Onset of Hydrodynamics as a Universal Attractor in Expanding QGP-Systems

Semester Project Report

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Contents

Abstract

The report divides conceptually into three parts. In the first part we present a review of hydrodynamics starting from Euler equations for ideal fluids to relativistic viscous hydrodynamics in the form of the M*ü*ller-Israel-Stewart (MIS) theory. In an article titled *Hydrodynamics beyond the Gradient Expansion: Resurgence and Resummation*, the authors Michal P. Heller and Michal Spalinski have identified hydrodynamics as an attractor type solution when the MIS theory is applied to a longitudinally expanding quark-gluon plasma system. In the second part we prepare the reader by introducing a few concepts in relativistic heavy-ion collisions before applying the knowledge of fluid dynamics to review and discuss some of the findings of this paper.

In the third part we take a quick look at the foundations of the kinetic theroy and then embark on yet another review. In an article titled *Onset of hydrodynamics for a quark-gluon plasma from the evolution of moments of distribution functions*, the authors Jean-Paul Blaizot and Li Yan, have shown that by studying the collisions in the frame work of kinetic theory, and constructing suitable moments of the distribution function, a direct correspondence with viscous hydrodynamics can be made.

1 Introduction

Fluid dynamics is one of the oldest and successful theory for describing dynamics of matter in bulk, provided that it exists in local equilibrium. Typically, a system should satisfy the condition that the mean free paths of the constituents of the bulk be much smaller than the system size, in order for the theory to be applicable. The theory itself is viewed as the macroscopic limit of an underlying kinetic theory so that hydrodynamics needs to concern itself with only the time-averaged quantities. In the coming sections we take a glance at the theory of hydrodynamics in various regimes of application.

2 Ideal Fluids

The dynamics of an ideal fluid body in non-relativistic regime, is governed by the following equations which relate the velocity field $\vec{v}(\vec{r}, t)$, pressure field $p(\vec{r}, t)$, and the fluid mass density $\rho(\vec{r}, t)$ by the equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0, \tag{2.1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\frac{1}{\rho}\nabla P. \tag{2.2}$$

While deriving the above equations, it is assumed that the fluid is close to local thermal equilibrium. That is, the various system variables described by the dynamical equations and the equation of state, vary slowly from place to place. Therefore the theory is applicable to processes where the macroscopic dynamical time scales are much longer than the microscopic collision time scale (also characterized by relaxation time). It is worth while to note that the equations are related to the conservation principles: (2.1) is called the continuity equation and represents conservation of mass. In an ideal fluid it is assumed that a fluid particle exerts forces equally in all directions, allowing us to characterize the effect with a single scalar quantity: force per unit area or pressure; (2.2) balances pressure differences to the acceleration of fluid particles, these are the Euler equations.

While deriving (2.2) and (2.1) we haven't talked about any exchange of energy between fluid particles. Indeed, for an ideal fluid, we assume total absence of internal friction and heat exchange between different part of the fluid. Essentially making any flow of an ideal fluid, adiabatic. Thus for a fluid particle, as it moves about in space, the entropy remians constant. Mathematically:

$$ds/dt = 0 \implies \frac{\partial S}{\partial t} + (\vec{v}.\nabla)S = 0. \tag{2.3}$$

The state of an ideal moving fluid can be completely described by 5 quantities: 3 velocity components and any two other state variables, for example density ρ and pressure p. A mathematically closed description therefore needs 5 equations, which in this choice of state variables are: the Euler equations, the continuity equation and an equation of state to relate ρ with P.

Important insights can be gained if we analyse flux of some of the dynamical quatities like energy and momentum. Momentum of a unit volume can be given as: $\rho\vec{v}$; let us determine the rate of change of momentum of such a unit volume fixed in space $\partial(\rho\vec{v})/\partial t$. Using the tensor notation, one can write:

$$\frac{\partial}{\partial t}(\rho v_i) = \rho\frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t}v_i.$$

On substituting 2.1 and 2.2 in the form:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_k)}{\partial x_k}, \qquad \frac{\partial v_i}{\partial t} = -v_k\frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i},$$

we obtain:

$$\begin{aligned}\frac{\partial}{\partial t}(\rho v_i) &= -\rho v_k\frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i\frac{\partial \rho v_k}{\partial x_k}, &\quad (2.4)\\ &= -\frac{\partial p}{\partial x_i} - \frac{\partial \rho v_i v_k}{\partial x_k}. &\quad (2.5)\end{aligned}$$

Rewriting the first term as $\frac{\partial p}{\partial x_i} = \delta_{ik}\frac{\partial p}{\partial x_k}$ obtain:

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k}. \tag{2.6}$$

where the term Π_{ik} is identified as the momentum flux density tensor, and characterises the i^{th} component of momenum flowing in unit time through unit perpendicular area in the k^{th} direction. It is defined as:

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k. \tag{2.7}$$

Real fluids have non-zero viscosity and their motion can not be described only with these equations. In the next section we discuss about the fluid evolution equations for viscous fluids.

3 Viscous Fluids

The tensor Π_{ik} is symmetric, and it accounts for the momentum transfer due to the pressure gradients and that due to moving fluid particles. For non-ideal fluids where dissipative processes are present, the Euler equation should be modified to include irreversible tranfer of momentum. This is achieved by adding the viscous stress tensor $\sigma^{'}_{ik}$ to Π_{ik}. The most general form $\sigma^{'}_{ik}$ is given as:

$$\sigma^{'}_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}. \tag{3.1}$$

where the quantities η, called coefficient of viscosity, and ζ, called second viscosity, are both scalars independent of fluid velocity, but in general dependent on temperature and density. With the *viscous stress tensor*, the Euler equation becomes:

$$\rho[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}] = -\nabla(p) + \eta\Delta\vec{v} + (\zeta + \frac{\eta}{3})\nabla(\vec{\nabla}.\vec{v}). \tag{3.2}$$

These equations are known as Navier-Stokes equations.

