DISCRETE MATHEMATICS

Handwritten Notes

VIVEKANAND VERNEKAR

GateWallah Parakram C 2024

(Hinglish Weekday)

Other Notes Uploaded Below

https://t.me/pwgatenotes

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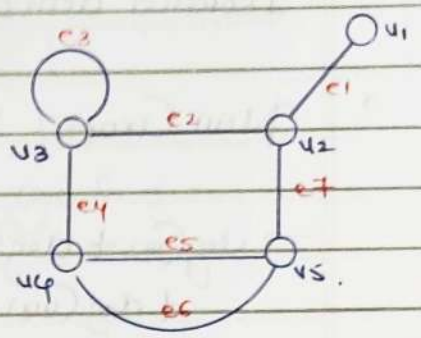
Graph Theory.

Basics of Graph:

* Point/joint/node → vertex
* line/arc/branch → edge



→ Graph 'G' = (V, E)

Set of vertices — Set of edges.

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2 \dots e_7\}$.

* Each edge must be associated with unordered pair of vertices.

$e_1 \rightarrow (v_1, v_2) \mid (v_2, v_1)$

$G = (V, E, \psi)$

$V = \{\dots\}$

$E = \{\dots\}$ → Each edge must be associated with unordered pair of vertices.

$\psi : E \rightarrow V \times V$.

* **Incident Point:** Meeting point of vertex and edge

* **End vertices:** each edge is associated with unordered pair of vertices called as end vertices.

* **Loop/self-loop:** if end vertices are same, edge is called loop.

* **Parallel edges:** two or more edges associated with same end vertices.

  eg: $e_5 \rightarrow (v_4, v_5)$
  $e_6 \rightarrow (v_4, v_5)$

* **Degree/valency ($d(v_i)$):** no. of edges incident with vertex.

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* **Pendant Vertex**: degree one vertex is called pendent vertex.

* **Isolated vertex**: degree '0' vertex is called isolated vertex.

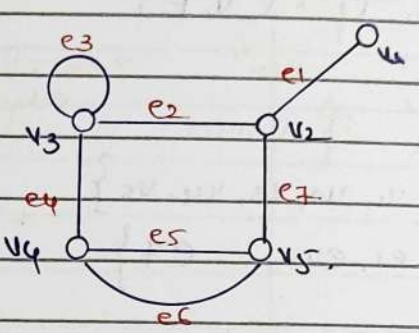
* **Null Graph**: Set of isolated vertices.

→ $\deg(v_1) + \deg(v_2) + \deg(v_3) + \deg(v_4) + \deg(v_5)$.

$= 1+3+4+3+3$

$= 14 = 2 \times 7$.

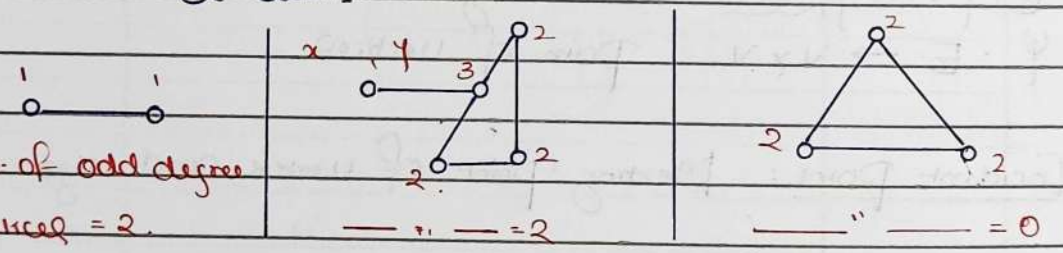
no. of edges

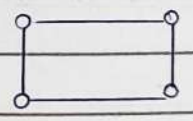
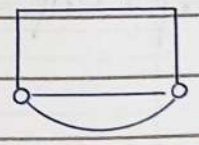
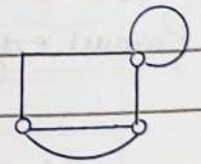


[Theorem 1] * Sum of degree of all vertices is equal to twice the no of edges.

[Theorem 5] * Sum of degree of all vertices will be even.

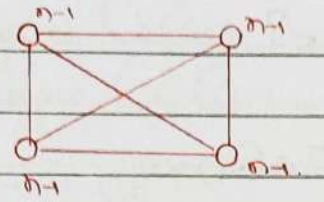
[Theorem 2]: No. of odd degree vertices in a graph can always be even.



	loop.	'||' edges.	
Simple Graph	×	×	
Multi Graph	×	✓	
Pseudo Graph.	✓	✓	

[Theorem 3]: Maximum no. of edges in Simple Graph $\leq \frac{n(n-1)}{2}$

$n=4$.



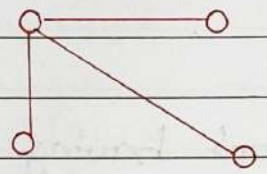
$\Sigma d(v_i) = 22$

$n \times (n-1) = 22$

$$\boxed{e = \frac{n(n-1)}{2}}$$

[Theorem 4]: Maximum degree in Simple graph $\leq n-1$.

$n=4$



Note: if degrees of all vertices are $n-1$, then it will have exactly $\frac{n(n-1)}{2}$ edges.

eg. $n=4$

Maximum edges $= \frac{4.3}{2} = 6$

n = total vertices

$\rightarrow$ Total no. of Graphs $= 2^{\frac{n(n-1)}{2}}$

Total no. of graphs $= 2^6 = 2^{\frac{4.3}{2}}$

Q1. How many graphs are possible with 4 vertices and 1 edge?

Q2. How many graphs possible with n vertices and 'e' edges.

$\rightarrow {}^{\frac{n(n-1)}{2}}C_e$.

Q3. How many graphs are possible with 4 vertices and atleast 2 edges

Ans. $m_1 \rightarrow {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$

$m_2 \rightarrow 2^6 - {}^6C_0 - {}^6C_1$.

Q4. A certain graph ~~ae~~ 'G' has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are 6 vertices of degree 4. How many vertices of G have degree 3 and how

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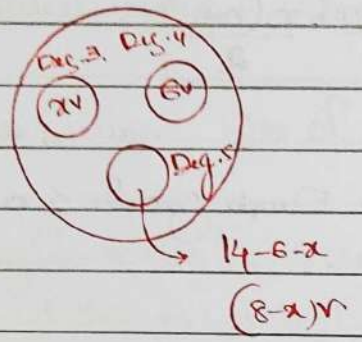
how many have degree 5?

Ans. $n=14$, $E=27$.



$\sum d(v_i) = 2e$

$6\times4 + x\times3 + (8-x)\times5 = 2.27$.

$\underline{x=5}$

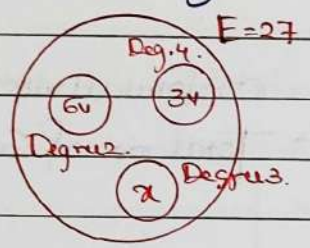
eg: Consider a graph having 27 edges

$6v \longrightarrow$ Degree $\rightarrow 2$

$3v \longrightarrow$ Degree 4.

Remaining vertices $\longrightarrow$ Degree 3.

Soln:



$\sum d(v_i) = 2e$

$6\times2 + 3\times4 + x\times3 = 2.27$

$12+12+3x = 54$

$3x = 54-24 = 30$

$3x = 30$

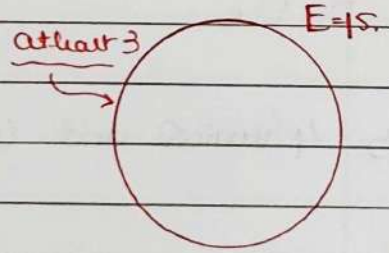
$\underline{x=10}$

Total vertices

$= 6+3+10$

$= \underline{19}$.

eg:: Consider a graph having 15 edges, degree of each vertex is atleast 3. What can be the number of vertices?



$\delta(G)=3$ $\quad n=?$

$\delta(G) \le \frac{2e}{n} \le \Delta(G) \le n-1$.

$\delta(G) \le \frac{2e}{n} \Rightarrow 3 \le \frac{2.15}{n} \Rightarrow \underline{n \le 10}$.

Maximum no. of vertices $= \underline{10}$.

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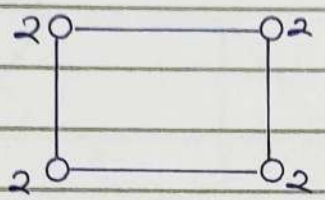
Minimum degree $(\delta(G))$
Maximum degree $(\Delta(G))$

Case I:

$$\boxed{\delta(G) = \frac{2e}{n} = \Delta(G)}$$



$\delta(G) = 2$

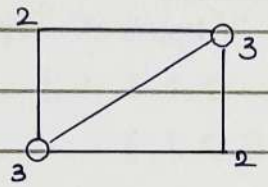
$\Delta(G) = 2$

$\frac{2e}{n} = 2$

avg. degree $= \frac{2+2+2+2}{\text{Total vertices}} = \frac{\sum d(v_i)}{n}$

$= 2$.

eg:



$\delta(G) = 2$

$\Delta(G) = 3$

Case II:

$$\boxed{\delta(G) < \frac{2e}{n} < \Delta(G)}$$

Theorem 5.

$$\boxed{\delta(G) \le 2e \le \Delta(G) \le n-1}$$

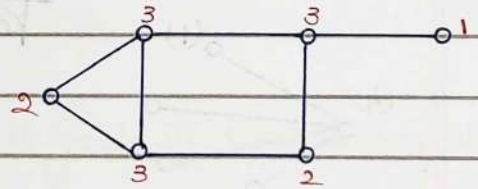
eg: G is Graph 25 degree, each vertex is having degree atleast 3. Maximum value of n is 16.

$E = 25 \quad \delta(G) = 3$

$\delta(G) \le \frac{2e}{n} = 3 \le \frac{2 \times 25}{n} \Rightarrow n \le \frac{50}{3} \quad \therefore n = 16$

Degree Sequence:

Writing degree of all vertices either in increasing or decreasing order is called "Degree Sequence".



$\rightarrow$ 3, 3, 3, 2, 2, 1 or 1, 2, 2, 3, 3, 3.

eg: What will be the no. of edge in 2S 5, 2, 2, 2, 1?

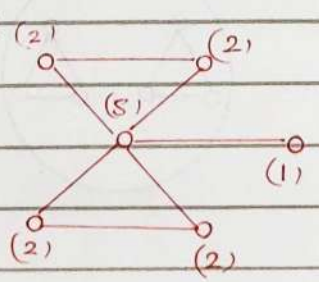
m1: $\sum d(v_i) = 2e$

$5+2+2+2+2+1$

$14 = 2e$

$e = 7$

m2:



$e = 7$

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Q2. What will be edges in 2.s of 3, 3, 3, 1?

m1:

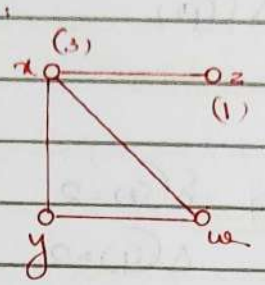
$\sum d(v) = 2e$

$3+3+3+1 = 2e$

$10 = 2e$

$e = 5$

m2:



$\boxed{\text{Degree Sequence} \longrightarrow \text{Simple Graph}}$

→ Graphical Sequence

5, 2, 2, 2, 2, 1 → Graphical Sequence

3, 3, 3, 1 → No Simple Graph.

Graphical?

(a) 5, 4, 3, 2, 1

Not Graphical

Reason 1: Total vertices = 5, $n = 5$

$\Delta(G) \le n-1$

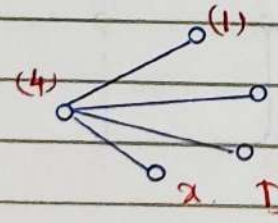
$\Delta(G) \le 4$.

Reason 2: Theorem 2

5, 4, 3, 2, 1.

(b) 4, 4, 3, 2, 1. Not Graphical.

Total vertices = 5



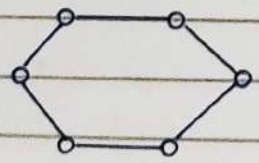
Demand : 3.

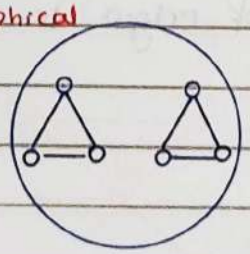
Total vertices = n

→ n-1, n-1, ... 1.

→ Not graphical. (2. (n-1) and a pendant cannot be graphical).

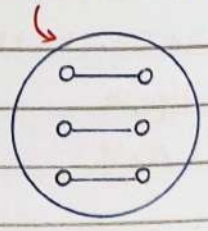
(c) 2, 2, 2, 2, 2, 2. Graphical





(d) 1, 1, 1, 1, 1, 1.

Graphical



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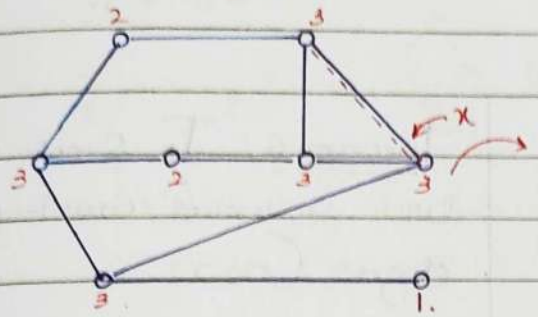
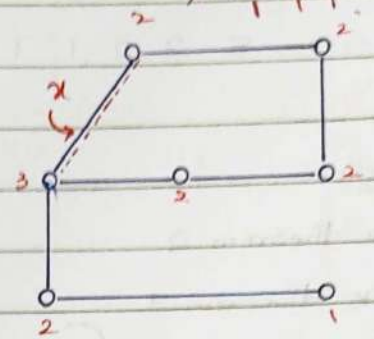
Havel-Hakimi

$\cancel{3}\ 3\ 3\ 3\ 3\ 2\ 2\ 1$



$\cancel{3}\ 2\ 2\ 2\ 2\ 2\ 1$

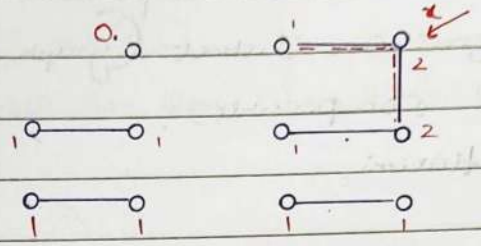


x (removing that vertex).

$\cancel{2}\ 2\ 1\ 1\ 1\ 1$

$\cancel{1}\ 1\ 1\ 1\ 0$

$1\ 1\ 0\ 0$



$\cancel{3},\ 3\ 3\ 3\ 3\ 2\ 2\ 1.$

$\cancel{3}\ 2\ 2\ 2\ 2\ 2\ 1.$ (ordering)

$\cancel{2}\ 2\ 1\ 1\ 1\ 1$ (ordering)

$\cancel{1}\ 1\ 1\ 1\ 0$ (ordering)

$1\ 1\ 0\ 0$

↳ if this can be a simple graph, it's previous one also can.

eg. Gate PYQ:

a. 7 6 5 (4 4) 3 2 1 ✓

b. 6 6 6 6 (3 3) 2 2 ✗

c. 7 (6 6) (4 4) 3 (2 2) ✓

d. 8 (7 7) 6 4 2 (1 1) ✗

(a) $\cancel{7}\ 6\ 5\ 4\ 4\ 3\ 2\ 1$

$\cancel{5}\ 4\ 3\ 3\ 2\ 1\ 0$

$\cancel{3}\ 2\ 2\ 1\ 0\ 0$

$1\ 1\ 0\ 0\ 0.$ ✓

(b). $\cancel{6}\ 6\ 6\ 6\ 3\ 3\ 2\ 2$

$\cancel{5}\ 5\ 5\ 2\ 2\ 2\ 1$

$\cancel{4}\ 4\ 1\ 1\ 1\ 1$

$3\ 0\ 0\ 0\ 1.$

no simple graph

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(c). 7 6 6 4 4 3 2 2
~~5~~ 5 3 3 2 1 1
5 2 2 1 1 0 ×

(d) 8 7 7 6 4 2 1 1 ×
$n = 8$
$\Delta(G) \leq 7$.

Steps:
1. Check theorem 2
2. Check theorem 3.
3. $n-1, n-1 \ldots 1$ (not possible)
4. All degree are distinct (Graph not possible).
5. Hamel Hakimi.

Theorem 6: In simple graph at least 2 vertices will have same degree ($n \geq 2$).

Q1. $0 \leq x \leq 5$ for which value of x it is graphical?
$x, 1, 2, 3, 5, 5$.

$5, 5, 3, 2, 1, x$.
$n-1. \; n-1. \ldots 1.$
Total vertices = 6. } not graphical

$\therefore$ for no value it will be graphical.

Q2. $0 \leq x \leq 7$. for which value of x it is graphical?
7 6 5 4 3 2 1 x.
x cannot be odd

$x = 0$ 7 6 5 4 3 2 1 0 × (all distinct)

$x = 2$. 7 6 5 4 3 2 2 1
~~5~~ 4 3 2 1 1 0.
3 2 1 0 0 0. × (not possible)

$x = 4$ 7 6 5 4 4 3 2 1
~~5~~ 4 3 3 2 1 0
~~3~~ 2 2 1 0 0
1 1 0 0 0. } Graphical ✓

$\therefore$ for $x = 4$ it is graphical

H.W { $x, 7, 7, 5, 5, 4, 3, 2$
$0 \leq x \leq 7$
Value for x for which it is graphical.

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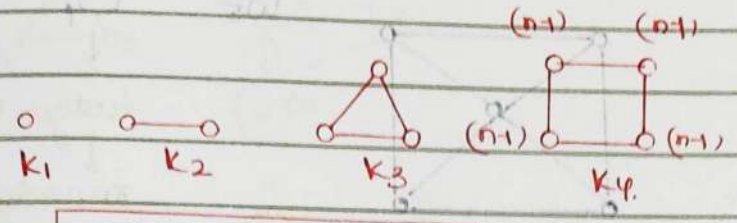
Complete Graph : (K_n). ($n \geq 1$)



* Degree of all vertices are $n-1$.

$\Sigma d(n) = 2e$

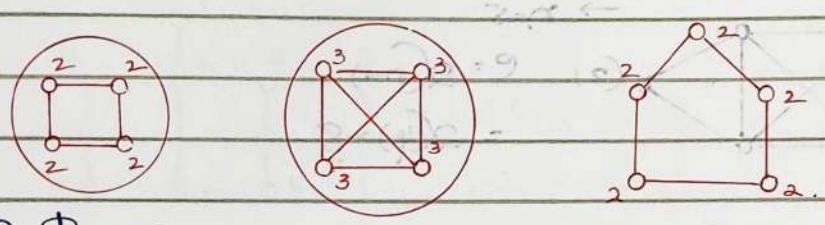
$n(n-1) = 2e$

$$\boxed{e = \frac{n(n-1)}{2}}$$

$$\boxed{\delta(G) = \frac{2e}{n} = \Delta(G) = n-1}$$

Regular Graph ($\delta(G) = \frac{2e}{n} = \Delta(G)$).

→ If degrees of all vertices are same then it is called Regular Graph.



2-Regular 3-Regular. 2-Regular.

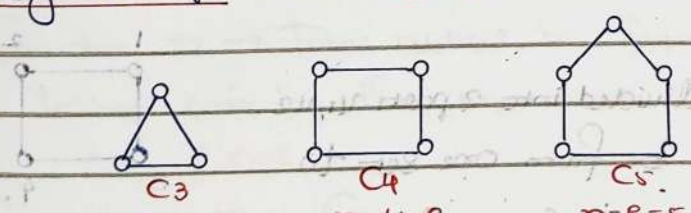
$\delta(G) = \frac{2e}{n} = \Delta(G) = k$

k-Regular Graph

Points about Complete Graph :

* $n-1$ Regular Graph is a Complete Graph
* all K_n are Regular Graph. (True)
* all Regular Graph are K_n. (False).

Cycle Graph : (C_n) ($n \geq 3$)



$n = e = 3$ $n = 4 = e$ $n = e = 5$.

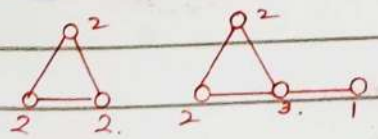
* Degree of all vertices are 2

$\Sigma d(v_i) = 2e$

$n \cdot 2 = 2 \cdot e$

* $n = e$.

* All Cycle graph are Regular graph
* All regular graph are Cycle graph.



Graph containing cycle

* if Graph is $C_n \rightarrow n = e$ (True)
* if $n = e \rightarrow$ then it is cycle graph (False)

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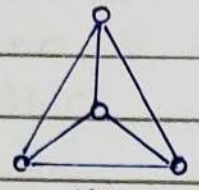
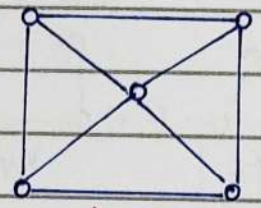
Wheel Graph (W_n) ($n \geq 4$)



W_4.



W_5.

W_5 → W_n.

C_4 → 4 edges + 4 edges → $n-1$ edges + $n-1$ edges.

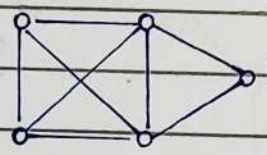
$e(W_n) = n-1 + n-1$

$\boxed{e(W_n) = 2(n-1)}$

→ if G is W_n → $e = 2(n-1)$ (True)

→ if G is having $e = 2(n-1)$ → G is wheel graph. (False)

eg:



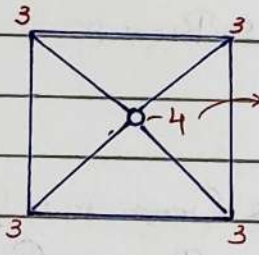
(2)

→ $n = 5$

$e = 2(n-1)$

$= 2(4) = 8$

eg::



→ $(n-1)$ W_5

W_5

4, 3, 3, 3, 3

W_6

5, 3, 3, 3, 3, 3

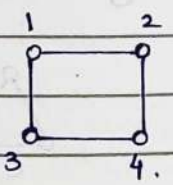
5 vertices

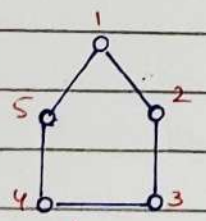
Properties of wheel graph.
- all degree are 3 (common)
- middle vertex has degree $(n-1)$.

Bipartite Graph.

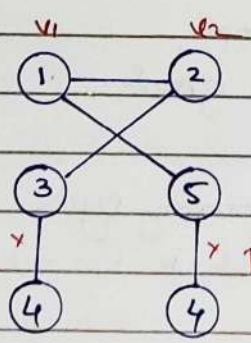
$G = (V, E)$

* Vertices can be divided into 2 parts V_1, V_2.
* Each edge must be from one set to another set but not in same set.

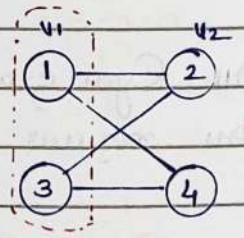




Not Bipartite Graph.



Because 4 is connected to both 3 & 5 so cannot do partition.



← Bipartite Graph.

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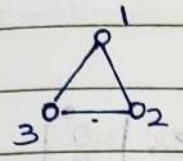
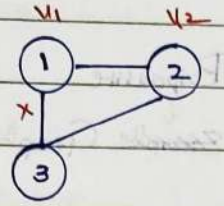
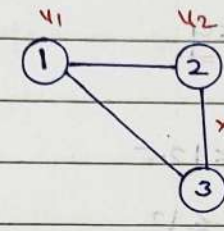
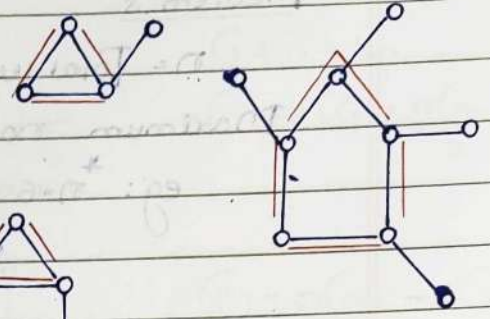


Not Bipartite Graph.





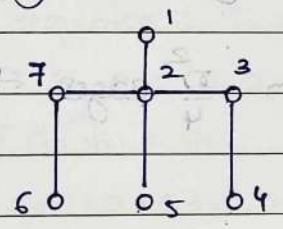


All have odd length cycle. so not Bipartite Graph.

Theorem 7.

Bipartite Graph does not contain odd length cycle.

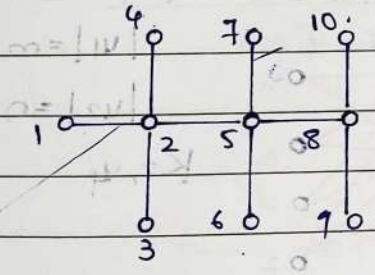
eg:

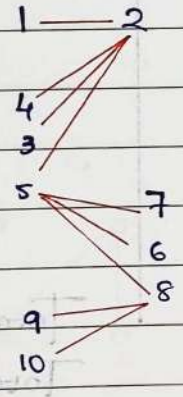


no odd length cycle

∴ Bipartite Graph

eg:





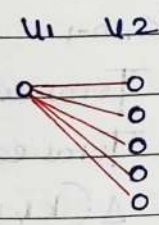
Bipartite Graph.

* Consider $n \rightarrow$ Total vertices in Bipartite Graph

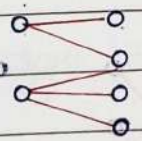
What will be the maximum no. of edges?

$n=6$.

Case 1:

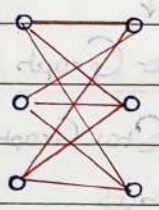


Case 2:



$e=5$.

Case 3:



$e=9$

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Theorem 8

n = Total vertices

Maximum no. of edges in ~~Simple~~ Bipartite Graph $e \le \left\lfloor \frac{n^2}{4} \right\rfloor$

eg: * $n = 6$

$$e \le \frac{6^2}{4} = e \le 9$$

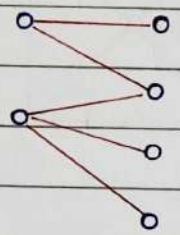
* $n = 7$ $\quad e \le \frac{7^2}{4} = e \le 12.5$

$$e \le 12$$

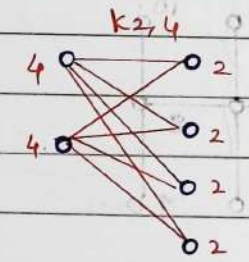
→ Maximum no. of edges in bipartite graph $e \le \frac{n^2}{4}$.

→ If bipartite graph contains more than $\frac{n^2}{4}$ edges then it contains odd length cycle.

Bipartite Graph



Complete Bipartite Graph $(K_{m,n})$



$|V_1| = m$

$|V_2| = n$

$K_{2,4}$

Total vertices $= 2+4 = 6$ $\quad V = m+n$

Total edges $= 2 \times 4 = 8$ $\quad E = m.n$

$\Delta(K_{2,4}) = 4$ $\quad \Delta(K_{m,n}) = \max(m,n)$

$\delta(K_{2,4}) = 2$ $\quad \delta(K_{m,n}) = \min(m,n)$

Complete Bipartite Graph

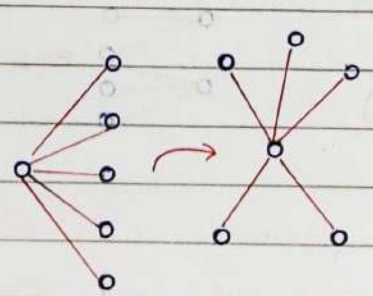
↓

Star Graph $(K_{1,n-1})$

Draw Star Graph of 6 vertices

$n = 6$

$K_{1,5}$.



$K_{1,n-1}$

Total vertices $= n$

Total edges $= n-1$

$\Delta(K_{1,n-1}) = n-1$

$\delta(K_{1,n-1}) = 1$.

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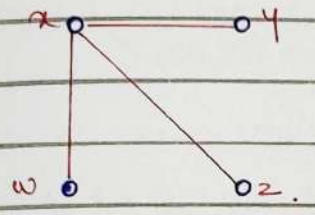
Compliment Graph ($\bar{G}$)

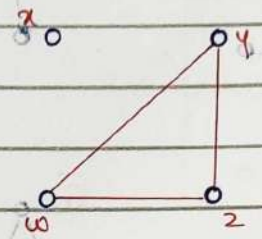
$G + \bar{G} = K_n$

$e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$

edge → present
edge → absent

edge → absent.
edge → present.





$$\boxed{e(\bar{G}) = \frac{n(n-1)}{2} - e(G)}$$

Q. What will be edge in the Compliment of the graph having degree sequence : 5, 2, 2, 2, 2, 1 ?

Ans. $G \rightarrow 5, 2, 2, 2, 2, 1$

$\sum d(v_i) = 2e$

$5+2+2+2+2+1 = 2e$

$14 = 2e$

$e = 7.$

$e(\bar{G}) = ?$

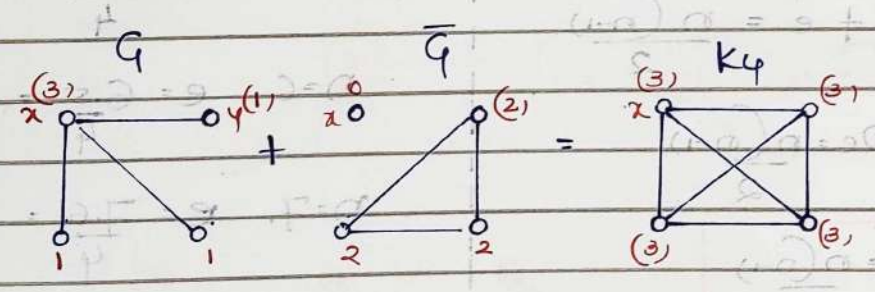
$e(G) = 7$

Total vertical = 6.

$\Rightarrow e(\bar{G}) + e(G) = \frac{n(n-1)}{2}$

$\Rightarrow 7 + e(\bar{G}) = \frac{6 \cdot 5}{2}$

$\rightarrow e(\bar{G}) = 15 - 7 = 8$



K_4	3	3	3	3
G	3	1	1	1
$\bar{G}$	0	2	2	2

$n = 4.$

General : Note:

→

K_n	$n-1$	$n-1$	$n-1$	....	$n-1$
G	d_1,	d_2,	d_3	....	d_n
$\bar{G}$	$n-1-d_1$,	$n-1-d_2$	....		$n-1-d_n$.

eg:: What will be no. of edge in the Compliment of the graph having De 5, 2, 2, 2, 2, 1.

→ Total vertical = 6.

K_6	5	5	5	5	5	5
G	5	2	2	2	2	1
$\bar{G}$	0,	3,	3,	3,	3,	4

$\sum d(v_i) = 2e$

$0+3+3+3+3+4 = 2e$

$e = 8$

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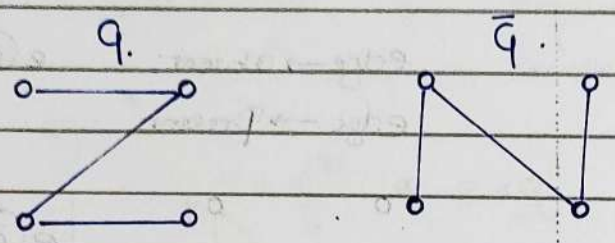
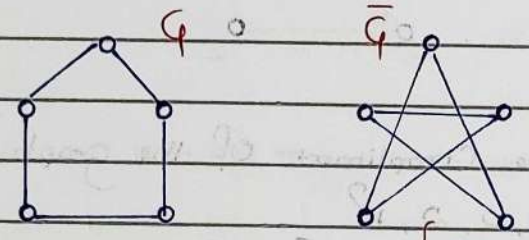
Self-Compliment Graph. ($G \equiv \bar{G}$)



* Graph is same as its own compliment.



C_5
$n = e = 5$

$n = 5 = e$

→ G is same as $\bar{G}$

$G + \bar{G} = K_n$

$e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$

$e + e = \frac{n(n-1)}{2}$

$2e = \frac{n(n-1)}{2}$

$$\boxed{e = \frac{n(n-1)}{4}}$$

$n = 4 \quad e = \frac{n(n-1)}{4} = \frac{4 \cdot 3}{4} = 3$ ✓

$n = 5 \quad e = \frac{n(n-1)}{4} = \frac{5 \cdot 4}{4} = 5$ ✓

$n = 6 \quad e = \frac{6 \cdot 5}{4} = \frac{15}{2} = 7.5$ ✗

$n = 7. \quad e = \frac{7 \cdot 6}{4} = \frac{42}{4}$ ✗ Should be divisible by 4.

→ $\underbrace{\frac{n}{4} \text{ or } \frac{n-1}{4}}_{\text{divisible by 4.}}$ then we get self compliment graph.

→ $a \equiv b \pmod{n}$

↳ $a \% n == b \% n$.

1 and 5 are having same remainder with respect to 4.

↳ $1 \equiv 5 \pmod{4}$

Or

$a \equiv b \pmod{n} = \frac{a-b}{n} \in z.$

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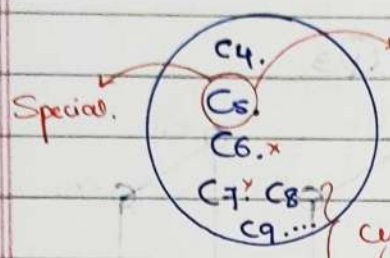
Cycle graph.

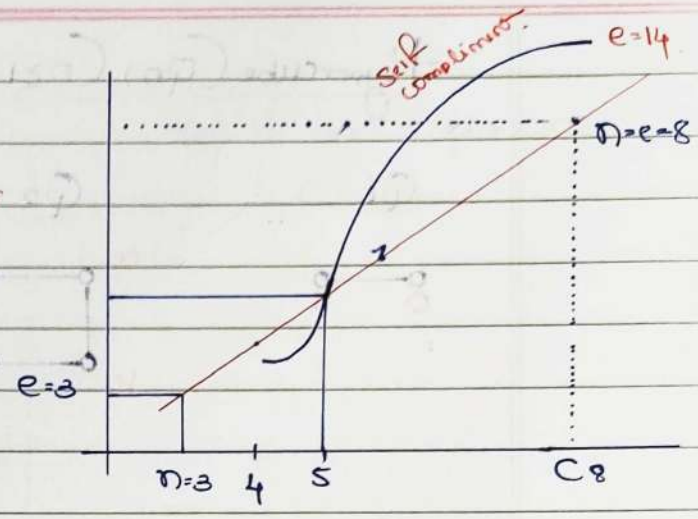


Special.

Self compliment e=5 cycle graph.

cycle graph = 8

Self compliment 14.



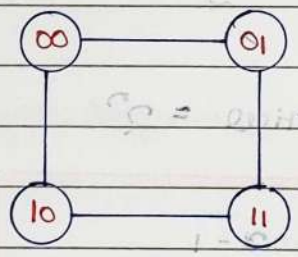
** Only for C5 will get Self Compliment graph. (n=5).

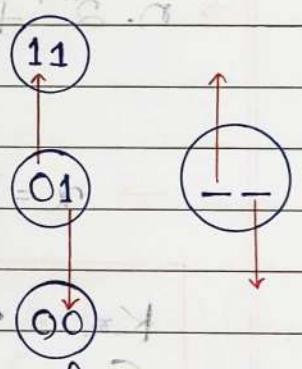
Q1. Graph vertices are represented as a n-bit signal, and 2 vertices are connected, if their bit position changes by 1 bit what will be the total edges?

Sol^n:

n = 2 bit

Total vertices $= 2^2$.
- 00
- 01
- 10
- 11

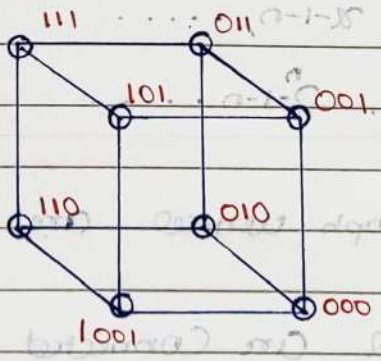


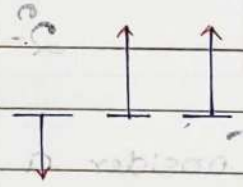


Degree of each vertex will be 2.

n = 3 bit.

Total vertices $= 2^3$
- 000
- 001
- ⋮
- 111





Degree of each vertex will be 3.

→ n → bit signal

Total vertices $(x) = 2^n$ ⇒ $\sum d(v_i) = 2e$

Degree of each vertex = n ⇒ $2^n \cdot n = 2e$.

$$\boxed{e = n \cdot 2^{n-1}}$$

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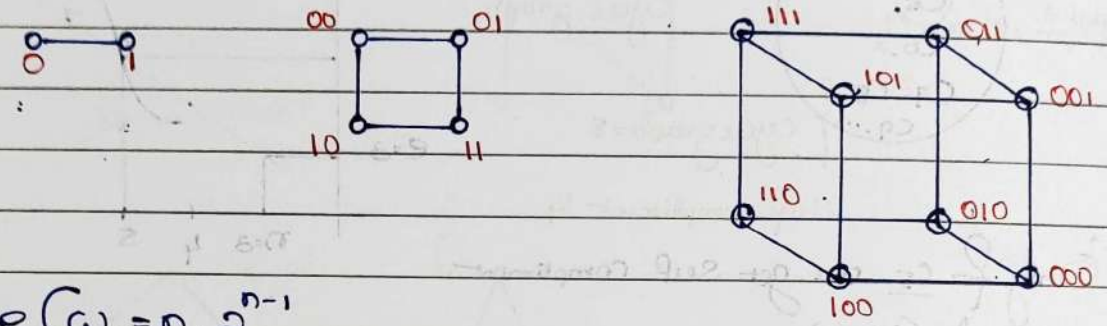
Hypercube (Q_n) ($n \geq 1$)

Q_1 $\quad$ Q_2 $\quad$ Q_3



$\rightarrow$ $e(G) = n \cdot 2^{n-1}$

$=$ Total vertices $= x = 2^n$

$= e(G) + e(\bar{G}) = K_x$ $\qquad$ $e(\bar{G}) = \dfrac{2^n(2^n-1)}{2} - n \cdot 2^{n-1}$

$= n \cdot 2^{n-1} + e(\bar{G}) = \dfrac{x(x-1)}{2}$

$\Rightarrow n \cdot 2^{n-1} + e(\bar{G}) = \dfrac{2n(2^n-1)}{2}$

$\rightarrow$ $x =$ Total vertices $= 2^n$

$K_x \rightarrow x-1 \quad x-1 \quad x-1$

$G \rightarrow n, \quad n, \quad n \ldots n$

$\bar{G} \rightarrow x-1-n, \; x-1-n, \ldots$

$2^n-1-n, \; 2^n-1-n \ldots$

Q2. Consider a graph vertices are represented as a no. from $\{1 \ldots 100\}$

two vertices are connected $|a-b| = 4$ eg $|1-5| = 4$ or $|5-9| = 4$

their difference is 4. like $|1-5| = 4$.

What will be the total edge in this?

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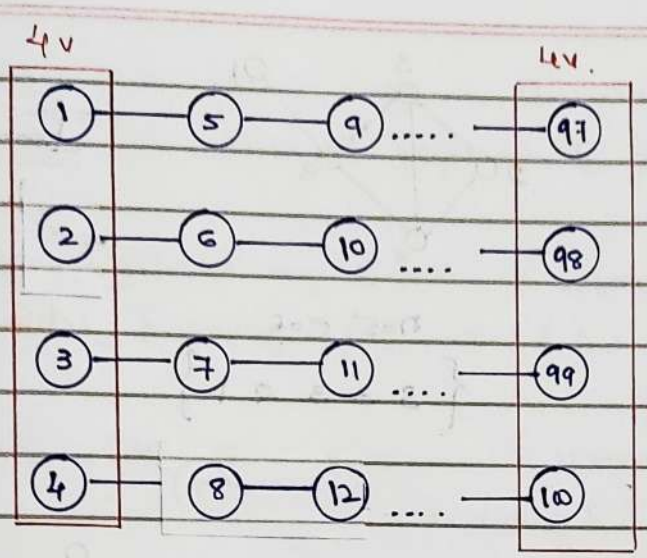
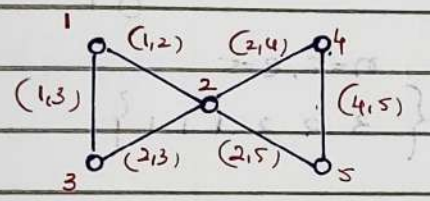
Ans:



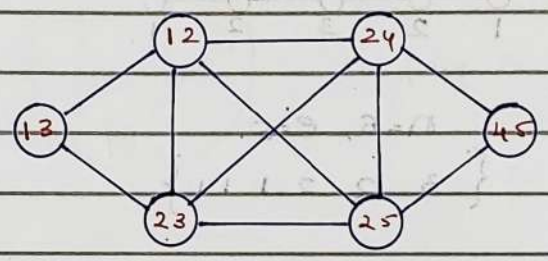
$4\times1 + 4\times1 + 92\times2 = 2e$

$e = 96.$

Line Graph - L(G)



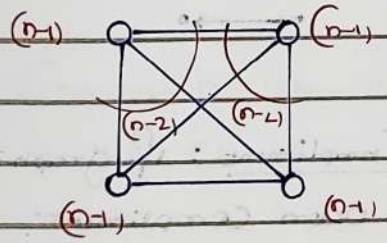
G



L(G).

Line Graph
↓
edge graph.

Q3. What will be the degree sequence of $L(K_n)$?



Degree of each vertex in $L(K_n)$ is $2(n-2)$.

Isomorphic Graph

G_1 and G_2 are isomorphic to each other when they have same incedent property

- no. of vertices
- no. of edges
- no. of degree sequence.

These properties are satisfied if two graphs are isomorphic.

If two graphs are isomorphic then they will have same vertices, edges and degree sequence and vice versa need not be true.

