Assignment 10

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1 NP-completeness proofs

For this assignment, consider the following language: QUAD3SAT = $\{\langle\phi\rangle \mid \phi$ is a 3-CNF formula having at least 4 different satisfying assignments$\}$

1. (30 points) Prove that QUAD3SAT $\in$ NP.

2. (70 points) Prove that 3SAT $\leq_P$ QUAD3SAT.

If we know that 3SAT is NP-complete (which is true), then together, the above proofs show that QUAD3SAT is also NP-complete.

1.1 Proof: QUAD3SAT $\in$ NP.

To show QUAD3SAT $\in$ NP let us Construct a poly-time verifier V for QUAD3SAT:

V = "On input $\langle\langle\phi\rangle, c\rangle$, where c is the collection of assignments for ϕ.

1. Check if the collection c has at least 4 different assignments.
2. Check if all the assignments in c have a boolean value assigned to them.
3. Set $count = 0$.
4. For each assignment b_i in c, where b_i is i^{th} element in the collection c:
   (a) Check if b_i has values for all variables in ϕ
   (b) Substitute the value of each literal with values assigned in b_i for positive literals and its negation for negative literals
   (c) Check if every clause evaluates to $true$
   (d) If all the above test passes, increase the count by 1
5. On completion, if $count \geq 4$, $ACCEPT$
6. Else $REJECT$."

Here, we have constructed a verifier V where each steps of its computation can be verified in polynomial time. The verifier V accepts $\langle\langle\phi\rangle, c\rangle$ if for some c, ϕ is satisfiable using 4 or more different assignments. V rejects if ϕ is unsatisfiable for any c. This construction of the verifier V suggests that the QUAD3SAT can be verified in polynomial time. Hence, QUAD3SAT $\in$ NP.

1.2 Proof: 3SAT $\leq_P$ QUAD3SAT.

Here we show that there exists a polynomial time reduction from 3SAT to QUAD3SAT. If such reduction exists than it shows that QUAD3SAT is NP-Hard and thus asserts QUAD3SAT is NP-complete.

Let F be the reduction function:
$F =$ "On input $\langle\phi\rangle$, where ϕ is a 3-CNF formula:

1. Introducing two variables x and y, such that $x, y \notin \phi$
2. Construct ϕ_1 as follows:
   $\phi_1 = \phi \wedge (x \vee x \vee \overline{x}) \wedge (y \vee y \vee \overline{y})$
3. Output $\langle\phi_1\rangle$"

Here, we have constructed ϕ_1 by addition of 2 different clauses with at least 4 different satisfying assignment to ϕ. The variables i.e. x and y are introduced such that $x, y \notin \phi$. If $\phi \in$ 3SAT then by definition of 3SAT, there exists at least one satisfiable assignment for ϕ. If there exist a satisfiable assignment for ϕ then by our above construction, the newly constructed ϕ_1 will have at least 4 satisfiable assignment as the newly added variables i.e. x and y in ϕ_1 are unique to the variables in ϕ. Also, if $\phi \notin$ 3SAT then $\phi_1 \notin$ QUAD3SAT as every clause of ϕ_1 has to be satisfiable for the ϕ_1 to be satisfiable and ϕ_1 is constructed by addition of clauses to ϕ.

From our construction above,

$$\langle\phi\rangle \in 3SAT \iff \langle\phi_1\rangle \in QUAD3SAT \text{ and } \langle\phi\rangle \notin 3SAT \iff \langle\phi_1\rangle \notin QUAD3SAT$$

Here, the following reduction is polynomial time as the addition of clauses and their variable takes polymonial time. Thus, $3SAT \leq_p QUAD3SAT$.