Assignment 7

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1 5.9) Let $T = \{\langle M\rangle | M$ is a TM that accepts w^R whenever it accepts w $\}$. Show that T is undecidable.

THEOREM 4.11, $A_{TM} = \{\langle M, w\rangle | M$ is a TM that accepts $w\}$.
A_{TM} is undecidable.

Proof sketch:

We approach the proof by showing a computable reduction function from A_{TM} to T. By defining a reduction function we will conclude that if A_{TM} is undecidable, so is T.

Proof:

Let us generate a computable function from A_{TM} to T.

S = "On input $\langle M, w\rangle$, here M is a TM and w is a string:
  Create a TM Q on $\sum = \{0, 1\}$ as follows:
    Q = "On input x:
      if $x =$ "0101", ACCEPT.
      if $x =$ "1010", Run M on w and do what it does.
      if $x \neq$ "1010", REJECT."
  Print $\langle Q\rangle$"

Above, we have shown a computable function from A_{TM} to T. When M accepts w, Q will accept both "0101" and "1010". This satisfies the criteria for T. Here when M accepts w, $\langle Q\rangle \in T$ and when M rejects or loops on $w, \langle Q\rangle \notin T$. This defines a mapping reduction from A_{TM} to T as only when $\langle M, w\rangle \in T, Q \in T$. Since, A_{TM} undecidable, we can conclude T is undecidable.

2 6.2)Show that any infinite subset of MIN_{TM} is not Turing-recognizable.

Proof:

THEOREM 6.7, MIN_{TM} is not Turing-recognizable.
Let us assume that there exists an infinite subset S of MIN_{TM} that is Turing-recognizable. If S exists then some enumerator E such that E enumerates S exists.

Let us construct a TM M:
M = "On input x:
- Obtain, via the recursion theorem, own description ⟨M⟩.
- Enumerate all possible members of S until a TM T such that $|\langle M\rangle| < |\langle T\rangle|$.
- Output $\langle T\rangle$."

Since S is infinite E will eventually find such T as the length of elements of S are not bounded by a certain length. Here we have found a TM T such that it behaves (accept, reject, loop) exactly as its own description, has a length greater than its own description, and $\in MIN_{TM}$. Since E enumerates all TM with Turing with the shortest description we cannot have two minimal TM that have different lengths and are equivalent. This gives us our contradiction. Hence, an infinite subset of MIN_{TM} is not Turing-recognizable.

3 6.23) Show that the function $K(x)$ is not a computable function. Hint: you could use a reduction for this problem (as usual), but it is much easier to prove if you consider what you would be able to do if $K(x)$ was computable, along with the idea that some strings of every length are incompressible (Theorem 6.29).

Proof by contradiction:

Let us assume $K(x)$ is a computable function.
Generate string b which is the first incompressible string of length n using machine F.
F = "On input n,
- Enumerate all strings x of length n in lexicographic order.
- For each string x, use a TM M to compute its descriptive complexity $K(x)$.
- If $K(x) > |x|$, outputs x.

The machine above produces the minimal description of x such that $d(x)$ is the shortest string $<M, w>$ where TM M on input w halts with x on its tape.

From above construction if s is the string generated, $d(s) = \langle F\rangle b_n$, where b_n is the binary string of integer n. Hence, $K(x) = |d(s)| = |\langle F\rangle b_n| = c + \log n$. For a long s, we have $|s| = n > \log n + c = |\langle F\rangle b_n|$ which is $K(s) < |s|$. This generates a contradiction as s is incompressible. Thus, $K(x)$ is not a computable function.

4 6.24)Show that the set of incompressible strings is undecidable.

Proof by contradiction:

Let us assume the set of incompressible strings is decidable.
If the set of incompressible strings is decidable then there exists some enumerator E such that E enumerates the incompressible strings in string order.

Let us construct a TM M which generates the first incompressible string of length n:
M = "On input b where b is the binary string of length n:
- Use E to enumerate the first incompressible string of length n.

The machine will halt as we set an upper bound with n. If x is the incompressible string generated by M.
$d(x) = \langle M \rangle b$
$K(x) = |\langle M \rangle b| = logn + c$
For a large value of n, $|x| = n > logn + c = K(x)$
Which implies, $K(x) < |x|$.
This generates a contradiction as s is incompressible. Thus, the set of incompressible strings is undecidable.