epsilon + P_T) u^\mu u^\nu - P_T g^{\mu\nu} + (P_L - P_T) Z^\mu Z^\nu \tag{2.7}
$$

To get a physical insight, imagine an expanding metric, that it $A(t)$, $B(t)$ and $C(t)$ are increasing functions of time. Expression (2.7) says that the pressures, as measured by the asymptotic observer, will decrease with time. This is expected because the same amount of radiation is now present in a larger volume. The above form can also be derived by imposing the transverse homogeneity on the general form given in (2.4). In the next few subsections we concern ourselves with taking the zeroth, first and second moments of the Boltzmann equation (2.3). The invariant integral measure is:

$$
\int dP \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\sqrt{-g}}{p^0}
$$

Note that the integral measure now has a coordinate dependent term $\sqrt{-g}$. Before we embark on taking moments of the Boltzmann equation, calculations for a some of the thermodynamical quantities are in order:

Number density, n defined as:

$$
n(\xi, \Lambda) \equiv \int \sqrt{-g} \frac{d^3p}{(2\pi)^3} f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x) p_z^2} / \Lambda(x) \right)
$$

After a change in parameter $p_z'^2 = (1+\xi) p_z^2$, the above expression for number density reads:

$$
n(\xi, \Lambda) = \frac{\sqrt{-g}}{\sqrt{1+\xi}} \int \frac{d^3p}{(2\pi)^3} f_{\text{eq}}(|\mathbf{p}|/\Lambda(x)) \Longrightarrow n(\xi, \Lambda) = \frac{\sqrt{-g}}{\sqrt{1+\xi}} n_{\text{eq}}(\Lambda) \tag{2.8}
$$

Energy density, ϵ defined as

$$
\begin{aligned}
\epsilon &= \int dP E^2 f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x) p_z^2} / \Lambda(x) \right) \\
&= \int \sqrt{-g} \frac{d^3p}{(2\pi)^3} \sqrt{p_x^2 + p_y^2 + p_z^2} f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x) p_z^2} / \Lambda(x) \right)
\end{aligned}
$$

After introducing parameter change similar to what was done for n, but in spherical coordinates and we also get another term under a square root.

$$= \frac{\sqrt{-g}}{\sqrt{1+\xi}} \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}| \sqrt{\sin^2\theta + \frac{\cos^2\theta}{1+\xi}} f_{\rm eq}(|\mathbf{p}|/\Lambda(x))$$

The integral over θ does not involve f_{eq}, so we take it out and multiply and divide by another $\int d\theta$ factor (this way integral over d^{3}p remains, and we get another 1/2 factor)

$$\epsilon = \left(\frac{\sqrt{-g}}{2\sqrt{1+\xi}} \int d(\cos\theta) \sqrt{\sin^2\theta + \frac{\cos^2\theta}{1+\xi}} \right) \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}| f_{\rm eq}(|\mathbf{p}|/\Lambda(x)) \tag{2.9}$$

We write the equation (2.9) in a simpler form: $\epsilon = \mathcal{R}(\xi)\epsilon_{\rm eq}(\Lambda)$, where $\mathcal{R}(\xi) = \frac{1}{2}\left[\frac{1}{1+\xi} + \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]$

Following a similar procedure, one can get

$$\begin{aligned} P_T &= \mathcal{R}_T(\xi) P_{\rm eq}(\Lambda) \\ P_L &= \mathcal{R}_L(\xi) P_{\rm eq}(\Lambda) \end{aligned} \tag{2.10}$$

where $\mathcal{R}_T(\xi) = \frac{3}{2\xi}\left[\frac{1+(\xi^2-1)\mathcal{R}(\xi)}{\xi+1}\right]$, $\mathcal{R}_L(\xi) = \frac{3}{\xi}\left[\frac{(\xi+1)\mathcal{R}(\xi)-1}{\xi+1}\right]$, and $P_{eq}(\Lambda) = \epsilon/3$ owing to the equation of state for conformal fluid.

2.3 Moments of the General Relativistic Boltzmann Equation

We define the $n^{\rm th}$ moment of the collision kernel, $\mathcal{C}[f]$ for a given r,

$$\mathcal{C}_r^{\mu_1\mu_2\cdots\mu_n} \equiv -\int dP (p.u)^r p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n} \mathcal{C}[f] \tag{2.11}$$

As is well known, the first moment of the Boltzmann equation gives energy-momentum conservation. While evaluating the first moment of the Boltzmann equation with collision kernel given by RTA, one has to impose the conservation by hand. This amounts to the condition

$$\mathcal{C}_0^\nu = \int dP p^\nu \frac{u\cdot p}{\tau_R} \left(f_{eq}(\xi,\Lambda) - f_{eq}(T)\right) = 0 \tag{2.12}$$

While, owing to the symmetries of $f(x,p)$, the above integral vanishes for $\mu = 1,2,3$, the vanishing of $\mathcal{C}_0^0$ gives the following condition:

$$T = \mathcal{R}^{1/4}(\xi)\Lambda \tag{2.13}$$

Zeroth Moment

The $0^{\rm th}$ moment of the Boltzmann equation is,

$$\partial_\mu \underbrace{\int dP(p^\mu f)}_{\mathrm{J}^\mu = nu^\mu} - (\partial_\mu\sqrt{-g}) \int \frac{d^3p}{(2\pi)^3 E}(p^\mu f) - \Gamma^\mu_{\rho\sigma} \int dP p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = \underbrace{-\int dP \frac{u\cdot p}{\tau_R}(f - f_{eq})}_{\mathcal{C}_0} \tag{2.14}$$

Where the second term on the left hand side enters because the integral measure is now coordinate dependent and hence one cannot just pull the partial derivative out of the integral. We have the identity:

$$\partial_\mu\sqrt{-g} = \frac{1}{2}\sqrt{-g}\; g^{\alpha\beta}\partial_\mu g_{\alpha\beta}$$

We use the above result and rewrite (2.14) as:

$$\partial_\mu(nu^\mu) - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) n(\xi,\Lambda) - \Gamma^\mu_{\rho\sigma} \int dP p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\mathcal{C}_0$$

Evaluating $\mathcal{C}_0$ using equations (2.11) and (2.8), one obtains:

$$\mathcal{C}_0 = \int \frac{p.u}{\tau_{eq}}[f - f_{eq}(T)]dP = \frac{n_{\rm eq}}{\tau_R}\left(\frac{1}{\sqrt{1+\xi}} - \mathcal{R}^{3/4}(\xi)\right)$$

Note that $\partial_\mu J^\mu = Dn + n\partial_\mu u^\mu$, where $D = u^\mu\partial_\mu$ and $\partial_\mu u^\mu = 0$ in the fluid rest frame. With these substitutions, and the definition of number density from (2.8), the zeroth moment of the Boltzmann equation becomes:

$$\partial_t\left(\sqrt{-g}\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right) + \left(\sqrt{-g}\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = -\sqrt{-g}\left[\frac{1}{\sqrt{1+\xi}} - R^{3/4}(\xi)\right]\frac{n_{eq}(\Lambda)}{\tau_R}$$

