MA 203 Project
Numerically Solving Lane-Emden Equations to Explore Stellar Structure

Problem statement

In the realm of astrophysical research, a fundamental endeavor is the exploration of the structure and evolution of stars. These celestial bodies are essential building blocks of the cosmos, and precise knowledge of their properties, such as radius, mass, and density, is crucial for a comprehensive understanding of the universe. To this end, this research focuses on numerically solving the Lane-Emden equations—an established theoretical framework—to delve into the intricate structure and evolution of stars, thereby enabling the precise determination of their essential parameters.

Stellar physics has long been a cornerstone of astrophysical research, offering profound insights into the lifecycle and behavior of stars. The Lane-Emden equations, formulated by **Jonathan Lane and Robert Emden**, provide a mathematical description of the density distribution within a star under the influence of self-gravity and pressure forces. These equations are instrumental in modeling the internal structure of stars, making them invaluable tools for astrophysicists.

The numerical solution of Lane-Emden equations offers a powerful means to delve deeper into the structure and evolution of stars. Numerical methods, such as Runge-Kutta and Euler's method, provide computational techniques to tackle these equations, enabling us to obtain highly accurate results that are difficult to achieve through analytical methods alone. The numerical approach also allows for the investigation of a broader range of stellar scenarios, including stars with varying compositions, masses, and evolutionary stages.

This research is driven by the following objectives:

1. Develop and implement efficient numerical algorithms, including the Runge-Kutta and Euler's methods, to solve the Lane-Emden equations for a diverse set of stellar models.
2. Explore the structural evolution of stars across different spectral classes, ages, and evolutionary phases, utilizing the numerical solutions to derive precise stellar parameters such as radius, mass, and density.
3. Validate the numerical results through comparisons with observational data and established theoretical models, ensuring the accuracy and reliability of the derived stellar parameters.

1. Physical Model

Stellar formation relies on the core's nuclear fusion reactions, which generate immense energy. This energy is crucial for maintaining a star's stability and brightness. Stars are in a constant struggle between gravity, attempting to collapse the star, and pressure, generated by high temperatures and nuclear reactions, pushing outward. The balance of these forces is essential for a star's survival.

Within a star, a pressure-temperature feedback mechanism acts as a thermostat. When fusion rates increase slightly, the core's temperature rises, which increases pressure. Surprisingly, gravity prevents the core from expanding too quickly, maintaining stable density and temperature. This feedback keeps the fusion rate in check and ensures a stable star.

If stars had no energy source like nuclear reactions, they wouldn't form or survive. Without nuclear fusion, gravity would cause rapid collapse, and the absence of the pressure-temperature feedback mechanism would prevent stabilization. In this scenario, stars as we know them, with their luminosity and longevity, would not exist.

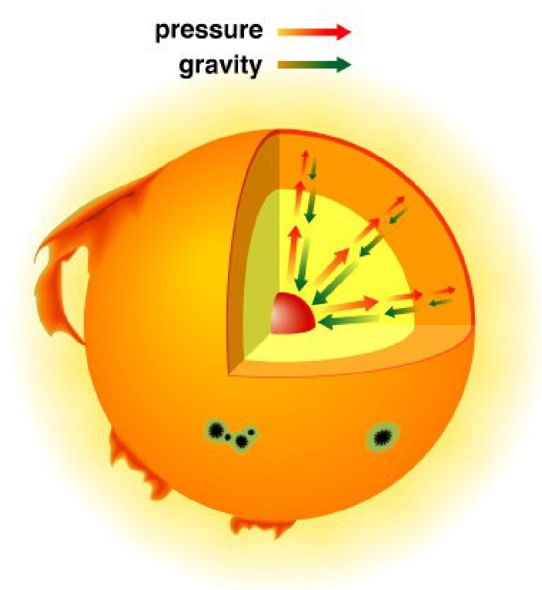


Fig.1 Direction of pressure and gravity inside a star

2. Stellar Equations

Stellar equations refer to a set of mathematical equations and models used in astrophysics and astronomy to describe the physical properties, behavior, and evolution of stars. These equations are fundamental tools for understanding how stars are structured, how they generate energy, and how they change over time. Stellar equations are essential for unraveling the mysteries of the cosmos, from the birth of stars to their eventual fates.
These are the four stellar equations that help us understand the mechanical and thermal properties of star formation and evolution.

2.1 Mass Conservation Equation:

The mass conservation equation, also known as the continuity equation, is a fundamental principle in astrophysics. It states that the total mass within a closed system remains constant over time unless there is an inflow or outflow of mass across the system's boundaries.

$$\frac{dm}{dr} = 4\pi r^2 \rho(r)$$

2.2 Hydrostatic Pressure Conservation Equation

The hydrostatic equilibrium equation describes the balance between the inward force of gravity and the outward pressure force within a star. It is crucial to understand how stars maintain their stable structures by resisting gravitational collapse.

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

2.3 Luminosity Conservation Equation:

This empirical relation describes the relationship between a star’s mass and its luminosity (brightness). It helps astronomers estimate the mass of a star based on its observed luminosity, aiding in the classification of stars.

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\,\varepsilon(r)$$

2.4 Energy Transport through Radiation and Convection:

These equations describe how energy generated in the core of a star is transported to its surface. There are two primary mechanisms: Radiative Diffusion, where energy is carried by photons, and convective transport, where energy is transported through the physical motion of hot gas.

$$\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2}\left(\frac{\rho(r)\,\kappa_R(r)}{T^3(r)}\right)L(r) \qquad\qquad \frac{P}{T}\left(\frac{dT}{dP}\right) = \frac{(\gamma - 1)}{\gamma}$$

through Radiation through Convection

2.5 Three supplement equations:

P = P (ρ, T, chemical composition) — Equation of State(EOS)
κR = κR (ρ, T, chemical composition)
ε = ε (ρ, T, chemical composition)

where the parameters resemble:

- *r = radius*
- *P = pressure at r*
- *m = mass of material within r*
- *ρ =density at r*
- *L = luminosity at r*
- *T = temperature at r*
- *κR = Rosseland mean opacity at r*
- *ε=energy release*

3. Assumptions

Firstly, we will neglect the rate of change of properties due to stellar evolution in the first instance; assume these are constant with time.

All stars are spherical and symmetric about their centers of mass that is, the basic assumption is to consider a self-gravitating, spherically symmetric fluid in hydrostatic equilibrium.

Unfortunately, unless some unrealistic assumptions are made, there are no analytical solutions to the stellar equations, given the complicated nature of the functions P, κ, and ε when all relevant processes are included.

As the first simplification, we assume a star is in hydrostatic and thermal equilibrium. In this case, the four partial differential equations for stellar structure reduce to ordinary, time-independent differential equations.

The four equations of stellar structure divide naturally into two groups: one describing the mechanical structure of the star and the other giving the thermal structure.

a. Equations defining mechanical structure:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

b. Equations defining thermal structure:

$$\frac{P}{T}\left(\frac{dT}{dP}\right) = \frac{(\gamma - 1)}{\gamma}$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\, \varepsilon(r)$$

However, the only contact between the mechanical variables and thermal equations is through the temperature dependence of the equation of state. If we can write the pressure in terms of the density alone, without reference to the temperature, then we can separate these two equations from the others and solve them by themselves. Solving two differential equations (plus one algebraic equation relating P and ρ is much easier than solving seven equations. We have already seen that under certain circumstances, the pressure can indeed become independent of temperature and only depend on density, i.e., degeneracy pressure, or the case where pressure and density are related adiabatically (convection).

A relation of the form $P = K(\varrho^{\gamma}) = K\varrho^{(1+1/n)}$ where K and γ are constants is assumed with polytropic conditions; this is called a polytropic relation, and the resulting models are called polytropic models.

