HYDRODYNAMICS IN GENERAL RELATIVISTIC SETUP

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DEDICATION

Dedicated to my parents Hemant Vyas and Jyoti Vyas.

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ABSTRACT

This work conceptually divides into three sections. First we present a brief review of the general relativistic Boltzmann equation, the relaxation time approximation and motivate the idea of deriving fluid dynamics from kinetic theory. The second part reports a failed attempt at calculating the second order transport coefficients from the Boltzmann equation using the Chapman-Enskog like expansion. Finally, in the third part we report an attempt at making a general relativistic extension of the anisotropic hydrodynamics (aHydro) formalism. We show that in the one of the two Kasner metric solutions, the aHydro equations obtained here match with those given by Strickland in his work on aHydro in flat spacetime. Extending on this work, we explore the anisotropic hydrodynamics evolution equations arising from different moments. The choice of second moment admits an attractor type solution which we present numerically as well as analytically. We obtain the Lyapunov coefficients for the two Kasner cases by performing a perturbation analysis. The hitherto existing reports[1] of attractors in anisotropic hydrodynamics rely on smallness of the anisotropy parameter. We have found that the attractor exists for any amount of anisotropy in the system. Apart from the attractors, we also report some failed attempts at kinetic theory derivation of a relaxation type equation used by C. W. Misner[2] in his 1968 paper.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 The General Relativistic Boltzmann Equation

The Boltzmann equation relates the phase-space derivatives of a single-particle distribution function to the net effect of non-reactive collisions occurring between the component particles. It derives from the Liouville equation which describes the phase-space evolution of a given ensemble of particles. The single particle distribution function is, in general, a function of phase-space coordinates. $f \equiv f(x^\mu, p^\nu)$. If λ is some affine parameter in the phase-space, the Boltzmann equation reads:

$$\frac{df}{d\lambda} = \mathcal{C}[f]$$

Where $\mathcal{C}[f]$ represents the phase-space density of collisions. Using the chain rule of derivatives, it can be rewritten as:

$$\frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dp^\mu}{d\lambda}\frac{\partial f}{\partial p^\mu} = \mathcal{C}[f] \tag{1.1}$$

For neutral particles, the trajectories are described by the following geodesic equation:

$$\frac{dp^\mu}{d\lambda} + \Gamma_\rho\sigma^\mu p^\rho p^\sigma = 0$$

Incorporating the above equation in (1.1), one obtains:

$$p^\mu \partial_\mu f - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = \mathcal{C}[f] \tag{1.2}$$

The exact form of $C[f]$ depends on the type of interactions between the particles. When there is a global equilibrium in the system, the function f has to be stationary;

that is $df/d\lambda = 0$ when $f = f_{eq}$, implying that the collision term becomes zero when $f = f_{eq}$. This equilibrium distribution function f_{eq} is called the Maxwellian and it is the only known exact solution of the Boltzmann equation used in practice (there is another one by Ikenberry and Truesdell but its use is limited to textbooks). The equilibrium distribution for a relativistic fluid at temperature T, chemical potential μ and fluid four-velocity u^μ is given by the Maxwell-Jüttner distribution:

$$f_{eq}(x, p) \sim exp\left(\frac{\mu}{k_B T} - \frac{u^\mu p_\mu}{k_B T}\right) \tag{1.3}$$

Where k_B is the Boltzmann constant. Throughout our exposition, we will be dealing with fluids with zero chemical potential and we will be working in natural units. Hence the equilibrium distribution function will look like

$$f_{eq} = exp(-\beta(u \cdot p)) \tag{1.4}$$

Where β is inverse temperature. Note that the temperature may be a function of coordinates, that is $T \equiv T(x)$. Used in this form, the Maxwellian is no longer an exact solution of the Boltzmann equation, but when imposed as a solution it gives constraint equations relating the gradients of the temperature and chemical potential fields with fluid velocity and acceleration. Therefore, only the global equilibrium distribution function is an exact solution which makes it uninteresting for practical applications.

If we require a description of more realistic non-equilibrium processes, we have to make use of approximation techniques. One such approximation technique is to linearize the collision kernel and then solve the Boltzmann equation for perturbations around the f_{eq}. One specific prescription for linearizing the collision integral is the relaxation time approximation, etymology of which will be clear in the next section.

1.2 Relaxation Time Approximation

The relaxation time approximation was introduced by J.L. Anderson and H.R. Witting[3] in 1974 following the non-relativistic version developed by P. L. Bhatnagar, E. P. Gross and M. Krook.[4] If the system under consideration is close to equilibrium, it is natural to linearize the collision kernel in the neighbourhood of a local equilibrium state. The relaxation time approximation specifies this linear approximation with one parameter τ_R, the relaxation time of the system which is interpreted as the time scale within which the system reaches equilibrium. With collision kernel given by the relaxation time approximation, the Boltzmann equation (1.2) becomes:

$$p^\mu \partial_\mu f - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\frac{u \cdot p}{\tau_R}(f - f_{eq}) \tag{1.5}$$

where $\tau_{eq} = 5\bar{\eta}/T$ is the relaxation time, and $\bar{\eta} = \eta/s$ is the ratio of shear viscosity to entropy density. For the rest of our work, we will be using the Boltzmann equation in the form of (1.5).

1.3 Deriving hydrodynamics from kinetic theory

The fluid dynamics equations often contain parameters of microscopic origin, for example the transport coefficients associated to various dissipative processes in a system that is out of equilibrium. With this picture in mind, one would like to start with a theory containing microscopic parameters and then perform a coarse graining to get to the macroscopic picture. In the transition from kinetic theory to fluid dynamics this coarse graining corresponds to integrating out the momentum dependencies from analog physical quantities. This paradigm will be best apparent with some examples which are in order.

- The energy momentum tensor for an ideal fluids is of the form:

$$T_{(0)}^{\mu\nu} = \varepsilon u^\mu u^\nu - p(g^{\mu\nu} - u^\mu u^\nu).$$

From kinetic theory, it is constructed as:

$$\int_p \frac{d^4 p}{(2\pi)^3} p^\mu p^\nu \delta(p^\alpha p_\alpha - m^2) 2\theta(p^0) f(p,x) \equiv T^{\mu\nu}.$$

- The particle current is similarly constructed with the first moment of the distribution function.

$$\int_p \frac{d^4 p}{(2\pi)^3} p^\mu \delta(p^\alpha p_\alpha - m^2) 2\theta(p^0) f(p,x) \equiv N^\mu. \tag{1.6}$$

While moments of the distribution function can generate coarse-grained hydrodynamical quantities, the moments of Boltzmann equation itself can generate their evolution. It is well known that the zeroth and first moments of the Boltzmann equation lead to particle conservation and energy-momentum conservation equations respectively. The higher moments, being sensitive to the high-momentum end of the distribution function also give independent equations in terms of higher order tensors. In the next section we present our attempts at calculating the second-order transport coefficients for the evolution of the shear stress tensor starting from the general relativistic Boltzmann equation.

Chapter 2

Transport coefficients

Having seen the scheme of deriving hydrodynamics from a microscopic theory, we now know that the parameters of the microscopic theory used will appear as coefficients of hydrodynamical quantities. Thus for example, the coefficient of viscosity will have an expression in terms of the microscopic parameters. We attempt at calculating these relations in the following section.

2.1 Attempts at calculating second order transport coefficients

2.1.1 From derivatives of the first order

We wish to calculate hydrodynamical quantities from distribution function up to first order (spacetime derivative) expansion around the equilibrium distribution function, $f \equiv f(u^\mu, T)$. We introduce the notation, $\delta f = (f - f_{eq})$ for the Chapman-Enskog (CE) like expansion around the equilibrium distribution function. In the CE expansion, one expands the distribution function in powers of spacetime derivatives around the equilibrium value. When this expansion contains terms which are first order in spacetime derivatives, we will call it δf_1, and δf_2 when the terms are second order in derivatives with respect to the spacetime coordinates. Since δf_1 is a scalar, the most general expression for δf_1 can be written with the scalars constructed out of first order derivatives of T and U^μ, which are listed as follows:

Table 2.1: First order derivatives of fluid degrees of freedom.

Type	Independent terms
Scalars	$\theta = \nabla_\mu u^\mu, \qquad \dot\beta = (u_\mu \nabla^\mu)\beta = \beta\theta/3$
Vectors	$\nabla_\mu^\perp \beta = \triangle_\mu^\alpha \nabla_\alpha \beta = -\beta\dot u_\mu, \qquad a_\mu \equiv \dot u_\mu$
Rank-two Tensors	$\sigma_{\mu\nu} = \triangle^{\mu\alpha}\triangle^{\nu\beta} \left[\frac{1}{2}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{1}{3}\theta g_{\alpha\beta} \right],$
	$\omega_{\mu\nu} = \frac{1}{2} \triangle^{\mu\alpha}\triangle^{\nu\beta} \left(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha \right)$

Notation

Throughout the text, 'dot' over a quantity represents its derivative with respect to time, or the action of the operator $(u^\mu \nabla_\mu)$ both of which are same (∇_μ is covariant derivative). The two-index projection operator $\triangle^{\mu\nu}$ is defined as

$$\triangle^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu$$

and the four-index projection operator orthogonal to the four-velocity as well as the two-index projection operator is given as:

$$\triangle_{\mu\nu}^{\alpha\beta} := \frac{1}{2} \left(\triangle_\alpha^\mu \triangle_\beta^\nu + \triangle_\alpha^\nu \triangle_\beta^\mu - \frac{2}{3} \triangle_{\mu\nu} \triangle^{\alpha\beta} \right)$$

Out of the five terms given in table 2.1, only three are independent owing to the equation of motion (EoM) $\nabla_\mu T^{\mu\nu} = 0$ that allows us to express the β derivatives in terms of first order derivatives of u^μ. Hence the most general first order deviation from the equilibrium distribution function is:

$$\delta f_1 = f_{eq} \left(a\theta + b p^\mu \dot u_\mu + c p^\mu p^\nu \sigma_{\mu\nu} \right) \tag{2.1}$$

Where a, b, c are scalar functions of $(u \cdot p)$ and β. Substituting the above form into the following Boltzmann equation (with the derivatives w.r.t. spacetime retained up

to first order)

$$\underbrace{p^\mu \partial_\mu f_{eq}}_{A} - \underbrace{\Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f_{eq}}{\partial p^\mu}}_{B} = \frac{u \cdot p}{\tau_R} \delta f_1 \tag{2.2}$$

Note that the right hand side is entirely dependent on the equilibrium distribution function f_{eq} only. This is because first order spacetime derivative enter through the $\partial_\mu f_{eq}$ and Christoffel terms.