As long as the underlying microscopic degrees of freedom of any fluid moves with non-relativistic speed the dynamics of the fluid can be described by the non-relativistic hydrodynamics equations as discussed above. But for many systems in nature (for example: hot plasma, matter produced in the high-energy heavy ion collisions, etc.) this approximation is violated as the speed involved in the problem is relativistic. In that case we need the relativistic formulation of hydrodynamics which we discuss in the next section.

4 Relativistic Ideal Hydrodynamics

An ideal fluid system is defined to be in local thermal equilibrium. This allows each fluid particle to be assigned a position, a temperautre and a chemical potential. However, for relativistic fluid systems, the fluid position and three velocities no longer serve as a good degree of freedom. This is because velocity vectors don't tranform properly under Lorentz transformations. Therefore each dynamical variable must be replaced by appropriate Lorentz 4-vectors. Each fluid particle is described in space-time coordinates using four-position $x^\mu = (ct, \vec{r})^T$ and the dynamics is captured in the four-velocity.

$$u^\mu = \frac{dx^\mu}{d\tau}.$$

Where the Greek indices denote Minkowski four-space, and τ is the proper time given by:

$$(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu = (dt)^2 - (d\vec{r})^2 = (dt)^2[1 - (\vec{v})^2],$$

where $g_{\mu\nu} = diag(1, -1, -1, -1)$ is the metric for the flat space time $\vec{v} = \frac{d\vec{r}}{dt}$ and as per convention we take c = 1. This implies that the four-velocity can be written as:

$$u^\mu = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \frac{1}{\sqrt{1-(\vec{v})^2}}\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} = \gamma(\vec{v})\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}. \tag{4.1}$$

Note that in the non-relativistic limit, the four-velocity reduces to $(1, \vec{v})$ and in the local rest frame (LRF) of the fluid, it takes the simplest form: $(1, \vec{0})$. The theory of relativistic hydrodynamics is then developed by writing down the conservation laws for the appropriate quantities among those that describe the state of a system. The state variables are $T^{\mu\nu}_{(0)}$, the energy momentum tensor for an ideal fluid and N^μ, the net particle four-current. $T^{\mu\nu}_{(0)}$ accounts for the currents and the densities associated with the conserved quatities: energy and momentum. The energy momentum tensor should be symmetric and should transform appropriately under Lorentz transformation. The most general for of $T^{\mu\nu}_{(0)}$, build up from hydrodynamic degrees of freedom is:

$$T^{\mu\nu}_{(0)} = \varepsilon(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + p(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu). \tag{4.2}$$

In local rest frame, the momentum densities should vanish $T^{0i}_{(0)} = 0$, and as per [1], the space-like components should vary proportionally with the pressure $T^{ij}_{(0)} = p\delta^{ij}$. In the local rest frame, $T^{\mu\nu}_{(0)}$ thus takes the form:

$$T^{\mu\nu}_{(0),LRF} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}.$$

When the conditions described in the previous paragraph are combined with the equation (4.2), one obtains the energy momentum tensor for the ideal fluids in the form:

$$T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu - p(g^{\mu\nu} - u^\mu u^\nu). \tag{4.3}$$

The term $g^{\mu\nu} - u^\mu u^\nu := \Delta^{\mu\nu}$, has some special properties. It is easy to verify that $\Delta^{\mu\nu} u_\mu = \Delta^{\mu\nu} u_\nu = 0$, and $\Delta^{\mu\nu}\Delta^\alpha_\nu = \Delta^{\mu\alpha}$. That is, the term acts as a projection operator, projecting four-vectors onto the space orthogonal to the fluid velocity four-vector. In absence of any external source, the conservation of energy momentum tensor allows to write.

$$\partial_\mu T^{\mu\nu}_{(0)} = 0. \tag{4.4}$$

Projecting equation (4.4) to direction parallel ($u_\nu \partial_\mu T^{\mu\nu}_{(0)}$) and perpendicular ($\Delta^\alpha_\nu \partial_\mu T^{\mu\nu}_{(0)}$) to the fluid four velocity gives:

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu}_{(0)} &= u^\mu \partial_\mu \varepsilon + \varepsilon(\partial_\mu u^\mu) + \varepsilon u_\nu u^\mu \partial_\mu u^\mu - p u_\nu \partial_\mu \Delta^{\mu\nu}, && (4.5) \\ &= (\varepsilon + p)\partial_\mu u^\mu + u^\mu \partial_\mu \varepsilon = 0. && (4.6) \end{aligned}$$

where we have used the identity $u_\nu \partial_\mu u^\nu = \frac{1}{2}\partial_\mu(u_\nu u^\nu) = \frac{1}{2}\partial_\mu 1 = 0$.

$$\begin{aligned} \Delta^\alpha_\nu \partial_\mu T^{\mu\nu}_{(0)} &= \varepsilon u^\mu \Delta^\alpha_\nu \partial_\mu u^\nu - \Delta^{\mu\alpha}(\partial_\mu p) + p u^\mu \Delta^\alpha_\nu \partial_\mu u^\nu, && (4.7) \\ &= (\varepsilon + p) u^\mu \partial_\mu u^\alpha - \Delta^{\mu\alpha}(\partial_\mu p) = 0. && (4.8) \end{aligned}$$

At this point we define new opertaors:

$$D = u^\mu \partial_\mu \text{ and } \nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu.$$

Equation (4.5) and (4.7) can then be written in a consice form as:

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu = 0, \tag{4.9}$$

$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p = 0. \tag{4.10}$$

These equation are the fundamental equations of relativistic ideal fluids. An insight into the equations can be obtained by analysing the behaviour of the individual terms in the non-relativistic limit. In the next section we broaden the theory by accounting for dissipative fluids too.