When the equation of state can be written in this form, the temperature does not enter at all into the equations, and the calculations of stellar structure simplify enormously. There are even analytical solutions for certain values of n(We will see that in a later section)

The pressure and density are related by a power-law $P = K(\varrho^{\gamma}) = K\varrho^{(1+1/n)}$ (it customary to adopt γ=(1+1/n), or n=1/(1- γ), where n is the polytropic index)

4. Derivations

4.1 Deriving the Lane Emden Equation:

Take the equation of Hydrostatic Equilibrium,

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

Multiply by $\frac{r^2}{\rho(r)}$ and differentiate with respect to r gives,

$$\frac{d}{dr}\left(\frac{r^2}{p(r)}\left(\frac{dP(r)}{dr}\right)\right) = -\frac{GdM(r)}{dr}$$

Now substitute the equation of Mass-Conversation on the right-hand side to obtain,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(v)}{dr}\right) = -4\pi G\rho(r)$$

Now we make use of our approximation. We use the power law approximation that is

$P = K(\varrho^{\gamma}) = K\varrho^{(1+1/n)}$ where n is the polytropic index.

$$\frac{K(n+1)}{r^2 n}\frac{d}{dr}\left(\frac{r^2}{\rho}\rho^{1/n}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

$$\frac{K(n+1)}{4\pi Gn}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{1/n^{-1}}\frac{d\rho}{dr}\right) = -\rho$$

Furthermore, we introduce a dimensionless variable θ in range 0 <θ<=1 by $\rho(r) = \rho_c\theta^n(r)$

where ρ(c) is the central density,

the equations become

$$\frac{K(n+1)\rho_c^{1/n^{-1}}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right)=-\theta^n$$

To simplify the equation further, we introduce the dimensionless radius $\xi = r/\alpha$, where

$$\alpha^2=\frac{K(n+1)}{4\pi G\rho_{\mathrm{c}}^{(1-1/n)}}$$

the equation finally becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n$$

The above equation is known as the **Lane-Emden** equation; it defines the rate of change of density within a stellar interior, and the solution $\theta = \theta_n(\xi)$ is called the Lane-Emden function.

4.2 Deriving the other quantities from the solutions of the Lane-Emden Equation

So once we obtain the solution of the Lane-Emden equation, we need to find the mass variation with respect to the radius of the star.

The radius at which mass density reaches zero is clearly the radius of the star, so $R = \alpha\xi_1$.

Similarly, given a $\theta(\xi)$ solution, we can also compute the mass of the star.

$$
\begin{aligned}
M &= \int_0^R 4\pi r^2 \rho dr \\
&= 4\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2 \Theta^n d\xi \\
&= -4\pi\alpha^3\rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right) d\xi \\
&= -4\pi\alpha^3\rho_c \xi_1^2 \left(\frac{d\Theta}{d\xi}\right)_{\xi_1}.
\end{aligned}
$$

From a polytropic model, we can derive other useful numbers and relationships. For example, it is often convenient to know how centrally concentrated a star is, i.e. how much larger its central density is than its mean density. We define this quantity as

$$
D_N \equiv \frac{\rho_c}{\bar{\rho}} = \frac{\rho_c 4\pi R^3}{3M} = \frac{4\pi}{3}\rho_c(\alpha\xi_1)^3\left[-4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1} = \left[-\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1}
$$

Another useful relationship is between mass and radius. We start by expressing the central density **ρ(c)** in terms of the other constants and our length scale α.

$$
\rho_c = \left[\frac{K(n+1)}{4\pi G\alpha^2}\right]^{n/(n-1)}
$$

Derived from using

$$
\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}
$$

Substitute this into the equation for the mass:

$$M = -4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1} = -4\pi\alpha^3\left[\frac{K(n+1)}{4\pi G\alpha^2}\right]^{n/(n-1)}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi}$$

Making the substitution α= R/ξ and rearranging, we arrive at

$$\left[\frac{GM}{-\xi_1^2(d\theta/d\xi)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

- n=1 is a special case, for which the radius is independent of mass and is uniquely determined by K:

$$R = \xi_1\left(\frac{K}{2\pi G}\right)^{1/2}$$

- Another important polytropic index is n = 3, for which the R dependence disappears. We find that

$$M = -\frac{4}{\sqrt{\pi}}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\left(\frac{K}{G}\right)^{3/2}$$

For other *n,* **mass and radius are related** by **M~ $R^{(n-3)/(n-1)}$.**

Another important relation is **obtained between the central pressure and the central density**. Substitute K from the **mass-radius** relation:

$$P_c = K\rho_c^{1+\frac{1}{n}} \rightarrow \left[\frac{GM}{-\xi_1^2(d\theta/d\xi)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

We obtain

$$P_c = \frac{(4\pi G)^{1/n}}{(n+1)}\left[\frac{GM}{M_n}\right]^{\frac{n-1}{n}}\left(\frac{R}{R_n}\right)^{\frac{3-n}{n}}\rho_c^{\frac{n+1}{n}}$$

Where $M_n = -\xi_1^2\,(d\theta/d\xi)_{\xi_1}$ and $R_n = \xi_1$

Now eliminating R, using $D_N \equiv \frac{\rho_c}{\bar{\rho}} = \frac{\rho_c 4\pi R^3}{3M}$, and assembling all n-dependent coefficients into one constant B_n, we get

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3}$$

$$B_n = -\left[(n+1)\left(\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right)^{2/3}\right]^{-1}$$

The remarkable property of this relation is that it depends on the polytropic equation of state only through the value of B(n), which varies very slowly with n.

n	**D_n**	**M_n**	**R_n**	**B_n**
1.0	3.290	3.14	3.14	0.233
1.5	5.991	2.71	3.65	0.206
2.0	11.40	2.41	4.35	0.185
2.5	23.41	2.19	5.36	0.170
3.0	54.18	2.02	6.90	0.157
3.5	152.9	1.89	9.54	0.145

5. Non-Dimensionalization of Governing Equations

The Lane-Emden Equation is a "dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid"

1. Central density: ρ_c
   Define $\rho(r) = \rho_c \theta^n(r)$ such that

   Then: $P(r) = K\rho^\gamma(r)$ where, $\gamma = \frac{n+1}{n} = 1 + \frac{1}{n}$

   $$= K\rho_c^\gamma \theta^{n\gamma}(r)$$

   $$= K\rho_c^{1+\frac{1}{n}} \theta^{n+1}(r)$$

   Given n, K, and ρ_c these two equations define the distribution of pressure and density in the star

2. We also define $r = \alpha\xi$ with a dimensional radius-like variable.

6. Boundary Condition

- At the center of the star where ε(0) =0, θ(0) = 1 so that $\rho = \rho_c$.
- Since $dP / dr \rightarrow 0$ as r → 0, $d\theta / d\varepsilon = 0$ at ε=0.
- The outer boundary (surface) is the first location where ρ = 0 or θ(ε) = 0; this location is referred to as ε_1.

