CP312: Algorithm Design & Analysis

Assignment #1

CP312, WLU, 2022

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Q1. Prove by induction that for all n>1 the following inequality holds:

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}$$

Base Case: n = 2

$$S(2)=\frac{1}{2+1}+\frac{1}{2+2}=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}=\boldsymbol{\frac{14}{24}}>\boldsymbol{\frac{13}{24}}$$

∴ Base case holds true

Asume the inequality holds for n = k

$$S(k)=\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k}>\boldsymbol{\frac{13}{24}}$$

Now we prove the case: n = k + 1

$$S(k+1)=\frac{1}{k+1+1}+\frac{1}{k+1+2}+\ldots+\frac{1}{2(k+1)}$$

$S(k+1)=\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$*we can assume $\frac{1}{2k}$ & $\frac{1}{2k+1}$will be in the summation*

$S(k+1)=\frac{1}{k+1}-\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$*add and sub $\frac{1}{k+1}$*

$S(k+1)=-\frac{1}{k+1}+\left(\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k}\right)+\frac{1}{2k+1}+\frac{1}{2k+2}$*this allows us to isolate S(k)*

$$S(k+1)=-\frac{1}{k+1}+S(k)+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$$S(k+1)=S(k)+\frac{1}{2k+1}+\frac{1}{2k+2}-\frac{1}{k+1}$$

$$S(k+1)=S(k)+\frac{1}{2k+1}-\frac{1}{2(k+1)}$$

$$S(k+1)=S(k)+\frac{1}{2(k+1)(2k+1)}$$

$S(k)+\frac{1}{2(k+1)(2k+1)}>$ S(k) > $\boldsymbol{\frac{13}{24}}$

$\therefore S(k + 1) > \frac{13}{24}$ and the inequality holds

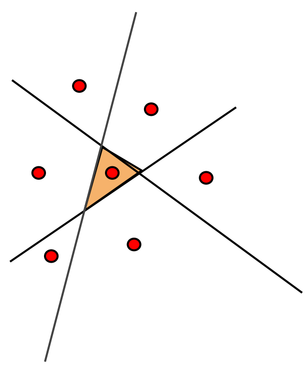
Q2. Suppose we have n >= 3 lines so that no two lines are parallel and no three lines intersect at a common point. Prove that at least one of the regions they form is a triangle. (hint: use induction)

Base Case: n = 3
Let L be the # of lines

The # of regions in the plane formed by n lines in general position states that no 2 lines are parallel and no 3 lines intersect at a common point then;

L = n(n+1)/2 + 1

L(3) = [3*(3+1) / 2] + 1
L(3) = 6 + 1
L(3) = 7



Note:
Red Dots -> Regions
Orange Triangle -> Triangle

We can see that a triangle will form at n=3, this triangle will still exist for any n >= 3, therefore for any n >= 3 a triangle will form in at least one of the regions.

Q3.For each of the following pairs of functions f(n) and g(n), either f(n) = O(g(n)) or g(n) = O(f(n)), but not both. Determine which is the case.

a) $f(n) = (n^2 - n) / 2$, $g(n) = 6n$

$$\lim_{n\to\infty} \to [\frac{\frac{1}{2} \cdot (n^2 - n)}{6n}]$$

$\lim_{n\to\infty} \to [\frac{1}{12} \cdot (n - 1)]$

$\frac{1}{12}[\lim_{n\to\infty} \to (n) - \lim_{n\to\infty} \to (1)]$

$\lim_{n\to\infty} \to (n) = \infty$

$\lim_{n\to\infty} \to (1) = 1$

$= \frac{1}{12}(\infty - 1) = \infty$

∴ Since the limits is ∞, then the following pair of functions is **g(n) = O(f(n))**

b) $f(n) = n + 2\sqrt{n}$, $g(n) = n^2$

$$\lim_{n\to\infty} \to [\frac{n + 2\sqrt{n}}{n^2}]$$

$$\lim_{n\to\infty} \to [\frac{1}{n} + \frac{2}{n^{\frac{3}{2}}}]$$

$\lim_{n\to\infty} \to [\frac{1}{n}] + \lim_{n\to\infty} \to [\frac{2}{n^{\frac{3}{2}}}]$

