MA 203

Numerically Solving Lane-Emden Equations to Explore Stellar Structure

Jayraj Dulange [22110106]
Mumuksh Jain [22110160]
Nishant Kumar [22110170]
Pratham Sharda [22110203]
Shreyans Jain [22110245]

Problem Statement

Stellar physics has long been pivotal in astrophysical research, shedding light on stars' lifecycles and behaviors. Astrophysical research focuses on understanding stars' structure and evolution, essential cosmic building blocks. Precise knowledge of properties like mass, radius, and density is vital for comprehensive universe understanding. Researchers numerically solve the Lane-Emden equations, mathematically describe how a star's density is distributed under self-gravity and pressure forces. These equations are invaluable tools for modeling stars' internal structures, aiding astrophysicists in their investigations.

Physical Model

Stellar birth relies on nuclear fusion in the core, creating vital energy for stability and brightness. Inside a star, there's a pressure-temperature feedback acting as a thermostat. A slight increase in fusion rates raises the core's temperature, boosting pressure. Interestingly, gravity keeps the core from expanding too fast, maintaining stable density and temperature. This feedback ensures a stable star by regulating fusion rates. Stellar equations refer to a set of mathematical equations and models used in astrophysics and astronomy to describe the physical properties, behavior, and evolution of stars. These equations are fundamental tools for understanding how stars are structured, how they generate energy, and how they change over time. We are using 4 stellar equations that are:

1. Mass Conservation
2. Hydrostatic Pressure Conservation
3. Luminosity Conservation
4. Energy Transport through Radiation & Convection

Lane-Emden Equation

In our model, we are assuming that the properties due to stellar evolution are remaining constant over time. All stars are spherical and symmetric about their centre of masses. The star is in hydrostatic and thermal equilibrium.

A relation of the form

$$P = K(\rho^{\gamma}) = K\rho^{(1+1/n)}$$

where K and γ are constants is assumed with polytropic conditions.

These assumptions lead us to the Lane Emden Equation i.e. $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$

where θ and ξ are dimensionless quantities.

At the center of the star, $\varepsilon(0) = 0$, $\theta(0) = 1$ so that $\rho=\rho_c$.

Since $dP / dr \to 0$ as $r \to 0$, $d\theta / d\varepsilon = 0$ at $\varepsilon=0$.

The outer boundary (surface) is the first location where $\rho = 0$ or $\theta(\varepsilon) = 0$

Solution Methodology

The analytical solutions of Lane-Emden Equation is possible only for n=0,1 and 5

When n=0, the sphere is called Homogeneous Sphere and $\xi=\sqrt{6}$

When n=1, $\xi=\pi$ and,

When n=5, the sphere is called Plummer Sphere and $\xi=\infty$

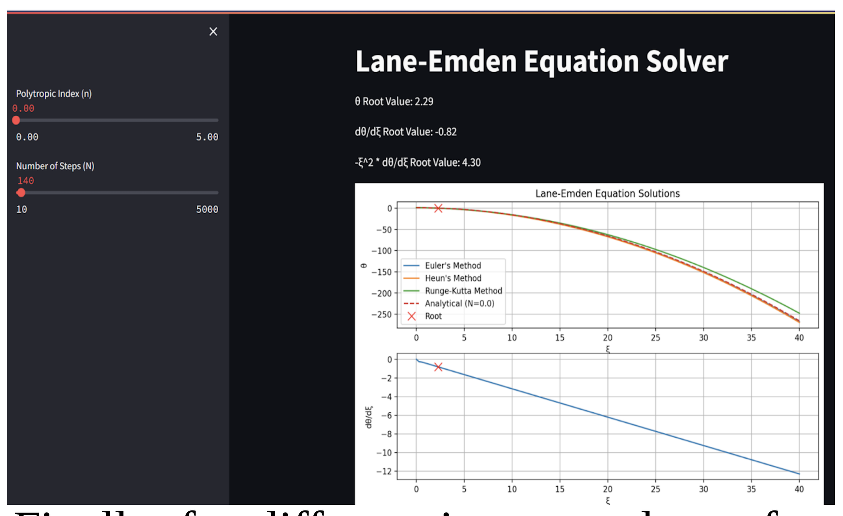
For $n \geq 5$, we do not obtain any solution as the mass keeps on increasing as we move outwards