6.1 Derivation for second boundary condition

$$\xi \equiv \frac{r}{\alpha} \text{ so } d\xi = \frac{1}{\alpha} dr \Rightarrow \frac{d\theta}{d\xi} = \frac{\alpha d\theta}{dr}$$

But $\rho \equiv \rho_c \theta^n$, so $\frac{d\rho}{dr} = n\rho_c \theta^{n-1} \frac{d\theta}{dr}$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{n\rho_c \theta^{n-1}} \frac{d\rho}{dr} \quad \text{and} \quad \frac{d\theta}{d\xi} = \frac{\alpha}{n\rho_c \theta^{n-1}} \frac{d\rho}{dr}$$

But in spherical symmetry for hydrostatic equilibrium, ρ is a local maximum at r=0, so

$$\frac{d\rho}{dr} = 0$$

7. Analytical Solution

7.1 For n=0 i.e Constant Density Case

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2 d\theta}{d\xi}\right) = -\theta^n = -1$$

$$\xi^2\frac{d\theta}{d\xi} = -\int \xi^2 = -\frac{\xi^3}{3} + c$$

$$\frac{d\theta}{d\xi} = \frac{-\xi}{3} + \frac{c}{\xi^2} \Rightarrow \int d\theta = -\frac{1}{3}\int \xi d\xi + \int \frac{c d\xi}{\xi^2}$$

`

$$\theta = \frac{-\xi^2}{6} - \frac{c}{\xi} + D$$

But, $\theta = 1 \ at \ \xi = 0$

So c=0 and D=1 and, $\theta = 1 - \frac{\xi^2}{6}$

7.2 For n=1,

$$\frac{1}{\xi^2}\left(\frac{d}{d\xi}\left(\frac{\xi^2 d\theta}{d\xi}\right)\right) = -\theta$$

The solution is to order zero spherical Bessel function.

$$\theta(\xi) = \frac{\sin\xi}{\xi} \Rightarrow \xi_1 = \pi \;\; at\ \theta = 0 \qquad\qquad \alpha = \frac{R}{\pi}$$

$$nb.\lim_{\xi \to 0}\left(\frac{\sin\xi}{\xi}\right) = 1 \qquad (\,l'Hopital's\ rule)$$

$$\alpha = \left[\frac{P_c(n+1)}{4\pi G\rho^2{}_c}\right]^{\frac{1}{2}} = \left[\frac{P_c}{2\pi \mathrm{G}\rho^2{}_{\mathrm{c}}}\right]^{\frac{1}{2}} = \frac{R}{\pi}$$

$$P_c = \frac{2G}{\pi}\rho_c^2 R^2$$

Also $\qquad \xi \equiv \alpha r = \dfrac{\pi r}{R} \Rightarrow \qquad \theta = \dfrac{\sin(\xi)}{\xi} = \dfrac{R}{\pi r}\sin\left(\dfrac{\pi r}{R}\right)$

$$P = P_c\theta^2 = P_c\left(\frac{R}{\pi r}\sin\left(\frac{\pi r}{R}\right)\right)^2$$

$$\rho = \rho_c\theta = \rho_c\left(\frac{R}{\pi r}\sin\left(\frac{\pi r}{R}\right)\right)$$

$$M = -4\pi\alpha^3\rho_c\left(\frac{d\theta}{d\xi}\right)_{\xi_1}$$

$$\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1} = -\pi \quad \text{for n} = 1$$

$$M = -4\pi\alpha^3\rho_2\xi_1^2, \left(\frac{d\theta}{d\xi}\right)_{\xi_1} = 4\pi^2\rho_c\left(\frac{R}{\pi}\right)^3 = \frac{4}{\pi}\rho_c R^3$$

But, $$P_c = \frac{2G}{\pi}\rho_c^{\ 2}R^2$$

So, $$R^3 = \left(\frac{\pi P_C}{2G\rho_c^2}\right)^{\frac{3}{2}}$$

And, $$P_c = K\rho_c^{\frac{n+1}{n}} = K\rho_c^2$$

So, $$R^3 = \left(\frac{\pi K}{2G}\right)^{\frac{3}{2}}$$

$$R = \left(\frac{\pi K}{2G}\right)^{\frac{1}{2}}$$

Independent of M!

N=	0	1	5
θ=	$1 - \frac{1}{6}\xi^2$	$\frac{sin\xi}{\xi}$	$\frac{1}{\sqrt{1+\xi^2/3}}$
ξ	√ 6	π	∞

8. Solution Methodology

8.1 Initialization and User Input:

- Initialize Streamlit and create a user interface for setting parameters.
- User can select the polytropic index n and the number of steps N using sliders.
- Define the range of t values (a to b) based on your problem requirements.

8.2 Numerical Methods Functions:

- Implement three numerical methods functions: Euler's Method, Heun's Method, and Runge-Kutta Method.
- Each method takes as input the Lane-Emden equation f, the range of t values, initial conditions IV, and the polytropic index polytropic_index.

8.3 Euler's Method:

- Calculate the step size h.
- Initialize arrays for t, theta, and dtheta.
- Iterate through the range of t values using Euler's method and store the results in t and theta arrays.

$$y_{i+1}=y_i+\Phi h \text{ and } \Phi=dy/dx$$

8.4 Heun's Method:

- Calculate the step size h.
- Initialize arrays for t, theta, and dtheta.
- Iterate through the range of t values using Heun's method and store the results in t and theta arrays.

$$y_{i+1}=y_i+\Phi h \text{ and } \Phi=(f(x_i,y_i)+f(x_{i+1},y^o_{i+1}))/2$$

8.5 Runge-Kutta Method (4th Order):

- Calculate the step size h.
- Initialize arrays for t, theta, and dtheta.
- Iterate through the range of t values using the 4th Order Runge-Kutta method and store the results in t and theta arrays.

$$k_1 = hf(t_i, y_i)$$

$$k_2 = hf(t_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$

$$k_3 = hf(t_i + \frac{h}{2}, y_i + \frac{k_2}{2})$$

$$k_4 = hf(t_{i+1}, y_i + k_3)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)) + \mathcal{O}(h^5)$$

8.6 Finding the Root:

- Assuming that the root is where theta_rk4 crosses zero, find the index root_index in the theta_rk4 array.
- Calculate the corresponding value of t at the root index, representing the root of the Lane-Emden equation.

8.7 Plotting the Results:

- Plot the solutions obtained using Euler's Method, Heun's Method, and Runge-Kutta Method as functions of t.
- If the polytropic index is 0, 1, or 5, also plot the analytical solution for comparison.
- Mark the root point on the graph using a red "x" marker.

8.8 Displaying the Results:

- Display the calculated root value and the graph in the Streamlit app.

8.9 User Interaction:

- Users can interact with the app by adjusting the polytropic index and the number of steps using sliders.
- The app dynamically updates the graph and root value based on user inputs.

8.10 Termination:

- The Streamlit app remains active for user interaction and terminates when the user closes the application.
- This methodology outlines the steps involved in solving the Lane-Emden equation numerically using three different methods and visualizing the results in a Streamlit web application.

9. Numerical Solution

The following graphs depict some of the solutions that we have written in our code for some particular value of n while solving the Lane-Emden Equation.
For detailed analysis, we have hosted a stream-lit website on our GitHub.

n	ξ_1	$-\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}$	$\rho_c/\bar{\rho}$
0.0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800.
5.0	∞	1.73205	∞

We have solved the Lane-Emden Equation using three methods: Euler's method, Huen's method, and RK4 method, and have found the necessary parameters as mentioned in the above table for finding other parameters such as mass, central pressure and central density of the star for different values of n.

