Outlines of CH-5

1. Simplex Method
2. Dual Linear Programming
3. Transportation (Only minimization case: excluding loop formation)
4. Assignment Model (Only minimization case)

Simplex Table

1. The ways of Converting Inequalities into Equalities in Forms

Types of Constraints	Additional Variables used in Subjective Equations	Additional Variables and Coefficients used in the Objective Equations	
		Max	**Min**
Less than or Equal to (≤)	Slack variables are used i.e. S, for 1st equation, S_2 for 2nd equation and so and so.	0	0
Greater than or Equal to (≥)	Surplus variables are used i.e. $-S_1 + A_1$ for 1st equation, $-S_2 + A_2$ for 2nd equations and so on.	0 for slack –M for artificial	0 for slack + M for artificial
Equal to (=)	Artificial variables are used i.e. A_1 for 1st equation A_2 for 2nd equation are so on.	–M	+M

Example-1

Consider the following LPP,

Maximize: $Z = 3x_1 + 2x_2 + 5x_3$

Subject to the constraints,

$x_1 + x_2 + x_3 \leq 9$

$2x_1 + 3x_2 + 5x_3 \leq 30$

$2x_1 - x_2 - x_3 \leq 8$

and $x_1, x_2, x_3 \geq 0$

- Solution: Inequalities of the subjective equations can be converted into equality in form adding slack variables such as:

Maximize $(Z) = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$

Subjective to the constraints:

$x_1 + x_2 + X_3 + S_1 + 0S_2 + 0S_3 = 9$

$2x_1 + 3x_2 + 5x_3 + 0S_1 + S_2 + 0S_3 = 30$

$2x_1 - x_2 - x_3 + 0S_1 + 0S_2 + S_3 = 8$

Simplex Table -1

C_j			3	2	5	0	0	0	Ratio Column
	B.V.	Const.	x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	9	1	1	1	1	0	0	$9/1 = 9$
0	S_2	30	2	3	5	0	1	0	$30/5 = 6 \leftarrow$
0	S_3	8	2	−1	−1	0	0	1	$8/-1 = -8$
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$	–	−3	−2	$-5 \uparrow$	0	0	0	

$$\text{New } R_2 \rightarrow \frac{\text{Old } R_2}{5}; \text{ New } R_1 \rightarrow \text{Old } R_1 - \text{New } R_2, \text{ New } R_3 \rightarrow \text{Old } R_3 + \text{New } R_2$$

Simplex Table -2

C_j			3	2	5	0	0	0	Ratio Column
	B.V.	Const.	x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	3	3/5	2/5	0	1	-1/5	0	5 ←
5	X3	6	2/5	3/5	1	0	1/5	0	15
0	S_3	14	12/5	-2/5	0	0	1/5	1	5.833
	Z_j	30	2	3	5	0	1	0	
	$Z_j - C_j$	-	-1↑	1	0	0	1	0	

$$R_1 \to \frac{5}{3} R_1;\ R_2 \to R_2 - \frac{2}{5} R_1;\ R_3 \to R_3 - \frac{12}{5} R_1$$

Simplex Table –3

C_j			3	2	5	0	0	0	Ratio Column
	B.V.	Const.	x_1	x_2	x_3	S_1	S_2	S_3	
3	X_1	5	1	2/3	0	5/3	–1/3	0	
5	X_3	4	0 2/5-2/5*1	11/25	1	–2/3	1/3	0	
0	S_3	2	0	–34/25	0	–4	1	1	
	Z_j	35	3	21/5	5	5/3	2/3	0	
	$Z_j - C_j$	–	0	11/5	0	5/3	2/3	0	

Since, all the values of $Z_j - C_j$ are in positive ($Z_j - C_j \geq 0$). It means solution is optimal and required answers for the variables are:

Max (Z) = 35; $x_1 = 5$; $x_3 = 4$ and $S_3 = 2$ and rest of the variables are zero.

Example 2

Solve the following problem by simplex method

Minimize (Z) = $3x_1 + 2x_2$

Subject to the constraints

$2x_1 + 4x_2 \geq 10$

$4x_1 + 2x_2 \geq 10$

$x_2 \geq 4$

and $x_1, x_2 \geq 0$

Solution

Inequalities of the subjective equations can be changed into equalities in forms adding surplus variables to the constraint signs of greater than and equal to (≥).