Term A $p^\mu \partial_\mu f_{eq} = -p^\mu f_{eq} ((u \cdot p)\partial_\mu \beta + \beta p^\nu \partial_\mu u_\nu)$

Converting the partial derivatives ∂_μ to covariant derivatives ∇_μ and using (1.4), we obtain:

$$p^\mu \partial_\mu f_{eq} = -p^\mu f_{eq} \left[(u \cdot p)(u_\mu \dot\beta + \nabla^\perp_\mu \beta) + \beta p^\nu \left(\triangle_\mu u_\nu + \Gamma^\rho_{\mu\nu} u_\rho \right) \right]$$

$$= -f_{eq} \left[(u \cdot p)^2 \frac{\beta\theta}{3} + \beta p^\mu p^\nu \nabla^\perp_\mu u_\nu + \beta p^\mu p^\nu \Gamma^\rho_{\mu\nu} u_\rho \right]$$

Term B $\Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f_{eq}}{\partial p^\mu} = \beta u_\mu \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma$

Combining terms A and B with (2.1) in 2.2,

$$\delta f_1 = \frac{\tau_R f_{eq}}{u \cdot p} \left[(u \cdot p)^2 \frac{\beta\theta}{3} + \beta p^\mu p^\nu \nabla^\perp_\mu u_\nu \right]$$

With the identity

$$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{\theta}{3} \triangle_{\mu\nu} + u_\mu (u \cdot \nabla) u_\nu \qquad \text{and the fact that} \qquad p_\mu p^\mu = 0$$

we get:

$$\delta f_1 = \frac{\tau_R f_{eq}}{u \cdot p} [\beta p^\mu p^\nu \sigma_{\mu\nu}] \tag{2.3}$$

We calculate the time evolution of the shear stress tensor in the following manner:

$$\dot\Pi^{<\mu\nu>} = \triangle^{\mu\nu}_{\alpha\beta} \int dP p^\alpha p^\beta (u \cdot \nabla)(\delta f_1) \tag{2.4}$$

This approach fails to give any curvature terms (like Ricci scalar, Ricci tensor, Riemann tensor or their contractions with fluid four velocity). But these are expected as shown in a previous study by S. Bhattacharyya.[5] Hence we attempt the calculation in yet another way.

2.1.2 From Second Order

Adopting an approach similar to that in the previous section, we list all possible second order terms, scalars or tensors that can be constructed from derivatives of T and U^μ, and that from the curvature of spacetime. Here we closely follow the development by S. Bhattacharyya.[5]

Table 2.2: Second order derivatives of fluid degrees of freedom.

Type	All terms, before imposing EoMs	Independent terms
Scalars	$(u^\mu \nabla_\mu)\theta, \quad \nabla^2\beta,$ $u^\mu u^\nu \nabla_\mu \nabla_\nu \beta$	$(u^\mu \nabla_\mu)\theta$
Vectors	$\triangle^{\mu\nu} (u^\rho \nabla_\rho)a_\nu, \quad \triangle^{\mu\nu} \nabla^2 u_\nu,$ $\triangle^{\mu\nu} \nabla_\nu\theta, \quad \triangle^{\mu\nu} \nabla_\nu\dot\beta$	$\triangle^{\mu\nu} \nabla_\nu\theta, \quad \triangle^\mu_\alpha \nabla_\beta \sigma^{\alpha\beta}$
Rank-two Tensors	$\triangle^{\mu\alpha}\triangle^{\nu\beta} (u^\mu \nabla_\mu)\sigma_{\alpha\beta},$ $\nabla_{<\mu}\nabla_{\nu>}\beta$	$\triangle^{\mu\alpha}\triangle^{\nu\beta} (u^\mu \nabla_\mu)\sigma^{\alpha\beta}$

The equations of motion (EoM) used to get rid of the inter-dependencies among the terms of column-two, are:

$$u_\nu(u \cdot \nabla)\nabla_\mu T^{\mu\nu} = 0 \quad \implies \quad u^\nu u^\mu (\nabla_\nu \nabla_\mu \beta) = \frac{\theta^2 \beta}{9} + \frac{\dot\theta \beta}{3} + a^2 \beta$$

$$\nabla_\mu \nabla_\nu T^{\mu\nu} = 0 \quad \implies \quad \nabla^2 \beta = \frac{\theta^2 \beta}{9} - \frac{2}{3}\beta\dot\theta + \frac{7}{3}\beta a^2 - \beta(\sigma^2 + \omega^2)$$

Similarly,

$$\triangle^\mu_\alpha (u \cdot \nabla)\nabla_\nu T^{\alpha\nu} = 0 \qquad u_\alpha \triangle^{\beta\mu} \nabla_\beta \nabla_\nu T^{\alpha\nu} = 0$$

Table 2.3: Second order curvature terms

Type	Independent terms
Scalars	$R \equiv R^{\mu\nu} g_{\mu\nu}, \quad R_{00} \equiv u_\mu u_\nu R^{\mu\nu}$
Vectors	$\triangle^{\mu\alpha} R^{\alpha\beta} u_\beta$

$$\text{Rank-two Tensors} \ \bigg| \ R_{<\mu\nu>}, \quad F_{<\mu\nu>} \equiv u^\mu u^\nu R_{\alpha\mu\beta\nu}$$

Table 2.4: Second order terms constructed out of combinations of two first order terms

Type	Independent terms
Scalars	θ^2, a^2, ω^2, σ^2
Vectors	θa^ν, $a_\nu \omega^{\mu\nu}$, $a_\nu \sigma^{\mu\nu}$
Rank-two Tensors	$\theta\sigma_{\mu\nu}$, $\sigma^\alpha_{<\mu}\sigma_{\alpha\nu>}$, $\omega^\alpha_{<\mu}\sigma_{\alpha\nu>}$, $\omega^\alpha_{<\mu}\omega_{\alpha\nu>}$, $a_{<\mu}a_{\nu>}$

Combining the independent terms from table (2.2), (2.3) and (2.4), one can construct the most general second order derivative expansion around the equilibrium distribution function as follows:

$$\delta f_2 = f_{eq}[a_2\theta^2 + b_2 a^2 + c_2\omega^2 + d_2\sigma^2 + e\theta p^\nu \dot{u}_\nu + g p^\mu \dot{u}^\nu \omega_{\mu\nu} + h p^\mu \dot{u}^\nu \sigma_{\mu\nu} + j p^\mu p^\nu \theta\sigma_{\mu\nu} + k p^\mu p^\nu \sigma^\alpha_\mu \sigma_{\alpha\nu}$$

$$+ l p^\mu p^\nu \omega^\alpha_y o\mu \sigma_{\alpha\nu} + m p^\mu p^\nu \omega^\alpha_\mu \omega_{\alpha\nu} + n p^\mu p^\nu a_\mu a_\nu + q(u\cdot\nabla)\theta + r p^\mu \bigtriangleup^\alpha_\mu \nabla_\alpha \theta + s p^\mu \bigtriangleup_{\alpha\mu} \nabla_\beta \sigma^{\alpha\beta}$$

$$+ t p^\mu p^\nu \bigtriangleup^\alpha_\mu \bigtriangleup^\beta_\nu (u\cdot\nabla)\sigma_{\alpha\beta} + vR + w u^\mu u^\nu R_{\mu\nu} + x p^\mu \bigtriangleup^\alpha_\mu R_{\alpha\beta} u^\beta + y p^\mu p^\nu R_{\mu\nu} + z p^\mu p^\nu F_{\mu\nu}].$$

$$(2.5)$$

Where the coefficients a to z are scalars yet to be determined. Up to second order, the Boltzmann equation (1.5) looks like:

$$p^\mu \partial_\mu (f_{eq} + \delta f_1) - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial(f_{eq} + \delta f_1)}{\partial p^\mu} = \frac{u\cdot p}{\tau_R}(\delta f_1 + \delta f_2) \qquad (2.6)$$

The first order terms 'A' and 'B' were calculated in the previous section, we calculate the second order terms on the left hand side of (2.6).

$$p^\mu \partial_\mu(\delta f_1) = (p^\mu \partial_\mu f_{eq})\left[a_1\theta + b_1 p^\nu \dot{u}_\nu + c_1 p^\mu p^\nu \sigma_{\mu\nu}\right] + f_{eq} p^\mu \partial_\mu \left(a_1\theta + b_1 p^\nu \dot{u}_\nu + c_1 p^\mu p^\nu \sigma_{\mu\nu}\right)$$

Converting partial derivatives to covariant derivatives wherever needed, and using the form of the equilibrium distribution function f_{eq} from (1.4), we obtain:

$$p^\mu \partial_\mu(\delta f_1) = -f_{eq}\beta \left[a_1\theta p^\mu p^\nu \sigma_{\mu\nu} + b_1 p^\nu \dot{u}_\nu p^\mu p^\nu \sigma_{\mu\nu} + c_1 p^\mu p^\nu \sigma_{\mu\nu} p^\sigma p^\rho \sigma_{\sigma\rho} + \gamma a_1\theta + \gamma b_1 p^\nu \dot{u}_\nu \right.$$

$$+ \gamma c_1 p^\mu p^\nu \sigma_{\mu\nu} + f_{eq}\theta p^\mu(\partial_\mu a_1) + f_{eq} p^\mu p^\nu \dot{u}_\nu(\partial\mu b_1) + f_{eq} p^\mu p^\nu \sigma_{\mu\nu}(\partial_\mu c_1)$$

$$+ f_{eq} p^\mu \left[a_1 u_\mu(u \cdot \nabla)\theta + a_1 \triangle_\mu^\alpha \nabla_\alpha \theta \right]$$

$$+ b_1 f_{eq} p^\mu p^\nu \left[\sigma_\mu^\alpha \sigma_{\alpha\nu} + 2\sigma_\mu^\alpha \omega_{\alpha\nu} + \frac{2}{3}\theta\sigma_{\mu\nu} + \omega_\mu^\alpha \omega_{\alpha\nu} + \frac{\theta^2}{9} \triangle_{\mu\nu} \right]$$

$$+ b_1 f_{eq} p^\mu p^\nu \left[u_\mu \dot{u}^\nu \sigma_{\mu\nu} + u_\mu \dot{u}^\nu \omega_{\mu\nu} + \frac{\theta}{3} u_\mu \dot{u}^\nu + R_{\theta\nu\beta\mu} u^\theta u^\beta + (u \cdot \nabla)\sigma_{\mu\nu} \right]$$

$$+ b_1 f_{eq} p^\mu p^\nu \left[\frac{\triangle_{\mu\nu}}{3}(u \cdot \nabla)\theta + a_\mu a_\nu + u_\mu(u \cdot \nabla)u_\nu + \Gamma_{\nu\mu}^\epsilon \dot{u}_\epsilon \right]$$

$$\left. + c_1 f_{eq} p^\mu p^\sigma p^\rho (\nabla_\mu \sigma_{\rho\sigma} + 2\Gamma_{\rho\nu}^\epsilon \sigma_{\epsilon\sigma}) \right]$$

$$. \quad (2.7)$$

Where γ is shorthand for $u_\mu \Gamma_{\rho\sigma}^\mu p^\rho p^\sigma$. The momentum derivative term in (2.6) contributes the following:

$$\frac{\partial(\delta f_1)}{\partial p^\mu} = -f_{eq}\beta u_\mu (a_1\theta + b_1 p^\nu \dot{u}_\nu + c_1 p^\mu p^\nu \sigma_{\mu\nu}) + f_{eq}(b_1 \dot{u}_\mu + 2c_1 p^\nu \sigma_{\mu\nu})$$

$$+ f_{eq}(\theta \frac{\partial a_1}{\partial p^\mu} + p^\nu \dot{u}_\nu \frac{\partial b_1}{\partial p^\mu} + p^\rho p^\sigma \sigma_{\rho\sigma} \frac{\partial c_1}{\partial p^\mu}) \quad (2.8)$$

If we plug the expressions (2.7) and (2.8) into (2.6) and compare coefficients of independent terms on the left hand side (of 2.6) to the unknown coefficients on the right hand side (from the δf_1 and δf_2 expressions), we obtain:

LHS	RHS
$-\beta p^\mu p^\nu \sigma_{\mu\nu} f_{eq} p^\mu p^\nu \sigma_{\mu\nu} (\partial_\mu c_1) + p^\rho p^\sigma \sigma_{\rho\sigma} \frac{\partial c_1}{\partial p^\mu}$	$-\frac{(u \cdot p)}{\tau_R} c_1$
$p^\mu \partial_\mu a_1 - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial a_1}{\partial p^\mu}$	$\frac{u \cdot p}{\tau_R} a_1$
$p^\mu \partial_\mu b_1 - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial b_1}{\partial p^\mu}$	$\frac{u \cdot p}{\tau_R} b_1$
.	
.	
.	