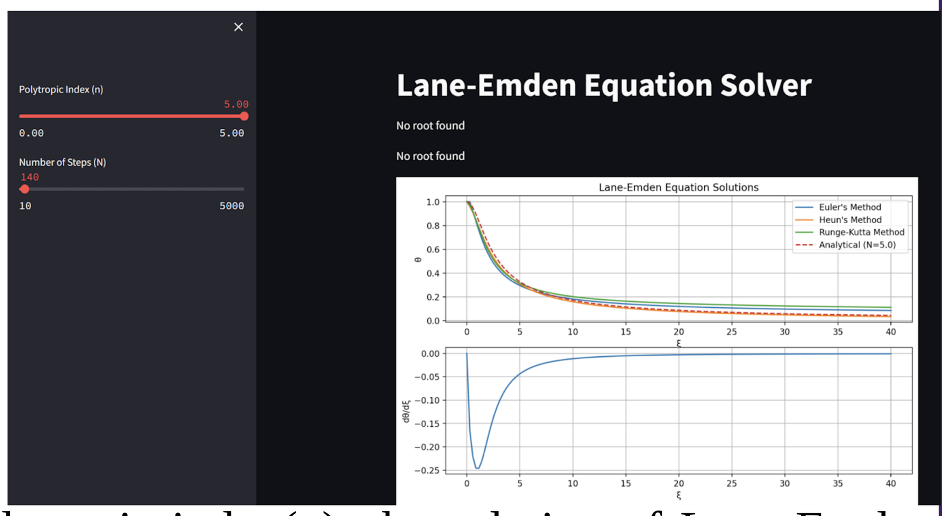
For other values of n, the equation is solved numerically. We have used 3 different methods to solve this equation numerically i.e.

1. Euler's Method: $y_{i+1}=y_i+\Phi h$ and $\Phi=dy/dx$
2. Heun's Method: $y_{i+1}=y_i+\Phi h$ and $\Phi=(f(x_i,y_i)+f(x_{i+1},y^o_{i+1}))/2$
3. Runge Kutta Method (4th Order): $y_{i+1}=y_i+(k_1+2k_2+2k_3+k_4)/6 + O(h^5)$

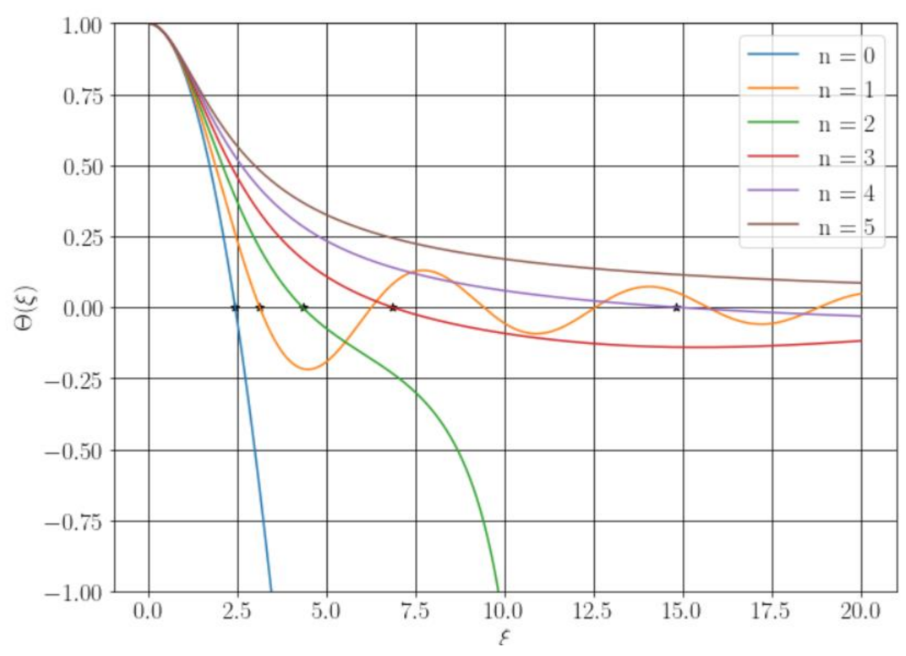
Where $k_1=hf(t_i,y_i)$, $k_2=hf(t_i+h/2,y_i+k_1/2)$, $k_3=hf(t_i+h/2,y_i+k_2/2)$, $k_4=hf(t_{i+1},y_i+k_3)$

These methods are used and the code for solving this problem is written in python. Streamlit is an open source web app framework and is used here to display the results. There is a slider which allows the user to indirectly choose the step size as well as a different slider for polytropic index n.





Finally, for different integer values of polytropic index(n), the solution of Lane Emden equation looks like this



Results & Conclusion

Solutions of Lane Emden Equation decrease monotonically (except for n=1) and have θ=0 at ξ= ξ1 (i.e. the stellar radius). With an increasing polytropic index, the star becomes more centrally condensed.

We have also analyzed some important values of n which specify different types of spherical formation as shown in table below.

Useful Polytropic Indices

n	γ	Description
0	—	Incompressible gas; constant density
0.42857	10/3	Thomas-Fermi EOS
1	2	analytic solution to Lane-Emden equqation; constant R_*
1.5	5/3	ideal monatomic gas EOS; convective; non-relativistic degenerate
2	3/2	Holzer & Axford's maximum γ for an accelerating solar wind
2.5	7/5	ideal diatomic gas EOS
3	4/3	Eddington's standard model; ultra-relativistic degenerate; constant M_*
3.25	17/13	Chandrasekhar's constant-ϵ Kramers model
5	6/5	Schuster sphere of infinite radius
∞	1	Isothermal gas; Bonnor-Ebert sphere

$$n=0 \quad \theta(\xi) = 1 - \frac{\xi^2}{6} \quad \xi_0 = \sqrt{6}$$

$$n=1 \quad \theta(\xi) = \frac{\sin\xi}{\xi} \quad \xi_1 = \pi$$

$$n=5 \quad \theta(\xi) = \left(1+\frac{\xi^2}{3}\right)^{-1/2} \quad \xi_5 = \infty$$

We also compared the n=3 polytrope with the Standard Solar Model, finding quite good agreement considering how simple the input physics was in the Lane Emden Equation as compared to very complicated equations of stellar formation.