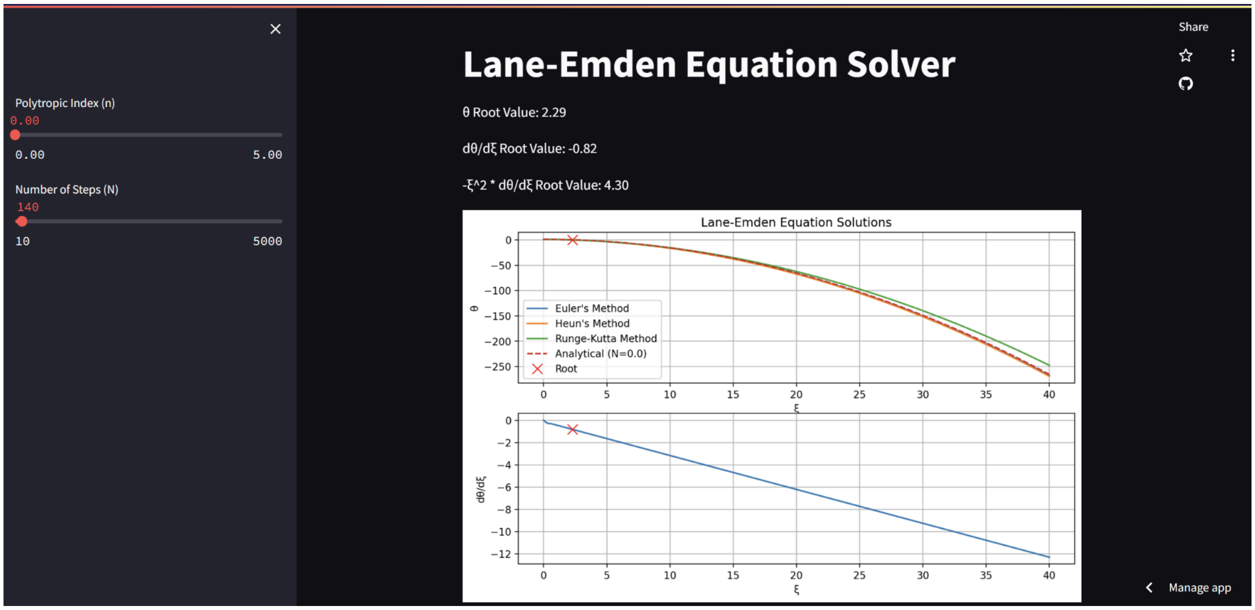


Fig.2 Direction of pressure and gravity inside a star

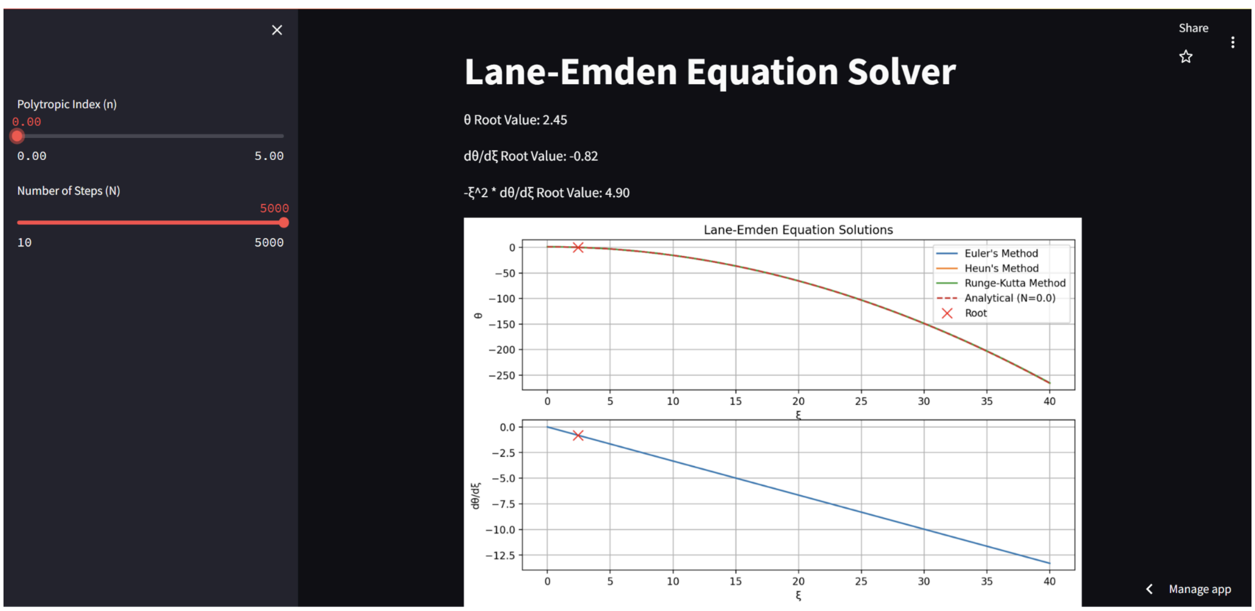


Fig.3 Direction of pressure and gravity inside a star

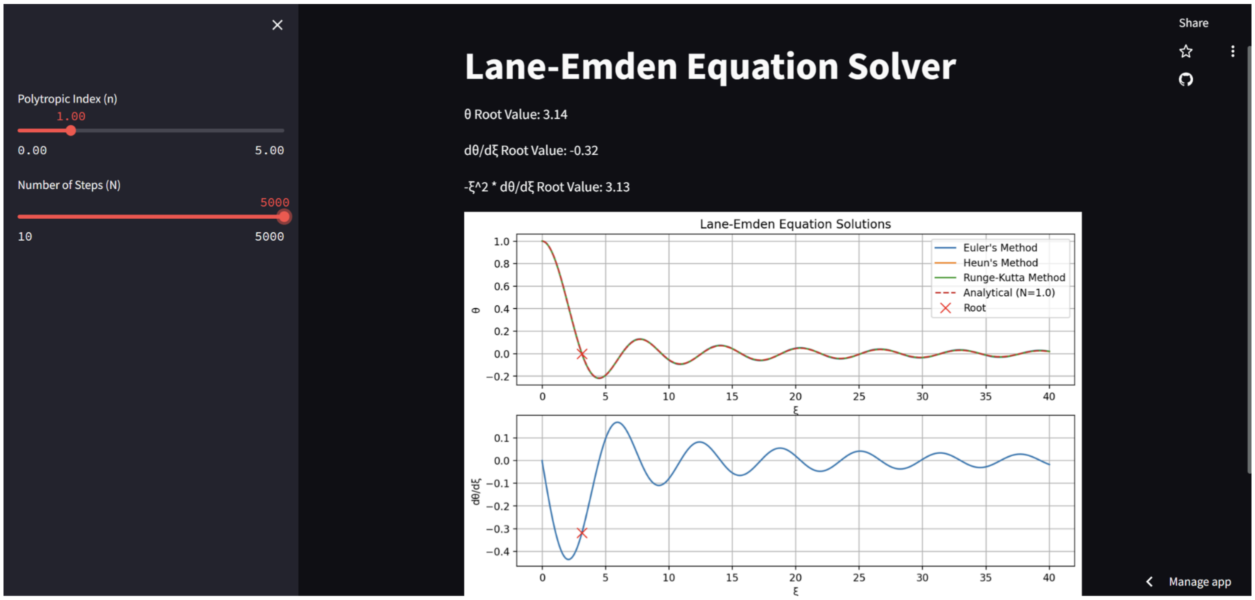


Fig.4 Direction of pressure and gravity inside a star

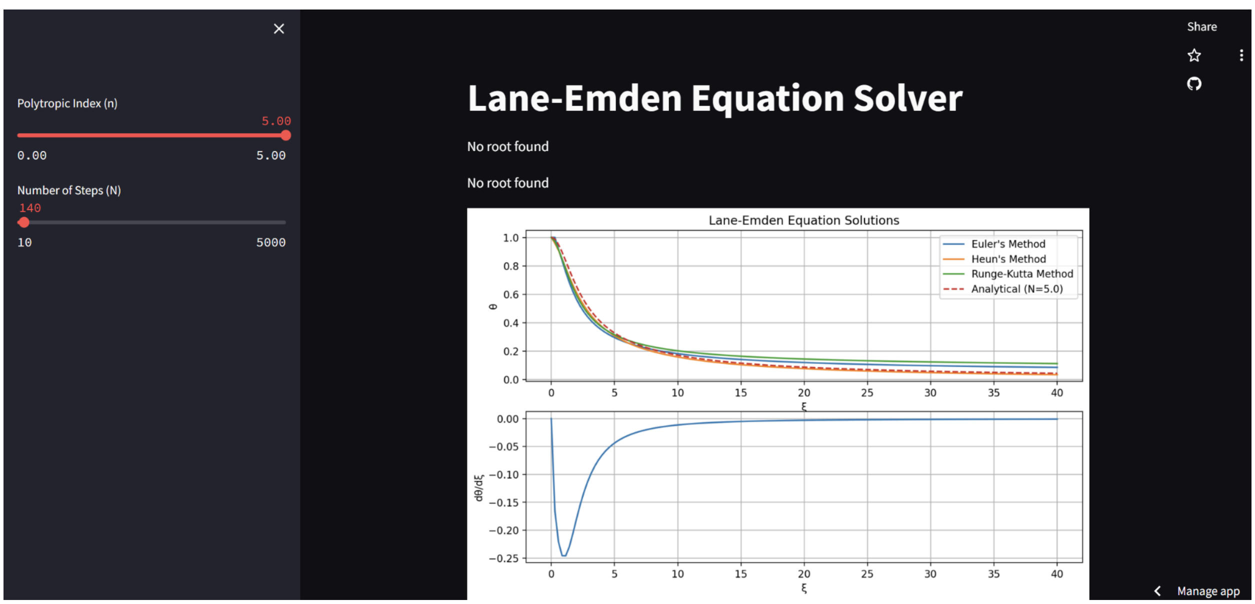


Fig.5 Direction of pressure and gravity inside a star

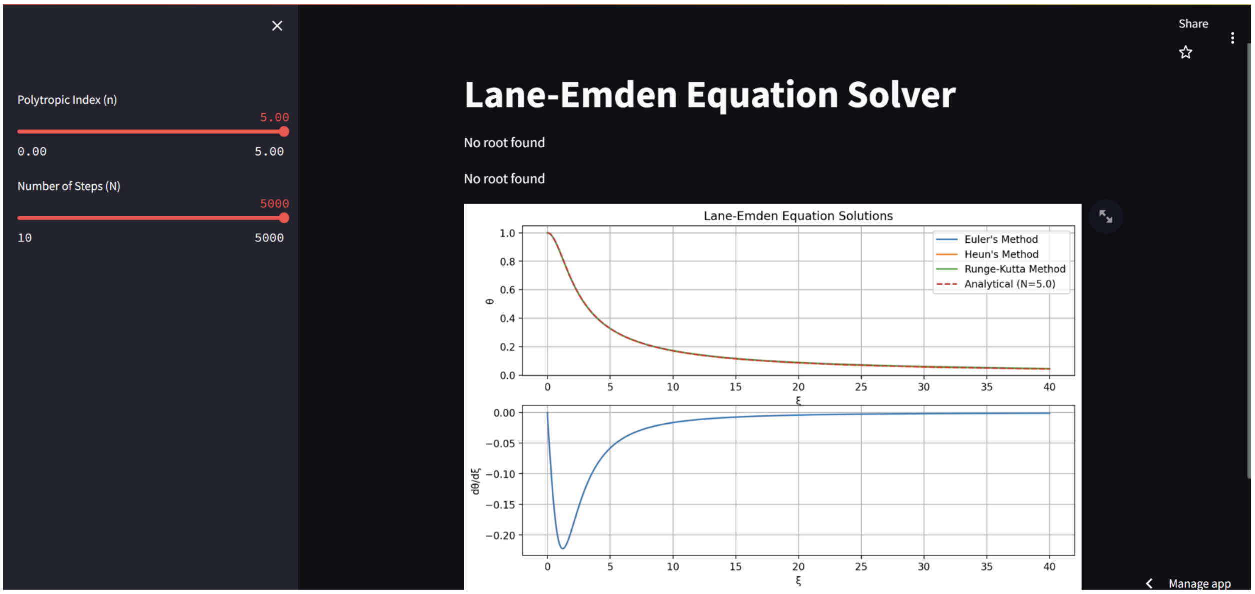


Fig.6 Direction of pressure and gravity inside a star

10. Algorithms Used

10.1 Algorithm for Euler's Method:

1. Define the Lane-Emden equation as f(r, u, theta, n) = -theta^n - 2*u/r.
2. Initialize the parameters, including the range of t values (a to b), the number of steps (N), the initial conditions (IV), and the polytropic index (polytropic_index).
3. Calculate the step size h as (b - a) / N.
4. Create an array t with values from a to b with step h.
5. Initialize arrays theta and dtheta with zeros.
6. Set the initial values for theta[0] and dtheta[0] based on IV.
7. Use Euler's method to solve the Lane-Emden equation iteratively:
   For each step i from 1 to N:
   - Calculate theta[i] as theta[i-1] + h * dtheta[i-1].
   - Calculate dtheta[i] as dtheta[i-1] + h * f(a + i*h, dtheta[i-1], theta[i-1], polytropic_index).
8. After the loop, we have arrays t and theta containing the solution.

10.2 Algorithm for Heun's Method (Improved Euler Method):

1. Initialize the parameters and arrays as described in Euler's Method.
2. Use Heun's Method to solve the Lane-Emden equation iteratively:
   For each step i from 1 to N:
   - Calculate theta_half as theta[i-1] + h * dtheta[i-1].