Table 2.5: Equations obtained by comparing both sides of (2.6)

We note from equation (2.7) that the coefficient of curvature term viz. the Riemann tensor is b_1, which is the coefficient of the acceleration term $\dot{u}_\mu \equiv a_\mu$ in the general expression for δf_1. Since all terms are mutually independent, we conclude that the only way to obtain curvature data is to have double derivatives of velocity like terms. The above table highlights three relations obtained by comparing coefficients of identical terms on either side of (2.6). There are as many equations as there are independent terms and they solve the system of equations with $b_1 = 0$. This implies that the presented calculations give a zero coefficient for the curvature terms.

Spotting the problem We came to know of the article by P. Romatschke et al.[6] where they discuss the properties of the RTA collision kernel. It is pointed out that RTA collision kernel has symmetries which do not allow any anti-symmetric tensor to be calculated from the Boltzmann equation. It is worthwhile to note that just like the anti-symmetric $\omega_{\mu\nu}$, the Riemann tensor is also anti-symmetric in its pair of indices. We therefore conclude this attempt by noting that a different choice of the collision kernel which doesn't have such symmetries might allow us to calculate the terms sought here.

Chapter 3

Anisotropic hydrodynamics

The fireball produced in the relativistic heavy-ion collision experiments has been analysed with scrutiny and it is now established that a very hot-dense plasma of quarks and gluons is indeed formed. The initial strongly interacting phase enjoys free dynamics due to asymptotic freedom and is described by relativistic ideal hydrodynamics. Since the framework of hydrodynamics is based on the assumption of local thermalization, and it successfully describes the evolution of quark-gluon plasma (QGP), it was believed that QGP achieves thermalization very quickly (on a timescale of $1fm/c$). However, dissipative hydrodynamic descriptions and QCD calculations showed that it is not possible to achieve thermalization in sub-fm/c scale. It was then suggested that thermalization is not needed and anisotropic hydrodynamics formalism can explain the QGP evolution by taking into account the large momentum-space anisotropy initially present in the QGP. The success of anisotropic hydrodynamics (henceforth referred as aHydro) at explaining the evolution of QGP without resorting to thermalization piques our interest in this formalism and motivates further exploration.

3.1 Brief Introduction to aHydro

In the derivation of aHydro equations, one starts with momentum space anisotropy incorporated into the one-particle distribution function. For example, for the simple pedagogical case of axis-symmetric longitudinal expansion of fluid, one assumes

$f(x, p)$ as prescribed by P. Romatschke and M. Strickland:[7]

$$f(x, p) = f_{\text{eq}} \left(\frac{\sqrt{\mathbf{p}^2 + \xi(x) p_z^2}}{\Lambda(x)}, \frac{\mu(x)}{\Lambda(x)} \right) \tag{3.1}$$

The moments of this distribution function substituted into the Boltzmann equation, provide the evolution equation for the anisotropy parameter $\xi(x)$ and the scaled temperature $\Lambda(x)$.

It was recognised by A. Jaiswal and A. Dash[8] that collisionless evolution of a conformal fluid in an anisotropic metric background, can lead to the same aHydro equations of the axis-symmetric case. This happens because the free-streaming particles adopt the anisotropies introduced by the metric. We wish to extend this work in presence of a non-zero collision kernel. The subsequent sections contain our attempts in this direction.

3.2   Our Setup

We work with a general Bianchi type-I metric where a line element looks like: $ds^2 = dt^2 - g_{ij} dx^i dx^j$. With a change of coordinates, the symmetric spatial part can be diagonalized and without loss of generality, we can take our background metric to be:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -A^2(t) & 0 & 0 \\ 0 & 0 & -B^2(t) & 0 \\ 0 & 0 & 0 & -C^2(t) \end{bmatrix} \tag{3.2}$$

where $A(t)$, $B(t)$, and $C(t)$ are scale factors dependent on time only. The determinant will be useful in defining the integral measure while taking moments of the Boltzmann equation. $\sqrt{-g} := \sqrt{-det(g_{\mu\nu})} = ABC$. The assumed metric has only six non-zero Christoffel symbols:

$$\Gamma^x_{xt} = \Gamma^x_{tx} = \frac{\dot{A}}{A}, \qquad \Gamma^y_{yt} = \Gamma^y_{ty} = \frac{\dot{B}}{B}, \qquad \Gamma^z_{tz} = \Gamma^z_{zt} = \frac{\dot{C}}{C}$$

Our setup consists of a transversely symmetric fluid, hence $A(t) = B(t)$. With the prescribed metric and transverse homogeneity, the Boltzmann equation (1.5) takes the form:

$$p^\mu \partial_\mu f - 2p^0 \left(\frac{\dot{A}}{A} p^x \frac{\partial f}{\partial p^x} + \frac{\dot{A}}{A} p^y \frac{\partial f}{\partial p^y} + \frac{\dot{C}}{C} p^z \frac{\partial f}{\partial p^z} \right) = -\frac{u \cdot p}{\tau_R} (f - f_{eq}) \qquad (3.3)$$

Next we define the basis vectors as follows:

$$u^\mu = (1, 0, 0, 0)$$

$$X^\mu = (0, 1/A(t), 0, 0)$$

$$Y^\mu = (0, 0, 1/A(t), 0)$$

$$Z^\mu = (0, 0, 0, 1/C(t))$$

Note that the basis vectors scale as metric expands or contracts in the respective directions. This is to be contrasted with the choice of basis where $X^\mu = (0, 1, 0, 0)$, $Y^\mu = (0, 0, 1, 0)$ and so on. The later choice introduces a co-moving coordinate frame where the observer is oblivious to the dynamics of the metric. Our choice of basis vector helps set up a proper frame where the observer is an asymptotic observer (assuming that the metric is asymptotically flat). Any tensor of any rank can be constructed out of these four vectors defined above. For example, it can be checked that the metric tensor can be decomposed as:

$$g^{\mu\nu} = u^\mu u^\nu - \sum_{i=1}^{3} X_i^\mu X_i^\nu$$

The general form of the current and energy-momentum tensor can be given as follows, where we only use the symmetric nature of these tensors.

$$J^\mu = n u^\mu + n^i X_i^\mu$$

$$T^{\mu\nu} = t_{00} g^{\mu\nu} + \sum_{i=1}^{3} t_{ii} X_i^\mu X_i^\nu + \sum_{i,j\neq 0, i>j} t_{ij} X_i^\mu X_j^\nu \qquad (3.4)$$

The spatial components of the current vanishes because of the spatial reflection symmetry, and therefore:

$$J^\mu = nu^\mu \tag{3.5}$$

Name the z axis as the longitudinal axis, and the xy plane, as the transverse plane then in the frame of a stationary (with respect to the fluid) asymptotic observer, the energy-momentum tensor assumes the following form:

$$T^{\mu\nu}_{asymp} = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_T/A^2(t) & 0 & 0 \\ 0 & 0 & P_T/A^2(t) & 0 \\ 0 & 0 & 0 & P_L/C^2(t) \end{bmatrix} \tag{3.6}$$

Where ϵ is the energy density, P_T and P_L are the transverse and longitudinal pressures. This motivates the following tensor decomposition for $T^{\mu\nu}$.

$$T^{\mu\nu} = (\epsilon + P_T)u^\mu u^\nu - P_T g^{\mu\nu} + (P_L - P_T)Z^\mu Z^\nu \tag{3.7}$$

To get a physical insight, imagine an expanding metric, that it $A(t)$, $B(t)$ and $C(t)$ are increasing functions of time. Expression (3.7) says that the pressures, as measured by the asymptotic observer, will decrease with time. This is expected because the same amount of radiation is now present in a larger volume. The above form can also be derived by imposing the transverse homogeneity on the general form given in (3.4). In the next few subsections we concern ourselves with taking the zeroth, first and second moments of the Boltzmann equation (3.3). The invariant integral measure is:

$$\int dP \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\sqrt{-g}}{p^0}$$

Note that the integral measure now has a coordinate dependent term $\sqrt{-g}$. Before we embark on taking moments of the Boltzmann equation, calculations for a some of the thermodynamical quantities are in order:

Number density, n defined as:

$$n(\xi, \Lambda) \equiv \int \sqrt{-g} \frac{d^3 p}{(2\pi)^3} f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x) p_z^2} / \Lambda(x) \right)$$

After a change in parameter $p_z'^2 = (1+\xi) p_z^2$, the above expression for number density reads:

$$n(\xi, \Lambda) = \frac{\sqrt{-g}}{\sqrt{1+\xi}} \int \frac{d^3 p}{(2\pi)^3} f_{\text{eq}}(|\mathbf{p}|/\Lambda(x)) \implies n(\xi, \Lambda) = \frac{\sqrt{-g}}{\sqrt{1+\xi}} n_{\text{eq}}(\Lambda) \qquad (3.8)$$

Energy density, ϵ defined as

$$\epsilon = \int dP E^2 f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x) p_z^2} / \Lambda(x) \right)$$

$$= \int \sqrt{-g} \frac{d^3 p}{(2\pi)^3} \sqrt{p_x^2 + p_y^2 + p_z^2} f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x) p_z^2} / \Lambda(x) \right)$$

After introducing parameter change similar to what was done for n, but in spherical coordinates and we also get another term under a square root.

$$= \frac{\sqrt{-g}}{\sqrt{1+\xi}} \int \frac{d^3 p}{(2\pi)^3} |\mathbf{p}| \sqrt{\sin^2 \theta + \frac{\cos^2 \theta}{1+\xi}} f_{\text{eq}}(|\mathbf{p}|/\Lambda(x))$$

The integral over θ does not involve f_{eq}, so we take it out and multiply and divide by another $\int d\theta$ factor (this way integral over d^3p remains, and we get another 1/2 factor)

$$\epsilon = \left(\frac{\sqrt{-g}}{2\sqrt{1+\xi}} \int d(\cos\theta) \sqrt{\sin^2 \theta + \frac{\cos^2 \theta}{1+\xi}} \right) \int \frac{d^3 p}{(2\pi)^3} |\mathbf{p}| f_{\text{eq}}(|\mathbf{p}|/\Lambda(x)) \qquad (3.9)$$

We write the equation (3.9) in a simpler form: $\epsilon = \mathcal{R}(\xi) \epsilon_{\text{eq}}(\Lambda)$, where $\mathcal{R}(\xi) = \frac{1}{2} \left[\frac{1}{1+\xi} + \frac{\arctan \sqrt{\xi}}{\sqrt{\xi}} \right]$

Following a similar procedure, one can get

$$P_T = \mathcal{R}_T(\xi) P_{\text{eq}}(\Lambda)$$

$$P_L = \mathcal{R}_L(\xi) P_{\text{eq}}(\Lambda)$$

$$(3.10)$$

16

where $\mathcal{R}_T(\xi) = \frac{3}{2\xi}\left[\frac{1+\left(\xi^2-1\right)\mathcal{R}(\xi)}{\xi+1}\right]$, $\mathcal{R}_L(\xi) = \frac{3}{\xi}\left[\frac{(\xi+1)\mathcal{R}(\xi)-1}{\xi+1}\right]$, and $P_{eq}(\Lambda) = \epsilon/3$ owing to the equation of state for conformal fluid.