5 Relativistic Viscous Hydrodynamics

5.1 Fundamental Equations

As done previously in the non-relativistic case, we account for dissipative (viscous) effects by adding a term to the energy-momentum tensor and to the net particle four-current. To start off with, we write:

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \Pi^{\mu\nu},$$

$$N^\mu = N^\mu_{(0)} + n^\mu.$$

where $\Pi^{\mu\nu}$ is the *viscous stress tensor*. Following a similar scheme, we find the fundamental equation of relativistic viscous hydrodynamics by taking projections of the conservation equations for the energy-momentum tensor. This yields:

$$u_\nu \partial_\mu T^{\mu\nu} = D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu + u_\nu \partial_\mu \Pi^{\mu\nu} = 0, \tag{5.1}$$

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = (\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0. \tag{5.2}$$

To simplify futher we use $u_\nu \partial_\mu \Pi^{\mu\nu} = \partial_\mu(u_\nu \Pi^{\mu\nu}) - \Pi^{\mu\nu}\partial_\mu(u_\nu)$ and $\partial_\mu = u_\mu D + \nabla_\mu$, and make a choice of reference frame in which $u_\mu \Pi^{\mu\nu} = 0$. The fundamental equations then take the form:

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_\mu(u_\nu) = 0, \tag{5.3}$$

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = (\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu}. \tag{5.4}$$

Note that we haven't yet derived any expression for the viscous stress tensor and there are several ways in the literature to do it. Earlies works on covariant formulation of viscous fluids were carried out by Ekart[2] and Landau and Lifshitz[1]. The Landau-Lifschitz formulation asserts that

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}. \tag{5.5}$$

where the η is shear viscosity and $\sigma^{\mu\nu} = 2\langle\nabla^{\mu}u^{\nu}\rangle$ ($\langle ... \rangle$ denotes symmetrization). The resulting theory, however does not have a well-possesed initial value problem. It has been found [8] that the failure shall persist no matter however many finite orders of derivatives of T and u^{μ} are added to the equation (5.5). These and other first order theories, so called because of atmost linear terms of dissipative currents in the viscous stress tensor that they adopt, essentially are relativistic generalisation of the Navier-Stokes theory for viscous fluids. It has been found that these have intrinsic instabilities due to their acausal nature. In the next section we take a look at how this probelm was resolved.

5.2 MIS Theory

One of the first and successful causal theory of viscous fluid dynamics is the M$\ddot{u}$ller-Israel-Stewart (MIS) theory. It resolves the causality problem by letting the *shear stress tensor* be an independent field which satisfies:

$$(\tau_{\Pi}u^{\alpha}\partial_{\alpha} + 1)\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + ... \quad . \tag{5.6}$$

Note that the differential equation satisfied by $\Pi^{\mu\nu}$ has a relaxation term τ_{Π} and elipses denote infinite terms as derived in [9]. The MIS theory contains relaxation times which give the time-scales in which the system responds to hydrodynamic gradients, which is in contrast to the Navier-Stokes theory where there are no such relaxation times and the system responds instantaneously.

5.3 An Overview

Each one of the hydrodynamic thoeries we talked about so far, can be viewed as a gradient expansion of the stress-energy tensor of the fluid $T^{\mu\nu} = T^{\mu\nu}_{(0)} + \Pi^{\mu\nu}$. For example:

1. Ideal hydrodynamics is of zero order.

$$\Pi^{\mu\nu} = 0.$$

2. Navier-Stokes equation has gradient upto first order

$$\Pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>}.$$

3. M$\ddot{u}$ller-Israel-Stewart theory contains gradients upto second order.

$$\Pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>} + \tau_\pi[\Delta^\mu_\alpha\Delta^\nu_\beta D\Pi^{\alpha\beta}...] + \Theta(\delta^2).$$

The zeroth order hydrodynamics is a complete description of ideal fluids. The Navier-Stokes equations also contain all the possible first order terms and offers a self consistent theory, though acausal. However, as seen in the previous section, the success MIS theory comes from letting the stress tensor be an independent field which introduces new parameters in the theory. These can only be determined by using a microscopic theory like kinetic theory. [11]

6 Heavy-Ion Collisions

Nuclear collisions are said to be relativistic if the collision energy, of the center of mass of the participating nuclei, is greater than the rest mass of the nuclei. The characterizing parameters for a nuclear collisions are: the center-of-mass collision energy per nucleon pair, $\sqrt{s}$ and the geomerty of the nuclei. Therefore, as per the definition, a collision is relativistic if $\sqrt{s}/2 > nucleon\ mass$. Lorentz factor can be given as:

$$\gamma = \frac{m\gamma c^2}{mc^2} = \frac{E_{total}}{m} \simeq \frac{\sqrt{s}}{2GeV},$$

It is possible to achieve collisions energies in TeV range with the current technology. Because of a high Lorentz contraction factor, this means that the nuclei suffer distortion to such an extent that they are pictured as 'pancakces' rather than an approximately spherical object.

Heavy-ion collisions are defined as relativistic collions of nuclie with an atomic weight greater than that of carbon. Since we are more interested in the bulk dynamics, we devide [3] the evolution of the matter created in a heavy ion collision, into four stages as per the proper time (described in section 6.2) in the following paragraph.