   - Calculate dtheta_half as dtheta[i-1] + h * f(a + i*h, dtheta[i-1], theta[i-1], polytropic_index).
   - Calculate theta[i] as theta[i-1] + 0.5 * h * (dtheta[i-1] + dtheta_half).
   - Calculate dtheta[i] as dtheta[i-1] + 0.5 * h * (f(a + i*h, dtheta[i-1], theta[i-1], polytropic_index) + f(a + (i+1)*h, dtheta_half, theta[i], polytropic_index).
3. After the loop, we have arrays t and theta containing the solution.

10.3 Algorithm for the Runge-Kutta Method (4th Order):

1. Initialize the parameters and arrays as described in Euler's Method.
2. Use the 4th Order Runge-Kutta Method to solve the Lane-Emden equation iteratively:
   For each step i from 1 to N:
   - Calculate intermediate values k1, l1, k2, l2, k3, l3, k4, and l4 using the following formulas:
   - k1 = h * dtheta[i-1]
   - l1 = h * f(a + i*h, dtheta[i-1], theta[i-1], polytropic_index)
   - k2 = h * (dtheta[i-1] + l1/2)
   - l2 = h * f(a + i*h + h/2, dtheta[i-1] + k1/2, theta[i-1] + l1/2, polytropic_index)
   - k3 = h * (dtheta[i-1] + l2/2)
   - l3 = h * f(a + i*h + h/2, dtheta[i-1] + k2/2, theta[i-1] + l2/2, polytropic_index)
   - k4 = h * (dtheta[i-1] + l3)
   - l4 = h * f(a + i*h + h, dtheta[i-1] + k3, theta[i-1] + l3, polytropic_index)
   - Update theta[i] and dtheta[i] using these intermediate values.
3. After the loop, we have arrays t and theta containing the solution.
   - theta[i] = theta[i-1] + (k1 + 2*k2 + 2*k3 + k4) / 6
   - dtheta[i] = dtheta[i-1] + (l1 + 2*l2 + 2*l3 + l4) / 6

These algorithms describe the step-by-step process of solving the Lane-Emden equation using the three numerical methods mentioned in our code.