3.3 Moments of the General Relativistic Boltzmann Equation

We define the nth moment of the collision kernel, $\mathcal{C}[f]$ for a given r,

$$\mathcal{C}_r^{\mu_1\mu_2\cdots\mu_n} \equiv -\int dP(p.u)^r p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}\mathcal{C}[f] \tag{3.11}$$

As is well known, the first moment of the Boltzmann equation gives energy-momentum conservation. While evaluating the first moment of the Boltzmann equation with collision kernel given by RTA, one has to impose the conservation by hand. This amounts to the condition

$$\mathcal{C}_0^\nu = \int dP p^\nu \frac{u \cdot p}{\tau_R}\left(f_{eq}(\xi,\Lambda) - f_{eq}(T)\right) = 0 \tag{3.12}$$

While, owing to the symmetries of $f(x,p)$, the above integral vanishes for $\mu = 1,2,3$, the vanishing of $\mathcal{C}_0^0$ gives the following condition:

$$T = \mathcal{R}^{1/4}(\xi)\Lambda \tag{3.13}$$

This relation can be obtained as follows:

$$\mathcal{C}_0^0 = \frac{\sqrt{-g}}{\tau_R}\int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\left(f_{eq}(\xi,\Lambda) - f_{eq}(T)\right) = 0 \tag{3.14}$$

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp\left(\sqrt{\mathbf{p}^2 + \xi p_z^2}/\Lambda\right) + a} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp(|\mathbf{p}|/T) + a}$$

The form we get is same as (3.9), hence with appropriate substitutions ($p_z'^2 = (1 + \xi)p_z^2$)), one obtains:

$$\mathcal{R}(\xi)\int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp\left(|\mathbf{p}|/\Lambda\right) + a} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp(|\mathbf{p}|/T) + a} \implies \mathcal{R}(\xi)\Lambda^4 = T^4$$

3.3.1 Zeroth Moment

The 0th moment of the Boltzmann equation is,

$$\partial_\mu \underbrace{\int dP(p^\mu f)}_{J^\mu = nu^\mu} - (\partial_\mu \sqrt{-g}) \int \frac{d^3p}{(2\pi)^3 E}(p^\mu f) - \Gamma^\mu_{\rho\sigma}\int dP p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\underbrace{\int dP \frac{u \cdot p}{\tau_R}(f - f_{eq})}_{\mathcal{C}_0}$$
(3.15)

Where the second term on the left hand side enters because the integral measure is now coordinate dependent and hence one cannot just pull the partial derivative out of the integral. We have the identity:

$$\partial_\mu \sqrt{-g} = \frac{1}{2}\sqrt{-g}\ g^{\alpha\beta}\partial_\mu g_{\alpha\beta}$$

We use the above result and rewrite (3.15) as:

$$\partial_\mu(nu^\mu) - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)n(\xi, \Lambda) - \Gamma^\mu_{\rho\sigma}\int dP p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\mathcal{C}_0$$

Evaluating $\mathcal{C}_0$ using equations (3.11) and (3.8), one obtains:

$$\mathcal{C}_0 = \int \frac{p.u}{\tau_{eq}}[f - f_{eq}(T)]dP = \frac{n_{\text{eq}}}{\tau_R}\left(\frac{1}{\sqrt{1+\xi}} - \mathcal{R}^{3/4}(\xi)\right)$$

The first term above comes trivially using equation (3.8), but the calculation for the second term involving f_{eq} (writing $T = \mathcal{R}^{1/4}(\xi)\Lambda$ and re-parameterizing p=p'$\mathcal{R}^{1/4}$, in spherical co-ordinates) is as shown below.

$$-\int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0}\frac{p.u}{\tau_R}\frac{1}{\exp(|\mathbf{p}|/T) + a} = -\frac{\mathcal{R}^{3/4}(\xi)}{\tau_R}\int \frac{d^3\mathbf{p}'|p'^2|}{(2\pi)^3}\frac{1}{\exp(|\mathbf{p}'|/\Lambda) + a} = -R^{3/4}(\xi)\frac{n_{eq}}{\tau_R}$$

Note that $\partial_\mu J^\mu = Dn + n\partial_\mu u^\mu$, where $D = u^\mu \partial_\mu$ and $\partial_\mu u^\mu = 0$ in the fluid rest frame. With these substitutions, and the definition of number density from (3.8), the zeroth moment of the Boltzmann equation becomes:

$$\partial_t\left(\sqrt{-g}\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right) + \left(\sqrt{-g}\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = -\sqrt{-g}\left[\frac{1}{\sqrt{1+\xi}} - R^{3/4}(\xi)\right]\frac{n_{eq}(\Lambda)}{\tau_R}$$

Using $\partial_t \sqrt{-g} = \sqrt{-g}\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)$, on the first term on the left

$$\partial_t\left(\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right) + \left(2\frac{n_{eq}(\Lambda)}{\sqrt{1+\xi}}\right)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = -\frac{n_{eq}(\Lambda)}{\tau_R}\left[\frac{1}{\sqrt{1+\xi}} - R^{3/4}(\xi)\right]$$

Again using equation (3.8), we can expand the left-hand-side of the above equation in terms of derivatives of ξ and Λ, this gives us:

$$\frac{1}{1+\xi}\partial_t\xi - \frac{6}{\Lambda}\partial_t\Lambda - 4\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{2}{\tau_R}\left(1 - R^{3/4}(\xi)\sqrt{1+\xi}\right) \tag{3.16}$$

In equation (3.16), the first and third term on the LHS, and the RHS are trivially obtained, but the second term on the LHS ($\frac{1}{\sqrt{1+\xi}}\partial_t n_{eq}(\Lambda) = \frac{6}{\Lambda}\partial_t\Lambda$) needs some explanation:

$$\frac{1}{\sqrt{1+\xi}}\partial_\tau n_{eq}(\Lambda) = \frac{\partial_\tau}{\sqrt{1+\xi}}\int\frac{d^3p}{(2\pi^3)}|p|^2\frac{1}{e^{|p|/\Lambda} + a}$$

$$\frac{\partial_t}{\sqrt{1+\xi}}\int\Lambda^3\frac{d^3y}{(2\pi^3)}|y|^2\frac{e^{-y}}{1+ae^y} \qquad \text{(Using y = p/}\Lambda\text{)}$$

Since only Λ is dependent on time, the rest of the calculation simple.

$$\frac{\partial_t(\Lambda^3)}{\sqrt{1+\xi}}\int\frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x} = \frac{3\Lambda^2\partial_t\Lambda}{\sqrt{1+\xi}}\int\frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x}$$

$$= \frac{3\partial_t\Lambda}{\Lambda\sqrt{1+\xi}}\int\Lambda^3\frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x} = \frac{3n_{eq}\partial_t\Lambda}{\Lambda\sqrt{1+\xi}}$$

3.3.2 First Moment

The 1st moment gives the equation of motion for the energy-momentum tensor,

$$\partial_\mu\int dP p^\mu p^\nu f - \Gamma^\mu_{\rho\sigma}\int dP p^\rho p^\sigma p^\nu \frac{\partial f}{\partial p^\mu} = 0 \tag{3.17}$$

$$T^{\mu\nu} := \int dP p^\mu p^\nu f = (\epsilon + P_T)u^\mu u^\nu - P_T g^{\mu\nu} + (P_L - P_T)Z^\mu Z^\nu$$

When (3.17) is projected along u_ν, one obtains,

$$u_\nu \partial_\mu T^{\mu\nu} = (\dot{\epsilon} + \dot{P}_T) + \underbrace{(\epsilon + P_T)\partial_\mu u^\mu}_{0} + \underbrace{(\epsilon + P_T)u_\nu u^\mu \partial_\mu u^\nu}_{0} - u^\mu \partial_\mu P_T + P_T(\Gamma^\mu_{\mu\alpha} g^{\nu\alpha} + \Gamma^\nu_{\mu\alpha} g^{\mu\alpha})u_\nu$$

$$+ \underbrace{(P_L - P_T)u_\nu Z^\mu \partial_\mu Z^\nu}_{0}$$

Performing a similar manipulation on the first moment equation of $\mathcal{C}_0$, we get a simplified form of energy conservation in $0+1$d, i.e.

$$\frac{\partial \epsilon(\tau)}{\partial \tau} + P_T\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = 0$$

$$\frac{d}{dt}\left(\epsilon_{eq}(\Lambda)R(\xi)\right) + \left[\frac{1}{\xi}\left(R(\xi) - \frac{1}{\xi+1}\right) + R(\xi)\right]\epsilon_{eq}(\Lambda)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = 0$$

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau \xi + \frac{4}{\Lambda}\partial_\tau \Lambda = \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right] \qquad (3.18)$$

3.3.3   Second Moment

The 2nd moment gives the equation of motion for $I^{\mu\delta\gamma}$,

$$\partial_\mu \underbrace{\int dP p^\mu p^\delta p^\gamma f}_{I^{\mu\delta\gamma}} - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\int dP p^0 p^\delta p^\gamma f - \Gamma^\mu_{\rho\sigma}\int dP p^\rho p^\sigma p^\delta p^\gamma \frac{\partial f}{\partial p^\mu}$$

$$= \underbrace{-\int dP p^\delta p^\gamma \frac{u \cdot p}{\tau_R}(f - f_{eq})}_{\mathcal{C}^{\delta\gamma}} \qquad (3.19)$$

Where the second term is a result of the partial derivative acting on the $\sqrt{-g}$ factor. A rank three tensor that respects the symmetries of the system can be written in its most general form in terms of the given basis as:

$$I^{\mu\delta\gamma} = I_u(u^\mu u^\delta u^\gamma) + I_z(u^\mu Z^\delta Z^\gamma + Z^\mu u^\delta Z^\gamma + Z^\mu Z^\delta u^\gamma)$$

$$+ I_y(u^\mu Y^\delta Y^\gamma + Y^\mu u^\delta Y^\gamma + Y^\mu Y^\delta u^\gamma) + I_x(u^\mu X^\delta X^\gamma + X^\mu u^\delta X^\gamma + X^\mu X^\delta u^\gamma)$$

Where

$$I_u = \mathcal{S}_u(\xi)I_{\text{eq}}(\Lambda); \qquad \mathcal{S}_u(\xi) = \frac{3+2\xi}{(1+\xi)^{3/2}}$$

$$I_x = I_y = \mathcal{S}_T(\xi)I_{\text{eq}}(\Lambda); \qquad \mathcal{S}_T(\xi) = \frac{1}{\sqrt{1+\xi}} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \quad I_{\text{eq}}(\Lambda) = \frac{1}{3}\int dP E^3 f_{eq}$$

$$I_z = \mathcal{S}_L(\xi)I_{\text{eq}}(\Lambda); \qquad \mathcal{S}_L(\xi) = \frac{1}{(1+\xi)^{3/2}}$$

zz projection

$$Z_\delta Z_\gamma \partial_\mu I^{\mu\delta\gamma} = \frac{dI_z}{dt}$$

Plugging this into (3.19),

$$\frac{dI_z}{dt} - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\int dP p^0 p^z p^z f - \Gamma^\mu_{\rho\sigma}\int dP p^z p^z p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} = -\int dP p^z p^z \frac{u \cdot p}{\tau_R}(f - f_{eq})$$

$$\frac{d}{dt}\left(\sqrt{-g}\, I_{eq}(\Lambda)S_L(\xi)\right) + \left(\sqrt{-g}\, I_{eq}(\Lambda)S_L(\xi)\right)\left(2\frac{\dot{A}}{A} + 5\frac{\dot{C}}{C}\right) = \frac{\sqrt{-g}}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda)S_L(\xi)\right]$$

$$\frac{d}{dt}\left(I_{eq}(\Lambda)S_L(\xi)\right) + 2\left(I_{eq}(\Lambda)S_L(\xi)\right)\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{1}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda)S_L(\xi)\right] \quad (3.20)$$

xx and yy projection

$$\frac{d}{dt}\left(I_{eq}(\Lambda)S_T(\xi)\right) + \left(I_{eq}(\Lambda)S_T(\xi)\right)\left(\frac{4\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{1}{\tau_R}\left[I_{eq}(T) - I_{eq}(\Lambda)S_T(\xi)\right] \quad (3.21)$$