6.1 A Timeline of Events

The various stages of a relativistic heavy ion collision is described [3] as follows:

Stage I The stage immediately after the collision is termed pre-equilibrium stage and it is characterized by strong gradients. It is usually assumed to last for about $1fm/c$, though the duration of the stage is theoretically unknown. **Stage II** is in near equilibrium regime and is seen to have small gradients. Hydrodynamics becomes applicable if the local temperature allows the matter to stay in a confined state. It last for about 5-10fm/c and successively becomes dilute to enter into the next stage. **Stage III** is the hadron gas regime, characterized by large viscosity and it is better described by kinetic theory than by hydrodynamics. In **stage**

IV, the matter stops interacting and and flies away straight to the detectors.

6.2 Bjorken Flow

In relativistic nucleus-nucleus collions, one sets up a coordinate system such that the collision axis is parallel to the longitudinal z-axis and assumes nuclie to be homogeneous, and extending infinitely in the transverse direction. This removes all dependencies on the transverse coordinates (x,y). One further assumes that the matter produced in the collision events is independent of any boost along the z-axis, this property is called "boost invariance", and the resulting "flow" is termed Bjorken flow [10].

Pertinent to the problem setup at hand would be Milne coordinates (see Appendix) with proper time given as $\tau = \sqrt{t^2 - z^2}$ and space-time rapidity $\xi = arctanh(z/t)$. The infinite transvers extent allows to write $u^x = u^y = 0$ and boost invariance allows to move to a frame in which $u^\xi = 0$, all the hydrodynamic fields are therefore, functions of proper time only. With the corresponding metric tensor for the Milne coordinates, the ideal fluid dynamics equations take the form:

$$D\varepsilon + (\varepsilon + p)\nabla_\mu u^\mu \quad = \quad \partial_\tau \varepsilon + \frac{\varepsilon + P}{\tau} = 0. \tag{6.1}$$

Note that the above equation is in two variables, and hence to solve it one would need an equation of state for the system. In 3 spatial dimensions, a relativistic gas satisfies: $P = c_s^2 \varepsilon$ where c_s^2 is the square of the speed of sound in the gas. For a massless gas it equals $\frac{1}{3}$.

For an equation of state where the speed of sound is constant, equation (6.1) can be solved analytically:

$$\varepsilon(\tau) = \varepsilon(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}. \tag{6.2}$$

The relation $c_s^2 = \frac{1}{3}$ for an ideal relativistic gas then implies that

$$\varepsilon \propto \tau^{-4/3}. \tag{6.3}$$

Viscous corrections to the equation 6.1 have been calculated upto first and second order but the Bjorken's hypothesis is not supported by the experimental data. However, due to its simplicity, it is used as a toy model.

7 Hydrodynamics as an Attractor Type Solution

The MIS equations can be complex to solve analytically for an arbitrary flow. In the article *Hydrodynamics beyond the Gradient Expansion: Resurgence and Resummation* [4], the authors have thus restricted to Bjorken flows in the case of which, due to high symmetry, the MIS equations reduce to a set of simpler ordinary differential equations [4]:

$$\tau\dot{\varepsilon} \;=\; \frac{-4}{3}\varepsilon + \phi, \tag{7.1}$$

$$\tau_\pi\dot{\phi} \;=\; \frac{4\eta}{3\tau} - \frac{\lambda\phi^2}{2\eta^2} - \frac{4\tau_\pi\phi}{3\tau} - \phi. \tag{7.2}$$

where the dot signifies a proper time derivative and $\phi = \Pi^y_y$ is (single) independent component of shear stress tensor. τ_π, λ, and η are the transport coefficients are given by:

$$\tau_\pi = \frac{C_{\tau\pi}}{T}, \quad \lambda = C_\lambda \frac{\eta}{T}, \quad \eta = C_\eta s.$$

where the constants $C_{\tau\pi}, C_\lambda, C_\eta$ are dimensionless and their value is obtained from the fluid-gravity duality in the case of $N = 4$ SYM theory.

$$C_{\tau\pi} = \frac{2 - log(2)}{2\pi}, \quad C_\lambda = \frac{1}{2\pi}, \quad C_\eta = \frac{1}{4\pi}.$$

Differentiating equation (7.1) with respect to proper time obtain:

$$\dot{\phi} = \tau\ddot{\varepsilon} + \frac{7}{3}\dot{\varepsilon}. \tag{7.3}$$

Now using the equations (7.1) and (7.3), eliminate ϕ in favour of τ from equation (7.2) to obtain:

$$\tau_\pi(\tau\ddot{\varepsilon} + \frac{7}{3}\dot{\varepsilon}) = \frac{4\eta}{3\tau} - \frac{\lambda}{2\eta^2}\left(\tau^2\dot{\varepsilon}^2 + \frac{16}{9}\varepsilon^2 + \frac{8}{3}\tau\varepsilon\dot{\varepsilon}\right) - \frac{4\tau_\pi}{3\tau}(\tau\dot{\varepsilon} + \frac{4}{3}\varepsilon) - (\tau\dot{\varepsilon} + \frac{4}{3}\varepsilon).$$

Put $C_\lambda = 0$ and substitute $\tau_\pi = C_{\tau\pi}/T$ to obtain:

$$\tau\tau_\pi\ddot{\varepsilon} + \dot{\varepsilon}(\frac{7C_{\tau\pi}}{3T} + \frac{4C_{\tau\pi}}{3T} + \tau) + \varepsilon(\frac{16\tau_\pi}{9\tau} + \frac{4}{3}) - \frac{4\eta}{3\tau} = 0.$$

Divide throughout by ε and get:

$$\frac{\ddot{\varepsilon}}{\varepsilon}(\tau\tau_\pi) + \frac{\dot{\varepsilon}}{\varepsilon}\left(\frac{11C_{\tau\pi}}{3T} + \tau\right) - \frac{4\eta}{3\tau\varepsilon} + \left(\frac{16\tau_\pi}{9\tau} + \frac{4}{3}\right) = 0. \tag{7.4}$$