These solutions are used to find the approximate mass, central pressure and central density of a star. We have solved this for n=3 which is a good approximation for sun-like stars.

Thus we can use this solution in the stellar solution for many applications such as formation of star formation, defining the conditions for formation of white dwarf planet, deriving the Chandrashekhar mass constant, Eddington's model, analyzing the stability of the stars and many other applications.

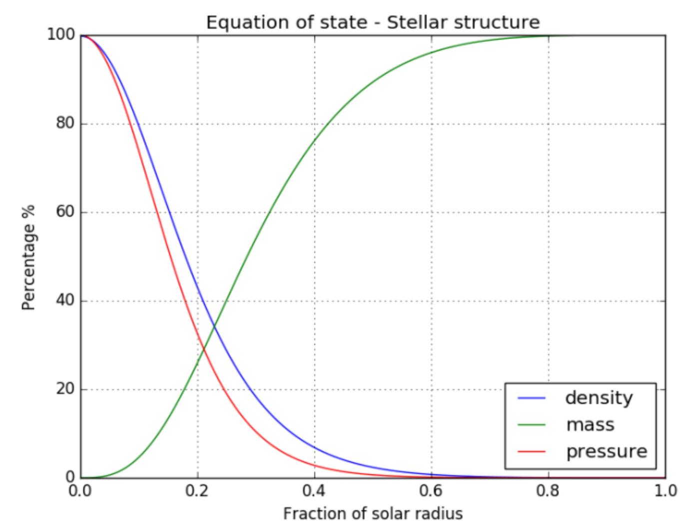
Analysis

Analysing the mass ,central pressure and central density of star

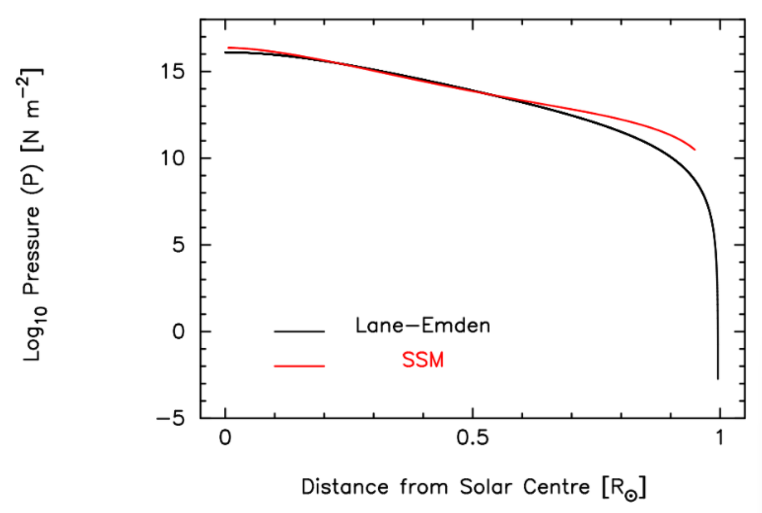
$$M = 4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_c^{\frac{3-n}{2n}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$$

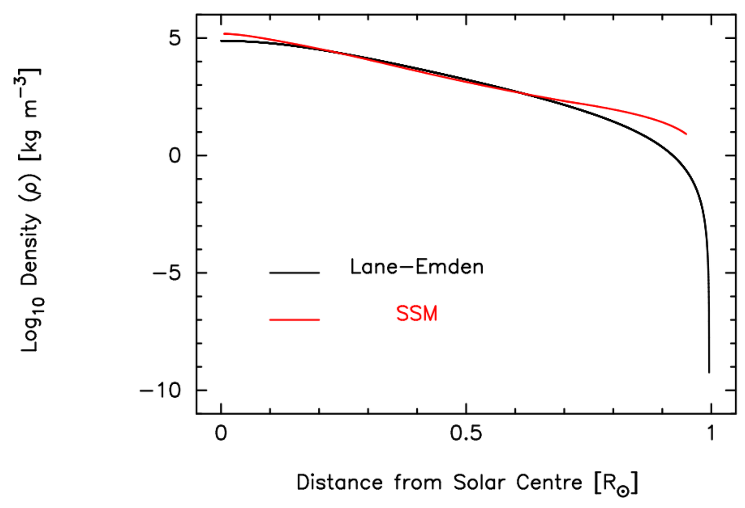
$$\rho_c = -\frac{1}{3}\left[\frac{\xi}{\dot{\theta}_n(\xi)}\right]_{\xi=\xi_1}\bar{\rho}$$

$$P_c = K\rho_c^{1+\frac{1}{n}} = \frac{1}{4\pi(n+1)\left[\dot{\theta}_n(\xi)\right]^2_{\xi=\xi_1}}\frac{GM^2}{R^4}$$



Analyzing the results and comparison with SSM





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