TOPIC Linear Algebra Assignment .... DATE ....

Q.1. Find the rank of the matrix A by reducing in Row Echelon Form.

Sol^n - Given :- $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}_{4\times 4}$. To find :- rank (A) = ?

1. $\Rightarrow R_2 \longrightarrow R_2 - R_1$

$$\Rightarrow A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$$

2. $\Rightarrow R_3 \longrightarrow R_3 - 3R_1$

$$\Rightarrow A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$$

3. $\Rightarrow R_4 \longrightarrow R_4 - 6R_1$

$$\Rightarrow A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{pmatrix}$$

4. $\Rightarrow R_2 \leftrightarrow R_4$

$$\Rightarrow A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

5. $\Rightarrow R_3 \longrightarrow R_3 - R_2$

$$\Rightarrow A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

6. $\Rightarrow R_4 \rightarrow R_4 + R_3$

$$\Rightarrow A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Since, no. of non-zero rows in Row Echelon Form of matrix A is 3

$\Rightarrow$ The rank (A) = 3

Q.2. Let W be the vector space of all symmetric 2x2 matrices & let $T: W \rightarrow P_2$ be the linear transformation defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b) + (b-c)x + (c-a)x^2$

Find the rank and nullity of T.

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Soln · finding Rank.

to find rank (T), the dimension of span of vectors formed by apply to all possible sym. matrices of 2 is needed.

· considering a basis for $W \rightarrow \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\Rightarrow T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = (1-0) + (0-0)x + (0-1)x^2 = 1-x^2$

$\Rightarrow T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = (0-1) + (1-0)x + (0-0)x^2 = -1+x$

$\Rightarrow T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (0-0) + (0-0)x + (1-0)x^2 = x^2$

$\Rightarrow \{1-x^2, -1+x, x^2\}$, forms basis of P_2, dimension of image of T. i.e. rank of T is $\underline{\underline{3}}$.

· finding Nullity

· 0 sym. matrix is mapped to 0 polynomial iff $T(M)=0$

$\Rightarrow a-b=0,\ b-c=0,\ c-a=0$

$\Rightarrow a=b=c=k$

· So, kernel of T consists sym. matrices of ord. 2 of form

$\begin{pmatrix} k & k \\ k & k \end{pmatrix}$, $k \in$ scalars

$\Rightarrow$ dimension of Kernel i.e. nullity is $\underline{\underline{1}}$

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Q.3. let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, find eigenvalues & eigenvectors of A^{-1} & $A+4I$

Soln:- Given :- $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{2\times 2}$, To find :- A^{-1}, $A+4I$, eigen values & eigenvectors

$\Rightarrow$ since, A is a sq. matrix of order '2'

$\Rightarrow A^{-1} = \dfrac{1}{4-1}\begin{pmatrix} 2 & +1 \\ +1 & 2 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$

- finding eigenvalues

$\Rightarrow \begin{pmatrix} 2/3-\lambda & 1/3 \\ 1/3 & 2/3-\lambda \end{pmatrix}$ $\Rightarrow (2/3-\lambda)^2 - (1/3)^2 = 0$ $\left\{ a^2-b^2 = (a+b)(a-b) \right\}$

$\Rightarrow (1/3-\lambda)(1-\lambda) = 0$

$\Rightarrow \boxed{\lambda = 1, 1/3}$

- finding eigenvectors

$\Rightarrow \lambda = 1$

$\Rightarrow \begin{pmatrix} -1/3 & +1/3 \\ 1/3 & -1/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$ $\Rightarrow -\frac{x}{3} + \frac{y}{3} = 0$ $\Rightarrow x = y = k$

$\Rightarrow$ for $\lambda = 1$, $\boxed{\text{eigen vector} = k[1,1]}$

$\Rightarrow \lambda = 1/3$

$\Rightarrow \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (0)$ $\Rightarrow \frac{x}{3} + \frac{y}{3} = 0$ $\Rightarrow x = -y = k$

$\Rightarrow$ for $\lambda = 1/3$, $\boxed{\text{eigen vector} = [k, -k] = k[1,-1]}$

$\Rightarrow A+4I = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ -1 & 6 \end{pmatrix}$

$\Rightarrow$ let $B = A+4I$ $\left\{ a^2-b^2 = (a+b)(a-b) \right\}$

$\Rightarrow$ finding eigenvalues

$\Rightarrow B-\lambda I = \begin{pmatrix} 6-\lambda & -1 \\ -1 & 6-\lambda \end{pmatrix}$ $\Rightarrow (6-\lambda)^2 - 1 = 0$

