Department of Mathematics

National Institute of Technology Jamshedpur

Mid Term Exam for B. Tech.- 1^{st} sem.(*All Branches*)

(Autumn semester, 2024-25)

Subject- Calculus (MA1101)

Timing: 08:30-10:30 AM | Date: October 05, 2024 | Max marks: 30

Attempt all questions

1. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function on $[a, b]$ such that $f(a) = f(b) = 0$. If $f(c) > 0$ for some $c \in (a, b)$, then prove that there exists $\xi \in (a, b)$ such that $f''(\xi) < 0$. (5)

2. Discuss the convergence of the following series:

$$\frac{2^2 4^2}{3^2 3^2} + \frac{2^2 4^2 5^2 7^2}{3^2 3^2 6^2 6^2} + \frac{2^2 4^2 5^2 7^2 8^2 10^2}{3^2 3^2 6^2 6^2 9^2 9^2} + - - - - - - -$$

(5)

3. Use Sandwich theorem to prove that:

$$\lim_{n \to \infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots\ldots \frac{1}{\sqrt{n^2+n}} \right) = 1.$$

(5)

$f(b) = f(a) + \frac{h f'(a)}{1!} + \frac{h^2 (f''(a))}{2!}$

4. Use Taylor's theorem to prove that:

$x - \frac{x^2}{2} < log(1 + x) < x$ for $x > 0$. (5)

5. Consider the function:

$$f(x) = \begin{cases} \frac{x^3+2y^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0), \end{cases}$$

Then:

(a) Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(0, 0)$. (2)

(b) Test whether the function is continuous at $(0, 0)$. (2)

(c) Show that the function is not differentiable at $(0, 0)$. (1)

6. If $\frac{x^2}{a^2+w} + \frac{y^2}{b^2+w} + \frac{z^2}{c^2+w} = 1$, where a, b, c are constants, and w is a function of x, y, z, then prove the following:

$(3)^4 (n+1)^4$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = 2\left(x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}\right).$$

NATIONAL INSTITUTE OF TECHNOLOGY JAMSHEDPUR

DEPARTMENT OF MATHEMATICS

MA1101: ENGINEERING MATHEMATICS-I

Autumn Semester 2024 - 2025 | End Semester Examination

Marks: 50 | Time: 3 Hrs

Answer any ten questions.

1. Test whether the sequence $\{u_n\}$ where $u_n = \dfrac{(2n+1)(2n+2)\cdots(2n+n)}{n^n}$ is convergent or divergent. What is the limit of $\{u_n\}$ if it is convergent? $\boxed{5}$

2. Determine the range of x, for which the series given below is divergent:
$$\frac{1^2}{2^2} + \frac{1^2\,3^2}{2^2\,4^2}x + \frac{1^2\,3^2\,5^2}{2^2\,4^2\,6^2}x^2 + \cdots\cdots\infty.$$
$\boxed{5}$

3. Consider the function
$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
(i) Is $f(x,y)$ continuous at $(0,0)$? Justify your answer.
(ii) Find the values of $f_x(0,0)$ and $f_y(0,0)$.
(iii) Is $f(x,y)$ differentiable at $(0,0)$? Justify your answer. $\boxed{2+2+1=5}$

4. If $f(x,y) = \tan^{-1}(xy)$, find an approximate value of $f(1.1, 0.8)$ using the Taylor's series quadratic approximation (i.e. upto 3rd term). $\boxed{5}$

5. If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u.$ $\boxed{5}$

6. Discuss the convergence of the improper integral
$$\int_0^\infty \frac{dx}{x^{\frac{1}{2}}(2+x)^{\frac{3}{4}}}$$
$\boxed{5}$

7. Show that the minimum value of $u = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$. $\boxed{5}$

8. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$. $\boxed{5}$

9. Find the general solution of the following equation:
$$\frac{d}{dx}\left[\int_{\frac{\pi}{6}}^{\sqrt{2x}} \left(\sin(t^2) + \cos(2t^2)\right) dt\right] = -\sqrt{\frac{2}{x}}, \qquad x \in \mathbb{R}.$$

10. Show that
$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}dx = \frac{\beta(m,n)}{a^n(1+a)^m},$$
where $\beta(m,n)$ denotes the Beta function.

11. Evaluate
$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}dydx.$$
$\boxed{5}$