Using $\partial_t\sqrt{-g} = \sqrt{-g}\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)$, on the first term on the left

$$\partial_t\left(\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right) + \left(2\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = -\frac{n_{eq}(\Lambda)}{\tau_R}\left[\frac{1}{\sqrt{1+\xi}} - R^{3/4}(\xi)\right]$$

Again using equation (2.8), we can expand the left-hand-side of the above equation in terms of derivatives of ξ and Λ, this gives us:

$$\frac{1}{1+\xi}\partial_t\xi - \frac{6}{\Lambda}\partial_t\Lambda - 4\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{2}{\tau_R}\left(1 - R^{3/4}(\xi)\sqrt{1+\xi}\right) \tag{2.15}$$

First Moment

The 1^{st} moment gives the equation of motion for the energy-momentum tensor,

$$\partial_\mu\int dPp^\mu p^\nu f - \Gamma^\mu_{\rho\sigma}\int dPp^\rho p^\sigma p^\nu\frac{\partial f}{\partial p^\mu} = 0 \tag{2.16}$$

$$T^{\mu\nu} := \int dPp^\mu p^\nu f = (\epsilon + P_T)u^\mu u^\nu - P_T g^{\mu\nu} + (P_L - P_T)Z^\mu Z^\nu$$

When (2.16) is projected along u_ν, one obtains,

$$u_\nu\partial_\mu T^{\mu\nu} = (\dot{\epsilon}+\dot{P}_T) + \underbrace{(\epsilon + P_T)\partial_\mu u^\mu}_{0} + \underbrace{(\epsilon + P_T)u_\nu u^\mu\partial_\mu u^\nu}_{0} - u^\mu\partial_\mu P_T + P_T(\Gamma^\mu_{\mu\alpha}g^{\nu\alpha} + \Gamma^\nu_{\mu\alpha}g^{\mu\alpha})u_\nu + \underbrace{(P_L - P_T)u_\nu Z^\mu\partial_\mu Z^\nu}_{0}$$

Performing a similar manipulation on the first moment equation of $\mathcal{C}_0$, we get a simplified form of energy conservation in $0+1$d, i.e.

$$\frac{\partial\epsilon(\tau)}{\partial\tau} + P_T\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = 0$$

$$\frac{d}{d\tau}\left(\epsilon_{eq}(\Lambda)R(\xi)\right) + \left[\frac{1}{\xi}\left(R(\xi) - \frac{1}{\xi+1}\right) + R(\xi)\right]\epsilon_{eq}(\Lambda)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = 0$$

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau\xi + \frac{4}{\Lambda}\partial_\tau\Lambda = \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right] \tag{2.17}$$

Second Moment

The 2^{nd} moment gives the equation of motion for $I^{\mu\delta\gamma}$,

$$\partial_\mu\underbrace{\int dPp^\mu p^\delta p^\gamma f}_{\mathrm{I}^{\mu\delta\gamma}} - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\int dPp^0 p^\delta p^\gamma f - \Gamma^\mu_{\rho\sigma}\int dPp^\rho p^\sigma p^\delta p^\gamma\frac{\partial f}{\partial p^\mu} = -\underbrace{\int dPp^\delta p^\gamma\frac{u\cdot p}{\tau_R}(f - f_{eq})}_{\mathcal{C}^{\delta\gamma}} \tag{2.18}$$

Where the second term is a result of the partial derivative acting on the $\sqrt{-g}$ factor. A rank three tensor that respects the symmetries of the system can be written in its most general form in terms of the given basis as:

$$\begin{aligned}I^{\mu\delta\gamma} &= I_u(u^\mu u^\delta u^\gamma) + I_z(u^\mu Z^\delta Z^\gamma + Z^\mu u^\delta Z^\gamma + Z^\mu Z^\delta u^\gamma)\\&\quad + I_y(u^\mu Y^\delta Y^\gamma + Y^\mu u^\delta Y^\gamma + Y^\mu Y^\delta u^\gamma) + I_x(u^\mu X^\delta X^\gamma + X^\mu u^\delta X^\gamma + X^\mu X^\delta u^\gamma)\end{aligned}$$

Where

$$\left.\begin{array}{rr} I_u = \mathcal{S}_u(\xi) I_{\rm eq}(\Lambda); & \mathcal{S}_u(\xi) = \dfrac{3+2\xi}{(1+\xi)^{3/2}} \\ I_x = I_y = \mathcal{S}_T(\xi) I_{\rm eq}(\Lambda); & \mathcal{S}_T(\xi) = \dfrac{1}{\sqrt{1+\xi}} \\ I_z = \mathcal{S}_L(\xi) I_{\rm eq}(\Lambda); & \mathcal{S}_L(\xi) = \dfrac{1}{(1+\xi)^{3/2}} \end{array}\right\} I_{eq}(\Lambda) = \frac{1}{3}\int dP E^3 f_{eq}$$

zz projection

$$Z_\delta Z_\gamma \partial_\mu I^{\mu\delta\gamma} = \frac{dI_z}{dt}$$

Plugging this into (2.18),

$$\frac{dI_z}{dt} - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\int dP p^0 p^z p^z f - \Gamma^\mu_{\rho\sigma}\int dP p^z p^z p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\int dP p^z p^z \frac{u\cdot p}{\tau_R}(f - f_{eq})$$

$$\frac{d}{dt}\left(\sqrt{-g}\; I_{eq}(\Lambda) S_L(\xi)\right) + \left(\sqrt{-g}\; I_{eq}(\Lambda) S_L(\xi)\right)\left(2\frac{\dot{A}}{A} + 5\frac{\dot{C}}{C}\right) = \frac{\sqrt{-g}}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda) S_L(\xi)\right]$$

$$\frac{d}{dt}\left(I_{eq}(\Lambda) S_L(\xi)\right) + 2\left(I_{eq}(\Lambda) S_L(\xi)\right)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{1}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda) S_L(\xi)\right]$$

xx and yy projection

$$\frac{d}{dt}\left(I_{eq}(\Lambda) S_T(\xi)\right) + \left(I_{eq}(\Lambda) S_T(\xi)\right)\left(\frac{4\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{1}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda) S_T(\xi)\right]$$

Using the definitions of the terms S_T, I_{eq} and S_L, obtain:

$$\begin{aligned}(\log \mathcal{S}_L)'\,\partial_t\xi + 5\partial_t \log\Lambda + \left(2\frac{\dot{A}}{A} + 3\frac{\dot{C}}{C}\right) &= \frac{1}{\tau_R}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_L} - 1\right]\\ (\log \mathcal{S}_T)'\,\partial_t\xi + 5\partial_t \log\Lambda + \left(\frac{4\dot{A}}{A} + \frac{\dot{C}}{C}\right) &= \frac{1}{\tau_R}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_T} - 1\right]\end{aligned}$$

