Assignment 4

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1 Exercise 2.2 a) Use the languages $A = \{a^m b^n c^n | m, n \geq 0\}$ and $B = \{a^n b^n c^m | m, n \geq 0\}$ together with Example 2.36 to show that the class of context-free languages is not closed under intersection.

Proof by Contradiction:

We have,
Example 2.36 proves using the pumping lemma that the language $B = \{a^n b^n c^n | n \geq 0\}$ is not context-free....(1)

Assumption:
Let us assume that the intersection of two context-free languages is a context-free language. If the languages A and B are the two context-free languages then by our assumption $A \cap B$ is also context-free language.

Let us prove the language A is context-free by generating a CFG (Context-free grammar) that generates it:
$S \rightarrow XY$
$X \rightarrow aX|\epsilon$
$Y \rightarrow bYc|\epsilon$
Here, S, X, and Y are variables, where S is the start variable, and a, b and c are the terminals.

Proving language B is context-free by generating a CFG(Context-free grammar) that generates it,
$S \rightarrow XY$
$X \rightarrow aXb|\epsilon$
$Y \rightarrow cY|\epsilon$
Here, S, X, and Y are variables, where S is the start variable, and a, b, and c are the terminals.

Here the language A contains a certain number of a's and an equal number of b's and c's. Also, the language B contains an equal number of a's and b's and a certain number of c's. Since the languages A and B operate on the same value of m and n, $A \cap B$ will have an equal number of a's, b's, and c's.
Example: Let's assume,
$m = 5$ and $n = 4$
$A = a^5 b^4 c^4$
$B = a^4 b^4 c^5$
$A \cap B = a^4 b^4 c^4$
This property holds for any value of m, $n \geq 0$.

Hence, $A \cap B$ can be defined as,
$A \cap B = \{a^n b^n c^n | n \geq 0\}$

The proof from Example 2.36 i.e. ...(1) states that, $A \cap B = \{a^n b^n c^n | n \geq 0\}$ is not context-free language which contradicts our assumption. Hence, by contradiction, we can state that the class of context-free languages is not closed under intersection.

2 Exercise 2.2 b) Use part (a) and DeMorgan's law (Theorem 0.20) to show that the class of context-free languages is not closed under complementation

Proof by Contradiction:

We have,

From 2.2 a): The class of context-free languages is not closed under intersection. .....(1)

De Morgan's laws state that,
$\overline{A \cup B} = \overline{A} \cap \overline{B}$,
$\overline{A \cap B} = \overline{A} \cup \overline{B}$,
where, $\overline{A}$ is the complement of A,
$A \cap B$ is the intersection, and
$A \cup B$ is the union. .....(2)

Assumption:
Let us assume the class of context-free languages is closed under complementation.

Let A and B be two context-free languages.
Then, By our assumption, we can assume, that
$\overline{A}$ and $\overline{B}$ are context-free languages.

We also have,
The class of context free language is closed under union. ....(3)
By statement (3),
$\overline{A}$ $\cup \overline{B}$ is a context-free language.

By our assumption,
$\overline{\overline{A} \cup \overline{B}}$ is a context-free language.
Using De Morgan's law (2),
$\overline{\overline{A} \cup \overline{B}} = \overline{\overline{A}} \cap \overline{\overline{B}}$
$= A \cap B$
From the deduction using De Morgan's law, $A \cap B$ should be context free language, but by (1), we have proved that $A \cap B$ is not a context-free language. This contradicts our assumption, hence we can conclude that the class of context-free languages is not closed under complementation.

3 Exercise 2.30 Part a) Use the pumping lemma to show that the following languages are not context free.
a. $\{0^n1^n0^n1^n | n \geq 0\}$

Proof by Contradiction:

Assumption:
Let us assume the given language, $A = \{0^n1^n0^n1^n | n \geq 0\}$ is context-free. If A is a context free language then it must satisfy the pumping lemma.

Let p be the pumping length and s be a string such that $s \in A$ and $|s| \geq p$.
According to the pumping lemma, s can be divided into five parts $uvxyz$ such that:
1. For each $i \geq 0$, $uv^ixy^iz \in A$
2. $|vy| > 0$, and
3. $vxy| \leq p$

Let s be the string $0^p1^p0^p1^p$ such that $s \in A$ and $|s| \geq p$.
To achieve a contradiction we show that no matter how we divide s into $uvxyz$, one of the three conditions of the lemma is violated.
We can divide the string s into $uvxyz$ in 4 such ways that it satisfies the condition(2) and condition(3) of the pumping lemma.

Case 1: v and y only contain 0's from initial 0^p or latter 0^p
When both v and y contain only 0's, the resulting string uv^ixy^iz (for $i \geq 2$) cannot contain equal numbers of 1's and 0's. This violates condition 1 of the pumping lemma as the resulting string $s \notin A$. Here we have reached a contradiction as the pumping lemma is not satisfied. Hence, the language A is not context free.

Case 2: v and y only contains 1's from initial 1^p or latter 1^p
When both v and y contain only 1's, the resulting string uv^ixy^iz (for $i \geq 2$) cannot contain equal numbers of 1's and 0's. This violates condition 1 of the pumping lemma as the resulting string $s \notin A$. Here we have reached a contradiction as the pumping lemma is not satisfied. Hence, the language A is not context free.

Case 3: v or y contains 0's followed by 1's
Here vxy straddles the boundary between 0's and 1's i.e. vxy contains 0's followed by 1's from initial 0^p1^p or latter 0^p1^p. The resulting string uv^ixy^iz (for $i \geq 2$) cannot have the alphabet symbols in the correct order as vxy can only occur in the initial or latter 0^p1^p. This violates condition 1 of the pumping lemma as the resulting string $s \notin A$. Here we have reached a contradiction as the pumping lemma is not satisfied. Hence, the language A is not context free.

Case 4: v or y contains 1's followed by 0's
Here vxy straddles the boundary between 1's and 0's i.e. vxy contains 1's followed by 0's from the middle 1^p0^p. The resulting string uv^ixy^iz (for $i \geq 2$) cannot have the alphabet symbols in the correct order as vxy only occurs in the middle and does not preserve the $0^p1^p0^p1^p$(i.e. equal value of p for all 0's and 1's) alphabet order. This violates condition 1 of the pumping lemma as the resulting string $s \notin A$. Here we have reached a contradiction as the pumping lemma is not satisfied. Hence, the language A is not context free.

As all 4 cases result in a contradiction, a contradiction is unavoidable. So through the contradiction obtained for the pumping lemma, we can conclude that the assumption language A is a CFL must be false. Hence, language A is not a CFL.