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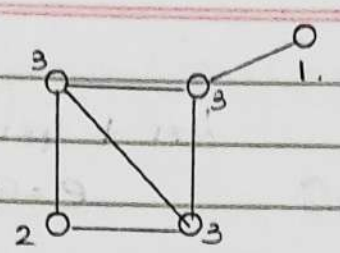
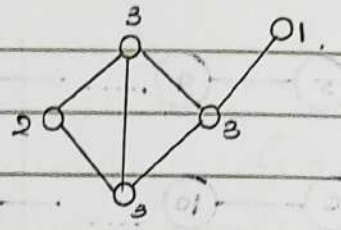
eg:



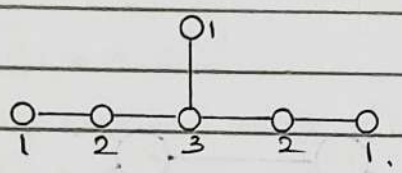


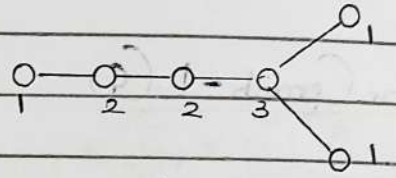
Isomorphic
↓
One Graph, but different representation

$n=5, e=6$
$\{3\ 3\ 3\ 2\ 1\}$

$n=5, e=6$
$\{3\ 3\ 3\ 2\ 1\}$

eg:





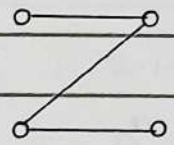
$n=6, e=5$
$\{3\ 2\ 2\ 1\ 1\ 1\}$

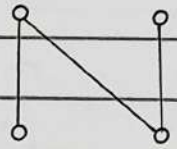
$n=6, e=5$
$\{3\ 2\ 2\ 1\ 1\ 1\}$

G_1, G_2 are isomorphic to each other.
$f: G_1 \to G_2$ are 1:1 correspondance.

$f: V_1 \to V_2$ $\quad$ $f: E_1 \to E_2$ $\quad$ $f: \varphi_1 \to \varphi_2$

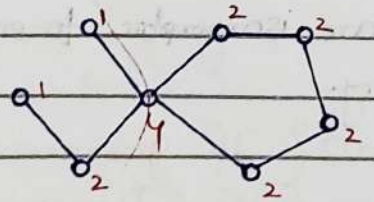
→



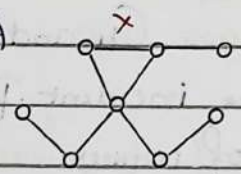


Self Compliment : if graph is isomorphic to its own compliment.

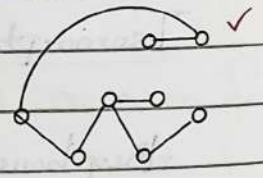
eg:



A.

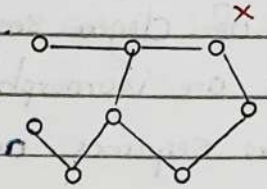


C.

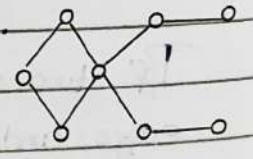


Cycle length = 5.

B.



D.



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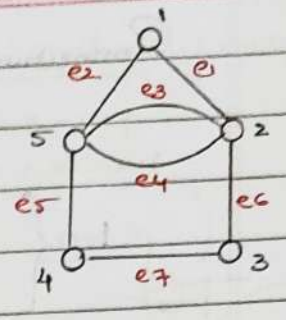
Walk: alternating sequence of vertices & edges.
(Repetition V / Repetation E.)

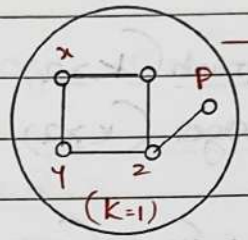


Trail: alternating sequence of vertices & edges
(Repeating V / Repeating E ×)

Path: alternating sequence of vertices & edges
(Repeating V × / Repeating E ×).

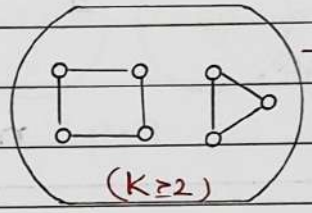
Graph

- **Connected**



→ Path is available between all pair of vertices

- **Disconnected**:



→ for atleast two vertices path is not available.

* Disconnected graph contains connected subparts (Component(K)).

Q1. Consider a star graph of 6 vertices, what will be the total edges in the compliment of the graph?

Ans. $e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$

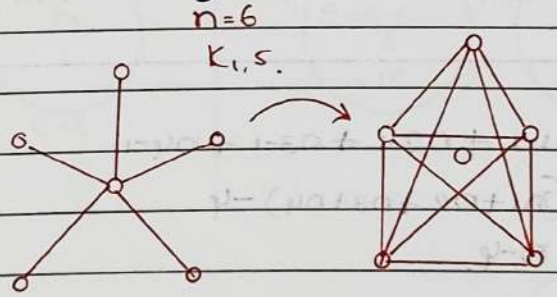
$n = 6$

$K_{1,5}$.

$5 + x = 6 \cdot 5/2$

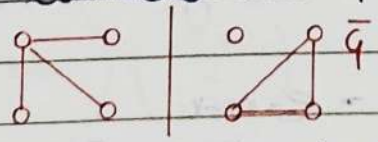
$x = 15 - 5$

$x = 10$



→ if G is connected then $\bar{G}$ will be connected. (False)

eg. G



→ if G is disconnected then $\bar{G}$ will be connected. (True).

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Connectivity in Graphs

1=2=3=4

1 or 2 or 3 or 4.

Connected Graph ($k=1$)

Range of edges ($k=1$)

Tree

1. $\left\{ \begin{array}{l} n-1 \\ + \\ \text{Connected} \end{array} \right. \quad \leq e \leq \frac{n(n-1)}{2}$ (K_n)
2. Connected + it does not contains cycle (acyclic graph)
3. Unique Path is available between all pairs of vertices.
4. Minimally connected.

Disconnected Graph ($k \geq 2$)

Range of edges ($k \geq 2$)

* Forest: Collection of trees

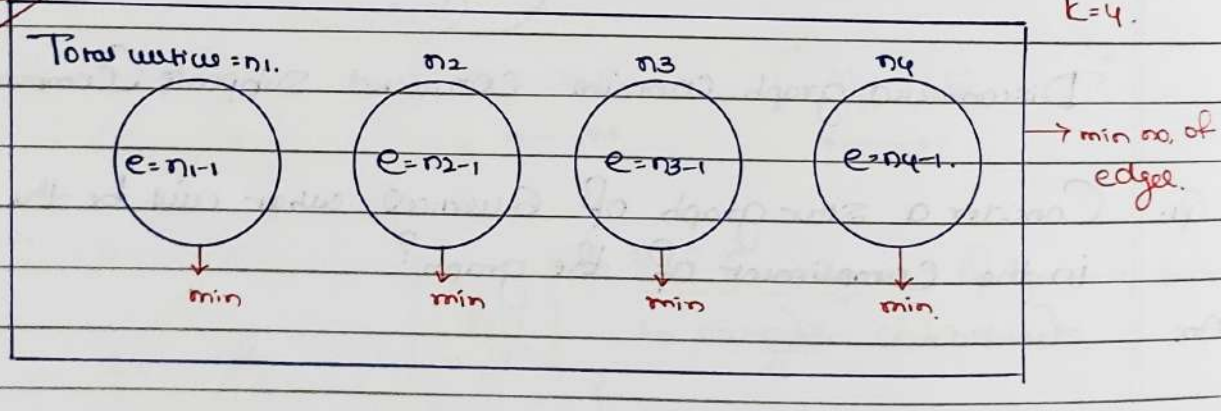
* $n-k \leq e \leq \frac{(n-k)(n-k-1)}{2}$

min no. of edges.

$n_1+n_2+n_3+n_4=n$

Disconnected $k=4$.

Forest



$e = n_1-1+n_2-1+n_3-1+n_4-1$

$= (n_1+n_2+n_3+n_4)-4$

$= n-4.$

$e=n-4$

$e=n-k$ (min no. of edges).

→ Consider undirected graph having 10 vertices and 2 components

$n=10,\ k=2$

min no. of edges $= e=n-k$

$= 10-2 = \underline{8}$

Max no of edges $= e=(n-k)(n-k+1)/2 = (10-2)(10-2+1)/2 = \underline{36}$

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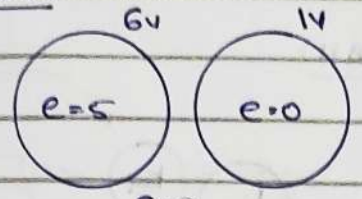
eg: Consider an undirected Graph having 7 vertices and 2 components.

$n=7,\ k=2.$

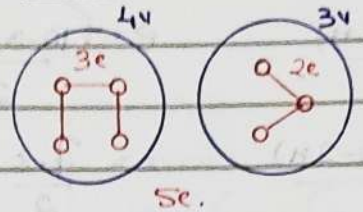
$e = n-k = 7-2 = 5$ (min no. of edges).

Case 1.



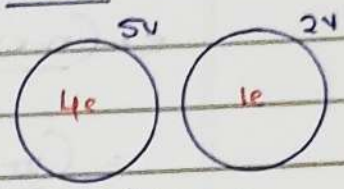
$e=5.$

Case 2



5e.

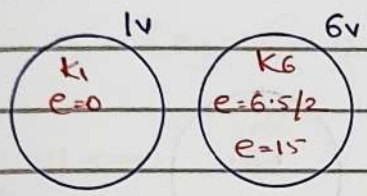
Case 3



$4+1=5e$

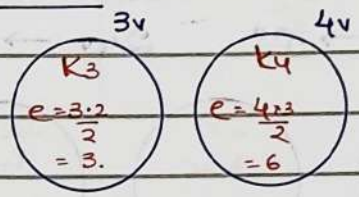
eg:: $n=7,\ k=2$ max no. of edges $e = \frac{(n-k)(n-k+1)}{2} = \frac{5\times 6}{2} = \underline{\underline{15e}}$

Case 1.



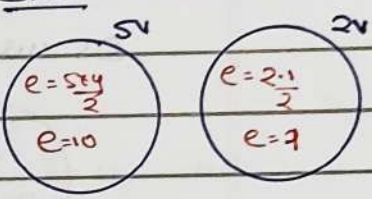
$e=15$

Case 2



$e=9$

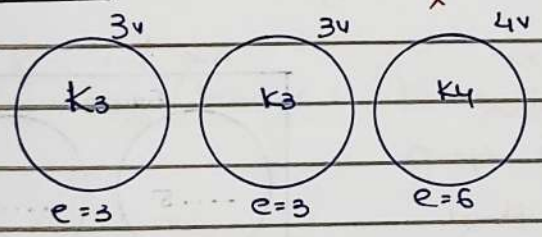
Case 3



$e=11.$

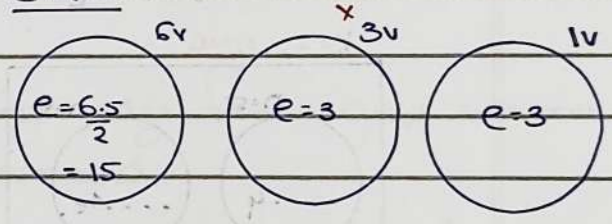
eg:: $n=10,\ k=3$ $e = \frac{(n-k)(n-k+1)}{2} = \frac{(10-3)(10-3+1)}{2} = \frac{7\cdot 8}{2} = \underline{\underline{28e}}$

Case 1:



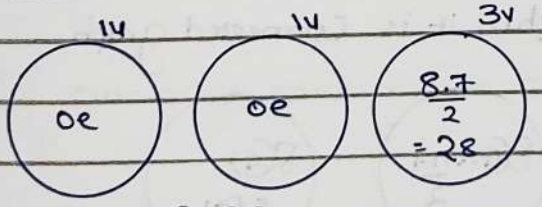
$e=12.$

Case 2:



$e=21.$

Case 3



$e=28.$

$n=10,\ k=3$

$e=28$

2v	2v	6v
3v	3v	4v
⋮		
1v	1v	8v

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eg Consider a disconnected Graph of 10 vertices, what will be the maximum no. of edges?

$n=10 \quad k \geq 2$

$k=2$

$e = \frac{(n-k)(n-k+1)}{2}$

$= \frac{(10-2)(10-2+1)}{2}$

$= 36e$

$k=3$

$e = \frac{(10-3)(10-3+1)}{2}$

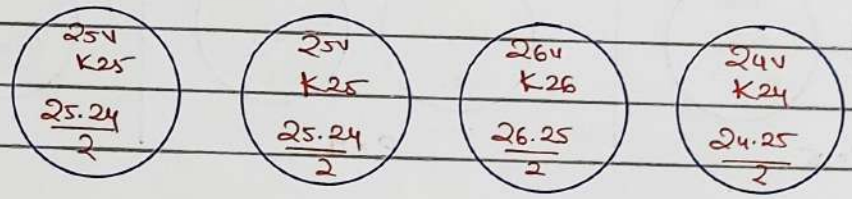
$= \frac{7 \cdot 8}{2}$

$= 28e$

eg. Consider a disconnected graph of four components what will be? - max no. of edges?

25 vertices $\rightarrow 1^{st}$ 26v $\rightarrow 3$

25 vertices $\rightarrow 2^{nd}$ 24v $\rightarrow 4$v



$\Rightarrow 1201$

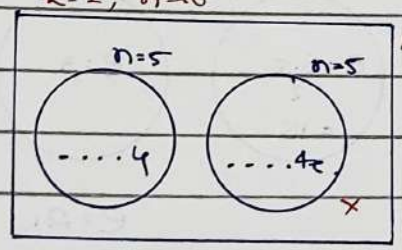
[True]

eg. Consider a graph of 10 vertices if $\delta(G) \geq 5$ then it is connected?

$n=10 \quad \delta(G) \geq 5$ degree of all vertices are atleast 5.

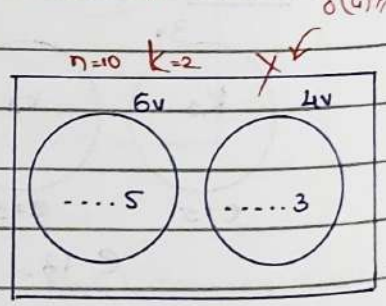
Assume: let's take disconnected. (wrong assumption)

$k=2, n=10$



$\delta(G) \geq 5$ Given

Degree should stay always 5.

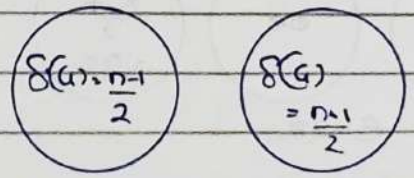


$\delta(G) \geq 5$

$\therefore$ Since the assumption of disconnected is wrong, it is connected

Theorem: If $\delta(G) \geq \frac{n-1}{2}$ then it is connected graph.

Asyume: it is disconnected.



$V_1 = \frac{n-1}{2} + 1 \qquad V_2 = \frac{n-1}{2} + 1$

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$V = V_1 + V_2$

$= \frac{n-1}{2} + 1 + \frac{n-1}{2} + 1$

$= \frac{2(n-1)}{2} + 2 \Rightarrow n-1+2$

$= n+1.$

Total vertices = n

∴ Assumption is wrong

It is connected

→ Max. no. of edges

$= \frac{(n-k)(n-k+1)}{2}$

$k = 2$ (2 components)

$= \frac{(n-2)(n-2+1)}{2}$

$= e = \frac{(n-2)(n-1)}{2}$

$n = 5$

$e = \frac{(n-1)(n-2)}{2}$

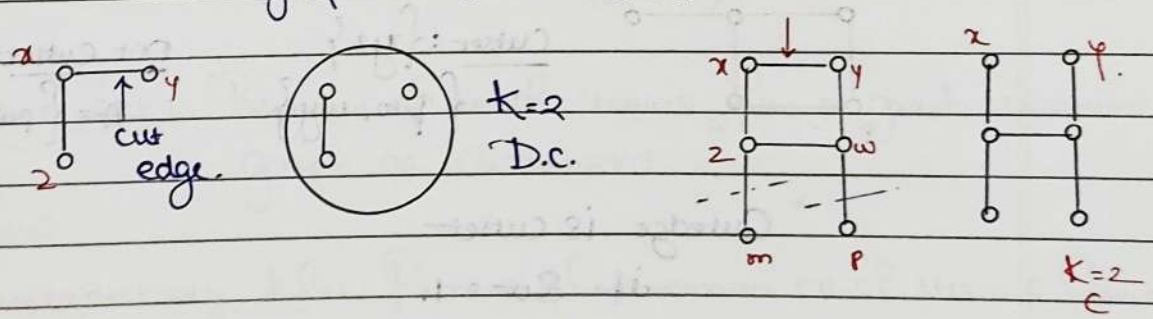
$e = \frac{(5-1)(5-2)}{2}$

$= \frac{4 \cdot 3}{2} \Rightarrow \underline{\underline{6}}$

* If a graph contains $\frac{(n-1)(n-2)}{2}$ edges → may be connected / may not be connected

* If a graph contains more than $\frac{(n-1)(n-2)}{2}$ no of edges then it will be connected.

<u>Cutedge/bridge</u> : Removal of single edge from a graph will make graph as disconnected.



* If cut edge exist then it does not belongs to cycle.

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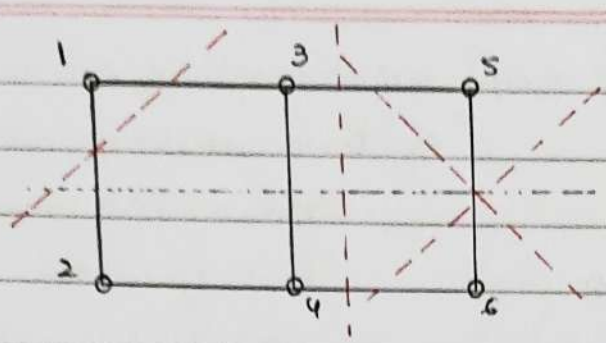


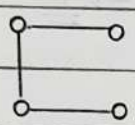
→ no cut edge
↳ 1 edge.

Cut edge Set / Cut Set
Removal of set of edges from a graph will make graph as disconnected

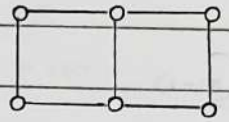
K=2
D.C.

Edge Connectivity / min-cut: $\lambda(G)$

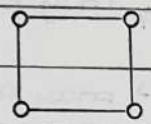
Removal of min. no. of edges from a graph will make graph as disconnected



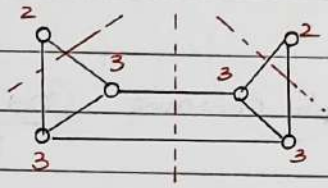
$\lambda(G)=1$



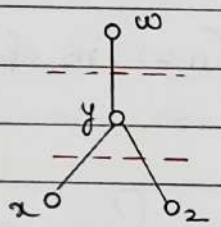
$\lambda(G)=2$



$\lambda(G)=2.$

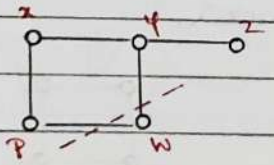


$\lambda(G)=2$



$\{e_1, e_2, e_3\}$

if Subset of any set is cutset then whole set is not cutset.



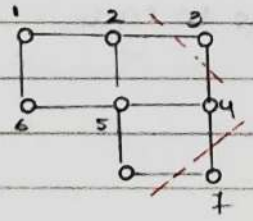
Cutset : $\{xyz\}$
$\{pw, wy\}$

Not cutset:
$A = \{pw, wy, yz\}$
↳ cutset

Cutset

Subset is cutset
$A \rightarrow$ not cutset.

* Cutedge is cutset if Sv = 1.



$\{23, 34\} \rightarrow$ cutset
$\{23, 34, 47\} \rightarrow$ not cutset
cutset.

Size 1 → cutedge
Size 2 } cutset

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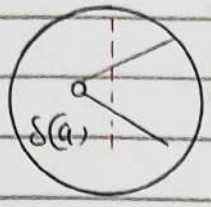
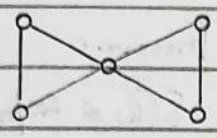
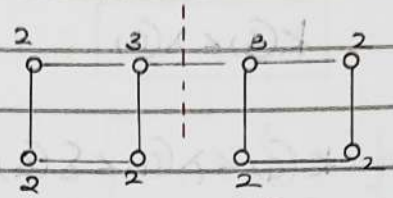


Every Graph $\longrightarrow \delta(G)$

$\downarrow$

edge connectivity $\left\{ \begin{array}{l} \text{min no of} \\ \text{edge} \end{array} \right.$





$\delta(G) = 2 \quad \lambda(G) = 2$

$\delta(G) = \lambda(G) \; (I)$

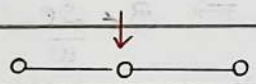
$\delta(G) = 2 \quad \lambda(G) = 1$

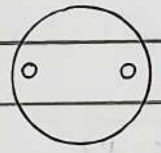
$\lambda(G) < \delta(G) \; — (II)$

$$\boxed{\lambda(G) \leq \delta(G)}$$

Cut vertex / Cut point / Articulation point

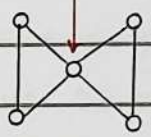
Removal of single vertex from a graph will make graph as a disconnected graph.





$K = 2$

cut vertex

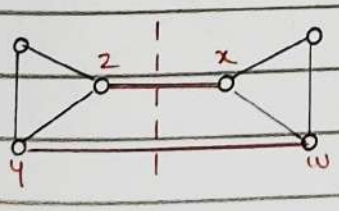


* If cut edge exist then cut vertex also exist for vertex ≥ 3.

$\hookrightarrow$ o——o (above statement is false for this example)

Cut vertex set: Removal of set of vertices from a graph will make graph as disconnected.

Vertex connectivity $K(G)$: Removal of minimum no. of vertices from a graph will make graph as disconnected.



$K(G) = 2$

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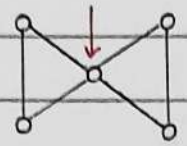
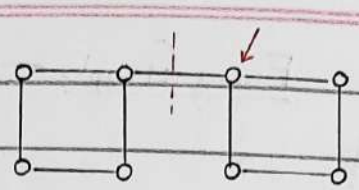


$\lambda(G) = 2 \quad K(G) = 1$

$K(G) < \lambda(G)$



$\lambda(G) = 1 \quad K(G)$

$K(G) = \lambda(G)$

$$\boxed{K(G) \le \lambda(G)}$$

edge conn.

$\lambda(G) \le \delta(G)$

$K(G) \le \lambda(G)$

vertex connectivity

$K(G) \le \lambda(G) \le \delta(G)$

Theorem:

$\delta(G) \le 2e/n \le \Delta(G) \le n-1.$

$$\boxed{K(G) \le \lambda(G) \le \delta(G) \le 2e/n \le \Delta(G) \le n-1}$$

Q1. G is connected graph with 10 vertices and vertex connectivity 3. min no. of edges in G?

Sol^n: $n = 10 \quad K(G) = 3$

$$K(G) \le \lambda(G) \le \delta(G) \le \frac{2e}{n} \le \Delta(G) \le n-1$$

$$K(G) \le \frac{2e}{n} \Rightarrow 3 \le \frac{2e}{10}$$

$$30 \le 2e$$

$$15 \le e$$

$$\Rightarrow 15 \le 15, 16, 17 \ldots$$

Q2. G is bipartite graph of 9 vertices and min no of edges & $K(G) = ?$

$K_{1,8}$

9 vertices

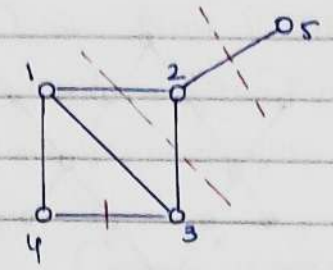
G is bipartite graph + max. no of edges

$K_{4,5}$ $\begin{cases} \lambda(K_{4,5}) = 4 \\ K(K_{4,5}) = 4. \end{cases}$

$K_{m,n}$

$$\boxed{\begin{aligned} K(K_{m,n}) &= \min(m,n) \\ \lambda(K_{m,n}) &= \min(m,n) \end{aligned}}$$

eg: 1

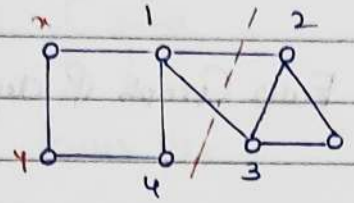


{1 2, 2 4} curser

{12, 24, 45}

curset not curset.

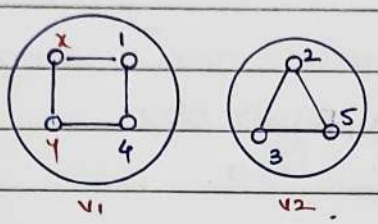
eg: 2



{12, 13} curset.

{12, 13, 14} not curset

Curset.

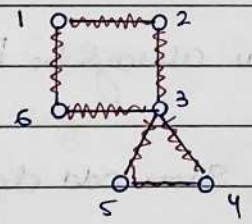


Euler Graph :

Trail : It is an alternating sequence of vertices and edges. R.v/R.X.E.

Closed Trail : Trail + starting = ending vertex.

Euler Circuit : Closed Trail + cover all edges exactly once.



1st end.

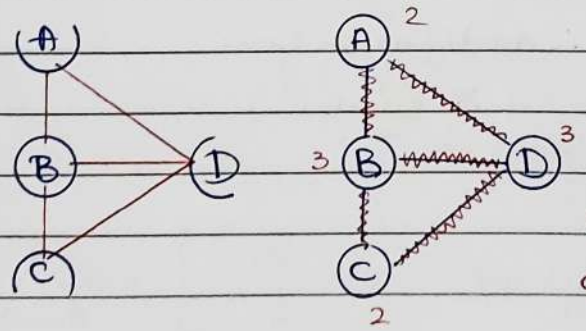
1 – 2 – 3 – 4 – 5 – 6 – 1

Representation of euler circuit.

$G = (V, E)$

Euler Graph.

eg:



– A

– bridge

Closed ✓

all edges ✗

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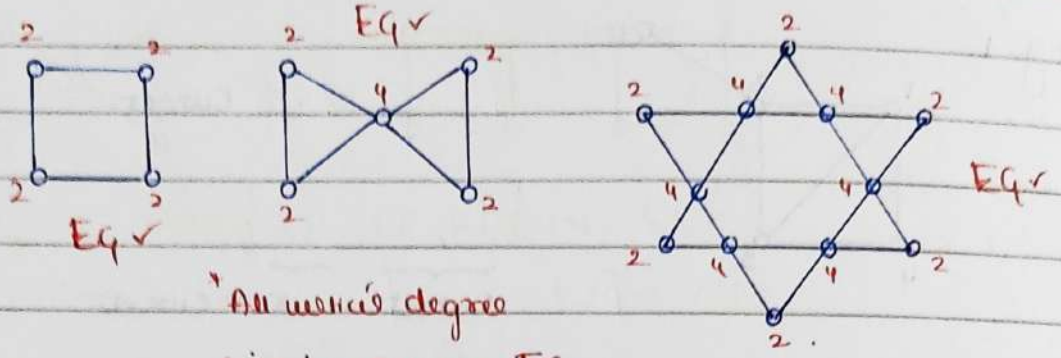
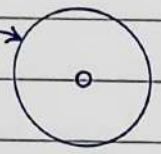


* All vertices degree will be even in EG.

* Non trivial connected graph is Euler Graph if degree of all vertices are even.

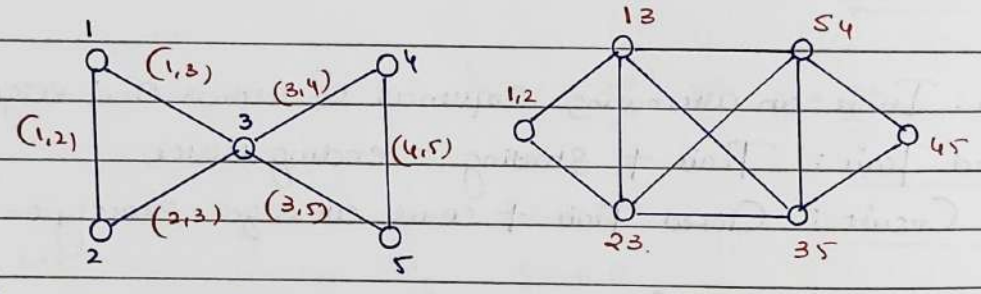
Trivial → Defaulter.

$\begin{cases} V \neq \phi \\ E = \phi \end{cases}$



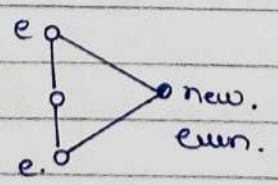
Trivial Graph.

$G(V, E)$
↓ set ↳ set.



* Line Graph of euler graph will always be Euler Graph

Q1. Consider a connected graph having some odd degree vertices, add new vertex make it connect to all odd degree vertices, then resultant graph would be Euler.



∴ all degree are even

∴ Euler.

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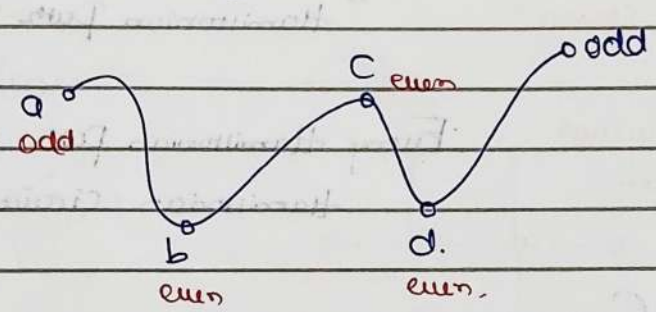
Open Trail: Trail + St ≠ ending vertex

Euler line: Open Trial + covering all edges exactly once
(Unicursal line / Euler path).

Covering all edges
- Closed → Euler ckt
- Open → Euler line / Euler path.

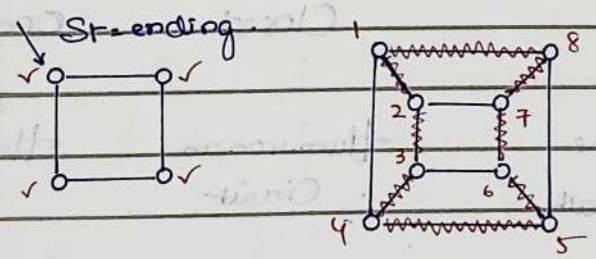
Theorem: Graph contains Euler line if it contains exactly two odd degree vertex.



Open
all edges stop → not on a.

Closed Path: Path + Starting vertex = ending vertex.

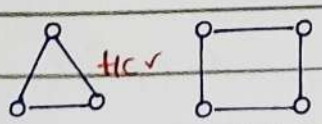
Hamiltonian Circuit: Closed path + cover all vertices exactly once



Graph contains Hamiltonian ckt then graph is called Hamiltonian Graph.

H.C { Closed + all vertices

HC
$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 1$
$G = (V, E)$



* All cycle graph are Hamiltonian Graph.

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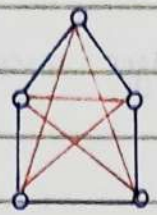
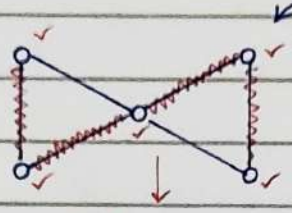


All $K_n \rightarrow$ H.G $(n \geq 3)$

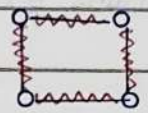


Hamiltonian Path:

One path + covers all vertices exactly once.

× Not Hamiltonian Graph

{ all vertices ✓
closed × }



all vertices
closed ✓

Every hamiltonian circuit contains hamiltonian path. ✓

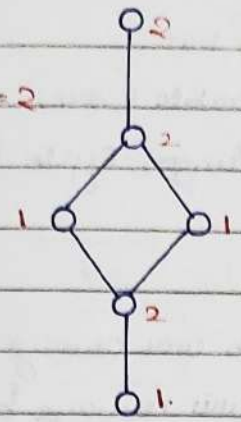
all vertices
open

Every Hamiltonian path contains hamiltonian circuit. (false)

Covering (Covering).

- All edges
  - Closed → Euler Circuit → Euler Graph.
  - Open → Euler line, Euler path.
- All vertices
  - Closed → Hamiltonian Circuit
  - Open → Hamiltonian path.

$\chi(G) = 2$

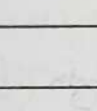


Properly coloring { Paint all vertices with different colors such that adjecent should not have same color.

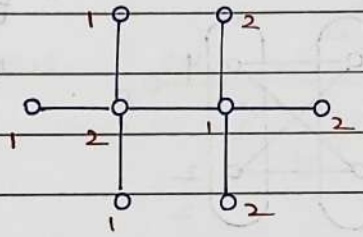
$\boxed{\chi(G) = \text{min} + \text{Properly coloring}}$

Chromatic no. ($\chi(G)$) { Paint all the vertices with minimum no. of colors such that adjecent should not have same color.

1. $\chi(G) = 1$

 $\chi(G) = 1$

2. Tree

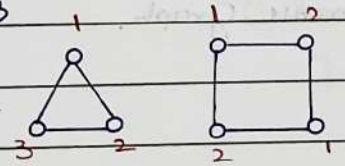


$\chi(\text{Tree}) = 2$

* Every tree will always be 2-colorable ✓ (True)
* Every 2-colorable will always be tree. ✗ (False)

3. Cycle Graph (C_n) ($n \geq 3$)

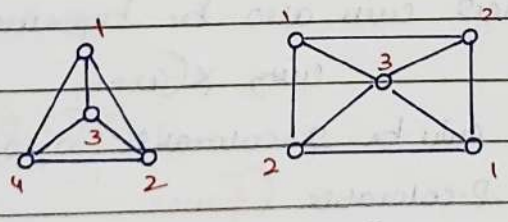
$\chi(C_3) = 3$



$\chi(C_4) = 2$

$\chi(C_n) = 2$	n is even
$\chi(C_n) = 3$	n is odd

4. Wheel Graph (W_n) ($n \geq 4$)



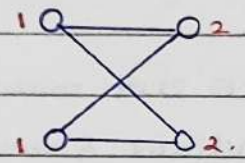
$\chi(W_n) = 4$	n is even
$\chi(W_n) = 3$	n is odd

$\chi(W_4) = 4$ $\chi(W_5) = 3$

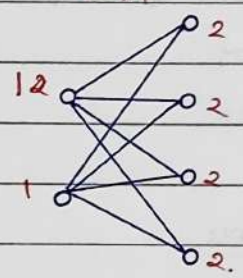
* Every Tree will always be 2-colorable (True)
* Every 2-colorable will always be tree (False)
* Every even length cycle will be 2-colorable (True)
* Every 2-colorable will always be even length cycle. (False)

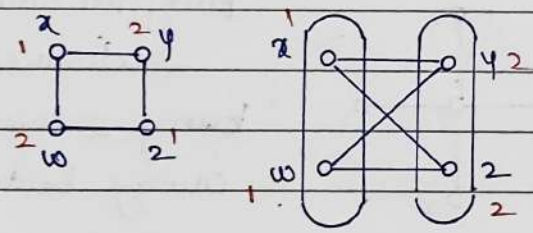
Bipartite Graph.

* Every bipartite graph will always be 2-colorable (True)
* Every 2-colorable will always be bipartite graph.

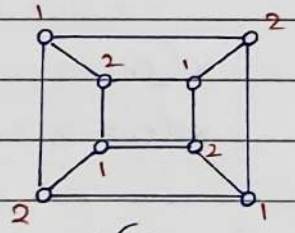


$K_{2,4}$





* Bipartite graph does not contain odd length cycle
  - → True (2-clr)
  - → Even length (2-clr)



$\chi(G) = 2$

* Hypercube will be 2-colorable so bipartite graph.

* Every 2-colorable will always be bipartite graph.
* 2- or more isolated vertices will also be bipartite Graph with $\chi(G) = 1$.

→ Connected bipartite graph will be 2-colorable ($n \geq 2$) (True)

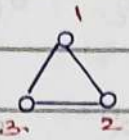
→ Bipartite graph will be 2-colorable (False)

eg: ⊙ ⊙ $\chi(G) = 1$

→ Every bipartite graph will be 2-colorable if connected + $n \geq 2$ (True)

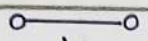
K_n $(n \geq 1)$

⊙ K_1 $\chi(G) = 1$



$\chi(K_3) = 3$.

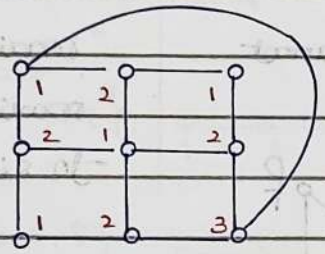
$\boxed{\chi(G) \leq n}$ any graph.



K_2 $\chi(G) = 2$

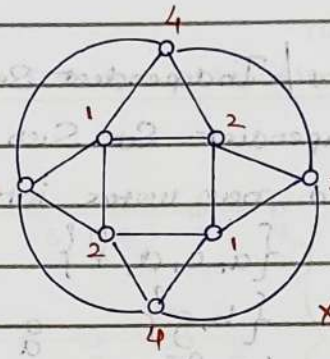
$\boxed{\chi(K_n) = n}$

(1)



$\chi(G) = 3$.

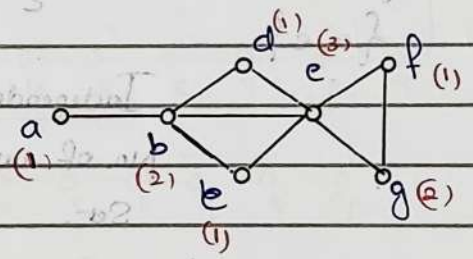
(2)



Triangle Cycle

$\chi(G) = 4$.

eg: Set of non-adjacent vertices.



$\chi(G) = 3$.

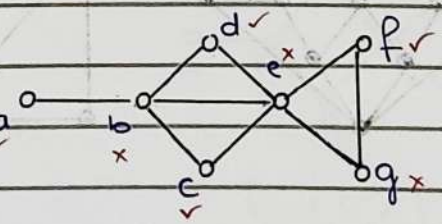
$1 \rightarrow \{a, c, d, f\}$

$2 \rightarrow \{b, g\}$

$3 \rightarrow \{e\}$ ⟶ Chromatic no: 3.

writing same color vertices in one set will give different partitions.

No. of partitions = $\chi(G) = 3$



Independent Set:

$\{a\}$ IS ✓

+c

$\{a, c\}$ IS ✓

+d

$\{a, c, d\}$ IS ✓

$\{a, c, d, f\}$ - IS ✓

↳ full = IS

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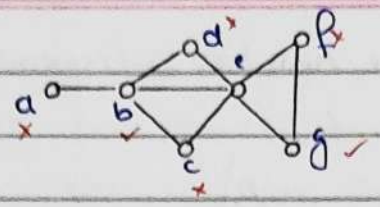
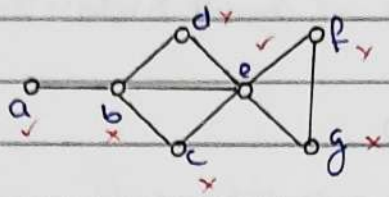
$\{b\}$ - IS

$\{b, g\}$ - IS



$\{a\}$ - IS.

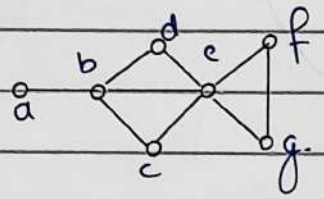
$\{a, e\}$ - IS



Maximal (Full) / Independent Set (MIS)

→ Independent Set such that we cannot add new vertex into this

(MIS) $\begin{cases} \{a, c, d, f\} \\ \{b, g\} \\ \{b, f\} \\ \{a, e\} \end{cases}$



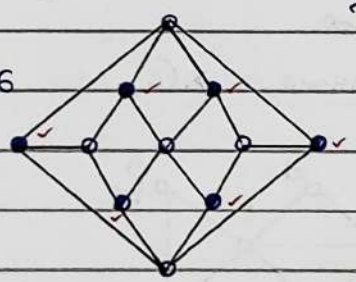
* maximal ≠ maximum.
* maximal is no related to size but a property

Independence No. ($\beta(G)$)

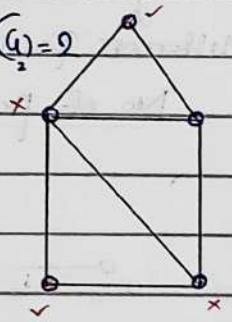
→ No. of vertices present in largest maximal independent set.

Determine the Chromatic number of the following graphs:

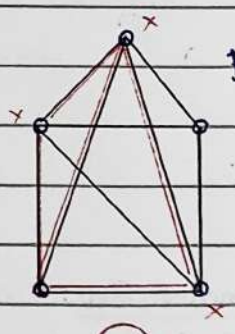
$\beta(G_1) = 6$



$\beta(G_2) = 2$



$\beta(G_3) = 2$



K_4

* 3 partitions = 3 independent sets (MIS)

MIS $\begin{cases} \{\dots\} \\ \{\dots\} \\ \{\dots\} \\ \vdots \end{cases}$

$\beta(G) \geq \frac{100}{3}$ → 100 vertices, $\chi(G) = 3$, 3 partitions

$$\boxed{\beta(G) \geq \frac{n}{\chi(G)}}$$

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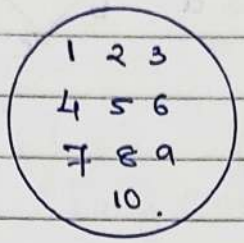
$B(G) = 4$

MIS $\begin{cases} \{\leftarrow 4 \rightarrow\} \\ \{\leftarrow 3 \rightarrow\} \\ \{\leftarrow 3 \rightarrow\} \end{cases}$

$n = 10$



$B(G) \geq \frac{10}{3}$ $\quad \chi(G)$ = no. of Partitions

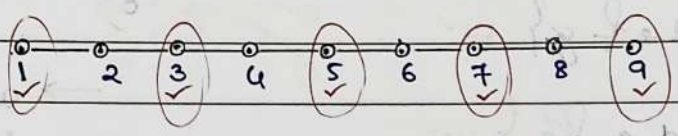
$B(G) \geq \frac{n}{\chi(G)}$

$$\boxed{B(G) \cdot \chi(G) \geq n}$$

eg: What will be the smallest size of maximal independent set?

