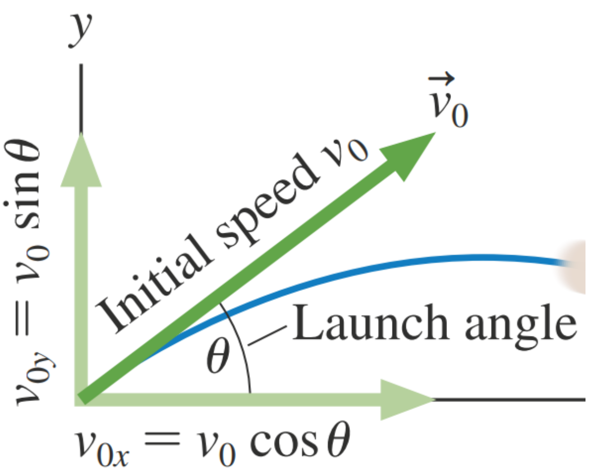
Projectile motion

One basic example of two-dimensional motion with constant acceleration is **projectile motion**. A projectile is an object that moves in two dimensions under the influence of only gravity and it is an extension of the free-fall motion. We will continue to neglect the influence of air resistance, leading to results that are a good approximation of reality for relatively heavy objects moving relatively slowly over relatively short distances.



The start of a projectile's motion, be it thrown by hand or shot from a gun, is called **the launch**, and the angle θ of the initial velocity $\vec{v}_0$ above the horizontal (i.e., above the x-axis) is called the **launch angle**.

From initial speed and launch angle we get *x*- and *y*-components for initial velocity with

$$\begin{cases} v_{0x} = v_0 \cos\theta \\ v_{0y} = v_0 \sin\theta \end{cases}$$

Launch angle can be also **zero** or **negative**: A projectile launched at an angle below the horizontal (such as a ball thrown downward from the roof of a building) has negative values for θ and v_{0y}.

Gravity acts downward, and we know that objects released from rest fall straight down. Hence a projectile has no horizontal acceleration, while its vertical acceleration is simply that of free fall. Therefore, in projectile motion

$$\begin{cases} a_x = 0 \\ a_y = -g \quad \left(= -9.81\ \dfrac{\text{m}}{\text{s}^2} \right) \end{cases}$$

Formulas of two-dimensional motion with constant acceleration apply here, so we get for velocity and position at timestamp *t*:

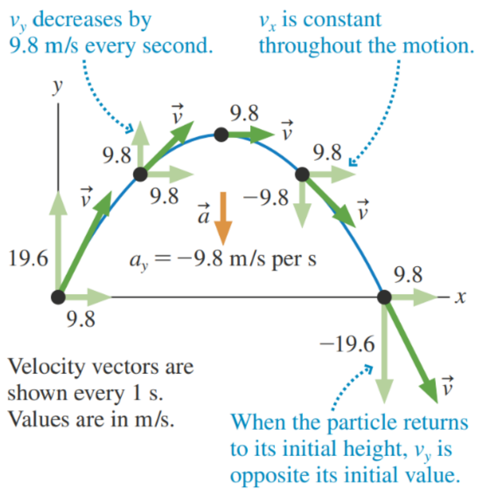
$$\begin{cases} v_{1x} = v_{0x} \quad (\text{constant}) \\ v_{1y} = v_{0y} - gt \end{cases} \qquad \begin{cases} x_1 = x_0 + v_{0x}t \\ y_1 = y_0 + v_{0y}t - \dfrac{1}{2}gt^2 \end{cases}$$

In problems x_0 can almost always be set zero but y_0 can be something else if projectile is not launched from the ground.

Figure shows a projectile launched from (0, 0) with initial velocity components: $\begin{cases} v_{0x} = 9.8\ \text{m/s} \\ v_{0y} = 19.6\ \text{m/s} \end{cases}$