Using the definitions of the terms S_T, I_{eq} and S_L, obtain:

$$(\log \mathcal{S}_L)' \partial_t \xi + 5\partial_t \log \Lambda + \left(2\frac{\dot{A}}{A} + 3\frac{\dot{C}}{C}\right) = \frac{1}{\tau_R}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_L} - 1\right]$$

$$(\log \mathcal{S}_T)' \partial_t \xi + 5\partial_t \log \Lambda + \left(\frac{4\dot{A}}{A} + \frac{\dot{C}}{C}\right) = \frac{1}{\tau_R}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_T} - 1\right]$$

The equation of motion for the second moment (equation 3.22) is thus obtained from the above equations, after solving further:

$$\frac{1}{1+\xi}\partial_t \xi - 2\left(\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right) + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_R}\xi\sqrt{1+\xi} = 0 \quad (3.22)$$

3.4   Consistency Checks

Checking for behaviour in free streaming limit

The free streaming case can be obtained by putting the $\tau_R \longrightarrow \infty$ in the equation (3.22). In this limit the equation solves exactly to give:

$$1 + \xi = (\frac{C}{A})^2 \tag{3.23}$$

This result matches with the one obtained by A. Jaiswal and A. Dash.[8]

Consistency with flat-space aHydro

<div align="center">Kasner Metric</div>

Restricting ourselves to a subclass of Bianchi type-I metric, we look at the evolution of anisotropy parameter ξ when

$$A = t^a, \ \ B = t^b, \ \ C = t^c \qquad \Longrightarrow \qquad \frac{\dot{A}}{A} = \frac{a}{t}, \ \ \frac{\dot{B}}{B} = \frac{b}{t}, \ \ \frac{\dot{C}}{C} = \frac{c}{t}$$

When this form of the metric is required to satisfy the Einstein's GR field equations, the parameters a, b, c are related to each other via the following equations:

$$a + b + c = 1, \qquad a^2 + b^2 + c^2 = 1$$

Requiring azimuthal symmetry as present in our setup, the only solution to the above set of equations are:

1. $(a, b, c) \equiv (0, 0, 1)$

2. $(a, b, c) \equiv (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$

In Case I, equations (3.22), (3.18) and (3.16) can be seen to reduce to the exact aHydro equations as derived by M. Strickland and M. Martinez,[9] and M. Strickand et al.[10] (see Appendix A for comparison).

Numerical Solution to the aHydro equations

We solve equation (3.22) numerically and evaluate the pressure anisotropy (P_T/P_L) till $\tau = 100 fm/c$. The behaviour of the direct solution of equation (3.22), that is $\xi(\tau)$ is also plotted to show that ξ does eventually converge to zero as expected. The convergence of ξ to zero naturally corresponds to convergence of pressure anisotropy to 1 (that is, the isotropic limit). The time taken for convergence depends on $\bar{\eta}$ among other variables as can be seen in figure (3.1). The plots were generated in python using mid-point method of solving differential equations, with the initial conditions: $T_0 = 0.3 GeV$, $\xi_0 \approx 0$, $\tau = 2.5 fm/c$. It is worthwhile to note that the pressure anisotropy converges to 1, the isotropic limit, at time scales much larger than $1 fm/c$.
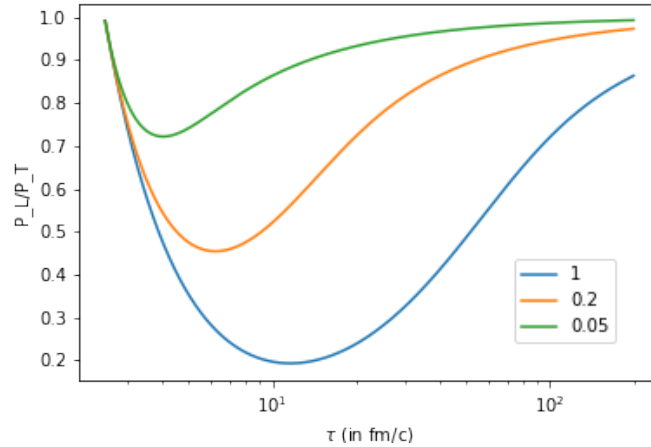

Figure 3.1: The 0+1d pressure anisotropy P_L/P_T predicted by the aHydro equation (3.22). The legend contains the $\bar{\eta}$ values used for each curve. Note that the free streaming limit would be a steep vertical line with negative slope, while the ideal hydro limit $\bar{\eta} \longrightarrow 0$ will be a horizontal line at $P_L/P_T = 1$.

Chapter 4

Misner's ideas and our follow up

4.1 A brief review

We now take a detour and look at some of the ideas presented by C. W. Misner in his 1968 paper titled *The Isotropy of the Universe*.[2] The idea is to separate out metric anisotropy and metric expansion by suitable re-parametrization of the metric. This allows construction of the energy-momentum tensor for a conformal fluid which further allows construction of other dynamical quantities like the shear stress tensor.

4.1.1 Re-parametrization of metric:

As earlier we start with a diagonal Bianchi type-I metric $g_{\mu\nu}$ such that $ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2$. Introduce parametrization as:

$$g_{ij}(t) = e^{2\alpha} \left(e^{2\beta}\right)_{ij}$$

where β is traceless matrix and α is a real scalar. Note that $det(exp(M)) = exp(trace(M))$, hence $g = det(g_{ij}) = exp(6\alpha)det(e^{2\beta})$ which implies:

$$g^{1/2} = e^{3\alpha}.$$

For our case,

$$g_{ij} = \begin{bmatrix} A^2 & 0 & 0 \\ 0 & B^2 & 0 \\ 0 & 0 & C^2 \end{bmatrix} = e^{2\alpha}e^{2\beta}.$$

Which gives the following equations:

$$e^{a+2a} = A^2, \quad e^{b+2\alpha} = B^2, \quad e^{c+2\alpha} = C^2.$$

Solving for a, b and c one obtains:

$$a = \log\left(\frac{A^{4/3}}{B^{2/3}C^{2/3}}\right), \quad b = \log\left(\frac{B^{4/3}}{A^{2/3}C^{2/3}}\right), \quad c = \log\left(\frac{C^{4/3}}{A^{2/3}B^{2/3}}\right), \quad \alpha = \frac{1}{3}\log(ABC)$$

It is imperative to note that the α term captures the net expansion while the other parameters are solely for anisotropy in the system. From energy considerations on a conformal fluid, one obtains the following relation between the metric and T^{00}.

$$T^{00} = T_c^4 \left\langle \left(n^\top e^{-2\beta} n\right)^{1/2} \right\rangle$$

Where $T_c = T_0 e^{-\alpha}$ is called the central temperature and the equation above relates it to the energy density of fluid by averaging the anisotropy over all directions.

$$\left\langle \left(n^\top e^{-2\beta} n\right)^{1/2} \right\rangle = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(n^\top e^{-2\beta} n\right)^{1/2} \sin\theta d\theta d\varphi$$

Where, $n = [\sin\theta\cos\varphi, \ \sin\theta\sin\varphi, \ \cos\theta]^\top$,

$$n^\top e^{-2\beta} n = n^\top \begin{bmatrix} e^{-a} & 0 & 0 \\ 0 & e^{-b} & 0 \\ 0 & 0 & e^{-c} \end{bmatrix} \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix}$$

$$= [\sin\theta\cos\varphi, \ \sin\theta\sin\varphi, \ \cos\theta] \begin{bmatrix} e^{-a}\sin\theta\cos\varphi \\ e^{-b}\sin\theta\sin\varphi \\ e^{-c}\cos\theta \end{bmatrix}$$

$$= e^{-a}\sin^2\theta\cos^2\varphi + e^{-b}\sin^2\theta\sin^2\varphi + e^{-c}\cos^2\theta$$

For the axis-symmetric case at hand (A = B),

$$\left\langle \left(nTe^{-2\beta} n\right)^{1/2} \right\rangle = \frac{1}{4\pi} \int d\Omega \left(e^{-a}\sin^2\theta + e^{-c}\cos^2\theta\right)^{1/2}$$

$$= \frac{1}{2} \int_1^\pi d\theta \sin\theta \left[\left(\frac{C}{A}\right)^{2/3}\sin^2\theta + \left(\frac{A}{C}\right)^{4/3}\cos^2\theta\right]^{1/2}$$

25

Performing the integral, one obtains:

$$\left\langle \left(n^\top e^{-2\beta} n\right)^{1/2} \right\rangle = \frac{1}{2}\left(\frac{A}{C}\right)^{2/3} + \frac{1}{2}\left(\frac{C}{A}\right)^{1/3}\left[\frac{\log\left[\frac{A}{C} + \sqrt{(\frac{A}{C})^2 - 1}\right]}{\sqrt{(\frac{A}{C})^2 - 1}}\right] \equiv 1 + V(\beta)$$

Where $V(\beta)$, termed as anisotropy potential has the following properties and comes handy in capturing the effects of anisotropic metric on a continuous media.

Properties

1. $d\left(g^{1/2}T^{00}\right) = \frac{1}{2}g^{1/2}T_{ij} \cdot dg_{ij} = \frac{1}{2}g^{1/2}\text{Trace}\left(T_- dg^-\right)$,

   where $g^- = e^{-2\alpha}e^{-2\beta} = \begin{bmatrix} 1/A^2 & 0 & 0 \\ 0 & 1/B^2 & 0 \\ 0 & 0 & 1/C^2 \end{bmatrix}$

2. Small β approximation,

   $V(\beta) = \frac{4}{15}\beta_{ab}\beta_{ab} = \frac{2}{5}\left(\beta_+^2 + \beta_-^2\right)$, where β_- and β_+ are the smallest and the largest eigenvalues of β matrix respectively.