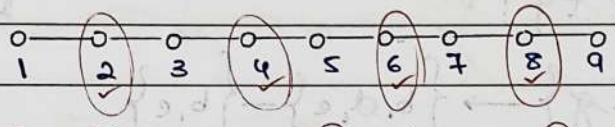
MIS

$\{1, 3, 5, 7, 9\}$

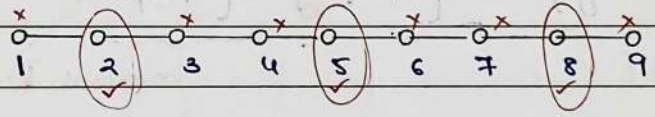


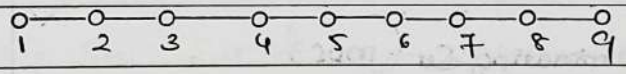
Ans = 3

$\{2, 4, 6, 8\}$



$\{2, 5, 8\}$

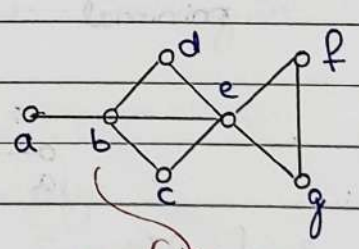




$B(G) \geq \frac{9}{2}$

$\begin{cases} \{1, 3, 5, 7, 9\} - \text{Clr } B(G) = 5 \\ \{2, 4, 6, 8\} - \text{Clr } 2. \end{cases}$

Dominating Set:



$G = (V, E)$

Dominating Set $D \subseteq V$

* If we take any vertex of V either that vertex belongs to D Or its adjacent belongs to D

$D = \{b, e, f\}$

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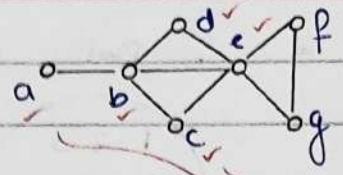
classmate
Date
Page

$D = \{b, e, f\}$ DS ✓



$D_2 = \{a, e\}$ DS ✓

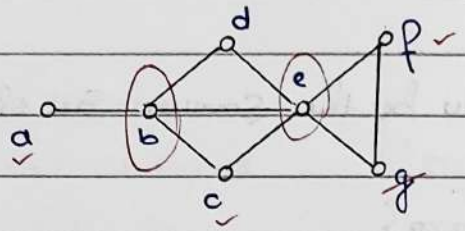
$D_3 = \{a, e, f\}$ DS ✓

$\{a, c, d\}$ not DS ✗

$\{a, c, d\}$ + f is not happy

$f \xleftarrow{e} g$

$\{a\,b\,c\,d\,e\,f\,g\}$ DS ✓



$\{b\,c\,d\,e\,f\,g\}$ DS ✓

$\{b\,d\,e\,f\,g\}$ DS ✓

→ all vertices of graph can be dominating set

$\{b\,d\,e\,g\} \rightarrow \{b, d, e\} - \underbrace{\{b, e\}}_{\text{minimal Dominating Set}}$

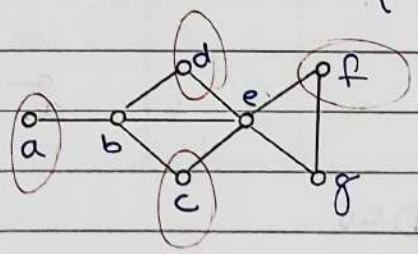
Minimal / Dominating Set (MDS)

Dominating set such that we cannot remove vertex from this

* minimal ≠ minimum
* minimal is not related to size but property.

maximal → Cannot add f to set

minimal → Cannot remove from set.



MDS:

$\{b, e\}$ ✓ $\{b, f\}$ ✓

$\{a\,e\}$ ✓ $\{b\,g\}$ ✓ $\{a, c, d, f\}$ ✓

Size = 2 ∴ $\alpha(G) = 2$

Domination no ($\alpha(G)$)

→ No. of vertices present in smallest minimal dominating set

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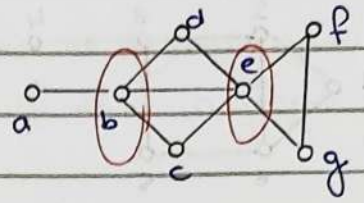
$\beta(G) = 4$

$\boxed{\alpha(G) \leq \beta(G)}$

MIS	MDS
{a, c, d, f}	{a, c, d, f}
{b, f}	{b, f}
{b, g}	{b, g}
{a, e}	{a, e}
	{b, e}

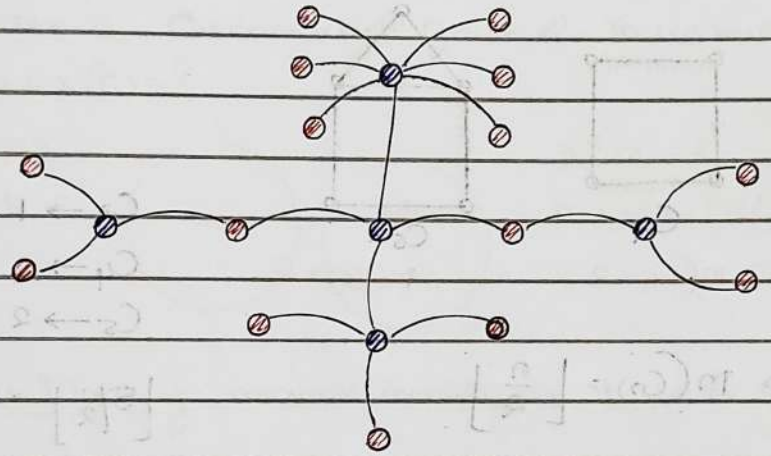


* Every MIS will always be MDS ✓
* Every MDS will be MIS (false)

eg: {b, e}

→ Domination ($\alpha(G)$) ≤ Independence no ($\beta(G)$).

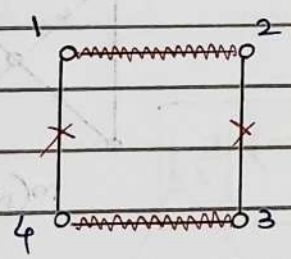
eg:



Independence No. $\beta(G) = 16$

Domination no. $\alpha(G) = 4$

Independent edge Set: Set of non adjacent edges.
(matching set)



{1, 2} – Matching set

{12, 43} – Matching Set.

maximal matching set.

Matching Set: Set of non-adjacent edges.

Maximal matching Set: matching set such that we cannot add new edge into this.

* Maximal is not related to size or property.

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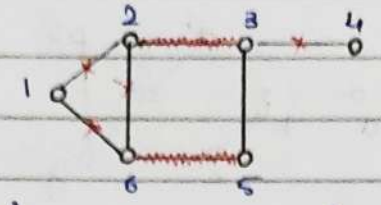
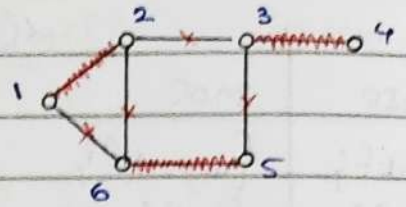




$\{23\}$ - ms

$\{23, 65\}$ - maximal matching set.

Matching No.

$m(G)$

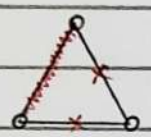
No. of edges Present in larger maximal matching set.

$\{34\}$ - ms

$\{34, 65\}$ - ms

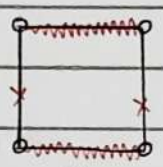
$\{34, 65, 12\}$ - mms

Cycle Graphs



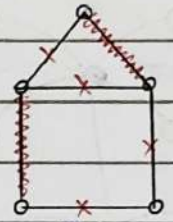
C_3

$m(C_3) = 1$



C_4

$m(C_4) = 2$



C_5

$m(C_5) = 2$

$C_3 \to 1$

$C_4 \to 2$

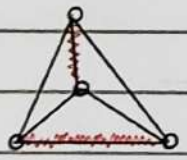
$C_5 \to 2$

$\Longrightarrow m(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$

$\lfloor 5/2 \rfloor = \lfloor 2.5 \rfloor$

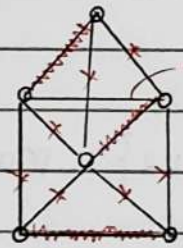
Wheel Graphs

W_4



$m(W_4) = 2$

W_6



not there

$m(W_6) = 3$

$\boxed{m(W_n) = \left\lfloor \frac{n}{2} \right\rfloor}$

W_5

$m(W_5) = 2$

$$\boxed{m(C_n) = m(K_n) = m(W_n) = \left\lfloor \frac{n}{2} \right\rfloor}$$

$K_{2,4}$

$m(K_{2,4}) = \min(2, 4) = 2$

$$\boxed{m(K_{m,n}) = \min(m, n)}$$

Q1 Concider graph vertices are represented as permutations of given set $\{1,2,3\}$, 2 vertices are connected if their adjecent elements are interchange, $m(G) = ?$

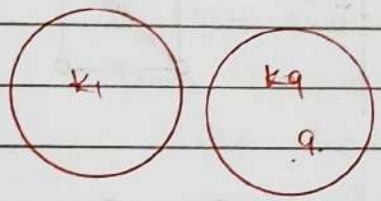
$S = \{1,2,3\}$
$= 3!$

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

C_6
$m(G) = \lfloor 6/2 \rfloor = 3$

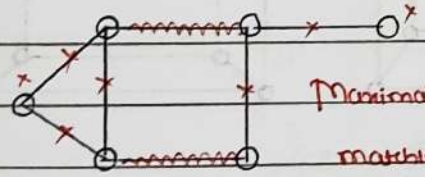
Q2 Consider a disconnected graph of 10 vertices and maximum edges $m(G) + \chi(G) = ?$

10 v
$k = 2$

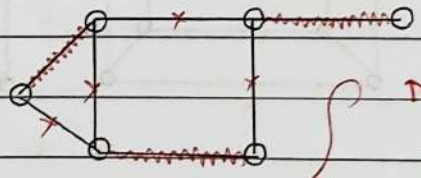


$\chi(K_9) = 9$
$m(K_9) = \lfloor 9/2 \rfloor = 4$

Perfect Matching: maximal matching set + which saturates all the vertices.

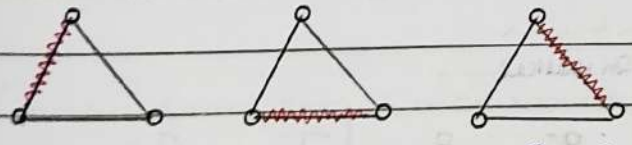


Maximal matching set ✓



P.m

Maximal matching set + covers all vertices



MMS MMS MMS

* If perfect matching exist then no of vertices will be even.

* P.m will not exist

Extra :- * If (perfect matching exist) then (no. of vertices will be even)
P. Q.
* If odd no. of vertices then P.m will not exist.

$P \rightarrow Q$
vice versa
$Q \not\rightarrow P$

Contra $\neg P \rightarrow \neg Q$

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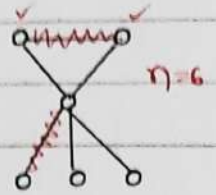
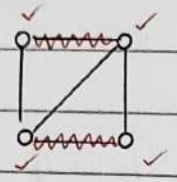
Stmt 2 - * If perfect matching exist then no. of vertices will be even.
— Vice versa is not true.

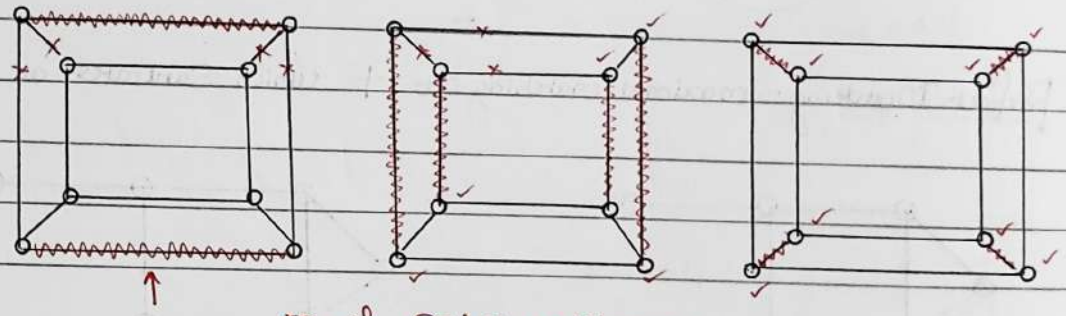
* If graph is even no. of vertices then perfect matching exist. (false)



Perfect match → maximal matching set

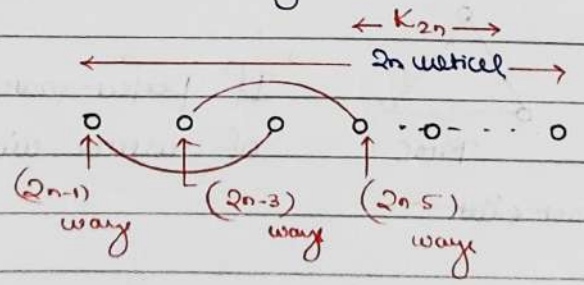


Maximal matching set ↛ Perfect matching

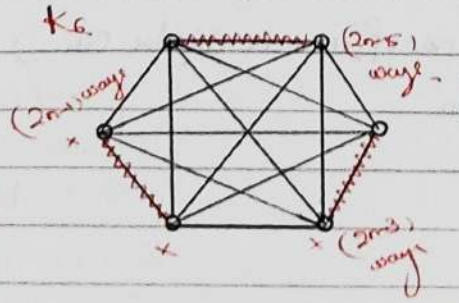


no. of perfect matching = 3

* No. of perfect matching in K_{2n}. * Total vertices = 2n



eg:



Total [2n vertices]

Total perfect matching

$= (2n-1) \times (2n-3) \times (2n-5) \cdots$

$= \frac{2n}{2n}(2n-1)\frac{2n-2}{2n-2}(2n-3)\cdots$

$= \frac{(2n)(2n-1)(2n-2)(2n-3)\cdots}{2n(2n-2)(2n-4)\cdots}$

$\Rightarrow \frac{2n!}{2^n \times n!}$

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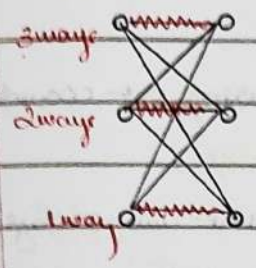
Q1. What will be P.m in complete graph of 6 vertices?

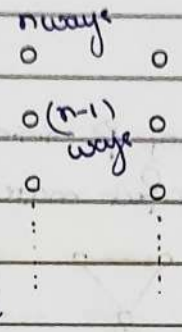
Ans $2n = 6$

$n = 3$ $\quad \frac{(2n)!}{2^n \cdot n!} = \frac{\cancel{720}^{15}}{\cancel{48}} = 15$

$K_{3,3}$.

eg.



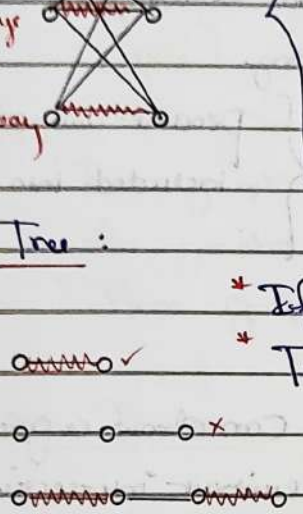


$n.(n-1).(n-2)\ldots = n!$

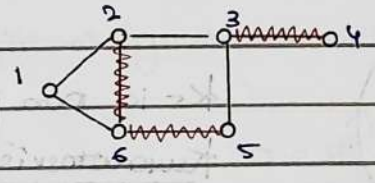
$K_{n,n} \rightarrow n!$

Tree :

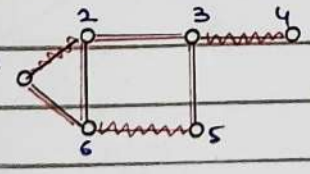
* If P.m exist in tree, it will be Unique.
* Total P.m in tree almost 1.
* P.m doesn't exist in all trees, but somehow if it exist there is no other representation so that is why it is unique.



Covering set (edge) : Set of edge, such that all vertices must incident on atleast 1 edge.

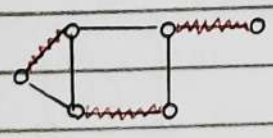


Covering vertex set : Set of vertices, such that all edges, must incident on atleast 1 vertex.



C.S $\{12, 16, 65, 23, 35, 34, 26\}$

Set of edges
↓
all vertices
incident
↓
atleast 1 edge



C.S $\{12, 16, 65, 23, 34, 26\}$

C.S $\{12, 65, 34\}$ → minimal Covering set.

$\{2, 3, 6\}$ → vertex covering set.

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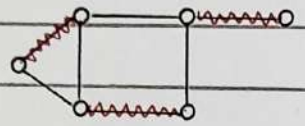
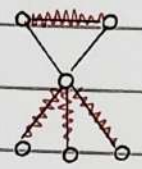
Minimal Covering Set: Covering set such that we cannot remove new edge from this.

Covering no ($C(G)$): no. of edges present in smallest minimal covering set.

* Every P.m will be minimal covering set.

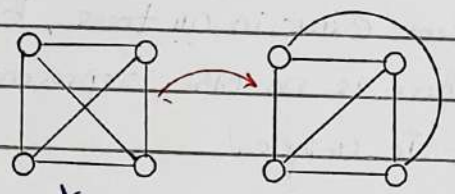


* Every ~~P~~ minimal covering set could be perfect matching (False).



{ Pendant will always be included into covering

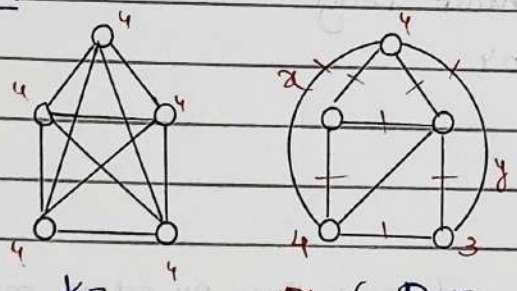
Planar Graph.



K_4 "Planar graph"

→ If we can draw a graph on a plane without intersection of its edges then that graph is called planar graph.
Otherwise its non-planar graph.

eg: 1



K_5

$n=5,\ e=10$

{ Planar { $P=5$
$e=9$ (nine)
$\{K_5-e\}$

K_5 is non planar graph
Kuratowski's 1st Graph
K_5 is non-planar graph.

$n=1$ ○

$n=2$ ○—○ ○ ○

$n=3$

$n=4$ ..... K_4

$n=5$ ..... $\{K_5-e\}$ ($n=5, e=9$) (K_5) → non planar.
(planar

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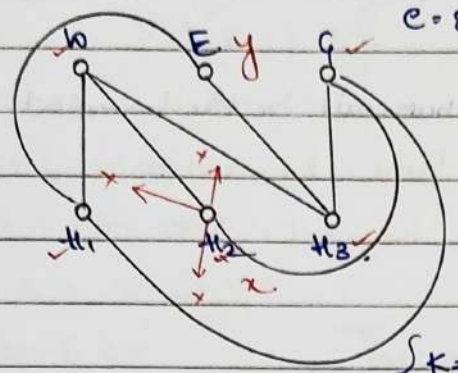
eg 2



$n=6$
$e=8$

$\{K_{3,3}-e\}$

$(K_{3,3})$
$\begin{cases} n=6 \\ e=9 \end{cases}$ $\delta(G) = \frac{2e}{n} = \Delta(G) = 3$

* Degree of each vertex will be 3:
* $K_{3,3}$ is non planar.

Note: (Pen, both examples)
* Both graphs are non-planar, removal of single edge from both the graphs will make Planar graph
* Both graphs are Regular Graph

Non Planar

	min	
K_5	$n=5$	$e=10$
$K_{3,3}$	$n=6$	$e=9$ ← min

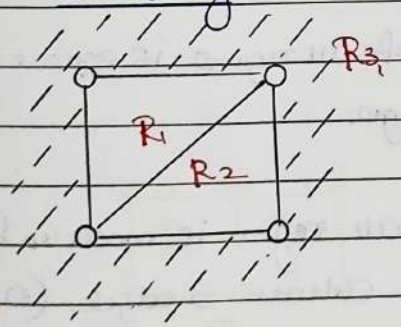
→ K_5 is non-planar graph with min. no. of vertices.
→ $K_{3,3}$ is non Planar with min no of edges.

Draw a planar graph on a plane.

embedding:

$\boxed{n-e+f=2}$

It creates or makes region / faces (r/f)



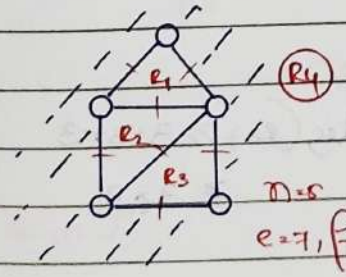
$n=4$ $f=3$
$e=5$

$4-5+3 \Rightarrow 2$

Closed
bounded
finite
eg: R_1, R_2

Open
bounded
infinite
eg: R_3

eg.



$n=5$
$e=7, f=4$

$n \rightarrow$ Total vertices
$e \rightarrow$ no. of edges
$f \rightarrow$ no. of faces
} Planar

Euler's Theorem
→ $n-e+f=2$

VIVEKANAND VERNEKAR

Q1. Consider a planar graph having 10 vertices & 15 edge. What will be no. of face?

$n=10, \quad e=15$

$n-e+f=2$

$10-15+f=2$

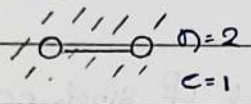
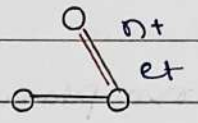
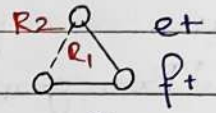
$f=7$

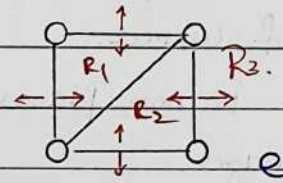
Q2. What will be the bounded region?

Total = bounded + 1

bounded = 7 − 1

= 6

Case 1.	Case 2	Case 3.
		
$n-e+f=2$	$n=3 \quad e=2$	$n-e+f=2 \quad (n \geq 2)$
$2-1+f=2$	$n-e+f=2$	$3-3+f=2$
$f=1.$	$3-2+f=2$	$f=2.$
	$f=1.$	



Degree of regions = no of edges involved into this

$e=5$

$\begin{cases} \deg(R_1)=3 \\ \deg(R_2)=3 \\ \deg(R_3)=4 \end{cases}$

$\deg(R_1)+\deg(R_2)+\deg(R_3)=3+3+4$

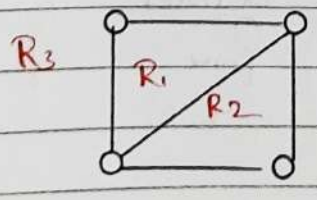
$\sum d(R_i)=10=2\times 5$

no. of edges.

$\boxed{\sum d(R_i)=2e}$

Sum of degree of all regions is equal to twice the no of edges.

eg:



$\begin{cases} \deg(R_1)=3 \\ \deg(R_2)=3 \\ \deg(R_3)=4 \end{cases}$

$\begin{cases} \text{all region is made up by} \\ \text{atleast 3 edges. } (n \geq 3) \end{cases}$

$\deg(R_1) \geq 3$

$\deg(R_2) \geq 3$

$\deg(R_3) \geq 3$

$\deg(R_1)+\deg(R_2)+\deg(R_3) \geq 3+3+3$

$\geq 3\times \underset{\text{no of regions}}{3}$

VIVEKANAND VERNEKAR

$\deg(R_1) + \deg(R_2) + \deg(R_3) \geq 3 \times 3$ no. of faces

$\sum d(R_i) \geq 3f$

$\boxed{2e \geq 3f}$

$n - e + f = 2$

$\boxed{f = 2 + e - n}$

$n \geq 3$ ↑ all edges are made by atleast 3 edges

$2e \geq 3(2+e-n)$

$2e \geq 6 + 3e - 3n$

$3n - 6 \geq 3e - 2e$

$3n - 6 \geq e$

$e \leq 3n - 6$

Theorem : If graph is planar then $e \leq 3n-6$

* Converse is not true.

→ if $e \leq 3n-6$ then G is planar (false).

eg: $K_{3,3}$

$e = 9,\ n = 6$

$e \leq 3n - 6$

$9 \leq 3(6) - 6$

$9 \leq 12$ (True)

Contrapositive:

if not $(e \leq 3n-6)$ then G is not planar.

***

$\boxed{\text{if } e > 3n-6 \text{ then G is non-planar.}}$

$p \rightarrow q$. Contrapositive

$\neg p \rightarrow \neg p$

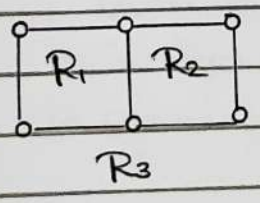
→ G is connected planar with 10 vertices. If the number of edges on each face is 3 then no. of edges in G.

$e = 3n - 6$

$e = 3(10) - 6$

$e = 24$

Region is made up of atleast 4 edges ($n \geq 4$)

→ 

$\deg(R_1) \geq 4$

$\deg(R_2) \geq 4$

$\deg(R_3) \geq 4$

$\deg(R_1) + \deg(R_2) + \deg(R_3) \geq 4+4+4$

$\sum d(R_i) \geq 4 \cdot 3$ no of region faces

$\boxed{2e \geq 4f}$

→ $2e \geq 4f$

$2e \geq 4(2+e-n)$

$2e \geq 8 + 4e - 4n$

$4n - 8 \geq 4e - 2e$

$4n - 8 \geq 2e$

$\underline{2n - 4 \geq e}$

$\begin{cases} n - e + f = 2 \\ f = 2 + e - n \end{cases}$

<u>Theorem</u>: If Graph is planar then $e \leq 2n-4$.

→ $2e \geq 5f$

$2e \geq 5(2+e-n)$

$2e \geq 10 + 5e - 5n$

$5n - 10 \geq 5e - 2e$

$5n - 10 \geq 3e$

$3e \leq 5n - 10$

$e \leq (5/3)n - (10/3)$.

$\begin{cases} n - e + f = 2 \\ f = 2 + e - n. \end{cases}$

Degree of each region is made up of atleast 5 edges.

<u>Theorem</u>: If Graph is planar then:

$e \leq (5/3)n - (10/3)$

→ If Graph is planar then $\underline{e \leq 3n-6}$

$\delta(G) \leq 2e/n \leq \Delta(G) \leq n-1$.

$\begin{cases} \text{Connected,} \\ \text{Planar,} \\ \text{Simple} \end{cases} \cap \begin{cases} \text{Connected} \\ \text{disconnected} \\ \text{Simple} \\ \text{Planar} \\ \text{non-planar} \end{cases}$

Connected
Planar
Simple

$\delta(G) \leq \frac{2e}{n}$

$\delta(G) \leq \frac{2(3n-6)}{n} \leq \frac{6n-12}{n} < \frac{6n}{n} - \frac{12}{n} \leq 6 - \frac{12}{n}$

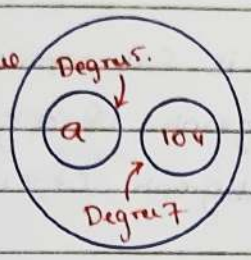
$\delta(G) \leq 6 - \frac{12}{n}$ $(\delta(G) \leq 5)$ $\begin{cases} \text{Connected} \\ \text{Planar} \\ \text{Simple.} \end{cases}$

<u>Theorem</u>: If G is Connected Planar graph then $\delta(G) \leq 5$.

VIVEKANAND VERNEKAR

Q1. A planar graph has vertices degree 5 & 7, if there are 10 vertices of degree 7. Prove that atleast 22 vertices of degree 5.

Total vertices $= 10 + a$



$\sum d(v_i) = 2e$

$5a + 10 \cdot 7 = 2e$

$e = \frac{5}{2}a + 35$

$e \leq 3n - 6$

$\frac{5}{2}a + 35 \leq 3(10 + a) - 6$

$\frac{5}{2}a + 35 \leq 30 + 3a - 6$

$35 - 30 + 6 \leq 3a - \frac{5}{2}a$

$11 \leq \frac{1}{2}a$

$\underline{\underline{22 \leq a}}$

Q2 $\delta \rightarrow$ minimum degree $(\delta \geq 3)$
for an planar of n. vertices
a) In any planar embedding, the no. of face is atleast $\frac{n}{2} + 2$

less than $\frac{n}{2} + 2$

$3n \leq 2e$

$3n < 2(n + f - 2)$

$3n \leq 2n + 2f - 4$

$3n - 2n + 4 \leq 2f$

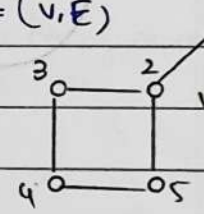
$n + 4 \leq 2f$

$\frac{n}{2} + 2 \leq f.$

$\begin{cases} n - e + f = 2 \\ n + f - 2 = e. \end{cases}$

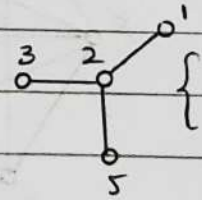
Subgraph: (C)
- edge disjoint Subgraph
- Vertex disjoint Subgraph
- induced Sub graph
- Spanning Sub graph (Spanning tree)

eg: $G = (V, E)$



$V = \{1, 2, 3, 4, 5\}$

$G_1 = (V_1, E_1)$ (Subgraph)



$\{1, 2, 3, 5\}$

$\rightarrow G_1$ is Subgraph of G

$\phi \neq V_1 \subseteq V \quad (V \subseteq V)$

$E_1 \subseteq E \quad (E \subseteq E)$

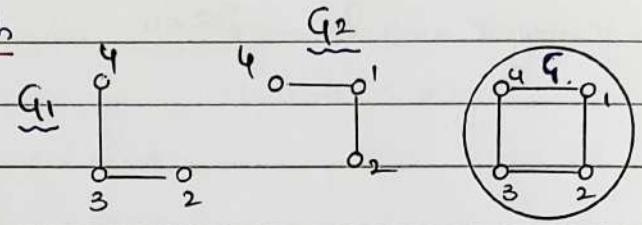
* each edge E_1 is incident on G_1 is same as in G

Key Points:

1. Every Graph is subgraph of itself.
2. $G_2 \subseteq G_1 \subseteq G$, $G_2 \subseteq G$
   Subgraph of a subgraph of a graph is subgraph of a Graph.
3. Every single vertex of a Graph is subgraph of a graph
4. Every single edge of a Graph is subgraph of a graph.

Edge disjoint Subgraph

→ Two graphs which does not have any common edge.

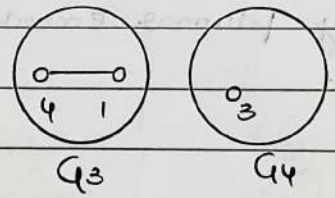


We may get common vertex.

$E_1 = \{43, 32\}$ $E_2 = \{14, 12\}$

Vertex disjoint Subgraph

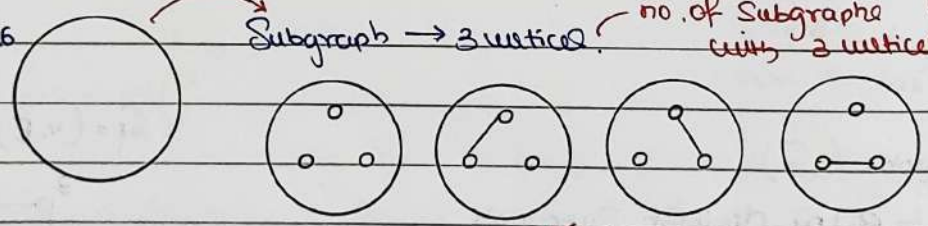
→ Two graphs, which does not have any common vertex.



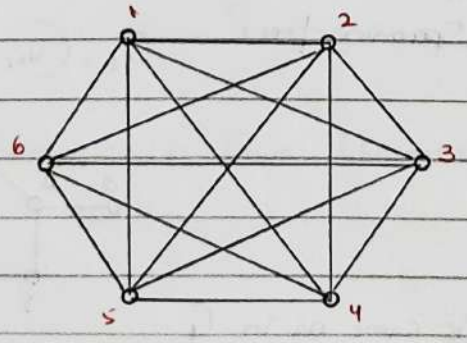
It will not have any common vertex so no point of having common edge.

Q3 How many subgraph of K_6 satisfy $|V| = 3$

K_6 → Subgraph → 3 vertices. no. of subgraphs with 3 vertices → $2^{\frac{3(3-1)}{2}}$



$^6C_3 \times 2^{\frac{3(3-1)}{2}}$



$V = \{1, 2, 3, 4, 5, 6\}$

In a class of 6 students, how many ways we can select 3 students

$^6C_3 \times 2^{\frac{3(3-1)}{2}}$

Q4. How many Subgraphs are possible with K_6?

→ 1 vertex, 2 vertices, 3 vertices, 4 vertices

$$^6C_1 \times 2^{\frac{1(1-1)}{2}} + {}^6C_2 \times 2^{\frac{2(2-1)}{2}} + {}^6C_3 \times 2^{\frac{3(3-1)}{2}} + {}^6C_4 \times 2^{\frac{4(4-1)}{2}}$$

5 vertices, 6 vertices

$$^6C_5 \times 2^{\frac{5(5-1)}{2}} + {}^6C_6 \times 2^{\frac{6(6-1)}{2}}.$$

$$\Rightarrow \sum_{i=1}^{n} {}^nC_i \times 2^{\frac{i(i-1)}{2}} = \sum_{i=1}^{6} {}^6C_i \times 2^{i(i-1)/2}.$$

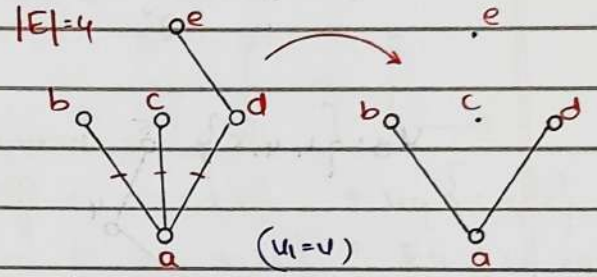
Spanning Subgraph

G_1 is Spanning Subgraph of G

$G_1 = (V_1, E_1)$ $G = (V, E)$

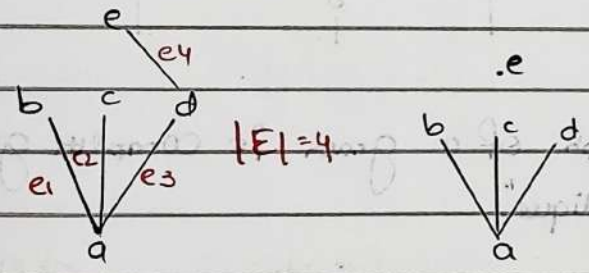
1. Subgraph
2. $V_1 = V$

$(E_1 \subseteq E)$

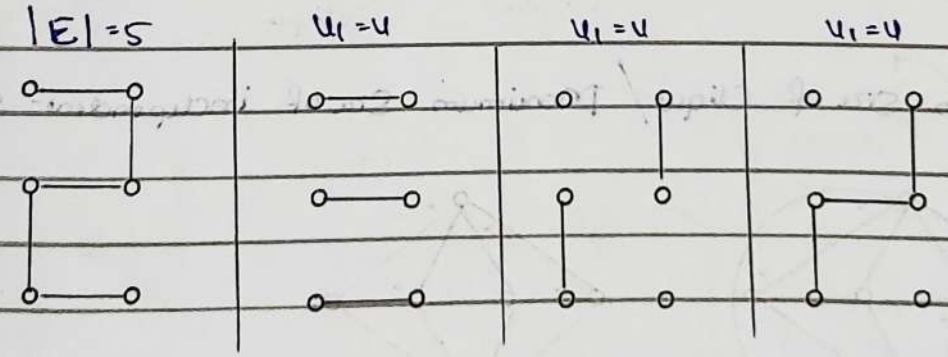


2^4 Spanning Subgraphs are possible

not Spanning Subgraph.



* all vertices must be present, edge may or may not.



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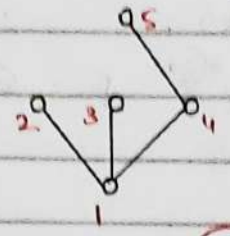
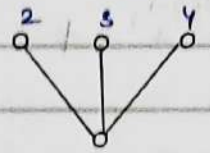
Induced Subgraph:



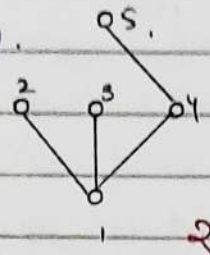
2^V

$V_1 = \{1, 2, 3, 4\}$



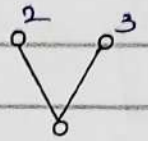
If we take any vertex set, all edge related to those vertices must be present.

eg: V.

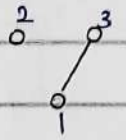


2^V

$V_1 \{1, 2, 3\}$



$V_1 \{1, 2, 3\}$

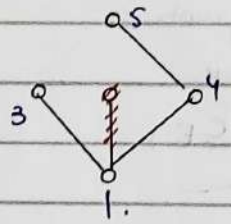


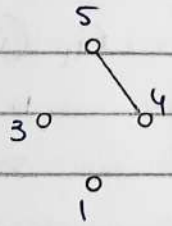
Not induced Subgraph
$\{12\}$ is missing.

$V_3 = \{1, 4, 5\}$

induced graph.

$V_4 = \{1, 3, 4, 5\}$

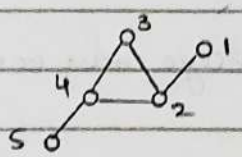




not induced graph.

Clique:

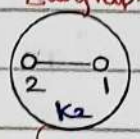
The subgraph of a graph is complete graph then it is called "Clique".



Subgraph

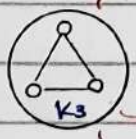
Subgraph

Subgraph



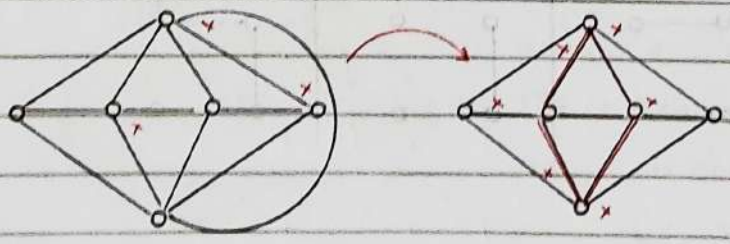
Clique

Clique no $\omega(G) = 3$



Subgraph
Clique.

Maximum size of Clique / Maximum size of independent set.
4 → 2.



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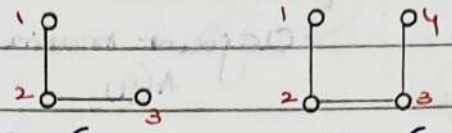
Graph Operations:

1. Union ($\cup$) →
2. Intersection ($\cap$)
3. Ring Sum ($\oplus$)

G_1: G_2

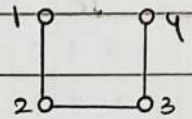


$G_3 = (V_3, E_3)$

$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$

$V_1 = \{1, 2, 3\}$ $\cup$ $V_2 = \{1, 2, 3, 4\} = V_3$

$E_1 = \{12, 23\}$ $\cup$ $E_2 = \{12, 23, 34\} = E_3$



$G_3 = (V_3, E_3)$

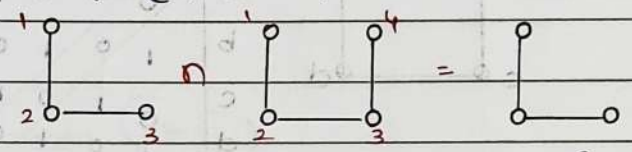
Intersection: ($\cap$)

$G_4 = (V_4, E_4)$

$V_4 = V_1 \cap V_2$

$E_4 = E_1 \cap E_2$

$G_1 = (V_1, E_2)$ $G_2 = (V_2, E_2)$



$V_1 = \{1, 2, 3\} \cap \{1, 2, 3, 4\} = V_4 \{1, 2, 3\}$

$E_1 = \{12, 23\} \cap \{12, 23, 34\} = E_4 \{12, 23\}$

Ring Sum ($\oplus$)

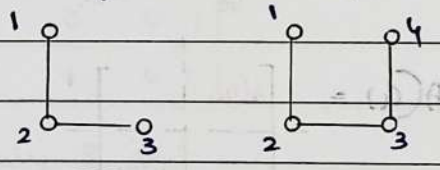
$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$

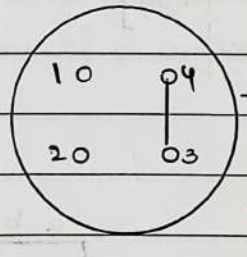
$G_6 = (V_6, E_6)$

$V_6 = V_1 \cup V_2$

$E_6 = (E_1 \cup E_2) - (E_1 \cap E_2)$

G_1 G_2

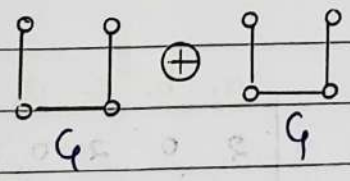




→ edge in G_1 or G_2 but not in both

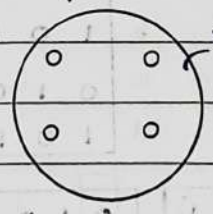
* $G \oplus G$ = null graph

→ $G \oplus G$ = null graph.



G G

G



null graph (set of isolated vertices).

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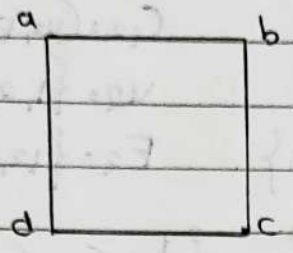
Graph representation : $n \times n$ – Symmetric binary matrix.

→ Adjacent matrix / Connection matrix
$A(G)$

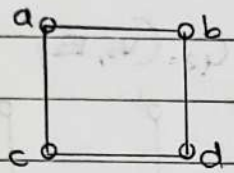
$x_{ij} = 1$ { connection between vertex i and j.



$$A(G) = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array} = 0 \qquad 4\times4.$$

$x_{ij} = x_{ji}$

$x_{ab} = x_{ba}$



$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array} \quad 4\times4$$

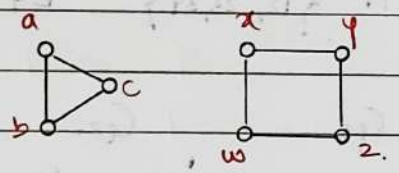
1.