$\lim_{n\to\infty} \to [\frac{1}{n}] = 0$

$\lim_{n\to\infty} \to [\frac{2}{n^{\frac{3}{2}}}] = 0$

∴ Since both limits are 0, then the following pair of functions is **f(n) = O(g(n))**

c) f(n) = n + log n, g(n) = n $\sqrt{n}$

$$\lim_{n\to\infty} \rightarrow [\frac{n + \ log\ n}{n\sqrt{n}}]$$

$$\lim_{n\to\infty} \rightarrow [\frac{1}{n^{1/2}} + \frac{ln(n)}{n^{\frac{3}{2}}}]$$

$$\lim_{n\to\infty} \rightarrow [\frac{1}{n^{1/2}}] + \lim_{n\to\infty} \rightarrow [\frac{ln(n)}{n^{\frac{3}{2}}}]$$

$$\lim_{n\to\infty} \rightarrow [\frac{1}{n^{1/2}}] = 0$$

$$\lim_{n\to\infty} \rightarrow [\frac{ln(n)}{n^{\frac{3}{2}}}] = 0$$

∴ Since both limits are 0, then the following pair of functions is **f(n) = O(g(n))**

d) f(n) = n^2 + 3n + 4, g(n) = n^3

$$\lim_{n\to\infty} \rightarrow [\frac{n^2\ +\ 3n\ +\ 4}{n^3}]$$

$$\lim_{n\to\infty} \rightarrow [\frac{1}{n} + \frac{3}{n^2} + \frac{4}{n^3}]$$

$$\lim_{n\to\infty} \rightarrow [\frac{1}{n}] + \lim_{n\to\infty} \rightarrow [\frac{3}{n^2}] + \lim_{n\to\infty} \rightarrow [\frac{4}{n^3}]$$

$$\lim_{n\to\infty} \rightarrow [\frac{1}{n}] = 0$$

$$\lim_{n\to\infty} \rightarrow [\frac{3}{n^2}] = 0$$

$$\lim_{n\to\infty} \rightarrow [\frac{4}{n^3}] = 0$$

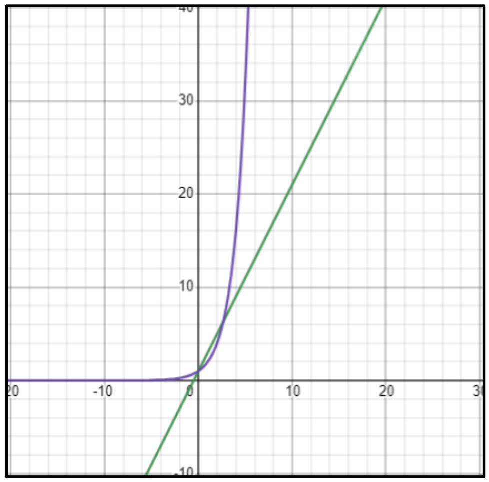
∴ Since all three limits are 0, then the following pair of functions is **f(n) = O(g(n))**

Q4. Prove the following:

$2n + 1 = O(2^n)$

Is this the best upper-bound for the function on the left? In other words, is it a tight upper-bound? If not, specify the tight upper-bound?

Size of n	2n + 1	2^n
1	3	2
2	5	4
3	7	8
4	9	16
5	11	32

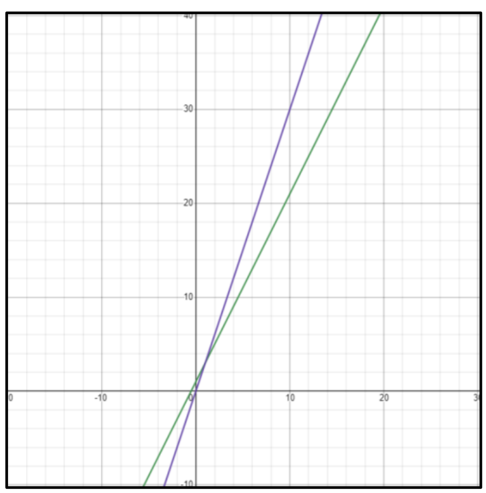


Looking at the outcomes for the increasing values of n, we can see that 2n + 1 grows linearly, whereas 2^n grows exponentially, meaning it is not a valid tight upper-bound. In order to find a tight upper bound, we must find a function f(n) that grows linearly and satisfies the following..

$$2n + 1 > f(n), n \in R$$

f(n) = 3n would be the obvious choice as it runs linearly and will be greater than or equal to 2n +1 for all instances of n.