3. Eigenvalues of β are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$

4. In the axis-symmetric case, $\beta_3 = \beta_+ = -\frac{2}{3}\log\left(\frac{A}{c}\right)$

$$\left.\begin{aligned} \beta_1 &= \frac{1}{2}\beta_+ + \frac{\sqrt{3}}{2}\beta_- = \frac{1}{3}\log\left(\frac{A}{C}\right) \\ \beta_2 &= \frac{1}{2}\beta_+ - \frac{1}{2}\sqrt{3}\beta_- = \frac{1}{3}\log\left(\frac{A}{c}\right) \end{aligned}\right\} \Rightarrow \beta_- = 0$$

Given the parametrization, we and the above properties, we note:

$$\frac{1}{2}dg^- = -g^- d\alpha - e^{-\alpha}e^{-\beta}d\beta e^{-\alpha}e^{-\beta}$$

After we plug-in the expressions for g^- and $\delta\beta = \frac{1}{2}\begin{bmatrix} da & 0 & 0 \\ 0 & db & 0 \\ 0 & 0 & dc \end{bmatrix}$, obtain:

$$g^{-1/2}\delta\left(g^{1/2}T^{00}\right) = -\text{trace}\left(T_{-g^-}\right)d\alpha - \text{trace}\left(e^{-\alpha}e^{-\beta}T_- e^{-\alpha}e^{-\beta}d\beta\right)$$

But, as seen in a previous relation,

$$T^{00} = aT_0^4 g^{-1/2}\left\langle\left(n^\top g^- \eta\right)^{1/2}\right\rangle = aT_c^4\left\langle\left(n^\top e^{-2\beta}n\right)^{1/2}\right\rangle$$

$$\underbrace{\left(T_{ij} - \frac{1}{3}\delta'_i j' T_{kl'}{}^k\right)}_{\text{Traceless part of } T_{\mu\nu}} d\beta_{ij'} = -aT_e^4 dV(\beta)$$

Note that the traceless spatial part of $T_{\mu\nu}$ is essentially the shear stress tensor, and hence:

$$\Pi = -\alpha T_0^4 e^{-4\alpha} dV(\beta)$$

$$(\Pi_{ij}) \equiv \left(T_{ij} - \frac{1}{3}\delta T_{ij}T_{k'}^{k'}\right) = -aT_a^4 e^{-4\alpha}\left(\frac{\partial V(\beta)}{\partial\beta}\right) = \frac{-8}{15}\alpha T_0^4 e^{-4\alpha}\beta_{ij} \qquad (4.1)$$

where the last equality follows from the small-β approximation from property (2) above. At this stage Misner argues that the in a collision-less fluid, the metric anisotropy will be frozen as it is in the momentum space anisotropy. Hence, if we denote the later by β', then $\beta = \beta'$. However, the author then goes on to argue that in presence of collision the two anisotropies will follow an interdependent, relaxation type equation. The author claims this equation to be:

$$\frac{\beta'}{dt} = \frac{\beta}{dt} - \frac{\beta'}{t_C} \qquad (4.2)$$

where t_C is a system dependent time-scale that should be related to the τ_R. Note that the steady state solution for equation 4.2 is $\beta' = t_C \times \frac{d\beta}{dt}$. Using equation 4.2

and 4.1, one can write:

$$\left(T_{ij} - \frac{1}{3}\delta T_{ij}T_{k'}^{k'}\right) = \frac{-8}{15}t_C T^{00}\frac{d\beta}{dt}$$

Recognising the proportionality coefficient as the viscosity, we get

$$\eta = \frac{2}{15}t_c T_0^4 \frac{1}{(A^3 C)}\left[\frac{A}{C} + \frac{\log\left(\frac{A}{C} + \sqrt{\left(\frac{A}{C}\right)^2 - 1}\right)}{\sqrt{\left(\frac{A}{C}\right)^2 - 1}}\right]$$

The idea of introducing anisotropy potential as a dynamical quantity and using it to generate other hydrodynamical quantities seems a plausible one to us. The ad hoc introduction of equation 4.2 is one of the first objection a careful reader may raise. However, it seems to work well enough and we hypothesised that it must be following from some moment of the Boltzmann equation. The next section contains our attempts to derive equation 4.2 from kinetic theory.

4.2 Other Moments

In order to obtain the equation 4.2 from kinetic theory, we list all possible moments and their combinations that might lead to it. Before embarking on this hit-and-trial calculation, we parametrize the momentum space anisotropy in analogy with the metric anisotropy parametrization introduced by Misner.

Parametrization

Note that, for our case of symmetric about the longitudinal axis, the momentum-space one particle distribution function looks like:

$$f_{RS} = \exp\left(-\sqrt{p_x^2 + p_y^2 + (1 + \xi)\,p_z^2}/\Lambda\right) \quad \Rightarrow \quad \exp\left(-\sqrt{p^\mu p^2 \Omega_{\mu\nu}}/\Lambda\right)$$

where we have introduced a matrix $\Omega_{\mu\nu}$ that accounts for the anisotropy. Ω is identity for isotropic medium. Now, taking inspiration from Misner's work, we parametrize Ω which looks like the following for our massless, axis-symmetric case.

$$\Omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+\xi \end{pmatrix} \overset{\text{parametrize}}{=} e^{2\alpha'}\left(e^{2\beta'}\right)$$

where α' is a scalar and β' is a 3×3 traceless matrix and the primes have been introduced to distinguish this parametrization from the metric parametrization in the previous section. If we let $\beta' = diag\ (a',b',c')$, the parameters a', b', c' in the exponent solve to give:

$$a' = b' = 0 \qquad c' = log(1+\xi)$$

We use the generalised form of moments of the Boltzmann equation derived by E. Molnár, H. Niemi and D. H. Rischke.[11] The expression looks like the following:

$$\frac{\partial}{\partial\tau}\left(\hat{I}_{i+j,i,0}\right) + \frac{1}{z}\left[(j+1)\hat{I}_{i+j,j,0} + (i-1)\hat{I}_{i+j,j+2,0}\right] = \hat{C}_{i-1,j} \tag{4.3}$$

where

$$\hat{C}_{i-1,j} = -\frac{1}{\tau_{iq}}\left[\hat{I}_{i+j,j,0} - I_{i+j,j,0}\right]$$

when the net particle number is not conserved, as in our case, we have the relations:

$$\hat{I}_{nrq}^{RS}(\beta_{Rs,\xi}) = I_{nq}(\beta_0)\frac{R_{nrq}(\xi)}{[R_{200}(\xi)]^{(n+2)/4}} \qquad I_{nq} = \frac{(-1)^2}{(2q+1)!!}\left\langle E_{k\psi}^{n-2q}\left(\Delta^{\alpha\alpha\beta_\alpha}k_{kq}\right)^q\right\rangle_0$$

Choosing different values of i and j, we obtain different moments and their projections. These have been summarised up to the fourth moment of the Boltzmann equation in the calculations follow. Some definitions and expressions that shall be

used frequently are listed hereby, the rest can be found in appendix B.

$$\left.\begin{array}{l} \hat{e} = \hat{I}_{200} = e_0 R_{200} \\[2mm] \hat{P}_l = e_0 R_{220} = \hat{I}_{220}^{PS} \\[2mm] \hat{P}_T = P_0 R_{201} = \hat{I}_{201}^{RS} \end{array}\right\} \begin{array}{l} R_{201}(\xi) = \dfrac{3}{2\xi}\left[\dfrac{1}{1+\xi} - (1-\xi)R_{200}(\xi)\right] \\[4mm] R_{200}(\xi) = \mathcal{R}(\xi) = \dfrac{1}{2}\left[\dfrac{1}{1+\xi} + \dfrac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right] \end{array}$$

Zeroth Moment $(i + j) = 1$

For the zeroth moment, there is only one possibility and that is: $i = 1$, $j = 0$. This leads to the equation:

$$\frac{\partial}{\partial z}\left(\hat{I}_{1,0,0}\right) + \frac{1}{\tau}\left(\hat{I}_{1,0,0}\right) = -\frac{1}{\tau_R}(\hat{n} - n_0)$$

First Moment $(i + j) = 2$

The first moment is just the energy-momentum conservation equations. There exists two independent projections as follows:

- $i = 0$, $j = 2$

$$\frac{\partial}{\partial\tau}\left(\hat{P}_l\right) + \frac{1}{\tau}\left(3\hat{P}_l - \hat{I}_{240}\right) = \frac{-1}{\tau_R}\left(\hat{P}_l - P_0\right)$$

- $(i = 2$, $j = 0)$ will give 3.18:

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau\xi + \frac{4}{\Lambda}\partial_\tau\Lambda = \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right]$$

Second Moment $(i + j) = 3$

The second moment ,one of which we have already seen in 3.22, gives rise to 2 equations:

- $i = 1$, $j = 2$

$$\frac{\partial}{\partial r}\left(\hat{I}_{3,2,0}\right) + \frac{1}{r}\left(3\hat{I}_{3,2,0}\right) = -\frac{1}{\tau_R}\left(\hat{I}_{320} - I_{320}\right)$$

- $i = 3$, $j = 0$

$$\frac{\partial}{\partial \tau}\left(\hat{I}_{3,0,0}\right) + \frac{1}{r}\left(\hat{I}_{3,0,0} + 2\hat{I}_{3,2,0}\right) = -\frac{1}{\tau_R}\left(\hat{I}_{3,0,00} - I_{3,00}\right)$$

Third Moment $(i + j) = 4$

The third moment would require more integrals to be calculated and these are listed in the appendix.

- $i = 0$, $j = 4$

$$\frac{\partial}{\frac{\partial}{\partial r}}\left(\hat{I}_{440}\right) + \frac{1}{r}\left(5\hat{I}_{440} - I_{460}\right) = -\frac{1}{\tau_R}\left(\hat{I}_{440} - I_{440}\right)$$

- $i = 2$, $j = 2$

$$\frac{6\partial_\pi \Lambda}{\Lambda} + \frac{R'_{420}}{R_{420}}\partial_\tau \frac{\pi}{4} + \frac{1}{\tau}\left(3 - \frac{R_{uu_0}}{R_{420}}\right) = \frac{-1}{\tau_R}\left(1 - \frac{R^{6/4}}{R_{420}}\right)$$

- $i = 4$, $j = 0$

$$\frac{\partial}{\partial \tau}\left(\hat{I}_{4,0,0}\right) + \frac{1}{2}\left[\hat{I}_{4,0,0} - 3\hat{I}_{420}\right] = \frac{-1}{\tau_R}\left[\hat{I}_{400} - I_{400}\right]$$

Fourth Moment $(i + j) = 5$

- $i = 1$, $j = 4$

$$\frac{\partial}{\partial r}\left(\hat{I}_{540}\right) + \frac{5}{r}\hat{I}_{540} = \frac{-1}{\tau_R}\left(\hat{I}_{540} - I_{540}\right)$$

31

- $i = 3$, $j = 2$

$$\frac{\partial}{\partial z}\left(\hat{I}_{520}\right) + \frac{1}{\tau}\left(3\hat{I}_{520} - 2\hat{I}_{540}\right) = \frac{-1}{\tau_R}\left(\hat{I}_{520} - I_{520}\right)$$

Note that the for odd value of j, the integral vanishes and hence those cases are omitted from the above table. We note that the equations originating as the same moment of the Boltzmann equation gives rise to similar integrals from the collision kernel on the right hand side. For this reason, we combined the various equations listed above with others of the same moment. However none of the moments of the Boltzmann equation listed above give rise to the equation used by Misner 4.2. However, it has not escaped our notice that equation 3.22 does reproduce equation 4.2 in small-ξ limit. But anisotropic hydrodynamics framework doesn't depend on smallness of the anisotropy parameter and hence we conclude that the relaxation type equation used by Misner cannot be derived from kinetic theory. However it contains all the essential features that it should if it is to be used as an input into evolution equation for dissipative quantities. An exemplary calculation is given below where we failed to obtain equation 4.2 by combining 3rd moment equations.