→ no self loop

→ diagonal elements can be zero

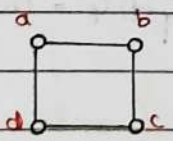
→ no. of 1's present in row or column can give degree

eg:



$$A(G) = \left[\begin{array}{c|c} A(G_1) & \\ \hline & A(G_2) \end{array}\right]$$

$$\left[\begin{array}{c|c} \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 1 & 0 & 1 \\ \vdots & & \vdots & \vdots \end{matrix} \end{array}\right]$$



$$A \quad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array}$$

$A^2(i,i)$ = degree of vertex v_i

$0\times0 + 1\times1 + 0\times0 + 1\times1 = 2$

$$A^2 = A \times A \quad \begin{array}{c} \boxed{0\ 1\ 0\ 1} \\ 1\ 0\ 1\ 0 \\ 0\ 1\ 0\ 1 \\ 1\ 0\ 1\ 0 \end{array} \quad \begin{array}{c} \boxed{\begin{matrix} 0 \\ 1 \\ 0 \\ 1 \end{matrix}}\ \begin{matrix} 1\ 0\ 1 \\ 0\ 1\ 0 \\ 1\ 0\ 1 \\ 0\ 1\ 0 \end{matrix} \end{array}$$

$$A^2 \quad \begin{array}{c|cccc} & a. & b. & c. & d. \\ \hline a & 2 & 0 & 2 & 0 \\ b & 0 & 2 & 0 & 2 \\ c & 2 & 0 & 2 & 0 \\ d & 0 & 2 & 0 & 2 \end{array}$$

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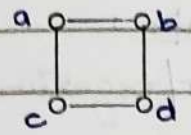
$a \to b$ (length 3) walk.



$$A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

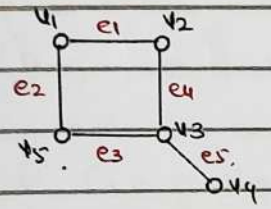
$A^3(a,b) = \{$ 4 walks of length 3 from a, b

$A^n(i,j)$ = no. of walks from i,j of length n.

a — d — a — b
a — d — c — b
a — b — a — b
a — b — c — b

$$A^3 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A^3 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 4 & 0 & 4 \\ b & 4 & 0 & 4 & 0 \\ c & 0 & 4 & 0 & 4 \\ d & 4 & 0 & 4 & 0 \end{array}$$



Incident matrix $(m(G))$

$$\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 & 1 & 1 \\ v_4 & 0 & 0 & 0 & 0 & 1 \\ v_5 & 0 & 1 & 1 & 0 & 0 \end{array}$$

$\text{rank}(m(G)) = n-1$

→ Every column can have only 2 one's.
→ Every row can give degree of a vertex.

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MATHEMATICAL LOGIC

Connectives:
- Conjunction
- disjunction
- single implication
- double implication.

Modifier:
- not/negation

Conjunction ($\wedge$) (AND/BUT).

S_1	S_2	C
T	T	T
T	F	F
F	T	F
F	F	F

$(F \wedge __) = F$

$(F \wedge __ \wedge __) = F$

OR: → Inclusive OR ($\vee$)
→ Exclusive OR ($\oplus$)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$(T \vee __) = T$

$(T \vee __ \vee __) = T$

Conditional Statement ($\rightarrow$)

$p \rightarrow q$

if p then q
if p, q
*q if p
q when p

q whenever p
*q unless ¬p
p implies q
*p only if q

eg: if you win the matches then i will give pizza.
p → q

$F \rightarrow T = T$

win(T)	pizza(T)	T
win(T)	pizza(F)	F
~~win~~(F)	pizza(T)	T
~~win~~(F)	~~pizza~~(F)	T

$p \rightarrow q$

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→ if perfect matching exist then no. of vertices would be even.

Theorem.

P.M (T)	Even (T)	T	$T \to T \equiv T$
P.M (T)	~~Even~~ (F)	F	$T \to F \equiv F$
~~P.M~~ (F)	Even (T)	True	$F \to T \equiv T$
~~P.M.~~	~~Even.~~	True	$F \to F \equiv T$

$p \to q$ $\qquad$ $p \to q \neq q \to p$ $\qquad$ $\boxed{p \to q = \tau q \to \tau p}$

p.m → Even. $\qquad$ $p \to q \neq \tau p \to \tau q$

(Inverse)

Inverse	Converse	Contrapositive
$\tau p \to \tau p$	$q \to p$	$\tau q \to \tau p$
not (p.m) → not (Even)	Even → p.m.	not (even) → not (p.m)

→ if G is planar then $e \leq 3n-6$

p $\qquad$ q.

$p \to q = \tau q \to \tau p$

not $(e \leq 3n-6)$ then G is not planar

if $e > 3n-6$ then G is non planar.

D			A			B	C
a	p	p	q	$p \to q$	τp	τq	$\tau q \to \tau p$	$\tau p \to \tau q$
T		T	T	T	F	F	T	T
T		T	F	F	F	T	F	T
F		F	T	T	T	F	T	F
T		F	F	T	T	T	T	T
				A			B	

* Expression A is logically equivalent to expression B.

$A \equiv B$ $\qquad$ or. 2. A, B are having same behaviour.

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Double Implication ($\leftrightarrow$)

p	q	$p \leftrightarrow q$
T	T	T
F	F	T
T	F	F
F	T	F

* E.G $\leftrightarrow$ degree of all vertices are even.

$A \equiv B$

1. Same column
2. Same behaviour

tautology: (1=2=3)

A	B	$A \leftrightarrow B$
T	T	T
F	F	T
T	T	T
F	F	T

1. $A \leftrightarrow B$ tautology
   $A \equiv B$
2. A, B are having same column
3. They are having same behaviour.

tautology/valid.

T
T
T
T

- all valid are satisfiable.
- all contigency are satisfiable.

Contradiction

F
F
F
F

Satisfiable

T

→ atleast 1 true.

Contigency

not tautology
not contradiction.

T
F

atleast.

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$$\overbrace{(\overset{T}{p} \wedge \overbrace{(p \to q)}^{T} \wedge \overbrace{(q \to m)}^{T}) \to \overset{F}{m}}^{T}$$

$T \to F = F$

$(T \to T) \wedge \underbrace{(T \to F)}_{F}$

$m = F$
$p = T$
$q = T$

False

True.

$$\overbrace{[\overset{T}{a} \wedge \overbrace{(a \to b)}^{T} \wedge \overbrace{(\neg b \vee c)}^{T} \wedge \overbrace{(c \to d)}^{T}] \to \overset{F}{d}}^{T}$$

(tautology)

$(T \to T)$ $(F \vee T)$ $\underbrace{(T \to F)}_{F}$

$d = F$
$q = T$
$b = T$
$c = T.$

False $\longrightarrow$

True

Type-2 (logical equivalence / $\equiv$)

$p \wedge p \equiv p$	$p \wedge T \equiv p$	$p \vee T \equiv F$
$p \vee p \equiv p$	$p \vee F \equiv p$	$p \wedge F = F$

$p \vee q = q \vee p$	$p \wedge (q \wedge R) = (p \wedge q) \wedge R$
$p \wedge q = q \wedge p$	$p \vee (q \vee R) = (p \vee q) \vee R$

logic	Set	Digital
$\wedge$	$\cap$	$\cdot$
$\vee$	$\cup$	$+$
T	U	1
F	$\emptyset$	0

$\to p \wedge (q \vee R) = (p \wedge q) \vee (p \wedge R)$

$p \wedge (q \vee R)$

$\to p \vee (q \wedge R) = (p \vee q) \wedge (p \vee R)$

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Absorption:

$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$
$\underline{a = T}$	$\underline{a = T}$
$T \vee (T \wedge _) = T$	$T \wedge (T \vee _) = T$
$\underline{a = F}$	$\underline{a = F}$
$F \vee (F \wedge _) = F$	$F \wedge (F \vee _) = F$

$$\boxed{\begin{aligned} p \to q &= \neg p \vee q \\ p \to q &= \neg q \to \neg p \end{aligned}}$$

Godi Rule:

$$\boxed{\begin{aligned} \neg(a \vee b) &= \neg a \wedge \neg b \\ \neg(a \wedge b) &\ \ \neg a \vee \neg b. \end{aligned}}$$

$(p \to q) \wedge (p \ \ R) = p \to (q \wedge R)$

$(\boxed{\neg p \vee} q) \wedge (\boxed{\neg p \vee} R)$

$\neg p \vee (q \wedge R)$

$p \to (q \wedge R)$

$(p \to R) \wedge (Q \to R) = (p \vee Q) \to R$

$(\neg p \boxed{\vee R}) \wedge (\neg Q \boxed{\vee R})$

$(\neg p \wedge \neg Q) \vee R$

$\neg(p \vee Q) \vee R$ Demorgan's Rule

$\hookrightarrow (p \vee Q) \to R$

$$\begin{cases} (p \to q) \wedge (p \to R) = p \to (q \wedge R) \\ (p \to R) \wedge (Q \to R) = (p \vee Q) \to R \end{cases}$$

$$\begin{cases} (p \to Q) \vee (p \to R) = p \to (Q \vee R) \\ (p \to R) \vee (Q \to R) = (p \wedge Q) \to R \end{cases}$$

$(p \to q) \wedge (\neg q \vee p)$

$p \leftrightarrow Q = (p \to q) \wedge (q \to p)$

$= (\neg p \vee q) \wedge (q \to p)$

$(\neg p \vee q) \wedge (\neg q \vee p)$

$(p \wedge q) \vee (\neg p \wedge \neg q)$

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$(\sim p \vee q) \wedge (\sim q \vee p)$

$A \wedge (\sim q \vee p)$

$(A \wedge \sim q) \vee (A \wedge p)$

$(\sim q \wedge A) \vee (p \wedge A)$

$(\sim q \wedge (\sim p \vee q)) \vee (p \wedge (\sim p \vee q))$

$(\sim q \wedge \sim p) \vee (p \wedge q)$

x: $\sim q \wedge (\sim p \vee q)$

$(\sim q \wedge \sim p) \vee (\sim q \wedge q)$

$(\sim q \wedge \sim p) \vee F$

$= \sim q \wedge \sim p.$

y: $p \wedge (\sim p \vee q)$

$(p \wedge \sim p) \vee (p \wedge q)$

$F \vee (p \wedge q) = p \wedge q.$

Q1. The simplest form of $(p \wedge (\sim r \vee q \vee \sim q)) \vee ((r \vee t \vee \sim r) \wedge \sim q)$ is

Ans

$p \wedge (\sim p \vee T)$

$p \wedge T$

$\downarrow$

$p.$

$\vee$

$(r \vee \sim p) \vee t) \wedge \sim q$

$(T \vee t) \wedge \sim q$

$\downarrow$

$T \wedge \sim q = \sim q$

$\Rightarrow \underline{\underline{p \vee \sim q}}$

Q2 The simplest form of $(p \vee (p \wedge q) \vee (p \wedge q \wedge \sim r)) \wedge ((p \wedge r \wedge t) \vee t)$

$p \vee (p \wedge q \wedge \sim p)$

$p \wedge t.$

$\Rightarrow \underline{\underline{p \wedge t}}$

Q3. P and Q are two propositions. Which of the following logical expressions are equivalent?

I. $P \vee \sim Q$

II. $\sim(\sim P \wedge Q)$

III. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

IV. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q).$

Ans $\Rightarrow$ I = II

I. $P \vee \sim Q$

II $\sim(\sim P \wedge Q) = P \vee \sim Q$

$(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

$P \wedge (Q \vee \sim Q) \vee (\sim P \wedge \sim Q)$

$(P \wedge T) \vee$

$P \vee (\sim P \wedge \sim Q)$

$P \vee (\sim P \wedge \sim Q)$

$(P \vee \sim P) \wedge$

$(P \vee \sim Q)$

$\sim P \vee (P \vee \sim Q)$

$(P \vee \sim Q)$

Q4. Which of the following is not equivalent to $P \leftrightarrow q$?

Ans. $(\sim p \wedge q) \vee (p \wedge \sim q).$

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Inference Rule (Types)

Premises $\longrightarrow$ Conclusion (Or) Premises
True Check $\therefore$ Conclusion

* take Premises as True, Check the Conclusion $\rightarrow$ True $\rightarrow$ V.R
$\rightarrow$ False $\rightarrow$ ~~V.R~~

eg: Premises — Conclusion or Premises
$\therefore$ Conclusion

eg2: $\boxed{\overset{\text{True}}{P_1 \wedge P_2 \wedge \dots P_n}} \longrightarrow \underset{\text{Conclusion}}{\text{Check}}$

P_1: mobile phone is in left or right pocket (True)
P_2: it is not in left pocket (True)
Q: it can be in right pocket (True)

$P_1: L \vee R$ $\quad [\overset{T.P_2}{(L \vee R)} \wedge \overset{T.P_2}{\neg L}] \longrightarrow R$
$P_2: \neg L$
Q: R

L	R	$L \vee R$	$\neg L$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

$((P \vee Q) \wedge \neg P) \longrightarrow Q$ $\quad \begin{array}{l} P \vee Q \\ \neg P \\ \hline \therefore Q. \end{array}$

$((P \vee Q) \wedge \neg Q) \longrightarrow P$

$P \vee Q$	$Q \vee P$
$\neg Q$	$\neg Q$
$\therefore P.$	$\therefore P$

P_1: if perfect matching exist then no. of vertices can be even (True)
P_2: Perfect matching exist (True)
Q: no. of vertices can be even (True).

$[\underbrace{(P \rightarrow Q) \wedge P}_{\text{True}}] \longrightarrow \underbrace{Q.}_{\text{Check.}}$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

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True Conc.

$[(P \to Q) \wedge P] \to Q$

P	Q	P Q	$(P\ Q) \wedge P$	$((P \wedge Q) \wedge P) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

P_1: if p.m exist then no. of wickets will be earn

P_2: no of wickets earn.

Q: p.m will exist.

$\overset{T}{((\overline{P \to Q) \wedge Q})} \to P$

$P \to Q$
Q
$\therefore P$

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

$P \vee Q$
$\neg P$
$\therefore Q$

OR

$P \vee Q$
$\neg Q$
$\therefore P$

OR

$\neg P \vee Q$
P
$\therefore Q$

$\equiv$

$P \to Q$
P
Q

$P \to Q$
Q
$\therefore P$ (fallacy)

P_1: if it raine then Cricket match will not be played.

P_2: Cricket match was Played.

$\therefore$ There was no rain

$R \to \neg P$
P
$\neg R$

$\equiv$

$P \to \neg R$
P
$\neg R$ (valid)

P_1: if it raine Cricket match will not be played.

P_2: it did not rain

$\therefore$ Cricket match was Played

$R \to \neg P$
$\neg R$
P

$\equiv$ ⊗

$P \to \neg R$
$\neg R$
P

1. $P \to Q$ or $((p \to Q) \wedge p) \to Q$
$\underline{P}$ (Modus ponens)
$\therefore Q$

Fallacy:
$P \quad Q$
$Q \quad \times$
$\therefore P$

2. $p \quad Q = \neg Q \to \neg P \quad ((p \to Q) \wedge \neg Q) \to \neg p$
$\underline{\neg Q}$ $\quad \underline{\neg Q}$ (modus tollens)
$\therefore \neg p \quad \neg P$

3. $P \to Q \quad ((p \to Q) \wedge (Q \to R)) \to (p \to R)$
$\underline{Q \to R}$ (Hypothetical Syllogism).
$P \to R$

4. $P \vee Q \quad ((p \vee Q) \wedge \neg p) \to Q$
$\underline{\neg P}$ (Disjunctive Syllogism).
$\therefore Q$

5. $\underline{P \wedge Q}$ $\quad P \wedge Q \to P$
P $\quad$ or
$P \wedge Q \to Q$.
(Simplification)

6. $\underline{P}$ $\quad P \to P \vee Q$
$\therefore P \vee Q$ (Addition).

7. $P \vee Q \quad ((p \vee Q) \wedge (\neg p \vee R))$
$\underline{\neg p \vee R}$ $\quad \to Q \vee R$
$Q \vee R$ (Resolution).

$(a \wedge (a \quad b) \wedge (\neg b \vee c) \to c$

1: a
2: $\underline{a \to b}$ (modus ponens)
b
3: $\underline{\neg b \vee c}$
c.

$(a \wedge (a \to b) \wedge (\neg b \vee c) \wedge (c \to x)) \to x$

1: a
2: $\underline{a \to b}$ (M.P)
b
3: $\underline{\neg b \vee c}$ (D.S)
c
4: $\underline{c \to x}$ (M.P)
x

Q1. The statement formula $\{(a \to c) \wedge (b \to d) \wedge (c \to \neg d)\} \to (\neg a \vee \neg b)$ is

$a \to c = \neg a \vee c$
$b \to d = \neg b \vee d$
$\underline{c \to \neg d} = \neg c \vee \neg d$
$\neg a \vee \neg b$

$\neg a \vee c$
$\underline{\neg b \vee \neg c}$
$\neg a \vee \neg b$

Q2 Establish the validity of the following arguments

(a) $[p \wedge (p \to q) \wedge (\neg q \vee r)] \to r$

(b) $p \to q \equiv \neg p \vee q$
$\neg p$
$\neg r$
$\therefore \neg(p \vee r)$

$\underline{\neg q}$ $\quad \neg p \wedge \neg r$
$\neg p$ $\quad \neg(p \vee r)$
$\wedge$
$\neg r$

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$

(c) $p \to q$ $\quad = \neg q \to \neg p$

$r \to \neg q$ $\quad \wedge q$

r $\quad \therefore r$

$\therefore \neg p$

(d) $p \wedge q$

$p \to (r \wedge q)$

$r \to (s \vee t)$

$\neg s$

$\therefore t.$

$p \wedge q$

$\downarrow$

p

$p \to (r \wedge Q)$

$r \wedge q \to r$

$r \to (s \vee t)$

$s \vee t$

$\neg s$

t

(e) 1. $p \to (q \to r)$

2. $p \vee s$

3. $t \to q$

4. $\neg s$

$\therefore \neg r \to \neg t$

2. $p \vee s$

4. $\neg s$

p

1. $p \to (Q \to R)$

$q \to R$

$t \to q$

$t \to R \equiv \neg R \to \neg t.$

Q3. $S_1: \{(\sim p \to (q \to \sim w)) \wedge (\sim s \to q) \wedge \sim t \wedge (\sim p \vee t)\} \to (w \to s)$

$S_2: \{(q \to t) \wedge (s \to r) \wedge (\sim q \to s)\} \to (\sim t \to r)$

S_2: $q \to t \equiv \neg q \vee t$

$s \to R \equiv \neg S \vee R$

$\neg q \to s \equiv q \vee s$

$\neg t \to R$

$\neg q \vee t$ $\quad t \vee R$

$q \vee R$

Predicate Logic:

$P(x)$: x is even number (open statement)

$P(0)$: 0 is even number (True) $\to$ Simple

$P(1)$: 1 is even number (False) $\to$ Simple.

$\hookrightarrow$ predicate variable.

Open Statement: It is a type of statement where we cannot directly define truth value. Once we provide input, it changes to simple statement.

Domain / Domain of discourse / Universe : Possible choices for open statement.

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Domain

I/P: $x=0$ → Open Smt → Simple Statement 1

$x=1$ → x is even no. → Simple Statement 2

Quantity → Truth value.

All (universal quantifier) ($\forall x$)

↳ Tool used to find out the truth value in terms of quantity (All)

$\forall x P(x)$

for all values of x, such that it satisfies $p(x)$

for all of x, such that $p(x)$

for every value of x, such that $p(x)$

T. T. T. T. T. → $\forall x(p(x))$ → True

F. atleast 1 el.n → $x\ p(x)$ ↳ False.

eg:

$D: \{1, 2, 3\}$

$p(x): x^2 \le 9$

$\forall x\ p(x)$

or.

$\forall x(x^2 \le 9)$

↳ True.

$D: \{1, 2, 3, 4\}$

$p(x): x^2 < 9$

1. 2. 3. T. T. T. T. F. → $\forall x\ p(x)$ ↳ False

$D: \mathbb{Z}$

$p(x): x^2 \ge 0$

$\forall x(x^2 \ge 0)$

↳ True.

T. T. T. T. ... → $\forall x\ P(x) \rightarrow$ True

- When it's true for all element in domain.

T. T. T. F. → $\forall x\ p(x) \rightarrow$ False.

- atleast 1 element is false.

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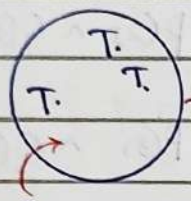
Existential Quantifier ($\exists x$) : Tool which is used to find out truth in terms of quantity.

$\exists x\, p(x)$

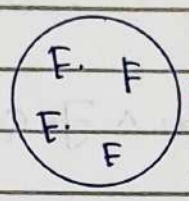
- there exist x, which satisfies $p(x)$
- there exist x, such that $p(x)$
  - Some of x, such that $p(x)$
  - atleast 1 value of x, such that $p(x)$.



True.

at least one value of x.

$\exists x\, p(x) \rightarrow$ True.



False.

all are false.

$\exists x\, p(x) \rightarrow$ False.

eg: $D: \{1, 2, 3, \ldots 10\}$

$p(x) = x^2 = 4$.

$\exists x\, p(x)$

or

$\exists x\, (x^2 = 4)$

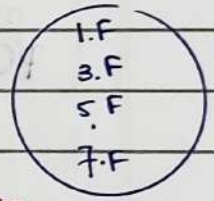
True

$D: \{1, 3, 5, 7\}$

$p(x): x^2 = 4$

$\exists x\, (x^2 = 4)$

False.



$D: \mathbb{Z}$	$D: \mathbb{Z}$
$p(x): x^2 \geq 0$	$p(x): x^2 \geq 0$
$\forall x\, p(x)$	$\exists x\, p(x)$
$\forall x\, (x^2 \geq 0)$	$\exists x\, (x^2 \geq 0)$
True.	

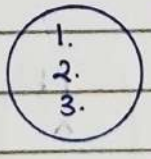
$\forall x\, (x^2 \geq 0) \rightarrow \exists x\, (x^2 \geq 0)$

True True.

$$\boxed{\underset{\text{True}}{\forall x\, p(x)} \rightarrow \underset{\text{True}}{\exists x\, p(x)}}$$

Negation of Quantifier

$$\boxed{\begin{aligned} \neg \forall x\, p(x) &= \exists x\, \neg p(x) \\ \neg \exists x\, p(x) &= \forall x\, \neg p(x) \end{aligned}}$$



$D: \{1, 2, 3\}$

$\neg \forall x\, p(x) = \neg \{p(1) \wedge p(2) \wedge p(3)\}$

$= \neg p(1) \vee \neg p(2) \vee \neg p(3)$

$= \exists x\, \neg p(x)$.

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$D: \{1, 2, 3\}$

$P(x): x+1 = 4$ $P(3)$

$x = 3$

$x=1 \quad 1+1=4$ (F)

$x=2 \quad 2+1=4$ (F)

$x=3 \quad 3+1=4$ (T)

$D: \{1, 2, 3\}$

$Q(x): 2x+1=7 \rightarrow x=3$

$x=1 \quad 2(1)+1=7$ (F)

$x=2 \quad 2(2)+1=7$ (F)

$x=3 \quad 2(3)+1=7$ (T)

$\exists x[P(x) \wedge Q(x)]$

$D: \{1, 2, 3\}$

$\exists x[P(x) \wedge Q(x)] \longrightarrow \exists x P(x) \wedge \exists x Q(x)$

$\overset{F}{P(1)} \wedge \overset{F}{Q(1)}$ F

$\vee$

$\overset{F}{P(2)} \wedge \overset{F}{Q(2)}$ F

$\vee$

$\overset{T}{P(3)} \wedge \overset{T}{Q(3)}$ T.

True

$\overset{F}{P(1)} \wedge \overset{F}{Q(1)}$ F

$\vee$

$\overset{F}{P(2)} \wedge \overset{F}{Q(2)}$ F

$\vee$

$\overset{T}{P(3)} \wedge \overset{T}{Q(3)}$ T

$P(1)$ F $\quad Q(1)$

$\vee \qquad \vee$

$P(2)$ F $\wedge$ $Q(2)$

$\vee \qquad \vee$

$P(3)$ T $\quad Q(3)$

True $\wedge$ True.

True.

$\exists x P(x) \wedge \exists x Q(x) \longrightarrow \exists x [P(x) \wedge Q(x)]$

$\boxed{\begin{matrix} T \\ \vee \\ F \end{matrix}} \wedge \boxed{\begin{matrix} F \\ \vee \\ T \end{matrix}}$

True $\quad$ True.

$\boxed{T \wedge F}$ F

$\vee \qquad \vee$

$\boxed{F \wedge T}$ F

False.

eg. $\forall x[P(x) \wedge Q(x)] = \forall x P(x) \wedge \forall x Q(x)$ $\quad D: \{1, 2\}$

$\boxed{P(1) \wedge Q(1)}$

$\wedge$

$\boxed{P(2) \wedge Q(2)}$

$\begin{pmatrix} P_1 \\ \wedge \\ P_2 \end{pmatrix} \begin{pmatrix} P_2 \\ \wedge \\ Q_2 \end{pmatrix}$

1. $\forall x[P(x) \wedge Q(x)] = \forall x(P(x)) \wedge \forall x\, Q(x)$
2. $\forall x[P(x) \vee Q(x)] = \forall x(P(x)) \vee \forall x\, Q(x)$
3. $\exists x[P(x) \vee Q(x)] = \exists x\, P(x) \vee \exists x\, Q(x)$
4. $\exists x[P(x) \wedge Q(x)] = \exists x\, P(x) \wedge \exists x\, Q(x)$
5. $\forall x[P(x) \rightarrow Q(x)] \rightarrow (\forall x\, P(x) \rightarrow \forall x\, Q(x))$
6. $\forall x[P(x) \leftrightarrow Q(x)] \rightarrow (\forall x\, P(x) \leftrightarrow \forall x\, Q(x))$.

Case 1 : $\overbrace{\forall x[P(x) \vee Q(x)]}^{\text{True}} \rightarrow \overbrace{\forall x\, P(x) \vee \forall x\, Q(x)}^{\text{False}}$

$$T \left\{ \begin{array}{l} \boxed{T \vee F}\ T \\ \quad \wedge \\ \boxed{F \vee T}\ T \end{array} \right. \qquad \begin{array}{c} \boxed{T \wedge F} \vee \boxed{F \wedge T} \\ \underbrace{F \vee F}_{F} \end{array}$$

$$\overbrace{\forall x\, P(x) \vee \forall x\, Q(x)}^{\text{True}} \rightarrow \overbrace{\forall x[P(x) \vee Q(x)]}^{\text{True}}$$

$$\begin{array}{c} \boxed{T \wedge T} \vee \boxed{\wedge} \\ \underbrace{T \vee F}_{T} \end{array} \qquad \begin{array}{l} \boxed{T \vee \quad}\ T \\ \quad \wedge \\ \boxed{T \vee \quad}\ T \end{array}$$

Q1. Let $p(x)$, $q(x)$ denote the following open statements:

$p(x): x \leq 3 \qquad q(x): x+1$ is odd

If the universe consist of all integers, what are the truth values of the following statements?

Ans. $\neg p(3) \rightarrow (T) = F$

$p(x): x \leq 3$

$p(3): 3 \leq 3$ (True). $\Rightarrow p(3) \wedge q(4)$

Q2. $p(x): x^2 - 7x + 10 = 0 \qquad r(x): x < 0$

$q(x): x^2 - 2x - 3 = 0$

Ans: $\forall x\,[P(x) \to \neg R(x)]$

$x=1$ $\boxed{\underset{F}{P(1)} \to \neg R(1)}$ T

$x=2$ $\boxed{\underset{T}{P(2)} \to \underset{T}{\neg R(2)}}$ T

$x=3$ $\boxed{P(3) \to \neg R(3)}$ T

Q3. $p(x): x^2 - 7x + 10 = 0$

$q(x): x^2 - 2x - 3 = 0$

$r(x): x < 0$

$\forall x\,(p(x) \to \neg R(x))$

$x=1$	$P(1) \to \neg R(1)$	T
$x=2$	$P(2) \to \neg R(2)$	T
$x=3$	$P(3) \to \neg R(3)$	T
$x=4$	$P(4) \to \neg R(4)$	T
$x=5$	$P(5) \to \neg R(5)$	T.

H.W

Q4. $p(x): x^2 - 8x + 15 = 0$

$q(x): x$ is odd

$r(x): x > 0$

Negation of Quantifier

$\neg \forall x\, p(x) = \exists x\, \neg p(x)$

$\neg \exists x\, p(x) = \forall x\, \neg p(x)$

Expression: $\forall x\,[p(x) \to Q(x)]$

negate it:

$\neg \forall x\,[P(x) \to Q(x)]$

$\neg \forall x\,[\neg p(x) \vee Q(x)]$

$\exists x\, \neg[\neg p(x) \vee Q(x)]$

$\exists x\,[\neg(\neg p(x)) \wedge \neg Q(x)]$

$\exists x\,[p(x) \wedge \neg Q(x)]$

Q5. Negate and simplify each of the following:

(a) $\exists x\,[p(x) \vee q(x)]$

$\neg \exists x\,[p(x) \vee q(x)]$

$\forall x\, \neg[p(x) \vee q(x)] \Rightarrow \forall x\,[\neg p(x) \wedge \neg Q(x)]$

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(b) $\exists x\,[p(x) \rightarrow q(x)]$

or

$\exists x\,[\neg p(x) \vee q(x)]$

(d) $\exists x\,[(p(x) \vee q(x)) \rightarrow r(x)]$

$\neg \exists x\,[p(x) \vee q(x) \rightarrow r(x)]$

$\neg \exists x\,[\neg (p(x) \vee q(x)) \vee R(x)]$

$\forall x\,[(p(x) \vee q(x) \wedge \neg R(x)]$

English Statement $\longrightarrow$ Logical Statement

eg: All mothers are female.

All of x, x is mother x is female.

$\forall x$ $\quad MT(x)$ $\quad F(x)$

$\forall x\,[MT(x) \quad F(x)]$ $\vec{\wedge}$

Case 1: $\forall x\,[MT(x) \wedge F(x)]$ $\qquad$ Case 2: $\forall x\,[MT(x) \rightarrow F(x)]$

False
- $\boxed{\text{False} \wedge}$ F
- $\wedge$
- $\boxed{F \wedge T}$ F
- $\wedge$
- $\boxed{T \wedge T}$ F

T
- $\boxed{F \rightarrow}$ T
- $\wedge$
- $\boxed{F \rightarrow}$ T
- $\wedge$
- $\boxed{F \rightarrow T}$ T

eg. Some cats are black

Some of x, x is cat, x is black

$\exists x$ $\quad C(x)$ $\quad bl(x)$

1. $\exists x\,[c(x) \rightarrow bl(x)]$ $\qquad$ $\exists x\,[c(x) \wedge bl(x)]$ $\qquad$ $\exists x\,[c(x) \leftrightarrow bl(x)]$

T
- $\boxed{T \rightarrow F}$ F
- $\vee$
- $\boxed{T \rightarrow T}$ T
- $\vee$
- $\boxed{F \rightarrow}$ T

T
- $\boxed{T \wedge F}$ F
- $\boxed{T \wedge T}$ T
- $\boxed{\wedge}$

T
- $\boxed{T \leftrightarrow F}$ F
- $\vee$
- $\boxed{T \leftrightarrow T}$ T
- $\vee$

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eg:1 1. All of my friends are perfect.
$\forall x [F(x) \rightarrow P(x)]$

2. Not all of my friends are perfect.
$\neg \forall x [F(x) \rightarrow P(x)]$
$\neg \forall x [\neg F(x) \vee P(x)]$
$\exists x [F(x) \wedge \neg P(x)]$

3. None of my friends are perfect. $\equiv$ 4. All of my friends are not perfect.
$\forall x [F(x) \rightarrow \neg P(x)]$ $\forall x [F(x) \rightarrow \neg P(x)]$

eg:2 All graphs are connected
$\forall x [G(x) \rightarrow C(x)]$

Not all graphs are connected
$\neg [\forall x [G(x) \rightarrow C(x)]]$

No graphs are connected. $\equiv$ All graphs are not connected.
$\forall x [G(x) \rightarrow \neg C(x)]$ $\forall x [G(x) \rightarrow \neg C(x)]$

Note:
1. not all $\Rightarrow \neg \forall x [\ \rightarrow\ ]$
2. no/none $\Rightarrow \forall x [\ \rightarrow\ ]$

$D: \{1, 2, 3\}$

$P(x, y): x \times y \leq 9$

$\forall x \forall y (x \times y \leq 9)$
for all values of x, all values of y
for every value of x, every value of y

for all values of x, all values of y
for $x = 1$ to 3 $x=1$, $y=1$ 2, 3
for $y = 1$ to 3 $x=2$, $y=1, 2, 3$
$x \times y \leq 9$ $x=3$ $y=1, 2, 3$.

$D: \{1, 2, 3\}$
$x=1$ $y=1$
$(x \times y \leq 9)$ $1 \times 1 \leq 9$ (T)
$x=1$ $y=2$ $1 \times 2 \leq 9$ (T)
$x=1$ $y=3$ $1 \times 3 \leq 9$ (T)
$x=2$ $y=1$ $2 \times 1 \leq 9$ (T)

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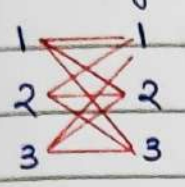
$x \times y \le 9$ $\quad x=1 \quad y=1 \quad 1\times 1 \le 9 \ (T)$



$\forall x \forall y \rightarrow$ True

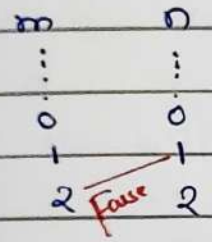
when all edges are True

$\forall x \forall y \rightarrow$ False

atleast 1 edge is false.

D: 2

$\forall m \forall n \ (m.n = n)$



$m=2, n=1$

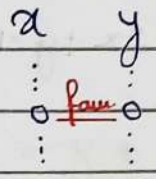
$m.n = n$

$2.1 \neq 1.$

D: 2

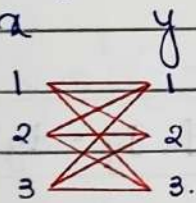
$\forall x \forall y \left(\frac{x}{y} \in Z\right)$

↳ false



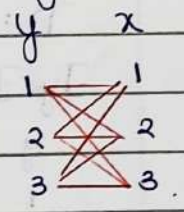
D: {1, 2, 3}

$\forall x \forall y \ (x \times y \le 9)$



D: {1, 2, 3}

$\forall y \forall x \ (x \times y \le 9)$



$\boxed{\forall x \forall y = \forall y \forall x}$

$\forall x \forall y \longleftrightarrow \forall y \forall x$

D: {1, 2, 3}

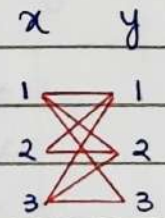
$\exists x \exists y \ (x \times y = 4)$

there exist x, there exist y

atleast 1 value of x, atleast 1 value of y.

for some value of x, some value of y.

$\exists x \exists y \ (x \times y = 4)$



$x=1 \ y=1$

$1 \times 1 \neq 4$

$x=1 \ y=2$

$1 \times 2 \neq 4$

$x=1 \ y=3$

$1 \times 3 \neq 4.$

→ $\exists x \exists y \rightarrow$ True

* atleast 1 edge is True

→ $\exists x \exists y \rightarrow$ False

* when all edges are false.

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D: Z

$\exists x \exists y \left(\frac{x}{y} \in Z\right)$

x	y
⋮	⋮
2 — True	2
3 — True	3
⋮	⋮

D: Z

$\exists x \exists y (x + y = 10)$

x	y
⋮	⋮
1 — True	9
2 — True	8
⋮	⋮

$x = 1 \quad y = 9$

$1 + 9 = 10$ (T)

$x = 2 \quad y = 8$

$2 + 8 = 10$ (T)

D: Z

$\exists x \exists y (x + y = 10 \wedge 2x + y = 15)$

x	y
⋮	⋮
5 — True	5

$$\begin{array}{r} x + y = 10 \\ 2x + y = 15 \\ \hline -x = -5 \end{array}$$

$x = 5$

$y = 5$

D: {1, 2, 3}

$\exists x \exists y (x \times y = 4)$

x	y
1	1
2	2
3	3

D: {1, 2, 3}

$\exists y \exists x (x \times y = 4)$

y	x
1	1
2 —	2
3	3

$\boxed{\exists x \exists y = \exists y \exists x}$

$\forall x \forall y = \forall y \forall x \qquad \exists x \exists y = \exists y \exists x$

$$\boxed{\begin{array}{l} \forall x \forall y \longrightarrow \exists x \exists y \\ \forall y \forall x \longrightarrow \exists x \exists y \\ \forall y \forall x \longrightarrow \exists y \exists x \\ \forall x \forall y \longrightarrow \exists y \exists x \end{array}}$$

Domain is fixed.

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D: Z (True)

$\forall x \exists y (x+y=10)$

for all value of x, there exist y

for every value of x, there exist y

x	y
1	9
2	8
3	7
⋮	⋮

$x+y=10$

$x=1$	$x=2$
$1+y=10$	$y=8$
$y=9$	

D: Z

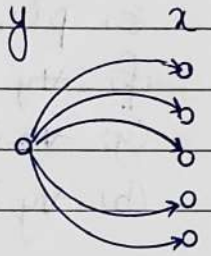
$\forall x \exists y (x+y=10) \not\rightarrow \exists y \forall x (x+y=10)$ (false)

D: Z

there exist y, all of x

Forcing ~~in~~ y to be Constant.

x	y
1	9
2	8
3	7
⋮	⋮

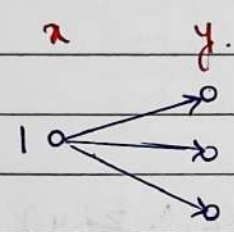


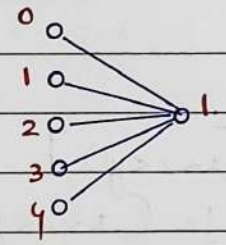
$x+y=10$

$y=10-x$

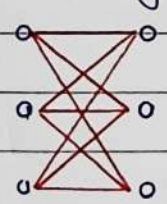
$y=10-$ ___

$\exists y \overset{T}{\forall x} \left(\frac{x}{y} \in Z\right) \longrightarrow \forall x \exists y \left(\frac{x}{y} \in Z\right)$

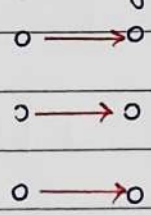




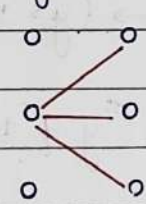
1. $\forall x \forall y$



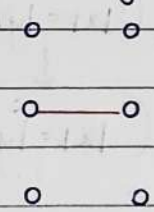
2. $\forall x \exists y$



3. $\exists y \forall x$



4. $\exists x \exists y$



O.S find

Domain: find.

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Date 4/7/22

1. $\forall x \forall y \equiv \forall y \forall x$ 5.

2. $\exists y \forall x \equiv \exists x \forall y$ 6.

3. $\forall x \exists y$ $\forall y \exists x$ 7.

4. $\exists x \exists y$ $\exists y \exists x$ 8.

1 → all

5 → all

all → 4, 8

2 → 3.

no relation between 2 & 6

Q1. Consider the open statement $p(x,y) : y-x = y+x^2$ where the universe for each of the variables x, y consists all integers. Determine the truth value for each of the following statements.

(a). $p(0,0)$ (T)

(b). $p(0,1)$ (T)

(c). $\exists y\, p(1,y)$ (F)

(d). $\exists y \forall x\, p(x,y)$ (F)

(e) $p(1,1)$ (F)

(f) $\forall y\, p(0,y)$ (T)

(g) $\forall x \exists y\, p(x,y)$ (F)

(h) $\forall y \exists x\, p(x,y)$ (T)

$p(x,y) : \cancel{y}-x = \cancel{y}+x^2$

$x^2+x=0$

$x(x+1)=0$

$\boxed{x=0 \quad x=-1}$

Q2. $\forall x \forall y\, [(x<y) \rightarrow \exists z\, (x<z<y)]$

$\neg \forall x \forall y\, [\neg(x<y) \vee \exists z\, (x<z<y)]$ $\neg(x<z \wedge z<y)$

$(x \geq z \vee z \geq y)$

$\exists x \exists y\, [\neg\neg(x<y) \wedge \neg \exists z\, (x<z<y)]$

$\forall z\, (x \geq z$ or $z \geq y)$

Q3. $\neg \forall x \forall y\, [(|x| = |y|) \rightarrow (y = \pm x)]$

$\exists x \exists y\, (|x| = |y| \wedge (y \neq \pm x)]$

Q4. $[\forall x, \alpha \to (\exists y, \beta \to (\forall u, \exists v, r))]$

Ans. $[\exists x, \alpha \wedge (\forall y, B \wedge (\exists u, \forall v, \neg r))]$

Q5. $p(x,y)$: x is divisor of y.

S_1: $\forall x \exists y\ p(x,y)$ (F)

S_2: $\forall y \exists x\ p(x,y)$ (T)

S_3: $\exists x \forall y\ p(x,y)$ (T)

S_4: $\exists y \forall x\ p(x,y)$ (T)

D: Z

$p(x,y)$: $\frac{y}{x}$

S_1: $\forall x \exists y \left(\frac{y}{x} \in Z\right)$.

all elements are ÷ by element called as 1.

$x \quad y$

$1 \to 2$

$2 \to 4$

Q6. D: 2

S_1: $\forall n \exists m\ (n+m=5)$ (T)

S_2: $\exists n \forall m\ (n.m=m)$ (T)

S_3: $\forall m \exists n\ (m.n=1)$ (F)

S_4: $\exists m \forall n\ (m+n=0)$ (F)

S_1: $\forall n \exists m (n+m=5)$ (True)

$n \quad m$

$1 \to 4$

$2 \to 3$

$4 \to 1$

$0 \to 0$

$0 \to 0$

S_2: $\exists n \forall m (n.m=m)$ (True)

$n \quad m$

$1 \to 2, 3, 4, 5$

S_4:

$\exists m \forall n\ (m+n=0)$ (False)

$m. \quad n.$

$0 \to 1, 2, 3, 4$

$m+n=0$

$n=2$

$m+2=0$

$m=-2$

$n=3$

$m=-3$

S_3: $\exists m \forall n (m.n=1)$

$2 \to \times$

(False)

$m=2$

$2.n=1$

$n=1/2$

arrow doesn't exist.