Size of n	2n + 1	3n
1	3	3
2	5	6
3	7	9
4	9	12
5	11	15



∴ O(3n) would be a valid tight upper bound for the function 2n +1

Q5. For each of the following six code segments:

a) Give Big-O analysis of the running time
b) Run the code and give the running time for several values of n
c) Compare your analysis with the actual runtimes obtained

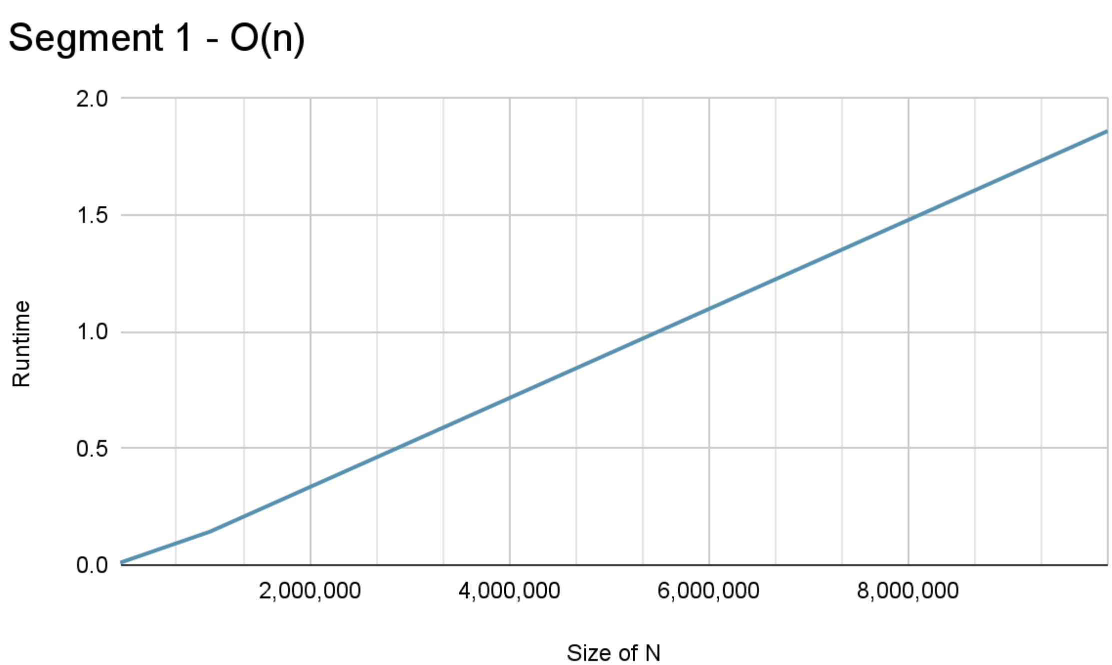
Segment 1

```
sum = 0
for i in range(n):
    sum += 1
```

Big-O: O(n)

Values:

Trial #	Size of N	Runtime (in seconds)
1	100,000	0.009996414184570312
2	1,000,000	0.14299726486206055
3	10,000,000	1.8609983921051025