The equation of motion for the second moment (equation 2.19) is thus obtained from the above equations, after solving further:

$$\frac{1}{1+\xi}\partial_t\xi - 2\left(\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right) + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_R}\xi\sqrt{1+\xi} = 0 \tag{2.19}$$

The aHydro evolution equations obtained above are general in nature up to an axis-symmetric metric. We now put more conditions on the metric in order to realise it in specific systems. We restrict ourselves to a subclass of Bianchi type-I metric, where the expansion factors are of the following form:
ξ when

$$A = t^a, \quad B = t^b, \quad C = t^c \qquad \Longrightarrow \qquad \frac{\dot{A}}{A} = \frac{a}{t}, \quad \frac{\dot{B}}{B} = \frac{b}{t}, \quad \frac{\dot{C}}{C} = \frac{c}{t}$$

When this form of the metric is required to satisfy the Einstein's GR field equations, the parameters a, b, c are related to each other via the following equations:

$$a + b + c = 1, \qquad a^2 + b^2 + c^2 = 1$$

Requiring azimuthal symmetry as present in our setup, the only solution to the above set of equations are:

1. $(a,b,c) \equiv (0,0,1)$ This case represents longitudinal expansion.
2. $(a,b,c) \equiv (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ Here the metric expands in the transverse direction but contracts longitudinally.

Case I produces the exact aHydro equations as derived in [2] and is seen to offer an exact matching with the equations derived by Strickland [3]. To point out the similarities, we present the final equations here:
Zeroth moment

$$\frac{1}{1+\xi}\partial_\tau\xi - \frac{6}{\Lambda}\partial_\tau\Lambda - \frac{2}{\tau} = \frac{2}{\tau_{\rm eq}}\left(1 - R^{3/4}(\xi)\sqrt{1+\xi}\right)$$

First moment, projected along u_μ

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau\xi + \frac{4}{\Lambda}\partial_\tau\Lambda = \frac{1}{\tau}\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right]$$

Second moment, zz and xx, yy projections

$$\left(\log\mathcal{S}_L\right)'\partial_\tau\xi + 5\partial_\tau\log\Lambda + \frac{3}{\tau} = \frac{1}{\tau_{\rm eq}}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_L} - 1\right]$$
$$\left(\log\mathcal{S}_T\right)'\partial_\tau\xi + 5\partial_\tau\log\Lambda + \frac{1}{\tau} = \frac{1}{\tau_{\rm eq}}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_T} - 1\right]$$

The last two equations can be combined to eliminate the Λ term, and one obtains:

$$\frac{1}{1+\xi}\partial_\tau\xi - \frac{2}{\tau} + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_{\rm eq}}\xi\sqrt{1+\xi} = 0$$

3 Misner's ideas and our follow up

We now take a detour and look at some of the ideas presented by Misner in his 1968 paper titled *The Isotropy of the Universe*. The idea is to separate out metric anisotropy and metric expansion by suitable re-parametrisation of the metric. This allows construction of the energy-momentum tensor for a conformal fluid which further allows construction of other dynamical quantities like the shear stress tensor.

Re-parametrisation of metric:

As earlier we start with a diagonal Bianchi type-I metric $g_{\mu\nu}$ such that $ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2$. Introduce parametrisation as:

$$g_{ij}(t) = e^{2\alpha}\left(e^{2\beta}\right)_{ij}$$

where β is traceless matrix and α is a real scalar. Note that $det(exp(M)) = exp(trace(M))$, hence $g = det(g_{ij}) = exp(6\alpha)det(e^{2\beta})$ which implies:

$$g^{1/2} = e^{3\alpha}.$$

For our case,

$$g_{ij} = \begin{bmatrix} A^2 & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & C^2 \end{bmatrix} = e^{2\alpha}e^{2\beta}.$$

Which gives the following equations:

$$e^{a+2a} = A^2, \quad e^{b+2\alpha} = B^2, \quad e^{c+2\alpha} = C^2.$$

Solving for a, b and c one obtains:

$$a = \log\left(\frac{A^{4/3}}{B^{2/3}C^{2/3}}\right), \quad b = \log\left(\frac{B^{4/3}}{A^{2/3}C^{2/3}}\right), \quad c = \log\left(\frac{C^{4/3}}{A^{2/3}B^{2/3}}\right), \quad \alpha = \frac{1}{3}\log(ABC)$$

It is imperative to note that the α term captures the net expansion while the other parameters are solely for anisotropy in the system. From energy considerations on a conformal fluid, one obtains the following relation between the metric and T^{00}.

$$T^{00} = T_c^4 \left\langle \left(n^\top e^{-2\beta} n\right)^{1/2} \right\rangle$$

Where $T_c = T_0 e^{-\alpha}$ is called the central temperature and the equation above relates it to the energy density of fluid by averaging the anisotropy over all directions.

$$\left\langle \left(n^\top e^{-2\beta} n\right)^{1/2} \right\rangle = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(n^\top e^{-2\beta} n\right)^{1/2} \sin\theta d\theta d\varphi$$

Where, $n = \begin{bmatrix} sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta \end{bmatrix}^\top$,

$$\begin{aligned} n^\top e^{-2\beta} n = n^\top \begin{bmatrix} e^{-a} & 0 & 0 \\ 0 & e^{-b} & 0 \\ 0 & 0 & e^{-c} \end{bmatrix} \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} &= [\sin 0 \cos\varphi, \sin \operatorname{sen}\varphi, \cos\theta] \begin{bmatrix} e^{-a}\sin\theta\cos\varphi \\ e^{-b}\sin\theta\sin\varphi \\ e^{-c}\cos\theta \end{bmatrix} \\ &= e^{-a}\sin^2\theta\cos^2\varphi + e^{-b}\sin^2\theta\sin^2\varphi + e^{-c}\cos^2\theta \end{aligned}$$

For the axis-symmetric case at hand (A = B),

$$\left\langle \left(nTe^{-2\beta} n\right)^{1/2} \right\rangle = \frac{1}{4\pi} \int d\Omega \left(e^{-a}\sin^2\theta + e^{-c}\cos^2\theta\right)^{1/2} = \frac{1}{2}\int_1^\pi d\theta \sin\theta \left[\left(\frac{C}{A}\right)^{2/3} \sin^2\theta + \left(\frac{A}{C}\right)^{4/3} \cos^2\theta \right]^{1/2}$$

Performing the integral, one obtains:

$$\left\langle \left(n^\top e^{-2\beta} n\right)^{1/2} \right\rangle = \frac{1}{2}\left(\frac{A}{C}\right)^{2/3} + \frac{1}{2}\left(\frac{C}{A}\right)^{1/3} \left[\frac{\log\left[\frac{A}{C} + \sqrt{(\frac{A}{C})^2 - 1}\right]}{\sqrt{(\frac{A}{C})^2 - 1}} \right] \equiv 1 + V(B)$$

Where $V(\beta)$, termed as anisotropy potential has the following properties and comes handy in capturing the effects of anisotropic metric on a continuous media.