Now in order to change the depended variable from ε to T, take help from the equation of state for the system. It is known to be $\varepsilon \sim T^4$. Let

$$\varepsilon = C_0 T^4,$$

then

$$\begin{aligned}\frac{d\varepsilon}{d\tau} &= 4C_0T^3\frac{dT}{d\tau},\\ \frac{d^2\varepsilon}{d\tau^2} &= 12C_0T^2\big(\frac{dT}{d\tau}\big)^2 + 4T^3\frac{d^2T}{d\tau^2}.\end{aligned}$$

With a little modification:

$$\frac{\dot\varepsilon}{\varepsilon} = 4\frac{\dot T}{T}, \tag{7.5}$$

$$\frac{\ddot\varepsilon}{\varepsilon} = 12\big(\frac{\dot T}{T}\big)^2 + 4\frac{\ddot T}{T}. \tag{7.6}$$

On combining the above equations with the equation (7.4) in ε we obtain:

$$\tau C_{\tau\pi}\big(\frac{\ddot T}{T}\big) + 3\tau C_{\tau\pi}\big(\frac{\dot T}{T}\big)^2 + \dot T\big(\frac{11C_{\tau\pi}}{3T} + \tau\big) - \frac{C_\eta sT}{3\tau\varepsilon} + \frac{4C_{\tau\pi}}{9\tau} + \frac{T}{3} = 0. \tag{7.7}$$

In order to eliminate s and ε from the equation (7.7), we note the following thermodynamic relation and the equation of state:

$$\frac{d\varepsilon}{T} = ds,$$

since $\varepsilon = C_0T^4$,

$$d\varepsilon = 4C_0T^3dT,$$

$$ds = 4C_0T^2dT,$$

$$s = \frac{4}{3}C_0T^3.$$

Which on substituting back into equation (7.7), yields:

$$\tau C_{\tau\pi}\big(\frac{\ddot T}{T}\big) + 3\tau C_{\tau\pi}\big(\frac{\dot T}{T}\big)^2 + \dot T\big(\frac{11C_{\tau\pi}}{3T} + \tau\big) - \frac{4C_\eta}{9\tau} + \frac{4C_{\tau\pi}}{9\tau} + \frac{T}{3} = 0. \tag{7.8}$$

Equation (7.8) is neither homogeneous nor is of first order, which can be dauting to solve. Authors therefore suggest a substitution:

$$\omega = \tau T,$$

$$f = \tau\frac{\dot\omega}{\omega}.$$

To change the dependent variable from T to f and the independent variable from τ to ω, we note the following derivations

$$\frac{d\omega}{d\tau} = \dot\omega = T + \tau\dot T,$$

$$f = \frac{T + \tau\dot{T}}{T},$$

which gives:

$$\frac{\dot{T}}{T} = \frac{f-1}{\tau}.$$

Now consider:

$$f' = \frac{df}{d\omega} = \frac{df}{d\tau}\frac{d\tau}{d\omega} = \frac{d}{d\tau}\big(\frac{T+\tau\dot{T}}{T}\big)\frac{1}{T+\tau\dot{T}},$$

$$f' = \frac{1}{T}\big[-\frac{\dot{T}}{T} + \frac{(2\dot{t})/T + (\tau\ddot{T}/T)}{1+\dot{T}\tau/T}\big],$$

and using $\tau T = \omega$, obtain:

$$\frac{\ddot{T}}{T} = \frac{\omega f f' + f(f-1) - 2(f-1)}{\tau^2},$$

which on substitution into equation (7.8) yields (on multiplication with τ^2):

$$C_{\tau\pi}\omega f f' + 4C_{\tau\pi}f^2 + f\big(-\frac{16}{3}C_{\tau\pi} + \omega\big) + \frac{16}{9}C_{\tau\pi} - \frac{4}{9}C_\eta - \frac{2}{3}\omega = 0. \tag{7.9}$$

where the prime denotes derivative with respect to ω. The eqution (7.9) is now of first order. The onset hydrodynamic behaviour is expected to start at large values of τ and for high temperatures at the center of the fireball created in a collision. This criteria is equivalent to a large value of ω ($:= \tau T$), and thus we look for a series solutions to (7.9) at large ω. This can be achieved by making a transformation of variables as:

$$z \longrightarrow \frac{1}{\omega}$$

$$\frac{d}{d\omega} = \frac{dz}{d\omega}\frac{d}{dz} = \frac{-1}{\omega^2}\frac{d}{dz}.$$

Which modifies the equation to:

$$-C_{\tau\pi}z f f' + 4C_{\tau\pi}f^2 + f\big(-\frac{16}{3}C_{\tau\pi} + \frac{1}{z}\big) + \frac{16}{9}C_{\tau\pi} - \frac{4}{9}C_\eta - \frac{2}{3z} = 0. \tag{7.10}$$

Note: f and f' are now functions of z ($:= 1/\omega$). Attempt a series solution of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ to equation (7.10).

$$f'(z) = \sum_{n=0}^{\infty}(n a_n z^{n-1}) = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n,$$

$$f(z)f'(z) = \sum_{n=0}^{\infty} z^n \sum_{i=0}^{n} a_i(n-i+1)a_{n-i+1},$$

$$f^2(z) = \sum_{n=0}^{\infty} z^n \sum_{i=0}^{n} a_i a_{n-i}.$$

Substitute the above relations in (7.10) to obtain:

$$C_{\tau\pi}\sum_{n=0}^{\infty}z^n\sum_{i=0}^{n}a_i(n-i+1)a_{n-i+1}+4C_{\tau\pi}\sum_{n=0}^{\infty}z^n\sum_{i=0}^{n}a_ia_{n-i}$$
$$+\sum_{n=0}^{\infty}a_nz^{n-1}-\frac{16}{3}C_{\tau\pi}\sum_{n=0}^{\infty}a_nz^n+\frac{16}{9}C_{\tau\pi}-\frac{4}{9}C_\eta-\frac{2}{3\tau}=0.$$

On camparision with the right hand side, we conclude that the coeffiecient of each power of z must be seperately equated to zero.