11. Analyzing the results of lane Emden solution for n=3 and comparing this polytropic model with the relativistic model of the sun(Standard Solar Model)

Predictions of the n = 3 polytropic model for the Sun of mass, density, pressure and temperature variations with radius are needed for comparison with the Standard Solar Model.

At the surface of the n = 3 polytrope where θ = 0

$$\alpha = \frac{R\odot}{\xi_1} = \frac{7\times 10^8}{6.9}\ m = 1.01\times 10^8\ m$$

The rate of change of mass with radius is given by the Equation of Mass Conservation

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

Integrating and substituting $\cdot = \alpha\xi$
And $\rho = \rho_c \theta^n$

gives

$$M_\odot = \int_0^{R_\odot} 4\pi r^2 \rho\, dr = 4\,\pi\,\alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n d\xi$$

The Lane-Emden equation may expressed in the form

$$\xi_1^2 \left| \frac{d\theta}{d\xi} \right|_{\xi=\xi_1} = -\int_0^{\xi_1} \xi^2 \theta^n d\xi$$

And substituting in the above expression for $M_\odot$ gives

$$M_\odot = 4\pi\alpha^3 \rho_c \xi_1^2 \left| \frac{d\theta}{d\xi} \right|_{\xi=\xi_1}$$

The Lane-Emden Equation for n=3 has a solution $(\theta = 0)$ relevant to stellar structure at

$$\xi_1 = 6.90 \quad \text{and} \quad \left|\frac{d\theta}{d\xi}\right|_{\xi=\xi_1} = -4.236\times 10^{-2}$$

Taking $M_{\odot} = 2\times 10^{30} kg$ and the Lane-emden equation for n=3, the expression for above gives an estimate for the central density of the sun of

$$\rho_c = 7.66\times 10^4 kg\ m^{-3}$$

And the dependence of density on the radial distance from the solar center immediately follows from

$$\rho = \rho_c \theta^n$$

Since θ varies from θ =1 at the center to θ71 =0 at the surface.

By definition
$$\alpha^2 = \frac{k(n+1)}{4\pi G p_c^{\left(1-\frac{1}{n}\right)}}$$

and as ρ_c and α are known, K = 3.85 × 10^{10} Nm kg−1 . It then follows since P = K ρ^{γ} that an estimate of the pressure at the center of the Sun (where $\rho = \rho_c$) is

$$P_c = 1.25\times 10^{16}\ Nm^{-2},$$

and the dependence of gas pressure on radial distance follows directly by substituting the appropriate ρ.

By a similar argument, the equation of state for a perfect gas

$$P_{gas} = \frac{k}{m_H \bar{\mu}} \rho T$$

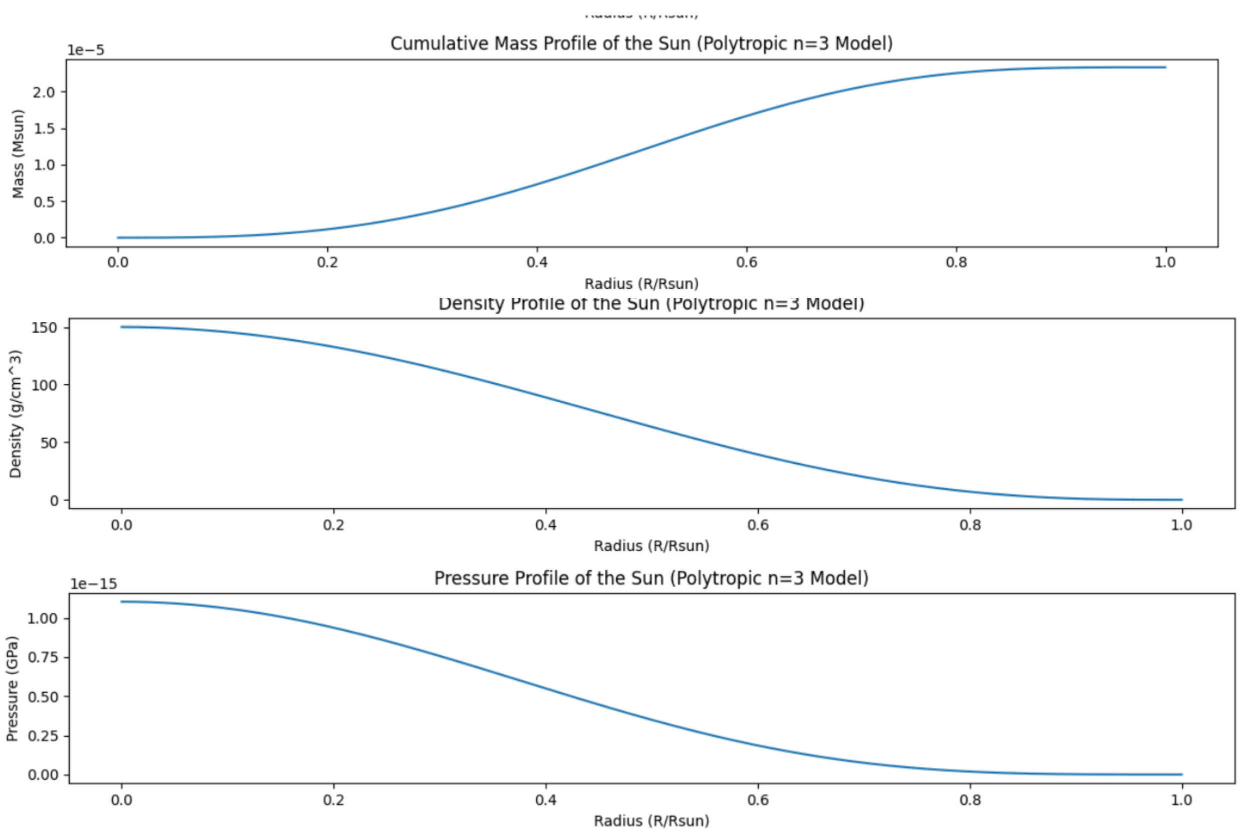
gives the dependence of T on radial distance (r) on substituting the P_{gas}(r) and adopting μ=0.6. In particular, setting $P_{gas}(r) = P_c$ gives a temperature at the solar centre of

