Assignment 1

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1 Theorem 0.20

For any two sets A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

2 Theorem 0.21

For every graph G, the sum of the degrees of all the nodes in G is an even number.

3 Theorem 0.22

For each even number n greater than 2, there exists a 3-regular graph with n nodes.

4 Theorem 0.24

$\sqrt{2}$ is irrational.

5 Theorem 0.25

For each $t \geq 0$,
$P_t = PM^t - Y(\frac{M^t-1}{M-1})$.

6 Definition 1.5

A finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$,where
1. Q is a finite set called the states,
2. Σ is a finite set called the alphabet,
3. $\delta : Q \times \Sigma \to Q$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

7 Definition 1.16

A language is called a regular language if some finite automaton recognizes it.

8 Definition 1.23

Let A and B be languages. We define the regular operations union, concatenation, and star as follows:
Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$.
Star: $A* = \{x_1 x_2 ... x_k \mid k \geq 0 \text{ and each } x_i \in A\}$.

9 Theorem 1.25

The class of regular languages is closed under the union operation. In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

10 Theorem 1.26

The class of regular languages is closed under the concatenation operation. In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

11 Definition 1.37

A non deterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where
1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta : Q \times \Sigma_\epsilon \rightarrow P(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

12 Theorem 1.39

Every non deterministic finite automaton has an equivalent deterministic finite automaton.

13 Corollary 1.40

A language is regular if and only if some non deterministic finite automaton recognizes it.

14 Theorem 1.45

The class of regular languages is closed under the union operation.

15 Theorem 1.47

The class of regular languages is closed under the concatenation operation.

16 Theorem 1.49

The class of regular languages is closed under the star operation.

17 Definition 1.52

Say that R is a regular expression if R is
1. a for some a in the alphabet Σ,
2. ϵ,
3. $\emptyset$,
4. $(R_1 \cup R_2)$,where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$,where R_1 and R_2 are regular expressions, or
6. (R_1^*),where R_1 is a regular expression.

In items 1 and 2, the regular expressions a and ϵ represent the language $\{a\}$ and $\{\epsilon\}$, respectively. In item 3, the regular expression $\emptyset$ represents the empty language. In items 4, 5, and 6, the expression represent the language obtained by taking the union or concatenation of the languages R_1 and R_2, or the star of the language R_1, respectively.

18 Theorem 1.54

A language is regular if and only if some regular expression describes it. This theorem has two directions. We state and prove each direction as a separate lemma.

19 Lemma 1.55

If a language is described by a regular expression, then it is regular.

20 Lemma 1.60

If a language is regular, then it is described by a regular expression.

21 Definition 1.64

A generalized non deterministic finite automaton is a 5-tuple,$(Q, \Sigma, \delta, q_{start}, q_{accept})$, where
1. Q is the finite set of states,
2. Σ is the input alphabet,
3. δ: $(Q$-$\{q_{accept}\})$ $\times(Q$-$\{q_{start}\}) \rightarrow R$ is the transition function,
4. q_{start} is the start state, and
5. q_{accept} is the accept state.

22 Claim 1.65

For any GNFA G, CONVERT$(G)$is equivalent to G.

23 Theorem 1.70

Pumping lemma: If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p,then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:
1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Recall the notation where $|s|$ represents the length of string s, y^i means that i copies of y are concatenated together, and y^0 equals ϵ.
When s is divided into xyz, either x or z may be ϵ, but condition 2 says that $y \neq \epsilon$. Observe that without condition2 the theorem would be trivially true. Condition 3 states that the pieces x and y together have length at most p. It is an extra technical condition that we occasionally find useful when proving certain languages to be non regular. See Example 1.74 for an application of condition 3.

1 Theorem 0.24

$\sqrt{2}$ is irrational.

1.1 Proof by Contradiction

Let us assume,
$\sqrt{2}$ is rational.
We can write,
$\sqrt{2} = \frac{m}{n}$
where m and n are integers.

Here, both m and n are reduced to their smallest form. That is, if m and n are both divisible by an integer greater than 1, divide them both by the largest such integer. This will preserve the value of the fraction. Since the integers m and n are reduced to their smallest form, at least one of these integers m and n must be an odd number.

We can re-write the above equation as,
$n\sqrt{2} = m$
Squaring both sides,
$2n^2 = m^2$

Since m^2 is 2 times the integer n^2, m^2 is even.
Since m^2 is even, m must be even as the square of an even number is always even. Using this conclusion, we can re-write $m = 2k$ for some integer k.

Replacing m with $2k$ in the above equation,
$2n^2 = (2k)^2$
$2n^2 = 4k^2$
Dividing both sides by 2, we obtain
n^2=$2k^2$

Since n^2 is 2 times the integer k^2, n^2 is even.
Since n^2 is even, n must be even as the square of an even number is always even.

From this we have deduced that both the integers m and n are even. Both the integers m and n cannot be even as we have already reduced m and n so that they were not both even. Thus we have come to a contradiction. Hence we have proved by contradiction that $\sqrt{2}$ is rational.

1 Exercises

0.2 Write formal descriptions of the following sets.

a. The set containing the numbers 1, 10, and 100
Answer: S = {1, 10, 100}
b. The set containing all integers that are greater than 5
Answer: S = $\{n | n \in N$ and $n > 5\}$
c. The set containing all natural numbers that are less than 5
Answer: S = {1, 2, 3, 4}
d. The set containing the string aba
Answer: S = {“aba”}
e. The set containing the empty string
Answer: S = $\{\epsilon\}$
f. The set containing nothing at all
Answer: S = $\emptyset$

0.5 If C is a set with c elements, how many elements are in the power set of C? Explain your answer

Answer: The power set of C has 2^c number of elements.
For a given set C with c elements, each element has two possibilities, possible number of subset is 2 multiplied with itself c times i.e., 2^c. Hence, power set contains 2^c elements.

0.6 Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. The unary function $f : X \to Y$ and the binary function $g : X \times Y \to Y$ are described in the following tables.

a. What is the value of $f(2)$?
Answer: 7
b. What are the range and domain of f?
Answer: Range Y, Domain X
c. What is the value of $g(2, 10)$?
Answer: 6
d. What are the range and domain of g?
Answer: Range Y, Domain $X \times Y$
e. What is the value of g(4,f(4))?
Answer: 8

1.3 The formal description of a DFA M is $\{q_1, q_2, q_3, q_4, q_5\}, \{u, d\}, \delta, q_3, \{q_3\}$

where δ is given by the following table.

	u	d
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_2	q_4
q_4	q_3	q_5
q_5	q_4	q_5

Give the state diagram of this machine.

Answer:
Here,
$\{q_1, q_2, q_3, q_4, q_5\}$ are the states.
$\{u, d\}$ are the alphabets.
δ is the transition function.
q_3 is the start state.
$\{q_3\}$ is the final state.