(True).

$\forall n \exists m\ (n^2<m)$

$n \quad m$

$1 \to 2$

$2 \to 5$

(True)

$\exists n \forall m\ (n<m^2)$

- there exist element which is less than square of any int.

$-1 \to 0, 1, 2, 3, 4$

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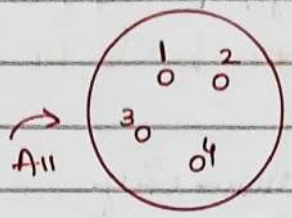
Quantifier with I.R:

Universal Specification:



$P(a)$

$a = 1, 2, 3, 4$

$\forall x P(x) \rightarrow P(a)$ (a is for all elements).

$p(x): x^2 \leq 9$

$\forall x P(x)$ $\quad D: \{1, 2, 3\}$

$\forall x (x^2 \leq 9) \rightarrow p(1) \; (1^2 \leq 9)$

$\rightarrow p(2) \; (2^2 \leq 9)$

$\rightarrow p(3) \; (3^2 \leq 9)$

$\forall x P(x) \rightarrow (p(a)$ a is for all).

Existential Specification:

$\exists x P(x) \rightarrow p(a)$

[a is specific]

$D: \{1, 2, 3\}$

$p(x) = x^2 = 4$

$\exists x P(x)$

$\exists x (x^2 = 4) \rightarrow p(1), p(2), p(3)$

Universal Generalization:	Existential Generalization:
$p(a) \rightarrow \forall x P(x)$	$P(a) \rightarrow \exists x P(x)$
a is for all.	a is fixed

eg: All students in class will go to IIT $\quad \forall x [C(x) \rightarrow IIT(x)]$

Some students in class are not solving problems $\quad \exists x [C(x) \wedge \neg Q(x)]$

Some students who are not solving questions will go to IIT. $\quad \exists x [\neg Q(x) \wedge IIT(x)]$

$$[\forall x [C(x) \rightarrow IIT(x)] \wedge \exists x [C(x) \wedge \neg Q(x)]] \rightarrow \exists x [\neg Q(x) \wedge IIT(x)]$$

1. $\forall x [C(x) \rightarrow IIT(x)]$ Premise (T)
2. $C(a) \rightarrow IIT(a)$ [U/ a is for all Students].
3. $\exists x [C(x) \wedge \neg Q(x)]$ Given (T) $(a = 5)$
4. $C(a) \wedge \neg Q(a)$ [a is fixed].

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5. $C(a)$ [4(A), $a=5$]
$(a=5)$
6. $IIT(a)$ [$a=5$]

4A. $C(a)$ [4, Simplification Rule]
$a=5$
4B. $\neg Q(a)$ [4, Simplification]
7. $\neg Q(a) \wedge IIT(a)$ [$a=5$]
8. $\exists x [\neg Q(x) \wedge IIT(x)]$

Q1. $(\forall x P(x) \wedge \forall x [P(x) \rightarrow Q(x)]) \rightarrow \forall x Q(x)$

1. $\forall x [P(x) \rightarrow Q(x)]$ [Given/T]
2. $P(a) \rightarrow Q(a)$ [U.S/ a is for all]
3. $\forall x P(x)$ [Given]
4. $P(a)$ [a is for all]
5. $P(a)$
6. $Q(a)$ [a is all/M.P]
7. $\forall x Q(x)$ [6, U.G].

Q2 $(\forall x [P(x) \rightarrow Q(x)] \wedge \exists x P(x)) \rightarrow \exists x Q(x)$

(1.) $\forall x (P(x) \rightarrow Q(x))$
(2.) $\exists x P(x)$
(1A.) $P(a) \rightarrow Q(a)$ [a = all/U.S].
(3.) $P(a)$ [a = Fixed]
(4.) $P(a)$ ([a is fixed]/3).
(4A) $Q(a)$ [a is fixed].
(4B) $\exists x Q(x)$ [Existance generalisation]

$\exists x [P(x) \wedge Q(x)] \rightarrow \exists x P(x) \wedge \exists x Q(x)$

1. $\exists x [P(x) \wedge Q(x)]$.
2. $P(a) \wedge Q(a)$ [a is fixed $a=3$].
3. $P(a)$ (2, Simpl)
4. $Q(a)$ (2, Simpl)

$\downarrow$ $\exists x P(x)$ [E.G]

$\downarrow$ $\exists x Q(x)$ [E.G]

$\exists x P(x) \wedge \exists x Q(x)$

$\exists x P(x) \wedge \exists x Q(x) \rightarrow \exists x [P(x) \wedge Q(x)]$

1. $\exists x P(x)$
2. $P(a)$ [a is fixed]
3. $\exists x Q(x)$
4. $Q(b)$ [b is fixed].

$\exists x\, P(x) \wedge \exists x\, Q(x) \not\rightarrow \exists x\, [P(x) \wedge Q(x)]$

$\boxed{T \wedge F}$ F

$\vee$

$\boxed{F \wedge T}$ F

$[\exists x\, [P(x) \rightarrow Q(x)] \wedge \exists x\, P(x)] \rightarrow \exists x\, Q(x)$

1. $\exists x\, [P(x) \rightarrow Q(x)]$
2. $P(a) \rightarrow Q(a)$ [a is fixed] $(a=5)$
3. $\exists x\, P(x)$
4. $P(a)$ [a is fixed] $(a=3)$

Q. $\forall x\, [p(x) \vee q(x)]$ $\longrightarrow$ $p(a) \vee q(a)$ [a = all]

$\exists x\, \neg p(x)$ $\longrightarrow$ $\neg p(a)$ [a = fixed]

$\forall x\, [\neg q(x) \vee r(x)]$ $\qquad$ $q(a)$ [a = fixed]

$\forall x\, [s(x) \rightarrow \neg r(x)]$

$\exists x\, \neg s(x)$.

$\neg q(a) \vee R(a)$ [a = all]

$R(a)$ [fixed]

$\neg s(a) \vee \neg R(a)$ [all]

$\neg s(a)$ (Fixed)

No mothers are male $\qquad$ $\forall x\, [mT(x) \rightarrow \neg mT(x)]$

Some male are politicians $\qquad$ $\exists x\, [mt(x) \wedge Pl(x)]$

Some politicians are not mother $\qquad$ $\exists x\, [Pl(x) \wedge \neg mT(x)]$.

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SET THEORY

Set : Collection of objects.

$A = \{1, 2, 3\}$ — Repetition is not allowed

— Order is not important

$1 \in A$

$4 \notin A$

element → Set

A:

eg: $\{\overset{e_1}{1}, \overset{e_2}{\{1\}}, \overset{e_3}{\{1, \{2\}\}}\}$

$\boxed{\text{element} \in \text{Set}}$

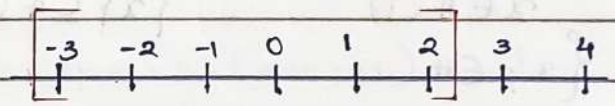
$1 \in A$ (T)

$\{1\} \in A$ (T)

Representation of Set:

1. Roster: $A = \{1, 2, 3\}$
2. Set builder: $\{x \mid \text{condition}\}$ eg: $\{x \mid 1 \le x \le 3\}$

Closed interval:



$[a, b] = \{x \mid a \le x \le b\}$

$[-3, 2] = \{-3, -2, -1, 0, 1, 2\}$

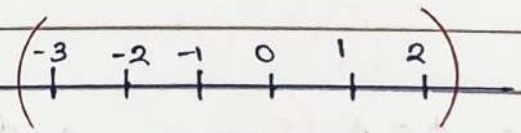
$A = \{1, 2, 3\}$

$A = \{x \mid 1 \le x \le 3\}$

$A[1, 3] = \{1, 2, 3\}$

Open interval:

$(-2, 3) = \{x \mid a < x < b\}$



$= \{-1, 0, 1, 2\}$.

Subset: ($\subseteq$)

$A \subseteq B$

$A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4\}$.

if $x \in A$ then $x \in B$

$A \subseteq B$ $\forall x (x \in A \rightarrow x \in B)$

$F \rightarrow$

T

Any Set:

$\phi \subseteq A$ ✓

$A \subseteq A$ ✓

$\forall x (x \in \phi \rightarrow x \in A)$

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Empty Set / null Set :

$A = \{\ \}$ $|A| = 0$ if cardinality is zero then empty set

$A\{\{\}\}$ $|A| = 1$

↳ not empty set.

Proper Subset ($\subset$)

$\forall x (x \in A \rightarrow x \in B \wedge \exists x (x \in B \wedge x \notin A)$

$A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4\}$

Subset:	Proper Subset
$A \subseteq B$	$A \subset B$
$\lvert A \rvert \leq \lvert B \rvert$	$\lvert A \rvert < \lvert B \rvert$.

eg. $A = \{ \overset{e1}{x}, \overset{e2}{\{x\}}, \overset{e3}{\{\{x\}\}} \}$

element $\in$ Set

Set $\subseteq$ Set

$x \in A$ (T) $\{x\} \subseteq A$ (T) $\{\text{element}\} \subseteq$ Set.

$\{x\} \in A$ (T)

Q.1 $A = \{1, \{1\}, \{2\}\}$

a. $1 \in A$ (T)
b. $\{1\} \in A$ (T)
c. $\{1\} \subset A$ (F)
d. $\{\{1\}\} \subseteq A$ (T)
e. $\{2\} \in A$ (T)
f. $\{2\} \subseteq A$ (F)
g. $\{\{2\}\} \subseteq A$ (T)
h. $\{\{2\}\} \subset A$ (T)

Except 'f' all are false.

Q2. a. $\phi \in \phi$ (False)
b. $\phi \subset \phi$ (False)
c. $\phi \subseteq \phi$ (True)
d. $\phi \in \{\phi\}$ (True)
e. $\phi \subset \{\phi\}$ (True)
f. $\phi \subseteq \{\phi\}$. (True)

$\phi \subset$ anything except ϕ

$\phi \subseteq$ anything.

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→ $A = \{1, 2, 3\}$ Powerset : $P(A) / 2^A$

ϕ $\{1,2\}$ ↳ Collection of Subset.

$\{1\}$ $\{2,3\}$

$\{2\}$ $\{3,1\}$ $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\} \{1,2,3\}\}$

$\{3\}$ $\{1,2,3\}$. Total no. of elements in $P(A) = 2^A$.

$\begin{cases} \phi \in P(A) \\ \phi \subseteq P(A) \end{cases}$

Operations in Set :

1. Set difference (−)
2. Symmetric difference (Δ / $\oplus$).

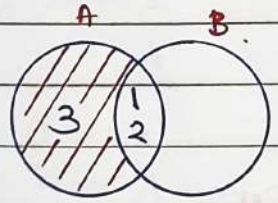
eg. $A = \{1, 2, 3\}$ $B = \{1, 2\}$

$A - B = \{3\}$. $B - A = \phi$.

Set ~~Symmetric~~ difference (−)

$A = \{1, 2, 3\}$ $B = \{1, 2\}$. $A - B = A - (A \cap B)$

$= \{3\}$

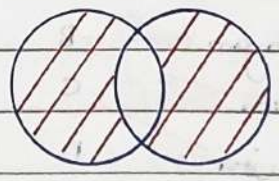


$B - A = \{\ \} = \phi$.

Symmetric difference (Δ / $\oplus$).

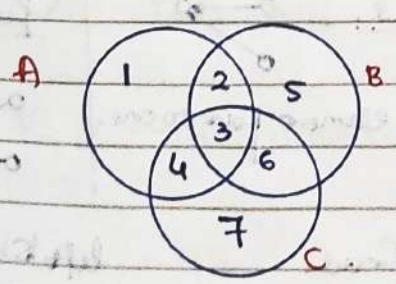
$A \Delta B = (A - B) \cup (B - A)$

$= (A \cup B) - (A \cap B)$



$A \Delta B = B \Delta A$

$A \Delta (B \Delta C) = (A \Delta B) \Delta C$.



$(A - B) - C = 1$.

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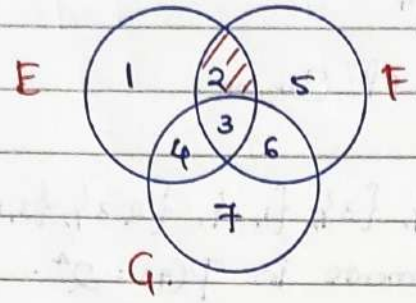


$x = (E \cap F) - (F \cap G)$

$y = (E - (E \cap G)) - (E - F)$

$\{E \cap F\} - (F \cap G) = \{2\}$. $x = y$

$E - (E \cap G) - (E - F) = \{2\}$

$A_n = [-2n, 3n]$

$n = 3$

$\bigcup_{i=1}^{7} A_i = A_7$

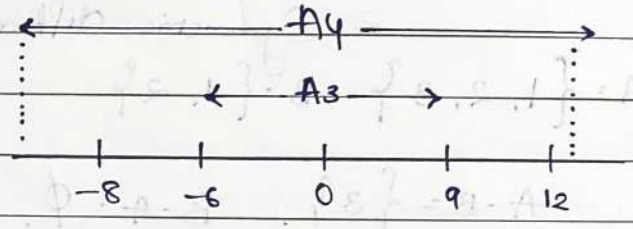
$\bigcap_{i=1}^{7} A_i = A_1$

$A_3 = [-6, 9]$ $A_4 = [-8, 12]$

$A_3 - A_4 = \phi$

$A_4 - A_3$

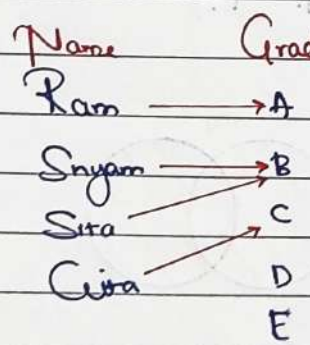
$= [-8, 6] \cup [9, 12]$.



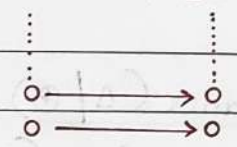
$\bigcup_{i=1}^{7} A_i = A_1 \cup A_2 \cup A_3 \ldots$ $A_7 = [-14, 21]$

$A_1 = [-2, 3]$.

Function / Assignment / Mapping



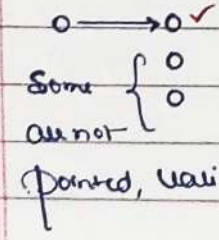
f: Set 1 $\longrightarrow$ Set 2



Set 1 is mapped to Set 2

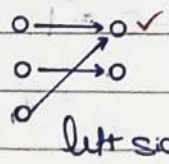
↳ Domain ↳ Co-domain.

Rule 1.



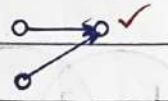
Some are not pointed, valid.

Rule 2.



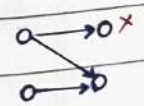
left side should always point to right side

Rule 3.

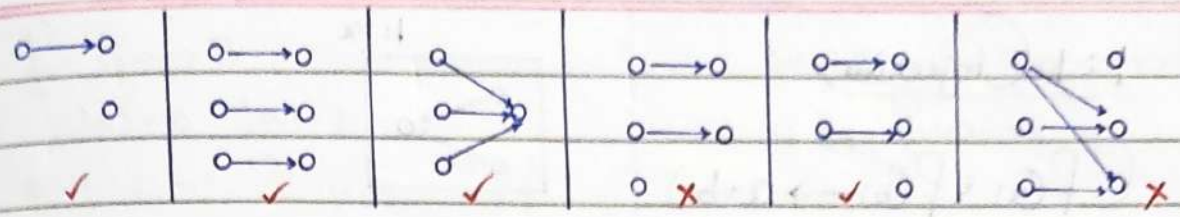


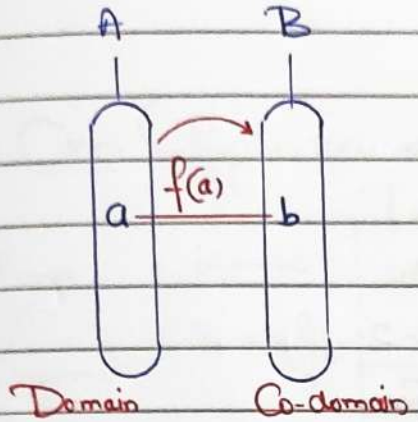
two element point to one.

Rule 4



left should not point multiple





$f: A \longrightarrow B$

$f(a) = b$

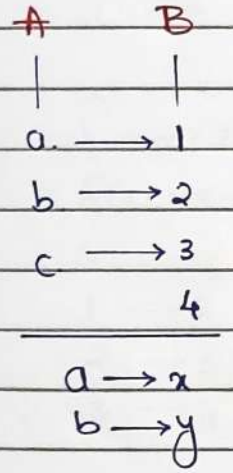
b is image of a

a is called Preimage of b.

Range = Set of image

$\boxed{\text{Range} \subseteq \text{Codomain}}$

eg.



$f: A \longrightarrow B$

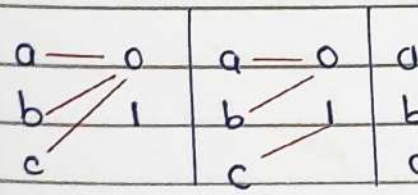
Range $= \{1, 2, 3\}$

Co-domain $= \{1, 2, 3, 4\}$

Range $= \{x, y\}$ Co-domain $= \{x, y\}$

eg. $f: A \longrightarrow B$ $|A| = 3$ $|B| = 2$

Total no. of functions = Total no of different arrow representations.



$= 8 = 2^3 = (R.S)^{L.S}$

000 001 010 111

Q1. $f: x \longrightarrow y$ Total no. of functions $= 97$

$(R.S)^{L.S} = y^x \longrightarrow |x| = 1 \wedge |y| = 97$

$(97)^1 = 97$

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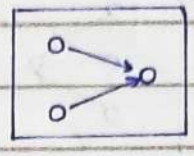
classmate
Date
Page

1:1 (injection)

$(f(a) = f(b) \rightarrow a = b)$

1:1x



eg: $f: z \rightarrow z$

$f(x) = x^2$

$0 \longrightarrow 0$

$1 \longrightarrow 1$

$a = 2$ $b = -2$

$f(a) = 4$ $f(b) = 4$

$a = 2 \rightarrow 4$

$b = -2 \rightarrow 4$

$f(a) = f(b) \rightarrow a = b$

$\underset{\text{True}}{\underline{4 = 4}} \longrightarrow \underset{\text{False.}}{\underline{2 = -2}}$

eg: $f: z \rightarrow z$ $f(x) = x+1$

$1 \longrightarrow 2$

$2 \longrightarrow 3$

$3 \longrightarrow 4$

$a \; 0 \longrightarrow 0 \; f(a)$

$b \; 0 \longrightarrow 0 \; f(b)$

$f(x) = x+1$

$f(w) = w+1$

$f(p) = p+1$

$f(w) = f(p)$

$w+1 = p+1$

$\downarrow$

$w = p$

$f: R \rightarrow R$ (1/2) $f(x) = \dfrac{4x}{2x-1}$ (1:1) ✓

$f(a) = f(b)$

$\dfrac{4a}{2a-1} = \dfrac{4b}{2b-1}$

$a(2b-1) = b(2a-1)$

$2ab - a = 2ab - b$

$-a = -b$

$\Rightarrow \underline{a = b}$

$f: A \rightarrow B$

$|A| = 5$ $|B| = 3$

$0 \text{———} 0$

$0 \text{———} 0$

$0 \text{———} 0$

$0 \text{———} 0$

Not 1:1 functions.

$f: A \rightarrow B$ $|A| \leq |B|$

$|A| = 3$ $|B| = 5$ $|L.S| \leq |R.S|$

$5 \; 0 \longrightarrow 0$

$4 \; 0 \longrightarrow 0$

$3 \; 0 \longrightarrow 0$

0

0

ways.

$= \dfrac{5!}{2!}$

$= \dfrac{5!}{(5-3)!}$ $= {}^5P_3 = {}^{R.S}P_{L.S}$

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$f: A \longrightarrow B$

$|A| = 3 \quad |B| = 7$

$1:1 \rightarrow$ Rs. $P_{L.S} = {}^7P_3$

Functions $= (R_e)^{L.S} = 7^3$

What is the probability of 1:1 functions?

$\rightarrow \dfrac{{}^7P_3}{7^3}$

<u>Onto</u> (Right side must be full)

Range $\subseteq$ Codomain	Range $\subseteq$ Codomain	Range = Codomain	
$\circ \longrightarrow \circ$	$\circ \longrightarrow \circ$	$\circ \longrightarrow \circ$	$\circ \longrightarrow \circ$
$\circ$	$\circ \longrightarrow \circ$	$\circ \longrightarrow \circ$	$\circ \longrightarrow \circ$
$\circ$	$\circ$	$\circ \longrightarrow \circ$	$\circ \longrightarrow \circ$
		$\circ \longrightarrow \circ$	
Function ✓			1:1 ✓
1:1 ✓	1:1 ✓	1:1 ×	Onto ✓
Onto ×	Onto ×	Onto ✓	

eg: $f: Z \rightarrow Z$

$f(x) = x+1$

$x \quad a \qquad a \in Z$

$1 \longrightarrow 2 \qquad a-1 \in Z$

$2 \longrightarrow 3$

$3 \longrightarrow 4$

$f(x) = x+1$

$y = x+1$

$y-1 = x$

eg 2: $f: Z \rightarrow Z$

$f(x) = x^2$

$1 \longrightarrow 1$

$2 \longrightarrow 4$

$\vdots$

$y = 2$

$f(x) = x^2$

$2 = x^2$

$\sqrt{2} = x$ Not Onto!

$\sqrt{2} \in Z$

1. $f(x) = x+7$
2. $f(x) = -x+5$
3. $f(x) = x^2+x$
4. $f(x) = 2x-3 \rightarrow$ not onto
5. $f(x) = x^2$
6. $f(x) = x^3$.

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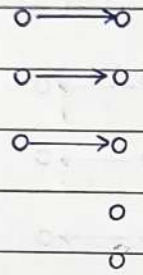
$f: A \to B$

$|A| = m \quad |B| = n$

$$\boxed{\text{Total Onto} = \sum_{i=0}^{n} (-1)^i \cdot {}^nC_i \cdot (n-i)^m.}$$

$f: A \to B$

$|A| = 3 \quad |B| = 5$



$|A| < |B|$

(onto is not possible)

$f: A \quad B$

$|A| = 5 \quad |B| = 3$

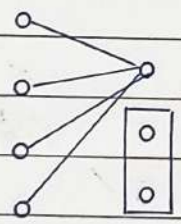
$|A| \geq |B|$

→ $|A| = 3 \quad |B| = 2$ Total onto functions = Total − Total non onto

$= 2^3 - 2$

$=$ <u>6 Functions</u> (onto)

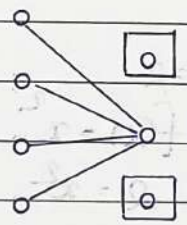
eg: $f: A \to B \quad |A| = 4 \quad |B| = 3$ Total Onto = Total − Total non-onto



1^4

Non Onto

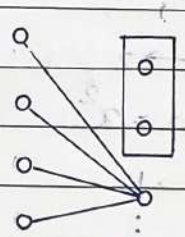
Skip 2.



1^4

Non Onto

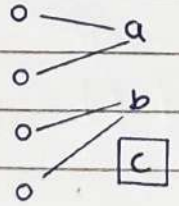
Skip 2



Non Onto

$3C_2 \; 1^4.$

Skipping 2 elements.



non onto

Skip 1 → C

Total functions if we are not pointing on C → 2^4

(R.S)$^{L.S.}$

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$f: A \to B$ Total onto = Total − Total non onto

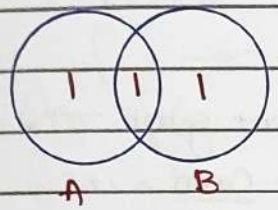
$|A| = 4 = m$ $= 3^4 - {}^3C_1 \cdot 2^4 + {}^3C_2 \cdot 1^4$

$|B| = 3 = n$

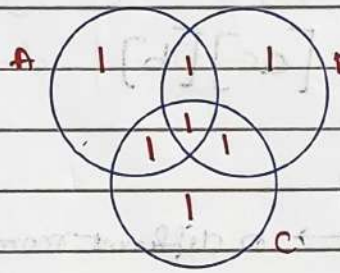
Skipping 1 Skipping 2 elements

$$\sum_{i=0}^{n} (-1)^i \cdot {}^nC_i (n-i)^m = {}^3C_0 (3-0)^4 - {}^3C_1 (3-1)^4 + {}^3C_2 (3-2)^4$$

$$= {}^nC_0 (n-0)^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - {}^nC_3 (n-3)^m$$

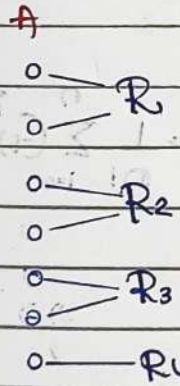


$\Rightarrow A \cup B = A + B - A \cap B$



$\Rightarrow A + B + C - A \cap B - B \cap C - A \cap C + A \cap B \cap C$

Q1. How many ways can we arrange 7 different guest to 4 different rooms, such that none of the rooms must be empty?



$f: A \to B$

$|A| = 7$ $|B| = 4$

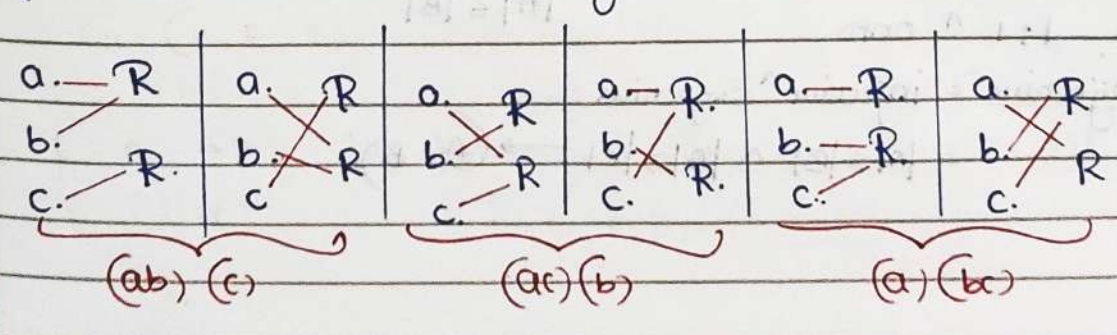
$m = 7$ $n = 4$ ${}^nC_0 (n-0)^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m -$

$= {}^4C_0 (4-0)^7 - {}^4C_1 (4-1)^7 + {}^4C_2 (4-2)^7 - {}^4C_3 (4-3)^7 + {}^4C_4 (4-4)^7$

Ans = 8400

Q2. $f: A \to B$ 3 different guest → 2 same room.

Total onto $= 2^3 - 2 = 6$.



3 different guest $\longrightarrow$ 2 different rooms Onto = 6

3 different guest $\longrightarrow$ 2 identical rooms $\frac{6}{2} = \frac{\text{Onto}}{2!}$

Onto

$[ab]_1\ [c]_2$ $\searrow$ $[ab][c]$ 3 guest

$[ab]_2\ [c]_1$ $\downarrow$

2 identical rooms

$[bc]_1\ [a]_2$ — $[bc][a]$

$[bc]_2\ [a]_1$

3 different guest $\longrightarrow$ 2 identical

$[ac]_1\ [b]_2$ — $[ac][b]$ $\frac{\text{Onto}}{2!}$

$[ac]_2\ [b]_1$

$\rightarrow$ m different guest $\longrightarrow$ n different room (none of the rooms should be empty)

$$\text{Onto} \rightarrow \sum_{i=0}^{n} (-1)^i \cdot {}^nC_i \cdot (n-i)^m.$$

$\rightarrow$ m different guest $\longrightarrow$ n identical rooms

Stering second kind no. $S(m,n) = \frac{\text{Onto}}{n!} = \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \cdot {}^nC_i (n-i)^m.$

diff. identical.

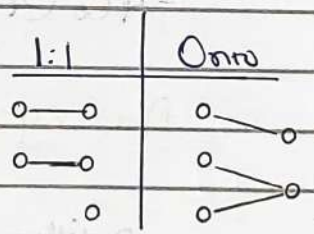
1:1 (injective)

Onto (Surjective)

1:1 Correspondance

$\downarrow$

1:1 $\wedge$ onto



$|A| \geq |B|$

$|A| \leq |B|$

bijection = injection $\wedge$ surjection

$= |A| \leq |B| \wedge |A| \geq |B| \longrightarrow (A = B)$

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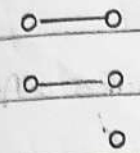
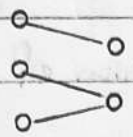
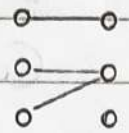


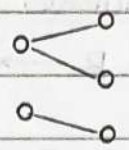
1:1 ✓
Onto ✗



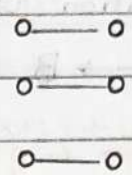
1:1 ✗
Onto ✓



1:1 ✗
Onto ✗
Function ✓



Not Function



1:1 ✓
Onto ✓
1:1C

$f(x) = x+1$

$f: z \to z$

$f(a) = f(b) \to a = b$

$a+1 = b+1 \to a = b$

1:1

$f(x) = x+1$

$y = x+1$

$y-1 = x$

Onto.

$1 \to 2$

$2 \to 3$

$3 \to 4$

$0 \to 0$

$0 \to 0$

eg: $\{1, 2, 3\}$

2^3

ϕ — 000 — FFF

$\{1\}$ — 001 — FFT

$\{2\}$ — 010

$\{3\}$ — 011

$\{1\ 2\}$ — 100

$\{1\ 3\}$ — 101

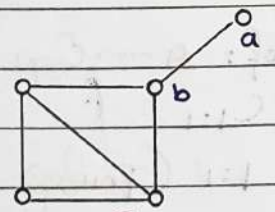
$\{2\ 3\}$ — 110

$\{1\ 2\ 3\}$ — 111 — TTT

$|x| = 30$

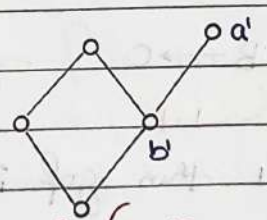
$f: x \to y$ (1:1C)

$|x| = |y| = 30$



$G_1 = (V_1, E_1)$

$G_1 = (V_1, E_1, \psi_1)$



$G_2 = (V_2, E_2)$

$G_2 = (V_2, E_2, \psi_2)$.

$f: G_1 \to G_2$

$f: V_1 \to V_2$ $\quad |V_1| = |V_2|$

$f: a \to a_1$ $\quad f: E_1 \to E_2$

$f: b \to b'$

$f: A \to B$ $\quad |A| = |B| = n.$ $\quad$ Total 1:1C Functions : $n!$

(1:1C)

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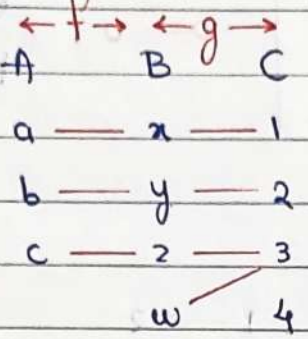
Composition of functions:

$f: A \longrightarrow B \qquad g: B \longrightarrow C \qquad$ Composition of function : $A \longrightarrow C$



$f(a) = x$
$f(b) = y$
$f(c) = z$

$g(x) = 1$
$g(y) = 2$
$g(z) = 3$
$g(w) = 3$

$g(f(a)) = 1$
$g(f(b)) = 2$
$g(f(c)) = 3.$

$g(f(\ ))$
$\downarrow$
gof

$f: A - B \qquad g: B - C$

$gof: A - C$

$gof \neq fog$

eg: $f: z \quad z \qquad g: z \quad z$

$f(x) = 2x+3 \qquad g(x) = x^2$

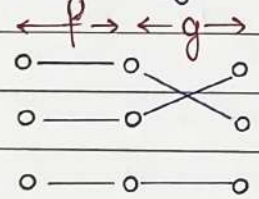
$g(a) = a^2$

$gof = g(f(a)) = g(2x+3)$

$gof = (2x+3)^2.$

$fog = f(g(x))$
$= f(x^2)$
$= 2x^2+3.$

1. If f and g are 1:1C then gof is 1:1C (True)

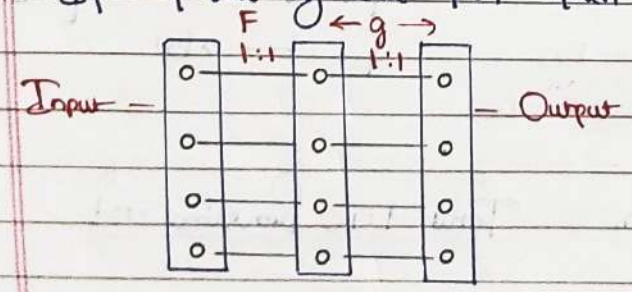


$gof \longrightarrow 1:1$

$f: A \longrightarrow B \qquad g: B \longrightarrow C \qquad gof: A \longrightarrow C$

$\quad 1:1 \qquad\qquad 1:1 \qquad\qquad 1:1$

→ If f and g are 1:1 then gof is 1:1 (True)



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$f: Z \to Z$ $\quad g: Z \to Z$ $\quad gof = g(f(x))$

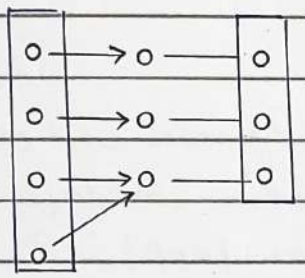
$f(x) = x+1$ $\quad g(x) = x+2$ $\quad = g(x+1)$

(1:1 $\quad$ (1:1 $\quad = (x+1)+2 = x+3.$

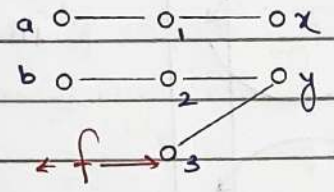
$$1 \xrightarrow{x+1} 2 \xrightarrow{x+2} 4 \qquad 1 \longrightarrow 4$$
$$2 \longrightarrow 3 \longrightarrow 5 \qquad 2 \longrightarrow 5$$
$$3 \longrightarrow 4 \longrightarrow 6 \qquad 3 \longrightarrow 6$$

→ If f and g are onto then gof is onto (True)

$\xleftarrow{f} \xrightarrow{g}$
onto $\quad$ onto



→ If gof is onto then f is onto (False).



gof : R.H Fun

a — x

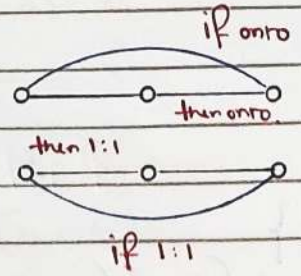
b — y

→ If gof is onto then g is onto. (True)

→ If gof is 1:1 then f is 1:1. (True)

→ If gof is 1:1 then g is 1:1 (False).

Summary.



Inverse function	Invertible
$f: A \longrightarrow B$	$f^{-1}: B \longrightarrow A$
$a \rightarrow x$ $\quad f(a) = x$	$a \leftarrow x$ $\quad f^{-1}(x) = a$
$b \rightarrow y$ $\quad f(b) = y$	$b \leftarrow y$ $\quad f^{-1}(y) = b$
$c \rightarrow z$ $\quad f(c) = z$	$c \leftarrow z$ $\quad f^{-1}(z) = c$

1:1 ✗ $\qquad$ Onto ✓

$f: A \quad B$ $\qquad$ $f^{-1}: B \quad A$

$a \circ\!\!\longrightarrow\!\!\circ\, x$ $\qquad$ $x \circ\!\!\longleftarrow\!\!\circ\, a$

$b \circ\!\!\longrightarrow\!\!\circ\, y$ $\qquad$ $y \circ\!\!\longleftarrow\!\!\circ\, b$

$\circ\, z$ $\qquad$ $z \circ$ ✗

$a \longrightarrow x$

$b \nearrow$

$f^{-1}: B \rightarrow A$

$x \rightarrow a$

$x \rightarrow b$ ✗

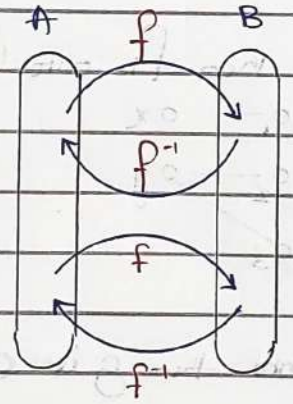
$i_A : A \longrightarrow A$

↳ Identity function.

$f: A \longrightarrow B$ (1:1 C)

$f \circ f^{-1} = i_B$

$f^{-1} \circ f = i_A$



$i_B : B \rightarrow B$

$i_A : A \rightarrow A$

Relations

$P_1 \; R \; P_2 \qquad G_1 \; R \; G_2$

$\begin{cases} e_1 \; R \; e_2 & G_1 \subset G_2 \\ 5 > 3 \end{cases}$

Set 1 R Set 2

$A \subseteq B$

eg: $A = \{1, 2, 3\} \quad B = \{2, 3\}$

$A \times B = \{(1,2)\ (1,3), (2,2)\ (2,3)\ (3,2)\ (3,3)\}$

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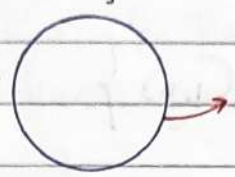
After passing through a condition:

Result: $\{(2,2)\ (3,3)\}$ — Relation

Result: $\{(1,2)\ (2,3)\ (1,3)\}$ — Relation $(a<b)$

eg: $A=\{1,2\}$ $B=\{1\}$

$A\times B=\{(1,1),\ (2,1)\}$



$R_1=\{\ \}=\phi$

$R_2=\{(1,1)\}$

$R_3=\{(2,1)\}$ ← $A\times B$

$R_4=\{(1,1)\ (2,1)\}$

2^{AB}

Relation: Subset of Cross product of A, B.

$|A|=m$ $|B|=n$

$|A\times B|=m.n$ Total Relations = Total no. of subsets

$=2^{m.n}$.

$|A|=n$, $|A\times A|=n^2$ Total Relations $=2^{n^2}$

↳ when only one set is given.

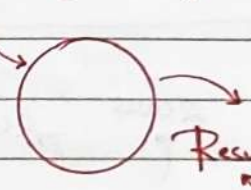
Symmetric Relation

$\forall a\forall b\,[(a,b)\in R \longrightarrow (b,a)\in R]$

if $(a,b)\in R$ then $(b,a)\in R$

$A=\{\ \}$

$A\times A=\{\ldots\ldots\}$



Result: Symmetric

$R_1=\{\ \}$ Symmetric:

$(a,b)\in R \rightarrow (b,a)\in R$

F

T

$R_2=\{(1,2)\ (2,1)\}$ Symmetric.

$(a,b)\in R \rightarrow (b,a)\in R$

$(1,2)\in R \rightarrow (2,1)\in R$

T T

$R_3=\{(1,2)\}$

$(a,b)\in R \rightarrow (b,a)\in R.$

$(1,2)\in R \rightarrow (2,1)\in R$

T F

$a=1,\ b=2$ Not Symmetric

$R_4 = \{(1,1)\}$ Symmetric.

$(a,b) \in R \rightarrow (b,a) \in R$

$\underset{T}{(1,1) \in R} \rightarrow \underset{T}{(1,1) \in R}$ $\quad a=1 \; b=1.$

$R_1 = \{\underset{\times}{(2,3)} \; \underset{\checkmark}{(2,2)}\}$ ×

$R_2 = \{\underset{\checkmark}{(1,1)} \; \underset{\checkmark}{(2,2)} \; \underset{\checkmark}{(3,3)} \; \underset{\times}{(1,3)}\}$

$R_3 = \{\underset{\checkmark}{(1,1)} \; \underset{\checkmark}{(2,2)} \; \underset{\checkmark}{(3,3)}\}$ ✓

$A = \{1, 2, 3\}$ $\quad |A| = n$

$n^2 = A \times A =$

(1,1)	(1,2)	(1,3)
(2,1)	(2,2)	(2,3)
(3,1)	(3,2)	(3,3)

$\rightarrow 2^n \times 2$

$A \times A$

$\{11 \; 22 \; 33 \; (12)(21)(13)(31)(23)(32)\}$

2 choice $\quad$ 2 choice $\quad$ 2 choice

2 choice

2^n

Total elements $= n^2$

Total diagonal $= n$

non-diagonal $= n^2 - n$

$\{\underbrace{(11)(22)(33)}_{n} \; \underbrace{\boxed{(12)(21)}}_{2} \; \underbrace{\boxed{(13)(31)}}_{2} \; \underbrace{\boxed{(23)(32)}}_{2}\}$

$n^2 - n$

n^2

2 case $\quad$ 2 case.

$\{(11) \; (12)\}$

$w_1 \quad w_2$

$4 \begin{cases} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{cases}$

$2^2 \begin{cases} \phi & — 00 \\ \{(11)\} & — 01 \\ \{(12)\} & — 10 \\ \{(11)(12)\} & — 11 \end{cases}$

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Reflexive: $A \to n$ or empty set.

$A = \{1, 2, 3\}$ $\qquad$ $A = \{1, 2, 3\}$

$\forall a \in A \ (a,a) \in R$ $\qquad$ $\forall a \in A \ (a,a) \in R$

$R_1 = \{(1,1)\ (2,2)\}$ not Reflexive

$A = \{1, 2, 3\}$ $R_2 = \{(1,1)\}$ ✗

$\forall a \in A \ (a,a) \in R$

$|A| = n$

$|A \times A| = n^2$

(1,1)	(1,2)	(1,3)
(2,1)	(2,2)	(2,3)
(3,1)	(3,2)	(3,3)

1 choice

Present 1^n

Total elements $= n^2$

Diagonal elements $= n$

non-diagonal elements $= n^2 - n$

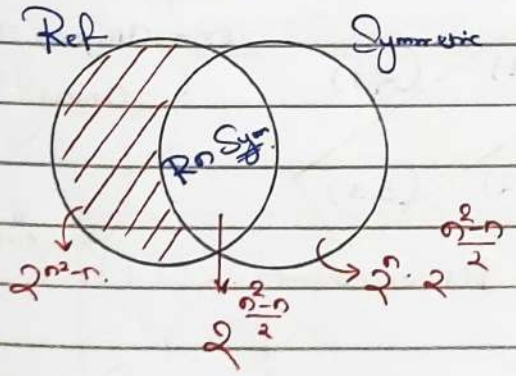
2 choice for non-diagonal

$\#R = 2^{n^2 - n}$.