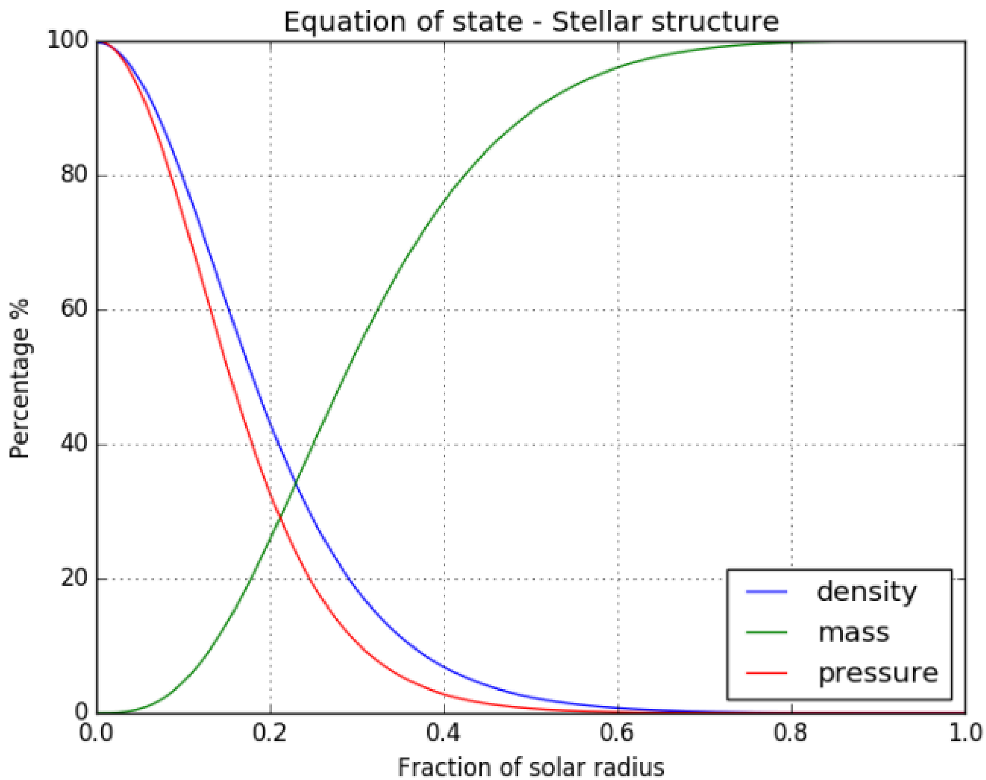
$$T_c = 1.19 \times 10^7 \text{ K}.$$

As previously discussed, the mass (M(r)) interior to some distance r from a stellar centre is given by the mass conservation equation, to which the Lane-Emden equation may be applied, to give

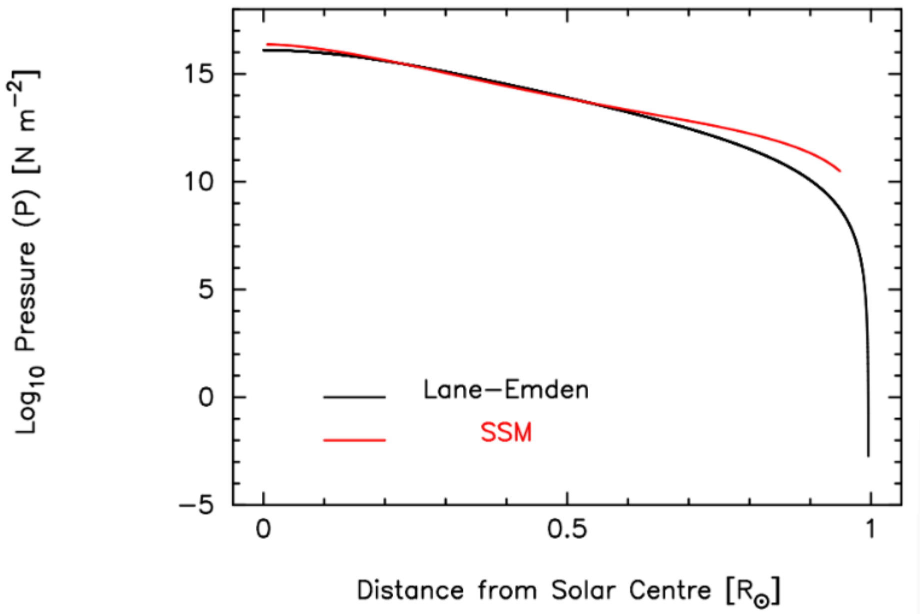
$$M(r) = -4\pi\alpha^3\rho_c\xi_r^2\left|\frac{d\theta}{d\xi}\right|_{\xi=\xi_r}$$

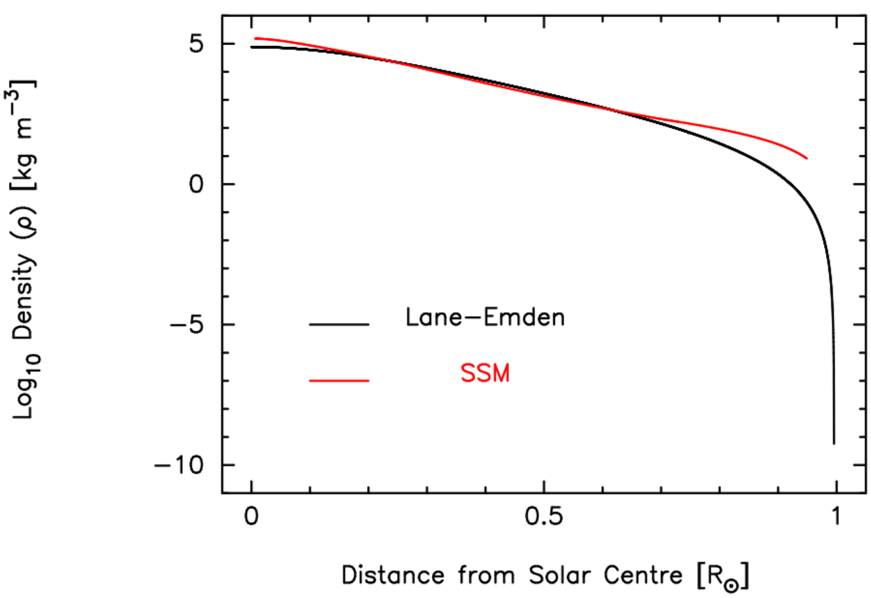
where ξ_r is the scaled radial distance r/α at distance r from the center of the Sun. Evaluating the right-hand side for successive values of ξ_r gives the mass interior to those points.