Properties

1. $d\left(g^{1/2}T^{00}\right) = \frac{1}{2}g^{1/2}T_{ij}\cdot dg_{ij} = \frac{1}{2}g^{1/2}\text{Trace}\left(T_- dg^-\right)$, where $g^- = e^{-2\alpha}e^{-2\beta} = \begin{bmatrix} 1/A^2 & 0 & 0 \\ 0 & 1/B^2 & 0 \\ 0 & 0 & 1/C^2 \end{bmatrix}$

2. Small β approximation,
   $V(\beta) = \frac{4}{15}\beta_{ab}\beta_{ab} = \frac{2}{5}\left(\beta_+^2 + \beta_-^2\right)$, where β_- and β_+ are the smallest and the largest eigenvalues of β matrix respectively.

3. Eigenvalues of β are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$

4. In the axis-symmetric case, $\beta_3 = \beta_+ = -\frac{2}{3}\log\left(\frac{A}{c}\right)$

$$\left.\begin{aligned} \beta_1 &= \frac{1}{2}\beta_+ + \frac{\sqrt{3}}{2}\beta_- = \frac{1}{3}\log\left(\frac{A}{C}\right) \\ \beta_2 &= \frac{1}{2}\beta_+ - \frac{1}{2}\sqrt{3}\beta_- = \frac{1}{3}\log\left(\frac{A}{c}\right) \end{aligned}\right\} \Rightarrow \beta_- = 0$$

Given the parametrisation, we and the above properties, we note:

$$\frac{1}{2}dg^- = -g^- d\alpha - e^{-\alpha}e^{-\beta}d\beta e^{-a}e^{-\beta}$$

After we plug-in the expressions for g^{-} and $\delta\beta = \frac{1}{2}\begin{bmatrix} da & 0 & 0 \\ 0 & db & 0 \\ 0 & 0 & dc \end{bmatrix}$, obtain:

$$g^{-1/2}\delta\left(g^{1/2}T^{00}\right) = -\text{trace}\left(T_{-g^{-}}\right)d\alpha - \text{trace}\left(e^{-\alpha}e^{-\beta}T_{-}e^{-\alpha}e^{-\beta}d\beta\right)$$

But, as seen in a previous relation,

$$T^{00} = aT_0^4 g^{-1/2}\left\langle\left(n^{\top}g^{-}\eta\right)^{1/2}\right\rangle = aT_c^4\left\langle\left(n^{\top}e^{-2\beta}n\right)^{1/2}\right\rangle$$

$$\underbrace{\left(T_{ij} - \frac{1}{3}\delta_i' j' T_{kl'}{}^{k}\right)}_{\text{Traceless part of } T_{\mu\nu}} d\beta_{ij'} = -aT_e^4 dV(\beta)$$

Note that the traceless spatial part of $T_{\mu\nu}$ is essentially the shear stress tensor, and hence:

$$\Pi = -\alpha T_0^4 e^{-4\alpha} dV(\beta)$$

$$(\Pi_{ij}) \equiv \left(T_{ij} - \frac{1}{3}\delta T_{ij} T_{k'}^{k'}\right) = -aT_a^4 e^{-4\alpha}\left(\frac{\partial V(\beta)}{\partial\beta}\right) = \frac{-8}{15}\alpha T_0^4 e^{-4\alpha}\beta_{ij} \tag{3.1}$$

where the last equality follows from the small-β approximation from property (2) above. At this stage Misner argues that the in a collision-less fluid, the metric anisotropy will be frozen as it is in the momentum space anisotropy. Hence, if we denote the later by β', then $\beta = \beta'$. However, the author then goes on to argue that in presence of collision the two anisotropies will follow an interdependent, relaxation type equation. The author claims this equation to be:

$$\frac{\beta'}{dt} = \frac{\beta}{dt} - \frac{\beta'}{t_C} \tag{3.2}$$

where t_C is a system dependent time-scale that should be related to the τ_R. Note that the steady state solution for equation 3.2 is $\beta' = t_C \times \frac{d\beta}{dt}$. Using equation 3.2 and 3.1, one can write:

$$\left(T_{ij} - \frac{1}{3}\delta T_{ij} T_{k'}^{k'}\right) = \frac{-8}{15} t_C T^{00}\frac{d\beta}{dt}$$

Recognising the proportionality coefficient as the viscosity, we get

$$\eta = \frac{2}{15} t_c T_0^4 \frac{1}{(A^3 C)}\left[\frac{A}{C} + \frac{\log\left(\frac{A}{C} + \sqrt{\left(\frac{A}{C}\right)^2 - 1}\right)}{\sqrt{\left(\frac{A}{C}\right)^2 - 1}}\right]$$

The idea of introducing anisotropy potential as a dynamical quantity and using it to generate other hydrodynamical quantities seems a plausible one to us. The ad-hoc introduction of equation 3.2 is one of the first objection a careful reader may raise. However, it seems to work well enough and we hypothesised that it must be following from some moment of the Boltzmann equation. The next section contains our attempts to derive equation 3.2 from kinetic theory.

3.1 Other Moments

In order to obtain the equation 3.2 from kinetic theory, we list all possible moments and their combinations that might lead to it. Before embarking on this hit-and-trial calculation, we parametrise the momentum space anisotropy in analogy with the metric anisotropy parametrisation introduced by Misner.

Parametrisation

Note that, for our case of symmetric about the longitudinal axis, the momentum-space one particle distribution function looks like:

$$f_{RS} = \exp\left(-\sqrt{p_x^2 + p_y^2 + (1+\xi)\, p_z^2}/\Lambda\right) \quad \Rightarrow \quad \exp\left(-\sqrt{p^\mu p^2 \Omega_{\mu\nu}}/\Lambda\right)$$

where we have introduced a matrix $\Omega_{\mu\nu}$ that accounts for the anisotropy. Ω is identity for isotropic medium. Now, taking inspiration from Misner's work, we parametrise Ω which looks like the following for our massless, axis-symmetric case.