1. Coefficient of 1/z

$$(a_0-\frac{2}{3})(\frac{1}{z})=0,$$
$$\implies a_0=2/3.$$

2. Consatants:

$$4C_{\tau\pi}a_0^2+a_1-\frac{16}{3}C_{\tau\pi}a_0-\frac{4}{3}C_\eta+\frac{16}{9}C_{\tau\pi}=0,$$
$$\implies a_1=\frac{4}{9}C_\eta.$$

3. coefficient of z

$$\left(-C_{\tau\pi}a_0a_1+4C_{\tau\pi}(2a_0a_1)+a_2-\frac{16}{3}C_{\tau\pi}a_1\right)z=0,$$
$$\implies a_2=\frac{8}{27}C_{\tau\pi}C_\eta.$$

And similarly, all the coefficients a_ns can be procured. The final power series soution of the equation (7.9) is thus:

$$f(\omega)=\frac{2}{3}+\frac{4C_\eta}{9\omega}+\frac{8C_{\tau\pi}C_\eta}{27\omega^2}+...\quad . \tag{7.11}$$

Stability of the Solution Let the original solution as given above be f_0, we analyse the stability of the solution (7.11) by feeding a perturbed solution to the equation (7.9) and studying its behaviour upto a first order perturbation. Let the perturbed solution be:

$$f(\omega)=f_0(\omega)+\alpha F(\omega).$$

where F is a perturbation term scalled by some arbitrary and small real number α.

$$f'(\omega)=f_0'(\omega)+\alpha F'(\omega).$$

Substituting the perturbed solution in equation (7.9):

$$C_{\tau\pi}\omega(f_0+\alpha F)(f_0'+\alpha F')+4C_{\tau\pi}(f_0+\alpha F)^2+(f_0+\alpha F)\big(\frac{16}{3}C_{\tau\pi}+\omega\big)+\frac{16}{9}C_{\tau\pi}-\zeta\frac{4}{9}C_\eta-\frac{2}{3}\omega=0.$$

Since f_0 is a solution to equation (7.9), the terms containing only f_0 seperate out to give:

$$C_{\tau\pi}(\alpha f_0'F+\alpha F'f_0+\alpha^2FF')+4C_{\tau\pi}(\alpha^2F^2+2\alpha f_0F)+\big(\omega-\frac{16}{3}C_{\tau\pi}\big)\alpha F=0.$$

Now we ignore terms in α which have order greater than one.

$$C_{\tau\pi}(\alpha f_0'F+\alpha F'f_0)+8C_{\tau\pi}\alpha f_0F+\big(\omega-\frac{16}{3}C_{\tau\pi}\big)\alpha F=0,$$

$$\frac{dF}{F}=-\frac{f_0'\omega C_{\tau\pi}+8C_{\tau\pi}f_0+\omega-(16/3)C_{\tau\pi}}{C_{\tau\pi}\omega f_0},$$

$$\frac{dF}{F}=\frac{-f_0'}{f_0}-\frac{8}{\omega}-\frac{1}{C_{\tau\pi}f_0}+\frac{16}{3\omega f_0}.$$

Approximate f and f' upto two terms and also put the variation equal to zero. Solving the resulting differential equation gives:

$$\delta f_0(\omega)\sim exp\big(-\frac{3\omega}{2C_{\tau\pi}}\big)\omega^{\frac{C_\eta-2C_\lambda}{C_{\tau\pi}}}\big(1+\Theta(\frac{1}{\omega})\big).$$

Hence, a perturbation given to the solution deacays exponentially with time and suggest that the solution is stable. Thus, the onset of hydrodynamics as an attractor type solution shows up as a solution to the MIS equations applied to a longitudinally expanding system.

8 Kinetic Theory

8.1 A Primer

Kinetic theory concerns itself with evolution of one-particle distribution function $f(\vec{p},t,\vec{x})$ which is related to the number of particles per unit phase space.

$$f(\vec{p},t,\vec{x})\propto\frac{dN}{d^3pd^3x}. \tag{8.1}$$

If there are no collisions in the system, the evolution of f follows from the Liouville's theorem [6] as:

$$\frac{df}{d\tau} = \frac{dt}{d\tau}\frac{\partial f}{\partial t}+\frac{d\vec{x}}{d\tau}.\frac{\partial f}{\partial\vec{x}}=0, \tag{8.2}$$

$$= \underbrace{m\frac{dt}{d\tau}}_{p^0}\frac{\partial f}{\partial t}+\underbrace{m\frac{d\vec{x}}{d\tau}}_{\vec{p}}\frac{\partial f}{\partial\vec{x}}=p^\mu\partial_\mu f=0. \tag{8.3}$$

When there are collisions, the theory takes them into account with a collision term, $C[f]$ that is a functional of f.

$$p^\mu \partial_\mu f = -C[f]. \tag{8.4}$$

The exact form of $C[f]$ depends on the type of interactions between the particles. When there is a global equilibrium in the system, the function f has to be stationary (say f_0) implying that the collision term becomes zero when $f = f_0$. Equation (8.4) is called the Boltzmann's equation.

From equation (8.1) it is clear that $\int_p d^3pf$ should give particle density, while adding a weight equal to particle energy: $\int_p p^0 d^3pf$ should give energy density. In a similar sense one can check that the stress-energy tensor can be given as:

$$\int_p \frac{d^4p}{(2\pi)^3} p^\mu p^\nu \delta(p^\mu p_\mu - m^2) 2\theta(p^0) f(p,x) \equiv T^{\mu\nu}. \tag{8.5}$$

Where θ is the step function to ensure only positive energies states are counted.