$\Rightarrow (6-\lambda-1) + (6-\lambda+1) = 0$

$\Rightarrow (5-\lambda)(7-\lambda) = 0$

$\Rightarrow \boxed{\lambda = 5, 7}$

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- finding eigenvectors

$\Rightarrow \lambda = 5$

$\Rightarrow \begin{pmatrix} 6-5 & -1 \\ -1 & 6-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$

$\Rightarrow x - y = 0 \Rightarrow x = y = k$

$\Rightarrow$ for $\lambda = 5$, $\boxed{\text{eigen vector} = k[1,1]}$

$\Rightarrow \lambda = 7$

$\Rightarrow \begin{pmatrix} 6-7 & -1 \\ -1 & 6-7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$

$\Rightarrow -x - y = 0 \Rightarrow x = -y = k$

$\Rightarrow$ for $\lambda = 7$, $\boxed{\text{eigen vector} = k[1,-1]}$

Q.4. Solve by Gauss-Seidel Method (3 iterations)

$3x - 0.1y - 0.2z = 7.85$

$0.1x + 7y - 0.3z = -19.3$

$0.3x - 0.2y + 10z = 71.4$

w/ initial values $x(0) = 0,\ y(0) = 0,\ z(0) = 0$.

Soln. – $\Rightarrow x = \frac{1}{3}(7.85 + 0.1y - 0.2z)$ — from given eqn ①

$\Rightarrow y = \frac{1}{7}(-19.3 - 0.1x + 0.3z)$ — from given eqn ②

$\Rightarrow z = \frac{1}{10}(71.4 - 0.3x + 0.2y)$ — from given eqn ③

$\Rightarrow$ Iteration 1

$\Rightarrow y = z = 0$

$\Rightarrow x(1) = (1/3)(7.85 + (0.1)(0) - (0.2)(0)) = 7.85/3 = 2.61$

$\Rightarrow x = 2.61,\ z = 0$

$\Rightarrow y(1) = (1/7)(-19.3 - (0.1)(2.61) + (0.3)(0)) = 2.79$

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$\Rightarrow x = 2.61, y = 2.79$

$\Rightarrow z(1) = (1/10)(71.4 - (0.3)(2.61) + (0.2)(2.79)) = 7.1175$

$\Rightarrow$ Iteration-2

$\Rightarrow x(2) = (7.85 - (0.1)(2.6167) - (0.2)(7.1408))(1/3) = 2.9255$

$\Rightarrow y(2) = (-19.3 - (0.1)(2.9255) - (0.3)(7.1408))(1/7) = -3.1049.$

$\Rightarrow z(2) = (71.4 - (0.3)(2.9255) - (0.2)(-3.1049))(1/10) = 7.1143$

$\Rightarrow$ Iteration-3

$\Rightarrow x(3) = (1/3)(7.85 - (0.1)(2.9255) - (0.2)(7.1143)) = 2.044$

$\Rightarrow y(3) = (1/7)(-19.3 - (0.1)(2.044) - (0.3)(7.1143)) = -3.0912$

$\Rightarrow z(3) = (1/10)(71.4 - (0.3)(2.044) - (0.2)(-3.0912)) = 7.1405$

Ans:- from above 3 iterations, the value of x, y & z to approximation can be estimated as $\underline{x = 2.0049}$, $\underline{y = -3.0912}$, $\underline{z = 7.1405}$

Q.5 Define consistent & inconsistent system of equations. Hence, solve the following system of equations if consistent.

$$x + 3y + 2z = 0$$
$$2x - y + 3z = 0$$
$$3x - 5y + 4z = 0$$
$$x + 17y + 4z = 0$$

Soln:- Given:- $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{pmatrix}$

Reducing to Row Echelon Form

1) $R_2 \rightarrow R_2 - 2R_1$
2) $R_3 \rightarrow R_3 - 3R_1$
3) $R_4 \rightarrow R_4 - R_1$
4) $R_3 \rightarrow R_3 - 2R_2$

$\Rightarrow A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 14 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

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$\Rightarrow$ The row reduced echelon form of A, corresponds to following system of equations

$\Rightarrow$ $x + 3y + 2z = 0$ (1) and $-7y - z = 0$ (2)

$\Rightarrow$ for (2) let $y = t$

$\Rightarrow$ $z = -7t$

$\Rightarrow$ for (1) w/ putting $y = t$ and $z = -7t$ (from above)

$\Rightarrow$ $x = -3t$

$\therefore$ The system of given linear equations is <u>consistent</u> & <u>dependent</u> w/ sol^n of form $(\underline{-3t, t, -7t})$ having <u>infinitely</u> <u>many</u> <u>solutions</u>.

Q.6. Determine whether the function $(T: P_2 \rightarrow P_2)$ is linear transformation / not.
where $T(a + bx + cx^2) = (a+1) + (b+1)x + (c+1)x^2$.