$A \times A = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,1)\ (2,3)\ (3,2)\ (1,3)\ (3,1)\}$

$R_1 = \{\boxed{11 \quad 22 \quad 33}\ (1,2)\}$

$R_2 = \{\boxed{11 \quad 22 \quad 33}\ (2,1)\}$



$|A| = n$

$\#R = 2^{n^2-n}$

$\#Sy. = 2^n \cdot 2^{\frac{n^2-n}{2}}$

↳ diagonal must be present.

$Ref \cap Sym = 1^n \cdot 2^{\frac{n^2-n}{2}}$

$= 2^{\frac{n^2-n}{2}}$

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Irrefuxive : (hate Same element)

$A \longrightarrow$ non empty set

$A = \{1, 2, 3\}$

$\forall a \in A \; (a,a) \notin R$

$R_3 = \{(1,2)\}$ IRR ✓

$R_4 = \{(2,3)\}$ IRR ✓

$R_1 = \{(1\,1)\}$ IRR ×

Ref ×

$R_2 = \{(1\,2)(2\,3)(2\,2)\}$ IRR ×

REF ×

$R_1 = \{\ \}$ ✓ IRR ✓

$R_2 = \{(1,2)\}$ ✓

$R_3 = \{(2,1)\}$ ✓

$R_4 = \{(1,1)\}$ ×

$R_5 = \{(1\,2)(2\,1)(2\,3)(3\,2)\}$ ✓

$R_6 = \{(1\,3)\ (3\,3)\}$ IRR ×

$$\#R = 2^{n^2-n} \begin{cases} \boxed{11 \quad 22 \quad 33} \ (1\,2) \\ \boxed{11 \quad 22 \quad 33} \ (1\,2)\ (2\,1) \end{cases}$$

$$\#IRR = 2^{n^2-n} \begin{cases} \boxed{\text{absent}} \ (1,2) \\ \boxed{\text{absent}} \ (1\,2)\ (2\,1) \end{cases}$$

$A = \{1, 2, 3\}$

(1,1)	(1,2)	(1,3)
(2,1)	(2,2)	(2,3)
(3,1)	(3,2)	(3,3)

1 choice absent.

Total $= n^2$

Diagonal $= n$

non-diagonal $= n^2 - n$

2 choice → absent

→ present

2^{n^2-n}.

→ If A is non empty set.

$R = \{\ \}$

Refuxive ×

Irrefuxive ✓

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Anti Symmetric : — Does not allow flipping
— Allows same element.

$\forall a \forall b [(a,b) \in R \wedge (b,a) \in R \rightarrow a=b]$

$R_1 = \{\ \} \longrightarrow$ Anti ✓

$(a,b) \in R \wedge (b,a) \in R \rightarrow a=b$

$\underbrace{F \wedge F}_{F}$

True.

$R_2 = \{(1,1)\} \rightarrow$ Anti ✓

$(a,b) \in R \wedge (b,a) \in R \rightarrow a=b$

$\underbrace{(1,1) \in R}_{T} \wedge \underbrace{(1,1) \in R}_{T} \longrightarrow 1=1$

$T \wedge T \longrightarrow$ True

$a=1,\ b=1.$

$R_3 = \{(1,2)(2,1)\} \rightarrow$ Anti ✗

$(a,b) \in R \wedge (b,a) \in R \rightarrow a=b$

$\underbrace{(1,2) \in R}_{T} \wedge \underbrace{(2,1) \in R}_{T} \rightarrow 1=2$

$T \rightarrow$ False

$T \rightarrow$ False

False.

$R_4 = \{(1,2)\}.$

$(a,b) \in R \wedge (b,a) \in R \rightarrow a=b$

$\underbrace{(1,2) \in R \wedge F}_{F}$

True.

$R_5 = \{\ \}$ ✓

$R_6 = \{(2,2)(1,1)\}$ ✓

$R_7 = \{(2,3)(1,1)(2,2)(3,3)(1,3)\}$ ✓

$R_8 = \{(2,3)\}$ ✓

$R_9 = \{(1,3)\}$ ✓

$R_{10} = \{(1,1)(2,2)(1,2)(2,1)\}$ ✗

not anti

$A = \{1,2,3\}$

$A \times A = \{(1,1)(2,2)(3,3) \mid (1,2)(2,1) \mid (1,3)(3,1) \mid (2,3)(3,2)\}$

$\longleftarrow n \longrightarrow$ $\longleftarrow n^2-n \longrightarrow$

$\longleftarrow n^2 \longrightarrow$

$2^n \times 3^{\frac{n^2-n}{2}}$

boxes $= \frac{n^2-n}{2}$

(1,1)	(1,2)	(1,3)
(2,1)	(2,2)	(2,3)
(3,1)	(3,2)	(3,3)

2 choices. 2^n.

Total $= n^2$ Diagonal $= n$

Non-diagonal $= n^2-n$

boxes $= \frac{n^2-n}{2}$

3 choices.

$\{ 2^n \times 3^{\frac{n^2-n}{2}}$.

→ $A = \{0, 1, 2, 3\}$

(i) What is smaller size of Anti symmetric → 0

(ii) larger size of Anti symmetric → $\left\{n + \frac{n-1}{2}\right\}$

(iii) Total no. of anti symmetric : $2^n \times 3^{\frac{n^2-n}{2}}$

(iv) Symmetric Smaller Size → 0

(v) Symmetric largest size → n^2.

(vi) Reflexive smallest → n

(vii) Reflexive larger → n^2

(ix) Irreflexive Smaller → 0

(x) Irreflexive largest → $n^2 - n$

Asymmetric – Does not allow same element

– Does not allow flipping.

$\forall a \forall b ((a,b) \in R \rightarrow (b,a) \notin R)$

$R_1 = \{\ \}$ Asymmetric ✓	$R_2 = \{(1,1)\}$ Asy ✓
$(a,b) \in R \rightarrow (b,a) \notin R$	$(1,1) \in R \rightarrow (1,1) \notin R$
F →	T F
True.	False.

$R_3 = \{(1,2)\}$ Asy ✓

$(a,b) \in R \rightarrow (b,a) \notin R$ – if (a,b) belongs to R then (b,a) should not belongs to R.

$(1,2) \in R \rightarrow (2,1) \notin R$

T T – if (a,b) is present then (b,a) must be absent.

(1,1)	(1,2)	(1,3)
(2,1)	(2,2)	(2,3)
(3,1)	(3,2)	(3,3)

1 Choice Absent

$\Rightarrow 3^{\frac{n^2-n}{2}}$

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Symmetric	AntiSymmetric	Asymmetric
* allowed same element.	* allowed same element.	* allowed same element.
* demands flipping	* no flipping	* no flipping.
$2^n \cdot 2^{\frac{n^2-n}{2}}$	$2^n \cdot 3^{\frac{n^2-n}{2}}$	$3^{\frac{n^2-n}{2}}$
$(a,b) \in R \rightarrow (b,a) \in R$	$(a,b) \in R \wedge (b,a) \in R \rightarrow a=b$	$(a,b) \in R \rightarrow (b,a) \notin R$.

Q1. $A = \{1,2,3,4\}$

$R = \{(1,1)\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (3,1)\ (3,2)\ (3,3)\ (3,4)\}$.

Refuxive × (4,4) is absent

Irrefexive × both same

Symmetric ×

Anti Symmetric ×

Asymmetric ×

	Ref 2^{n^2-n}	IRR 2^{n^2-n}	Sym. $2^n \cdot 2^{\frac{n^2-n}{2}}$	Anti $2^n \cdot 3^{\frac{n^2-n}{2}}$	Asy. $3^{\frac{n^2-n}{2}}$
Ref.	×	0	$1^n \cdot 2^{\frac{n^2-n}{2}}$	$1^n \cdot 3^{\frac{n^2-n}{2}}$	0
Irr		×	$1^n \cdot 2^{\frac{n^2-n}{2}}$	$1 \cdot 3^{\frac{n^2-n}{2}}$	$1^n \cdot 3^{\frac{n^2-n}{2}}$
Sym.			×	2^n	1.
Anti				×	$3^{\frac{n^2-n}{2}}$
Asy.					×

$A \rightarrow$ non empty set.

$R_1 \rightarrow$ refuxive relation on A

$R_2 \rightarrow$ refuxive relation on A.

$A = \{1,2,3\}$.

$R_1 = \{(11)\ (22)\ (33)\ (\ )..(\ )..(\ )...\}$

$R_2 = \{(11)\ (22)\ (33)\ (\ )(\ )....\}$

$R_1 \cup R_2 = \{\ \}$ } Both

$R_1 \cap R_2 = \{\ \}$ } Refuxive

	Union ($\cup$)	Intersection ($\cap$)
R_1, R_2 reflexive.	✓	✓
R_1, R_2 Irreflex.	✓	✓
R_1, R_2 Symmetric	✓	✓
Anti Symmetric	✗	✓
Asymmetric	✗	✓

$R_1 = \{(1,2)\}$ $R_2 = \{(2,1)\}$

Anti ✓ Anti ✓

$R_1 \cup R_2 = \{(1,2)\ (2,1)\}$

Not Anti

$R_1 \cap R_2 = \{\ \}$

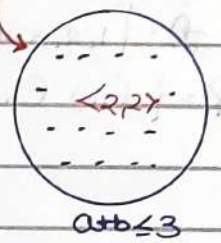
$R_1 = \{a,b \mid a+b \leq 3\}$

Anti ✗

Asy. ✗

Set: 2

2×2



Reflexive ✗

aRa

$a+a \leq 3$

$2a \leq 3$

Symmetric:

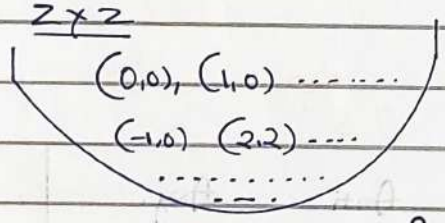
$(a,b) \in R \rightarrow (b,a) \in R$

$aRb \rightarrow bRa$

$\underset{T}{\underline{a+b \leq 3}} \rightarrow \underset{T}{\underline{b+a \leq 3}}$

Reflex: $(0,0)\ (1,1)$

2×2



$a+b \leq 3 \rightarrow \{(1,2)\ (2,1)\ (0,3)\ (3,0) \ldots\}$

$R_2 = \{(a,b) \mid a = b+1\}$

Reflexive: ✗

aRa

$a = a+1$

Symmetric:

$aRb \rightarrow bRa$

$a = b+1 \rightarrow b = a+1$

$(2,1) \rightarrow 1 \neq 2+1$

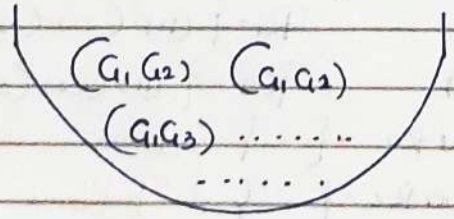
Anti Symmetric ✓

Asymmetric ✓

$G = \{G_1, G_2, G_3 \ldots\}$

$R = \{(G_1, G_2) \mid G_1$ is isomorphic to $G_2\}$

$G \times G$



$R = G_1 R G_1$ ✓

Symmetric: $G_1 R G_2 \rightarrow G_2 R G_1$ ✓

Anti ✓

Asy ✓ Irr ✓

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$A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\} \ldots \{1,2,3\}\}$

$P(A) \times P(A) = \{(\phi\ \phi)\ (\phi, \{1\}$
$(\{1\}\{2\}) \ldots\ldots$
$\ldots\ldots(\{1\},\{12\})\}$

$R: \{(A,B) \mid A \subseteq B\}$

Ref: $A R A \quad A \subseteq A$

Sym: $A R B \longrightarrow B R A$ (no flipping)

$A \subseteq B \longrightarrow B \subseteq A$

$\{1\} \subseteq \{12\} \rightarrow \{12\} \subseteq \{1\}$

Anti ✓

Asymmetric ✗

Transitive

$\{(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R\}$.

$R_1 = \{\ \}$

$\underbrace{\underbrace{(a,b) \in R}_{F} \wedge \underbrace{(b,c) \in R}_{F} \rightarrow (a,c) \in R}_{F}$

F

$R_2 = \{(1,1)\}$

$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

$\underbrace{\underbrace{(1,1) \in R}_{T} \wedge F}$

F

$R_3 = \{(1,2)\}$

$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

$\underline{(1,2) \in R \wedge F}$

F

$\underline{a=1, b=2}$

True.

$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

$(2,1) \in R \wedge (1,2) \in R \rightarrow (2,2) \in R$

$R_4 = \{(1,2)\ (2,1)\ (1,1)\}$ Not Transitive

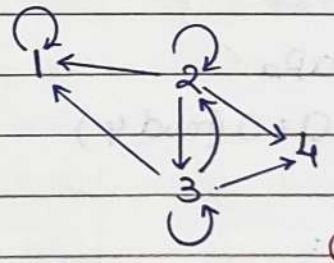
$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

$\underbrace{(1,2) \in R}_{T} \wedge \underbrace{(2,1)}_{T} \longrightarrow \underbrace{(1,1) \in R}_{T}$

→ $\{(1,1)\ (2,1)\ (2,2)\ (2,3)$
$(2,4)\ (3,1)$
$(3,2)\ (3,3)\ (3,4)\}$

1 → (1,1)

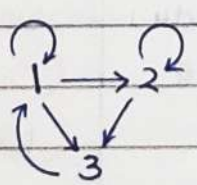
2 → (2,2) (2,3) (2,4) (2,1)



3 → (3,1) (3,2) (3,3) (3,4)

→ $A = \{1, 2, 3\}$

$R: \{(1,2)\ (2,3)\ (1,3)$
$(1,1)\ (2,2)$
$(3,1)\}$.



R'
$1 \rightarrow \{(1,1)\ (1,2)\ (1,3)$
$2 \rightarrow (2,2)\ (2,1)\ (2,3)\}$
$3 \rightarrow (3,1)\ (3,2)\ (3,3)$

if $R = R'$ then R is transitive.

Equivalence Relation :: $A = \{1, 2, 3\}$

Reflexive
Symmetric
Transitive
} (RST) then Equivalence R.

$R_1 = \{ \underbrace{(1,1)\ (2,2)\ (3,3)}_{RST}\ \boxed{(1,2)\ (2,1)} \}$

$R_1 = \{ (a,b) \mid a+b = \text{even} \}$

Re: aRa

$a+a = \text{even}$

$2a = \text{even}$

Symmetric:

$aRb \rightarrow bRa$

$a+b = \text{even} \rightarrow b+a = \text{even}$

if $a+b = \text{even}$ then $b+a = \text{even}$

$(0,4) \in R \rightarrow (4,0) \in R$

$0+4 = \text{even} \rightarrow 4+0 = \text{even}$

Transitive ✓

$aRb \wedge bRc \rightarrow aRc$

$a+b = \text{even} \wedge b+c = \text{even} \rightarrow a+c = \text{even}$

$(0,4) \in R \wedge (4,6) \in R$

$0+4 = \text{even} \wedge 4+6 = \text{even} \rightarrow 0+6 = \text{even}$

$1+3 = \text{even} \wedge 3+5 = \text{even} \rightarrow 1+5 = \text{even}$

$R_2 = \{ (a,b) \mid a = b \pmod 4 \}$

$R = aRa$ ✓

$a = a \pmod 4$

Sym:

$aRb \rightarrow bRa$

$a = b \pmod 4 \rightarrow b = a \pmod 4$

$(0,4) \in R \rightarrow (4,0) \in R$.

$aRb \wedge bRc \rightarrow aRc$

$a = b \pmod 4 \wedge b = c \pmod 4$

$\rightarrow a = c \pmod 4$

a, b are having same remainder w.r.t 4.

$(0,4) \in R$ ✓ $0 = 4 \pmod 4$

$(1,5) \in R$ $1 = 5 \pmod 4$

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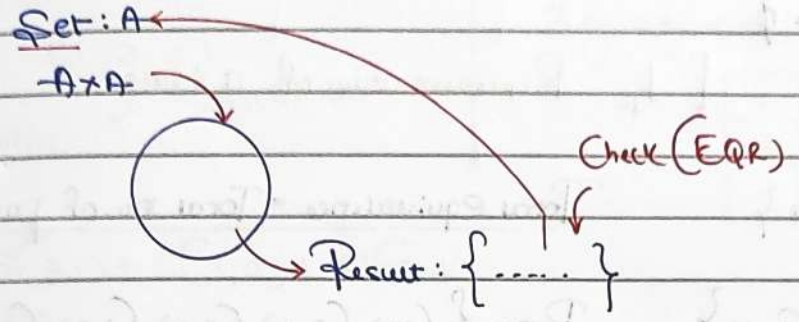
$(0,4) \in R \wedge (4,8) \in R \rightarrow (0,8) \in R$

$\underbrace{0 = 4 \pmod 4}_{\to 0} \wedge \underbrace{4 = 8 \pmod 4}_{\to 0} \rightarrow \underbrace{0 = 8 \pmod 4}_{\to 0}$

→ Equivalence Relation Create Partition on set.



eg: $A = \{1,2,3,4\}$

$R = \{(1,2)\ (3,4)\}$

How to Create a Partition?

$\{(a,b) \in R$ or $aRb\}$

a,b will go in one Partition.

$(1,2) \in R$

1,2 will go in one partition.

Set: A

1,2 → P_1
3,4 → P_2

eg: Set: Z

Z×Z

partition.

$R = \{(a,b) \mid a+b = \text{even}\}$

(RST)

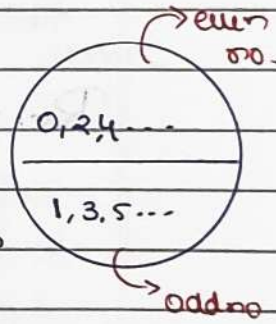
EQR ✓

$(1,3) \in R$ $1+3 = \text{even}$

$(0,0) \in R$ $0+0 = \text{even}$

$(0,1) \in R$ $0+1 \neq \text{even}$

$(0,2) \in R$ $0+2 = \text{even}$



$R_2 = \{(a,b) \mid a = b \pmod 4\}$

(RST) EQR ✓

$(0,4) \in R$ $0 = 4 \pmod 4$

$(1,5) \in R$ $1 = 5 \pmod 4$

0,4
1,5
2,6
3,7

$R = \{(s_1, s_2) \mid s_1, s_2$ will go in same state$\}$

$R = sRs$

$Sy = s_1 R s_2 \rightarrow s_2 R s_1$

$T: s_1 R s_2 \wedge s_2 R s_3$

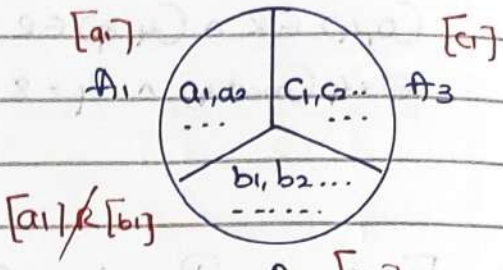
011 R 01 ∧ 01 R 0111 → 011 R 0111.

→ Set: A

EQR Create partition on set A

(equivalence classes)

(A_1, A_2, A_3)



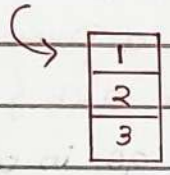
1) $A_1 \cup A_2 \cup A_3 = A$

2) $A_1 \cap A_2 \cap A_3 = \phi$

$[\ ]_R$: Representation of a class

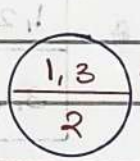
→ Set: $A = \{1, 2, 3\}$ Total equivalence = Total no. of partitions

$R_1 : \{(11)(22)(33)\}$

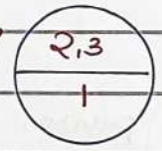


$R_2 : \{(11)(22)(33)(12)(21)\}$

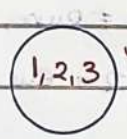
$R_3 = \{(11)(22)(33)(13)(31)\}$



$R_4 : \{(11)(22)(33)(23)(32)\}$



$R_5 : \{(11)(22)(33)(12)(21)(23)(32)(31)(13)\}$



∴ Total no of partitions = 5

Total Equivalence relations = Total Partitions

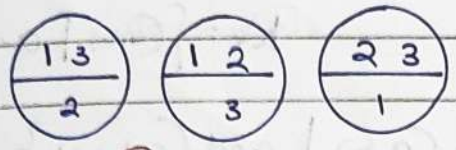
$= S(3,1) + S(3,2) + S(3,3)$

$= \sum_{i=1}^{B} S(3,i)$

$B(n) = \sum_{i=1}^{n} S(m,n)$

VIVEKANAND VERNEKAR

How many ways can we can adjust 3 guest 2 identical rooms?

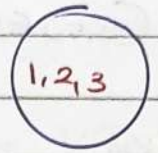


$S(3,2)$

identical

diff.

How many ways can can adjust 3 guest to 1 room.



$S(3,1)$

3 guest and 3 identical rooms.

$S(3,3)$

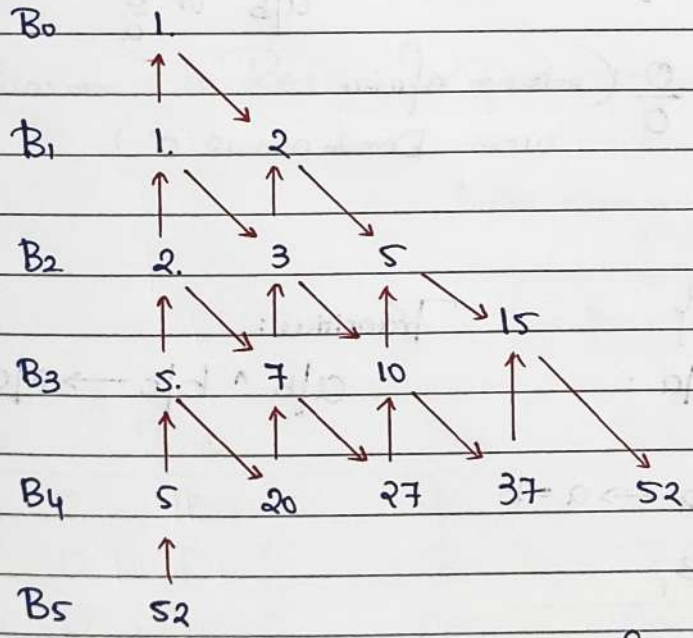
$S(3,1) + S(3,2) + S(3,3)$

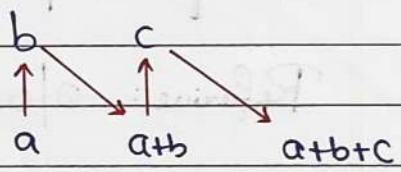
$\Rightarrow \sum_{i=1}^{3} S(3,i)$

$B(n) = \sum_{i=1}^{n} S(n,i)$

Bell Number

Triangle





$R : \{(a,b) \mid a \le b\}$

Partial Order Relations :

$R : aRa \quad a \le a$ (T)

(RAT) { Reflexive, Antisymmetric, Transitive }

Anti: $aRb \wedge bRa \rightarrow a=b$

$a \le b \wedge b \le a \rightarrow a=b$

$\underbrace{2 \le 3}_{T} \wedge \underbrace{3 \le 2}_{F} \rightarrow T$

I. $a \le b \wedge b \le c \rightarrow a \le c$

$2 \le 3 \wedge 3 \le 5 \rightarrow 2 \le 5$

$(2,3) \in R \wedge (3,5) \in R \rightarrow (2,5) \in R$

$R_1 : \{(a,b) \mid a \le b\}$ RAT ✓

$R_2 : \{(a,b) \mid a < b\}$ RAT ✗

(RAT.)

$R : \{(A,B) \mid A \subseteq B\}$

R: A R A $A \subseteq A$

Anti: $A \subseteq B \wedge B \subseteq A \rightarrow A = B$

T: $A \subseteq B \wedge B \subseteq A \rightarrow A \subseteq C$.

$R : \{(a,b) \mid \gcd(a,b) = 1\}$

× R: $\gcd(2,2) = 2 \ne 1$

× Anti: Relation allows flipping So not anti.

× Transitive:

$\gcd(3,5) \wedge \gcd(5,9) \rightarrow \gcd(3,9)$

$= 1 \quad = 1 \quad = 3 \ne 1$

$R : \{(a,b) \mid a \text{ divides } b\}$

$R : \{(a,b) \mid a|b\}$

a divides b

$\Downarrow$

$a|b$ or $\frac{b}{a}$

Reflexive: $0|0$ $\frac{0}{0}$ (it is not reflexive when Domain allows '0')

$D : 2^+$

$R : \{(a,b) \mid a|b\}$

R: aRa $a|a$

Transitive:

$a|b \wedge b|c \rightarrow a|c$.

Anti: $a|b \wedge b|a \rightarrow a = b$

$3|6 \wedge 6|3$ F

F

Set : 2

Relation: $\{(a,b) \mid a \le b\}$

Set: 2^+

Relation: $\{(a,b) \mid a|b\}$.

VIVEKANAND VERNEKAR

Set: $\{\ldots a, b, \ldots\}$

aRb or bRa

a, b are called comparable.

Relation: 1

Set: 2^+ $\{1, 2, 3, 4 \ldots\ldots\}$

(only partial elements are comparable)

$a=3$ $b=6$

aRb or bRa

$3R6$ or $6R3$

$3|6$ or $\cancel{6R3}$

T

3, 6 are called comparable

Domain: $Z = \{\ldots -2, -1, 0, +1, 2 \ldots\}$

$R: \{(a,b) \mid a \le b\}$

Total Order Relation (linear order relation)

$a=3$ $b=5$	$a=100$ $b=-1$
aRb or bRa	$a \le b$ or $b \le a$
$a \le b$ or $b \le a$	$100 \le -1$ or $-1 \le 100$
$3 \le 5$ or $5 \le 3$	T
T	

Relation $\longrightarrow$ RAT $\longrightarrow$ Partial order relation

$\downarrow$

all elements are comparable

$\downarrow$ (YES)

$\boxed{\text{Total Order Relation}}$

POR		POR	POR		POR
R_1	$\cup$	R_2	R_1		R_2
R	$\cup$	R ✓	R	$\cap$	R ✓
Anti	$\cup$	Anti ×	Anti	$\cap$	Anti ✓
Tr.	$\cup$	T ×	T	$\cap$	T ✓

Equivalence Relation = RST

Partial Order Relation : RAT

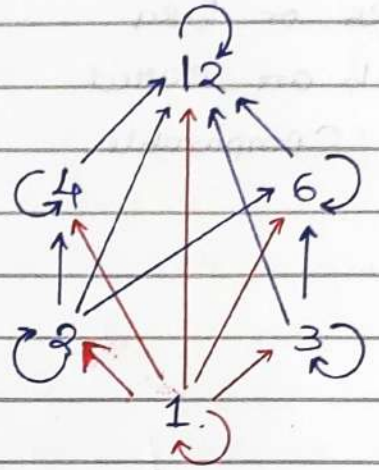
$A: \{1, 2, 3\}$

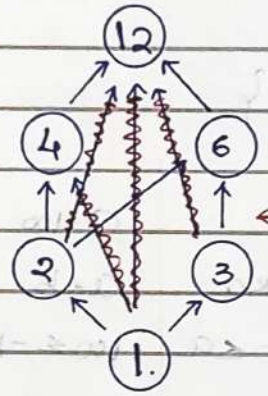
$R_1: \{(1,1)\ (2,2)\ (3,3)\ (1,2)\}$ POR

$R_2: \{(1,1)\ (2,2)\ (3,3)\ (2,1)\}$ POR

Set: $\{1, 2, 3, 4, 6, 12\}$ — Partial Order Relation

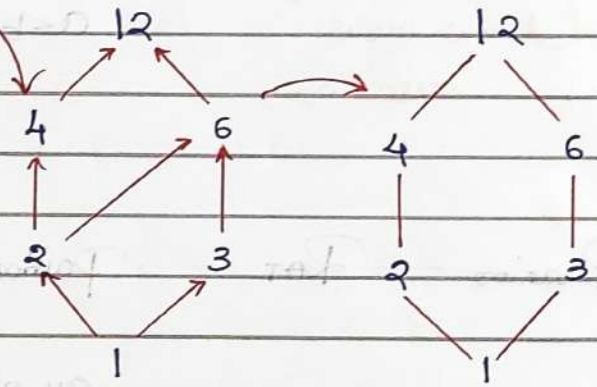
R: $\{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,6)\ (1,12)$
$(2,2)\ (2,4)\ (2,6)\ (2,12)$
$(3,3)\ (3,6)\ (3,12)$
$(4,4)\ (4,12)$
$(6,6)\ (6,12)$
$(12,12)\}$.





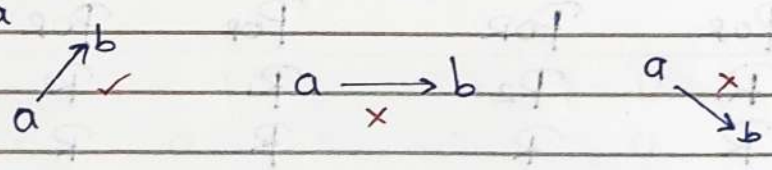
Simplification:
Remove Self loop
Remove transitive

Hasse Diagram.



aRb or $(a,b) \in R$

a, can be at lower level.
b, can be at higher level.



Set: Z^+
Relation: $\leq$ — POR
(Set, Relation) (poset)
Partial order set

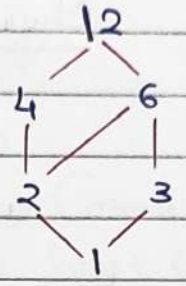
Total order relation
$(Z, \leq)$
Total order set
(TOSET)

VIVEKANAND VERNEKAR

Set: $\{1, 2, 3, 4, 6, 12\}$

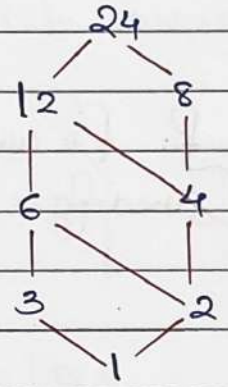
$R: \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,6)\ (1,12)$
$(2,2)\ (2,4)\ (2,6)\ (2,12)$
$(3,3)\ (3,6)\ (3,12)$
$(4,4)\ (4,12)$
$(6,6)\ (6,12)$
$(12,12)\}$

(Set, Relation)
$(\{1,2,3,4,6,12\}, |)$
Divisor of 12
$(D_{12}, |)$



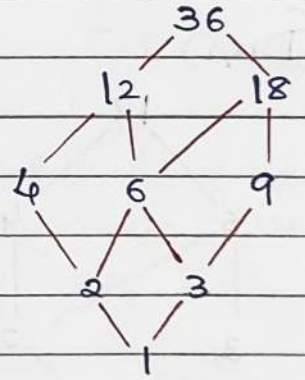
(Set, Relation)
$(\{1,2,3\} \leq)$

$(D_{24}, |)$
$(\{1,2,3,4,6,8,12,24\})$

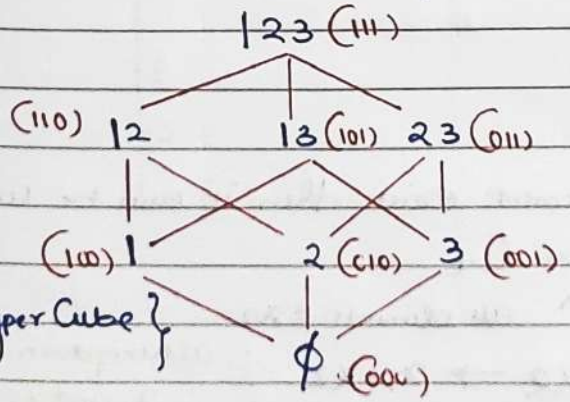


#edge = 10

$(D_{36}, |)$
$(\{1,2,3,4,6,9,12,18,36\}, |)$



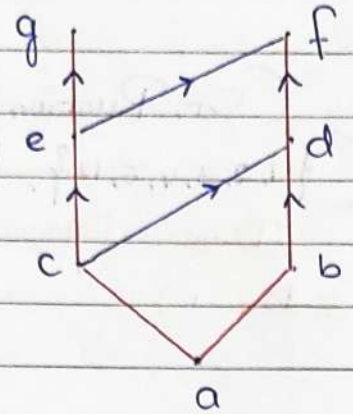
$(P(A), \subseteq)$ $A = \{1,2,3\}$.



{Hyper Cube}

Powerset : $P(A) / 2^A$
$A = \{1,2,3,4\}$ $2^n \cdot n = ee$
$(P(A), \subseteq)$
$n = 4$
$e = 2^{n-1} \cdot n = 4 \cdot 2^{4-1} = \underline{\underline{32}}$

VIVEKANAND VERNEKAR



(A, R) poset

Greatest Element (GE)

x is called Greatest Element of (A, R)

all elements $\leq x \in R$

Check for g: g is not GE

$abcdefg \leq g$

$a \leq g$ (T) $b \leq g$ (False)

Note:

$(a,b) \in R$ (or) $\begin{matrix} b \\ | \\ a \end{matrix}$ (or) aRb

$[a \leq b]$

aRb

$a \leq b$

a will be at lower level

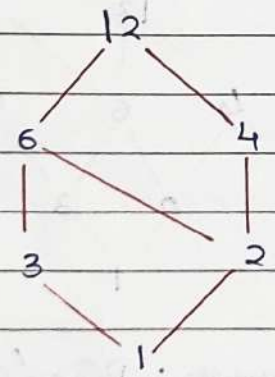
b will be at higher level

Check for f: f is also not GE

$abcdefg \leq f$ $\begin{cases} a \leq f \text{ (T)} & c \leq f \text{ (T)} & e \leq f \text{ (T)} \\ b \leq f \text{ (T)} & d \leq f \text{ (T)} & f \leq f \text{ (T)} \end{cases}$

$g \leq f$ (False)

$(D_{12}, |)$



Greatest Element

all elements $\leq x \in A$ (12 is G.E)

Check for 12:

$1, 2, 3, 4, 5, 6, 12 \leq 12$ (True)

Theorem: If greatest element exist then it will be unique

Assume x_1 and x_2 are G.E

all elements $\leq x_1$ $\wedge$ all elements $\leq x_2$

$x_2 \leq x_1 \wedge x_1 \leq x_2 \rightarrow x_1 = x_2$ $\therefore$ assumption is wrong

$\therefore$ x_1 and x_2 are unique

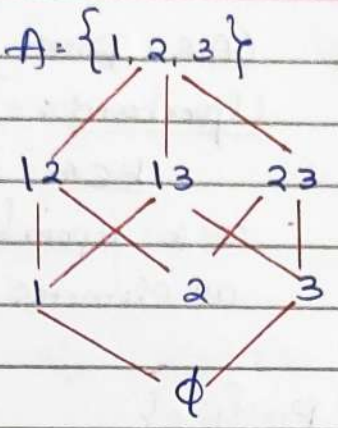
VIVEKANAND VERNEKAR

eg: $A = \{1, 2, 3\}$

$(2^A, \subseteq)$

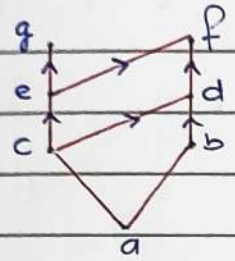
or

$(P(A), \subseteq)$



GE : $\{1, 2, 3\}$

$\boxed{\text{GE : } A}$



(A, R) poset

<u>Least Element</u> (minimum element)

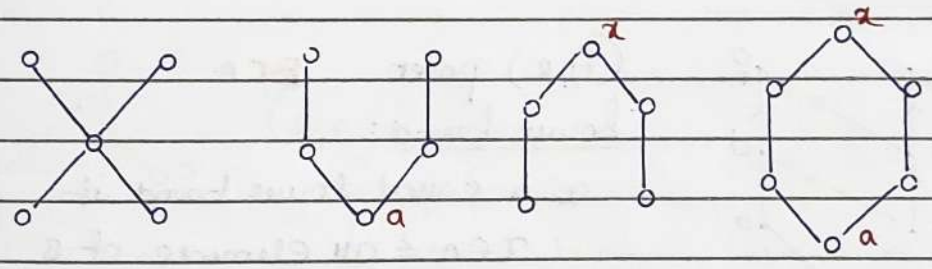
x is called least element if:

$x \in A \leq$ all elements of A.

<u>Check for 'a':</u>

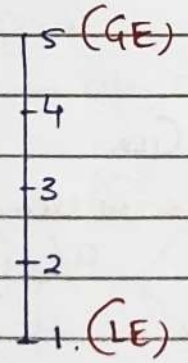
$a \leq abcdefg$ $\therefore$ a is least element.

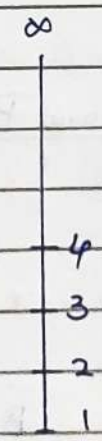
* If least element exist it will be unique.



L.E. ×	LE : a	LE ×	L.E : a
G.E ×	G.E ×	G.E a	G.E : x.

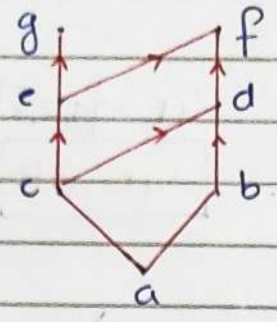
$(\{1, 2, 3, 4, 5\} \leq)$ $(2^+, \leq)$ $(\mathbb{Z}, \leq)$





LE : 1
GE : NA

LE : NA
GE : NA



(A, R) poset

<u>Upperbound</u>:

$B \subseteq A$

x is upperbound of B if:

all elements $\in B \leq x \in A$.

$bc \leq e$

$c \leq e$ (T)

$b \leq e$ (F)

eg: $B = \{b, c\}$. $\rightarrow$ UB's of $\{b, c\}$ is $\{d, f\}$

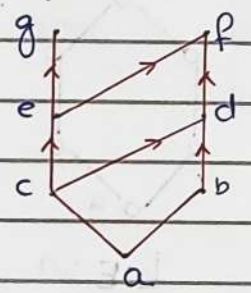
$bc \leq d$	$bc \leq f$
$b \leq d$	$b \leq f$
$c \leq d$	$c \leq f$

$B = \{a, b\}$ UB's: $\{b, f, d\}$

$B = \{e, b\}$ UB's: f

$B = \{c, d\}$ UB's: $\{a, f\}$

$B = \{a, g\}$ UB's: g



(A, R) poset $B \subseteq A$

<u>lower bound</u>:

x is called lower bound if

$x \in A \leq$ all elements of B

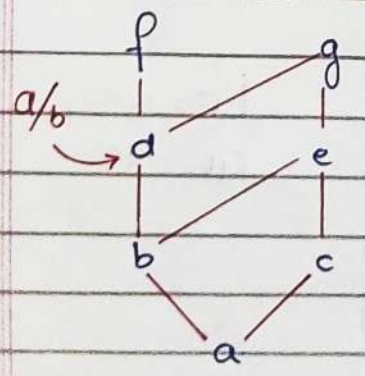
$B = \{e, f\}$

LB's of B is $\{e, c, a\}$

$e \leq ef$ | $c \leq ef$ | $a \leq ef$

$B_1 = \{g, f\}$ LB: $\{e, c, a\}$

$B_2 = \{e, b\}$ LB: $\{a\}$



<u>Greatest/lower Bound</u>: GLB

$\hookrightarrow x \in A \leq$ all elements of B

a/b can be unique

all lower bounds of B $\leq x \in A$

$B = \{f, g\}$ lower bound: $\{d, b, a\}$

d is a/b of $\{f, g\}$.

VIVEKANAND VERNEKAR

$(D_{12}, |)$

$a|b(2,3) = 1$

$a|b(2,6) = 2$

$a|b(3,6) = 3$

$a|b(2,12) = 2$

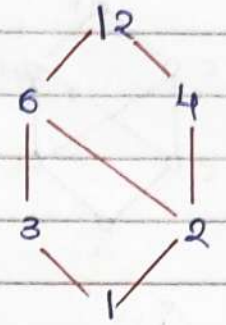
$\hookrightarrow /b$

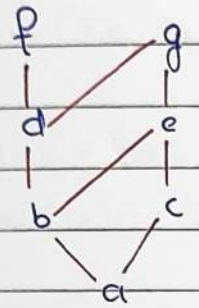
$2 \le 2 \mid 1 \le 2$

$2 \le 12 \mid 1 \le 12$

$lb: a|b \{1, 2\}$.

$\hookrightarrow a|b \rightarrow \{2\}$.





Least/Upper Bound : LUB

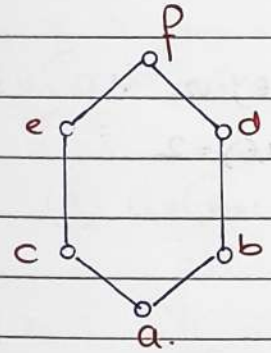
$\hookrightarrow$ all elements $\le x \in A$

$\hookrightarrow x \in A \le$ all UB's of B.

$B = \{a, c\}$

$\hookrightarrow$ U.B of $\{a, c\} = \{c, e, g\}$

$\hookrightarrow$ LUB: $\{c\}$.



(A, R) Lattice

$\begin{cases} a|b(a,b) = a \\ lub(a,b) = b \end{cases}$

$b \leftarrow$ lub

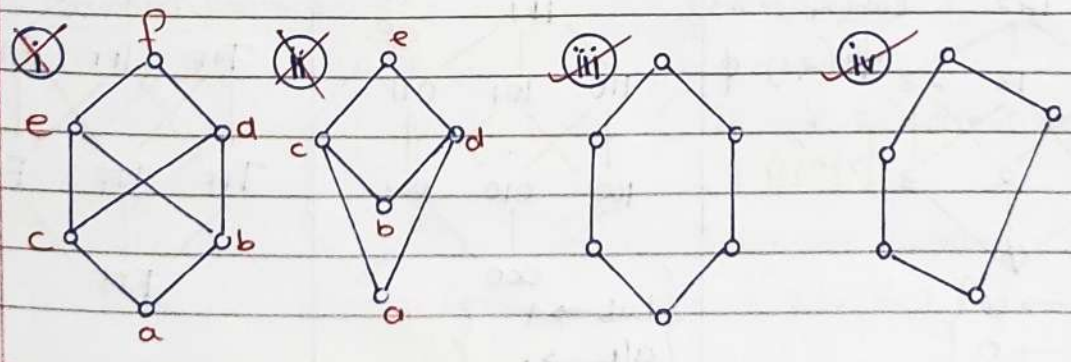
$a \leftarrow a|b$.