Minimize (Z) = $3x_1 + 2x_2 + 0S_1 + MA_1 + 0S_2 + MA_2 + 0S_3 + MA_3$

Subject to the constraints

$$2x_1 + 4x_2 - S_1 + A_1 + 0S_2 + 0A_2 + 0S_3 + 0A_3 = 10$$

$$4x_1 + 2x_2 + 0S_1 + 0A_1 - S_2 + A_2 + 0S_3 + 0A_3 = 10$$

$$0x_1 + x_2 + 0S_1 + 0A_1 + 0S_2 + 0A_2 - S_3 + A_3 = 4$$

Simplex Table 1

$C_j \rightarrow$			3	2	0	M	0	M	0	M	Ratio Column
$\downarrow$	B.V.	Const.	x_1	x_2	S_1	A_1	S_2	A_2	S_3	A_3	
M	A_1	10	2	4	–1	1	0	0	0	0	$10/4 = 2.5 \leftarrow$
M	A_2	10	4	2	0	0	–1	1	0	0	$10/2 = 5$
M	A_3	4	0	1	0	0	0	0	–1	1	$4/1=4$
	Z_j	24m	6m	7m	–m	m	–m	m	–m	m	
	$Z_j–C_j$	-	6m – 3	$7m – 2 \uparrow$	–m	0	–m	0	–m	0	

$$R_1 \rightarrow \frac{R_1}{4}, R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - R_2$$

Simplex Table 2

$C_j \rightarrow$			3	2	0	M	0	M	0	M	Ratio Column
$\downarrow$	B.V.	Const.	x_1	x_2	S_1	A_1	S_2	A_2	S_3	A_3	
2	x_2	3/2	5/2	1	–1/4	1/4	0	0	0	0	5
M	A_2	5	3	0	1/2	–1/2	–1	1	0	0	5/3←
M	A_3	3/2	–1/2	0	1/4	–1/4	0	0	–1	1	–3
	Z_j	$\frac{13m}{2}+5$	$\frac{5m}{2}+1$	2	$\frac{3m}{4}-\frac{1}{2}$	$\frac{-3m}{4}+\frac{1}{2}$	–m	m	–m	m	
	$Z_j–C_j$	-	$\frac{5m}{2}-2\uparrow$	0	$\frac{3m}{4}-\frac{1}{2}$	$\frac{-7m}{4}+\frac{1}{2}$	–m	0	–m	0	

$$R_2 \rightarrow \frac{R_2}{3};\ R_1 \rightarrow R_1 - \frac{1}{2}R_2;\ R_3 \rightarrow R_3 + \frac{1}{2}R_2$$

Simplex Table 3

$C_j \rightarrow$			3	2	0	M	0	M	0	M	Ratio Column
$\downarrow$	B.V.	Const.	x_1	x_2	S_1	A_1	S_2	A_2	S_3	A_3	
2	x_2	5/3	0	1	–1/3	1/3	1/6	–1/6	0	0	–5
3	x_1	5/3	1	0	1/6	–1/6	–1/3	1/3	0	0	10
M	A_3	7/3	0	0	1/3	–1/3	–1/6	1/6	–1	1	7$\leftarrow$
	Z_j	$\frac{7m}{3}+\frac{4}{3}$	3	2	$\frac{m}{3}-\frac{1}{6}$	$\frac{m}{3}+\frac{1}{6}$	$\frac{-m}{6}-\frac{2}{3}$	$\frac{m}{6}+\frac{2}{3}$	–m	m	
	$Z_j – C_j$	-	0	0	$\frac{m}{3}-\frac{1}{6}\uparrow$	$\frac{-2m}{3}+\frac{1}{6}$	$\frac{-m}{6}-\frac{2}{3}$	$\frac{-5m}{6}+\frac{2}{3}$	–m	0	

$$R_3 \rightarrow 3R_3;\ R_1 \rightarrow R_1 + \frac{1}{3}R_3,\ R_2 \rightarrow R_2 - \frac{1}{6}R_3$$

Simplex Table 4

$C_j \rightarrow$			3	2	0	M	0	M	0	M	Ratio Column
$\downarrow$	B.V.	Const.	x_1	x_2	S_1	A_1	S_2	A_2	S_3	A_3	
2	x_2	4	0	1	0	0	0	0	-1	1	
3	x_1	1/2	1	0	0	0	$-1/4$	1/4	1/2	$-1/2$	
0	S_1	7	0	0	1	-1	$-1/2$	1/2	-3	3	
	Z_j	9.5	3	2	0	0	$\frac{-3}{4}$	$\frac{3}{4}$	$\frac{-1}{4}$	$\frac{1}{4}$	
	Z_j-C_j	-	0	0	0	$-m$	$-\frac{3}{4}$	$-m+\frac{3}{4}$	$-\frac{1}{4}$	$-m+\frac{1}{4}$	