Sol^n:- Given :- $T(a + bx + cx^2) = (a+1) + (b+1)x + (c+1)x^2$

- To find, if $T: P_2 \rightarrow P_2$ is a linear transformation or not.

$\Rightarrow$ 2 properties are to be checked :-

1) Additivity :- $T(U+V) = T(U) + T(V)$
2) Homogenity of Degree :- $T(kU) = k\,T(U)$ for $\forall\, U \in$ domain, $k \in$ scalars

$\Rightarrow$ 1) Additivity check :- $T(U+V) = T(U) + T(V)$

$\Rightarrow$ $T(U+V) = T\left((a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)\right)$

$= T\left((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2\right)$

$= (a_1 + a_2 + 1) + (b_1 + b_2 + 1)x + (c_1 + c_2 + 1)x^2$

$\Rightarrow$ $T(U) + T(V) = (a_1 + 1) + (b_1 + 1)x + (c_1 + 1)x^2 + (a_2 + 1) + (b_2 + 1)x + (c_2 + 1)x^2$

$= (a_1 + a_2 + 2) + (b_1 + b_2 + 2)x + (c_1 + c_2 + 2)x^2$

$\{T: P_2 \rightarrow P_2\}$

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- Comparing values of $T(U+V)$ (LHS) and $T(U)+T(V)$ (RHS), since, they are equal.

$\Rightarrow$ Additivity Holds

$\Rightarrow$ 2) Homogenity of Degree :- $T(kU) = kT(U)$

$\Rightarrow$ LHS: $T(kU) = T(k(a+bx+cx^2)) = T(ka+kbx+kcx^2)$

$= T((ka+1) + (kb+1)+x+ (kc+1)x^2)$

$= k(a+1) + k(b+1)x + k(c+1)x^2$

$= k((a+1)+(b+1)+(c+1))$

$= kT(U)$

$=$ RHS

- Since, LHS = RHS.

$\Rightarrow$ Homogenity of Degree holds

Ans:- Since, both, properties of Additivity & Homogenity holds, the $T(P_2 \to P_2)$ is a linear transformation.

Q.7 Determine that wether set $S = \{(1,2,3), (3,1,0), (-2,1,3)\}$ is a basis of $V_3(R)$. In case S isn't a basis determine the basis of sub space spanned by S.

Soln:- $S = \{(1,2,3), (3,1,0), (-2,1,3)\}$

$\Rightarrow A^T = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{pmatrix}_{3\times3}$ $\Rightarrow A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}_{3\times3}$

- Performing row reduction on A, to get reduced echelon form

1) $R_2 \to R_2 - 2R_1$

2) $R_3 \to R_3 - 3R_1$

$\Rightarrow A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \\ 0 & 9 & 9 \end{pmatrix}$

3) $R_3 \to R_3 + 9/5 \cdot R_2$

$\Rightarrow A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

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⇒ As R_3 is a zero row, this indicates that vectors in S are L.D. (linearly dependent) for basis of subspace spanned by S.

⇒ $\begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \end{pmatrix}$ ⇒ $(1,3,-2)$ and $(0,-5,5)$ form a basis for subspace spanned by S.

⇒ Dimension of subspace spanned by S = 2.

⇒ Set S isn't a basis of R^3 because the row reduced echelon form has a zero row.

∴ Basis for subspace spanned by S is $\{(1,3,-2), (0,-5,5)\}$.

Q.8. Using Jacobi's method (perform 3 iterations), solve.

- $3x - 6y + 2z = 23$
- $-4x + y - z = -15$
- $x - 3y + 7z = 16$

w/ initial values $x(0) = 1$, $y(0) = 1$ and $z(0) = 1$.

Sol^n:- # Iteration 1

⇒ $x(1) = (23 + 6(1) - 2(1))(1/3) = (27)/(3) = 9.0$

⇒ $y(1) = (-15 + 4(9) + (1))(1/1) = 22.0$

⇒ $z(1) = (16 - 9 + 3(21))(1/7) = 10.0.$

iteration-2

⇒ $x(2) = (23 + 6(22) - 2(10))(1/3) = 45.0$

⇒ $y(2) = (-15 + 4(45) + 10)(1/1) = 175.0$

⇒ $z(2) = (16 - 45 + 3(175))(1/7) = 70.85$

iteration-3

⇒ $x(3) = (23 + 6(175) - 2(70.85))(1/3) = 310.43$

⇒ $y(3) = (-15 + 4(310.43) + 70.85)(1/1) = 1297.57$

⇒ $z(3) = (16 - 310.43 + 3(1297.57))(1/7) = 514.04.$

Approximate ans:- $x \approx 310.43$, $y \approx 1297.57$ & $z \approx 514.04$.