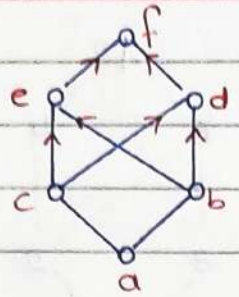
$lub(b,c) = f$ | $lub(e,b) = f$

$a|b(b,c) = a$ | $a|b(e,b) = a$

(A, R) Poset

$\hookrightarrow$ when a|b and lub exist for all pairs $\rightarrow$ (A, R) lattice.



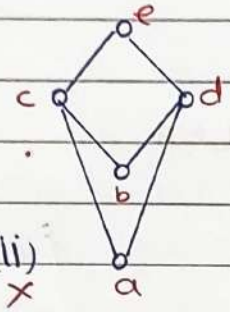


(i) ×

$lub(c,d) = NA$

$\hookrightarrow ub\{c,d\} \rightarrow \{e,d,f\}$

Check (e)	Check (d)	Check (f)
× $e \leq d f$	$d \leq e d f$	$f \leq e d f$
$e \leq e$ (T)	$d \leq e$ (F)	$f \leq e$ (F)
$e \leq d$ (F)		

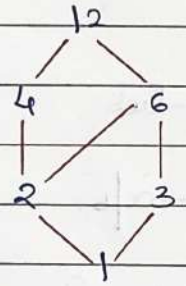


(ii) ×

$a|b(c,d) =$

$\hookrightarrow |b\{c,d\} \rightarrow \{a,b\}$

Check (a)	Check (b)
$ab \leq a$	$ab \leq b$
$a \leq a$ (T)	$ab \leq b$ (F)
$b \leq a$ (F)	



$(D_{12}, |)$

$lub(4,6) = 12$

$a|b(4,6) = 2$

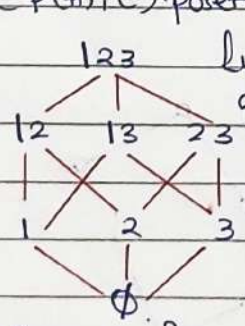
$\left\{ \begin{array}{l} lcm(4,6) = 12 \\ gcd(4,6) = 2 \end{array} \right.$

$$\boxed{\begin{array}{l} (D_n, |) \\ \quad a|b \rightarrow gcd \\ \quad lub \rightarrow lcm \end{array}}$$

$A = \{1,2,3\}$

$(P(A), \subset)$ Poset.

$\{2\} \cup \{3\} = \{23\}$



$lub(2,3) = 23$

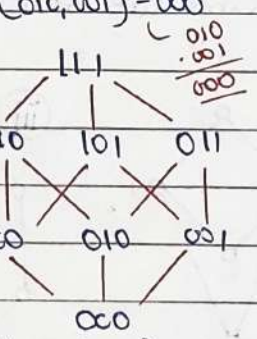
$a|b(2,3) = \phi$

$\{2\} \cap \{3\} = \phi$

$\left\{ \begin{array}{l} lub \rightarrow \cup \\ a|b \rightarrow \cap \end{array} \right\}$

$lub(010, 001) = 011$ (010, 001, 011)

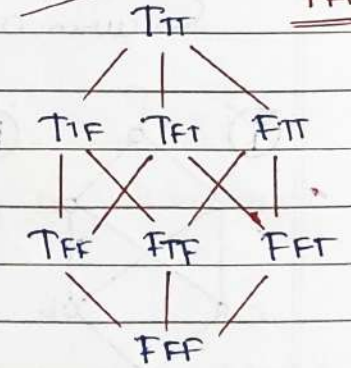
$a|b(010, 001) = 000$ (010, 001, 000)



$\left\{ \begin{array}{l} Lub \rightarrow + \\ a|b \rightarrow \cdot \end{array} \right\}$

FTF
∨ FFT
FTT

FTF
∧ FFT
FFF



VIVEKANAND VERNEKAR

$\vee$	$\cup$	$+$
$\wedge$	$\cap$	$\cdot$
T	$\vee$	1
F	ϕ	0

* Every TOSET is lattice (chain)

[L, ∨, ∧] lattice

(∨) lub
(∧) glb

1. $a \vee a = a \qquad a \wedge a = a$

   $lub(a,a) = a \qquad glb(a,a) = a$

2. $a \vee b = b \vee a \qquad a \wedge b = b \wedge a$

   $lub(a,b) = lub(b,a) \qquad glb(a,b) = glb(b,a)$

3. $a \vee (b \vee c) = (a \vee b) \vee c \qquad a \wedge (b \wedge c) = (a \wedge b) \wedge c$

   $lub(a, lub(b,c)) = lub(lub(a,b), c)$

4. $a \vee (a \wedge b) = a. \qquad a \wedge (a \vee b) = a.$

③ $a \vee (b \vee c) = (a \vee b) \vee c$

$a \vee (b \vee c) = (a \vee b) \vee c$

$\hookrightarrow lub(b,c) \qquad \hookrightarrow lub(a,b)$

$a \vee b \qquad\qquad b \vee c$

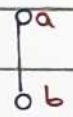
$lub(a,b) \qquad\qquad lub(b,c)$

$\hookrightarrow b \qquad\qquad \hookrightarrow b$

$a \vee (a \wedge b) = a \qquad a \wedge (a \vee b) = a.$

Case 1:

Case 2:

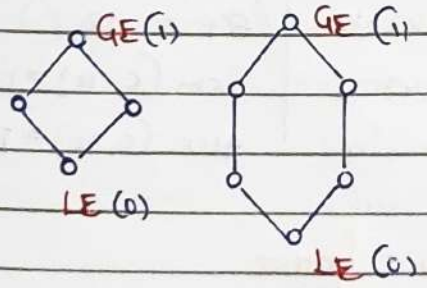


$a \vee (a \wedge b) = a$

$\hookrightarrow glb(a,b)$

$\hookrightarrow (a \vee a) = a.$

Bounded lattice:



Lattice [L, ∨, ∧]

↓

(G.E & L.E) are existing

↓

Bounded lattice

[L, ∨, ∧, 1, 0]

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$(Z^+, \leq)$ toset → lattice

↳ G.E → N.A

L.E → 1.

* Every finite lattice can be bounded lattice

$(Z^-, \leq)$ → Toset → lattice

↳ G.E → N.A

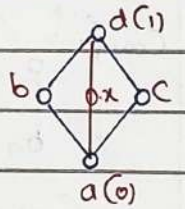
L.E → N.A

$a+b=1$	$a \vee b = G.E$	$1 \rightarrow GE$
$a \cdot b = 0$	$a \wedge b = L.E$	$0 \rightarrow LE$
a, b are compliment to each other		

<u>Compliment lattice</u> : $a+b=1$ | $lub(a,b) = GE(1)$

$a \cdot b = 0$ | $glb(a,b) = LE(0)$

↳ Bounded lattice, atleast 1 ~~element~~ Compliment exist for all elements.



$lub(b,x) = d$

$glb(b,x) = a$

$lub(\ ) = G.E$

$glb(\ ) = LE$

$b' = \{x, c\}$

$a' = d$.

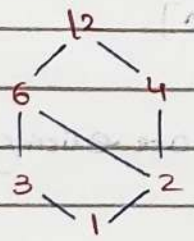
$lub(b,c) = d(1)$ $b^1 = c$

$glb(b,c) = a(0)$

$lub(a,d) = d(1)$ $a+d=1$

$glb(a,d) = a(0)$ $a \cdot d = 0$.

$(D_{12}, |)$ Compliment lattice?



$3' = ?$

$3+ = 12$ } $Lcm(3,4) = 12$

$3 \cdot = 1$ } $HCF(3,4) = 1$

$\therefore 3' = 4$

$6' = NA$

$6+ = 12$

$lcm(6, 12) = 12$ ✓

$HCF(6, 12) = 1$ ✗

$\therefore$ It is not compliment lattice.

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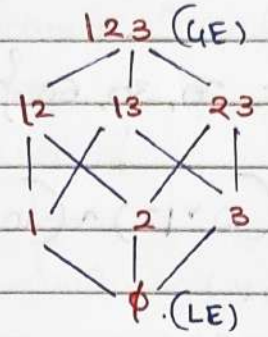
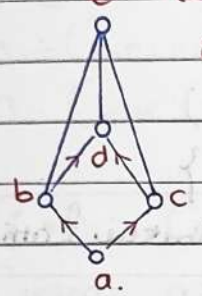
$A = \{1, 2, 3\}$
$(P(A), \subseteq)$



$a \cap b = 1$ $\quad a \cup b = GE$

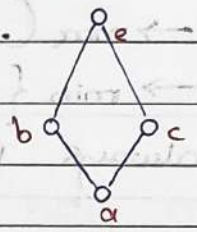
$\begin{cases} a \cup b = \{123\} \\ a \cap b = \phi \end{cases}$

$\{1\} \cup \{23\} = \{123\}$
$\{1\} \cap \{23\} = \phi$
$\{1\}' = \{23\}$



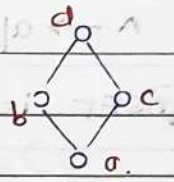
(1)

$lub(b,c) = d$
$glb(b,c) = a.$



(2)

$lub(b,c) = e$
$glb(b,c) = a.$



(3)

B is Sublattice of A

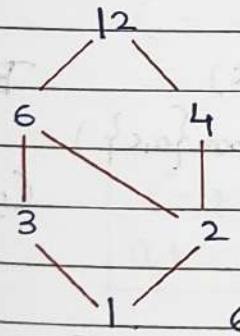
1) $B \subseteq A$
2) glb of lub of each pair of B must be same as in A

Distributive lattice:

$\forall a, \forall b, \forall c$ $\begin{cases} a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \checkmark \\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \checkmark \end{cases}$

$\wedge \rightarrow glb$
$\vee \rightarrow lub.$

$(D_{12}, |)$ Distributive Lattice / Not Compliment lattice



$6 \vee (4 \wedge 2) = (6 \vee 4) \wedge (6 \vee 2)$

$\downarrow$ $gcd(4,2)$ $\rightarrow$ $6 \vee 2$ $\rightarrow$ $lub(6,2)$

$lcm(6,4) \wedge lcm(6,2)$
$12 \wedge 6$
$gcd(6,12)$
$\rightarrow \underline{\underline{6}}$

$6' = NA.$ $\quad lcm(6,2) = 6.$

$(D_n, |) \rightarrow$ lattice
$\rightarrow$ distributive lattice
$\begin{cases} a|b \rightarrow gcd \\ lub \rightarrow lcm \end{cases}$

$(P(A), \subseteq)$
$\rightarrow$ Distributive lattice
$lub \rightarrow \cup \rightarrow \vee$
$a|b \rightarrow \cap \rightarrow \wedge$

eg: $(D_{60}, |)$

$(\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}, |)$

$10 \vee (12 \wedge 15) = (10 \vee 12) \wedge (10 \vee 15)$

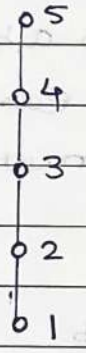
$\hookrightarrow a|b(12,15)$

$\hookrightarrow 10 \vee 3$

$\hookrightarrow lcm(10,3) = 30$

$(\{1,2,3,4,5\} \leq)$ $\vee \rightarrow lub \rightarrow max(\ldots,\ldots)$

$\wedge \rightarrow a|b \rightarrow min\{\ ,\ \}$



$\boxed{\text{Every POSET is always Distributive lattice.}}$

$lub(1,2) = max(1,2) = 2$

$a|b(1,2) = min(1,2) = 1$

$1 \vee (2 \wedge 3) = (1 \vee 2) \wedge (1 \vee 3)$

$max(1, min(2,3)) = (max\{1,2\}, max\{1,3\}) min$

$\hookrightarrow max(1,2) = 2 = min(2,3) = 2$

$\vee \rightarrow lub \rightarrow max$

$\wedge \rightarrow a|b \rightarrow min$

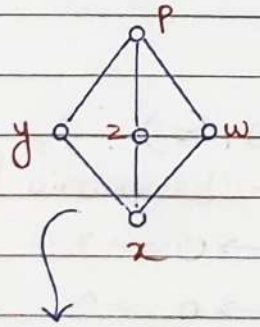
G.L.B / L.B.

$a \vee (b \wedge c)$ $\quad (a \vee b) \wedge (a \vee c)$

$max(a, min\{b,c\})$ $\quad min(max\{a,b\}, max\{a,c\})$

Bounded

Compliment



$y \wedge (z \vee w) = (y \wedge z) \vee (y \wedge w)$

$lub(z,w)$ $\quad a|b(y,z) \vee a|b(y,w)$

$\hookrightarrow y \wedge p$ $\quad x \vee x$

$a|b(y,p)$ $\quad x$

y

Not distributive lattice.

$y \neq x$

Compliment ✓

distributive ✗

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Boolean Algebra = lattice ^ Compliment ^ Distributive.

lattice	Compliment	Distributive
$a+a=a$	$a+b=1$	$a+(b\cdot c)=(a+b)(a+c)$
$a\cdot a=a$	$a\cdot b=0$	$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$
$a+b=b+a$		
$a\cdot b=b\cdot a$		

$a+(b+c)=(a+b)+c$

$a\cdot(b\cdot c)=(a\cdot b)\cdot c$

$a+(b\cdot c)=a$

$a\cdot(b+c)=a$

→ $(G,*)$ is Group

1. Closed: $a\in G,\ b\in G,\ a*b\in G$
2. Associative: $a*(b*c)=(a*b)*c$
3. Identity: $a*e=a$.
   ↳ identity.

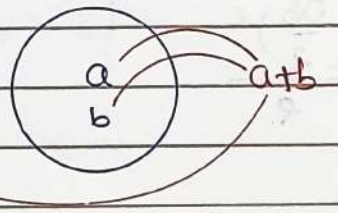
$*$ is any random operation.

Closed:

$(G,*)$ → $a\in G,\ b\in G\ \ a*b\in G$

$(Z,+)$ → $a\in Z,\ b\in Z,\ a+b\in Z$

Set: Z



$\boxed{\begin{array}{l}(Z,+)\\ a+0=a\end{array}}$

↳ identity.

Associative:

$a+(b+c)=(a+b)+c.$

$(D_{12}, a|b)$ $(D_n,1)$ (Set Operation)

$a\in D_{12},\ b\in D_{12}$

$\gcd(a,b)\in D_{12}$

$(\{1,2,3,4,6,12\}\ a|b)$ → Associative

$(\{1,2,3,4,6,12\}, \gcd)$

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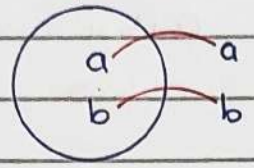
→ $(G, *)$

$(Z, 1)$

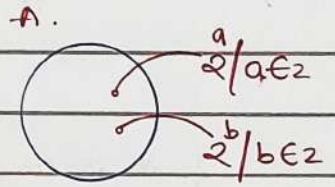
Set: Z



$a/b \notin Z$

→ $(G, *)$

$(A, \times)$ $\quad A = \{2^n \mid n \in Z\}$



$\boxed{2^a \times 2^0 = 2^a}$

→ Identity

Associative:

$2^a \cdot (2^b \cdot 2^c) = (2^a \cdot 2^b) \cdot 2^c$

$2^a (2^{b+c})$

$2^{a+b+c} = 2^{a+b+c}$

$2^a \times 2^b$

$\left\{ 2^{a+b} \;\middle|\; \begin{matrix} a \in Z \\ b \in Z \end{matrix} \; a+b \in Z \right\}$

→ $(R, *)$ $\quad a * b = \frac{ab}{2}$

$a * (b * c) = (a * b) * c$

$a * \left(\frac{bc}{2}\right) \quad \left(\frac{ab}{2}\right) * c$

$\frac{a\left(\frac{bc}{2}\right)}{2} \quad \frac{\left(\frac{ab}{2}\right)c}{2}$

Associative:

$a * e = a$

$\frac{ae}{2} = a$

$\boxed{e = 2}$

→ unique

→ $(A, \times)$

$A = \{1, \omega, \omega^2\}$

$\omega^3 = 1$

$1 \times (\omega \times \omega^2) = (1 \times \omega) \times \omega^2$

$1 \times \omega^3 = \omega \times \omega^2$

$1 \times 1 = \omega^3$

$1 = 1$

$\boxed{\begin{matrix} \omega \times 1 = \omega \\ \omega^2 \times 1 = \omega^2 \end{matrix}}$

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→ $(A, \times)$ $\qquad i^2 = -1$.

$A = \{1, i, -i, -1\}$

$\boxed{i \times 1 = i}$

Closed.

Associative.

$1 \times (i \times -i) = (1 \times i) - i$

$1 \times (-1)^2 = i \times -i$

$1 \times (-(-1)) = -i^2$

$1 = -(1)$

$1 = 1$.

→ (Set, Operation) $\quad A = [\ ]_{2\times2} \quad B = [\ ]_{2\times2}$ $\quad \boxed{A \times B \neq B \times A}$

$(M_{2\times2}, \times) \quad A \times B = [\ ]_{2\times2} \in M_{2\times2}$.

$(\{[\ ]_{2\times2}, [\ ]_{2\times2} \ldots\}, \times)$ $\quad A.(B.C) = (A.B).C$ $\quad \boxed{A \times I = A}$

→ (Set, *)

$(\{0,1,2,3,4,5\}, \oplus_6)$

Operation: addition mod 6

$\oplus_6$

$a \oplus_6 b = (a+b) \bmod 6$

$2 \oplus_6 3 = (2+3) \bmod 6 = 5 \bmod 6 = 5$.

$3 \oplus_6 (4 \oplus_6 5) = (3 \oplus_6 4) \oplus_6 5$

$3 \oplus_6 (3) = (1) \oplus_6 5$

$0 = 0$ (Associative)

$\boxed{3 \oplus_6 0 = 3}$

→ $A = \{1, 2, 3\}$

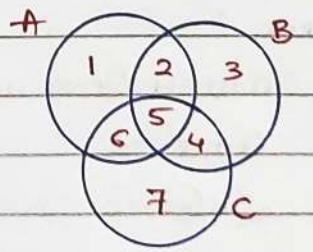
$(P(A), \Delta)$ or $(2^A, \Delta)$

or $\qquad$ or

$(P(A), \oplus)$. $\qquad (2^A, \oplus)$.

$A \oplus B = (A \cup B) - (A \cap B)$

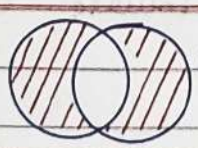
$A \Delta (B \Delta C) = (A \Delta B) \Delta C$.



$\boxed{A \oplus \phi = A}$

$\boxed{(A \cup \phi) - (A \cap \phi) = A}$

$\boxed{A \oplus B \quad (A \cup B) - (A \cap B)}$



VIVEKANAND VERNEKAR

$A = \{1, 2, 3\}$.

$1 \to 1$	$1 \to 1$	$1 \to 2$	$1 \to 2$	$1 \to 3$	$1 \to 3$
$2 \to 2$	$2 \to 3$	$2 \to 1$	$2 \to 3$	$2 \to 1$	$2 \to 2$
$3 \to 3$	$3 \to 2$	$3 \to 3$	$3 \to 1$	$3 \to 2$	$3 \to 1$
f_1	f_2	f_3	f_4	f_5	f_6

(Set, Operation)
$(\{f_1, f_2, f_3, f_4, f_5, f_6\}, \circ)$

$f_5 \circ f_1 = f_5$.

(Set, Operation)
$(\{f_1, f_2, f_3, f_4, f_5, f_6\}, \circ)$

Closed

$f_2 \in A$

$f_3 \in A$

$f_2 \circ f_3 = f_5$

$1 \to 2 \to 3$
$2 \to 1 \to 1$
$3 \to 3 \to 2$

$f_3 \circ f_2$

$f_2 \circ (f_3 \circ f_5) = (f_2 \circ f_3) \circ f_5$

$1 \to 3 \to 3$
$2 \to 1 \to 2$
$3 \to 2 \to 1$

$f_2 \circ f_6$

Group $(G, *)$

1.> Closed $\to a \in G,\ b \in G,\ a * b \in G$
2.> Associative $\to a * (b * c) = (a * b) * c$
3.> Identity $\to a * e = a$
4.> Inverse $\to a * a^{-1} = e$.

$(Z, +)$ ✓	$(Z, \times)$ × (not group)
1. Closed	1. Closed
2. Associative	2. Associative
3. Identity $a + 0 = a$ $(e = 0)$ unique	3. Identity $a \times 1 = a$
4. Inverse $a * a^{-1} = e$	4. Inverse $a \times \frac{1}{a} = 1$.
$2 + (-2) = 0$	

VIVEKANAND VERNEKAR

$(Q, \times)$ $\quad (Q \neq 0, \times) \rightarrow$ Group

1. Closed
2. Associative
3. $a \times 1 = a$ (identity)
4. $a \times \frac{1}{a} = 1$ $(a \neq 0)$

Group

Infinite Groups: $(Z, +)$, $(Q \neq 0, \times)$

Finite Groups: $(\{1, \omega, \omega^2\}, \times)$, $(\{1, i, -i, -1\}, \times)$

Cayley Table

*	e_1	e_2	.....
e_1			
e_2			
⋮			

$\omega^3 = 1$

$(\{1, \omega, \omega^2 \mid \times\})$

$\omega^2 \times 1 = \omega^2$.

$1' = 1$
$\omega' = \omega^2$
$\omega^{2'} = \omega$.

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

1. Closed
2. Associative
3. Identity
4. Inverse

$1 \rightarrow 1$	$1 \rightarrow 1$	$1 \rightarrow 2$	$1 \rightarrow 2$	$1 \rightarrow 3$	$1 \rightarrow 3$
$2 \rightarrow 2$	$2 \rightarrow 3$	$2 \rightarrow 1$	$2 \rightarrow 3$	$2 \rightarrow 1$	$2 \rightarrow 2$
$3 \rightarrow 3$	$3 \rightarrow 2$	$3 \rightarrow 3$	$3 \rightarrow 1$	$3 \rightarrow 2$	$3 \rightarrow 1$
f_1	f_2	f_3	f_4	f_5	f_6

o	f_1	f_2 .....
f_1		
f_2		
⋮		

f_4 of $f_5 = f_1$

2	3	1
3	1	2
1	2	3.

$\oplus_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$(\{0, 1, 2, 3, 4, 5\} \oplus_6)$

1. Closed ✓
2. Associative ✓
3. Identity ✓
4. Inverse ✓

$0' = 0$ $\quad 2' = 4$ $\quad 4' = 2$

$1' = 5$ $\quad 3' = 4$ $\quad 5' = 1$

Subgroup: H is called sub group of G if

(i) $H \subseteq G$

(ii) H should also satisfy:

a. Closed c. Identity

b. Associativity d. Inverse

eg: $\{\{0,1,2,3,4,5\} \oplus 6\}$ $H = \{1,3,5\}$

→ $H \subseteq G$ ✓

→ identity ×

↳ not subgraph because identity element is absent.

eg: $\{\{0,1,2,3,4,5\} \oplus 6\}$ $H = \{0,3,5\}$

→ $H \subseteq G$ ✓ – not group

→ Closed × – not closed

– not sub group.

eg: $\{\{0,1,2,3,4,5\} \oplus 6\}$ $H = \{0,2,4\}$.

→ $H \subseteq G$ ✓

→ H is also Group.

	0	2	4
0	0	2	4
2	2	0	4
4	4	0	2

Theorem: if H is Subgroup of G, then $\frac{|G|}{|H|}$. And vice versa not true.

→ $H = \{0,2,4\}$ $|G| = 6$

H is subgroup of G ⟶ $\frac{|G|}{|H|} = \frac{6}{3}$

$|H| = 3$

Every group contain 2 trivial subgroup:

1) $e \leq G$

2) $G \leq G$.

Closed

associativity

identity $e*e=e$

inverse $e*e=e$

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Q1. If $|G| = 84$, then what will be maximum size of subgraph? $\rightarrow 84$
- what will be the maximum size of proper subgraph? $\rightarrow 42$

G be the group with subgroup H & K. $|G| = 660$ $|K| = 66$

$$K \subset H \subset G$$
$$66 \quad\quad 660$$

$$66 \begin{cases} 66 \times 2 \\ \text{or} \\ 66 \times 5 \end{cases} \quad (66 \times 10)$$

*Subgroup of Cyclic group is also cyclic group.

$2^1 = 2$

$2^2 = 2 \oplus_6 2 = 4$

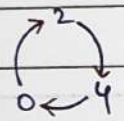
$2^3 = \underline{2 \oplus_6 2} \oplus_6 2 = 0$

$4 \oplus_6 2$. $\{0, 2, 4\}$ — Subgroup.

$= 0$

$2^4 = 0 \oplus_6 2 = 2$, 2 has generated $\{0, 2, 4\}$

$\langle 2 \rangle = \{0, 2, 4\}$



$3^1 = 3$

$3^2 = 3 \oplus_6 3 = 0$

$3^3 = 0 \oplus_6 3 = 3$

$\langle 3 \rangle = \{0, 3\}$

$1^1 = 1$

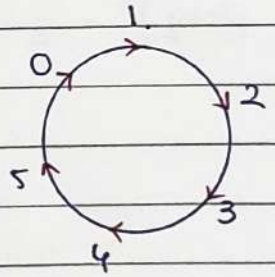
$1^2 = 1 \oplus_6 1 = 2$ $\quad \langle 1 \rangle = \{0, 1, 2, 3, 4, 5\}$.

$1^3 = 2 \oplus_6 1 = 3$

$1^4 = 3 \oplus_6 1 = 4$

$1^5 = 4 \oplus_6 1 = 5$

$1^6 = 5 \oplus_6 1 = 0$



$5^1 = 5$

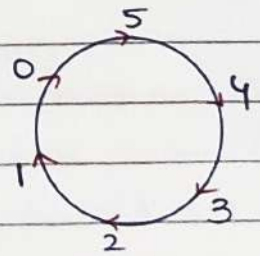
$5^2 = 5 \oplus_6 5 = 4$ $\quad \langle 5 \rangle = \{5, 4, 3, 2, 1, 0\}$

$5^3 = 4 \oplus_6 5 = 3$

$5^4 = 3 \oplus_6 5 = 2$

$5^5 = 2 \oplus_6 5 = 1$

$5^6 = 1 \oplus_6 5 = 0$



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* In previous example '1' has generated all value/elements in the group.
* 1 is called "Generator".
* Group → Generator → Cyclic Group.

Group + Commutative = Albelian Group.

Closed	Associative	Identity	Inverse.
Algebric Structure.			

Semi Group.

Monoid

Group.

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COMBINATORICS

Product Rule:

Given a Task.

$n_1 \rightarrow T_1 \quad T_2 \leftarrow n_2$

$K=0$	$K=n_1 \times n_2$
for $i=1$ to n_1	$i=1, j=1-n_2$
for $j=1$ to n_2	$i=2, j=1-n_2$
$K=K+1$.	⋮ ⋮
	$j=1-n_2, i=n_1$

n_1 ways to do task T_1

n_2 ways to do task T_2.

each of n_1 ways we can perform n_2 both the tasks are happening simultaneously.

Q1. How many code we can generate if we have 2 bits?

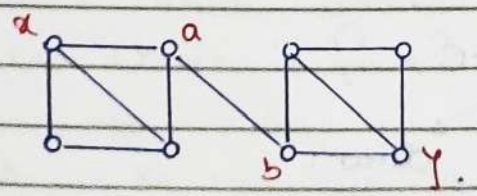
T_1: 0, 1 $\quad$ T_2: 0, 1.

00
01
10
11

0 → 0, 1
1 → 0, 1

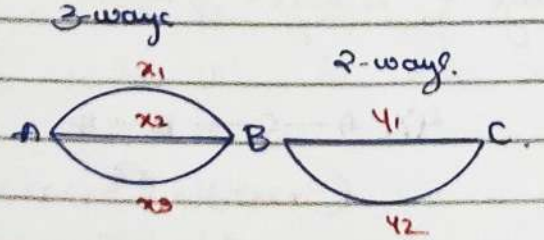
Q2. How many ways we can generate a number plate if number plate consist of first 4 characters followed by 2 digits.

$26 \times 26 \times 26 \times 26 \times 10 \times 10 = 26^4 \times 10^2$.



$x \rightarrow a \rightarrow b \rightarrow y$

3 ways $\times 1 \times 3 = 9$ paths from $x \rightarrow y$.



$A \rightarrow C$

$3 \times 2 = 6$ ways.

Sum Rule:

Task T

n_1 ways $\rightarrow T_1 \quad T_2 \leftarrow n_2$ ways.

Both tasks not happening simultaneously.

$K=0$. for $i=1$ to n_1 : $K=K+1$
for $i=1$ to n_2 : $K=K+1$ $\Big\}$ $K=n_1+n_2$

Q3. How many number plates we can generate if 4 characters followed by 2 digit or 3 digit?

— — — — | — — (OR) — — — — | — — —

$$26^4 \cdot 10^2 + 26^4 \cdot 10^3 = 26^4(10^2+10^3)$$

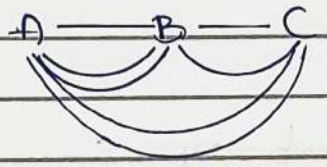
Q4. How many ways we can generate if one or two or 3 or 4 characters followed by 1 or 2 or 3 or 4 digit?

1C	1
(or)	(or)
2C	2
(or)	(or)
3C	3
(or)	(or)
4C	4.

$$26(10+10^2+10^3+10^4)$$
$$+$$
$$26^2(10+10^2+10^3+10^4)$$
$$+$$
$$26^3(10+10^2+10^3+10^4)$$
$$+$$
$$26^4(10+10^2+10^3+10^4)$$

$$\Longrightarrow (26+26^2+26^3+26^4) \times (10+10^2+10^3+10^4)$$

Q5



1) A — B — C : 3 ways × 2 ways = 6 ways

2) A — C

via B direct :

6 ways + 2 ways = 8 ways.

3) A — B — C — B — A

$3 \times 2 \times 2 \times 3 = 6 \times 6$

$= 36$ ways.

4). A — C — B — A

$(3\times2+2) \times (3\times2+2)$

$= 8 \times 8 = 64$ ways

Q6 How many 4 digit integer contains exactly one 6.

$\underline{6}\ _\ _\ _ + \underline{\text{not } 6}\ _\ _\ _ + \underline{\text{not } 6}\ _\ _\ _ + \underline{\text{not } 6}\ _\ _\ _\ \underline{6}$

$9 \cdot 9 \cdot 9 + 8 \cdot 9 \cdot 9 + 8 \cdot 9 \cdot 9 + 8 \cdot 9 \cdot 9 = 9 \cdot 9 \cdot 9 + 3 \cdot (8 \cdot 9 \cdot 9)$

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n-objects → r places.

$n \times (n-1) \ldots \times (n-R+1) \left\{ \frac{(n-R)!}{(n-R)!} \right.$

$^nP_r \Rightarrow \frac{n!}{(n-R)!}$ } Permutations.

eg: {a, b, c}

→ {a, b}

→ {b, c}

→ {a, c}

* This is an example of combination

$^nC_r \Rightarrow \frac{n!}{r!\,(n-r)!}$

Combination: Selecting 2 elements is same as shifting 2 lines.

{a, b, c}.

→ Place C line → $^3C_2$

110 – {a, b} → {|, |, –}

011 – {b, c} → {–, |, |}

101 – {a, c} → {|, –, |}

eg. A man which contains 3 containers, A, P, O fruits how many ways can he can choose 4 fruits?

4 A	3A	1O	2A	2O	2A	1O	1P
4 O	3A	1P	2A	2P	2O	1A	1P
4 P.	3O	1A	2P	2O.	2P	1A	1O
	3O	1P					
	3P	1O					
	3P	1A.					

When done manually.

3 + 6 + 3 + 3 ⇒ 15

3A1O – { A̅ A̅ A̅ | O̅ | }

2A2O – { A̅ A̅ | O̅ O̅ | } A | O | P.

⋮ ⋮

4 fruits + 2 line = 6 places., line = 2.

∴ $^6C_2 = 15$

Places C line.

Q1. How many ways we can distribute 4 chocolate among 2 girls?

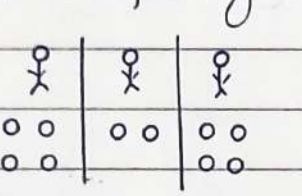
1	2
4	0
3	1
2	2
1	3
0	4.

5 ways.

$\longrightarrow \{ \bar{o}\,\bar{o}\,\bar{o}\,\bar{o} \mid \bar{o} \}$ Place 1 line

$\Rightarrow {}^5C_1 \Rightarrow \underline{\underline{5}}$

Q2. How many ways we can distribute 10 chocolate among 3 student?



$\bar{o}\,\bar{o}\,\bar{o}\,\bar{o} \mid \bar{o}\,\bar{o} \mid \bar{o}\,\bar{o}\,\bar{o}\,\bar{o}$

${}^{12}C_2 = \underline{\underline{66}}$ ways

How many non negative solutions are possible:

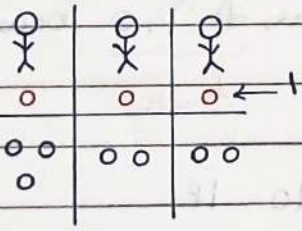
$x_1 + x_2 + x_3 = 10 \quad x_i \geq 0$

$x_1 = 4 \quad x_2 = 2 \quad x_3 = 4$

$x_1 = 3 \quad x_2 = 3 \quad x_3 = 4 \quad$ Ans $= \underline{\underline{{}^{12}C_2}}$

$\vdots \quad \vdots \quad \vdots$

Q3. How many ways we can distribute 10 chocolate among 10 students, each child gets atleast 1 chocolate?



← 1 chocolate already given.

$10 - 3 = 7.$

$\Rightarrow \underline{\underline{{}^{7+2}C_2}}$

– how many non-negative solutions?

$x_1 + x_2 + x_3 = 10 \quad x_i \geq 1$

$y_i = x_i - 1 \qquad \underbrace{x_i - 1}_{y_i} \geq 0$

$y_i + 1 = x_i$

$y_1 + 1 + y_2 + 1 + y_3 + 1 = 10 \quad y_i \geq 0$

$y_1 + y_2 + y_3 = 10 - 3 \quad y_i \geq 0$

$= \underline{7}$

– If 20 chocolate are there

$x_1 + x_2 + x_3 = 20$

$x_1 \geq 2 \qquad x_2 \geq 3 \qquad x_3 \geq 4$

$\underbrace{x_1 - 2}_{y_1} \geq 0 \qquad \underbrace{x_2 - 3}_{y_2} \geq 0 \qquad \underbrace{x_3 - 4}_{y_3} \geq 0$

$y_1 + 2 = x_1, \quad y_2 + 3 = x_2, \quad y_3 + 4 = x_3$

$y_1 + 2 + y_2 + 3 + y_3 + 4 = 20$

$y_1 + y_2 + y_3 = 20 - 9$

$= \underline{11}$

$\therefore$ Ans $= {}^{11+2}C_2 \Rightarrow \underline{\underline{{}^{13}C_2}}$

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- If 40 chocolate are there and 10 students, 10 chocolate to be distributed among first 3 students.

$x_1 + x_2 + x_3 \ldots\ldots x_{40} = 40 \qquad x_i \geq 0$

$x_1 + x_2 + x_3 = 10$

$$\underbrace{\boxed{x_1 + x_2 + x_3}}_{^{10+2}C_2} + \underbrace{\boxed{x_4 + x_5 \ldots\ldots + x_{10}}}_{^{30+6}C_6} = 40$$

$$\boxed{\therefore \text{Ans} = {}^{12}C_2 \times {}^{36}C_2}$$

Q4. How many ways we can distribute m identical balls among n boxes such that each box contains atleast k balls $(m \geq kn)$?

Ans.

$K + K + K \ldots = n.K$

after distributing

Remaining balls $= m - nK$

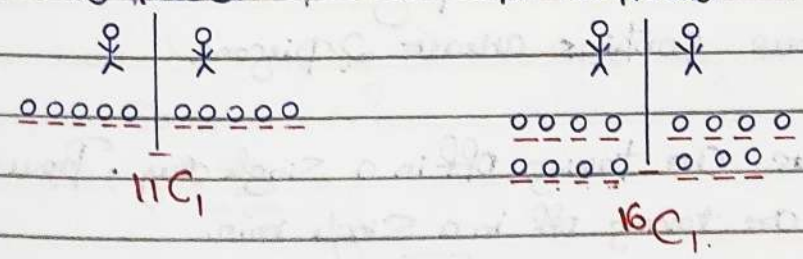
lines $= n-1$

$$\text{Ans} = {}^{m-nK+n-1}C_{n-1}.$$

Q5. How many ways we can distribute 10 Roses, 15 daffodils among 2 girls?

Ans

10 Roses, 15 Daffodils

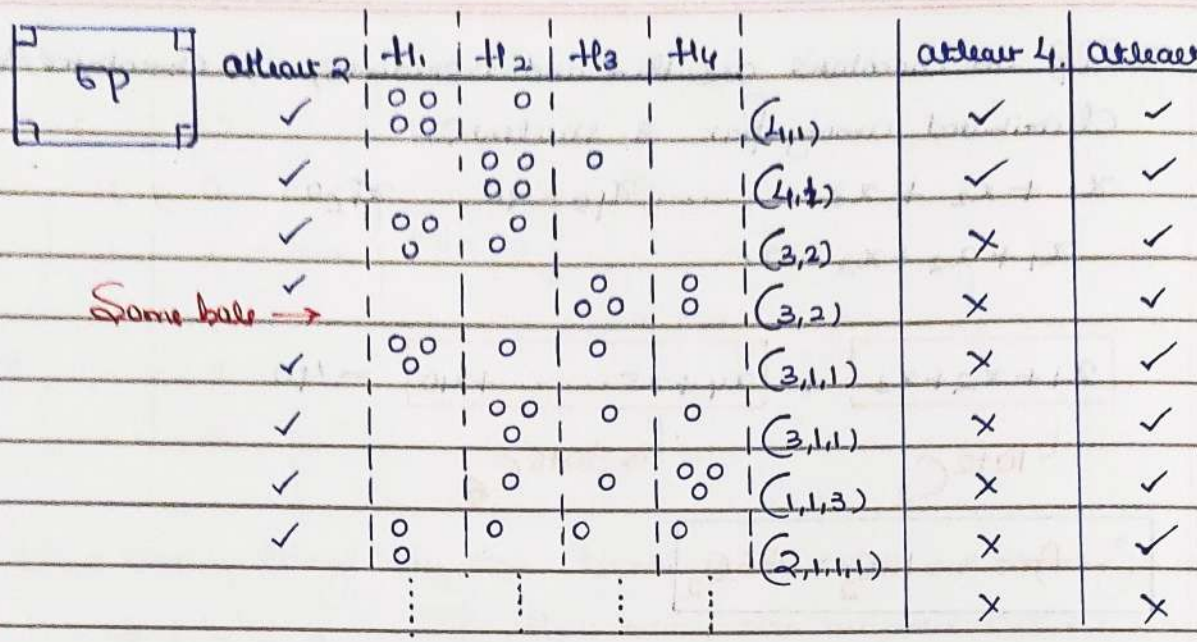


$$\boxed{\text{Ans} = {}^{11}C_1 \times {}^{16}C_1}$$

$\rightarrow x_1 + x_2 + x_3 \leq 10, \quad x_i \geq 0$

$x_1 + x_2 + x_3 = 0$	$x_1 + x_2 + x_3 + a = 10$ ($a \geq 0$)
$x_1 + x_2 + x_3 = 1$	$x_1 + x_2 + x_3 = 9, \; a = 1$
⋮	8, $a = 2$
10.	3 ⋮

$\Rightarrow {}^{10+3}C_3 \Rightarrow {}^{13}C_3$

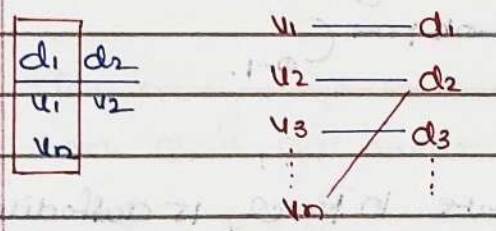


Theorem: In simple graph atleast 2 vertices will have same degree. $(n \geq 2)$.

Assumption: Let's take all vertices will have different degrees.

Degree possibilities : $\{1, 2, 3, 4 \ldots n-1\}$ Total different degrees $= n-1$.

Total vertices $= n$



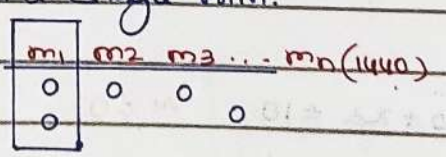
* Our assumption is wrong! and so is theorem.

Note: ** If we have $n+1$ pigeons, n holes, then some of the holes contains atleast 2 pigeons.

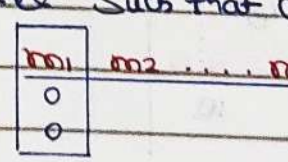
Q1. 1500 flights are taking off in a single day. Prove that atleast 2 flights are taking off in a single min.

Ans 1 Day = 24 hrs

$24 \times 60 = 1440$ minutes.



Q2. What will be min no of students required such that atleast 2 students will have same month birthday.



Q3. What will minimum no. of Students require? Such that atleast 6 will have Same grade in School, where it has five different grade System.

Ans

	A	B	C	D	E
atleast 6	o	o	o	o	o
	o	o	o	o	o
	o	o	o	o	o
	o	o	o	o	o
	o	o	o	o	o
	o				

$5 \times 5 + 1$

$\Rightarrow \underline{\underline{26}}$

Q4. What will be the minimum no. of cards will be drawn such that atleast 3 belongs to Same Suite?