Since, all the values of $Z_j - C_j$ are in zero or negative (i.e. $Z_j - C_j \leq 0$). It means optimum feasible solutions has been obtained. The required answers for the variables are:

$$z_j = 9.5,\ X_2 = 4,\ x_1 = \frac{1}{2} \text{ and } S_1 = 7$$

Dual Linear Programming

Write the dual of the following LPP:

Primal Equations	Dual Equations (Solution)
i. Maximize (Z) = $4X_1 + 3X_2$ Subject to the linear constraints $2X_1 + 3X_2 \leq 12$ $2X_1 + X_2 \leq 4$ $3X_1 + 7X_2 \leq 21$ $X_1, X_2 \geq 0$	i. Minimize (Z) = $12y_1 + 4y_2 + 21y_3$ Subject to the linear constraints $2y_1 + 2y_2 + 3y_3 \geq 4$ $3y_1 + y_2 + 7y_3 \geq 3$ $y_1, y_2, y_3 \geq 0$
ii. Minimize (Z) = $4X_1 + 5X_2$ Subject to the constraints $X_1 + X_2 \geq 2$ $2X_1 + X_2 \geq 6$ $3X_1 + 5X_2 \geq 15$ $X_1, X_2 \geq 0$	ii. Maximize (Z) = $2y_1 + 6y_2 + 15y_3$ Subject to the constraints $y_1 + 2y_2 + 3y_3 \leq 4$ $y_1 + y_2 + 5y_3 \leq 5$ $y_1, y_2, y_3 \geq 0$
iii. Maximize (Z) = $2X_1 + 3X_2$ Subject to linear constraints $X_1 + X_2 \leq 5$ $2X_1 + 3X_1 \geq 6$ $X_1, X_2 \geq 0$	iii. Max (Z) = $2x_1 + 3x_2$ Subject to $x_1 + x_2 \leq 5$ $-2x_1 - 3x_2 \leq -6$ Now, Min (Z) = $5y_1 - 6y_2$ Subject to $y_1 - 2y_2 \geq 2$ $y_1 - 3y_2 \geq 3$ $y_1, y_2 \geq 0$

Transportation Problem

Example 1

Find the optimal transportation schedule from following with the objective of minimizing the cost. [T.U. 2058]

Factory	Quantity requirements per day in kg.	Ware house	Quantity available per day in kg.
P	450	X	350
Q	500	Y	400
R	200	Z	400

Cost of transportation per kg is given in the following table.

From	To factory		
	P	Q	R
Warehouse X	10	20	20
Warehouse Y	40	60	40
Warehouse Z	10	16	24

Step 1. Getting an initial basic feasible solution by VAM

To / From	P	Q	R	Available	Cost Difference		
					I	II	R_i
X	350 10 x	x 20	x 20	350	← 10	–	$R_1 = 0$
Y	100 40	100 60	200 40 x	400	0	0	$R_2 = 30$
Z	10 x	400 16 x	24 x	400	6	6	$R_3 = -14$
Required	450	500	200	1150			
Cost difference I	0	4	4				
II	30	44↑	16				
Kj	$K_1 = 10$	$K_2 = 30$	$K_3 = 10$				

Step 2: Initial transportation cost = 10 × 350 + 40 × 100 + 60 × 100 + 40 × 200 + 16 × 400 = Rs. 27900

Step 3. Test of degeneracy

No. of occupied cells = No. of rows + No. of columns – 1

or, 5 = 3 + 3 – 1

or, 5 = 5

It is the case of non degeneracy.

Step 4. Calculation of values of occupied cells.

$C_{ij} = R_i + K_j$

Assuming $R_1 = 0$, we generate other values for row and column using the above relation.

For cell (1, 1)	$C_{11} = R_1 + K_1$	$10 = 0 + K_1$	∴	$K_1 = 10$
For cell (2, 1)	$C_{21} = R_2 + K_1$	$40 = R_2 + 10$	∴	$R_2 = 30$
For cell (2, 2)	$C_{22} = R_2 + K_2$	$60 = 30 + K_2$	∴	$K_2 = 30$
For cell (2, 3)	$C_{23} = R_2 + K_3$	$40 = 30 + K_3$	∴	$K_3 = 10$
For cell (3, 2)	$C_{32} = R_3 + K_2$	$16 = R_3 + 30$	∴	$R_3 = -14$

Step 5. Calculation of values unoccupied cells or further improvement test.

$$\Delta_{ij} = C_{ij} - R_i - K_j$$

- $\Delta_{12} = C_{12} - R_1 - K_2 = 20 - 0 - 30 = -10$
- $\Delta_{13} = C_{13} - R_1 - K_3 = 20 - 0 - 10 = 10$
- $\Delta_{31} = C_{31} - R_3 - K_1 = 10 + 14 - 10 = 14$
- $\Delta_{33} = C_{33} - R_3 - K_3 = 24 + 14 - 10 = 28$

$\therefore$ Since, above calculated total cost of Rs. 27,900 is not optimal because Δ_{12} is in negative. Loop path should be formulated to minimize above cost.