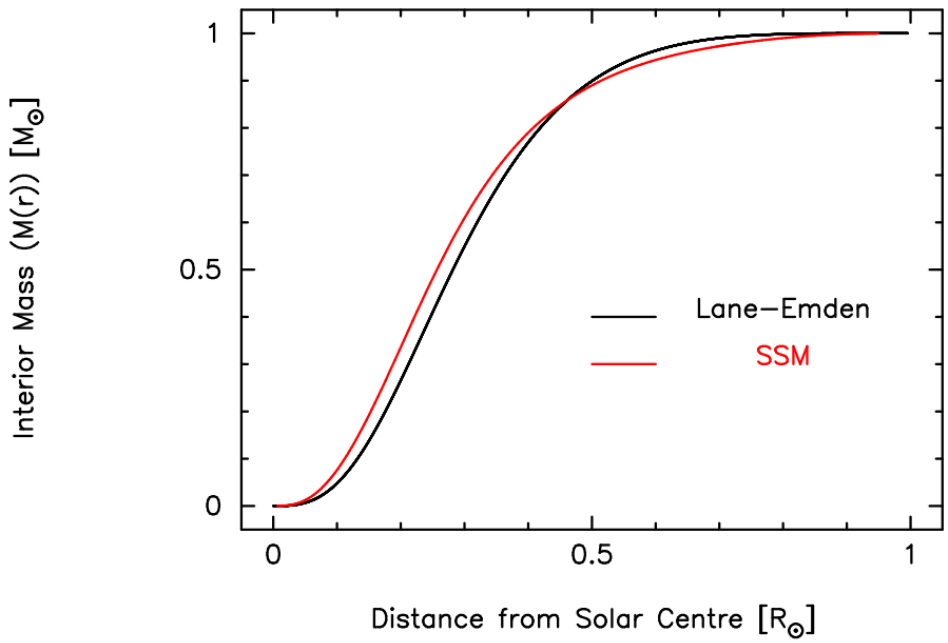


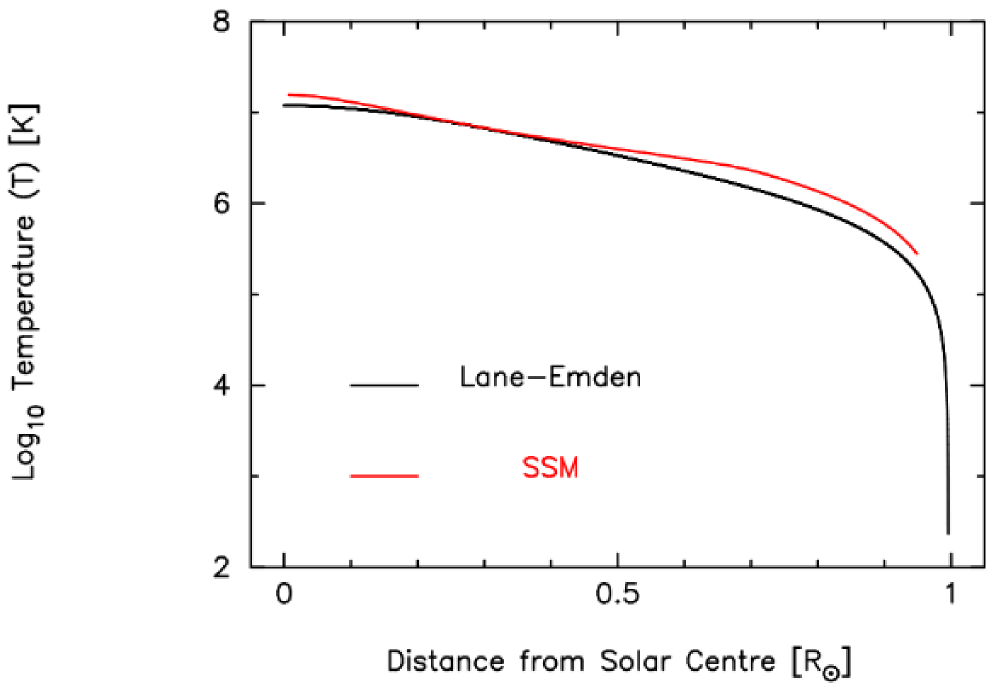


11.1 Comparison of Polytropic Model with Actual Standard Sun Model


Log$_{10}$ Pressure (P) [N m^{-2}]
15
10
5
0
−5
0
0.5
1
Lane−Emden
SSM
Distance from Solar Centre [R$_\odot$]


Log$_{10}$ Density (ρ) [kg m^{-3}]
5
0
−5
−10
0
0.5
1
Lane−Emden
SSM
Distance from Solar Centre [R$_\odot$]


Interior Mass (M(r)) [M⊙]
1
0.5
0
Lane−Emden
SSM
0
0.5
1
Distance from Solar Centre [R⊙]


Log10 Temperature (T) [K]
8
6
4
2
Lane−Emden
SSM
0
0.5
1
Distance from Solar Centre [R⊙]

12. Results and Conclusions

- Solutions of Lane Emden Equation decrease monotonically (except for n=1) and have $\theta=0$ at $\xi=\xi_1$ (i.e. the stellar radius). With an increasing polytropic index, the star becomes more centrally condensed.

- We find that for the Lane Emden equation we can find analytical solutions only for n=0, 1, and 5 while we can use numerical approximation to get roots of the equation for all values of n.

- We have also analyzed some important values of n which specify different types of spherical formation as shown in table below

Useful Polytropic Indices

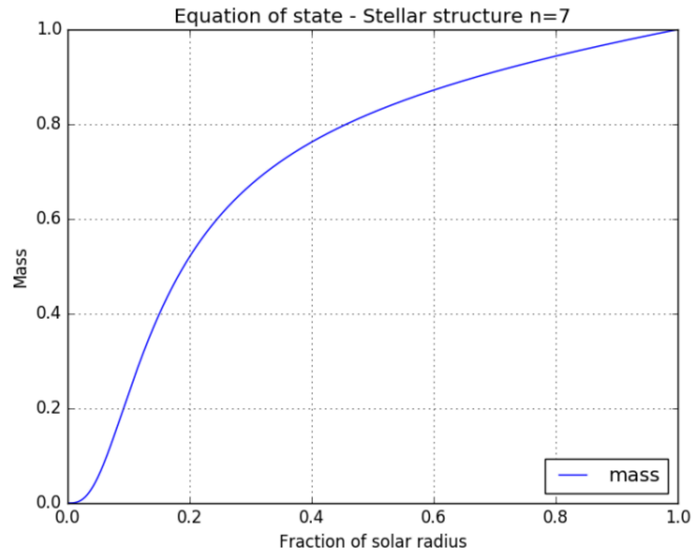
n	γ	Description
0	—	Incompressible gas; constant density
0.42857	10/3	Thomas-Fermi EOS
1	2	analytic solution to Lane-Emden equqation; constant R_*
1.5	5/3	ideal monatomic gas EOS; convective; non-relativistic degenerate
2	3/2	Holzer & Axford's maximum γ for an accelerating solar wind
2.5	7/5	ideal diatomic gas EOS
3	4/3	Eddington's standard model; ultra-relativistic degenerate; constant M_*
3.25	17/13	Chandrasekhar's constant-ϵ Kramers model
5	6/5	Schuster sphere of infinite radius
∞	1	Isothermal gas; Bonnor-Ebert sphere

$$n=0 \qquad \theta(\xi)=1-\frac{\xi^2}{6} \qquad \xi_0=\sqrt{6}$$

$$n=1 \qquad \theta(\xi)=\frac{\sin\xi}{\xi} \qquad \xi_1=\pi$$

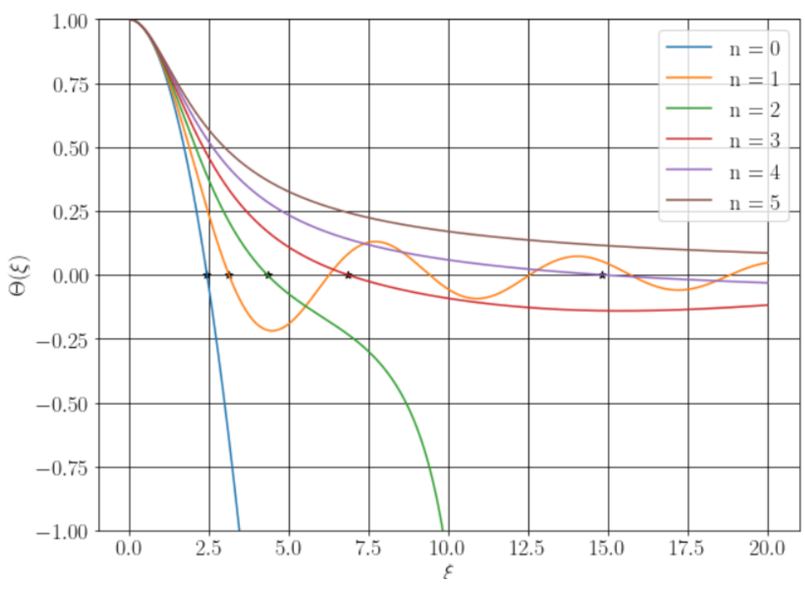
$$n=5 \qquad \theta(\xi)=\left(1+\frac{\xi^2}{3}\right)^{-1/2} \qquad \xi_5=\infty$$

- We also find out that for n=5 we get root at ξ=∞ , that is , it forms a star of infinite radius and thus for n=5 we get the solution at ξ=∞ and for n>5 we do not obtain any solution as the mass keeps on increasing as we move outwards (as shown in the diagram below for n=7)



- We also compared the n=3 polytrope with the Standard Solar Model, finding quite good agreement considering how simple the input physics was in the Lane Emden Equation as compared to very complicated equations of stellar formation.

- Thus, finally we plot the solution of Lane Emden Equation



- These solutions are used to find the approximate mass, central pressure and central density of a star. We have solved this for n=3 which is a good approximation for sun-like stars.

Thus we can use this solution in the stellar solution for many applications such as formation of star formation, defining the conditions for formation of white dwarf planet, deriving the Chandrashekhar mass constant, eddington's model, analyzing the stability of the stars and many other applications.

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