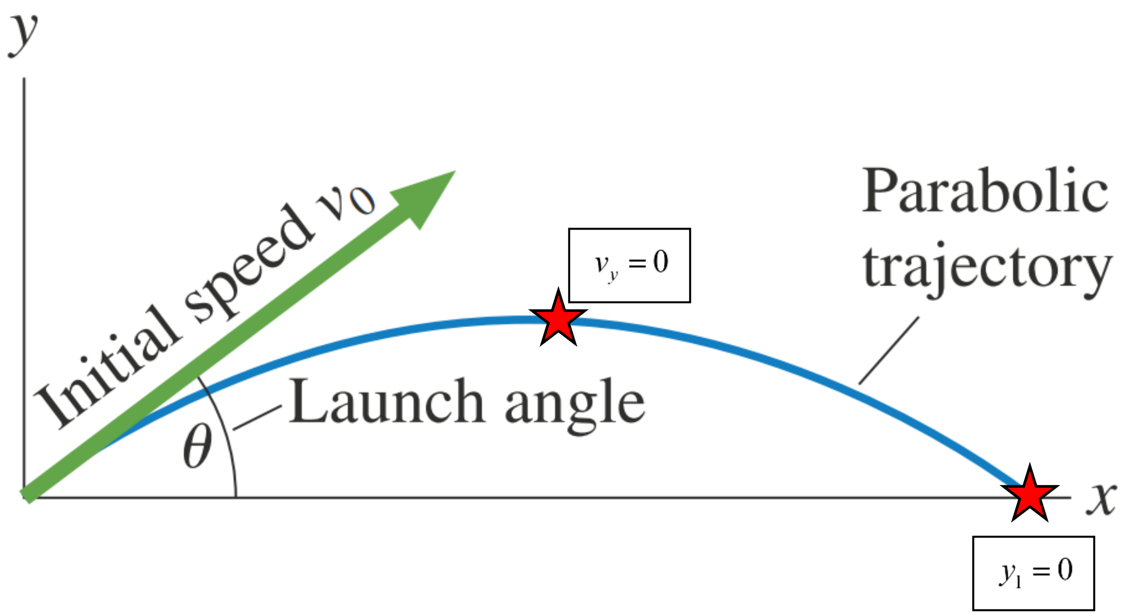
The value of v_x never changes because there's no horizontal acceleration, but v_y decreases by 9.8 m/s every second.

A projectile follows **a parabolic trajectory**.



Problems often ask questions like "how far projectile flies" or "how high it reaches". In these problems we'll need to find **a timestamp** when projectile hits the ground or reaches highest point of its trajectory. These are found with:

- hits the ground: $y_1 = 0$
- reached highest point: $v_y = 0$



Example: Ball is thrown with 30 m/s at 40° angle from height 2 m above the ground.

a) How far it travels horizontally?
b) What is its trajectory's maximum height?

components of initial velocity $\begin{cases} v_{0x} = 30\ \dfrac{\text{m}}{\text{s}} \cdot \cos 40° = 22.98...\dfrac{\text{m}}{\text{s}} \\ v_{0y} = 30\ \dfrac{\text{m}}{\text{s}} \cdot \sin 40° = 19.28...\dfrac{\text{m}}{\text{s}} \end{cases}$

initial position $\begin{cases} x_0 = 0 \\ y_0 = 2\ \text{m} \end{cases}$, formulas $\begin{cases} v_{1x} = v_{0x} \\ v_{1y} = v_{0y} - gt \end{cases}$ $\begin{cases} x_1 = x_0 + v_{0x}t \\ y_1 = y_0 + v_{0y}t - \dfrac{1}{2}gt^2 \end{cases}$

a) At first we'll need to find a timestamp when it hits the ground, i.e. when $y_1 = 0$

$$y_1 = y_0 + v_{0y}t - \frac{1}{2}gt^2 \Rightarrow 0 = 2\ \text{m} + 19.28...\frac{\text{m}}{\text{s}} \cdot t - \frac{1}{2} \cdot 9.81\frac{\text{m}}{\text{s}^2} \cdot t^2$$

This quadratic equation is solved with calculator or formula $ax^2 + bx + c = 0 \Leftrightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

t = 4.03... s (or *t* = -0.10 s)
Then *x*-position at *t* = 4.03... s

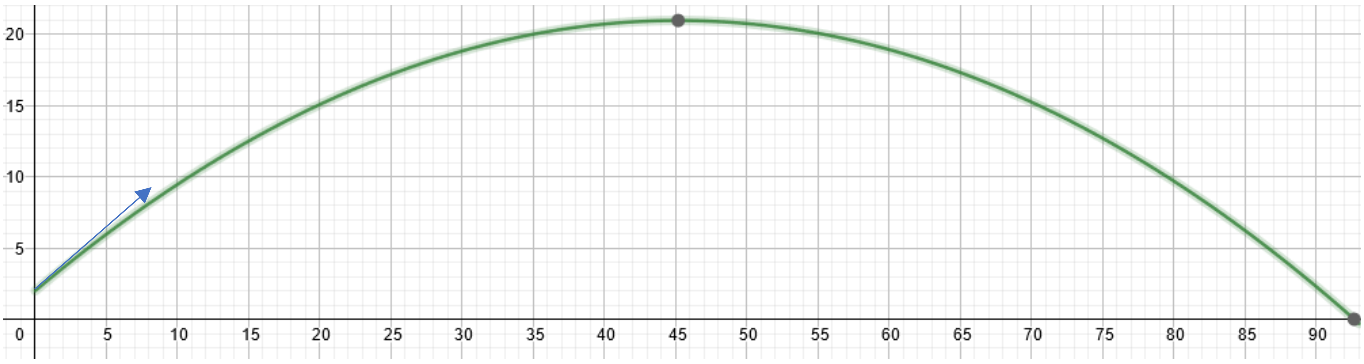
$$x_1 = x_0 + v_{0x}t = 0 + 22.98...\frac{\text{m}}{\text{s}} \cdot 4.03...\ \text{s} = \underline{92.67...\text{m}}$$

b) Timestamp when maximum height is reached = timestamp when $v_y = 0$

$$v_y = v_{0y} - gt \Rightarrow 0 = 19.28...\ \frac{\text{m}}{\text{s}} - 9.81\ \frac{\text{m}}{\text{s}^2} \cdot t \Rightarrow t = 1.965...\text{s}$$

then y-coordinate at t = 1.965... s

$$y_1 = 2\ \text{m} + 19.28...\frac{\text{m}}{\text{s}} \cdot 1.965...\ \text{s} - \frac{1}{2} \cdot 9.81\ \frac{\text{m}}{\text{s}^2} \cdot (1.965...\ \text{s})^2 = 20.95...\text{m}$$