$$\Omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+\xi \end{pmatrix} \overset{\text{parametrize}}{=} e^{2\alpha'}\left(e^{2\beta'}\right)$$

where α' is a scalar and β' is a 3×3 traceless matrix and the primes have been introduced to distinguish this parametrisation from the metric parametrisation in the previous section. If we let $\beta = diag\ (a', b', c')$, the parameters $a',\ b',\ c'$ in the exponent solve to give:

$$a' = b' = 0 \qquad\qquad c' = log(1+\xi)$$

We use the generalised form of moments of the Boltzmann equation derived by Molnar in [5]. The expression looks like the following:

$$\frac{\partial}{\partial \tau}\left(\hat{I}_{i+j,i,0}\right) + \frac{1}{z}\left[(j+1)\hat{I}_{i+j,j,0} + (i-1)\hat{I}_{i+j,j+2,0}\right] = \hat{C}_{i-1,j} \tag{3.3}$$

where

$$\hat{C}_{i-1,j} = -\frac{1}{\tau_{iq}}\left[\hat{I}_{i+j,j,0} - I_{i+j,j,0}\right]$$

when the net particle number is not conserved, as in our case, we have the relations:

$$\hat{I}^{RS}_{nrq}\left(\beta_{Rs,\xi}\right) = I_{nq}\left(\beta_0\right)\frac{R_{nrq}(\xi)}{\left[R_{200}(\xi)\right]^{(n+2)/4}} \qquad\qquad I_{nq} = \frac{(-1)^2}{(2q+1)!!}\left\langle E^{n-2q}_{k\psi}\left(\Delta^{\alpha\alpha\beta_\alpha} k_{kq}\right)^q\right\rangle_0$$

Choosing different values of i and j, we obtain different moments and their projections. These have been summarised up to the fourth moment of the Boltzmann equation in the table that follows. Some definitions and expressions that shall be used frequently are listed hereby:

$$\left.\begin{aligned} \hat{e} &= \hat{I}_{200} = e_0 R_{200} \\ \hat{P}_l &= e_0 R_{220} = \hat{I}^{PS}_{220} \\ \hat{P}_T &= P_0 R_{201} = \hat{I}^{RS}_{201} \end{aligned}\right\} \begin{aligned} R_{201}(\xi) &= \frac{3}{2\xi}\left[\frac{1}{1+\xi} - (1-\xi)R_{200}(\xi)\right] \\ R_{200}(\xi) &= \mathcal{R}(\xi) = \frac{1}{2}\left[\frac{1}{1+\xi} + \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right] \end{aligned}$$

Zeroth Moment $(i+j) = 1$ For the zeroth moment, there is only one possibility and that is: $i = 1,\ j = 0$ This leads to the equation:

$$\frac{\partial}{\partial z}\left(\hat{I}_{1,0,0}\right) + \frac{1}{\tau}\left(\hat{I}_{1,0,0}\right) = -\frac{1}{\tau_R}\left(\hat{n} - n_0\right)$$

First Moment $(i+j) = 2$ The first moment is just the energy-momentum conservation equations. There exists two independent projections as follows:

- $i = 0,\ j = 2$

$$\frac{\partial}{\partial \tau}\left(\hat{P}_l\right) + \frac{1}{\tau}\left(3\hat{P}_l - \hat{I}_{240}\right) = \frac{-1}{\tau_R}\left(\hat{P}_l - P_0\right)$$

- $(i = 2,\ j = 0)$ will give 2.17:

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau \xi + \frac{4}{\Lambda}\partial_\tau \Lambda = \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right]$$

Second Moment $(i+j)=3$ The second moment ,one of which we have already seen in 2.19, gives rise to 2 equations:

- $i=1,\ j=2$

$$\frac{\partial}{\partial r}\left(\hat{I}_{3,2,0}\right)+\frac{1}{r}\left(3\hat{I}_{3,2,0}\right)=-\frac{1}{req}\left(\hat{I}_{320}-I_{320}\right)$$

- $i=3,\ j=0$

$$\frac{\partial}{\partial \tau}\left(\hat{I}_{3,0,0}\right)+\frac{1}{r}\left(\hat{I}_{3,0,0}+2\hat{I}_{3,2,0}\right)=-\frac{1}{req}\left(\hat{I}_{3,0,00}-I_{3,00}\right)$$

Third Moment $(i+j)=4$ The third moment would require more integrals to be calculated and these are listed in the appendix.

- $i=0,\ j=4$

$$\frac{\partial}{\frac{\partial}{\partial r}}\left(\hat{I}_{440}\right)+\frac{1}{r}\left(5\hat{I}_{440}-I_{460}\right)=-\frac{1}{\tau_{qq}}\left(\hat{I}_{440}-I_{440}\right)$$

- $i=2,\ j=2$

$$\frac{6\partial_\pi\wedge}{\Lambda}+\frac{R'_{420}}{R_{420}}\partial_\tau\frac{\pi}{4}+\frac{1}{\tau}\left(3-\frac{R_{uu_0}}{R_{420}}\right)=\frac{-1}{\tau_{2q}}\left(1-\frac{R^{6/4}}{R_{420}}\right)$$

- $i=4,\ j=0$

$$\frac{\partial}{\partial \tau}\left(\hat{I}_{4,0,0}\right)+\frac{1}{2}\left[\hat{I}_{4,0,0}-3\hat{I}_{420}\right]=\frac{-1}{\tau_{aq}}\left[\hat{I}_{400}-I_{400}\right]$$

Fourth Moment $(i+j)=5$

- $i=1,\ j=4$

$$\frac{\partial}{\partial r}\left(\hat{I}_{540}\right)+\frac{5}{r}\hat{I}_{540}=\frac{-1}{r_{eq}}\left(\hat{I}_{540}-I_{540}\right)$$

- $i=3,\ j=2$

$$\frac{\partial}{\partial z}\left(\hat{I}_{520}\right)+\frac{1}{\tau}\left(3\hat{I}_{520}-2\hat{I}_{540}\right)=\frac{-1}{\tau_{20}}\left(\hat{I}_{520}-I_{520}\right)$$

Note that the for odd value of j, the integral vanishes and hence those cases are omitted from the above table. We note that the equations originating as the same moment of the Boltzmann equation gives rise to similar integrals from the collision kernel on the right hand side. For this reason, we combined the various equations listed above with others of the same moment. However none of the moments of the Boltzmann equation listed above give rise to the equation used by Misner 3.2. However, it has not escaped our notice that equation 2.19 does reproduce equation 3.2 in small-ξ limit. But anisotropic hydrodynamics framework doesn't depend on smallness of the anisotropy parameter and hence we conclude that the relaxation type equation used by Misner cannot be derived from kinetic theory. However it contains all the essential features that it should if it is to be used as an input into evolution equation for dissipative quantities. An exemplary calculation is given below where we failed to obtain equation 3.2 by combining 3^{rd} moment equations.