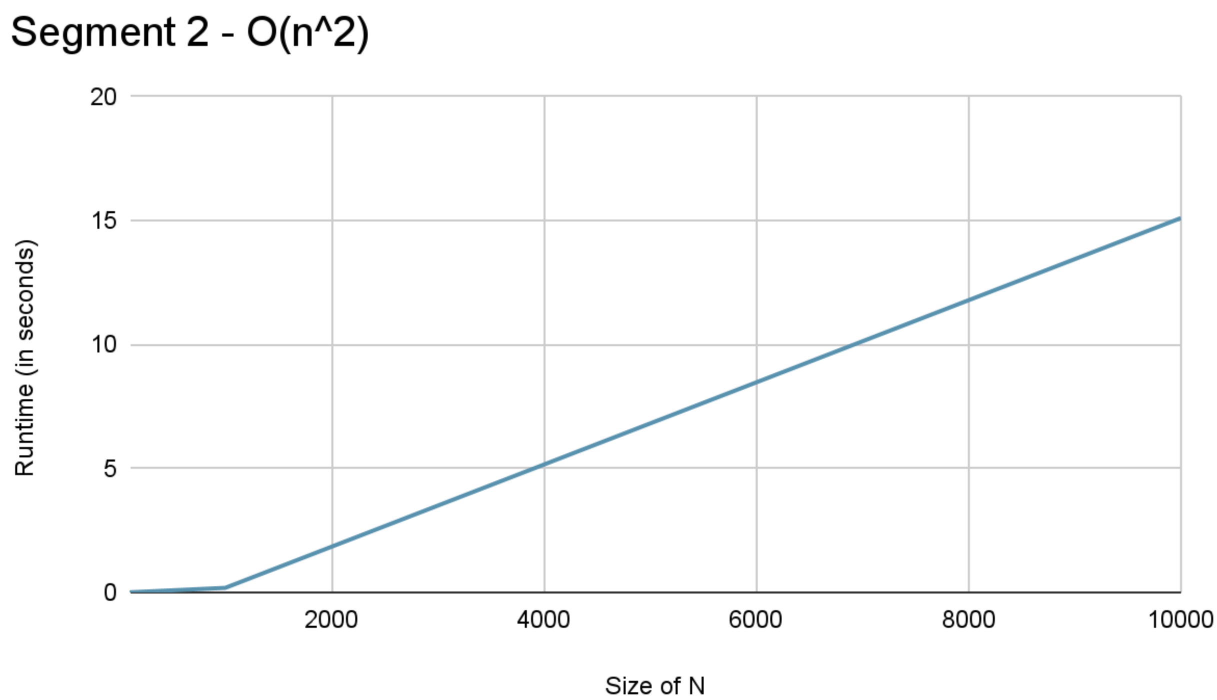
Segment 2

```
sum = 0
for i in range(n):
    for j in range(n):
        sum += 1
```

Big-O: O(n^2)

Values:

Trial #	**Size of N**	**Runtime (in seconds)**
1	100	0.0019991397857666016
2	1,000	0.1819934844970703
3	10,000	15.102467060089111



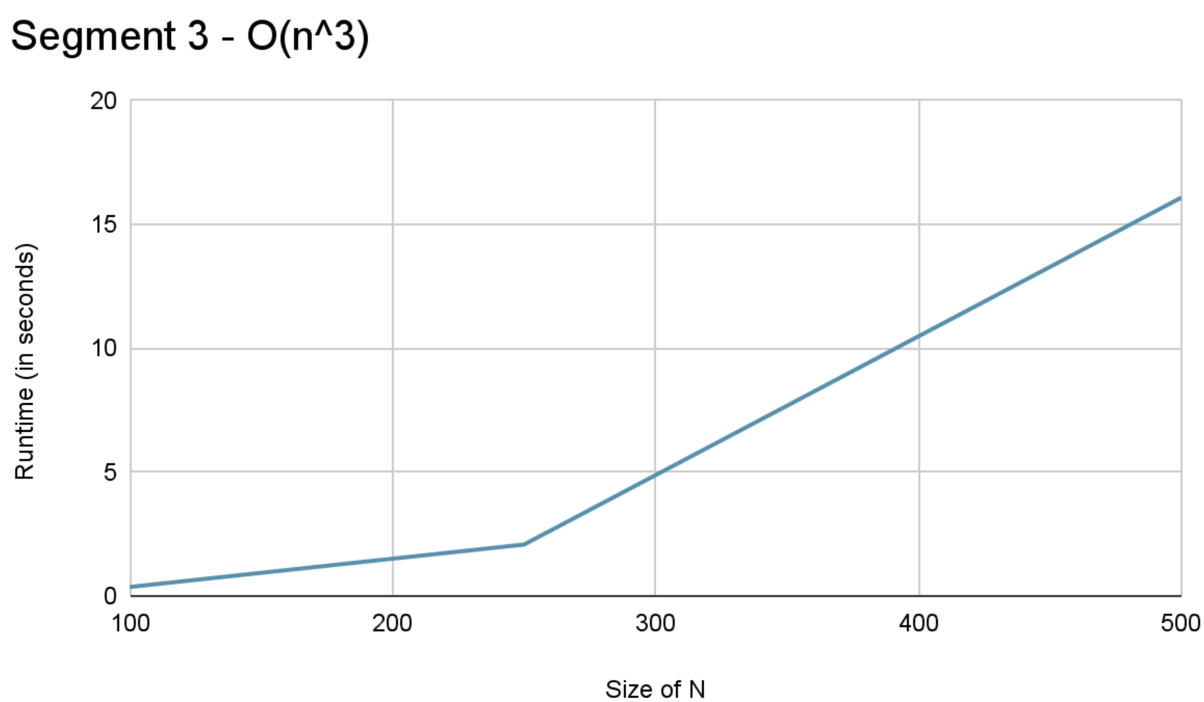
Segment 3

```
sum = 0
for i in range(n):
    for j in range(n*n):
        sum += 1
```