Example 2

Determine the minimum transportation cost from the following matrix. [T.U. 2061]

Warehouse	Stores				Supply
	P_1	P_2	P_3	P_4	
	Cost per unit				
W_1	45	60	45	30	70
W_2	35	15	35	35	60
W_3	30	25	45	55	90
Demand	60	40	60	20	180 \ 220

Solution

The given transportation problem is unbalanced as supply exceeds demand by 220 –180 = 40 units. So, we need to create a dummy store for supplying excess units. The unit transportation cost will be taken as Rs. 0. The balanced transportation will be as follows:

Step 1. Getting an initial basic feasible solution by VAM.

Warehouse		Stores					Supply	Row cost difference			
		P_1	P_2	P_3	P_4	D_p		I	II	III	IV
W_1		X 45	X 60	$\boxed{10}$ 45	$\boxed{20}$ 30	$\boxed{40}$ 0	~~70~~ ~~30~~ ~~10~~	← 30	15	15	← 15
W_2		X 35	$\boxed{40}$ 15	$\boxed{20}$ 35	X 35	X 0	~~60~~ ~~20~~	15	← 20	0	0
W_3		$\boxed{60}$ 30	X 25	$\boxed{30}$ 45	X 55	X 0	~~90~~ ~~30~~	25	5	← 15	10
Demand		~~60~~	~~40~~	~~60~~ ~~40~~ ~~10~~	~~20~~	~~40~~	220				
Column cost difference	I	5	10	10	5	0					
	II	5	10	10	5	–					
	III	5	–	10	5	–					
	IV	–	–	10	5	–					

Step 2.

Initial transportation cost = $10 \times 45 + 20 \times 30 + 40 \times 0 + 40 \times 15 + 20 \times 35 + 60 \times 30 + 30 \times 45$

$= 450 + 600 + 0 + 600 + 700 + 1800 + 1350 =$ Rs. 5500

Step 3. Test of degeneracy

Since, number of occupied cells = No. of rows + No. of columns – 1

$7 = 3 + 5 – 1$

This is the case of non-degeneracy.

Step 4. Testing the optimality condition

Step 4.1 Calculation of row values and column values for occupied cells using the relation.

$C_{ij} = R_i + K_j$

Let's assume $R_1 = 0$ as it has many occupied cells.

$C_{13} = R_1 + K_3 \Rightarrow 45 = 0 + K_3 \Rightarrow K_3 = 45$

$C_{14} = R_1 + K_4 \Rightarrow 30 = 0 + K_4 \Rightarrow K_4 = 30$

$C_{15} = R_1 + K_5 \Rightarrow 0 = 0 + K_5 \Rightarrow K_5 = 0$

$C_{23} = R_2 + K_3 \Rightarrow 35 = R_2 + 45 \Rightarrow R_2 = - 10$

$C_{33} = R_3 + K_3 \Rightarrow 45 = R_3 + 45 \Rightarrow R_3 = 0$

$C_{31} = R_3 + K_1 \Rightarrow 30 = 0 + K_1 \Rightarrow K_1 = 30$

$C_{22} = R_2 + K_2 \Rightarrow 15 = -10 + K_2 \Rightarrow K_2 = 25$

Step 4.2 Calculation of improvement indices for unoccupied cells by using the relation,

$$\Delta_{ij} = C_{ij} - (R_i + K_j)$$

Here, $\Delta_{11} = C_{11} - (R_1 + K_1) = 45 - (0 + 30) = 15$

$\Delta_{12} = C_{12} - (R_1 + K_2) = 60 - (0 + 25) = 35$

$\Delta_{21} = C_{21} - (R_2 + K_1) = 35 - (-10 + 30) = 15$

$\Delta_{24} = C_{24} - (R_2 + K_4) = 35 - (-10 + 45) = 0$

$\Delta_{25} = C_{25} - (R_2 + K_5) = 0 - (-10 + 0) = 10$

$\Delta_{32} = C_{32} - (R_3 + K_2) = 25 - (0 + 25) = 0$

$\Delta_{34} = C_{34} - (R_3 + K_4) = 55 - (0 + 30) = 25$

$\Delta_{35} = C_{35} - (R_3 + K_5) = 0 - (0 + 0) = 0$

Since, $\Delta'ij's \geq 0$, the optimal solution has been obtained.