	H	D	S	C
atleast 3.	o	o	o	o
	o	o	o	o
	o			

$2 \times 4 + 1$

$\Rightarrow \underline{\underline{9}}$

Q5. In a graph of 9 vertices, Prove that Some of the vertices will contain atleast 4 degrees if sum of degrees of all vertices is atleast 27.

Ans

	v_1	v_2	....	v_9
atleast 4.	o	o		o
	o	o	.....	o
	o	o	....	o
	o			

$9 \times 3 = 27$

$\sum d(v_i) = \text{even}$

$\sum d(v_i) = \text{atleast } 27$

Q6. What will be min no. of Students require atleast 9 will be born on Same month?

Ans $12 \times 8 + 1 \Rightarrow \underline{\underline{97}}$

Q7. A bag contains 6 Red, 8 blue, 10 Green, 15 white, 20 yellow min no of balls Such that atleast 6 balls of Same color?

Ans

	6R	8B	10G	15W	20Y
atleast 6	o	⋮	⋮	⋮	⋮
	o	⋮	⋮	⋮	⋮
	o	⋮	⋮	⋮	⋮
	o	⋮	⋮	⋮	⋮
	o	⋮	⋮	⋮	⋮
	o				

$5 \times 5 + 1 \Rightarrow \underline{\underline{26}}$

Q8 A bag contains 6 Red, 8 Blue, 10 Green, 15 White, 20 Yellow, min no. of balls such that atleast 9 balls of same color?

Ans

6R	8B	10G	15W	20Y
6	6	6	6	6

$6+5+8+1 \Rightarrow \underline{\underline{39}}$

atleast 9.

Q9. $S=\{1,2,3 \dots 24,25\}$ What min be no. of elements we min choose, such that one element divides another?

Ans. $13+1 \Rightarrow \underline{\underline{14}}$

→ if $S=\{1 \dots 200\}$ $\Rightarrow \left\lceil \frac{n}{2} \right\rceil + 1$

Ans: 101.

In such questions apply

Ceiling of $n/2+1$

Q10. $S=\{0,1 \dots 9\}$ What is the smallest positive integer k such that any subset of S of size k contains two distinct subset of size two $\{\{x_1,x_2\},\{y_1,y_2\}\}$

Ans $\{\{x_1,x_2\},\{y_1,y_2\}\}$ $[x_1+x_2 = y_1+y_2 = 9]$

$\{\{0,9\},\{1,8\}\}$ $k=4$.

Q11. How many elements are divisible by 2 ~~and~~ or 5 in set $\{1,2 \dots 10\}$?

Ans $P(2)$: no. of elements ÷ by 2 $= \left\lfloor \frac{10}{2} \right\rfloor = \underline{5}$

$P(5)$: no. of elements ÷ by 5 $= \left\lfloor \frac{10}{5} \right\rfloor = \underline{2}$

$P(2 \text{ or } 5) = P(2)+P(5)-P(2 \cap 5)$ $\{1,2,3 \dots 9,10\}$

$= 5+2-\left\lfloor \frac{10}{10} \right\rfloor$

$= 7-1$

$= \underline{\underline{6}}$

Counted by both 2 & 5 therefore $-(2 \cap 5)$.

$\rightarrow |A_1 \cup A_2 \cdots \cup A_n| = \sum n_i - \sum A_i \cap A_j + \sum A_i \cap A_j \cap A_k + (-1)^{n+1} |A_i \cap A_j \cdots A_n|$.

Q12 No. of elements which are not divisible by 2, 3 or 5 in over $\{1 \ldots 123\}$?

Ans 33

Q13 How many elements are ÷ by 4, 6, or 8 in a set $\{1 \ldots 100\}$?

Ans $n(4 \text{ or } 6 \text{ or } 8) = n(4) + n(6) + n(8) - n(4 \cap 6) - n(4 \cap 8) - n(6 \cap 8) + n(4 \cap 6 \cap 8)$

$$\Rightarrow \left\lfloor \frac{100}{4} \right\rfloor + \left\lfloor \frac{100}{6} \right\rfloor + \left\lfloor \frac{100}{8} \right\rfloor - \left\lfloor \frac{100}{L(4,6)} \right\rfloor - \left\lfloor \frac{100}{L(4,8)} \right\rfloor - \left\lfloor \frac{100}{L(8,6)} \right\rfloor - \left\lfloor \frac{100}{L(4,6,8)} \right\rfloor \Rightarrow 333$$

Lcm.

Q14 A, B, C, D

$n(A) = 42$

$n(B) = 36$

$n(C) = 28$

$n(D) = 24$.

Intersection of any two of the 4 sets contains 12 elements

Intersection of any three of the contains 8 elements.

$A \cup B \cup C \cup D = A + B + C + D - {}^4C_2 + {}^4C_3 - 4$

$\times 12 \quad \times 8$

$\Rightarrow 130 - 6 \times 12 + 4 \times 8 - 4 \Rightarrow 86$

eg:

$A_1: x_1 + x_2 + x_3 = 11 \quad x_1 \geq 4$	$A_2: x_1 + x_2 + x_3 = 11$	$A_3: x_1 + x_2 + x_3 = 11 \quad x_3 \geq 7$
$7 + 2C_2 = 9C_2$	$x_2 \geq 5$	$4 + 2C_2 = 6C_2$
	$6 + 2C_2 = 8C_2$	

$A_1 A_2 . \; x_1 + x_2 + x_3 = 11$	$A_1 A_3 \quad x_1 + x_2 + x_3 = 11$	$A_2 A_3 \quad x_1 + x_2 + x_3 = 11$
$x_1 \geq 4$, $x \geq 5$ $\Big\}$ $2 + 2C_2 = 4C_2$	$x_1 \geq 4$, $x_3 \geq 7$ $\quad 0 + 2C_2 = 2C_2$	$x_2 \geq 5 \quad x_3 \geq 7$
		Ans = 0.

$A_1 . A_2 . A_3 .$	* $x_1 + x_2 + x_3 = 11 \quad \{x_1 \geq 4, x_2 \geq 5, x_3 \geq 7\}$. $\quad {}^{13}C_2 - [\text{Ans}]$
Ans = 0	$A_1 \cup A_2 \cup A_3 = A_1 + A_2 + A_3 - A_1A_2 - A_2A_3 - A_1A_3 + A_1A_2A_3$
	$= 9C_2 + 8C_2 + 6C_2 - 4C_2 - 2C_2 - 0 + 0$
	$\Rightarrow 72$

Total $= {}^{13}C_2 \quad \therefore \quad 13C_2 - 72 \Rightarrow 78 - 72 \Rightarrow 6$

Derangement : Nothing is at right place (D_n). $(n \geq 1)$	$D_1 = 0$
	$D_2 = 1 \left\{ \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \right.$ Only possibility of derangement

$D_3 = 3! - (\text{atleast 1 element @ right})$

$D_4 = 4! - (\text{atleast 1 element @ right})$

$= \text{Total Arrangement} - (A_1 \cup A_2 \cup A_3 \cup A_4)$

$A_1 \cup A_2 \cup A_3 \cup A_4 = A_1 + A_2 + A_3 + A_4 - \sum A_i \cap A_j + \sum A_i \cap A_j \cap A_k - |A_1 \cap A_2 \cap A_3 \cap A_4|$

$$= 4! - \left(4! - \frac{4!}{2!} + \frac{4!}{3!} - \frac{4!}{4!}\right) \Rightarrow \underline{\underline{9}}$$

$$\boxed{D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \frac{(-1)^n}{n!}\right]}$$

$D_1 = 0$ $\quad D_5 = 44$

$D_2 = 1$ $\quad D_6 = 265$

$D_3 = 2$

$D_4 = 9$

$$\boxed{D_n \simeq 0.36 \times n! \quad (n \geq 7)}$$

expansion

$$e^x = 1 + \frac{(x)^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$e = 2.7$

$1/e = 0.36$

$e^{-1} = 0.36$

$x = -1$

$$e^{-1} = 1 + \frac{(-1)}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} \cdots$$

$$0.36 = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5} \cdots$$

$$0.36 \simeq \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5} \cdots$$

$$D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \frac{(-1)^n}{n!}\right]$$

$$\boxed{D_n \simeq n!\,[0.36]}$$

VIVEKANAND VERNEKAR

COMBINATORICS (Continued).

$(a+b)^2 = a^2 + 2ab + b^2$.
$= 1.a^2 + 2.ab + 1.b^2$.

$(a+b)$
$\downarrow$ a or b
$1.a^2 = 1$ way
we can take a^2 out.

$(a+b) \times (a+b)$ | $(a+b) \times (a+b)$
$a \times a$ | $a \times b + b \times a$
$2ab$.

* $2(ab)$
→ 2 ways we can take ab out.

* 2 boxes $(a+b)$
how many ways we can take '1b' out from 2 boxes.
$(_ + b) \times (_ + b)$
$\Rightarrow {}^2C_1\, ab$.

$(a+b)^2 = a^2 + 2ab + b^2$
$= 1.a^2 + 2ab + 1.b^2$.
$\Rightarrow {}^2C_0 . a^2b^0 + {}^2C_1\, ab + {}^2C_2\, a^0b^2$.

${}^2C_1$: 2 boxes, 1b out.
${}^2C_2$: 2 boxes, 2b's out.

→ Out of 2 boxes how many ways we take '2 b' at same time:
$(a+b) \times (a+b)$
$b \times b$ ✓ $\Rightarrow {}^2C_2$

* Power of a is decreasing & power of b is increasing.
* Power of a and b will always be 'n'.

$\Rightarrow (a+b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1}b^1 + {}^nC_2\, a^{n-2}b^2 \ldots\ldots {}^nC_n\, a^0b^n$.

$$\boxed{(a+b)^n = \sum_{i=0}^{n} {}^nC_i \times a^{n-i} \times b^i}$$

→ binomial co-efficient: no. of ways we can take b^i outside in (n) boxes.

Note:
${}^nC_x = {}^nC_{n-x}$

$\{a, b, c\}$ → no. of ways to take 1 element out $= {}^3C_1$. } no of ways we left '2' elements inside.

Q1 How many ways we can select 6 students in a class of 10?

Ans: $^{10}C_6$

S_1

or

Select → b | Not Select → a

$(b + a)$

S_2

Select → $\rightarrow {}^nC_i\, a^{n-i} b^i$

not select → $\rightarrow {}^{10}C_6\, a^4 b^6$

$(a+b) \times (a+b) \times (a+b) \ldots$

$b \times b \times b$

* Selecting 6 students means selecting 6b out of 10 bone.

$(a+b)^{10}$

$(a+b)^n = \sum_{i=0}^{n} {}^nC_i\, a^{n-i} b^i$

$\Rightarrow {}^{10}C_6\, a^4 b^6$

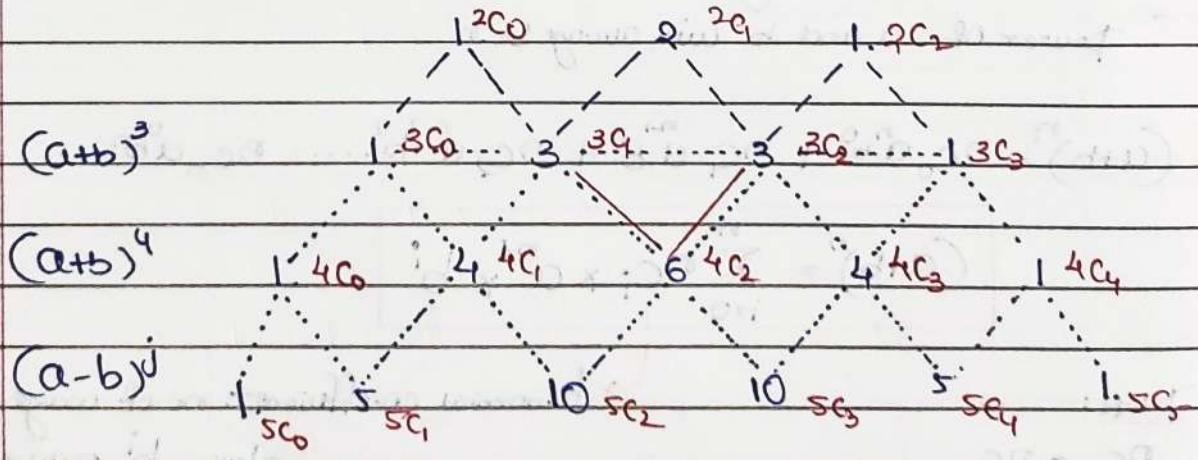
* Selecting 6 student
* Coefficient of b^6.

eg: $(a+b)^n = \sum_{i=0}^{n} {}^nC_i \cdot a^{n-i} \cdot b^i$

$a=1 \quad b=1$

$(1+1)^n = \sum_{i=0}^{n} {}^nC_i\, (1)^{n-i} (1)^i$

$2^n = \sum_{i=0}^{n} {}^nC_i = \boxed{{}^nC_0 + {}^nC_1 + {}^nC_2 \ldots {}^nC_n = 2^n}$



$^3C_2 + {}^3C_1 = {}^4C_2$

$3 + 3 = 6$

$\Rightarrow {}^nC_x + {}^nC_{x+1} = {}^{n+1}C_{x+1}$

→ $(x+y)^{30}$ co-efficient of $y^{10} = {}^{30}C_{10} = {}^{30}C_{20}$

→ $\underline{{}^{30}C_{10}\, x^{20} y^{20}}$

→ $(2x+3y)^{30}$ co-efficient of y^{10}. ${}^{30}C_{10}\, 2^{10}\, 3^{10}$

${}^{30}C_{10}(2x)^{20}(3y)^{10} \rightarrow \underline{{}^{30}C_{10}\, 2^{20} \cdot 3^{10} \cdot x^{20} \cdot y^{10}}$

Extended Binomial Co-efficient:

→ $^{11}C_2 = \frac{(11)!}{2! \times 9!} = \frac{11 \times 10}{2 \times 1} \Longrightarrow {}^{11}C_2 = \frac{11 \cdot 10}{2 \times 1}$

→ $^{11}C_3 = \frac{11 \times 10 \times 9}{3 \times 2 \times 1}$

$$^{n}C_k = \frac{n(n-1)\cdot(n-2)\cdots(n-k+1)}{k!}$$

$$^{-n}C_k = \frac{(-n)(-n-1)(-n-2)\cdots(-n-k+1)}{k!}$$

$$= \frac{(-n-k+1)\cdots(-n-2)(-n-1)(-n)}{k!} = (-1)^k \frac{(n+k-1)\cdots(n+2)(n+1)(n)}{k!}$$

$$= (-1)^k \frac{(n+k-1)!}{k! \times (n-1)!}$$

$\frac{(n-1)!}{(n-1)!}$ { ÷ by n/b by $(n-1)!$ }

$$\boxed{^{-n}C_k = (-1)^k \cdot {}^{n+k-1}C_k} \rightarrow \frac{(-1)^k (n+k-1)!}{k! \times (n+k-1-k)!} = \frac{(-1)^k (n+k-1)!}{k!\,(n-1)!}$$

eg: $^{-11}C_2 = {}^{12}C_2$ | $^{-11}C_2 = {}^{-13}C_2$ | $^{-1}C_0 = 1$ | $^{-1}C_1 = -1$

$^{-11}C_2 = {}^{-n}C_k$ → even (total +ve)
→ odd (total −ve)

$n=11$ $k=2$ $= (-1)^k\, {}^{n+k-1}C_k$

$= (-1)^2 \Rightarrow \underline{{}^{12}C_2}$

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Extended Binomial Co-efficient : (Continued)

$-n_{C_k} = (-1)^k \quad n+k-1_{C_k}$.

even → odd

Co-efficient → +ve. | Co-efficient → −ve.

eg: $-^{11}C_2 = +$

$-^{11}C_3 = -$

$$(a+b)^n = \sum_{i=0}^{n} {}^nC_i \times a^{n-i} \times b^i$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4.$$

$$\frac{1}{1-x} = (1-x)^{-1} = (1+(-x))^{-1} = {}^nC_0\, a^{n-0} b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 \ldots$$

$a=1$, $b=-x$, $n=-1$.

$$(a+b)^n = {}^{-1}C_0 (1^{-1-0})(-x)^0 + {}^{-1}C_1\, 1^{-1-1}(-x)^1$$

$$1 \times 1 \times 1 + (-1)\, 1\, (-x)$$

$$\hookrightarrow 1 + x + x^2 + (-1)(-1)$$

$$\frac{1}{1-ax} = (1+(-ax))^{-1}.$$

$$= \overset{+ve}{{}^{-1}C_0\, 1^{-1-0}(-ax)^0} + \overbrace{{}^{-1}C_1 (1)^{-1-1}(-ax)^{+1}}^{ax} + \overset{+ve}{{}^{-1}C_2 (1)^{-1-2}(-ax)^{+2}}.$$

$$1 + (-1).1 \times (-1)(ax) + (ax)^2 + (ax)$$

$$\Rightarrow \frac{1}{1-ax} = 1 + ax + (ax)^2 + (ax)^3 + (ax)^4 \ldots$$

→ $a=1$. $\frac{1}{1-x} = 1 + x + x^2 \ldots$ | → $a=2$ $\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 \ldots$

$$\Rightarrow \frac{1}{1+ax} = (1+ax)^{-1} = {}^{-1}C_0\, 1^{-1}(ax)^0 + {}^{-1}C_1\, 1^{-1-1}(ax)^1$$

$$\frac{1}{1+ax} = 1 - ax + (ax)^2 - (ax)^3 + (ax)^4 \ldots$$

classmate
Date
Page

Q1. How many ways to distribute 10 similar coins among 3 children such that each child gets atleast 2 coins and atmost 4 coins?

Ans atleast : 2 coins. } $2 \le x_i \le 4$
atmost : 4 coins.

♀	♀	♀
x^2	x^2	x^2
x^3	x^3	x^3
x^4	x^4	x^4

♀	♀	♀
2	2	2
3	3	3
4	4	4.

Giving 2 coins is represented as x^2,
3 coins — x^3...
4 coins — x^4.

One possible soln.
$= 2+4+4 = 10 \; (2 \le x_i \le 4)$.

Co-efficient of $x^{10} = ?$

$(x^2+x^3+x^4) \times (x^2+x^3+x^4) \times (x^2+x^3+x^4)$

$x^2+x^4+x^4 = x^{10}$ $(2+4+4)$. | $x^{e_1}+x^{e_2}+x^{e_3} = x^{10}$ | $e_1+e_2+e_3 = 10$

→ $(x^2+x^3+x^4)^3$.

*take x^2 common.

$(x^2)^3 [1+x+x^2]^3 \rightarrow x^6 \left[\frac{1-x^3}{1-x}\right]^3 = x^6 (1-x^3)^3 (1-x)^{-3}$.

$= x^6 [1-3x^3+3x^6-x^9][1-x]^{-3}$.

(expanding the equation).

$\Rightarrow x^6 [1-3x^3+3x^6-x^9] [{}^{-3}C_0 1^{-3} (-x)^0 + {}^{-3}C_1 1^{-3-1} (-x)^1 + {}^{-3}C_2 1^{-3-2} (x)^2 + \dots$

Co-efficient of x^{10}

$\Rightarrow [x^6 - 3x^9 + 3x^{12} - x^{15}] [\dots\dots]$

$\Rightarrow x^6 [{}^{-3}C_4 (1)(-x)^4] - 3x^9 [{}^{-3}C_1 \; 1 \cdot (-x)^1]$ $\boxed{\text{Ans} = 6}$

$\Rightarrow x^6 (15x^4) - 3x^9 (3(x)) \Rightarrow (15-9)\, x^{10}$.

$\Rightarrow 6$.

Q2. How many ways to distribute 12 coins among 4 students where each student get atleast 2 coins and atmost 5 coins?

Ans $(x^2+x^3+x^4+x^5)^4$ — children.
atleast, atmost

Co-efficient of $x^{12} = ?$

$(x^2+x^3+x^4+x^5)^4$

*take x^2 common

$(x^2)^4 [1+x+x^2+x^3]^4 \Rightarrow x^8 \left[\frac{1-x^4}{1-x}\right]^4$

$$\Rightarrow x^8 \frac{(1-x^4)^4}{(1-x)} = x^8(1-x^4)^4(1-x)^{-4}$$

$$\Rightarrow x^8(1-4x^4+6x^8\ldots)(1-x)^{-4}$$

$$(x^8-4x^{12}+\ldots)(1-x)^{-4}$$

$$\boxed{\text{Ans} = 31}$$

$$\Rightarrow x^8\left({}^{-4}C_4 \times 1(-x)^4\right) - 4x^{12} \Rightarrow \underline{31x^{12}}$$

Q3. How many ways we can distribute 15 coins among 4 children such that each student gets atleast 2 coins & atmost 6 coins?

Ans $(x^2+x^3+x^4+x^5+x^6)^4$ ← children

atleast atmost

Co-efficient of x^{15}

$$(x^2+x^3+x^4+x^5+x^6)^4$$

$$(x^2)^4(1+x+x^2+x^3+x^4)^4$$

$$x^8\left(\frac{1-x^5}{1-x}\right)^4 = x^8(1-x^5)^4(1-x)^{-4}$$

$$x^8\left({}^{-4}C_7(-x)^7\right) - 4x^{13}\left({}^{-4}C_2 1(-x)^2\right) = x^8(1-4x^5+6x^0\ldots(1-x)^{-4}$$

$$\left({}^{10}C_3(x^{15}) - 4\times{}^8C_2x^{15}\right) = (x^8-4x^{13}+\ldots)(1-x)^{-4}$$

$$\boxed{\text{Ans} \Rightarrow 80}$$

eg: x^{12}

$$(x^3+x^4+x^5+x^6\ldots)^3.$$

$$(x^3)^3.(1+x+x^2+\ldots)^3.$$

$$x^9\left(\frac{1}{1-x}\right)^3. \quad x^9.(1-x)^{-3}. \Rightarrow x^9\left({}^{-3}C_3(-x)^3\right)$$

$$= (-1)^3.\ {}^{3+3-1}C_3(-1)^3x^3.$$

$$= {}^5C_3.x^{12}$$

$$\Rightarrow \underline{\underline{10}}$$

Q4. There are 300 identical envelopes

25 envelope/package → 4 students

atleast $-\frac{150}{25} \rightarrow 6.$

atmost $1000 = \frac{1000}{25} = 40$

Co-efficient of x^{120}
$(x^6 + \dots x^{40})^4$

$(x^6)^4 [1 + x \dots x^{34}]^4$

$x^{24} \left[\frac{1 - x^{35}}{1 - x} \right]^4$

$x^{24} [1 - x^{35}]^4 [1 - x]^{-4} \Rightarrow x^{24} [1 - 4x^{35} + 6x^{70} - 4x^{105}]$

$(x^{24-4} C_{96} x^{96}) - 4x^{59}$

$\Rightarrow -4C_6 (-x^6) \Rightarrow +6x^{94} (-4C_{26} (-x)^{26})$

Generating Functions

$\langle a_0, a_1, a_2, a_3 \dots \rangle$

$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots$ closed form

$$\boxed{G(x) = \sum_{i=0}^{\infty} a_i x^i}$$

$\langle 1, 1, 1, 1, \dots \rangle$
$a_0 \quad a_1 \quad a_2 \quad a_3 \dots$

$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4$
$= 1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + 1 \cdot x^4$
$= 1 + x + x^2 + x^3 + x^4 \dots$

$$\boxed{G(x) = \frac{1}{1-x}}$$

$\rightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \dots$

$\frac{d}{dx}\left(\frac{1}{1-x}\right)$

$\frac{-1}{(1-x)^2} \frac{d}{dx}(-x)$

$= \frac{-1}{(1.x)^2}(-1) = \frac{1}{(1-x)^2}$

$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 \dots$

$\rightarrow 1, -1, 1, -1, 1 \dots$

$G(x) = \frac{1}{1+x}$

eg: 0, 0, 0, 6, −6, 6, −6

$G(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 \ldots$

$G(x) = \frac{6x^3}{1+x}$

$6x^3 - 6x^4 + 6x^5 - 6x^6 \ldots$

take $6x^3$ common,

$6x^3[1 - x + x^2 - x^3 + x^4 - x^5 \ldots]$

$\Rightarrow \frac{6x^3}{1+x}$

eg: $0, 0, 1, a, a^2, a^3 \ldots (a \neq 0) \rightarrow \frac{x^2}{1-ax}$

$1, 0, 1, 0, 1 \ldots \rightarrow \frac{1}{(1-x^2)}$

7, 8, 9, 10 .....

$G(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 \ldots$

$= 7.x^0 + 8x + 9x^2 + 10x^3 \ldots$

$= 7 + 8x + 9x^2 + 10x^3.$

$= 1 + 6 + 6x + 2x + 6x^2 + 3x^2 + 6x^3 + 4x^3.$

$\Rightarrow 6 + 6x + 6x^2 + \ldots + \boxed{1 + 2x + 3x^2 + 4x^3 \ldots}$

$\hookrightarrow \frac{6}{1-x}.$ $\hookrightarrow \frac{1}{(1-x)^2}.$

$\Rightarrow \frac{6}{1.x} + \frac{1}{(1-x)^2}.$

$\rightarrow G(x) = \frac{x^3}{1-x^2}$

$= x^3[1 + x^2 + x^4 + x^6 \ldots]$

$= x^3 + x^5 + x^7 + x^9 \ldots$

$G(x) = 0.x^0 + 0.x^1 + 0.x^2 \ldots \Rightarrow \langle 0, 0, 0, 1, 0, 1, 0 \ldots \rangle$

$\rightarrow G(x) = \frac{1}{3-x} = \frac{1}{3[1-x/3]} = \frac{1}{3} \times \frac{1}{(1-x/3)}$

$= 1/3\,[1 + (x/3) + (x/3)^2 + (x/3)^3 \ldots]$

$= \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} x + \frac{1}{3} \times \frac{1}{3^2} x^2 + \frac{1}{3} \times \frac{1}{3^3} x^3 + \dots$

$\Rightarrow \langle \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \dots \rangle$.

$\rightarrow G(x) = \frac{1}{1-x} + 3x^7 - 11.$

$= [1 + x + x^2 + x^3 + x^4 \dots] + [3x^7 - 11]$

$= (1-11) + x + x^2 \dots x^6 + (x^7 + 3x^7) + \dots$

$= -10 + \dots + 4x^7 + \dots$

$a_0 = -10 \quad \langle -10, 1, 1 \dots 4 \cdot 1 \dots \rangle$.

Recurrence Relation.

In a colony, there are 5 bacteria which is at present at time 0. Bacteria are increasing 2 times as the previous what will be the total bacteria at time 100?

a_0 : no. of bacteria at time : $0 \rightarrow 5$

a_{100} : no. of bacteria at time $100 \rightarrow$

Initial Condition : $a_0 = 5$.

$a_1 = 2 \times a_0$	$a^3 = 2 \times a_2$
$a_2 = 2 \times a_1 = 2(a_0 \times 2)$	$= 2(2^2 \cdot a_0)$
$= 2^2 \cdot a_0$	$a^3 = 2^3 \cdot a_0$

$\therefore \underline{a_n = 2^n \cdot a_0} \rightarrow O(2^n)$

Type-1 : $a_n = d \cdot a_{n-1}$

a_0 = initial condition

$\boxed{a_n = d^n \cdot a_0}$

$\rightarrow (a_{n+2}) = (3a_{n+1} - 2a_n) \quad (n \geq 0)$

$\sum a_{n+2} \cdot x^{n+2} = 3 \cdot \sum a_{n+1} \cdot x^{n+2} - 2 \sum a_n \cdot x^{n+2}$

$\sum_{n \geq 0} a_{n+2} \cdot x^{n+2} = 3 \cdot x \sum_{n \geq 0} a_{n+1} \cdot x^{n+1} - 2x^2 \sum_{n \geq 0} a_n \cdot x^n$

$G(x) - a_0 x^0 - a_1 x^1 = 3x(G(x) - a_0 x^0) - 2x^2 \cdot G(x)$

$$G(x) - 0x^2 - x = 3x(G(x)) - 2x^2 G(x)$$

$$G(x) - x = (3x - 2x^2)\, G(x)$$

$$G(x) + (3x + 2x^2) G(x) = x$$

$$G(x)\left[1 - 3x + 2x^2\right] = x$$

$$G(x) = \frac{x}{1 - 3x + 2x^2}.$$

$\rightarrow$ $$G(x) = \sum_{i \geq 0} a_i x^i = G(x) = \sum_{n \geq 0} a_n x^n$$

$$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 \ldots$$

$$G(x) = a_0 x^0 + a_1 x^1 + \sum_{n \geq 0} a_{n+2} \cdot x^{n+2}$$

$\rightarrow$ $$G(x) = \frac{x}{1 - 3x + 2x^2} = \frac{x}{(1-x)(1-2x)} = \left\{\frac{A}{1-x} + \frac{B}{1-2x}\right\} = \frac{\ldots\ldots}{(1-x)(1-2x)}$$

$$1 - 3x + 2x^2$$
$$1 - x - 2x + 2x^2$$
$$(1-x) - 2x(1-x)$$
$$(1-x)(1-2x).$$

$$= A(1-2x) + B(1-x)$$
$$= A(-2Ax) + B \cdot -Bx$$
$$\frac{x}{(1-x)(1-2x)} = \frac{(-2Ax - Bx) + (A+B)}{(1-x)(1-2x)}$$

$$G(x) = \frac{A}{1-x} + \frac{B}{1-2x} \qquad A = -1,\ B = 1.$$

$$= \frac{-1}{1-x} + \frac{1}{1-2x}$$

$$\boxed{G(x) = \left(\frac{1}{1-2x}\right) - \left(\frac{1}{1-x}\right)}$$

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 \ldots$$

$$= \sum_{n \geq 0} 2^n \cdot x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 \ldots\ldots$$

$$= \sum_{n \geq 0} 1^n \cdot x^n$$

$$\sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} 2^n \cdot x^n - \sum_{n \geq 0} 1^n \cdot x^n$$

$$\boxed{a_n = 2^n - 1}$$

→ $a_n = 2n+3$

$\sum a_n x^n = 2 \cdot \sum n x^n + 3 \sum x^n$ $(x \cdot \sum x^n)$

$G(x) = 2 \cdot \frac{x}{(1-x)^2} + \frac{3}{(1-x)}$

$\sum_{n \geq 0} n \cdot x^n$

$\frac{x}{(1-x)^2} = 0x^0 + 1x + 2x^2 + 3x^3$

$= x + 2x^2 + 3x^3 + 4x^4$

$x(1 + 2x + 3x^2 + 4x^3)$

→ $a_{n+2} = 3a_{n+1} - 2a_n$

$x^2 = 3x - 2$

$x^2 - 3x + 2 = 0$

Roots = 2, 1.

$n = 0$

$a_0 = 2^0 c_1 + 1^0 c_2$

Characteristics equation: $a_n = 2^n c_1 + 1^n c_2$

$a_n = 2^n \cdot 1 - 1^n$

$a_n = 2^n - 1$

→ $a_n = 5a_{n-1} - 6a_{n-2}$

$x^2 = 5x - 6$

$x^2 - 5x + 6$

Roots : 3, 2 Characteristic equation : $a_n = 3^n c_1 + 2^n c_2$

Type - 2 :

$a_n = X a_{n-1} + Y a_{n-2}$

Roots. R_1, R_2

Characteristic Eqn : $a_n = (R_1)^n c_1 + (R_2)^n c_2$

Type - 3

$a_n = X a_{n-1} + Y a_{n-2}$

Roots : R, R

C.E : $a_n = R^n c_1 + n \cdot R^n c_2$

$a_n = a_{n-1} + a_{n-2}$ $a_0 = 0, a_1 = 1$

$x^2 = x + 1$

$x^2 - x + 1 = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

→ $\frac{1 \pm \sqrt{5}}{2}$

C.E = $a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n c_1 + \left(\frac{1-\sqrt{5}}{2}\right)^n c_2$

$c_1 = \frac{1}{\sqrt{5}}$ $c_2 = -\frac{1}{\sqrt{5}}$

$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$

$a_n = 6a_{n-1} - 9a_{n-2}$

CE: $a_n = 3^n c_1 + n\,3^n c_2$

$n=0$

$a_0 = 3^0 c_1 + 0.3^0.c_2$

$1 = 1.c_1$

$c_1 = 1.$

$n=1$

$a_1 = 3^1 c_1 + 1.3^1 c_2$

$2 = 3.1 + 3c_2$

$-1 = 3c_2 \quad c_2 = -1/3.$

$$\boxed{a_n = 3^n + n.3^n(-1/3)}$$

Type 4:

$a_n = Xa_{n-1} + Y a_{n-2} + Z a_{n-3}$

Roots: $R_1, R_2, R_3,$ CE: $a_n = R_1^n c_1 + R_2^n c_2 + R_3^n c_3.$

Roots: $R_1\ R_1\ R_1$ CE: $a_n = R^n c_1 + R^n c_2 + R_1^n c_3.$

Roots: $R_1\ R_1\ R_1$ CE: $a_n = R^n c_1 + R^n c_2 + R^n c_3$

Euler-totient-Function: $\phi(n)$

no. of relative prime less than n

$\phi(4) = 2.$

1. $\gcd(1,4) = 1$ ✓
2. $\gcd(2,4) = 2$ ✗
3. $\gcd(3,4) = 1$ ✓
4. $\gcd(4,4) = 4$ ✗

$\phi(3) = 2$

1 ✓ 2 ✓ 3 ✗

2.

$\gcd(a,a) = a.$

Relative prime

Coprime

$\gcd(a,b) = 1$

$\gcd(3,5) = 1$

3,5 are relative prime

$\phi(12) = 4$

1 ✓	5 ✓	9 ✗
2 ✗	6 ✗	10 ✗
3 ✗	7 ✓	11 ✓
4 ✗	8 ✗	12 ✗

$\phi(12) = \phi(3 \times 4)$

$= \phi(\phi(3) \times \phi(4))$

$\gcd(3,4) = 1.$

condition to be satisfied.

$= 2 \times 2 \rightarrow 4$

$\phi(AB)$

$\phi(A) \cdot \phi(B)$

$\gcd(A,B) = 1$

$$\boxed{\begin{array}{c}\phi(AB) = \phi(A).\phi(B) \\ \gcd(A,B) = 1.\end{array}}$$

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$\phi(7) = 6$

$\left.\begin{matrix}1\\2\\3\\4\\5\\6\\\cancel{7}\end{matrix}\right\}$ $\gcd(_,7)=1$.

$\phi(p) = p-1$ { p i.e Prime no. }

$\phi(49) =$ Total elements − non relative prime

$= 49 - 7 = 49 - \left(\frac{49}{7}\right)$

$\phi(49) = 49 - \frac{49}{7}$

$\gcd(7, 49) = 7$ (non relative prime)

$\phi(7^2) = 7^2 - \frac{7^2}{7}$

$$\boxed{\phi(p^a) = p^a - \frac{p^a}{p} \begin{cases} a \geq 0 \\ \text{Prime} \end{cases}}$$

$n = P_1^a \cdot P_2^b \cdot P_3^c \ldots \begin{cases} a, b, c \ldots \geq 0 \\ P_1, P_2, P_3 \ldots \text{ Prime no.} \end{cases}$

$4 = 2^1 \cdot 3^0 \cdot 5^0 \ldots$

$5 = 2^0 \cdot 3^0 \cdot 5^1 \ldots$

$$\boxed{n = 2^k y \begin{cases} k \geq 0 \\ y \to \text{odd no.} \end{cases}}$$

$n = P_1^a \cdot P_2^b \cdot P_3^c \ldots$

$\phi(n) = \phi(P_1^a \cdot P_2^b \cdot P_3^c \ldots)$

$= \phi(P_1^a) \times \phi(P_2^b) \times \phi(P_3^c)$

$= \left(P_1^a - \frac{P_1^a}{P_1}\right) \times \left(P_2^b - \frac{P_2^b}{P_2}\right) \times \left(P_3^c - \frac{P_3^c}{P_3}\right)$

$= P_1^a \left(\frac{P_1 - 1}{P_1}\right) P_2^b \left(\frac{P_2 - 1}{P_2}\right) P_3^c \left(\frac{P_3 - 1}{P_3}\right)$

$$\boxed{\phi(n) = \frac{n(P_1-1)(P_2-1)(P_3-1)\ldots}{P_1 \cdot P_2 \cdot P_3}}$$

eg: $\phi(110)$.

$110 = 2 \cdot 5 \cdot 11$.

$\phi(n) = n \frac{(P_1-1)(P_2-1)(P_3-1)}{P_1 P_2 P_3}$.

$= 110 \frac{(2-1)(5-1)(11-1)}{2 \times 5 \times 11}$

$= 1 \cdot 4 \times 10 = \underline{40}$

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Recurrence Relations.

homogenous

$a_{n+1} = 6a_{n+1} - 7a_{n-2} + f(n)$

$f(n) = 0$

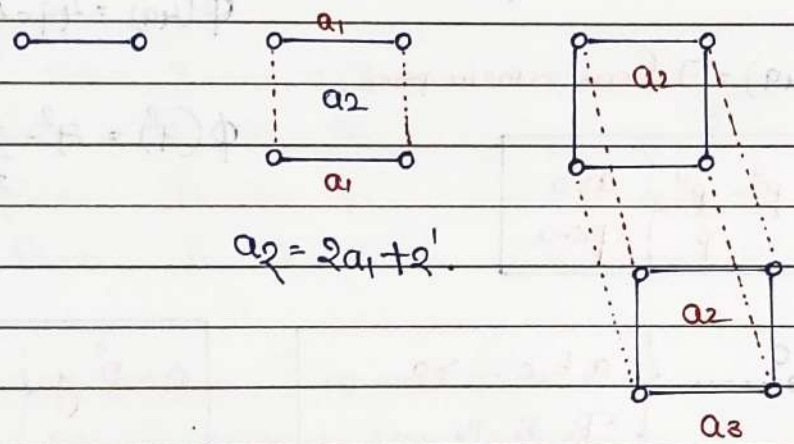
$a_n = 2a_{n-1} + f(n)$

non-homogenous

$a_n = x\,a_{n-1} + y\,a_{n-2} + f(n)$

$f(n) = 2^n$

$f(n) = n \cdot 2^n$



$a_2 = 2a_1 + 2^1$

$a_3 = 2a_2 + 2^2$

$a_2 = 2a_1 + 2^1$

$a_3 = 2a_2 + 2^2$

$a_{n+1} = 2a_n + 2^n$

$f(n) = 2^n$

→ $\boxed{a_n - 3a_{n-1} = 5(7^n)}$ $a_0 = 2$.

Total soln $= a_n = a_n^H + a_n^P$

$a_n - 3a_{n-1} = 0$

$x - 3 = 0$

$x = 3$

Roots:

$a_n^H = 3^n c$.

$\boxed{a_n^P = A(7^n)}$

$\frac{A(7^n)}{7^{n-1}} - \frac{3A(7^{n-1})}{7^{n-1}} = \frac{5(7^n)}{7^{n-1}}$

$\frac{A(7^n)}{7^n/7} - 3A = \frac{5(7^n)}{7^n/7}$

$7A - 3A = 5 \cdot 7$

$4A = 35$

$A = 35/4$

$a_n^P = \frac{35}{4}(7^n)$

$\boxed{a_n = a_n^H + a_n^P}$

$a_n = 3^n c + \frac{35}{4} \cdot 7^n$

$n = 0$

$a_0 = 3^0 c + \frac{35}{4} \cdot 7^0$

$2 = c + \frac{35}{4}$.

$\boxed{a_n = -\frac{27}{4} \cdot 3^n + \frac{35}{4} \cdot 7^n}$

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$a_{n+1} = 2a_n + 2^n \qquad a_0 = 0.$

$a_n = a_n^H + a_n^P.$

Root : 3 $\quad f(n) = 5(7^n) \quad A(7^n)$

Root : 3 $\quad f(n) = 3^n. \quad An(3^n)$

$a_{n+1} = 2a_n$

$a = 2$

Root : 2

$a_n^H = 2^n c.$

$a_n^P = An(2^n)$

$\frac{A(n+1)\cdot 2^{n+1}}{2^n} = \frac{2(An\,2^n)}{2^n.} + \frac{2^n}{2^n.}$

$2A(n+1) = 2An + 1$

$2An + 2A = 2An + 1$

$2A = 1.$

$A = 1/2.$

$a_n^P = \frac{1}{2} n (2^n)$

$a_n = a_n^H + \frac{1}{2} n (2^n)$

$a_n = 2^n c + 1/2\, n (2^n)$

$0 = 2^0 c + 1/2 \cdot 0 (2^0)$

$c = 0$

$a_n = \frac{1}{2} n (2^n)$

$\boxed{a_n = n \cdot 2^{n-1}}$

$a_n - 3a_{n-1} = 5(3^n) \qquad a_0 = 2.$

Homogeneous:

$a_n - 3\,a_{n-1} = 0$

$x - 3 = 0$

Root : 3

$a_n^H = 3^n c$

$a_n^P = An(3^n)$

$An(3^n) - 3A(n-1)\,3^{n-1} = 5.3^n.$

$\div$ by 3.

$\frac{An(3^n)}{3^n.} - \frac{3A(n-1)3^{n-1}}{3^n.} = \frac{5.3^n}{3^n}$

$An - A(n-1) = 5$

$An - An + A = 5.$

Root : 3 $\quad f(n) = 5(3^n)$

$\hookrightarrow a_n^P = An(3^n)$

$a_n^P = 5n(3^n)$

$\boxed{a_n = 3^n \cdot 2 + 5n \cdot 3^n}$

$a_n = a_n^H + a_n^P$

$a_n = 3^n c + 5n(3^n)$

$n = 0$

$2 = 3^0 c + 5\,0(3^n)$

$c = 2$

	$f(n)$	a_n^P
	5	A
Root : 3	$5(7^n)$	$A(7^n)$
Root : 3	$5(3^n)$	$An(3^n)$
	$5(3^n)$	$An^2(3^n)$

$f(n)$	a_n^P
n	$An+b$
n^2	An^2+bn+c
$n(7^n)$	$An+b(7^n)$

Root : 3,3

$a_n^H : 3^n c_1 + 3^n n . c_2$

$a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$

Root : 3,3

$a_n^H = 3^n c_1 + n . 3^n c_2$

$a_n^P = A(2^n) + Bn^2(3^n)$