$$\frac{6\partial_\tau\Lambda}{\Lambda} + \frac{R'_{400}(\xi)}{R_{400}(\xi)}\partial_\tau\xi + \frac{1}{\tau}\left(1 - 3\frac{R_{420}}{R_{400}}\right) = -\frac{1}{\tau_{eq}}\left[1 - \frac{R^{6/4}}{R_{400}}\right]$$

$$\frac{6\partial_\tau\Lambda}{\Lambda} + \frac{R'_{420}(\xi)}{R_{420}(\xi)}\partial_\tau\xi + \frac{1}{2}\left(3 - \frac{R_{440}}{R_{420}}\right) = \frac{-1}{\tau_{eq}}\left[1 - \frac{R^{6/4}}{R_{420}}\right]$$

$$\frac{6\partial_\tau\Lambda}{\Lambda} + \frac{R^1_{440}(\xi)}{R_{440}(\xi)}\partial_\tau\xi + \frac{1}{\tau}\left(5 - \frac{R_{460}}{R_{440}}\right) = \frac{-1}{\tau_{\text{eq}}}\left[1 - \frac{R^{6/4}}{R_{440}}\right]$$

If the above equations are to be combined to lead to 4.2, the constant coefficients (a,b,c) they need to be multiplied with, need to satisfy:

$$a + b + c = 0 \qquad\qquad a + 3b + 5c = 0$$

These conditions come from the observation that there are no stray $1/\tau$ like terms in 4.2, and hence the three equations above should add up to give a net-zero coefficient of $1/\tau$. Even after satisfying these constraints, the three equations on suitable combination give:

$$\partial_\tau \xi \left[\frac{R'_{400}}{R_{400}} - 2\frac{2R'_{420}}{R_{420}} + \frac{R'_{440}}{R_{410}} \right] + \frac{1}{2} \left(\frac{-3R_{420}}{R_{400}} + 2\frac{R_{440}}{R_{420}} - \frac{R_{460}}{R_{440}} \right)$$

$$= \frac{-1}{\tau_{eq}} \left[-\frac{R^{6/4}}{R_{400}} + \frac{R^{6/4}}{R_{420}} \times 2 - \frac{R^{614}}{R_{440}} \right]$$

which cannot be reduced to the Misner's equation 4.2. We therefore conclude this part of the project and embark on the next one.

Chapter 5

Attractor

5.1 Existence of attractor

Consider the ξ (the anisotropy parameter) evolution equation obtained in subsection 3.3.3,

$$\frac{\partial_\tau \xi}{1 + \xi} \pm \frac{2}{\tau} + \frac{R^{5/4}(\xi)\xi\sqrt{1 + \xi}}{\tau_R} = 0 \tag{5.1}$$

Consider a change of variables, we intend to go from the variables (ξ, τ) to $(\xi, \bar{\tau})$ because the later set is dimensionless and hence independent of the system parameters.

$$\bar{\tau} = \frac{\tau}{\tau_R} = \frac{\tau T}{5\bar{\eta}} = \frac{\omega}{5\bar{\eta}}$$

Multiplying 5.1 throughout by $(1 + \xi)$ and τ_R,

$$\tau_R \partial_\tau \xi \pm \frac{2(1 + \xi)}{\bar{\tau}} + R^{5/4}(\xi)\,\xi\,(1 + \xi)^{3/2} = 0$$

The first term changes as follows, using chain rule of derivatives:

$$\tau_R \frac{\partial}{\partial \tau} = \tau_R \frac{\partial \omega}{\partial \tau} \frac{\partial}{\partial \omega} = \tau_R \frac{\partial \bar{\tau}}{\partial \tau} \frac{\partial}{\partial \bar{\tau}}$$

From the expression for $\bar{\tau}$ in terms of τ, we have,

$$\tau_R \frac{\partial}{\partial \tau} = \frac{\tau_R}{5\bar{\eta}} \left[T + \tau \frac{\partial T}{\partial \tau} \right] \frac{\partial}{\partial \bar{\tau}} = \left[1 + \frac{\tau}{T} \frac{\partial T}{\partial \tau} \right] \frac{\partial}{\partial \bar{\tau}}$$

Where we used the relation: $\tau_R = \frac{5\bar{\eta}}{T}$ above. Note that we need time-evolution for the temperature field. We shall obtain it as follows:

$$T = \Lambda R^{1/4}(\xi), \qquad \frac{\partial T}{\partial \tau} = (\partial_\tau \Lambda)\, R^{1/4}(\xi) + \frac{\Lambda}{4} R'(\xi)\, (\partial_z \xi)$$

$$\frac{1}{T}\frac{\partial T}{\partial \tau} = \frac{\partial_\tau \Lambda}{\Lambda} + \frac{1}{4}\frac{R'(\xi)}{R(\xi)}\partial_\tau \xi$$

From the first moment 3.18, we have:

$$\frac{4\partial_\tau \Lambda}{\Lambda} + \frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau \xi = \frac{1}{\tau}\left[\frac{1}{\xi(1+\xi)R(\xi)} - \frac{1}{\xi} - 1\right]$$

Combining the above equations, we get:

$$\frac{1}{4}\left[\frac{1}{\xi(1+\xi)R(\xi)} - \frac{1}{\xi} + 3\right]\frac{\partial \xi}{\partial \bar{\tau}} \pm \frac{2(1+\xi)}{\bar{\tau}} + R^{5/4}(\xi)\xi(1+\xi)^{3/2} = 0$$

$$\frac{1}{4}\left[1 - (1+\xi)R(\xi) + 3\xi(1+\xi)R(\xi)\right]\frac{\partial \xi}{\partial \bar{\tau}} \pm \frac{2}{\bar{\tau}}\xi(1+\xi)^2 R(\xi) + R^{9/4}(\xi)\xi^2(1+\xi)^{5/2} = 0$$

$$(5.2)$$

Having obtained the ξ evolution equation in terms of another dimensionless parameter τ, we now solve the equation numerically to find that the equation admits attractor type solutions for the two Kasner cases. This claim can be verified analytically by solving the equation and then performing a perturbation analysis about the solution. This approach seems to be unfeasible because of the complexity and highly non-linear nature of the equation. However, we take the ansatz $\xi = a/\bar{\tau}$ as a plausible approximate solution and perturb the equation about it. This is the method of analysing what is known as Lyapunov stability. Before attempting that, we first look plot the numerical solution, identify the fixed points and also make an attempt at slow-roll expansion of the differential equation.
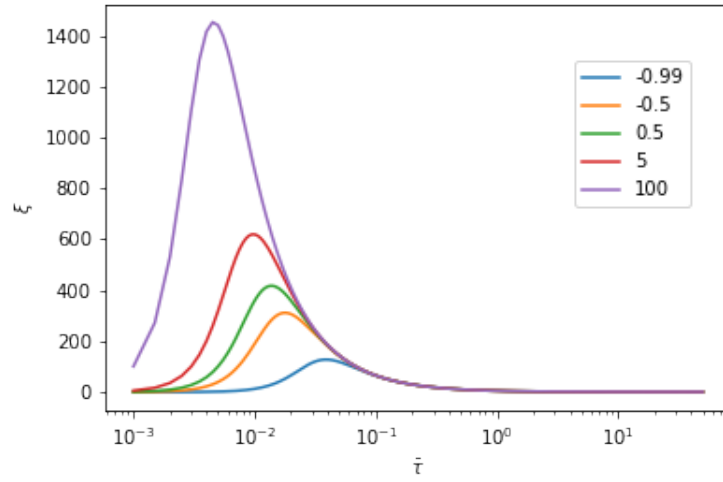
For the first Kasner case $(0, 0, 1)$

Figure 5.1: ξ vs $\bar{\tau}$. The legend contains initial values of the various curves. Notice that higher the initial value earlier the curve merges with the curve coming from ∞ which is the stable fixed point for the Kasner's first case $(0, 0, 1)$.
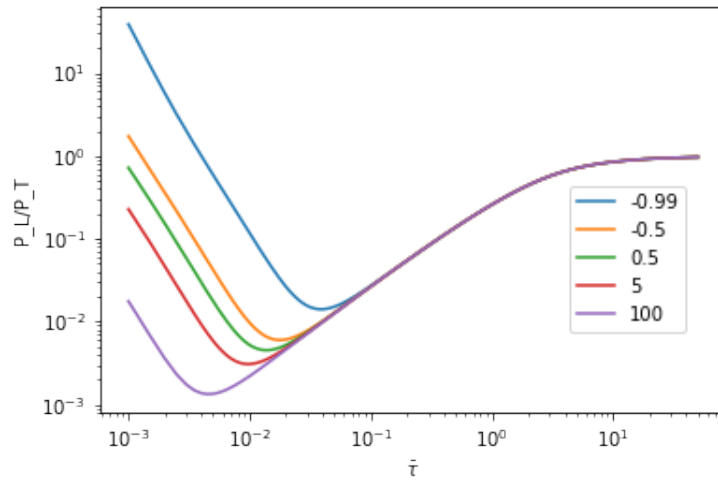

Figure 5.2: Same evolution equation but the derived quantities P_L and P_T are plotted. The ratio approaches 1 as expected.

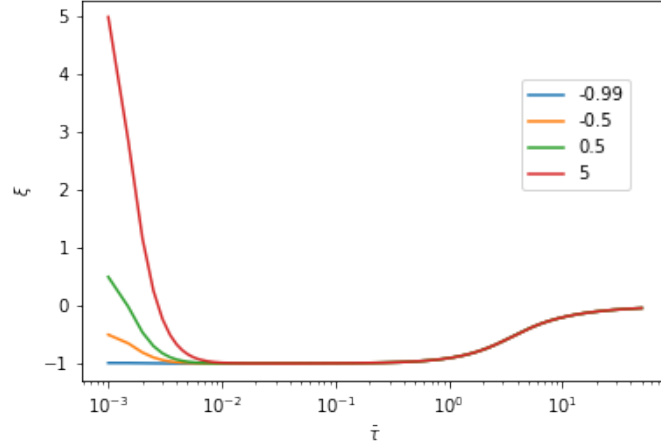**For the second Kasner case** $(2/3, 2/3, -1/3)$**.**

Figure 5.3: Notice the evolution of ξ vs. $\bar{\tau}$, all the curves first reach the stable fixed point -1 and then follow the attractor solution to reach the hydrodynamical equilibrium point $\xi = 0$
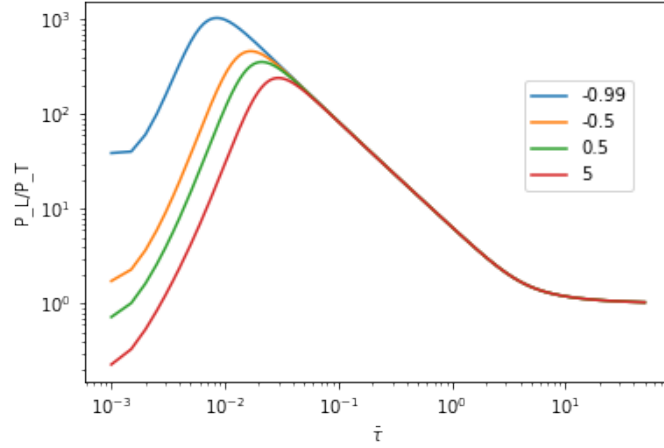

Figure 5.4: Same evolution equation but the derived quantities P_L and P_T are plotted.