8.2 From Kinetic to Hydro

If we take a simplifying assumption of the case of ultrarelativistic limit ($\gamma \gg 1$), the particles' masses can be assumed to be zero which implies $T^\mu_\mu = 0$; termed as vanishing conformal anomaly. Eased by this assumption, we make some quick observations. Let $d\xi = \frac{d^4p}{(2\pi)^3}\delta(p^\mu p_\mu)2\theta(p^0)$ be a shorthand for convenience of writing the coming equations. Note the first moment of the Boltzmann equation (8.4):

$$\partial_\mu \int d\xi p^\nu p^\mu f(p,x) = \int d\xi p^\nu p^\mu \partial_\mu f(p,x) = -\int d\xi p^\nu C[f] = \partial_\mu T^{\mu\nu}. \tag{8.6}$$

The integral over $C[f]$ vanishes if the particle interactions conserve energy and momentum. Therefore, the first moment of the Boltzmann equation (8.4) gives the fundamental equations of ideal fluid dynamics. Note that we obtain fundamental equations for ideal fluids because 8.5 was interpreted as $T^{\mu\nu}$.

Since the equilibrium distribution finction $f_0(\vec{p})$ does not transform appropriately under Lorentz transformations, we adopt $f_{eq}\left(\frac{p^\mu u_\mu}{T}\right)$ in place of $f_0(\vec{p})$ and move to the rest frame and expand the stress energy tensor in the $\{u^\mu u^\nu, \Delta^{\mu\nu}\}$ basis as:

$$T^{\mu\nu}_{(0)} = \int d\xi p^\mu p^\nu f_{eq}\left(\frac{p^\mu u_\mu}{T}\right) = a u^\mu u^\nu + b\Delta^{\mu\nu}. \tag{8.7}$$

Note that only two tensors suffice to represent $T^{\mu\nu}$ because: it is symmetric, reducing the degree of freedom (DoF) to 10; ultra-relativistic limit $\implies$ $T^\mu_\mu = 0$, and we are in local rest frame, reducing the DoF to 2. The coefficients 'a' and 'b' are just temperature dependent and can be obtained by contracting equation (8.7) with $\{u^\mu u^\nu$ and $\Delta^{\mu\nu}\}$. For an ideal fluid, as in equation (4.3), $a = \varepsilon$ and $b = -p$.

Any departure from equilibrium state involves gradients of hydrodynamical variables, and can be accounted only by dissipative fluid dynamics. Let us inspect correspondences between viscous hydrodynnamics and kinetic theory for a system not in equilibrium, by giving a perturbation to the equilibrium distribution function.

$$f(p^\mu, x^\mu) = f_{eq}\left(\frac{p^\mu u_\mu}{T}\right)[1 + \delta f(p^\mu, x^\mu)], \tag{8.8}$$

From the definition of energy momentum tensor (8.5), it then follows that:

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \int d\xi p^\mu p^\nu f_{eq}\delta f = T^{\mu\nu}_{(0)} + \Pi^{\mu\nu}, \tag{8.9}$$

$$\Pi^{\mu\nu} = \int d\xi p^\mu p^\nu f_{eq}\delta f. \tag{8.10}$$

Using the Boltzmann equation (8.4) one can write:

$$C[f] = -p^\mu \partial_\mu [f_{eq}(1 + \delta f)] = -p^\mu \partial_\mu f_{eq} + \Theta(\delta^2). \tag{8.11}$$

Hence the collision term can be calculated if the phase space distribution function is known.

9 A Kinetic Theory Approach for Heavy-Ion Collisions

Success of hydrodynamics in describing the evolution of quark gluon plasma (QGP), shortly after a collision event, suggests that the system achieves local thermal equilibrium in a relatively shorter time span than expected. As noted in section 6.1, hydrodynamics becomes applicable at stage II of the evolution, while the previous stage is governed by field equations from QCD. The question about the conditions and mechanism for the onset of hydrodynamics then naturally arises. Kinetic theory, being at the gap between the dynamics of fields and the dynamics of dissipative fluids, offers a possible explaination. In this framework it might be possible to understand the evolution of the system from pre-equilibrium stage to a state of quasi local equilibrium.

As expected, the transition from kinetic theory to hydrodynamics is to be done by averaging out the dynamical quantities of the former to obtain those of the later. This is achieved by taking moments of kinetic equations, where it is seen that the lower moments reproduce the conservation laws while higher moments account for the dissipative effects. In the paper [4], the authors introduce a specific set of moments L_ns defined as weighted integrals of momentum distribution function $f(p)$. If P_{2n} is Legendre polynomial of degree $2n$ and $cos\theta = p_z/p$ where p_z is the longitudinal momentum of a particle with total momentum p; then $L_n \propto \int_p p^2 P_{2n}(cos\theta) f(p)$.