$$\frac{6\partial_\tau\Lambda}{\Lambda}+\frac{R'_{400}(\xi)}{R_{400}(\xi)}\partial_\tau\xi+\frac{1}{\tau}\left(1-3\frac{R_{420}}{R_{400}}\right)=-\frac{1}{\tau_q}\left[1-\frac{R^{6/4}}{R_{400}}\right]$$
$$\frac{\sigma\partial_\tau\Lambda}{\Lambda}+\frac{R'_{420}(\xi)}{R_{420}(\xi)}\partial_2\xi+\frac{1}{2}\left(3-\frac{R_{440}}{R_{420}}\right)=\frac{-1}{\tau_{eq}}\left[1-\frac{R^{6/4}}{R_{420}}\right]$$
$$\frac{6\partial_\tau\Lambda}{\Lambda}+\frac{R^{1}_{440}(\xi)}{R_{440}(\xi)}\partial_\tau\xi+\frac{1}{\tau}\left(5-\frac{R_{460}}{R_{440}}\right)=\frac{-1}{\tau_{\rm eq}}\left[1-\frac{R^{6/4}}{R_{440}}\right]$$

If the above equations are to be combined to lead to 3.2, the constant coefficients (a,b,c) they need to be multiplied with, need to satisfy:

$$a+b+c=0 \qquad\qquad a+3b+5c=0$$

These conditions come from the observation that there are no stray $1/\tau$ like terms in 3.2, and hence the three equations above should add up to give a net-zero coefficient of $1/\tau$. Even after satisfying these constraints, the three equations on suitable combination give:

$$\partial_2\xi\left[\frac{R'_{400}}{R_{400}}-2\frac{2R'_{420}}{R_{420}}+\frac{R'_{440}}{R_{410}}\right]+\frac{1}{2}\left(\frac{-3R_{420}}{R_{400}}+2\frac{R_{440}}{R_{420}}-\frac{R_{460}}{R_{440}}\right)=\frac{-1}{\tau_{eq}}\left[-\frac{R^{6/4}}{R_{400}}+\frac{R^{6/4}}{R_{420}}\times 2-\frac{R^{614}}{R_{440}}\right]$$

which cannot be reduced to the Misner's equation 3.2. We therefore conclude this part of the project and embark on the next one.

4 Existence of attractor

Consider the ξ (the anisotropy parameter) evolution equation obtained in subsection 2.3,

$$\frac{\partial_\tau\xi}{1+\xi}\pm\frac{2}{\tau}+\frac{R^{5/4}(\xi)\xi\sqrt{1+\xi}}{\tau_R}=0 \tag{4.1}$$

Consider a change of variables, we intend to go from the variables (ξ,τ) to $(\xi,\bar{\tau})$ because the later set is dimensionless and hence independent of the system parameters.

$$\bar{\tau}=\frac{\tau}{\tau_R}=\frac{\tau T}{5\bar{\eta}}=\frac{\omega}{5\bar{\eta}}$$

Multiplying 4.1 throughout by $(1+\xi)$ and τ_R,

$$\tau_R\partial_\tau\xi\pm\frac{2\,(1+\xi)}{\bar{\tau}}+R^{5/4}\,(\xi)\,\xi\,(1+\xi)^{3/2}=0$$

The first term changes as follows, using chain rule of derivatives:

$$\tau_R\frac{\partial}{\partial\tau}=\tau_R\frac{\partial\omega}{\partial\tau}\frac{\partial}{\partial\omega}=\tau_R\frac{\partial\bar{\tau}}{\partial\tau}\frac{\partial}{\partial\bar{\tau}}$$

From the expression for $\bar{\tau}$ in terms of τ, we have,

$$\tau_R\frac{\partial}{\partial\tau}=\frac{\tau_R}{5\bar{\eta}}\left[T+\tau\frac{\partial T}{\partial\tau}\right]\frac{\partial}{\partial\bar{\tau}}=\left[1+\frac{\tau}{T}\frac{\partial T}{\partial\tau}\right]\frac{\partial}{\partial\bar{\tau}}$$

Where we used the relation: $\tau_R=\frac{5\bar{\eta}}{T}$ above. Note that we need time-evolution for the temperature field. We shall obtain it as follows:

$$T=\Lambda R^{1/4}(\xi),\qquad \frac{\partial T}{\partial\tau}=(\partial_\tau\Lambda)\,R^{1/4}(\xi)+\frac{\Lambda}{4}R'(\xi)\,(\partial_z\xi)$$

$$\frac{1}{T}\frac{\partial T}{\partial \tau} = \frac{\partial_\tau \Lambda}{\Lambda} + \frac{1}{4}\frac{R'(\xi)}{R(\xi)}\partial_\tau \xi$$

From the first moment 2.16, we have:

$$\frac{4\partial_2 \Lambda}{\Lambda} + \frac{R}{R}(\xi)\partial_2 \xi = \frac{1}{2}\left[\frac{1}{\xi(1+\xi)R(\xi)} - \frac{1}{\xi} - 1\right]$$

Combining the above equations, we get:

$$\frac{1}{4}\left[\frac{1}{\xi(1+\xi)R(\xi)} - \frac{1}{\xi} + 3\right]\frac{\partial \xi}{\partial \bar{\varepsilon}} \pm \frac{2(1+\xi)}{\bar{\varepsilon}} + R^{5/4}(\xi)\xi(1+\xi)^{3/2} = 0$$

$$\frac{1}{4}\left[1-(1+\xi)R(\xi) + 3\xi\left(1+\xi\right)R\left(\xi\right)\right]\frac{\partial \xi}{\partial \bar{\tau}} \pm \frac{2}{\bar{\tau}}\xi(1+\xi)^2 R(\xi) + R^{9/4}\left(\xi\right)\xi^2\left(1+\xi\right)^{5/2} = 0 \tag{4.2}$$

Having obtained the ξ evolution equation in terms of another dimensionless parameter τ, we now solve the equation numerically to find that the equation admits attractor type solutions for the two Kasner cases. This claim can be verified analytically by solving the equation and then performing a perturbation analysis about the solution. This approach seems to be unfeasible because of the complexity and highly non-linear nature of the equation. However, we take the ansatz $\xi = a/\bar{\tau}$ as a plausible approximate solution and perturb the equation about it. This is the method of analysing what is known as Lyapunov stability. Before attempting that, we first look plot the numerical solution, identify the fixed points and also make an attempt at slow-roll expansion of the differential equation.

4.1 Fixed point analysis

Multiply the equation throughout by $\bar{\tau}$ and analyse the roots of the resulting algebraic equation. These are -1, 0, and ∞. We shall see that the ξ evolution given by 4.2 has -1 and ∞ as its stable/unstable point depending on which Kasner case we are looking at.