<u>Big-O:</u> O(n^3)

<u>Values:</u>

Trial #	**Size of N**	**Runtime (in seconds)**
1	100	0.36200380325317383
2	250	2.0759949684143066
3	500	16.08599352836609



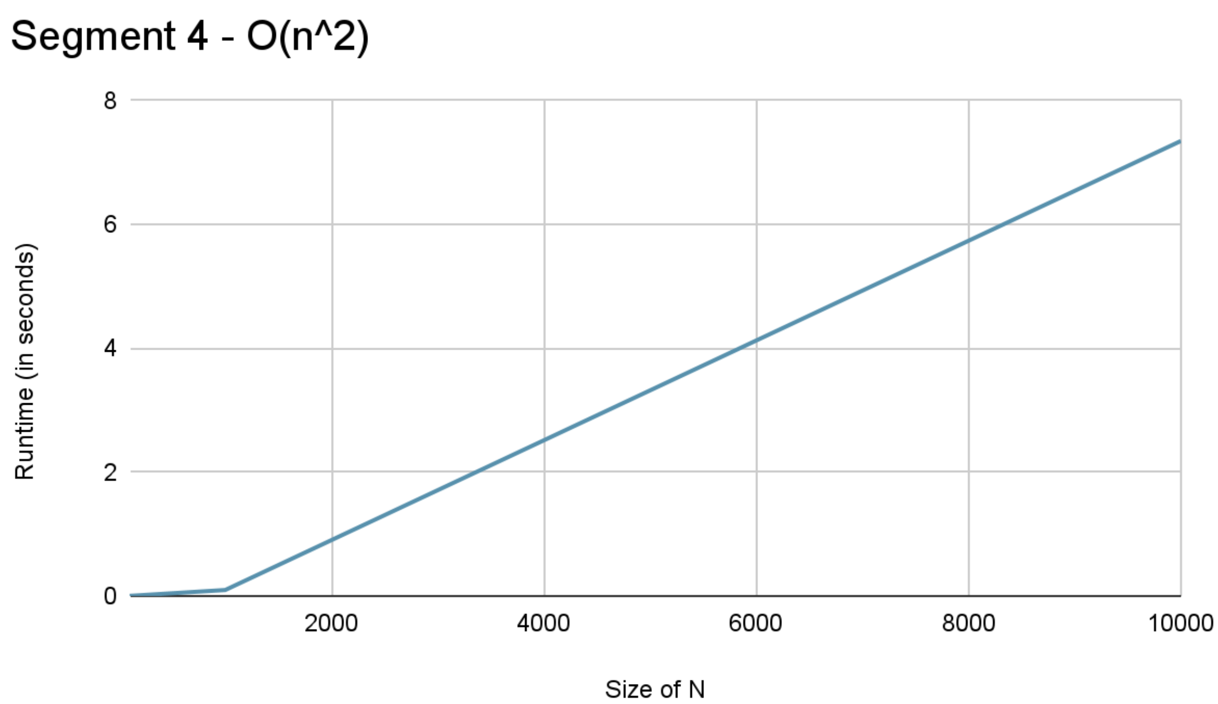
Segment 4

```
sum = 0
for i in range(n): #O(n)
    for j in range(i): #(n^2)
        sum += 1
```