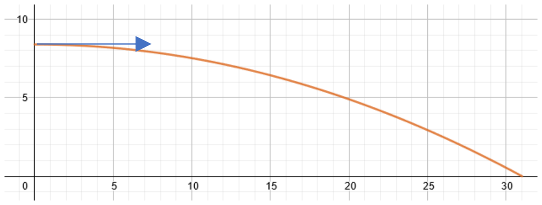


NOTE: some sources have formula $range = \dfrac{v_0^2 \sin(2\alpha)}{g}$ but that applies only if **a projectile lands at the same elevation from which it was launched** (i.e. $y_0 = 0$). So it isn't usable here.

Example: A ball thrown horizontally at 23.7 m /s from the roof of a building lands 31.0 m from the base of the building. How high is the building?

components of initial velocity $\begin{cases} v_{0x} = 23.7\ \text{m/s} \\ v_{0y} = 0 \end{cases}$ (launch angle is zero)

initial position $\begin{cases} x_0 = 0 \\ y_0 = ? \end{cases}$



final position $\begin{cases} x_1 = 31.0\ \text{m} \\ y_1 = 0 \end{cases}$

formulas $\begin{cases} v_{1x} = v_{0x} \\ v_{1y} = v_{0y} - gt \end{cases}$ $\begin{cases} x_1 = x_0 + v_{0x}t \\ y_1 = y_0 + v_{0y}t - \frac{1}{2}gt^2 \end{cases}$

timestamp when x = 31.0 m: $31.0\ \text{m} = 0 + 23.7\ \frac{\text{m}}{\text{s}} \cdot t \Rightarrow t = 1.308...\text{s}$

y_0 solved with info that y = 0 at t = 1.308... s:

$$0 = y_0 + 0 \cdot 1.308...\ \text{s} - \frac{1}{2} \cdot 9.81\ \frac{\text{m}}{\text{s}^2} \cdot (1.308...\ \text{s})^2 \Rightarrow y_0 = 8.39...\ \text{m}$$

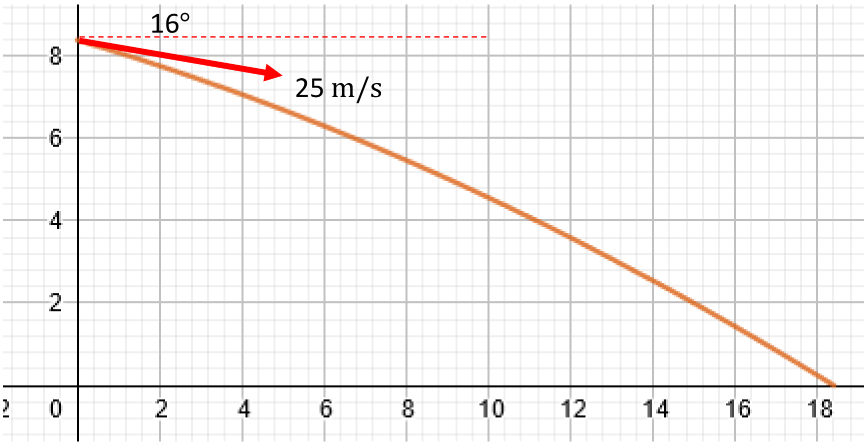
If ball would have been thrown downward, i.e. at an angle below the horizontal, launch angle would have been **negative**.

Example:

thrown 25 m/s 16° below horizontal

-> $\alpha = -16°$

-> $\begin{cases} v_{0x} = 25\ \dfrac{\mathrm{m}}{\mathrm{s}} \cdot \cos(-16°) = 24.0...\ \dfrac{\mathrm{m}}{\mathrm{s}} \\ v_{0y} = 25\ \dfrac{\mathrm{m}}{\mathrm{s}} \cdot \sin(-16°) = -6.89...\ \dfrac{\mathrm{m}}{\mathrm{s}} \end{cases}$



after that all same formulas apply

$$\begin{cases} v_{1x} = v_{0x} \\ v_{1y} = v_{0y} - gt \end{cases} \qquad \begin{cases} x_1 = x_0 + v_{0x}t \\ y_1 = y_0 + v_{0y}t - \dfrac{1}{2}gt^2 \end{cases}$$