Note the anisotropy parameter is allowed to take values in the range $(-1, \infty)$ but the expression for $R(\xi)$ contains a square root. This is overcome by taking note of the following identity.

$$tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad tan(x) = \frac{sin(x)}{cos(x)} = \frac{e^{ix} - e^{-ix}}{i\left(e^{ix} + e^{-ix}\right)}$$

$\frac{\tan^{-1}(\sqrt{\varepsilon})}{\sqrt{\varepsilon}}$ $-y = \tanh(x) \Rightarrow \tan^{-1} -1(-y) = x \quad \Rightarrow \tan^{-1}(iy) = itanh^{-1}(y)$. Hence:

$$R(\xi) = \begin{cases} \frac{1}{2}\left[\frac{1}{1+\xi} + \frac{\tan^{-1}(\sqrt{\xi})}{\sqrt{\xi}}\right] & \text{for } \xi > 0 \\ \frac{1}{2}\left[\frac{1}{1+\xi} + \frac{\tanh^{-1}(\sqrt{|\xi|})}{\sqrt{|\xi|}}\right] & \text{for } \xi < 0 \end{cases}$$

4.2 Lyapunov exponents

The

We perturb the equation 4.2:

$$\frac{1}{4}\left[1-\left(\xi+1\right)R(\xi) + 3\xi\left(1+\xi\right)R(\xi)\right] \times \bar{\tau}\frac{\partial \xi}{\partial \bar{\tau}} \pm 2\xi\left(1+\xi\right)^2 R(\xi) + R^{9/4}\xi^2\left(1+\xi\right)^{5/2}\omega = 0\right]$$

with $\xi = \xi_0 + \delta\xi$ and solve for $\delta\xi$. This gives:

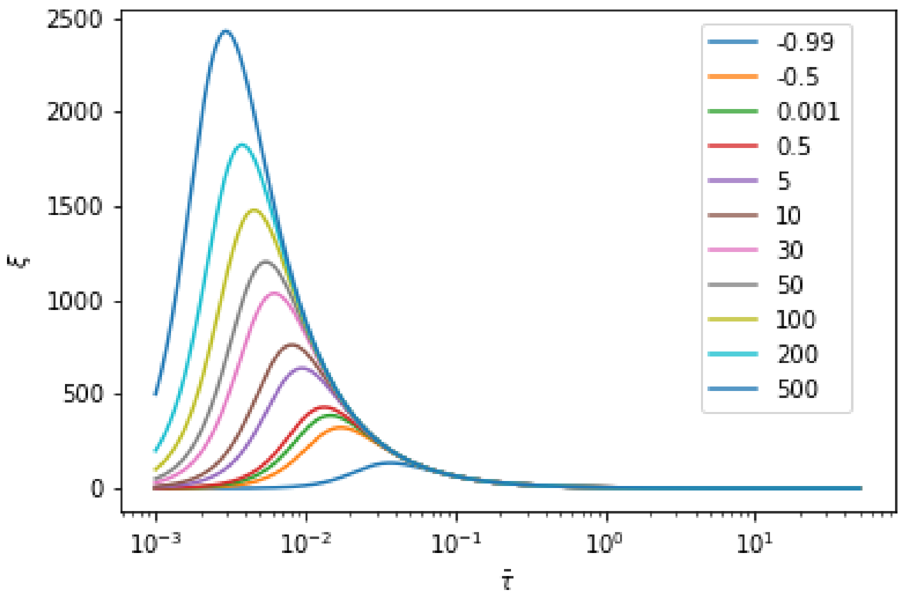

Figure 4.1: ξ vs $\bar{\tau}$. The legend contains initial values of the various curves. Notice that higher the initial value earlier the curve merges with the curve coming from ∞ which is the stable fixed point for the Kasner's first case $(0, 0, 1)$.

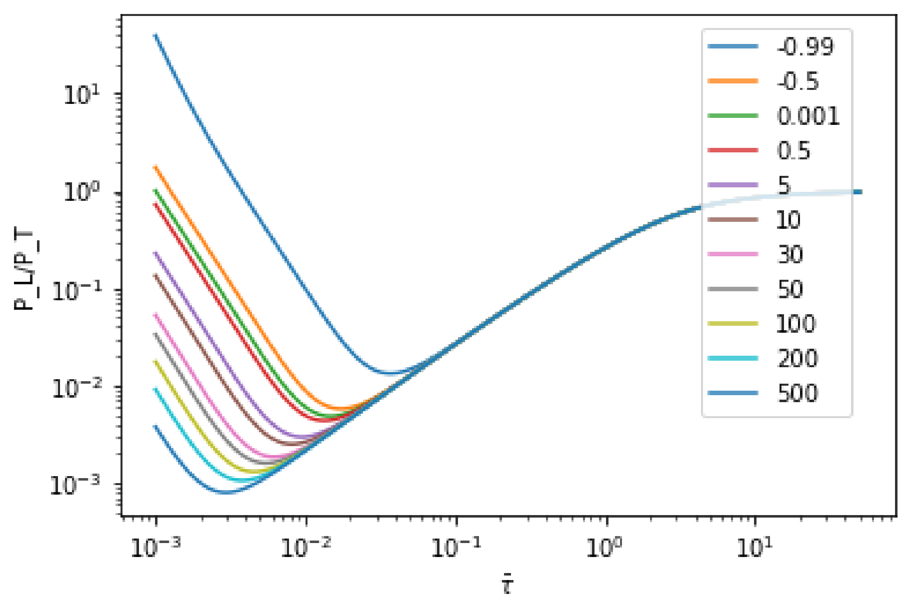

Figure 4.2: Same evolution equation but the derived quantities P_L and P_T are plotted.