5.1.1 Fixed point analysis

Multiply the equation throughout by $\bar{\tau}$ and analyse the roots of the resulting algebraic equation. These are -1, 0, and ∞. We shall see that the ξ evolution given by 5.2

37

has -1 and ∞ as its stable/unstable point depending on which Kasner case we are looking at. Note that the anisotropy parameter is allowed to take values in the range $(-1, \infty)$ but the expression for $R(\xi)$ contains a square root. This is overcome by taking note of the following identity.

$$tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad tan(x) = \frac{sin(x)}{cos(x)} = \frac{e^{ix} - e^{-ix}}{i\left(e^{ix} + e^{-ix}\right)}$$

$\frac{\tan^{-1}(\sqrt{\varepsilon})}{\sqrt{\varepsilon}} - y = \tanh(x) \Rightarrow \tan^{-1}{-1}(-y) = x \quad \Rightarrow \tan^{-1}(iy) = itanh^{-1}(y)$. Hence:

$$R(\xi) = \begin{cases} \dfrac{1}{2}\left[\dfrac{1}{1+\xi} + \dfrac{\tan^{-1}(\sqrt{\xi})}{\sqrt{\xi}}\right] & \text{for } \xi > 0 \\[4mm] \dfrac{1}{2}\left[\dfrac{1}{1+\xi} + \dfrac{\tanh^{-1}(\sqrt{|\xi|})}{\sqrt{|\xi|}}\right] & \text{for } \xi < 0 \end{cases}$$

5.1.2 Slow roll

We envisage that the exact solution to equation 5.2 must be a power series in $1/\bar{\tau}$. The slowest of these terms in the long-time limit must be $a/\bar{\tau}$ where the constant 'a' shall be fixed in the following text. This is because the hydrodynamical equilibrium value for ξ is 0 which rules out any constant terms in the power series solution for 5.2. The ansatz $\xi = a/\tau$ can be fixed by analysing equation 5.2 by approximating it up to second order in the anisotropy parameter ξ. On deploying this technique, equation 5.2 looks like:

$$\frac{2}{3}\left(\xi + \frac{2\xi^2}{5}\right)\bar{\tau}\frac{\partial\xi}{\partial\bar{\tau}} \pm 2\xi + \xi^2\left(\frac{8}{3}\right) = 0 \tag{5.3}$$

Substituting $\xi = \frac{a}{\bar{\tau}}$, we get:

$$-\frac{2}{3}\left(\frac{a}{\bar{\tau}} + \frac{2}{5}\frac{a^2}{\bar{\tau}^2}\right)\frac{a}{\bar{\tau}} \pm \frac{2a}{\bar{\tau}} + \frac{a^2}{\bar{\tau}^2}\left(\bar{\tau} \pm \frac{8}{3}\right) = 0$$

Comparing coefficients of $1/\bar{\tau}$ on both sides, we have:

$$a = \pm 2$$

5.1.3 Lyapunov exponents

We perturb the equation 5.2:

$$\frac{1}{4}\left[1 - (\xi + 1)R(\xi) + 3\xi(1 + \xi)R(\xi)\right] \times \bar{\tau}\frac{\partial \xi}{\partial \bar{\tau}} \pm 2\xi(1 + \xi)^2 R(\xi) + R^{9/4}\xi^2(1 + \xi)^{5/2}\bar{\tau} = 0$$

with $\xi = \xi_0 + \delta\xi$ and solve for $\delta\xi$. This gives:

$$\frac{1}{4}\left[1 - (1 + \xi_0)R(\xi_0) + 3\xi_0 R(\xi_0)(1 + \xi_0)\right]\bar{\tau}\frac{\partial(\delta\xi_0)}{\partial\bar{\tau}}$$

$$+ \frac{1}{4}\left[2R'(\xi_0)\xi_0 + 2R(\xi_0) - R'(\xi_0) + 6R(\xi_0)\xi_0 + 3R'(\xi_0)\xi_0^2\right](\delta\xi)\bar{\tau}\frac{\partial\xi_0}{\partial\bar{\tau}}$$

$$\pm 2(\delta\xi_0)\left[2\xi_0(1 + \xi_0)R(\xi_0) + (1 + \xi_0)^2(R'(\xi_0)\xi_0 + R(\xi_0))\right]$$

$$+ R^{9/4}(\xi_0)(1 + \xi_0)^{5/2}\xi_0^2\left(\frac{9}{4}\frac{R'(\xi_0)}{R(\xi_0)} + \frac{2}{\xi_0} + \frac{5}{2(1 + \xi_0)}\right)\bar{\tau}(\delta\xi) = 0$$

Note that $\xi_0 = \frac{a}{\bar{\tau}}$, $\frac{\partial\xi_0}{\partial\bar{\tau}} = \frac{-a}{\bar{\tau}^2}$. We begin by simplifying the terms by retaining them only up to the first order in $1/\bar{\tau}$. Renaming $\delta\xi$ as y, we have:

$$\left(\frac{2}{3}a\right)\frac{\partial y}{\partial \bar{z}} + \frac{y}{\bar{\tau}}\left(-\frac{2a}{3} \pm \frac{16a}{3} + 3a^2\right) \pm 2y + 2ay = 0 \tag{5.4}$$

One by one, let's consider the two cases of the above equation which correspond to the two Kasner metric solutions.

Case I

$$\frac{\partial y}{\partial \bar{\tau}} = -3y\frac{1 + a}{a} - \frac{y}{\bar{\tau}}\left(7 + \frac{9a}{2}\right)$$

This gives:

$$\log y = \int \frac{-3(1 + a)}{a}d\bar{\tau} + \int -\frac{(7 + 9a/2)}{\bar{\tau}}d\bar{\tau} \implies y = exp\left(-3\frac{1 + a}{a}\bar{\tau}\right) \times \bar{\tau}^{-(7+9a/2)}$$

Hence, for one of the Kasner case ($a = -2$), the perturbation decays as:

$$y = exp(-3\bar{\tau}/2) \times \bar{\tau}^2 \tag{5.5}$$

39

Case II

$$\frac{\partial y}{\partial \bar{\tau}} = -3y\frac{a-1}{a} - \frac{y}{\bar{\tau}}\left(-9 + \frac{9a}{2}\right)$$

This gives:

$$\log y = \int \frac{-3(a-1)}{a}d\bar{\tau} + \int -\frac{(-9+9a/2)}{\bar{\tau}}d\bar{\tau} \implies y = exp\left(-3\frac{a-1}{a}\bar{\tau}\right) \times \bar{\tau}^{(9-9a/2)}$$

Hence, for the other Kasner case $(a = 2)$, the perturbation decays as:

$$y = exp(-3\bar{\tau}/2) \times \bar{\tau}^0 \tag{5.6}$$

Note that the Lyapunov coefficient for both the cases is $-3/2$ because 'a' takes values ± 2 respectively. This is in agreement with the Lyapunov exponent of $-3/2$ obtained by S. Jaiswal et al.[12] We conclude this chapter by remarking that the Lyapunov coefficients of $-3/2$ obtained here are reliable. However, the power laws need to be verified with more scrutiny. This is because of the approximate (only valid for large times) nature of ξ_0 admitted as the solution to equation 5.2.

Chapter 6

Discussion and conclusion

We derived the anisotropic hydrodynamics evolution equations and saw that with suitable choice of metric, they reproduce the aHydro equations of flat space. This feature, we realise, is powerful for it allows one to separate the metric dynamics from the fluid dynamics and analyse either of the problem in the other picture. For example, a fluid undergoing Bjorken flow with transverse homogeneity and longitudinal symmetry is the same as a fluid at rest, immersed in a metric that expands like the Kasner's first case.

The above example naturally incites enquiry into the second Kasner case, which was possible to explore because of the generality of our aHydro evolution equations. We looked into the solution of 5.2 and found that the role of the two fixed points: -1 and ∞ reverses when one goes from the Kasner's first case to the second. Our result is in clear agreement with L. Yan and J. P. Blaizot's[13] claim that an attractor should be defined as the curve which originates at free-streaming initial condition and evolves to the hydrodynamic equilibrium point. In the first Kasner case, this free-streaming initial condition is $\xi = \infty$ because at this value of anisotropy, P_T becomes ∞. It is clear from 5.1 that the curve coming from the ∞ is the stable fixed point. On the other hand the second Kasner case reverses the picture, since here the free-streaming initial condition is $\xi = -1$ because it is this value of ξ at which P_L becomes ∞. The stable fixed point in this case is -1.

The analytical proof for the existence of attractor solution of equation 5.2 is attempted via perturbation analysis. First we note that the large time behaviour of the evolution of the anisotropy parameter ξ is to converge to zero. Hence, in a power series solution of equation 5.2, there must not be any constant term and the large time behaviour must be dominated by the term of the least order in $1/\bar{\tau}$. Recognising this as the solution in the equilibrium regime, a perturbed solution reveals that any perturbation about this solution dies out at a rate governed by a power law and an exponential. The negative exponent of $-3/2$ obtained for both the Kasner cases tells us that the attractor indeed exists and it does for any amount of anisotropy in the system for no assumption about the smallness of ξ was made.

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Appendix A

Equations describing the evolution of the anisotropy parameter as obtained from the aHydro formalism are derived in detail in a review paper by M. Alqahtani, M. Nopoush and M. Strickland.[10] To point out the similarities, we present the final equations here:

Zeroth moment

$$\frac{1}{1+\xi}\partial_\tau\xi - \frac{6}{\Lambda}\partial_\tau\Lambda - \frac{2}{\tau} = \frac{2}{\tau_{\text{eq}}}\left(1 - R^{3/4}(\xi)\sqrt{1+\xi}\right)$$

First moment, projected along u_μ

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau\xi + \frac{4}{\Lambda}\partial_\tau\Lambda = \frac{1}{\tau}\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right]$$

Second moment, zz and xx, yy projections

$$(\log\mathcal{S}_L)'\,\partial_\tau\xi + 5\partial_\tau\log\Lambda + \frac{3}{\tau} = \frac{1}{\tau_{\text{eq}}}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_L} - 1\right]$$

$$(\log\mathcal{S}_T)'\,\partial_\tau\xi + 5\partial_\tau\log\Lambda + \frac{1}{\tau} = \frac{1}{\tau_{\text{eq}}}\left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_T} - 1\right]$$

The last two equations can be combined to eliminate the Λ term, and one obtains:

$$\frac{1}{1+\xi}\partial_\tau\xi - \frac{2}{\tau} + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_{\text{eq}}}\xi\sqrt{1+\xi} = 0$$

Appendix B

Here we list some of these integrals explicitly,

- $R_{240}(\xi) = \frac{1}{\xi^2}\left[\frac{3+\xi}{1+\xi} - 3R_{200}(\xi)\right]$

- $R_{320}(\xi) = \frac{1}{3(1+\xi)^{3/2}}$

- $R_{440}(\xi) = -\frac{1}{8\xi^2}\left(\frac{3+5\xi}{(1+\xi)^2} - 3\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)$

- $R_{460} = \frac{1}{86^3}\left(\frac{15+25\xi_6+8\varepsilon^2}{(1+5)^2} - \frac{15\tan^{-1}(\sqrt{6})}{\sqrt{6}}\right)$

- $R_{420} = \frac{1}{8\xi}\left(\frac{arctan(\sqrt{\xi})}{\sqrt{\xi}} + \frac{\xi-1}{(\xi+1)^2}\right)$

- $R_{400} = \frac{1}{8}\left[\frac{3arctan(\sqrt{\xi})}{\sqrt{\xi}} + \frac{3\xi+5}{(\xi+1)^2}\right]\mathrm{k}$

- $R_{500} = \frac{8\xi^2+20\xi+15}{15(\xi+1)^{5/2}}$

- $R_{520} = \frac{2\xi+5}{15(\xi+1)^{5/2}}$

- $R_{460}(\xi) = \frac{1}{8\xi^3}\left(\frac{15+25\xi+8\xi^2}{(1+\xi)^2} - 15\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)$

- $R_{540}(\xi) = \frac{1}{5(1+\xi)^{5/2}}$