i.e. $X_{13} = 10, X_{14} = 20, X_{15} = 40, X_{22} = 40, X_{23} = 20, X_{31} = 60, X_{33} = 30$

and minimum transportation cost = Rs. 5500

Assignment

Example 1

The ABC Company has three jobs to be done on three machines. Each job must be done on one and only one machine. The cost of each job on each machine is given in the following table.

Cost information			
Jobs	**X**	**Y**	**Z**
A	4	6	8
B	2	3	4
C	4	8	5

Give the job assignments, which will minimize cost. [T.U. 2059]

Solution

Step 1: Subtract the lowest element of each row from all the elements of corresponding row.

i.e. $R_1 \rightarrow R_1 - 4$ $R_2 \rightarrow R_2 - 2$ $R_3 \rightarrow R_3 - 4$

Jobs	Machines		
	X	**Y**	**Z**
A	0	2	4
B	0	1	2
C	0	4	1

Step 2: Subtract the smallest elements of each column from all the elements of corresponding column.

i.e., $C_2 \to C_2 - 1$ $C_3 \to C_3 - 1$

Jobs	Machines		
	M_1	M_2	M_3
A	0	1	3
B	0	0	1
C	0	3	0

Step 3: Minimum numbers of lines to cover all zeros are three which is equal to number of rows. Therefore we need not further processing. The optimal solution has been obtained and presented as below:

Jobs	Machines		
	M_1	M_2	M_3
A	$\boxed{0}$	1	3
B	$\cancel{0}$	$\boxed{0}$	1
C	$\cancel{0}$	3	$\boxed{0}$

Step 4: The minimum cost is:

Job	Machine	Assignment Cost
A	M_1	Rs 4
B	M_2	Rs 3
C	M_3	Rs 5
Minimum assignment cost		Rs 12

Example 2

Four children in a household were assigned three different household chores to be done. The children are motivated to get pocket money for the job. Assign the jobs to the children in such a way that their pocket money income is maximum. [T.U. 2062]

Children	Clean the house	Wash clothes	Cook dinner
Ram	1	4	5
Laxman	2	3	3
Bharat	3	3	3
Shatrughan	5	1	2

Solution

Step 1: First of all, we should make square matrix to given non square matrix for solution. Therefore, we introduce dummy column to convert 4×4 square matrix.

	Jobs			
Children	Clean house	Wash clothes	Cook dinner	Dummy
Ram	1	4	5	0
Laxman	2	3	3	0
Bharat	3	3	3	0
Shatrughan	5	1	2	0

Step 2: Subtract all elements of initial matrix from the largest element 5 to convert the given maximization problem to minimization problem.

Children	Clean house	Wash clothes	Cook dinner	Dummy
Ram	4	1	0	5
Laxman	3	2	2	5
Bharat	2	2	2	5
Shatrughan	0	4	3	5

Step 3: Subtracting the smallest elements of each row from all element of corresponding row.

i.e. $R_2 \to R_2 - 2$, $R_3 \to R_3 - 2$,

Children	Clean house	Wash clothes	Cook dinner	Dummy
Ram	4	1	0	5
Laxman	1	0	0	3
Bharat	0	0	0	3
Shatrughan	0	4	3	5

Step 4: Subtracting the smallest elements of each column from all the elements of corresponding column.

i.e. $C_4 \to C_4 - 3$

Children	Clean house	Wash clothes	Cook dinner	Dummy
Ram	4	1	0	2
Laxman	1	0	0	0
Bharat	0	0	0	0
Shatrughan	0	4	3	2

Step 5: Here, there is at least one zero for each row and column. Now, we have to cross zeros. Minimum number of lines to cover all zeros is equal to number of rows. Therefore, the optimal solution has been obtained.

Children	Clean House	Wash clothes	Cook dinner	Dummy
Ram	4	1	$\boxed{0}$	2
Laxman	1	$\cancel{0}$	$\cancel{0}$	0
Bharat	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
Shatrughan	$\boxed{0}$	4	3	2

Step 6: Optimal Assignment

Children	Job	Income
Ram	Cook dinner	5
Laxman	D	0
Bharat	Wash the cloth	3
Shatrughan	Clean the house	5
	Total Income	Rs. 13