Kinetic Theory Applied to Bjorken Flow

For the QGP produced in a relativistic heavy-ion collisions, the authors assume a simplified model (Bjorken flow) with translational symmetry in transvers direction and boost invariance in the longitudinal direction, which is the direction of collisions. As described in the section (6.2), the above freedoms can be used to move to $z = 0$ plane and then the distribution function $f(t, p_T, p_z)$ satisfies a simpler form of the kinetic equation (8.4):

$$\frac{\partial f}{\partial t} - \frac{p_z}{t}\frac{\partial f}{\partial p_z} = -C[f], \tag{9.1}$$

Multiplying the above equation by p^2 and integrating, it is easy to obtain (given that the collision term conserves energy):

$$\frac{d\varepsilon}{dt} + \frac{\varepsilon(t) + P_L(t)}{t} = 0. \tag{9.2}$$

where the longitudinal pressure $P_L(t) \equiv \langle\langle p_z^2 \rangle\rangle$ can be obtained from (9.2) as:

$$P_L(t) \equiv \langle\langle p_z^2 \rangle\rangle = -\frac{d(t\varepsilon(t))}{dt}. \tag{9.3}$$

And similarly:

$$P_T(t) \equiv \frac{1}{2}\langle\langle(p_x^2 + p_y^2)\rangle\rangle = \frac{1}{2}\frac{d(t^2\varepsilon(t))}{dt}. \tag{9.4}$$

Now consider the moment $L_1 = \int_p p^2 P_2(cos\theta) f(p)$.

$$\begin{aligned} L_1 = \int p^2 \frac{1}{2}(3cos\theta - 1) f \frac{d^3p}{(2\pi)^3 p^0} &= \frac{3}{2}\int p_z^2 \frac{d^3p}{(2\pi)^3 p^0} - \frac{1}{2}\int p^2 \frac{d^3p}{(2\pi)^3 p^0} \\ &= \frac{3}{2}\langle\langle p_z^2 \rangle\rangle - \frac{1}{2}\langle\langle p^2 \rangle\rangle = \langle\langle p_z^2 \rangle\rangle - \frac{1}{2}\langle\langle p_x^2 + p_y^2 \rangle\rangle = P_L - P_T. \end{aligned}$$

We have already seen in section (8.1) in equation (8.5) that ideal hydrodynamics results from zeroth order moments of the distribution function. In this section we follow the first order correction as done by the author. When a system is not in equlibrium, there are collisions between the particles which irreversibly transfer momentum to isotropize the system. In the case of longitudinally expanding system considered here, the expansive forces work against this isotropization. Hence to characterise the onset of quasi local equilibrium, the ratio P_T/P_L is investigated. When the ratio is close to one, or equivalentely, when $L_1 = 0$, local equilibration is said to be complete.

It is shown in [5] that with a linear ansatz ($\delta f / f_{eq} \approx p/T$) for the equation (8.11) a first order viscous correction can be obtained for the distribution function with the help of which, it can be shown that $L_1 \approx -2\frac{\eta}{t}$. Hence,

$$L_1 = P_L - P_T \approx -2\frac{\eta}{t}. \tag{9.5}$$

Hence it is exemplified that the first moment of the phase space distribution function weighted with Legendre polynomials gave first order correction to the ideal hydrodynamics. It is shown in [5] how can higher moments be used to obtain a heirarchy of corrections.

10 Conclusion

We started off with a review of the theory of hydrodynamics in various regimes of application. A fundamental assumption which also serves as a criterion for the applicability of hydrodynamics is existence of local equilibrium throughout the system. The success of hydrodynamics in modelling quark-gluon plasma (QGP) formed in heavy-ion collision has been taken to suggest that the system of quarks and gluons achieves local equilibrium relatively earlier, that is while the system is still expanding longitudinally. In order to understand mechanisms which lead to such a thermalization, the authors Michal P. Heller and Michal Spalinski start from the MIS equations and find the existence of hydrodynamics as an attractor type solution; while the authors Jean-Paul Blaizot and Li Yan, in a seperate Letter, investigate specific moments of the phase space distribution function and discuss their behaviour as a system approaches local thermal equilibrium. Charaterizing momentum anisotropies in those moments of distribution function, they produce a hierarchy of corrections to ideal fluid dynamics.

Appendix I: Coordinate Systems

Minkowski Space time coordintes

Minskowski space-time is a four dimensional space equipped with coordinates $x^\mu = (t, x, y, z)$ and characterised by the metric tensor: $diag(-1, 1, 1, 1)$. All the Christoffel symbols vanish and therefore the Riemann tensor $(R^\mu_{\nu\rho\sigma})$ and the Ricci tensor $(R_{\mu\nu})$ defined as follows, vanish.

$$R^\mu_{\nu\rho\sigma} := \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\tau\rho}\Gamma^\tau_{\nu\sigma} - \Gamma^\mu_{\tau\sigma}\Gamma^\tau_{\nu\rho}.$$

$$R_{\mu\nu} := R^\rho_{\mu\rho\nu}.$$

Milne Coordinates

Milne coordinates, in reltion to Minkowski coordinates are defined as:

$$\text{Propertime}: \ \tau = \sqrt{t^2 - z^2}, \quad \text{and} \quad \text{spacetime rapidity}: \ \xi = arctanh(z/t).$$

Hence the full set of Milne coordinaates is $\overline{x}^\mu = (\tau, x, y, \xi)$. Transformation matrices from Milne coordinates to Minkowski can be obtained using the relation: $R^\nu_\mu = \frac{dx^\nu}{d\overline{x}^\mu}$

$$\begin{pmatrix} cosh\xi & 0 & 0 & \tau sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ sinh\xi & 0 & 0 & \tau cosh\xi \end{pmatrix}.$$

The metric tensor $\overline{g}_{\mu\nu}$ can be obtained either from the definition $\overline{g}_{\mu\nu} = \overline{e_\mu}.\overline{e_\nu}$ or with the relation

$$g_{\mu\nu} = (R^\lambda_\mu)^T g_{\lambda\kappa}(R^\kappa_\nu),$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \tau^2 \end{pmatrix}.$$

Non-zero Christoffel symbols for $\overline{g}_{\mu\nu}$ are

$$\bar{\Gamma}^\xi_{\xi\tau} = \bar{\Gamma}^\xi_{\tau\xi} = \frac{1}{\tau},$$

$$\bar{\Gamma}^\tau_{\xi\xi} = \tau.$$

However, the Ricci and Riemann tensors are identically zero, that is the Milne space-time is also flat.

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