Big-O: O(n^2)

Values:

Trial #	**Size of N**	**Runtime (in seconds)**
1	100	0.0009996891021728516
2	1,000	0.09599757194519043
3	10,000	7.347998142242432



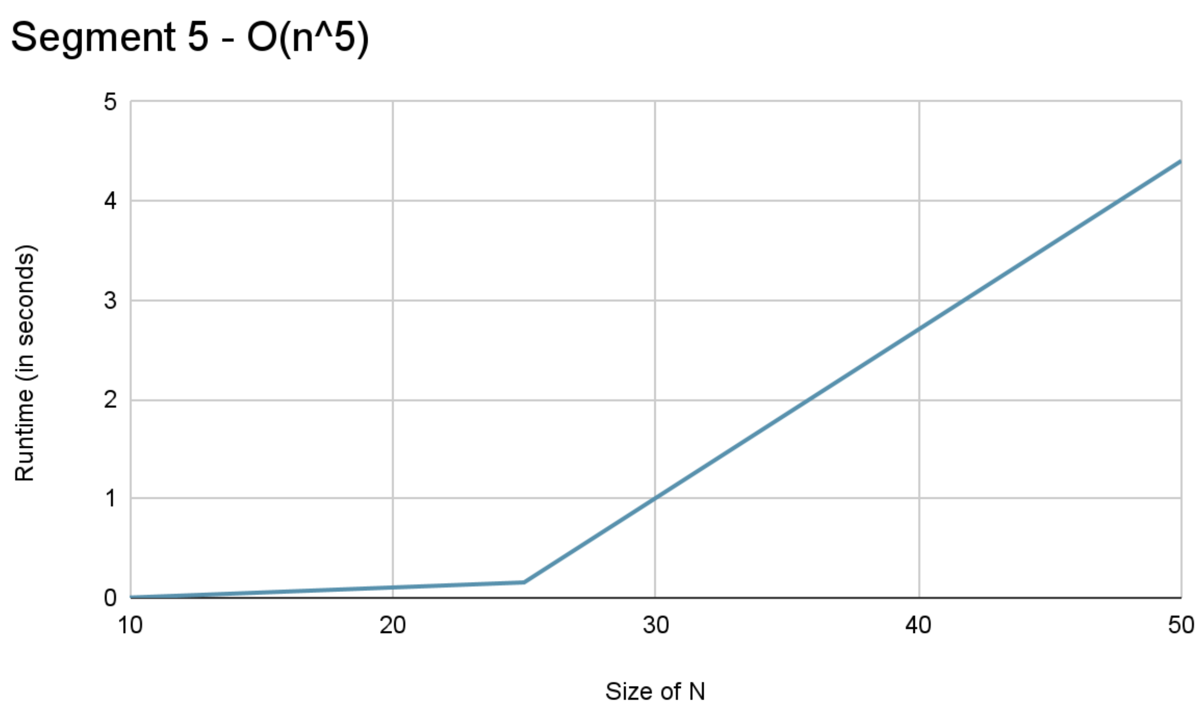
Segment 5

```
sum = 0
for i in range(n):
    for j in range(i*i):
        for k in range(j):
            sum += 1
```

Big-O: $O(n^5)$

Values:

Trial #	**Size of N**	**Runtime (in seconds)**
1	10	0.002000093460083008
2	25	0.1549985408782959
3	50	4.404094696044922



Segment 6

```
sum = 0
for i in range(n):
    for j in range(i*i):
        if (j%i == 0):
            for k in range(j):
                sum += 1
```

Big-O: O(n^5)

Values:

Trial #	**Size of N**	**Runtime (in seconds)**
1	50	0.1810002326965332
2	100	1.6929974555969238
3	150	8.793999910354614