Figure 4.3: For the first Kasner case $(0, 0, 1)$

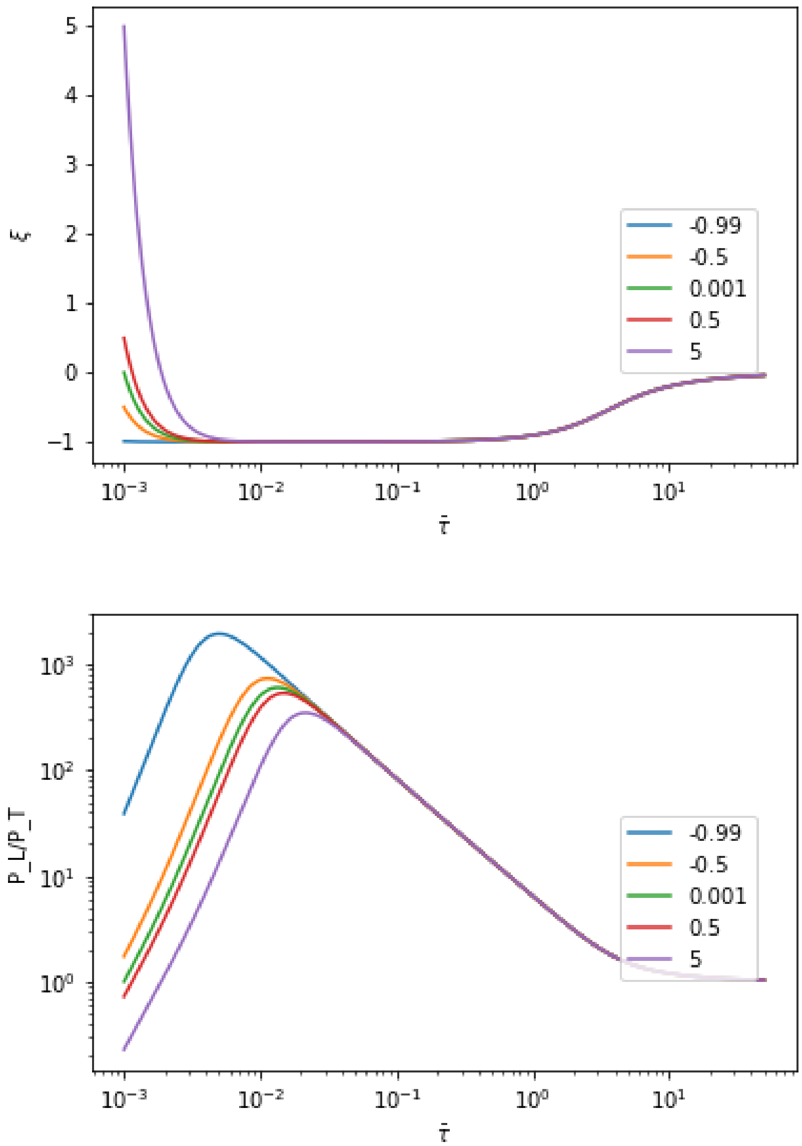

Figure 4.4: For the second Kasner case $(2/3, 2/3, -1/3)$. Notice the evolution of ξ vs. $\bar{\tau}$, all the curves first reach the stable fixed point -1 and then follow the attractor solution to reach the hydrodynamical equilibrium point $\xi = 0$

$$
\begin{aligned}
\frac{1}{4}\left[1-\left(1+\xi_0\right)R\left(\xi_0\right)+3\xi_0 R\left(\xi_0\right)\left(1+\xi_0\right)\right]\bar{\tau}\frac{\partial\left(\delta\xi 0\right)}{\partial\bar{\tau}} \\
+\frac{1}{4}\left[2R'\left(\xi_0\right)\xi_0+2R\left(\xi_0\right)-R'\left(\xi_0\right)+6R\left(\xi_0\right)\xi_0+3R'\left(\xi_0\right)\xi_0^2\right](\delta\xi)\bar{\tau}\frac{\partial\xi_0}{\partial\bar{\tau}} \\
\pm 2\left(\delta\xi_0\right)\left[2\xi_0\left(1+\xi_0\right)R\left(\xi_0\right)+\left(1+\xi_0\right)^2\left(R'\left(\xi_0\right)\xi_0+R\left(\xi_0\right)\right)\right] \\
+R^{9/4}\left(\xi_0\right)\left(1+\xi_0\right)^{5/2}\xi_0^2\left(\frac{9}{4}\frac{R'\left(\xi_0\right)}{R\left(\xi_0\right)}+\frac{2}{\xi_0}+\frac{5}{2\left(1+\xi_0\right)}\right)(\delta\xi)=0
\end{aligned}
$$

To solve this equation and to obtain the Lyapunov coefficients for the two Kasner cases is a challenge I haven't yet been able to finish. However, it the numerical results are almost convincing that the perturbation analysis will yield a stable solution and there indeed are attractors in the general relativistic aHydro framework as well.

5 Discussion

In the previous semester's project work, we derived the anisotropic hydrodynamics evolution equations and saw that with suitable choice of metric, they reproduce the aHydro equations of flat space. This feature, we realise, is powerful for it allows one to separate the metric dynamics from the fluid dynamics and analyse either of the problem in the other picture. For example, a fluid undergoing Bjorken flow with transverse homogeneity and longitudinal symmetry is the same as a fluid at rest, immersed in a metric that expands like the Kasner's first case.

The above example natureally incites enquiry into the second Kasner case, which was possible to explore because of the generality of our aHydro evolution equations. We looked into the solution of 4.2 and found that the role of the two fixed points: -1 and ∞ reverses when one goes from the Kasner's first case to the second. Our result is in clear agreement with Yan and Blaizot's [6] claim that an attractor should be defined as the curve which originates at free-streaming initial condition and evolves to the hydrodynamic equilibrium point. In the first Kasner case, this free-streaming initial condition is $\xi = \infty$ because at this value of anisotropy, P_T becomes ∞. It is clear from 4.3 that the curve coming from the ∞ is the stable fixed point. On the other hand the second Kasner case reverses the picture, since here the free-streaming initial condition is $\xi = -1$ because it is this value of ξ at which P_L becomes ∞. The stable fixed point in this case is -1.

The analytical proof for the existence of attractor solution of equation 4.2 will require more inquiry into solving the equation in the last section and obtaining the Lyapunov coefficients. The future prospect of this work is clear, at least for the immediate future: to prove the existence of attractors analytically.

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Appendix I: Integrals

Here we list some of these integrals explicitly,

- $R_{240}(\xi) = \frac{1}{\xi^2}\left[\frac{3+\xi}{1+\xi} - 3R_{200}(\xi)\right]$
- $R_{320}(\xi) = \frac{1}{3(1+\xi)^{3/2}}$
- $R_{440}(\xi) = -\frac{1}{8\xi^2}\left(\frac{3+5\xi}{(1+\xi)^2} - 3\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)$
- $R_{460} = \frac{1}{86^3}\left(\frac{15+25\xi_6+8\varepsilon^2}{(1+5)^2} - \frac{15\tan^{-1}(\sqrt{6})}{\sqrt{6}}\right)$
- $R_{420} = \frac{1}{8\xi}\left(\frac{arctan(\sqrt{\xi})}{\sqrt{\xi}} + \frac{\xi-1}{(\xi+1)^2}\right)$
- $R_{400} = \frac{1}{8}\left[\frac{3arctan(\sqrt{\xi})}{\sqrt{\xi}} + \frac{3\xi+5}{(\xi+1)^2}\right]$
- $R_{500} = \frac{8\xi^2+20\xi+15}{15(\xi+1)^{5/2}}$
- $R_{520} = \frac{2\xi+5}{15(\xi+1)^{5/2}}$
- $R_{460}(\xi) = \frac{1}{8\xi^3}\left(\frac{15+25\xi+8\xi^2}{(1+\xi)^2} - 15\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)$
- $R_{540}(\xi) = \frac{1}{5(1+\xi)^{5/2}}$

  where $\mathrm{P}_0 \equiv n_0/\beta_0 = e_0/3$, and $I_{30}\left(\alpha_0, \beta_0\right) \equiv I_{300} = \lambda_0 \frac{96\pi A_0}{\beta_0^5}$ $I_{31}\left(\alpha_0, \beta_0\right) \equiv I_{301} = I_{320} = \frac{I_{30}(\alpha_0,\beta_0)}{3}$